HEAT CONDUCTION IN DISTRIBUTED SYSTEMS

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LIST OF SYMBOLS

Q	heat flow in Btu
t	time
T	temperature
x	distance
k	thermal conductivity
A	cross sectional area of the media
q	rate of flow of heat (dQ/dt)
V	volume
c_p	specific heat
ρ	density
α	$k/c_p\rho$ = thermal diffusivity
$T'(x)$	variation of temperature as a function of x only
$q'(x)$	variation of rate of heat flow as a function of x only
$f(t)$	time dependent characteristic function of the temperature distribution
$g(t)$	time dependent characteristic function of the rate of heat flow
ω	angular velocity of the driving function
Y	shunt admittance
Z	series impedance
T_s	source temperature
q_s	heat flow at source

Z_c characteristic impedance

Y_c characteristic admittance

γ propagation constant

T_r temperature at the receiving end

q_r heat flow at the receiving end

T^- reflected temperature wave

q^- reflected heat flow wave

T^+ incident temperature wave

q^+ incident heat flow wave

T_s^- reflected component of temperature source

T_s^+ incident component of temperature source

q_s^- reflected component of heat flow at source

q_s^+ incident component of heat flow at source

σ attenuation constant

β phase constant

λ space period or wave length

τ time period

v phase velocity

Θ characteristic impedance phase angle

t_{lag} time lag

Z_t total impedance

Z_r receiving end impedance

Γ_r reflection coefficient

ψ_r phase angle associated with the reflection coefficient

h_c coefficient of heat transfer due to connection

h_r coefficient of heat transfer due to radiation

h_t combined coefficient of heat transfer

P perimeter

S $h_t(P\alpha/kA)$

T_a average temperature about which the periodic variation takes place

T_e temperature of the surrounding environment

T_{actual} absolute temperature with respect to 0°F

Z_{tr} transfer impedance

Z_{tro} overall transfer impedance

n number of wave lengths on the receiving end

INTRODUCTION

The solution of heat conduction equations, when the applied heat source is a periodically varying function, is ordinarily obtained by one of two methods. The first of these is solution by Fourier Series. Although the accuracy obtained by this method is limited only by the pains a person is willing to take, the method is usually quite tedious and drawn out. The method also requires several integrations. The second method in common use is that of the separation of variables. In this method a solution is usually obtained in terms of a negative exponential term, the possibility of a positive exponential term being discarded. This method requires resolving for each individual problem with its own characteristic boundary conditions and leads to some degree of error if the body under consideration is of the finite class.

An attempt will be made to obtain one solution, applicable to all systems in which the driving function is periodic, in terms of the boundary conditions and apply the reflection type of analysis usually associated with electrical transmission lines. The method should have certain advantages and use could possibly be made of many transmission line techniques.

The nature of this thesis is purely investigative, of which the purpose is to see if the method is applicable and to see if it offers

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CHAPTER I

REVIEW OF THE LITERATURE AND STATE OF THE ART

1.1 Historical Notes

The transfer of heat has always been of primary concern to man since he comes into direct contact with it every day. The transfer of heat in our atmosphere and through our buildings affects the physical comfort of man, and man has learned that knowledge of the properties of bodies undergoing temperature variations can be a powerful tool, whether possessed by a blacksmith or a design engineer.

The theory of heat transfer has been revised several times as man has uncovered experimental evidence to support new theories. One of the first theories was the superstitious belief that heat was the evidence of the presence of an angry spirit. A more scientific theory was proposed by Lavoisier in the 18th century. This theory held that heat was a substance (caloric) that got between the particles of a body and made it hot. Still later Count Rumford proposed that heat was made manifest by the vibrations of the molecules of a body, and this theory is presently accepted.

It is now understood that heat may flow by three distinct mechanisms; radiation, convection, and conduction. The study in this thesis will be restricted to a study of the latter of these. The mathematical theory of heat conduction in solids is due primarily to Jean Baptiste

Joseph Fourier (1768-1830) and was set forth by him in his "Theorie Analytique de la Chaleur." (1). It was Fourier who first brought order out of the confusion in which the early experimental physicists had left the subject. While Fourier treated a great number of problems, his work was extended and applied to more complicated problems by his contemporaries, LaPlace and Poisson, and later by others including Lame and Thomson (2). To date, extensive work has been done, and there have appeared many fine texts on the subject of heat conduction. Of the recent works, one of the most elegant and authoritative is that done by Carslaw and Jaeger (3) in 1947. Some approximate methods are in use, and more recently the area of analogous systems has been explored and found to be of considerable value.

1.2 Basic Heat Conduction Equations

Fourier's law for the conduction of heat states that the instantaneous rate of heat flow dQ/dt is equal to the product of three factors: 1) The area A of the section, taken at right angles to the direction of flow; 2) The temperature gradient $-\partial T/\partial x$, which is negative due to the fact that temperature decreases in the direction of flow and which is a partial derivative, since heat flow and temperature variation as a function of time as well as position will be considered; and 3) The thermal conductivity (k) of the solid. Expressed mathematically this is:

$$q = -kA\partial T/\partial x \tag{1.1}$$

Fourier's equation expresses the conditions that govern the flow of heat in a body. It can be derived as follows, from the basic laws of heat conduction.

Consider the differential element of volume $dV = dx \cdot dy \cdot dz$ which has as one end the differential element of area $dA = dy \cdot dz$ and a temperature T at its center. This element is illustrated in Figure 1.

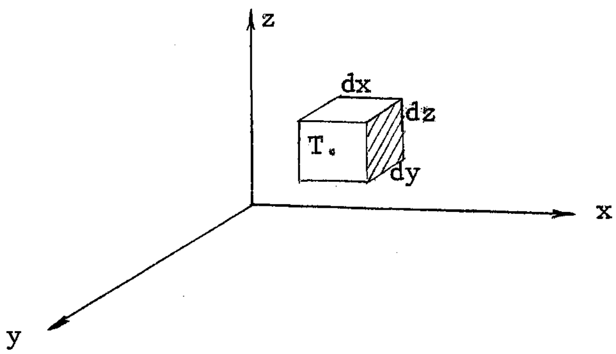

Figure 1. Differential Element of Volume

Considering temperature distribution in the x direction only, as will be done throughout this thesis, the temperature difference between the center and the two faces would be:

$$T = \partial T/\partial x \cdot dx/2$$

Then the temperature of the face nearest the source would be:

$$T_i = T + \partial T/\partial x \cdot dx/2$$

and that farthest from the source would be:

$$T_o = T - \partial T/\partial x \cdot dx/2$$

Therefore the heat flux into the volume would be:

$$q_i = k\ dA\ \partial T/\partial x = k\ dy\ dz\ \partial(T + \partial T/\partial x \cdot dx/2)/\partial x$$

The heat flux out through the other face would be:

$$q_o = k\ dy\ dz\ \partial(T - \partial T/\partial x \cdot dx/2)/\partial x$$

The difference in these two would then evidently give the rate of change of thermal energy of the differential volume.

$$dq/dt = k\ dy\ dz\ dx\ \partial^2T/\partial x^2 = d(dQ/dt)/dt$$

This change must also equal

$$dq/dt = c_p\ \rho\ dx\ dy\ dz\ \partial T/\partial t = d(dQ/dt)/dt$$

where c_p is the specific heat of the material

ρ is the density, and

$\partial T/\partial t$ is the temperature gradient with respect to time.

Therefore

$$k\ dx\ dy\ dz\ (\partial^2T/\partial x^2) = c_p\ \rho\ dx\ dy\ dz\ \partial T/\partial t$$

such that
$$\partial T/\partial t = \alpha(\partial^2T/\partial x^2) \tag{1.2}$$

where
$$\alpha = k/c_p\ \rho$$

and is called thermal diffusivity.

If heat flow in three dimensions were being considered, it would be similarly obtained that

$$\partial T/\partial t = \alpha(\partial^2T/\partial x^2 + \partial^2T/\partial y^2 + \partial^2T/\partial z^2) \tag{1.3}$$

Here, of course, it is assumed that k is constant over the temperature variation and that there are no sources nor sinks within the material.

1.3 State of the Art

Equations 1.1 and 1.3 are the two basic equations from which heat conduction equations are derived. The usual method of solution is to solve differential equation 1.3 in conjunction with various available boundary and initial conditions. If the rate of heat flow is then desired, this solution may be used in conjunction with equation 1.1 to obtain the desired equation. The common methods of solution of equation 1.3 are limited somewhat since certain boundary conditions must be known before a satisfactory solution can be obtained. This limitation leads to the necessity of finding a solution for every particular problem with its own characteristic boundary and initial conditions. These solutions have been treated extensively in the literature, and most cases have been developed thoroughly. The most commonly encountered methods of solution are the separation of variables method, the LaPlace transform method, and solution by Fourier sine and cosine series expansions. Each of the methods has its advantages and disadvantages. These will be discussed more fully in a later section.

Recently a number of approximate methods for solving heat conduction problems have been developed. These are especially useful when a quick evaluation is desired and accuracy is not too critical. The accuracy with these methods usually depends upon the pains one is willing to take. When very accurate results are desired however, the approximate methods lose their value. The approximate methods are useful also in that some of them help to get a better physical picture of the situation. This is true especially with respect to the method of isothermal surfaces and flow lines. (4). Other approximate methods are the Schmidt (5) method, the relaxation method (6), and the step method, to name only a few.

The similarity between certain physical phenomena in other areas and the phenomena encountered in heat conduction leads to the concept of mathematically analogous systems. These systems are termed mathematically analogous since the equations describing them have the same form. It follows that once a solution has been found for one system, the solution for the analogous system has the same form and can be found by changing the terms in the solution, to the corresponding terms in the desired system.

The use of analogous systems has led to a more refined study of heat transfer in certain areas. This is due to the fact that by the nature of other systems, it is easier to study certain phenomena than it is in heat conduction. Thus analogous systems lead us to a better understanding of the nature of heat conduction. When an experimental or analytical solution is not easily obtained in the system in which one is working, use can be made of this technique to convert to a system which has an easily obtained experimental solution.

Analog computers are a result of this work, and they have proved themselves of great value. Several elaborate analogs have been designed for use, solely for heat transfer studies. One of these is the Paschkis model at Columbia University, and others were made by Gelissen in Holland, Fisher and Muller in Germany, Miroux in France, Jackson and Lowson in England, and McCann in the United States. Probably the first of these though was constructed in 1934 by Beuken in Holland.

Many analogous systems exist, a few of which are the fluid flow analogy, the membrane analogy, and of course the electrical analogy. As an example of one analogous system, the corresponding terms for electrical and thermal systems are shown in Table I.

1.4 Definition of Steady State

More will be said about the electrical analogies later.

At this point, it might be well to make some distinguishing remarks as to terminology. It has been common practice in the field of

TABLE I

ANALOGOUS ELECTRICAL AND THERMAL TERMS

Electrical	Thermal
Charge = Q (coulomb)	Heat = Q (Btu)
Voltage = e (volt)	Temperature = t (°F)
Resistance = R (ohm)	Resistance = R (hr-°F/Btu)
Current = i (amps.)	Flow = q (Btu/hr)
Capacitance = C (farad)	Unit Capacity = $C_p \omega V$ (Btu/°F)

heat transfer, to refer to "steady state" as those conditions under which temperature or heat flow does not vary with respect to time. Departure will be made from this practice in this thesis, in that when reference is made to the steady state solution of a differential equation, the technical meaning is what is otherwise called a particular solution. Steady state conditions are now the conditions in the system after all transient or decaying components have become negligible. Thus it will be possible to encounter "periodic steady state" conditions. In this light, there may be steady state conditions, regardless of the form of the driving function.

With these thoughts in mind, advance can be made toward obtaining a steady state solution of the equations.

CHAPTER II

STEADY STATE SOLUTION OF BASIC EQUATIONS

2.1 Solution

Taking the partial derivative of equation 1.1 with respect to x,

$$\partial q/\partial x = -kA\,\partial^2 T/\partial x^2$$

Substituting for $\partial^2 T/\partial x^2$ from equation 1.2,

$$\partial q/\partial x = -kA\ (c_p\rho/k)\ \partial T/\partial t$$

or

$$\partial q/\partial x = -c_p\rho A\,\partial T/\partial t \qquad (2.1)$$

Now solving equation 1.1 for $\partial T/\partial x$,

$$\partial T/\partial x = -q/\ kA \qquad (2.2)$$

As mentioned previously, the usual method of solution is to solve equation 1.2. Attempt will be made instead to keep equations 2.1 and 2.2 separate and obtain steady state solutions. This method of solution will have advantages for this method of analysis, as will become evident later.

It is evident that solutions of these equations will give equations for q and T as functions of both time (t) and distance (x) or q(x,t) and T(x,t). Representing these functions as products of two characteristic functions,

$$T(x,t) = T'(x)\ f(t) \tag{2.3}$$

$$q(x,t) = q'(x)\ g(t) \tag{2.4}$$

The basic underlying assumption in obtaining the form of these two equations is that the steady state solution will be expressible as some temperature, which varies with time ($T'\ \ (x)\ \ $), plus a constant temperature term (T_a), about which $T'\ (x)$ varies. Throughout this thesis the assumption will be made that T_a is zero, since this term adds nothing to the analysis of the time variant distributions in the bar. In applications, T_a may be some non-zero value, and in this case, the actual temperature distribution may be found by,

$$T_{actual} = T(x,t) + T_a \tag{2.5}$$

This is valid, since with linear differential equations the principle of superposition holds. Note that heat flow is not affected by the value of T_a, since in equation 1.1, the constant term would drop out. Then

$$q_{actual} = q(x,t) \tag{2.6}$$

For this reason the primary concern will be only with an analysis of $T(x,t)$. The fact that equations 2.5 and 2.6 are solutions of equations 1.1 and 1.2, if equations 2.3 and 2.4 are, is easily seen by substitution.

The assumption is now made that the driving function is periodic with angular velocity ω, and if the transient effects are neglected

$$T(x,t) = T'(x)\ e^{i\omega t} \qquad (2.7)$$

$$q(x,t) = q'(x)\ e^{i\omega t} \qquad (2.8)$$

since the periodic driving function would result in a periodic response. (7). $T'(x)$ and $q'(x)$ represent the maximum instantaneous values of temperature and heat flow at any point x, or the amplitude as $T(x,t)$ and $q(x,t)$ vary periodically.

It should be noted that

$$e^{i\omega t} = \cos\omega t + i \sin\omega t$$

and then

$$\mathrm{Re}\ [T'(x)\ e^{i\omega t}] = \text{Real part of } T'(x)e^{i\omega t} = T'(x)\ \cos\omega t$$

and

$$\mathrm{Im}\ [T'(x)e^{i\omega t}] = \text{Imaginary part of } T'(x)e^{i\omega t} = T'(x)\sin\omega t$$

Then if a solution was desired for $T'(x)\ \sin\omega t$ or $T'(x)\ \cos\omega t$, it could be obtained merely by taking the imaginary or the real part respectively of the $T'(x)e^{i\omega t}$ solution. The same is true of course for $q(x,t)$.

The restriction to periodic driving functions is not a very limiting one since there can be obtained an approximation of any form of driving function with Fourier series, which is merely a series composed of sinusoidal functions. This gives the equations the desired flexibility.

Substituting equations 2.7 and 2.8 into equations 2.1 and 2.2,

$$\partial q/\partial x = -c_p\rho A\ [\partial(T'(x)e^{i\omega t})/\partial t] = -ic_p\rho A\omega T'(x)e^{i\omega t} = e^{i\omega t}[\partial q'(x)/\partial x]$$

and

$$\partial T/\partial x = [-q'(x)/kA]\ e^{i\omega t} = e^{i\omega t}\ [\partial T'(x)/\partial x]$$

Then

$$\partial q'/\partial x = -ic_p\rho A\omega T' = -YT' \tag{2.9}$$

and

$$\partial T'/\partial x = -1/kA\ q' = -Zq' \tag{2.10}$$

where

$$Y = i\ c_p\rho A\omega \tag{2.11}$$

and

$$Z = 1/kA \tag{2.12}$$

In the future reference will be made to $T'(x)$ and $q'(x)$ merely as T' and q'. The constant Z represents series impedance or a measure of the opposition to a change in temperature due to the heat flow at a point. Z has the form of what is commonly called thermal resistance in the literature. Similarly, Y represents the shunt admittance, or is a measure of the loss in heat flow at a point, as a result of the temperature varying with time at that point. Y has the form of thermal capacity. The usefulness of the concept of thermal resistance has been proven in the steady state case in which temperature at a point does not vary with respect to time, but little has been done with respect to an impedance analysis in the periodic case. It would now appear that such an analysis might be possible. This thought will be developed in section 8.2.

It should be noted that, due to the introduction of the time dependent function, the equations now become ordinary differential equations.

Proceeding with the solution, by differentiating equations 2.9 and 2.10 with respect to x,

$$d^2 q'/dx^2 = -Y\ dT'/dx$$

and

$$d^2T'/dx^2 = Z\ dq'/dx$$

Now substituting from 2.9 and 2.10

$$d^2q'/dx^2 = YZ\ q' \tag{2.13}$$

$$d^2T'/dx^2 = YZ\ T' \tag{2.14}$$

Both of these equations are of the form $(D^2 - YZ)\ T' = 0$, which has the solution

$$T' = A\ e^{\sqrt{ZY}\ x} + B\ e^{-\sqrt{ZY}\ x} \tag{2.15}$$

Similarly

$$q' = C\ e^{\sqrt{ZY}\ x} + D\ e^{-\sqrt{ZY}\ x}$$

But since

$$q' = -1/Z\,[dT'/dx] = -1/Z\ \sqrt{ZY}\ A\ e^{\sqrt{ZY}\ x} + \sqrt{ZY}/Z\ B\ e^{-\sqrt{ZY}\ x} = -\sqrt{Y/Z}\ A\ e^{\sqrt{ZY}\ x} + \sqrt{Y/Z}\ B\ e^{-\sqrt{ZY}\ x} \tag{2.16}$$

then

$$C = -\sqrt{Y/Z}\ A \text{ and } D = +\sqrt{Y/Z}\ B$$

2.2 Assumption of Boundary Conditions

There has now been obtained a general form for the solutions. If boundary conditions are assumed, evaluation can be made of the coefficients A and B and a more workable form found that lends itself to analysis. The assumption has already been made that the driving function will be periodic, so if the magnitude of this variation is indicated, there will then be sufficient boundary conditions to permit evaluation of A and B. Call this source amplitude, T_s. Then at $x = 0$

$$T' = T_s = A + B$$

$$q' = q_s = \sqrt{Y/Z}\ (-A + B)$$

Remember that T_s is the amplitude of the periodic variation, which takes place about some average value T_a, taken here as zero. For example, if the periodic source varies from 50°F to 150°F, then

$$T_a = 100^{\circ}F$$

and take T_s as

$$T_s = 50^{\circ}F$$

Solving for A and B,

$$q_s = \sqrt{Y/Z}\ (-A-A+T_s) = -2A\sqrt{Y/Z} + \sqrt{Y/Z}\ T_s$$

$$A = -[q_s - \sqrt{Y/Z}\ T_s]\ /2\sqrt{Y/Z} = \tfrac{1}{2}(-\sqrt{Z/Y}\ q_s + T_s)$$

$$B = T_s - A = T_s - \tfrac{1}{2}(-\sqrt{Z/Y}\ q_s + T_s) = \tfrac{1}{2}(T_s + \sqrt{Z/Y}\ q_s)$$

Giving

$$T' = \tfrac{1}{2}(T_s - \sqrt{Z/Y}\ q_s)\ e^{\sqrt{ZY}\ x} + \tfrac{1}{2}(T_s + \sqrt{Z/Y}\ q_s)e^{-\sqrt{ZY}\ x}$$

and

$$q' = \tfrac{1}{2}(q_s - \sqrt{Y/Z}\ T_s)e^{\sqrt{ZY}\ x} + \tfrac{1}{2}(q_s + \sqrt{Y/Z}\ T_s)e^{-\sqrt{ZY}\ x}$$

Now let

$$Z_c = \sqrt{Z/Y} = \text{Characteristic impedance} \qquad (2.17)$$

$$\gamma = \sqrt{ZY} = \text{Propagation constant} \qquad (2.18)$$

$$Y_c = 1/Z_c = \text{Characteristic admittance} \qquad (2.19)$$

giving

$$T' = \tfrac{1}{2}(T_s - Z_c q_s)e^{\gamma x} + \tfrac{1}{2}(T_s + Z_c q_s)e^{-\gamma x} \qquad (2.20)$$

and

$$q' = \tfrac{1}{2}(q_s - 1/Z_c\ T_s)e^{\gamma x} + \tfrac{1}{2}(q_s + 1/Z_c\ T_s)e^{-\gamma x} \qquad (2.21)$$

Expressions for T' and q', in terms of T_r, q_r, the temperature and heat flux at the receiving end, could be similarly obtained if such was desired. These would be found to be

$$T' = \tfrac{1}{2}(T_r + Z_c q_r)e^{\gamma x} + \tfrac{1}{2}(T_r - Z_c q_r)e^{-\gamma x}$$

and

$$q' = \tfrac{1}{2}(q_r + 1/Z_c T_r)e^{\gamma x} + \tfrac{1}{2}(q_r - 1/Z_c T_r)e^{-\gamma x}$$

where, in this particular case, x is measured from the receiving end. A discussion of γ and Z will be presented in the next section.

An equivalent, and sometimes more useful, form for equation 2.20 and 2.21 could be found by expressing them in hyperbolic notation.

$$\begin{aligned} T' &= T_s\ \tfrac{1}{2}(e^{\gamma x} + e^{-\gamma x}) - Z_c q_s\ \tfrac{1}{2}(e^{\gamma x} - e^{-\gamma x}) \\ &= T_s \cosh \gamma x - Z_c q_s \sinh \gamma x \end{aligned} \qquad (2.22)$$

and

$$\begin{aligned} q' &= q_s\ \tfrac{1}{2}(e^{\gamma x} + e^{-\gamma x}) - T_s/Z_c\ \tfrac{1}{2}(e^{\gamma x} - e^{-\gamma x}) \\ &= q_s \cosh \gamma x - T_s/Z_c \sinh \gamma x \end{aligned} \qquad (2.23)$$

These equations are the desired solution of equations 2.9 and 2.10. It should be recalled that equations 2.22 and 2.23 are functions of x only and represent the maximum amplitudes of T and q at any point as they vary periodically with respect to time. The complete solution of equations 2.1 and 2.2 may now be had merely by multiplying T' and q' by $e^{i\omega t}$, as mentioned in the first part of section 2.1. These are then

$$T = [T_s \cosh \gamma x - Z_c q_s \sinh \gamma x]e^{i\omega t} \tag{2.24}$$

$$q = [q_s \cosh \gamma x - 1/Z_c\, T_s \sinh \gamma x]e^{i\omega t} \tag{2.25}$$

which represent a periodic steady state solution.

CHAPTER III

ANALYSIS OF SOLUTION

3.1 The Wave Equation

In their present form, a casual inspection of equations 2.24 and 2.25 does not present a clear picture of the response. Therefore, further investigation and analysis is in order.

It is interesting to note at this point, that equations 2.9 and 2.10 are of the same form as the basic electrical transmission line equations

$$dE/dx = ZI$$

and

$$dI/dx = YE$$

where, as mentioned previously in connection with analogous systems, E is analogous to T' and I is analogous to q'. These equations do not have the negative sign that the equations used in this study do, because these were derived with x taken from the receiving end. Except for this difference, the solution is found to be exactly the same as the solution for the transmission line equations. One would then expect an analysis of the solution to closely parallel the transmission line analysis (8), and this is found to be true. It will also be found that the solution and analysis are very similar to that obtained by Norton and Freeny in their sucker rod research. (9).

The sucker rod solutions differ from the equations in this study due only to the effect of mass in the equations. The inclusion of mass in these equations affects the coefficients of the solution, however, and does not affect the form of the solution. Therefore, the analysis remains essentially the same.

The concept of a wave analysis of the heat conduction equations should not be hard to accept for recalling equations 2.1 and 2.2,

$$\partial q/\partial x = -c_p \rho A(\partial T/\partial t)$$

$$\partial T/\partial x = -q/kA$$

differentiate them with respect to x,

$$\partial^2 q/\partial x^2 = -c_p \rho A[\partial/\partial x(\partial T/\partial t)] = -c_p \rho A[\partial/\partial t(\partial T/\partial x)]$$

$$\partial^2 T/\partial x^2 = -1/kA\ \partial q/\partial x$$

and substitute from equations 2.1 and 2.2,

$$\partial^2 q/\partial x^2 = c_p \rho/k\ \partial q/\partial t \qquad (3.1)$$

and

$$\partial^2 T/\partial x^2 = c_p \rho/k(\partial T/\partial t) \qquad (3.2)$$

These two equations are recognized as special cases of the wave equation in its general form.

$$\nabla^2 \Phi = k_1{}^2(\partial^2 \Phi/\partial t^2) + k_2\ \partial \Phi/\partial t + g(x,y,z,t)$$

For systems in which there are no interior sources, the last term is not present and

$$\nabla^2 \Phi = k_1{}^2(\partial^2 \Phi/\partial t^2) + k_2\ \partial \Phi/\partial t \qquad (3.3)$$

When this general form of the wave equation is solved by the method of separation of variables, the space function must satisfy

$$\nabla^2 F = \kappa^2 F$$

where F is the space function or the function of Φ as a variable of x only. This is identical to equations 2.13 and 2.14, and therefore the solutions would be identical.

If the transient state were being considered, there would be some difference in the solution to the equations and the solution of equation 3.3, since the solution for the time functions would be different. However, since here the consideration is only with the steady state solution with an externally applied periodic driving function, both time solutions would be the same; i.e., of the form $e^{i\omega t}$ since the steady state response has the same time variation as the forcing function. (10). It is seen then that the equations 3.1 and 3.2 have steady state solutions which are identical with the steady state solution of the general wave equation. The equations should then lend themselves to a wave type of analysis.

3.2 Wave Analysis

In keeping with the line of thought of the previous section, advance will be made toward an interpretation of our solution as traveling waves. In order to facilitate this interpretation, rewrite equations 2.20 and 2.21 as

$$T' = T^- + T^+ = T_s^- e^{\gamma x} + T_s^+ e^{-\gamma x} \qquad (3.4)$$

and

$$q' = q^- + q^+ = q_s^- e^{\gamma x} + q_x^+ e^{-\gamma x} \qquad (3.5)$$

where

$$T_s^- = \tfrac{1}{2}(T_s - Z_c q_s) = A \qquad \text{a)}$$
$$T_s^+ = \tfrac{1}{2}(T_s + Z_c q_s) = B \qquad \text{b)} \qquad (3.6)$$

$$q_s^- = \tfrac{1}{2}(q_s - Y_c T_s) = -Y_c A \qquad \text{c)}$$
$$q_s^+ = \tfrac{1}{2}(q_s + Y_c T_s) = Y_c B \qquad \text{d)} \qquad (3.6)$$

Realize that each term of equation 3.4 is a temperature since their sum must be a temperature, and each term of equation 3.5 is heat flow since their sum must give heat flow. Recalling that at $x = 0$, $T' = T_s$, and $q' = q_s$, then

$$T_s = T_s^- + T_s^+, \quad q_s = q_s^- + q_s^+$$

It is now evident that T_s^- and T_s^+ are components of the driving function temperature, and q_s^- and q_s^+ are components of the heat flow at the source. All of these terms are complex numbers and have associated with them magnitudes and phase angles. This is due to the complex form of Z_c and will become more apparent as progress is made. Since these terms are independent of x, the manner in which temperature and heat flow is distributed along a bar is determined entirely by the $e^{+\gamma x}$ and $e^{-\gamma x}$ terms.

From its definition in section 2.2, γ will be a complex number and can be represented by

$$\gamma = \sigma + i\beta$$

Then multiplying equations 3.4 and 3.5 by $e^{i\omega t}$,

$$T = T' e^{i\omega t} = T_s^- e^{\sigma x + i(\omega t + \beta x)} + T_s^+ e^{-\sigma x + i(\omega t - \beta x)} \qquad (3.7)$$

and

$$q = q' e^{i\omega t} = q_s^- e^{\sigma x + i(\omega t + \beta x)} + q_s^+ e^{-\sigma x + i(\omega t - \beta x)} \qquad (3.8)$$
$$= Y_c T_s^- e^{\sigma x + i(\omega t + \beta x)} + Y_c T_s + e^{-\sigma x + i(\omega t - \beta x)}$$

Since the equations for T and q are so similar, there is need to treat only the temperature equation throughout the rest of this section and realize that the analysis for T will apply equally as well to q.

Consider first the term $T_s^+ e^{-\sigma x + i(\omega t - \beta x)}$ or $T_s^+ e^{-\sigma x} e^{i(\omega t - \beta x)}$. Since both σ and x are real and x increases from the sending end at x = 0, it is seen that the term $T_s^+ e^{-\sigma x}$ decreases as one moves away from the sending end or as x increases. The $e^{i(\omega t - \beta x)}$ term is simply the product of two harmonic functions, one of which varies as a function of time and the other as a function of space. The time function is the reproduction of the source variation in the response, as previously discussed. The space function may be represented by

$$e^{-i\beta x} = \cos\beta x - i \sin\beta x = 1 \underline{|-\beta x}$$

which always has a magnitude of 1, but an associated angle which decreases as x increases. Therefore, as one moves away from the source at x = 0 and moves toward the receiving end, the T^+ term is decreased in magnitude and retarded in phase.

A traveling wave is characterized by a retardation of phase and usually by a decrease in magnitude in the direction of travel. Therefore the second term of equation 3.7 may be interpreted as a temperature traveling wave, traveling toward the receiving end, and the second term of equation 3.8 as a heat flow traveling wave, traveling in the same direction.

This wave, as it travels along the rod, or the distribution in the body at some instant of time t is shown in Figure 2. The periodic response must be enclosed within the envelope formed by the $T_s^+ e^{-\sigma x}$ term.

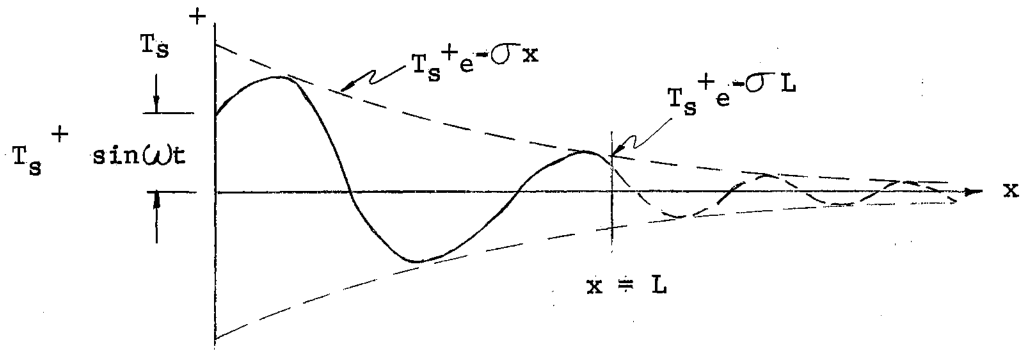

Figure 2. Traveling Wave

While the temperature at the source varies periodically with time as shown in Figure 3, the temperature at some distance along the x axis, x, in the body also varies periodically with a maximum amplitude, determined by $T_s^{+} e^{-\sigma x}$ as in Figure 4 and lagging behind the source temperature by some angle determined by βx.

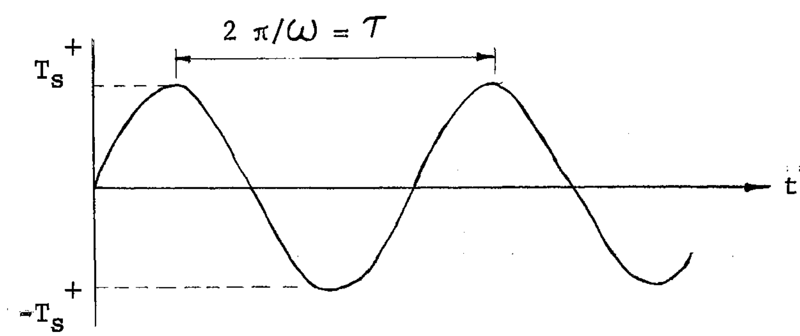

Figure 3. Temperature Variation at the Source

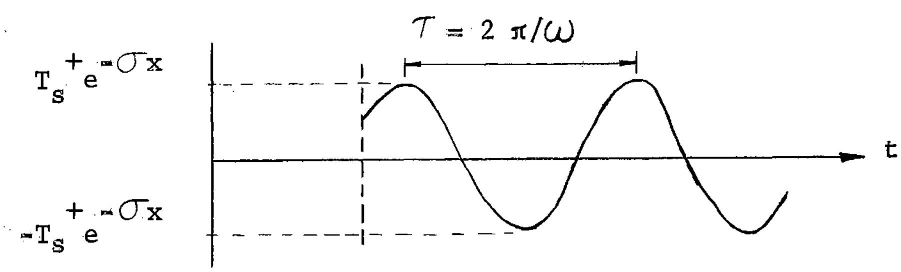

Figure 4. Temperature Variation at Some Point x

Examination of the first term of equations 3.7 and 3.8 shows that as x increases, T^- is increased in magnitude by $e^{\sigma x}$ and increased in phase by the $e^{i\beta x}$ term. However, if this term is examined as one moves from the receiving end at x = L toward the source at x = 0, it is seen that T^- is decreased in magnitude and decreased in phase which is just the reverse process from moving from x = 0 to x = L. Therefore, it can be interpreted that the first term in these equations is a traveling wave, traveling from the receiving end at x = L. This wave has a distribution in x at any time t as shown in Figure 5. This wave is enclosed within the envelope formed by $T_r^- e^{\sigma(x - L)}$.

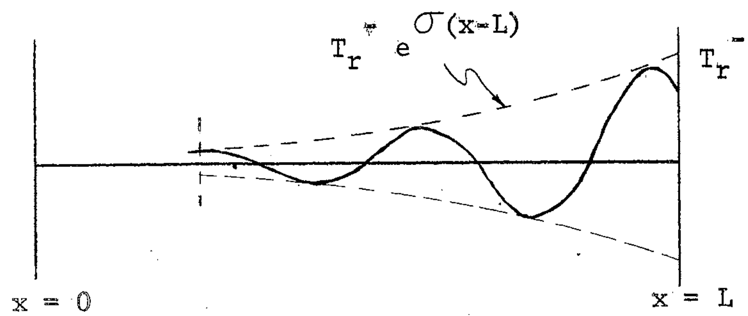

Figure 5. Distribution of the Reflected Wave

The waves traveling toward the receiving end are called incident waves, and those traveling toward the source, are called reflected waves. These waves have properties which are commonly encountered in other types of waves, such as electromagnetic waves on a transmission line, water waves, sound waves, and light waves. Some of these properties are phase shift, wave length, frequency, velocity of propagation, and reflection.

3.3 Propagation Constant

The complex quantity γ, which determines the change in phase and magnitude of each component wave per unit distance traveled, is called the propagation constant. The real part of the propagation constant σ, determines the rate at which the magnitude is decreased or attenuated as x increases and is called the attenuation constant. The imaginary part of the propagation constant β, determines the change in phase of the wave per unit distance and is called the phase constant.

$$\sigma = \mathrm{Re}\,\gamma$$
$$\beta = \mathrm{Im}\,\gamma$$

Since the exponent of e in the term $e^{i(\omega t-\beta x)}$ must be radians, it follows that ω must have units of radians per unit time, and β must have units of radians per unit length.

To evaluate σ and β, recall that

$$\gamma = \sqrt{ZY} = \sqrt{ic_p\rho\,\omega/k}$$

Then

$$\gamma^2 = ZY = \sigma^2 - \beta^2 + 2i\sigma\beta = ic_p\rho\omega 1/k$$

equating real and imaginary parts

$$\sigma^2 - \beta^2 = 0$$

or $$\sigma = \pm\beta$$

and $$\sigma\beta = \tfrac{1}{2}\, c_p\rho\omega 1/k$$

substituting $$\pm\beta^2 = \tfrac{1}{2}\, c_p\rho\omega 1/k$$

Since σ and β are real numbers by definition, take the positive sign, and then

$$\beta = \sigma = \sqrt{c_p\rho\,\omega/2k} \tag{3.9}$$

Then

$$\gamma = \sqrt{c_p \rho \omega / 2k}\ (1 + i) \tag{3.10}$$

If a particular point is picked on a traveling wave at time t, and distance x, and neglect attenuation for the moment, then

$$T^+ = T_s^+ e^{i(\omega t_1 - \beta x_1)}$$

If a second point is picked x_2, t_2, such that

$$\omega t_1 - \beta x_1 = \omega t_2 - \beta x_2$$

it follows that

$$T_s^+ e^{i(\omega t_1 - \beta x_1)} = T_s^+ e^{i(\omega t_2 - \beta x_2)}$$

These two points represent the same wave condition, or they are equiphase. This means that a fixed point on the traveling wave has moved from point x_1 to point x_2 in time $t_2 - t_1$. Therefore, the phase velocity or velocity of propagation of our traveling wave is defined by

$$\omega t - \beta x = \text{constant}$$

$$\omega dt - \beta dx = 0$$

$$dx/dt = v = \omega/\beta = \text{phase velocity} \tag{3.11}$$

This is illustrated by Figure 6.

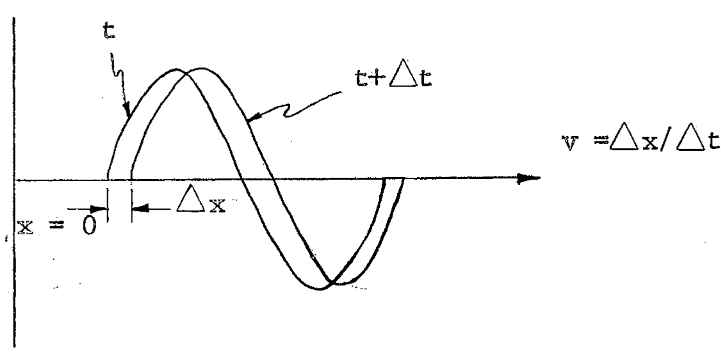

Figure 6. Displacement of Wave in Time Δt

The traveling wave $T_s^+ e^{i(\omega t - \beta x)}$ is then traveling toward the receiving end with a velocity v. Upon investigating the reflected wave, it is seen that it travels from the receiving end to the source with the same velocity v. This result proves the validity of our assumption of two traveling waves.

To visualize more completely the physical picture, it is instructive to re-introduce the damping term ($e^{-\sigma t}$) and sketch Figure 6 again, both for the incident wave and the reflected wave as shown in Figures 7 a) and 7 b).

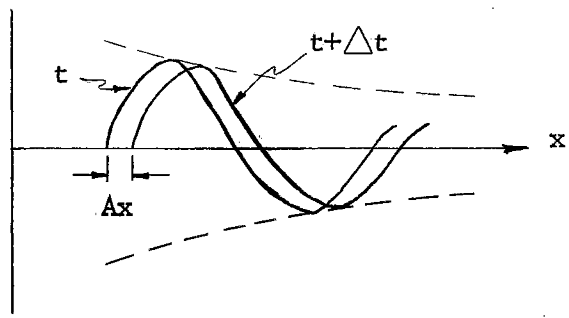

Figure 7 a). Incident Wave

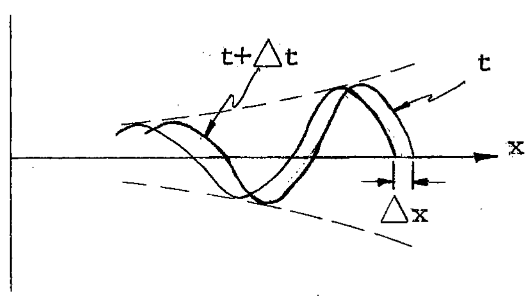

Figure 7 b). Reflected Wave

The temperature at any point x at any time t is then the sum of these two components.

From the definition of β, the change in phase in distance x is βx radians. Since wave length is the distance required for the phase to change a whole revolution of 2π radians, then

$$\beta \lambda = 2\pi$$

or

$$\lambda = \text{wave length} = 2\pi/\beta \qquad (3.12)$$

λ might also be referred to as the space period. Similarly, there is a time period given by

$$\tau = 2\pi/\omega \tag{3.13}$$

The time period, of course, is determined entirely by the driving source. The time period is indicated on Figure 3 and Figure 4.

3.4 Characteristic Impedance

As previously defined,

$$\begin{aligned} Z_c &= \sqrt{Z/Y} = \sqrt{1/kc_p\rho A^2 i\omega} = \sqrt{-i/kc_p\rho A^2\omega} \\ &= \sqrt{1/2kc_p\rho A^2\omega}\ (1-i) \end{aligned} \tag{3.14}$$

using the identity $\sqrt{-i} = (1-i)/\sqrt{2}$. If the relationships are investigated among the equations 3.6 a, b, c, and d, it is found that the ratio of the incident temperature wave to the incident heat flow wave is

$$T^+/q^+ = T_s^+ e^{-\gamma x}/q_s^+ e^{-\gamma x} = T_s^+/q_s^+ = Z_c \tag{3.15}$$

and the ratio of the reflected temperature wave to the reflected heat flow wave is

$$T^-/q^- = T_s^- e^{\gamma x}/q_s^- e^{\gamma x} = T_s^-/q_s^- = -Z_c \tag{3.16}$$

This shows that the ratio of incident temperature to incident heat flow at any point x is a constant and is independent of any terminal conditions. This constant, as previously noted, is called the characteristic impedance of the body. The reciprocal of characteristic impedance is called the characteristic admittance, Y_c.

Since Z_c is a complex number, it must have associated with it a magnitude and a phase angle. From equation 3.14, this form is

$$Z_c = \sqrt{1/kc_p\rho A^2\omega}\ \angle{-45^\circ}$$

Therefore the incident temperature wave lags behind the incident heat flow wave by a phase angle of 45° or $\pi/4$ radians. Do not mistakenly suppose that this is always true in all systems, because if more complicated cases are considered, such as an uninsulated bar with surface losses, it is definitely not true. However it is always true for the case of the uniform, perfectly insulated body that is considered at present. The more complicated cases are approached in a later section.

The concept of lag in heat conduction systems has some potential value in the analysis of these systems. For example, one could determine the time lag between the time that the maximum temperature outside a building was obtained and the time that the maximum heat flow into the interior of the building was obtained. There are many similar applications some of which will be investigated later. The time lag is the time required for the wave to move from the surface to a point x.

When the time lag at a point x is desired and the phase velocity is known, it can be expressed as

$$t_{lag} = x/v = x\,\beta/\omega \qquad (3.17)$$

The minus sign associated with equation 3.16 is due to the fact that the positive direction for heat flow is the same for both waves; that is, the direction of travel of the incident wave. The minus sign arises since the temperature is always positive and the heat flow in the reflected component is in the direction x = L to x = 0, acting as interference to the incident wave; the negative sign is necessary.

3.5 Reflection Coefficient

Traveling waves going in both directions could be produced by a source at both ends. However, from man's experience with sound waves being reflected from a cliff and light rays being reflected by a mirror, it would be reasonable that the backward waves or reflected waves could also be a reflection of the incident wave. In the considered case, the incident wave is due to the source, and the reflected wave is a reflection of the incident wave as it is seen at the receiving end. It follows that, if the body is infinitely long, there will be no reflection and no reflected wave. This follows since at $x = \infty$, the incident wave has decayed to zero, and there can be no reflection. For this case

$$Z_t = T/q = T'/q' = T_s^+/q_s^+ = Z_c \tag{3.18}$$

from equation 3.15. Then for the infinite rod, the temperature is equal to the characteristic impedance times the heat flow, or the total impedance is equal to the characteristic impedance.

If a finite insulated rod is then perfectly connected to an infinite insulated rod with characteristic impedance Z_{c2}, as in Figure 8,

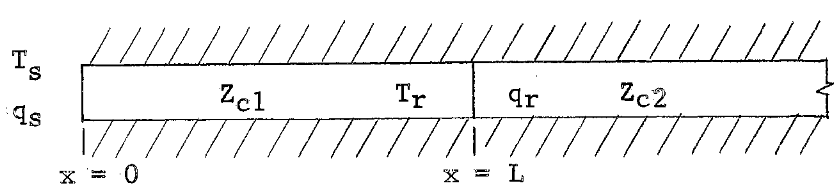

Figure 8. Composite Rod

the temperature at $x = L$ is

$$T_r = Z_{c2} q_r = Z_r q_r \tag{3.19}$$

Here Z_{c2} will be referred to as the terminating impedance or the receiving end impedance Z_r. If the conditions at the point $x = L$ are examined, there can be found significant information concerning the reflected component. At point $x = L$ there is an incident temperature T_r^+ and a reflected temperature T_r^-, as discussed previously. Since under steady state conditions there should be continuity of heat flow at any point, it can be written at point $x = L$,

$$q_{r\ in} = q_{r\ out} \tag{3.20}$$

The total temperature at $x = L$ will be the sum of the two components

or $$T_r = T_r^+ + T_r^-$$

Since in the infinite bar

$$q_r Z_r = T_r$$

then $$q_{r\ out} = T_r/Z_r = (T_r^+ + T_r^-)/Z_r \tag{3.21}$$

Then recalling equation 3.15 and 3.16,

$$q_{r\ in} = q_r^+ + q_r^- = T_r^+/Z_c - T_r^-/Z_c \tag{3.22}$$

Substituting 3.21 and 2.22 into equation 3.20,

$$(T_r^+ - T_r^-)/Z_c = (T_r^+ + T_r^-)/Z_r$$

and

$$T_r^-(-1/Z_c - 1/Z_r) = T_r^+(1/Z_r - 1/Z_c)$$

$$= T_r^-(-Z_r - Z_c/Z_rZ_c)$$

Then

$$T_r^-/T_r^+ = Z_r - Z_c/Z_r + Z_c = \Gamma_r \tag{3.23}$$

Γ_r is called the reflection coefficient and depends entirely upon the characteristic impedance of the body and the terminating impedance. As is shown above, the reflection coefficient is the ratio of the reflected temperature wave to the incident temperature wave.

Considering again the case of the infinite rod, which can be obtained, in this case, by saying that the infinite section is merely an extension of the finite bar, it is found that

$$Z_r = Z_c$$

and $$\Gamma_r = (Z_c - Z_c)/2Z_c = 0$$

or in an infinite rod, there is no reflection. This result confirms the previous analysis of the infinite rod.

If in Figure 8 the infinite rod is not of the same material as the finite rod but has other parameters, such that

$$Z_{c2} = Z_r = Z_{c1}$$

and $$\Gamma_r = 0$$

there is again no reflection in the finite bar. In this composite bar, there is a simulation of the conditions for the infinite bar, although the two do not necessarily propagate and attenuate their traveling waves in the same manner. The point is, however, that the temperature and heat flow distribution in the finite rod is identical with the length from x = 0 to x = L in the infinite case.

There is another critical value of the reflection coefficient when the receiving end is perfectly insulated or $Z_r = \infty$. In this case,

$$\Gamma_r = (\infty - Z\ )/\infty + Z \ = \infty/(\infty + Z) - 0 = 1\underline{|0°}$$

and there is now perfect reflection or the case in which the reflected wave at x = L is exactly equal to the incident wave at x = L. Of course, it is physically impossible to obtain perfect insulation and thus perfect reflection, but such conditions can be approximated if Z_r is made very large with respect to Z_c. This case is analogous to the case of an open circuit in the transmission line analysis. Since Γ_r is a complex number, it may be expressed as a magnitude times a phase angle or

$$\Gamma_r = |\Gamma_r| \angle \psi_r \tag{3.24}$$

The corresponding ratio of reflected heat flow to incident heat flow would be

$$q_r^-/q_r^+ = Z_c/-Z_c \; T_r^-/T_r^+ = -\Gamma_r = |\Gamma_r| \angle -\psi_r \tag{3.25}$$

The difference in sign between equations 3.24 and 3.25 is necessary in order for them to be consistent with equations 3.15 and 3.16.

For the case where the receiving end is perfectly insulated, then

$$\Gamma_r = 1 = 1 \angle 0°$$

so that the reflected temperature wave is of the same magnitude and phase as the incident wave at x = L. The ratio of the reflected heat flow to the incident heat flow is

$$q_r^-/q_r^+ = -1 = 1 \angle 180°$$

so that the reflected temperature wave is of the same magnitude as the incident wave, but 180° out of phase with it. This situation is illustrated in Figures 9 a) and b).

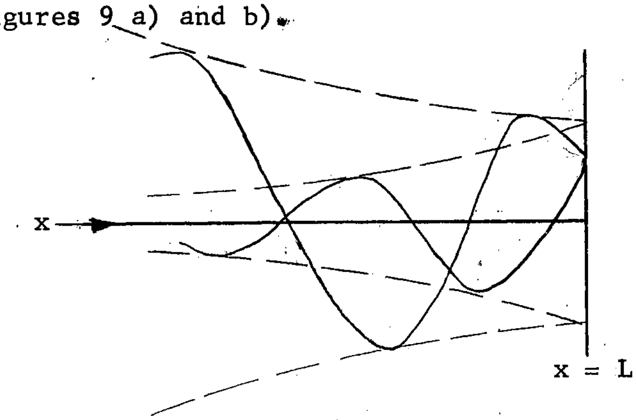

Figure 9 a). Reflected Temperature Wave for $\Gamma_r = 1$

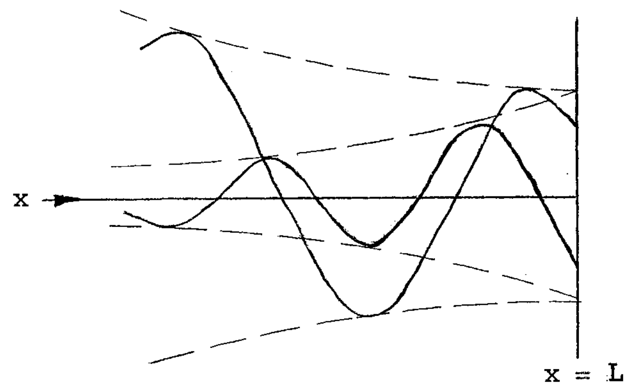

Figure 9 b). Reflected Heat Wave for $\Gamma_r = 1$

In general it can be said that the reflected temperature wave has a magnitude $|\Gamma_r|\ T_r^+$ and is shifted in phase from the incident wave by ψ_r. The reflected heat flow wave has a magnitude $|\Gamma_r|\ q_r^+$ and is shifted in phase from the incident wave by $\psi_r + \pi$.

Now consider the other extreme, or the case where $Z_r = 0$. This case corresponds to the case of a short circuit in transmission line analysis. This case in this problem is physically impossible, but it can be approximated by making Z_r very small with respect to Z_c. For this case,

$$\Gamma_r = -1$$

This indicates that perfect reflection is again present, except that this time, there is phase reversal of temperature and no phase change in the heat flow.

As a summary of the critical cases:

If $Z_r = Z_c$(or $L = \infty$), then $\Gamma_r = 0$. There is no reflected wave.

If $Z_r = \infty$ (perfect insulation), then $\Gamma_r = +1$. The incident wave is totally reflected with phase reversal of heat flow, but no phase change in temperature.

If $Z_r = 0$, then $\Gamma_r = -1$. The incident wave is totally reflected with phase reversal of temperature, but no phase change in heat flow. This condition is illustrated in Figures 10 a) and b).

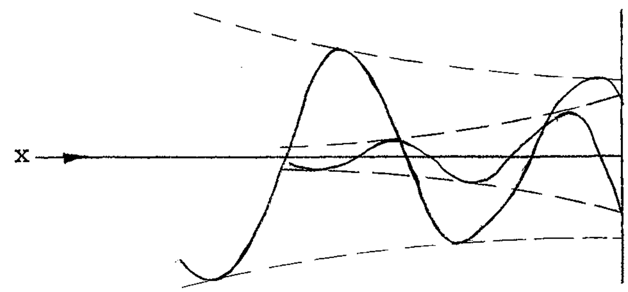

Figure 10 a). Reflected Temperature Wave for $\Gamma_r = -1$

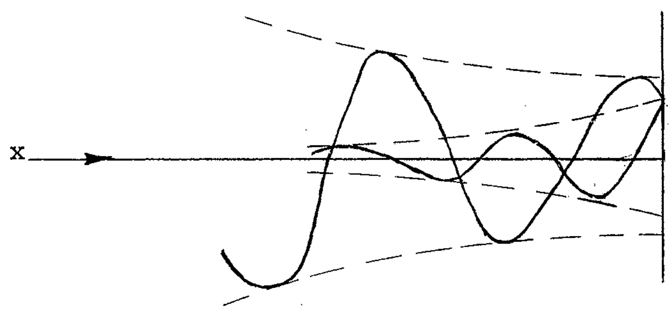

Figure 10 b). Reflected Heat Flow Wave for $\Gamma_r = -1$

CHAPTER IV

INFINITE BAR EQUATIONS

4.1 Form of the Equations

There has been previously mentioned the special conditions for an infinite insulated bar. There will be developed in this section a further analysis of the infinite bar and an investigation of the temperature and heat flow distribution in this bar.

For completeness recall the conditions that have been associated with the infinite insulated bar.

$$Z_r = Z_c$$

$$\Gamma_r = 0$$

$$T/q = Z_c$$

Then equations 3.6 become

$$T_s^- = 0$$

$$T_s^+ = T_s$$

$$q_s^- = 0$$

$$q_s^+ = T_s/Z_c$$

Substituting these values in equations 3.7 and 3.8,

$$T = T_s e^{-\sigma x + i(\omega t - \beta x)} \tag{4.1}$$

and

$$q = T_s/Z_c \; e^{-\sigma x + i(\omega t - \beta x)} = T/Z_c \tag{4.2}$$

4.2 Interpretation of Equations

Equation 4.1 can be rewritten as

$$T = T_s e^{-\sigma x - i\beta x} e^{i\omega t} \tag{4.3}$$

A sketch of the $T_s e^{-\sigma x - i\beta x}$ portion of this equation in three dimensions, as a function of the complex plane and x is shown in Figure 11.

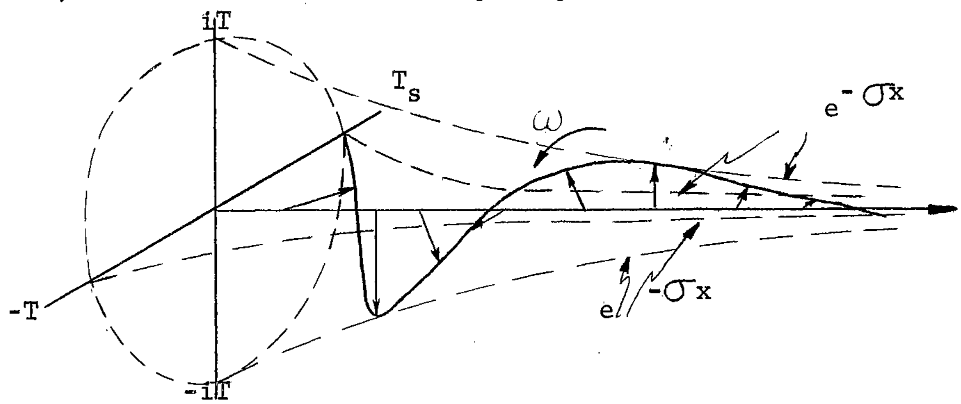

Figure 11. Traveling Wave at $t = 2n\pi/\omega$

The form of this sketch becomes evident if it is noted that

$$T_s e^{-\sigma x} e^{-i\beta x} = T_s e^{-\sigma x} (\cos \beta x - i \sin \beta x) \tag{4.4}$$

This plot is independent of time, although it is the plot of T at $t = 2n\pi/\omega$, where n is an integer sufficiently large to assure steady state conditions. This could be considered as a series of phasors separated by a distance Δx. It is recalled from alternating circuit theory that a phasor is merely a vector in the complex plane that has some magnitude and phase angle associated with it. Any two adjacent phasors then have a difference in phase angle of $\beta\Delta x$. Then as x increases, the phasors spiral around the x axis, and they are decreased in magnitude according to $e^{-\sigma x}$. This curve has the form of the thread of a left hand wood screw as $\Delta x \rightarrow 0$.

Reintroducing the time variant portion,

$$T = T_s e^{-\sigma x}[\cos (\beta x - \omega t) - i \sin (\beta x - \omega t)] \tag{4.5}$$

which corresponds to equation 4.4. Then as t increases, the curve rotates in a counterclockwise direction about the x axis. This motion rotates the curve in this direction with an angular velocity, given by ω as shown in Figure 11. This may now be visualized as the thread of a left hand wood screw rotating with an angular velocity ω about its axis, in the right hand or counterclockwise direction.

Since the angular velocity ω is constant and the same at all points, x, the phasors rotate about the x axis with this velocity and thus become sinors. Again referring to alternating circuit theory, it is recalled that a sinor is a phasor with some angular velocity about its origin. To facilitate the study of the temperature variation at any point x, note that it can be represented as a phasor with magnitude $T_s e^{-\sigma x}$ and initial angle of $-\beta x$. This concept agrees with equation 4.5. If $x = 0$

$$T = T_s[\cos(-\omega t) - i\sin(-\omega t)]$$

$$= T_s[\cos\omega t + i\sin\omega t] = T_s e^{i\omega t}$$

This result agrees with the assumption of a periodic driving function as was discussed in section 2.1. If the temperature at the source is a sine function, $T_s \sin \omega t$, the temperature at any point x may be obtained by taking the imaginary part of equation 4.5.

$$\mathrm{Im}[T] = T_s e^{-\sigma x} \sin(\omega t - \beta x)$$

If the source temperature is a cosine function, $T_s \cos \omega t$, the temperature at any point x may be obtained by taking the real part of equation 4.5.

$$\mathrm{Re}[T] = T_s e^{-\sigma x} \cos(\omega t - \beta x)$$

This process of taking the real or imaginary part of equation 4.5 is the same as taking the projection of the curve in Figure 11 on to the real or imaginary planes, respectively, as the curve rotates about the x axis. These projections are shown in Figure 12 and Figure 13 for any time t.

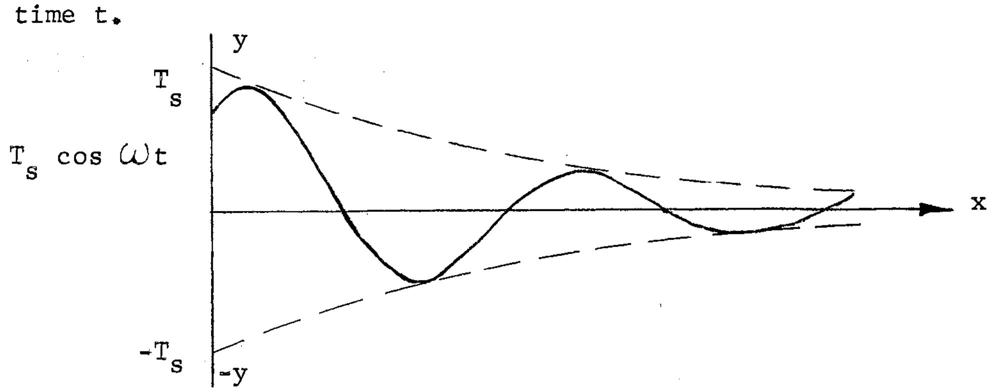

Figure 12. Projection on to the Real Plane

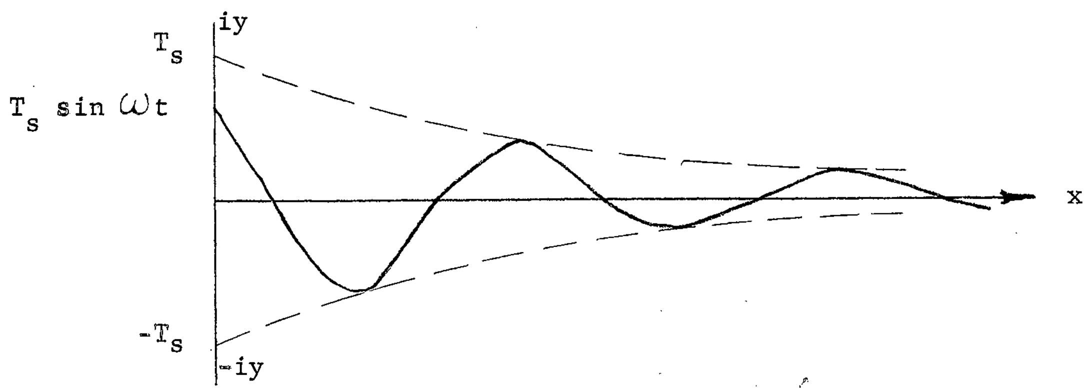

Figure 13. Projection on to the Complex Plane

Both of these curves are enclosed by the envelope formed by $e^{-\sigma x}$. They represent the temperature distribution in the infinite rod at any time t.

The heat flow at any point in the bar will be of the same form except that it will lead the temperature wave by an angle of θ and will have a magnitude determined by T_s/Z_c as we discussed previously in section 3.4.

4.3 Comparison With Existing Equations and Simple Example

Before moving on to more complex applications of the equations, a comparison should be made of these results against existing formulas and apply our equations to a simple example which has been solved by ordinary methods.

For purposes of comparison, several of the variables which have been derived are rewritten.

$$\beta = \sigma = \sqrt{c_p \rho \omega / 2k} = \sqrt{\omega / 2\alpha}$$

$$v = \omega/\beta = \omega\sqrt{2\alpha/\omega} = \sqrt{2\alpha\,\omega}$$

$$\lambda = 2\pi/\beta = 2\pi\sqrt{2\alpha/\omega} = \pi\sqrt{8\alpha/\omega}$$

$$t_{lag} = x/v = x\sqrt{1/2\alpha\,\omega}$$

These equations are identical with those found in the literature, (11), and require no further investigation. They have been rewritten in order to be compatible with the form in which they are usually found.

For purposes of illustration and comparison, an example will be worked illustrating the concepts presented to this point. This example will be concerned only with the case for an infinite body.

The temperature variation at the earth's surface at a given place is from -10°F to 10°F over a 24 hour period. If this temperature is assumed to vary sinusoidally, find a) the amplitude of the temperature oscillation at a depth of 1 foot, b) the time lag of the temperature wave at a depth of 1 foot, and c) the temperature at a depth of 1 foot, five hours after the surface temperature reaches the minimum temperature. Assume $k = 0.3$ Btu hr^{-1} ft^{-1} F^{-1}, $c_p = 0.47$ Btu $lb^{-1}{}_m$ F^{-1}, and $\rho =$ 100 lb ft^{-3}.

$$\sigma = \beta = \sqrt{c_p \rho \omega / 2k}$$

$$\omega = 2\pi/24 \text{ rad/hr} = 2\pi/24 \cdot 3600 = 7.88 \times 10^{-5} \text{ rad/sec}$$

$$= .262 \text{ rad/hr}$$

$$\sigma = \beta = \sqrt{0.47 \cdot 100 \cdot .262/2 \cdot .3} = \sqrt{20.5} = 4.53$$

a) $T'(x) = 10e^{-4.53 \times 1} = 10(.011) = 0.11°F$

b) $t_{lag} = x\,\beta/\omega = 1(4.53/.262) = 17.3$ hr.

Assume

$$T(0,t) = T_s \sin(\omega t + n\,3\pi/2)$$

which gives the minimum at $t = 0$ when n and $(n+1)/2$ are odd, the temperature at 1 foot is

$$T(1,t) = 0.11 \sin(\omega t - \beta x + 3n\pi/2)$$

$$= 0.11 \sin(\omega t - 4.53 + 3n\pi/2)$$

Five hours after the minimum is reached at the surface or at $t = 5$ hours,

c) $T(1,5) = 0.11 \sin(.262 \times 5 - 4.53 + 4.72)$

$$= 0.11 \sin(1.50) = 0.11(.997)$$

$$= 0.11°F$$

This example is identical to example IV-7 page 70, Jakob and Hawkins (12). The results are identical with those obtained in the reference by classical methods. This confirms the validity of the equations derived.

Since the equations for velocity of propagation, wave length, and time lag have already been proved to be the same as those found in the literature, an example illustrating their use does not seem feasible.

CHAPTER V

APPLICATION OF EQUATIONS TO THE UNINSULATED CASE

5.1 Re-analysis of Solution

A commonly encountered situation in practice is the case of an uninsulated body or a body which loses heat from its sides or transmits heat across its boundaries. This heat loss from the sides of a body could be due partially to the mechanism of convection and partially to the mechanism of radiation. In order to retain a general form which is applicable to most conditions, it will be assumed that heat is lost due to both these processes.

The heat flow from a surface of area A_1 and temperature T_1 into a surrounding environment of temperature T_e, due to the combined effects of convection and radiation, is

$$q = h_t A_1(T_1 - T_e) \tag{5.1}$$

where

$$h_t = h_c + h_r$$

Here h_c is the coefficient of heat transfer due to convection, h_r is the coefficient of heat transfer due to radiation, and h_t is referred to as the combined coefficient of heat transfer. Values for these constants depend upon the surrounding conditions, the material of which the body is composed, and the temperature difference. These constants are tabulated for various conditions in many heat transfer books.

In the derivation of equation 1.2, the heat flow through a differential volume dx dy dz was considered. In this case then,

$$A_1 = 2dx(dy + dz)$$

The rate of heat flow through the differential volume then becomes

$$dq/dt = (kdxdydz)\ \partial^2T/\partial x^2 - h_t 2dx(dydz)(T-T_e)$$
$$= (kdxdydz)\ \partial^2T/\partial x^2 - 2h_t(T-T_e)dxdy^2dz+dxdydz^2/dydz$$

Setting this equal to $c_p\rho$ dxdydz $(\partial T/\partial t)$,

$$c_p\rho(\partial T/\partial t) = k(\partial^2T/\partial x^2)-[h_t(T-T_e)]\ 2(dy+dz)/dydz$$

or
$$\partial T/\partial t = \alpha(\partial^2T/\partial x^2) - h_t(\alpha P/kA)(T-T_e)$$

where P is the perimeter of the body to be considered and A is the cross sectional area. The use of these parameters is justified by the assumption of a homogeneous body and consideration of heat flow and temperature variation only in the x direction. It is convenient to define

$$S = h_t(P\alpha/kA) = h_t\ P/c_p\rho A$$

giving
$$\partial T/\partial t = \alpha\ (\partial^2T/\partial x^2) - ST+ST_e$$

or
$$\partial^2T/\partial x^2 = 1/\alpha\ \partial T/\partial t + S/\alpha\ T - S/\alpha T_e \tag{5.2}$$

This equation may be used to obtain a solution in the same manner as was equation 1.2. For the sake of completeness, the following will proceed with a solution of the heat conduction equations as they have been modified to account for surface losses.

Differentiating equation 1.1 with respect to x and substituting equation 5.2 into it,

$$\partial q/\partial x = -kA(1/\alpha(\partial T/\partial t) + (S/\alpha)\ T - (S/\alpha)\ T_e)$$

or
$$\partial q/\partial x = -Ac_p l(\partial T/\partial t + ST-ST_e) \tag{5.3}$$

Solutions must be assumed of the form

$$T(x,t) = T'(x)e^{i\omega t} + D \tag{5.4}$$

and

$$q(x,t) = q'(x)\ e^{i\omega t} + E \tag{5.5}$$

since there has been added the effect of a surrounding constant temperature to our system. Now substitute the equations 5.4 and 5.5 into equations 1.1 and 5.3, obtaining

$$\partial T'/\partial x = -1/kA(q' + Ee^{-i\omega t}) \tag{5.6}$$

and

$$\partial q'/\partial x = e^{-i\omega t}\left\{-Ac_p\rho[i\omega e^{i\omega t}T' + S(T'e^{i\omega t} + D) - ST_e]\right\}$$

$$= -Y[T' + (S/i\omega)T' + (S/i\omega)e^{-i\omega t}(D - T_e)] \tag{5.7}$$

Now since 5.6 and 5.7 are functions of x only by hypothesis, it is required that

$$E = 0 \text{ and } D = T_e$$

This result indicates, as should be expected, that the temperature varies at all points in the body about the environmental temperature, T_e. T_e represents an average value about which the periodic temperature variation takes place. The temperature at the sending end then has a form as shown in Figure 14.

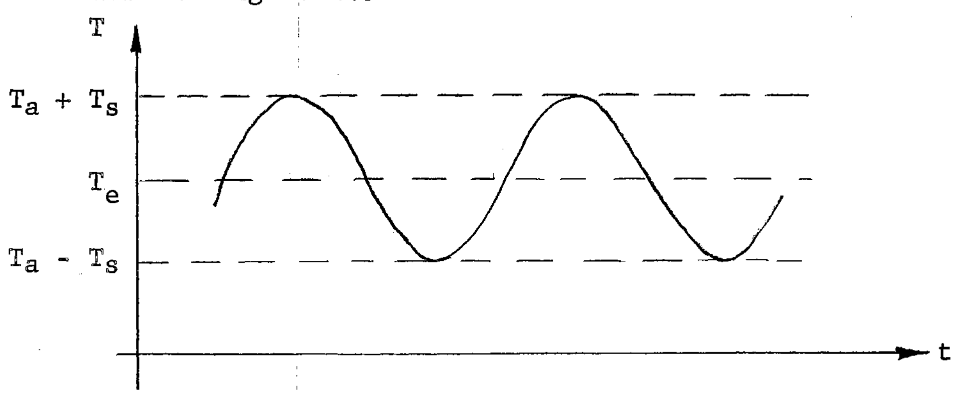

Figure 14. Actual Sending End Temperature

This is identically the situation discussed previously in section 2.1

in connection with the insulated body. It would seem then, that the uninsulated case could be treated similarly. Since the variation takes place about T_e, this value is the same as T_a that was mentioned in section 2.1. Setting $T_e = 0$ in equations 5.4 and 5.5 and solving for

$$T(x,t) = T'(x)\ e^{i\omega t}$$

and

$$q(x,t) = q'(x)\ e^{i\omega t}$$

the actual temperature could be found as before, or

$$T_{actual} = T(x,t) + T_a$$

and

$$q_{actual} = q(x,t)$$

This is actually the same solution that was obtained before except the notation has been changed to retain compatibility and to facilitate the analysis. Then equations 5.6 and 5.7 can be rewritten as

$$dT'/dx = -1/kA\ q' = -Zq' \qquad (5.8)$$

and

$$dq'/dx = -Y(T'+(S/i\omega)T') \qquad (5.9)$$

Differentiating 5.8 and 5.9 with respect to x again

$$d^2T'/dx^2 = ZY(T'+(S/i\omega)T')$$

and

$$d^2q'/dx^2 = ZY(1+S/i\omega)q'$$

Solutions to these two equations are of the form

$$T' = Ae^{\gamma x} + Be^{-\gamma x} \qquad (5.10)$$

$$q' = Ce^{\gamma x} + Fe^{-\gamma x}$$

or

$$q' = (\gamma/Z)(Be^{-\gamma x} - Ae^{\gamma x}) \qquad (5.11)$$

where γ has now been redefined as

$$\gamma = \sqrt{ZY(1+S/i\omega)} \qquad (5.12)$$

from equation 5.8

$$\text{at } x = 0,\ T_s = T' = A+B$$

and $$q_s = q' = (\gamma/Z)(B-A)$$

then $$B = T_s - A$$

$$q_s = (\gamma/Z)(T_s - 2A)$$

or $$A = \tfrac{1}{2}((-Z/\gamma)(q_s + T_s))$$

$$B = \tfrac{1}{2}((Z/\gamma)(q_s + T_s))$$

which gives

$$T'(x) = \tfrac{1}{2}((-Z/\gamma)q_s + T_s)e^{\gamma x} + \tfrac{1}{2}((Z/\gamma)q_s + T_s)e^{-\gamma x} \tag{5.13}$$

$$q'(x) = \tfrac{1}{2}[q_s - (\gamma/Z)(T_s)]e^{\gamma x} + \tfrac{1}{2}[q_s + (\gamma/Z)(T_s)]e^{-\gamma x} \tag{5.14}$$

If Z_c is redefined as

$$Z_c = Z/\gamma = \sqrt{Z/Y(1+S/i\omega)} \tag{5.15}$$

the complete steady state solution is

$$T(x,t) = \left\{\tfrac{1}{2}[T_s - Z_c q_s]e^{\gamma x} + \tfrac{1}{2}[T_s + Z_c q_s]e^{-\gamma x}\right\} e^{-i\omega t} \tag{5.16}$$

$$q(x,t) = \left\{\tfrac{1}{2}[q_s - (1/Z_c)(T_s)]\ e^{\gamma x} + \tfrac{1}{2}[q_s + (1/Z_c)(T_s)]e^{-\gamma x}\right\} e^{i\omega t} \tag{5.17}$$

Equations 5.16 and 5.17 are of the same form as equations 2.11 and 2.12. The only requirement in order to correct for surface losses is to use the more general values for Z_c and γ just derived. Note that if $h_t = 0$, Z_t and γ return to the values as defined for the insulated case. In both cases, insulated and uninsulated, to obtain the real temperature T_a must be added to the values given by equations 5.16 and 5.17. T_a will be assumed to be equal to zero unless otherwise indicated because it adds nothing to the analysis.

5.2 Analysis of Coefficients

The coefficients in the equations, that have been redefined for the general case of an uninsulated body, and their new definitions are as follows:

$$Z_c = Z/\gamma = \sqrt{Z/Y(1+S/i\omega)} = \sqrt{1/kA^2 c_p\rho\,(i\omega+S)} = \sqrt{(-i\omega+s)/kA^2 c_p\rho\,(S^2+\omega^2)} \qquad (5.18)$$

The impedance angle is not now as easily defined as it was for the uninsulated case. It is now a function of the physical parameters.

$$\gamma = \sqrt{ZY(1+S/i\omega)} = \sqrt{(c_p\rho)/k(i\omega+S)} = \sqrt{i\omega/\alpha + h_t P/kA} \qquad (5.19)$$

$$S = h_t(P/C_p\rho A) = h_t(P\alpha)/kA$$

With these values it is possible to re-evaluate the attenuation and phase constants. Remembering that

$$\gamma^2 = \sigma^2 - \beta^2 + 2i\sigma\beta = (i/\alpha)(i\omega+s) = h_t(P/kA) + i\omega/\alpha$$

Equating real and imaginary parts, this becomes

$$\sigma^2 - \beta^2 = h_t P/kA \qquad (5.20)$$

and

$$2i\sigma\beta = i\,\omega/\alpha \qquad (5.21)$$

then

$$\beta = \omega/2\alpha\sigma \qquad (5.22)$$

$$\beta^2 = \omega^2/4\alpha^2\sigma^2 \qquad (5.23)$$

Then substituting equation 5.23 into equation 5.20,

$$\sigma^2 - \omega^2/4\alpha^2\sigma^2 = h_t P/kA$$

$$\sigma^4 - h_t\ P/kA\sigma^2 - \omega^2/4\alpha^2 = 0$$

Solving by the quadratic formula,

$$\sigma^2 = h_t P/2kA \pm \tfrac{1}{2}\sqrt{h_t^2 P^2/k^2A^2 + \omega^2/\alpha^2}$$

$$\sigma^2 = 1/2kA(h_t P \pm 1/\alpha\sqrt{h_t^2 P^2\alpha^2 + \omega^2 k^2 A^2})$$

$$\sigma = \sqrt{1/2kA(h_t P + 1/\alpha\sqrt{h_t^2 P^2\alpha^2 + \omega^2 k^2 A^2})} \qquad (5.24)$$

Since σ and β are real by definition and are both positive by convention, the negative signs in front of the square root radicals can be eliminated. Substituting equation 5.24 into equation 5.20, there is obtained an expression for β.

$$\beta^2 = (1/2kA)(h_t P + 1/\alpha\sqrt{h_t^2 P^2 \alpha^2 + \omega^2 k^2 A^2}) - h_t\ P/kA$$
$$= (1/2kA)(-h_t P + 1/\alpha\sqrt{h_t^2 P^2 \alpha^2 + \omega^2 k^2 A^2})$$
$$\beta = \sqrt{(1/2kA)(-h_t P + 1/\alpha\sqrt{h_t^2 P^2 \alpha^2 + \omega^2 k^2 A^2})} \qquad (5.25)$$

From inspection, it is seen that this expression for β gives a real number, which agrees with the initial assumption.

The equations for the characteristics of the solution, such as phase velocity, wave length, period, and time lag, remain the same as previously defined in sections 3.3 and 3.4 except that the σ and β used in their evaluation must be the ones that have just been derived.

Notice that if $h_t = 0$, $\sigma = \sqrt{\omega/2\alpha}$, and $\beta = \sqrt{\omega/2\alpha} = \sigma$, which is the same as obtained for the insulated case.

From inspection of equations 5.24 and 5.25, it is seen that the introduction of surface losses into our equations has increased the value of σ and decreased the value of β. This means that the attenuation of the traveling waves is greater or that they decrease in magnitude faster than they did in the insulated case. A smaller value of β means that the phase change as the traveling waves travel down the bar will be less, thus the phase velocity is greater, the wave length is greater, and the time lag is less. All these results are reasonable and to be expected when there is an additional energy loss.

The same effect, as far as the change in phase velocity, wave length, and time lag are concerned, could be attained by decreasing c_p, increasing k, or both in the insulated case. These variations are not sufficient to realize an approximation of the uninsulated case, however, since in this case σ decreases also, instead of increasing.

CHAPTER VI

UNINSULATED INFINITE ROD

It is now desirable to apply the equations for an uninsulated body (5.16 and 5.17) to the particular case of an uninsulated infinite rod. If it is recalled from section 3.5 that an infinite rod has no reflected wave, then the coefficient of the $e^{\gamma x}$ terms in equations 5.16 and 5.17 must be zero.

$$T_s - Z_c q_s = 0$$

or

$$Z_t = Z_c = T_s/q_s \tag{6.1}$$

The equations for the infinite rod must then be

$$T(x,t) = \tfrac{1}{2}(T_s + T_s)\, e^{-(\sigma + i\beta)x} e^{i\omega t} \tag{6.2}$$

and

$$q(x,t) = (1/2Z_c)(T_s + T_s)\, e^{-(\sigma + i\beta)x} e^{i\omega t} \tag{6.3}$$

Then

$$T(x,t)/q(x,t) = Z_c$$

In this light it is needed only to discuss the solution for $T(x,t)$.

$$T(x,t) = T_s e^{-\sigma x} \cos(\omega t - \beta x) + i \sin(\omega t - \beta x) = T_s e^{-\sigma x + i(\omega t - \beta x)} \tag{6.5}$$

Since equation 6.5 is exactly of the same form as equation 4.1, the analysis previously performed for the insulated infinite rod will apply equally as well to the uninsulated infinite rod. The effect of the T_e term, if the actual temperature was desired, is merely to cause a transformation in coordinates in Figure 11 or to give the attenuated, phase shifted sinors a non-zero temperature about which to rotate.

This case is illustrated in Figure 15.

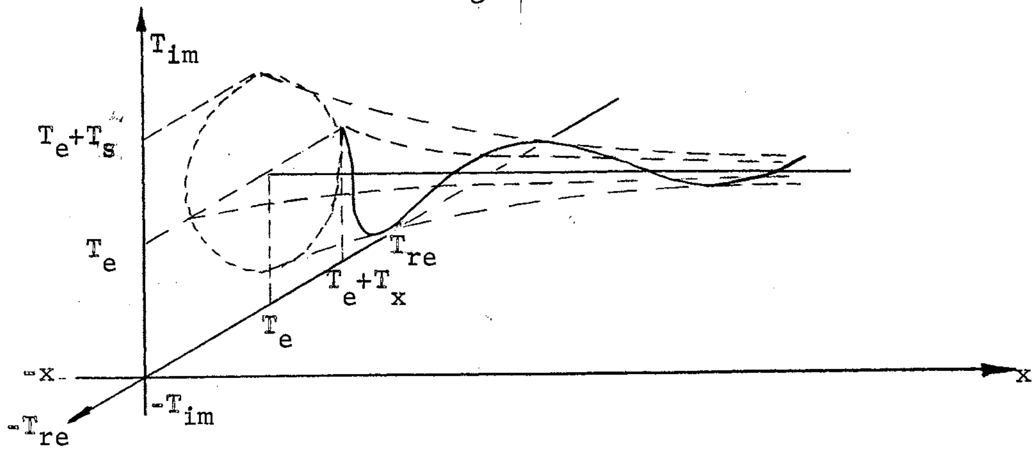

Figure 15. Actual Position of Temperature Traveling Wave

Wave length, phase velocity, period, and time lag may be computed using the values of σ and β given by equations 5.24 and 5.25. The analysis of the infinite rod then is essentially the same whether it is insulated or uninsulated.

CHAPTER VII

THE FINITE ROD

7.1 Derivation of Equations for a Finite Body

The distinction between finite and infinite bodies, as far as the equations are concerned, is that a finite body is one in which the reflected wave becomes significant in determining the variation at any point. The approximate length a body must be in order to neglect the reflected wave is to be the subject of a later section.

Consider for purposes of analysis a homogeneous rod of the configuration shown in Figure 16.

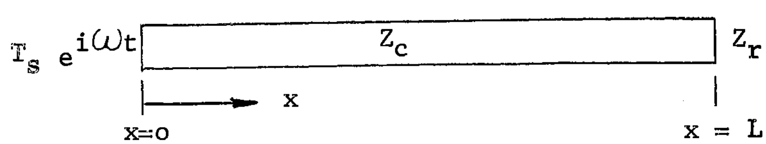

Figure 16. Finite Rod

There is an applied temperature source $T_s e^{i\omega t}$ at $x = 0$. The rod has a characteristic impedance Z_c, and an end impedance Z_r.

Assuming for the moment that the forward wave has an amplitude at $x = 0$ of T_s^+, there can be derived equations for temperature and heat flow at any point in terms of this value. From the boundary conditions and the equations to be derived, T_s^+ can be determined, and the solution will be complete.

The amplitude of the forward wave at the receiving end (x = L) is

$$T_r^+ = T_s^+ e^{-\gamma L} \qquad (7.1)$$

This follows from the discussion in section 3.2. Using the definition of the reflection coefficient,

$$\Gamma_r = (Z_r - Z_c)/(Z_r + Z_c) \qquad (7.2)$$

Using the reflection coefficient, there is obtained the amplitude of the reflected wave at the receiving end (x = L).

$$T_r^- = \Gamma_r T_r^+ = T_s^+ e^{-\gamma L} \Gamma_r \qquad (7.3)$$

The reflected wave after it reaches the sending end would then be

$$T_s^- = T_r^- e^{-\gamma L} = T_s^+ e^{-2\gamma L} \Gamma_r \qquad (7.4)$$

The resultant temperature at the sending end, which was given as T_s, is

$$T_s = T_s^+ + T_s^- = T_s^+(1+\Gamma_r e^{-2\gamma L}) \qquad (7.5)$$

Using this equation, T_s^+ can be obtained since we know T_s. T_s^+ would then be given as

$$T_s^+ = T_s[1/(1+ \Gamma_r e^{-2\gamma L})] \qquad (7.6)$$

If the resultant temperature at the receiving end were desired, it would be

$$\begin{aligned} T_r = T_r^+ + T_r^- &= T_s^+ e^{-\gamma L} + T_s^+ e^{-\gamma L} \Gamma_r \\ &= T_s^+ e^{-\gamma L} (1+ \Gamma_r) \end{aligned} \qquad (7.7)$$

It is likely that the temperature at any point x along the rod would be desired. The forward wave at any point x is

$$T^{+} = T_s^{+}\, e^{-\gamma x} \qquad (7.8)$$

and the reflected wave at any point x is

$$T^{-} = (T_s^{+}\, e^{-\gamma L})\, e^{-\gamma (L-x)}\, \Gamma_r$$
$$= T_s^{+}\, e^{\gamma (x-2L)}\, \Gamma_r \qquad (7.9)$$

Then either by adding these two terms, or by substituting equation 7.4 into equation 3.4, we get for the temperature at any point

$$T' = T_s^{+}\, e^{-\gamma x} + T_s^{+}\, e^{\gamma (x-2L)}\, \Gamma_r$$
$$= T_s^{+}\, (e^{-\gamma x} + e^{-\gamma (x-2L)})\, \Gamma_r \qquad (7.10)$$

The temperature as a function of both time and position would be given by

$$T(x,t) = T_s^{+}(e^{-(\sigma+i\beta)x} + \Gamma_r\, (e^{-(\sigma+i\beta)(2L-x)})\, e^{i\omega t}$$
$$= T_s^{+}(e^{-\sigma x+i(\omega t-\beta x)} + \Gamma_r\, (e^{-\sigma(2L-x)+ i(\omega t+\beta(x-2L))}) \qquad (7.11)$$

The heat flow equation corresponding to equation 7.10 is found by evaluating equation 3.5.

$$q' = T_s^{+}/Z_c\; e^{-\gamma x} - T_s^{+}/Z_c\;\; \Gamma_r\;\; e^{-2\gamma L}\, e^{\gamma x}$$
$$= (T_s^{+}/Z_c)(e^{-\gamma x} - \Gamma_r\, e^{-2\gamma L}\, e^{\gamma x}) \qquad (7.12)$$

The heat flow at any point as a function of both x and t is

$$q(x,t) = (T_s^{+}/Z_c)(e^{-\sigma x+i(\omega t-\beta x)} - \Gamma_r\, e^{-\sigma(2L-x)+i(\omega t+\beta(x-2L))}) \qquad (7.13)$$

The magnitude of the heat flow at each boundary (x=0 and x =L) would then be

$$q_s = (T_s^{+}/Z_c)(1 - \Gamma_r\, e^{-2\gamma L}) \qquad (7.14)$$

and $$q_r = (T_s^+/Z_c)(e^{-\gamma L} - \Gamma_r e^{-\gamma L}) = (T_s^+/Z_c)e^{-\gamma L}(1 - \Gamma_r) \quad (7.15)$$

Note that these equations for the finite bar are perfectly general and can be applied either to the insulated case or the uninsulated case. This is true, of course, only if the coefficients derived in section 5.2 are used and since $h_t = 0$ for the insulated case.

7.2 Total Impedance

Interesting results can be obtained by investigating the impedance for a finite body as a function of x. As a first step in this direction, T' will be plotted in the complex plane as x varies from 0 to L. The expression for T' in the finite rod was found to be

$$T' = T^+ + T^- = T_s^+ e^{-\gamma x} + T_s^+ e^{-2\gamma L + \gamma x} \Gamma_r$$

Plotting the incident wave and the reflected wave, T' can be found as the vector sum of these two curves. This is done in Figure 17.

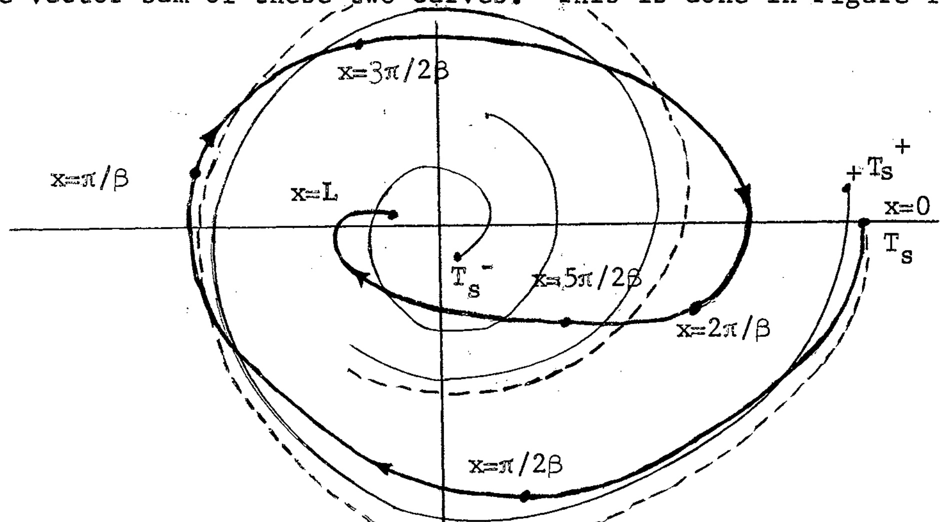

Figure 17. Projection of the Traveling Temperature Waves on to the Complex Plane and Their Sum

The resultant curve for T' appears as a spiral flattened somewhat about the real axis. Actually the spiral is flattened about an axis which is rotated by some small angle from the real axis in the clockwise direction. The distribution in a finite bar may be more readily visualized if there is included in Figure 17 the distribution in an infinite bar with T_s applied at the sending end. This is plotted as a dotted line in Figure 17. Recall that in the discussion of Figure 11 for the infinite body that as the curve rotated about the x axis, which is perpendicular to the page in Figure 17, the temperature at any point x varied sinusoidally with an amplitude dependent upon x. Similarly in the finite rod, the temperature at any point x varies sinusoidally as the curve found above rotates about the x axis. The distinction between the finite and the infinite bodies is that in the finite case, the amplitude of the temperature variation is not decreased with x according to a simple exponential decay. In fact, it is possible to find some point or points, x such that the temperature variation at this point or points is actually greater than the amplitude of variation at some points nearer the source than the point or points being investigated. In other words, the decay seems to be according to an exponential with some periodic term superimposed upon it. This becomes even more evident if the magnitude of the variation as a function of x is plotted. This has been done in Figure 18.

At points one quarter wave length or 90° apart, the vector sum becomes alternately the arithmetic sum and the arithmetic difference.

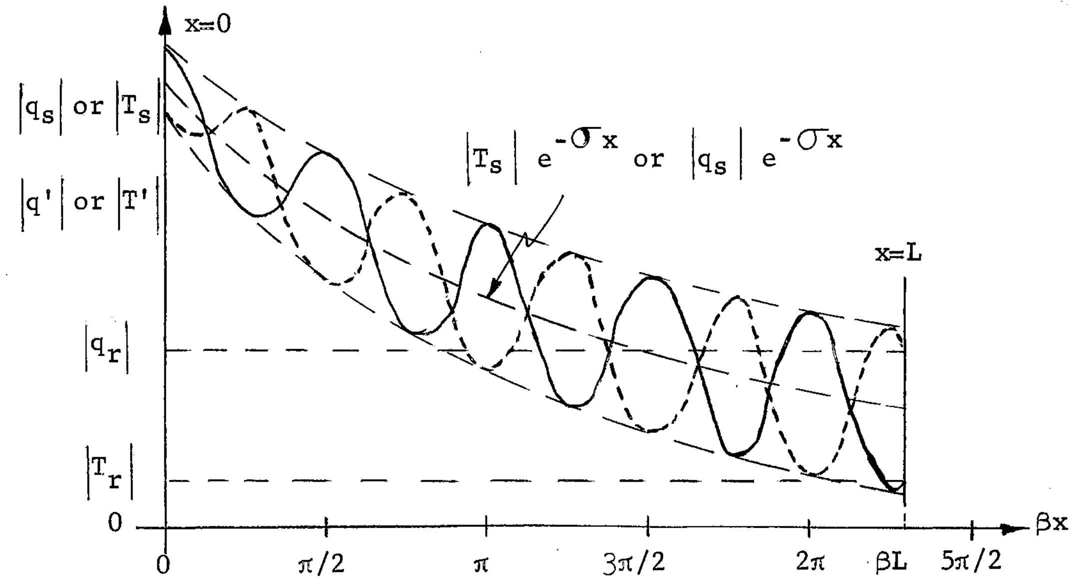

Figure 18. Magnitude Distribution

These points are not quite the maximum and minimum points, but closely approximate them.

This type of temperature response is unique to traveling waves traveling in opposite directions. Therefore, if this type of response could be verified in an actual physical situation, it would confirm the validity of the traveling wave type of solution that has been developed. Significant data is not readily available, but it is intended to obtain such data from an actual test and present it in one of the latter sections of this thesis. This type of response, due to the interference of traveling waves, is not readily observable due to the rapid decay of the incident wave in most systems. The decay usually is so great that at a distance greater than about one wave length, the variation is negligible. This subject will be developed more fully in Chapter IX.

A sketch of q' in the complex plane can be obtained in the same manner as was obtained the sketch in Figure 17 for T'. This curve appears in Figure 19.

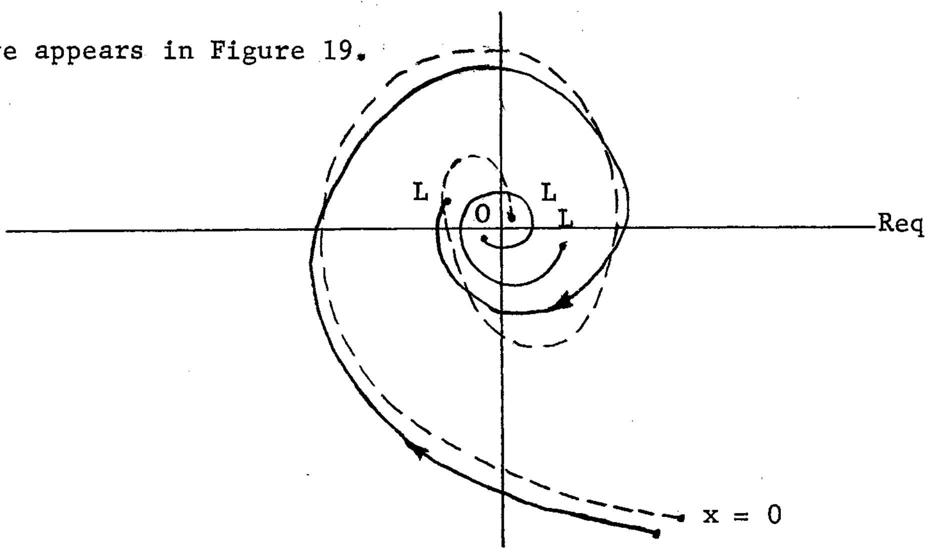

Figure 19. Projection of the Traveling Heat Flow Waves on to the Complex Plane and Their Sum

To obtain this plot, recall the expression for q'

$$q' = T_s^{+}/Z_c \; e^{-\gamma x} - T_s^{+}/Z_c \; \Gamma_r \; e^{-2\gamma L} \; e^{\gamma x}$$

Note that

$$q' = T^{+}/Z_c - T/Z_c = T^{+}/|Z_c| \; \underline{|-\theta} \quad - T^{-}/|Z_c| \; \underline{|-\theta} \qquad (7.16)$$

Figure 19 can be obtained from Figure 17 merely by rotating the component vectors by $-\theta$, dividing them by $|Z_c|$, and finding their vector differences. Remember that for the insulated body θ has a valueof 45°, but for the uninsulated case θ will be different from 45°. This curve is also a flattened spiral. It is flattened about an axis dependent upon θ. If θ were 90°, the spiral would be flattened about an axis only slightly rotated from the imaginary axis.

If one assumes θ = 90° and plots the magnitude of q' in Figure 18, one sees that the minimums of q' occur at the miximums of T' and vice versa. If this were the insulated case where θ = 45°, this would not be true, but the shape of the curve would be the same.

The preceding results indicate that the temperature and heat flow at the receiving end depend critically upon the length of the body. It is possible, if the length of the bar is some odd multiple of a quarter wave length, to increase the magnitude of the temperature variation and decrease the magnitude of the heat flow variation by either increasing or decreasing the length of the bar by some amount less than a quarter wave length. If the length of the rod is an even multiple of a quarter wave length, the magnitude of the temperature variation may be decreased and the magnitude of the heat flow may be increased at the receiving end by either increasing or decreasing the length by some values less than a quarter wave length. Those facts and other variations are observable in Figure 18. The effect on temperature of varying the length remains essentially the same under all conditions. The relative effects on heat flow, however, would depend upon the impedance angle, θ, which would determine the amount of shift in the heat flow magnitudes from the position shown in Figure 18. Figure 18 was formed on the basis of an uninsulated bar or $\theta = 90°$. If θ had some other value, as in the insulated case, the heat flow curve would be shifted along the $|q_s| \; e^{-\sigma x}$ curve. If the body were uninsulated, then the effect on the heat flow of varying the length would have to be determined for each particular case.

This technique suggests certain ramifications which could be of value in practical analysis. For example, the temperature variation or heat flow variation in a building, due to heat transfer through the walls, could be reduced by this process of either thickening or reducing the width of the walls, depending upon building requirements and the feasibility of the change in design under certain circumstances. At least an optimum thickness could be found to satisfy certain requirements or desires.

It is possible that other suitable building materials could be selected so as to adjust the wave length in the material to optimize the amount of heat flowing into the building. These are only a few of the possibilities and are mentioned only to point out the potential of this type of analysis. A further study than will be possible in this thesis, in conjunction with the many transmission line techniques that have been developed would undoubtedly be very rewarding and would probably suggest many methods of reducing undesirable effects in the application to heat transfer problems.

The feasibility of this analysis depends upon the magnitude of the attenuation constant. The attenuation determines the amount of reflection and thus the magnitude of the variations about the simple exponential decay. If attenuation is great and these variations in Figure 18 are small, then there can be little gained by any adjustments in length.

It is also interesting to notice the value of the impedance as a function of x. The total impedance at a point has been defined as

$$Z_t = T'/q' = T_s^+[e^{-\gamma x} + e^{-\gamma(2L-x)}\,\Gamma_r]/[T_s^+/Z_c][e^{-\gamma x} - \Gamma_r\, e^{-\gamma(2L-x)}] \quad (7.17)$$

This can be rewritten as

$$Z_t = Z_c[1+e^{-2\gamma(L-x)}\,\Gamma_r]/[1-e^{-2\gamma(L-x)}\,\Gamma_r] \quad (7.18)$$

or

$$Z_t = |Z_c|\,\underline{|\theta}\,[1+e^{+2\sigma(x-L)}\,|\Gamma_r|\,\underline{|\psi_r+2\beta(x-L)}]/[1-e^{2\sigma(x-L)}\,|\Gamma_r|\,\underline{|\psi_r+2\beta(x-L)}] \quad (7.19)$$

The numerator and the denominator of the fraction in this expression can be represented as shown in Figure 20. For convenience, rewrite equation 7.19 as

$$Z_t = |Z_q|\,\underline{|\theta}\,N(x)/D(x) = |Z_c|\,\underline{|\theta}\,(1+N_1)/(1-N_1) \quad (7.20)$$

where $N_1 = e^{2\sigma(x-L)}\,|\Gamma_r|\,\underline{|\psi_r + 2\beta(x-L)}$

Notice that since $|\Gamma_r|$ has a maximum value of 1 and as x increases to L, $e^{2\sigma(x-L)}$ approaches a maximum value of 1, then the magnitude

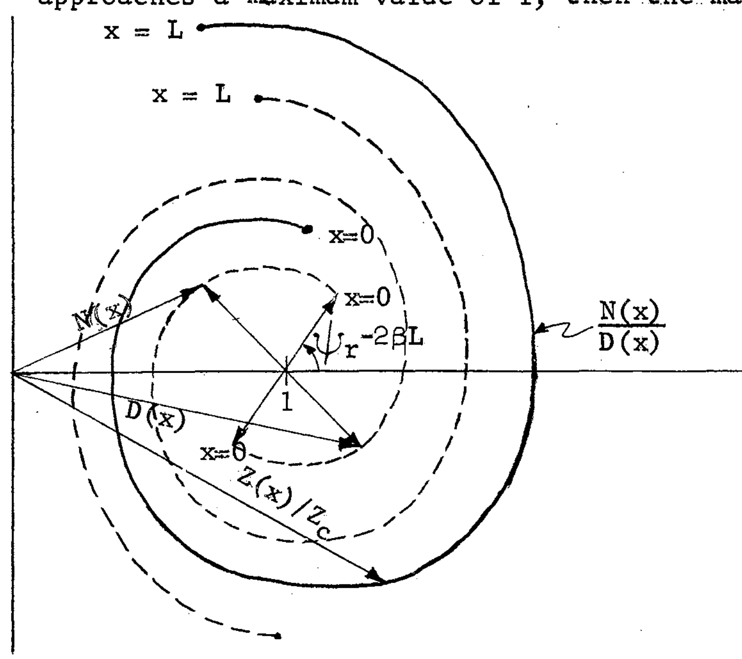

Figure 20. Ratio of Total Impedance to Characteristic Impedance

of N_1 has a maximum value of 1 and as a result all vectors representing both N and D will lie in the right half plane in Figure 20 for all values of $x \leq L$.

The vectors shown in the figure are of an assumed position and magnitude to represent N_1 and $-N_1$ at $x = 0$. As x increases, the vectors increase in magnitude and rotate in a counterclockwise direction at the same magnitude. The loci of their end points would then be an increasing spiral rotating in the counterclockwise direction as shown in the figure. Note that in the plane the end points of these vectors, which rotate as x increases, represent the vector sum of 1 and N_1, and 1 and $-N_1$ or N(x) and D(x). This is true since the base of these vectors has been placed at the point $1\underline{|0}$ in the plane. Thus the loci of these vectors as x increases also represents the loci of N(x) and D(x) as x increases.

For the special case where there is no attenuation, the loci in Figure 20 become circles and the figure becomes the bicycle or crank diagram encountered in connection with lossless transmission lines.

The total impedance may now be found as a function of x by plotting the quotient of N(x)/D(x) as x varies from 0 to L, and then multiplying this quotient by $|Z_c|$ and rotating the whole diagram about the origin through an angle θ. N(x)/D(x) is plotted in Figure 20. It is interesting to note that if $\Gamma_r = 1$, as x approaches L, N_1 approaches 1 or -1 depending upon ψ_r, and thus D approaches 0, making Z_t approach infinity, or N approaches 0 making Z_t approach 0.

This is actually the case because if $Z_r = \infty$, $\Gamma_r = 1$ or if $Z_r = 0$, $\Gamma_r = 1$. Likewise, as either σ increases or L increases, the spiral becomes tighter about the point $1\underline{|0}$. If σ or L were increased enough, the point at x = 0 would be approximately the point $1\underline{|0}$ and the impedance at x = 0 would be approximately Z_c. Of course if L became infinite, the entire spiral would become the point $1\underline{|0}$, and the impedance at all points x would be Z_c, which is the condition that was previously found for the infinite body.

The concept of total impedance becomes a little clearer if the magnitude of Z_t is plotted as a function of x. This is done in Figure 21.

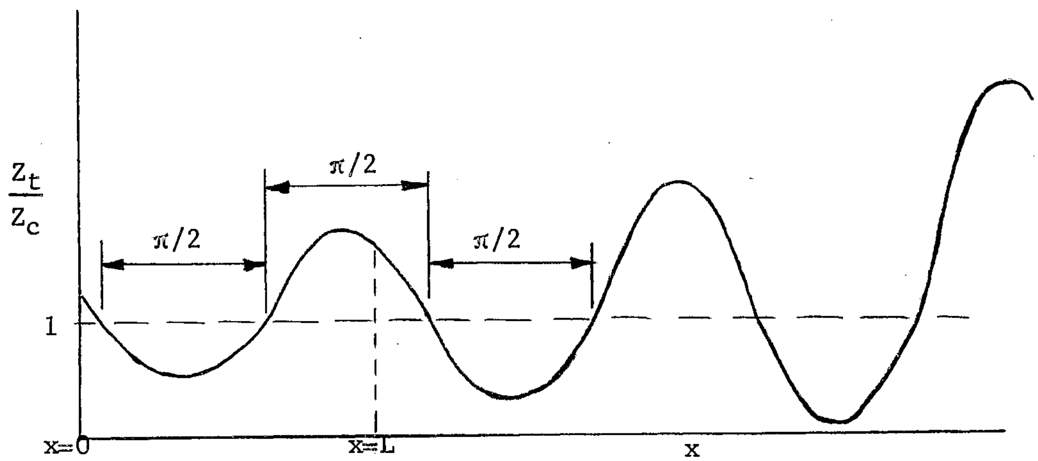

Figure 21. Total Impedance as a Function of x

The curve has been extended beyond the point x = L in order to more fully illustrate the variation.

The point to be made here is essentially the same as before. That is, by adjusting the length of the body the most desirable value of Z can be obtained and thus of q' and T'. It is evident that varying the length to effect the impedance has exactly the same effect that was discussed previously in analyzing the T' and q' curves. Note that Z_s is a function of the length and is given by

$$Z_s = Z_c[(1+e^{-2\gamma L}\ \Gamma_r)/(1-e^{-2\gamma L}\ \Gamma_r)] \tag{7.21}$$

This indicates that as L changes, the position of the curve in Figure 21 is translated parallel to the x axis. The length also has a direct effect upon the magnitude of Z_t as well as on the magnitudes of T' and q'. It appears now that the amount of heat flow at the source could be regulated merely by changing the length of body. This also would find its place in practical applications.

7.3 End Impedance for Finite Rods

The study of finite bodies brings us to another problem which is the determination of the receiving end impedance for these bodies. Consider first a composite rod such as is represented in Figure 22. Reference will be made to the two sections as rod 1 and rod 2. Consideration will be directed toward finding the receiving end impedance Z_{r1} of rod 1. If the most general case of an uninsulated rod of finite length is considered, Z_{r1} can be determined for other conditions as special cases of this one.

Actually, there is no problem in this case since the receiving end impedance of rod 1 will be the sending end impedance of rod 2, or

$$Z_{r1} = Z_{s2} = Z_{c2}[1+e^{-2\gamma_2 L_2}\ \Gamma_{r2}/1-e^{-2\gamma_2 L_2}\ \Gamma_{r2}] \tag{7.22}$$

This expression says that the amount of reflection in rod 1 and the amount of heat flowing into rod 2 can be affected by changing the length of rod 2.

If rod 2 were of infinite length, equation 7.22 would still be valid for if $L_2 = \infty$

$$Z_{r1} = Z_{s2} = Z_{c2} \tag{7.23}$$

The receiving end impedance of rod 1 then becomes the characteristic impedance of rod 2. The expression for Z_{r1} where either rod 1, rod 2, or both are insulated would be obtained by setting either $h_1 = 0$, $h_2 = 0$, or both.

If the end of rod 1 is exposed to some medium such as a gas or liquid in which conduction is not the significant method of heat transfer, the heat transfer from the end can be computed and thus Z_r, by using the combined coefficient of heat transfer, h_t, and writing

$$q = h_t \; TA_1 \tag{7.24}$$

where A_1 is the cross sectional area of the end of bar 1. Then the end impedance would be

$$Z_{r1} = T/q = 1/h_t A_1 \tag{7.25}$$

This has as a special case, the case of the perfectly insulated end. In this case $h_t = 0$, and

$$Z_{r1} = 1/0 = \infty \tag{7.26}$$

This result agrees with previous conclusions.

The receiving end impedance has now been derived for several general cases. Most systems may be adequately described using these expressions. However, there are potentially several conditions which might not be covered by these equations, but the process is clear and a suitable approximation of Z_r can usually be made.

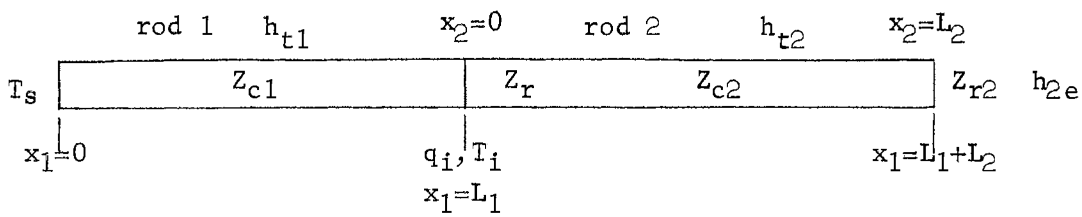

Figure 22. Composite Rod, Two Sections

CHAPTER VIII

THE COMPOSITE ROD

8.1 Analysis

The next case of interest is the composite, finite, homogeneous rod. Figure 22 is a suitable representation of this case. The assumption will be made, of course, that the two rods are perfectly joined at the point $x = L_1$ and that there is no heat loss there due to the connection. The general case where both sections are uninsulated will be considered from which the insulated case may be obtained by setting $h_t = 0$ for the insulated section or sections. Assume also that the receiving end of section 2 is exposed to still air or some other media such that equation 7.25 applies. This can be specialized to include the insulated case also, by setting $h_{2e} = 0$, where h_{2e} is the combined coefficient of heat transfer at the end of section 2.

The solution of this system will be represented by two equations, one for section 1 and one for section 2. The equation for section 1 will merely be the finite rod equation that has already been derived.

For section 1,

$$0 \leq x_1 \leq L_1$$

$$T_1(x_1, t) = T_s \left[e^{-\sigma_1 x + i(\omega_1 t - \beta_1 x_1)} + \Gamma_{r1} e^{-\sigma_1 (2L_1 - x_1) + i(\omega_1 t + \beta_1 (x_1 - 2L_1))} \right.$$

and $$q_1(x_1,t) = T_1(x,t)/Z_{t1} \tag{8.2}$$

where

$$\Gamma_{r1} = (Z_{r1} - Z_{c1})/(Z_{r1} + Z_{c1}) \tag{8.3}$$

and

$$Z_{r1}=Z_{c2}[(1+e^{-2\gamma_2 L_2}\ \Gamma_{r2})/(1-e^{-2\gamma_2 L_2}\ \Gamma_{r2})] \tag{8.4}$$

$$\Gamma_{r2}=(Z_{r2}-Z_{c2})/(Z_{r2}+Z_{c2}) \tag{8.5}$$

$$Z_{r2}= 1/(h_{2e}\ A_2) \tag{8.6}$$

For section 2 the sending end temperature is the receiving end temperature of section 1, or

$$T_{s2} = T_{r1} \tag{8.7}$$

then

$$T_{s2}^{+}(1+e^{-\gamma_2(2L_2)}\ \Gamma_{r2}) = T_{s1}^{+}(e^{-\gamma_1 L_1})(1+\Gamma_{r1})$$

or

$$T_{s2}^{+}= T_{s1}^{+}e^{-\gamma_1 L_1}[(1+\Gamma_{r1})/(1+\Gamma_{r2}\ e^{-2\gamma_2 L_2})] \tag{8.8}$$

Using this value of T_{s2}^{+}, return to the finite rod equations and find the solution for section 2.

For section 2,

$$L_1 \le x_1 \le L_2 + L_1,\ 0\le x_2 \le L_2$$

$$T_2(x_2,\ t) = T_{s2}^{+}[e^{-\sigma_2 x_2+i(\omega_2 t-\beta_2 x_2)}$$
$$+ \Gamma_{r2}\ e^{-\sigma_2(2L_2-x_2)+\ i(\omega_2 t+\beta_2(x_2-2L_2))}] \tag{8.9}$$

$$q_2(x_2,t) = T_2(x_2,t)/Z_{t2} \tag{8.10}$$

If it is preferred to have these expressions in terms of x_1, merely replace x_2 by x_1-L_1, giving

$$T_2(x_1,t)=T_{s2}^{+}[e^{-\sigma_2(x_1-L_1)+i(\omega_2 t-\beta_2(x_1-L_1))}$$
$$+ \Gamma_{r2}\ e^{-\sigma_2(2L_2+L_1-x_1)+i(\omega_2 t+\beta_2(x_1-L_1-2L_2))}] \tag{8.11}$$

and

$$q_2(x_1,t)=T_{s2}^{+}/Z_{c2}\left[e^{-\sigma_2(x_1-L_1)+i(\omega_2 t-\beta_2(x_1-L_1))}\right.$$
$$\left.\Gamma_{r2}\ e^{-\sigma_2(2L_2+L_1-x_1)+i(\omega_2 t+\beta_2(x_1-L_1-2L_2))}\right] \qquad (8.12)$$

The equations for the composite rod actually present nothing new. The analysis of these equations is essentially the same as the analysis of the finite rod equations. These equations have been derived for the purpose of illustrating an application of the finite rod equations. The only additional point that could be made here is that in situations such as this, the length, area, and all the physical constants associated with one bar affect the temperature and heat flow in the other. An analysis of temperature, heat flow, and impedance variation could be performed on this system exactly as was done in section 7.2 for the finite bar. This analysis would have essentially the same results that obtained before and would add nothing new to our investigation. Keep in mind that there is as much or more potential value in this type of analysis for the composite bar as there is for the finite bar. Further investigation in this area could be very profitable, but it is not the purpose of this thesis to go into such detail. The possibilities that are apparent here indicate that it is a subject worthy of further pursuit.

8.2 Transfer Impedance

Many times it is not of interest to find the temperature and heat flow distribution in a body, but merely to find the terminal values. It would then be of interest to derive an expression for the system transfer impedance, so that the terminal values could be found with a minimum of effort. Transfer impedance is to be defined as

$$Z_{tr} = T_{in}/q_{out} \tag{8.13}$$

Consider first the finite rod. The transfer impedance for this case is

$$Z_{tr}=T_s/q_r= T_s^{+}(1+ \Gamma_r\, e^{-2\gamma L})/[(T_s^{+}/Z_c)(1- \Gamma_r)e^{-\gamma L}]$$
$$= Z_r \cosh \gamma L + Z_c \sinh \gamma L \tag{8.14}$$

Notice that if $L = 0$, $Z_{tr} = Z_r$.

This indicates that if the rod were removed, the driving function would be acting directly into the end conditions. In this case the sending end temperature and heat flow would be the same as the receiving end temperature and heat flow. Notice also that the transfer impedance and thus q_r are independent of the receiving end temperature T_r in this expression.

This type of approach is not of particular value in the finite case, since q_r can be more easily computed from equation 7.15. This approach becomes of value when composite bodies of two or more sections are considered. If the heat flow at the receiving end of a composite body was desired, and not the intermediate temperature and heat flow distribution, it would be a very laborious process to grind through the equations for each section separately. In a composite section, the concept of transfer impedance could be applied to each section repeatedly to obtain terminal values. If an overall transfer impedance for the entire composite section could be determined, it would greatly simplify the calculations by offering the advantages of compactness and a simpler form.

Consider now the composite section shown in Figure 22. This figure could represent either a composite rod or a section of a composite wall or plate. The transfer impedance of section 1 of this composite body may be expressed as

$$Z_{tr1} = T_s/q_i \tag{8.15}$$

Likewise the transfer impedance of section two can be expressed as

$$Z_{tr2} = T_i/q_r \tag{8.16}$$

Realizing that

$$q_i = T_i/Z_{r1} = q_r\ Z_{tr2}/Z_{r1}$$

equation 8.15 becomes

$$Z_{tr1} = T_s Z_{r1}/q_r Z_{tr2}$$

or

$$T_s/q_r = Z_{tro2} = Z_{tr1}\ Z_{tr1}/Z_{r1} \tag{8.17}$$

Z_{tro2} represents the overall transfer impedance for the composite body with two sections.

Proceeding now to a composite body of three sections such as shown in Figure 23, the transfer impedance for the first section and third section are the same as the transfer impedances for the first section and second section of the body just considered. From previous considerations, it is known that

$$T_s/q_i = Z_{tr1}\ Z_{tr2}/Z_{r1}$$

Then, since

$$q_i = T_i/Z_{r2} = q_r Z_{tr3}/Z_{r2}$$

giving $$Z_{tro3} = T_s/q_r = Z_{tr1}\ Z_{tr2}\ Z_{tr3}/Z_{r1}\ Z_{r2} \qquad (8.18)$$

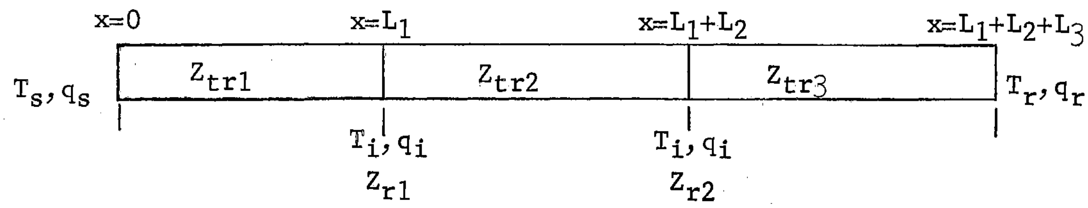

Figure 23. Composite Rod - Three Sections

Similarly, a composite body consisting of n sections was considered as shown in Figure 24 and an expression for the overall transfer impedance could be obtained which would be of the form

$$\begin{aligned} Z_{tron} &= (Z_{tr1}\ Z_{tr2} \ldots Z_{trn})/(Z_{r1}\ Z_{r2} \ldots Z_{rn-1}) \qquad (8.19) \\ &= T_s/q_r \end{aligned}$$

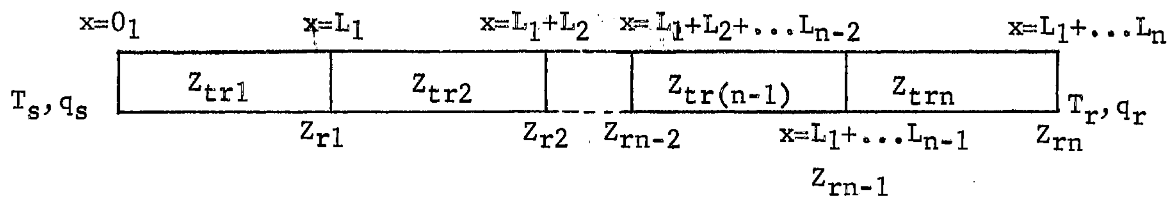

Figure 24. Composite Rod - n Sections

Use of this equation would permit simplified calculation of the terminal values of a composite body, such as a wall where one might know the outside temperature and be concerned only with finding the heat flow into the inside of the structure. Although this form does not greatly reduce calculations, it does reduce error and increase the speed of calculation due to the compact form.

8.3 Equivalent Four Terminal Networks

In accordance with the line of thought of the previous section, it is possible to represent a finite section by an equivalent four terminal network, for the purpose of investigation of terminal values only. The only restriction that will be placed upon this equivalent network is that the input and output temperature and heat flow be the same as in the actual system. This equivalent network may be found as either a π section or a T section. An equivalent π section as shown in Figure 25 will now be considered. In this equivalent network, the T's represent temperature but are treated as voltages, and the q's represent heat flow but are treated as currents.

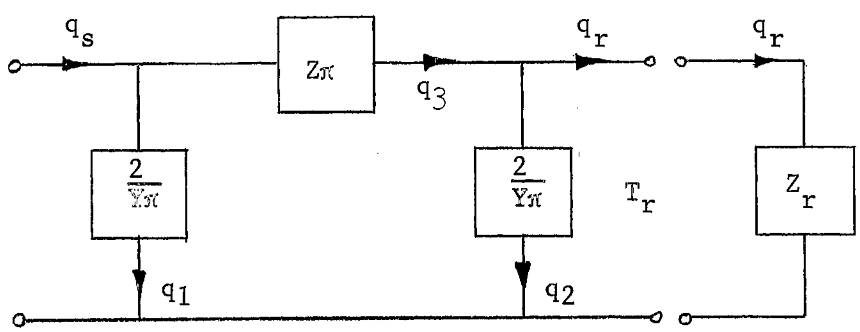

Figure 25. Equivalent π Section

The current in the left hand "pillar" is

$$q_1 = T_s \, Y\pi/2$$

The current in the "architrave" is

$$q_3 = q_s - q_1 = q_s - T_s(Y\pi/2)$$

The temperature at the right hand side is

$$T_r = T_s \; Z\pi q_3 = T_s(1+Z\pi Y\;\pi/2) - Z\pi q_s$$

We also know from equation 2.22

$$T_r = T_s \cosh\gamma L - Z_c q_s \sinh\gamma L$$

Equating coefficients of T_s and q_s we obtain

$$Z\pi = Z_c \sinh\gamma L \tag{8.20}$$

and

$$Y\pi/2 = (\cosh\gamma L - 1)/Z\pi = \frac{(e^{\gamma L/2} - e^{-\gamma L/2})^2}{Z_c(e^{\gamma L/2} - e^{-\gamma L/2})(e^{\gamma L/2} + e^{-\gamma L/2})}$$

$$= (\tanh\gamma L/2)/Z_c \tag{8.21}$$

or

$$2/Y\pi = Z_c \coth\gamma L/2 \tag{8.22}$$

If an equation for q_r had been obtained instead of T_r and solved for $Z\pi$ and $Y\pi/2$, the same results would have been obtained. This π circuit is equivalent to a distributed system with constants Z_c and γL if its components have the calculated values.

It is very simple to show that the transfer impedance for the equivalent π circuit is the same as previously obtained in equation 8.14, so time need not be taken to show this. It should be evident that this would be true since the terminal values remain unaffected.

Since the equivalent π in Figure 25 represents a finite body, a composite body could be represented as a certain number of these equivalent circuits connected in cascade. The equivalent of a composite body is shown in Figure 26. Using this configuration and the standard methods of network analysis, the terminal values can be easily determined.

The equivalent system method of approach is not as simple, and

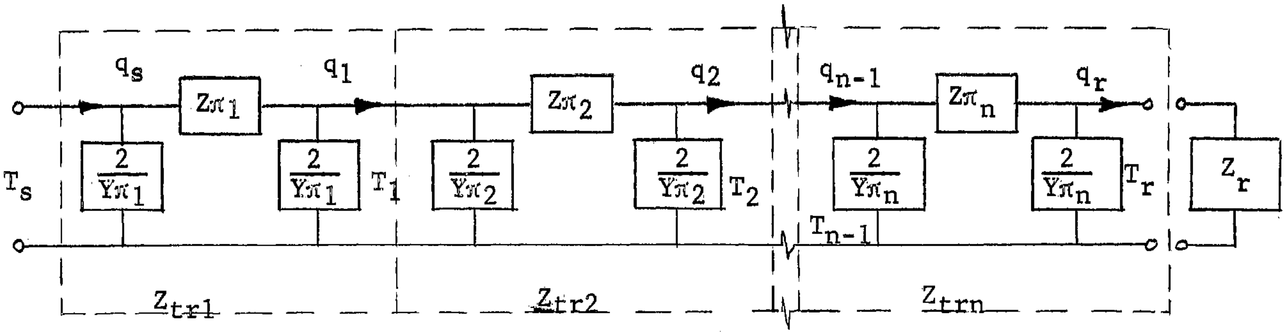

Figure 26. Equivalent of Composite Rod

does not yield itself to calculation as readily as does the transfer impedance method previously discussed; but it does introduce the possibility of design and analysis according to the principles set forth in most transmission line texts with regard to equivalent systems. It is not in accordance with the purpose of this thesis to pursue the subject further, but it should be mentioned that this type of analysis is developed fully in transmission line literature. The application of this method to heat transfer should be the subject of a later work.

It would be possible, according to network analysis, to obtain an expression comparable to equation 8.19 in terms of $Z\pi_n$ and $Y\pi_n/2$; but such an expression is sure to be much more complex than equation 8.19 and would be of no value to us.

The equivalent systems are especially interesting because they give a physical concept of the impedance approach to a finite or a composite body.

8.4 Impedance in Parallel

There is only one more composite system desirable to attack at this time. This would be a body of the composition shown in Figure 27. These two materials have a common temperature source at x = 0. The receiving end heat flow will be considered to be the sum of the heat

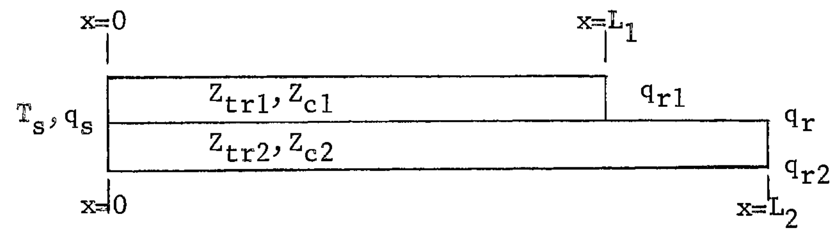

Figure 27. Composite Sections in Parallel

flow from the end of each of the sections. Remembering the definition of transfer impedance,

$$q_r = q_{r1} + q_{r2}$$

$$= T_s/Z_{tr1} + T_s/Z_{tr2} \quad (8.23)$$

$$Z_{tr} = T_s/q_r = Z_{tr1}\ Z_{tr2}/(Z_{tr1} + Z_{tr2}) \quad (8.24)$$

which is the standard equation for the total impedance of two impedances in parallel.

If either or both of the two sections of this body are composite, the respective transfer impedances can be calculated according to equation 8.19 and then substituted into equation 8.24 to give the total overall transfer impedance for the body.

Many different combinations of the cases presented, infinite, finite, insulated, uninsulated, simple, and composite, are possible. These cases, too numerous to discuss, are all subject to the basic attack presented for each class of problem and should be fairly simple to evaluate, using these methods or combinations of these methods and the basic principles of heat conduction.

CHAPTER IX

ATTENUATION ANALYSIS AND CRITERION FOR INSULATED BODIES

From previous investigation, it appears that the decay of a temperature or heat flow wave, be it the incident or reflected wave in a finite body or the single wave in an infinite body, as a function of n is the same for any insulated body irrespective of the material or any characteristic properties. Define now

$$n = x/\lambda = \text{number of wave lengths from the sending end.}$$

By this it is meant that a plot of $e^{-\sigma x}$ as a function of n would be independent of any of the properties of the propagating medium as long as it is an insulated body. It was mentioned previously in section 3.2 that the decay of the waves was due to the $e^{-\sigma x}$ term. As a function of n the exponential becomes

$$e^{-\sigma x} = e^{-\sigma n \lambda} = e^{-\sigma 2n\pi/\beta}$$

Since for an insulated body, it was previously found that

$$\sigma = \beta$$

then $$e^{-\sigma x} = e^{-2n\pi}$$

This is indeed independent of any properties of the medium, and may be plotted as shown in Figure 28. Notice that this is a very rapid decay with respect to wave length. By the time one wave length is reached, the magnitude has been reduced to a value of 0.00188. Due to the rapid decay of this term, a more correct representation of

Figure 10 is as shown in Figure 29 with quarter wave length values as indicated.

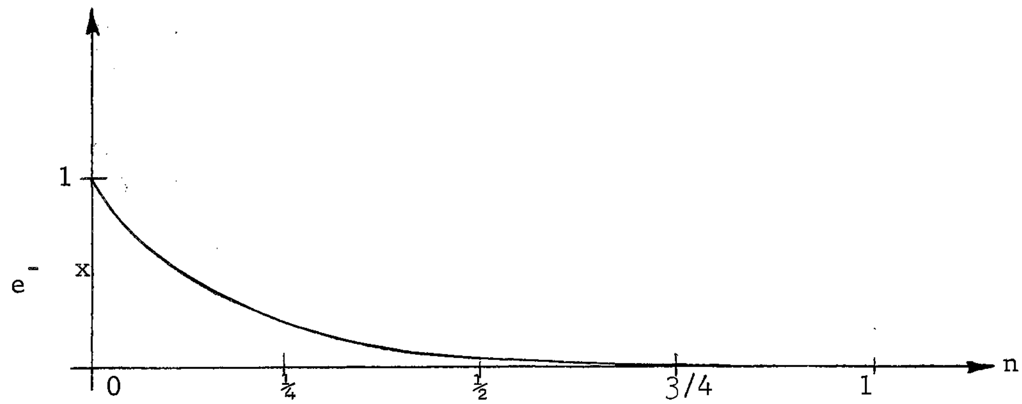

Figure 28. Exponential Attenuation

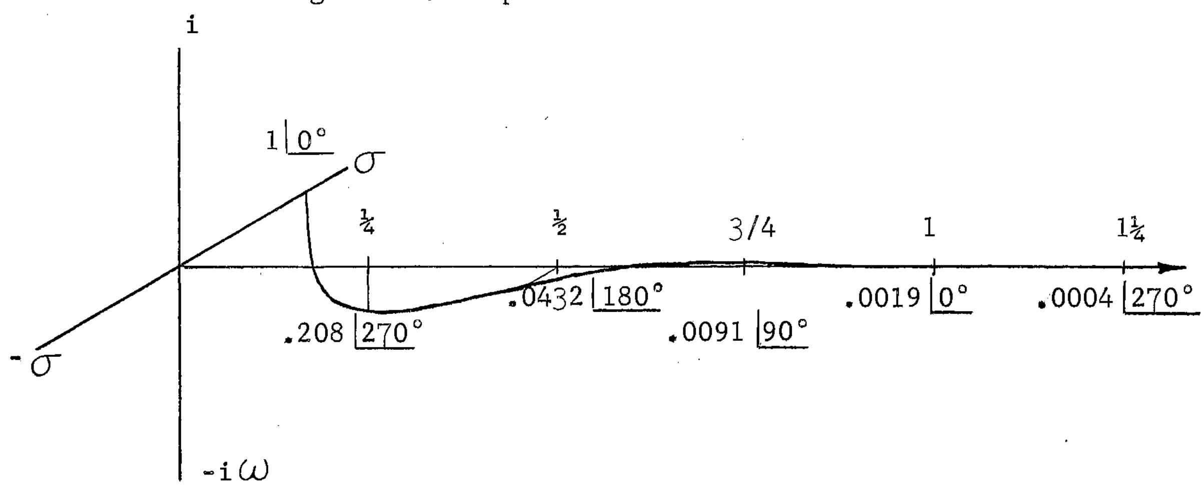

Figure 29. Attenuation of Traveling Wave

In most cases it could be said that 3/4 of a wave length would be sufficient length to insure that there would be no reflection if the body were terminated at that point. It would depend, of course, upon the accuracy desired, but generally 3/4 of a wave length could be taken as a criterion for reflection. If the body under study had a length greater than 3/4 wave length, it could safely be said that reflection would be negligible. If the body had a length less than 3/4 wave length, it might be desirable to compute the magnitude of the reflection. The criterion could be varied to suit the demands of any particular problem and applied in a similar manner.

From equations 3.12 and 5.25, it is seen that the wave length is a function of the properties of the conducting medium and the frequency of the driving function. It would be possible to draw a straight line curve for any particular material desired with λ and ω as the coordinates. A family of curves such as this, for several different materials, could be a valuable design tool in particular types of problems. If a designer was given the frequency of the driving function and the required dimensions of the body, he could go to this family of curves and pick the most suitable material to use by considering the wave lengths indicated. There could be many variations of this procedure.

The 3/4 wave length criterion, as mentioned above, need not be restricted to insulated bodies as was the discussion associated with Figure 29. The damping that occurs in an uninsulated body is greater than that in an insulated body, so this criterion would be more conservative in the uninsulated case. A more accurate criterion could be developed for uninsulated bodies if so desired, but such a criterion would be more complex, since it would no longer be a simple function of the input frequency and the properties of the medium, but would also be a function of the overall heat transfer coefficient.

It would be interesting to investigate several materials and determine if it might be possible to have reflection in some common physical system in which these materials are present.

Air at 0°F and a frequency of 1/24 cycles per hour gives a wave length of 4.58 feet. Three fourths of this wave length is 3.44 feet. The given frequency corresponds to the frequency of atmospheric temperature variation due to radiation from the sun. A common physical object in which there could be reflection in air would be a wall or roof of a building which has an enclosed air space. This air space in most cases would not be greater than six inches, which is much less than 3.44 feet. This physical situation would then give rise to considerable reflection. Reflection analysis could then be an important concept to an air conditioning or heating engineer.

In frame houses, heat transfer through the walls is determined not only by reflection in the air space in the walls but also by reflection in the wood. In southern yellow pine with 13.8% moisture and heat flowing perpendicular to the grain, the wave length is 1.34 feet at a frequency of 1/24 cycles per hour. Since the maximum thickness usually encountered in the walls of a frame building is about two inches, which is much less than 1.00[3/4(1.34) = 1] foot, reflection would certainly be significant.

If heat transfer through a concrete wall of dam was of concern, reflection might also be significant here. At a frequency of 1/24 cycles per hour, the wave length in average stone concrete is 2.38 feet. Three fourths of this value is 1.79 feet. In most concrete dams and some large concrete buildings, reflection could be neglected. In small concrete buildings, with a wall thickness less than 1.79 feet, reflection would be a significant factor.

Considering the steam engine referred to by Ingersoll and Zobel (7) in their paragraph 5.13 there exists a frequency of 100 cycles per minute or 6000 cycles per hour. Considering cylinder walls of 1% carbon steel, a wave length of .382 inches is obtained. Three fourths of this value is 0.286 inches. The thickness of the cylinder wall could possibly be less than 0.286 inches, and thus give rise to reflection. In a quarter inch steel plate, a frequency as slow as 1/24 cycle per hour would evidently give rise to considerable reflection.

These examples are only a few of the possibilities, but these few point out the fact that many common physical systems give rise to reflection and deserve approach by this method.

CHAPTER X

COMPARISON WITH EXPERIMENTAL RESULTS

Some interesting and especially applicable work has been done with reference to heat conduction through roof panels heated in a periodic manner by solar radiation by Houghten, Blackshaw, Pugh, and McDermott. (13). It is interesting to observe a comment made by them in this paper. "Research, carried on by the Research Laboratory of the American Society of Heating and Ventilation at the Pittsburgh Station of the United States Bureau of Mines, has shown that a large error may be introduced into the calculations by failure to consider the periodic character of heat flow resulting from the diurnal movement of the sun and the heat capacity of the structure, which determines the timing and magnitude of the heat wave flowing through the walls into a building on a hot sunny day." This quotation is compatible with observations made in the last chapter.

This team obtained much data for temperature and heat flow through various types of roof panels over a series of days as the external temperature varied periodically. They found that the external or applied temperature did not vary as a simple sine wave but that it could be closely approximated by a two term Fourier analysis approximation. Upon this basis attempt will be made to verify the data obtained by them for a four-inch gypsum panel. In their presentation, the team also solved this particular problem by mathematical analysis. After a solution is obtained, it will be compared with the solution they obtained, as well as the data presented by them, in order to evaluate this method of approach. The data as presented by them for this panel is

$$k = 0.1203 \text{ Btu/ft-hr-}^{\circ}\text{F}$$

$$\rho = 64.9 \text{ lb/ft}^3$$

$$c_p = 0.234$$

$$L = 4.188 \text{ in} = 0.348 \text{ ft.}$$

The fundamental frequency is of course one cycle per 24 hours. Then

$$\omega_f = 2\pi f = 2\pi 1/24 = \pi/12 = .262 \text{ rad/hr}$$

$$\beta_f = \sigma_f = \sqrt{c_p \rho \omega/2k} = \sqrt{(.234)(64.9)(\pi/12)/(2)(.1203)}$$

$$= 3.94 \text{ feet}^{-1}$$

$$Z_{cf} = \sqrt{1/2(.1203)(.234)(64.9)(\pi/12)} \quad (1-i)$$

Here the cross sectional area considered will be one square foot. The wavelength in gypsum at the fundamental frequency is

$$\lambda f = 6.28/3.94 = 1.59 \text{ feet}$$

$$3/4\ \lambda f = 1.19 \text{ feet}$$

Since three-fourths of the wavelength is much greater than 4.2 inches, some degree of reflection could be expected at this frequency. Therefore equations 7.11 and 7.13 are the ones that should be applied. The overall heat transfer coefficient for the end conditions was given as

$$h_t = 1.9 \text{ Btu/ft}^2\text{-hr-}^{\circ}\text{F}$$

Then

$$Z_r = 1/(1.9 \times 1) = 0.526 \quad \text{hr-}^{\circ}\text{F/Btu}$$

and

$$\Gamma_{rf} = (.526-1.022\ (1-i))/(.526 + 1.022\ (1-i)) = .613\ \underline{|149.4^{\circ}} = -0.528 + i(0.312)$$

The frequency of the first harmonic will be twice that of the fundamental or one cycle every twelve hours. This gives

$$\omega_h = 2\pi\ 1/12 = 0.525 \text{ rad/hour}$$

$$\sigma_h = \beta_h = \sqrt{2}\ (3.94) = 5.58 \text{ feet}^{-1}$$

$$Z_{ch} = (1/\sqrt{2})\ (1.022)(1-i) = 0.723\ (1-i) = 1.025\ \underline{|-45^{\circ}}$$

$$\lambda_h = 6.28/5.58 = 1.148 \text{ feet}$$

$$3/4\ \lambda_h = 0.86 \text{ feet} = 10.3 \text{ inches}$$

Three-fourths of the wavelength at the frequency of the first harmonic is still large enough for significant reflection. Equations 7.11 and 7.13 must be used for the first harmonic also. The reflection coefficient is

$$\Gamma_{rh} = (.526-.723(1-i))/(.526+.723(1-i)) = .52\ \underline{|135.3^{\circ}} = -0.370 + i\ 0.368$$

The Fourier approximation of the temperature at the outer surface was found by the team to be

$$T(0,t) = -27.186 \cos\omega t + 20.929 \sin\omega t - 4.308 \cos 2\omega t - 12.9 \sin 2\omega t + \ldots$$

This can be reduced to the form

$$T(0,t) = 34.3 \sin(\omega t - 52.4^{\circ}) + 13.7 \sin(2\omega t - 161.55^{\circ}) + \ldots$$

$$= \text{Im}\left(34.3\ e^{i(\omega t - 52.4^{\circ})} + 13.7\ e^{i(2\omega t - 161.55^{\circ})} + \ldots\right)$$

The amplitude of the fundamental and the first harmonic are then

$$T_{sf} = 34.3^{o}F$$

$$T_{sh} = 13.7^{o}F$$

The outer surface temperature variation takes place about an average value of $87.14^{o}F$. Zero time is taken as 4:00 A.M.

An evaluation will now be made of equation 7.11 for the fundamental component and then for the first harmonic component. Equation 7.6 gives

$$T^{+}_{sf} = 34.3 \ [1/[1+(.613 \underline{|149.4^{o}})(e^{-2(3.94)(1-i).348})]] = 33 \underline{|0.3^{o}}$$

and

$$T^{+}_{sh} = 13.7[[1/[1+1.52 \underline{|135.3^{o}})(e^{-2(5.58)(1-i)(.348)})]] = 13.7 \underline{|0.62^{o}}$$

Assuming an input of the form $e^{i(\omega t - 52.4^{o})}$ for the fundamental and of the form $e^{i(2\omega t - 161.55^{o})}$ for the first harmonic, equation 7.11 gives

$$T_{f}(x,t) = 33 \ e^{-3.94x + i(.262t - 3.94x - 52.1^{o})} + 1.295 \ e^{3.94x + i(.262t + 3.94x - 60.2^{o})}$$

and

$$T_{h}(x,t) = 13.7 \ e^{-5.58x + i(.525t - 5.58x - 160.93^{o})} + 0.148 \ e^{5.58x + i(.525t + 5.58x - 248.13^{o})}$$

Since the input was taken as a sine wave, the actual expressions for the fundamental and first harmonic components may be obtained by taking the imaginary parts of these equations. If this is done,

$$T_f(x,t) = 33\, e^{-3.94x} \sin(0.262t - 3.94x - 52.1^\circ) + 1.295\, e^{3.94x} \sin(0.262t + 3.94x - 60.2^\circ)$$

and

$$T_h(x,t) = 13.7\, e^{-5.58x} \sin(0.525t - 5.58x - 160.93^\circ) + 0.173\, e^{5.58x} \sin(0.525t + 5.58x - 248.13^\circ)$$

The average integrated temperature at the inner surface may be computed by the usual steady state methods in which the driving function is constant. An expression for the heat flow at the receiving end gives

$$q_r = [(T_{sa} - T_{ra})/L]\, kA = (T_{ra} - T_{air})\, h_t$$

The temperature of the air inside is known to be $69.6^\circ F$, so the above expression can be solved for T_{ra}. In doing this, one obtains

$$T_{ra} = 72.3^\circ F$$

The temperature distribution in the steady state with a constant applied temperature will be a straight line function, and can be written as

$$T_c(x) = 87.14 - (87.14 - 72.3/0.348)x = 87.14 - 42.7x$$

The total temperature in the bar is then the sum of this distribution and the fundamental and first harmonic components, or

$$T(x,t) = T_c(x) + T_f(x,t) + T_h(x,t)$$

The fundamental and first harmonic components of the heat flow may be computed in the same manner as were the corresponding components of the temperature. These are easily seen from equation 7.13 to be

$$q_f(x,t) = 22.8\, e^{-3.94x} \sin(0.262t - 3.94x - 7.1^\circ) - 0.894\, e^{3.94x} \sin(0.262t + 3.94x - 15.2^\circ)$$

and

$$q_h(x,t) = 13.35\ e^{-5.58x} \sin (0.525t - 5.58x - 115.93^{\circ})$$
$$-0.1442\ e^{5.58x} \sin (0.525t + 5.58x - 203.13^{\circ})$$

The steady state heat flow with an applied constant temperature of $87.14^{\circ}F$ and an average inside surface temperature of $72.3^{\circ}F$ is

$$q_c = ((T_{sa} - T_{ra})/L)kA = 5.14 \text{ Btu/hr}$$

The total heat flow at any point is then the sum of these three components, or

$$q(x,t) = q_c + q_f(x,t) + q_h(x,t)$$

The main interest in a problem such as this is the conditions at the receiving end or at the interior of the building. If x is set equal to L in the foregoing equations and they are reduced to a simpler form, then

$$T_f(L,t) = 4.82 \sin (0.262t - 97.4^{\circ})$$
$$T_h(L,t) = 1.435 \sin (0.525t - 241.83^{\circ})$$
$$T_c(L) = 72.3^{\circ}$$
$$q_f(L,t) = 8.98 \sin (0.262t - 97.25^{\circ})$$
$$q_h(L,t) = 2.76 \sin (0.525t - 212.13^{\circ})$$

The form of the input was assumed to be

$$T_f(0,t) = 34.3 \sin (0.262t - 52.4)$$

for the fundamental, and

$$T_h(0,t) = 13.7 \sin (0.525t - 161.55)$$

for the first harmonic. The lag in the temperature at the receiving end for the fundamental and first harmonic components respectively are

$$t_{lag\ Tf} = 97.4 - 52.4/(57.3)(0.262) = 3 \text{ hours}$$

and

$$t_{lag\ Th} = 241.83 - 161.55/(57.3)(0.525) = 2.67 \text{ hours}$$

This compares favorably with 3.216 hours and 2.837 hours as computed by the investigating team. Note that their values were not verified by experimental data, so it is not known whether our values were better than theirs or not. The lag of the total temperature wave at the receiving end could be obtained by differentiating the sum of the fundamental and harmonic components, setting this equal to zero, and solving for t.

The expressions for the total temperature and heat flow at the receiving end will now be given as

$$T(L,t) = 72.3^{\circ} + 4.82 \sin(0.262t-97.4^{\circ}) + 1.435 \sin(0.525t-241.83^{\circ})$$

and

$$q(L,t) = 5.14 + 8.98 \sin(0.262t-97.25^{\circ}) + 2.76 \sin(0.525t-212.13^{\circ})$$

In the paper (13), the team presented a plot of the observed heat flow at the inner surface, and a superimposed plot of the heat flow at the inner surface as calculated from their equations. This figure will now be reproduced here, and there will be plotted on it the heat flow at the inner surface as calculated from our equations. This should provide an evaluation of the equations and method of solution.

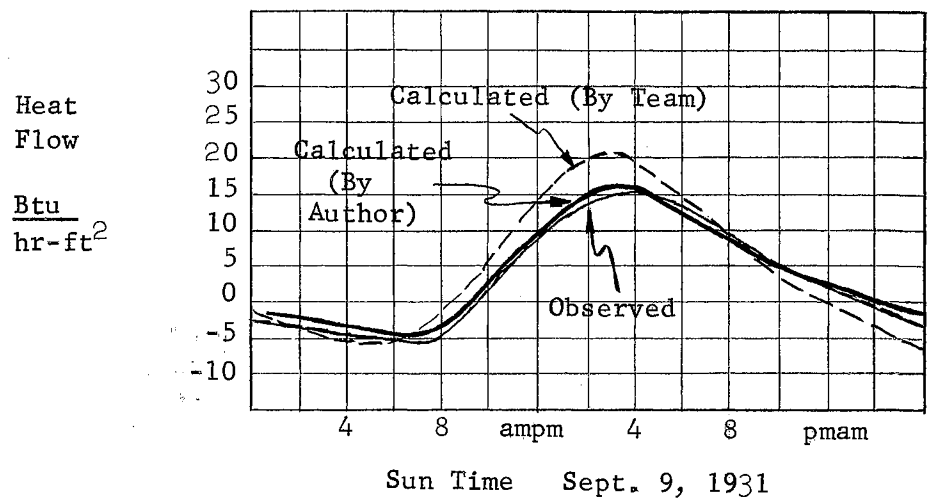

Figure 30. Heat Flow at the Inner Surface

This figure shows that the method outlined in this thesis gives an almost perfect fit to the observed curve. The fact that this solution gives a much better approximation than the solution obtained by the investigators, should be sufficient proof of the validity of these methods. The time lag of the heat flow as computed from the equations, obtained by setting the derivative of q (L,t) equal to zero, should be fairly accurate since the maximum of the author's computed curve seems to coincide with the maximum of the observed curve. At any rate, it is much more accurate than would have been obtained using the solution obtained by the investigators since its maximum deviates somewhat from the time of the observed maximum.

As a whole, the results of this comparison have been gratifying in that they indicate the desirability of using the equations and method of approach listed here in many cases involving the periodic flow of heat. There are many other areas in which the application and desirability of these methods could be shown, but the one presented here is a sufficient indication.

CONCLUSIONS

Since the object of this thesis was to see if the proposed method offered any particular advantages over conventional methods, it could be considered a success. In the relatively little space devoted to each aspect of the method, several advantages became evident in most cases. Although time and purpose did not permit full development of the analysis in many areas, sufficient background has been laid to form the basis and incentive for further investigation.

The form of the equations for temperature and heat flow themselves offer several advantages. The first of these is the fact that the solution has been obtained in one form which is applicable to all problems involving sinusoidal periodic driving functions. The basic differential equations do not have to be resolved for each individual problem. However, there is no restriction to purely sinusoidal driving functions. Since the equations are linear and the principal of superposition applies, there can be obtained a Fourier expansion of a given waveform in terms of sine and cosine functions, to be applied to our equations for a sufficient number of terms, and then the results superimposed. Thus there has been obtained a solution which is applicable to virtually any heat conduction problem concerned with periodic flow of any type.

The value of having a closed form type of solution which accurately describes the system is evident. It leads to simplified calculation and likewise a reduced possibility of error. The fact that the approach does give a closed form solution which is evidently much more accurate than conventional methods leads to a better physical understanding of the problem. The time lag, which is usually of great interest, is readily computed in most cases, and the factors affecting time lag are better understood.

The accuracy of the method is evident from the example presented in Chapter X. The little errata that was present in this example was probably due to the two term approximation. If higher order terms had been taken to approximate the driving function, such errata would probably not have been present.

The portion of this thesis with the most potential to the engineer and designer is the valuable design techniques which are inherent in the method. Some of these techniques have been presented here as time would allow, and many others worth investigating can be obtained from the transmission line field. The possibility of varying the output temperature or heat flow by varying the length of the body as discussed in section 7.2 could be a very valuable tool, especially to the heating or air conditioning engineer in the design of walls or insulation. The three-fourths of a wavelength criteria presented in Chapter IX provides a quick idea of the significance of reflection in a body. If a designer wished to reduce reflection as much as possible, he could apply this criteria, and then vary the frequency, change certain properties of the body, or change the material in the body, until he obtained the desired degree of reflection or absence of reflection.

The ease of computing temperature, heat flow, and lag in a composite body regardless of the number of sections or the configuration, makes the method very valuable. The number of different problems and configurations that could be handled is almost limitless. The concept of transfer impedance has much potential in this area.

In summary, it can be concluded that the investigation has been worthwhile. There have been pointed out several advantages inherent in this type of approach and certain areas suggested as being especially indicative of bearing fruit if subjected to further study. Whether or not this further study is performed, the ideas, equations, and techniques presented here should be sufficient to allow a complete and critical analysis and design in any problem concerning the periodic flow of heat by the mechanism of conduction with which an engineer might be confronted.

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APPENDIX

An experiment has been performed for the purpose of illustrating the presence of reflected waves in an actual physical situation. The experiment was run in three parts.

The setup for the first part consisted of a three foot long, insulated aluminum rod, three-fourths of an inch in diameter. This rod was driven at the sending end by a periodic temperature and was left exposed at the receiving end to slightly moving air at room temperature. The periodic driving function was approximately sinusoidal and could be assumed to be such. Thermocouples were placed at points 0 inches, 6 inches, and 11 inches from the sending end, and the temperature at those points were recorded as a function of time. This is shown in the following plate.

In the second part, the original rod was cut off at 13 inches, and a two foot long aluminum rod of 1/2-inch diameter was connected to it at that point. The original thermocouples were left in place and one more was located at the junction of the two rods. The receiving end conditions of the second rod were approximately the same as they were in the first case. A constant temperature source was then applied, and the resulting response was recorded as shown on the plate.

The third part used the same physical setup as was used in part two. The difference was that in this case the same periodic source was applied that was applied in part one. This was done in order that the results of part one might be compared with the results of part three where there should be measurable reflection.

The response in part one was exactly as expected. The periodic source gave rise to periodic variations of some smaller magnitude and lagging behind the source variation. The decrease in magnitude and the time lag increased as points were observed farther from the source. This part of the experiment was run only to obtain some basis on which to compare the response obtained from part three. Since the exact properties of the aluminum alloy used were unknown, the attenuation and phase constants for the material were determined from the data recorded for this case. To do this, the magnitude of the variation was noted to be 92.5°F at x=0, and 39°F at x=½ ft. It could be safely assumed that the attenuation near x=0 approximates the infinite case, then

$$e^{\sigma \frac{1}{2}} = \frac{92.5}{39}$$

$$\text{or} \qquad \sigma = 1.726$$

From the measured time lag, β can be found

$$\beta = \frac{\omega t}{.5} = 1.365$$

Before proceeding, it should be noted that this method of determining the properties of a material could be very interesting to an individual interested in obtaining such values. The method is very simple, and should provide practical values for these constants. To obtain reliable values the body under study should have a length of at least one wave length in order to insure a simulation of the infinite case. With a little care in measurement and calculations this method should provide accurate results.

The response to the constant temperature which was applied in part two gives convincing evidence of reflection. Consider first the response at thermocouple one. The curve starts at O with some initial rate of change determined by the incident wave, and continues until at point A on the curve the point receives a reflected wave which has come from the head which was placed on the rod to contain the heating element. This wave causes a small discontinuity, and then after reaching point B, the curve continues on at essentially the initial rate of change. The curve continues on until at C the point receives another reflected wave through the head, but this time about the time D is reached the reflected wave from the receiving end reaches the source. This component is of sufficient magnitude and of proper phase to change the rate of change of the curve, which is evident from point D on. The curve then continues on essentially uninterrupted except for an occasional reflection from the head.

The response at thermocouple number two follow a curve typical of a critically overdamped system until it reaches point A'. At this time the reflected wave has reached the six-inch point and changes the rate of change of the curve. This change is obvious from inspection of the curve. The curve then continues on at this rate, since there are no other reflections significant enough to affect it.

The thermocouples three and four, follow what is a typical response for an overdamped system. There is significant reflection at thermocouple three, but it is so close to the source of the reflection that there is not enough lag between the incident and reflected waves for a change to be noticeable.

The experimental response obtained for this case cannot be simply explained unless reflections are considered. These results then should be convincing evidence of the presence of reflection.

In order to compare part one of the experiment with part three, it is useful to compute what would be expected and then compare this with the actual results. Using the values of σ and β computed in part one, expressions may be found for the temperature at various points. Assuming a frequency of one cycle per hour, and using the previously given values, it is found that

$$T(\tfrac{1}{2}, t) = 38 \cos(\omega t - 39.2^\circ) + 2.06 \cos(\omega t - 122.8^\circ)$$

and $$T(\tfrac{11}{12}, t) = 18.35 \cos(\omega t - 72^\circ) + 4.24 \cos(\omega t - 90^\circ)$$

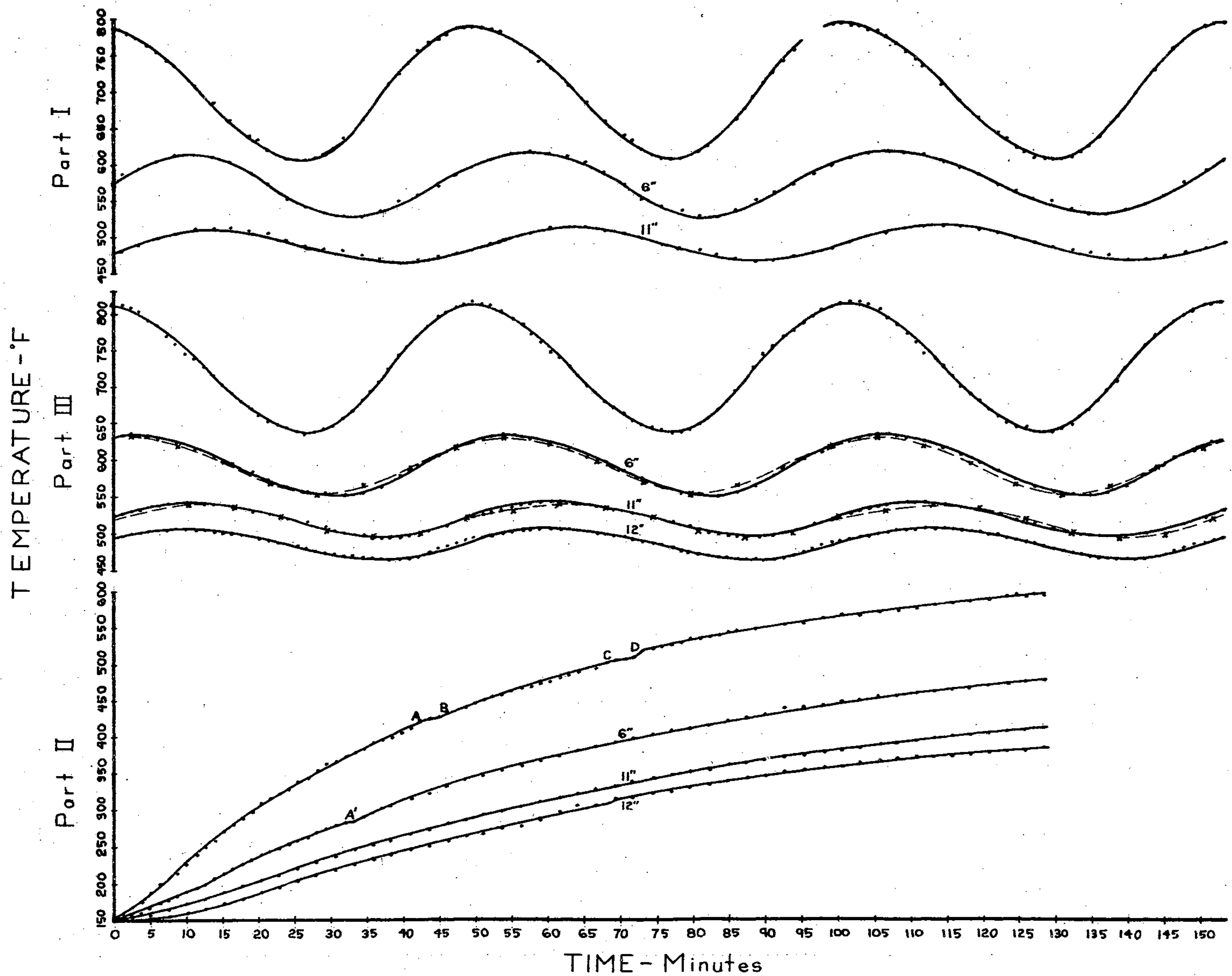
assuming that the input is a cosine function with a magnitude of 90°F. This gives maximum values for the temperatures at 6 inches and 11 inches of 39°F and 22°F which lag behind the input by 7 minutes and 10 minutes. This compares with the experimentally obtained maximums of 42°F and 19°F and lags of 7 minutes and 10 minutes. These results are very good in view of the several approximations which were made. The values of σ and β for the second section of the rod were obtained from those for the first section by proportion. This, together with the fact that the temperature at $x = 0$ was not exactly a sine wave and the fact that the frequency was not quite one cycle per hour, assures us that these values are sufficiently accurate. The values given by these two equations are plotted on the plate for the purpose of comparison.

The magnitudes computed for the system in part one are 38°F and 18.35°F as compared with the experimental values of 39°F and 18.5°F. These computed values are excellent considering the approximations made.

The significant point to be made here is that in the stepped or composite rod the magnitude of variation is greater than the magnitude of variation in the simple rod. This in itself is conclusive proof of the presence of reflection in the composite rod, for if there was no reflection the magnitude of variation at any point would be the same for both cases. The increase in variation at the 6-inch point was 3.5°F, and at the 11-inch point the increase was 8°. The increase was greatest at the point nearest the receiving end. This fact removes any doubt that the increase was due to reflection, because if the increase was due to some external effect it would no doubt have been uniform, whereas, considering reflection you would expect the greatest increase at the receiving end.

The purpose of this experiment was for the confirmation of the existence of reflected waves in heat conduction systems. Considerable effort, time, and money could have been put into this project, and some very elaborate, conclusive results could probably have been obtained. However, it was the purpose here to only confirm the existence of traveling waves in order to supplement the development presented in the body of the thesis, and to lay the foundation for further investigations.

Plate I. Recorded Response of Experimental Systems



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