VIBRATIONS IN LIQUID CYLINDERS AND LIQUID FILLED PIPES

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PREFACE

Vibrations from reciprocating pumps have created problems in the petroleum pipeline industry for years. Much work has been done with empirical studies of the vibrations by making measurements on the pipelines and pumps but little has been accomplished on the problem of determining the fundamental conditions for the transmission of vibrations along pipelines.

The purpose of this study was to provide a sound mathematical basis for the description of the phase velocities and frequencies of vibration which are transmitted as guided waves along pipelines. Equations were derived for steady state guided waves in liquid filled pipelines in space and liquid filled pipelines buried in an elastic medium.

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NOMENCLATURE

Vector Notation

Vectors are denoted by a bar above them, for example, $\bar{A}$.

$\nabla\Phi$ . . . . Gradient Φ

$\nabla\cdot\bar{A}$ . . . . Divergence $\bar{A}$

$\nabla \times \bar{A}$ . . . . Curl $\bar{A}$

$\nabla(\nabla\cdot\bar{A})$ . . . . Gradient Divergence $\bar{A}$

Unit vectors are denoted by a caret over them, for example, $\hat{r}$

Latin Characters

$\bar{A}$ . . . . Vector Potential

A_r . . . . Component of $\bar{A}$ in r direction

A_θ . . . . Component of $\bar{A}$ in θ direction

A_z . . . . Component of $\bar{A}$ in z direction

A_0, A_1 -- A_5 . . . . . . Constants of Integration

a . . . . Radius of Liquid Cylinder or Radius of Pipe

B_0, B_1 -- B_5 . . . . . . Constants of Integration

b . . . . Outside Radius of Pipe

C_0, C_1, C_2 . . . . . . Constants of Integration

D_0, D_1, D_2 . . . . . . Constants of Integration

e . . . . Elastic Strains

f . . . . Frequency

g_1 . . . . Ratio of Compressional Wave Velocity in Medium 2 to Compressional Wave Velocity in Medium 1

g_2 . . . . Ratio of Compressional Wave Velocity in Medium 2 to the Shear Wave Velocity in Medium 2

g_3 . . . . Ratio of Compressional Wave Velocity in Medium 2 to Compressional Wave Velocity in Medium 3

g_4 . . . . Ratio of Compressional Wave Velocity in Medium 2 to the Shear Wave Velocity in Medium 3

$H_p^{(1)}(x)$ . . . . Hankel Function

$H_p^{(2)}(x)$ . . . . Hankel Function

h_1 . . . . Ratio of Density in Medium 1 to Density in Medium 2

h_2 . . . . Ratio of Density in Medium 3 to Density in Medium 2

$I_n(x)$. . . . . Bessel Function

i . . . . Imaginary Number of Absolute Value Unity

$J_n(x)$ . . . . Bessel Function

$K_n(x)$ . . . . Bessel Function

K . . . . Dimensionless Form of Eigenvalue

k . . . . Eigenvalue

$\widehat{k}$ . . . . Unit Vector in z direction

L . . . . Dimensionless Form of Eigenvalue

ℓ . . . . Eigenvalue

M . . . . Dimensionless Form of Eigenvalue

m . . . . Eigenvalue

N . . . . Dimensionless Form of Eigenvalue

n . . . . Eigenvalue

P . . . . Dimensionless Form of Eigenvalue

p . . . . Eigenvalue

$\overline{Q}$ . . . . Laplace Transformed Form of Vibrational Displacement

Q_r . . . . Component of $\overline{Q}$ in r Direction

Q_z . . . . Component of $\overline{Q}$ in z Direction

$\overline{q}$ . . . . Vibrational Displacement

q_r . . . . Component of $\overline{q}$ in r Direction

q_z . . . . Component of $\overline{q}$ in z Direction

r . . . . Radial Coordinate

$\hat{r}$ . . . . Unit Vector in r Direction

s_{rr} . . . . Radial Stress

s_{rz} . . . . Shear Stress

s . . . . Laplace Variable

T . . . . Dimensionless Time

t . . . . Time

V_c . . . . Compressional Wave Velocity

V_s . . . . Shear Wave Velocity

V_{c_1} . . . . Compressional Wave Velocity in Medium 1

V_{s_1} . . . . Shear Wave Velocity in Medium 1

V_{c_2} . . . . Compressional Wave Velocity in Medium 2

V_{s_2} . . . . Shear Wave Velocity in Medium 2

V_{c_3} . . . . Compressional Wave Velocity in Medium 3

V_{s_3} . . . . Shear Wave Velocity in Medium 3

W . . . . Dimensionless Form of Angular Frequency

x . . . . Variable

$Y_n(x)$ . . . . Bessel Function

z . . . . Axial Coordinate

Greek Characters

A . . . . Dimensionless Form of Propagation Constant

α . . . . Propagation Constant

β . . . . Propagation Constant

γ . . . . Propagation Constant

δ . . . . Propagation Constant

ϵ . . . . Propagation Constant

θ . . . . Angular Coordinate

$\hat{\theta}$ . . . . Unit Vector in Angular Direction

Λ . . . . Wave Length

$\lambda, \lambda_1, \lambda_2, \lambda_3$ . . . . . Lamé' s Elastic Constant

μ, μ_1, μ_2, μ_3 . . . . . Lamé' s Elastic Constant

$\rho, \rho_1, \rho_2, \rho_3$ . . . . . Density

Φ . . . . Scalar Potential

ω . . . . Angular Frequency ($\omega = 2\pi f$)

Special Symbols

A prime means differentiation with respect to time, for example, $\overline{q}'$ is equivalent to $\frac{d\overline{q}}{dt}$.

CHAPTER I

INTRODUCTION

The study of vibrations connected with the flow of liquids through pipes and tubes is of interest in several fields of science and engineering. For example, in the study of medicine the vibrations introduced by the heart in the otherwise steady flow are through tubes with elastic and yielding walls. On the other hand, in the petroleum industry the vibrations or variations in flow are generally in steel pipes. These pipes may in some cases be considered as being rigid.

In many cases concerned with vibrations in liquid cylinders there is also a steady state flow superimposed on the vibrational motion. If, however, the velocity of the steady flow is slow in comparison to the wave velocity in the liquid the effect of the steady flow may be neglected when considering vibrations traveling in the liquid and in the containing walls.

The Purpose of the Study

The purpose of this study was (1) to provide an operational mathematical basis for setting up equations for the study of vibrations in liquid cylinders and the surrounding solid media, (2) to resolve typical problems by operational methods that had already been solved by classical mathematical methods, (3) to provide a procedure to solve more complex problems in liquid cylinder vibrations, and (4) to show how other circular symmetric problems concerned with steady state vibrations may be solved by these methods.

Previous Work

The most notable paper recently published on steady state vibrations in liquid cylinders is by William J. Jacobi (1949) (1). Only the simple problems concerned with the propagation of vibrations along liquid cylinders are considered. These are (1) liquid cylinder with rigid walls, (2) liquid cylinder with pressure release walls, (3) liquid cylinder embedded in an infinite liquid, (4) liquid cylinder with liquid walls, and (5) an approximate solution for a liquid cylinder with thin solid walls.

The approach used by Jacobi to the problems does not include the part of the general elastic equations concerned with shearing stresses in the materials considered. The present work contains the results obtained by Jacobi and extends the solutions to the case of a liquid cylinder in an infinite solid medium. The method developed in this paper also extends the mathematical equations to the case of a liquid cylinder in an elastic pipe. This case is treated both for the pipe in free space and buried in another medium.

Conditions for the Physical Application of Mathematical Results

In the study of vibrations in liquid filled pipes and tubes one encounters all variations of wall conditions from that which may be called pressure release walls to almost perfectly rigid walls. In some cases the outside walls are in air so that little vibrational energy is lost by the outside wall. In other cases the pipes are buried in another medium so that vibrational energy is lost by the pipes to the surrounding medium.

Fundamental assumptions were made as to the physical properties of the liquid and pipe walls. The liquids were considered to be ideal and to have no viscosity. Perfect elasticity was assumed to hold for both the liquids and solids unless otherwise stated. In all cases the materials were considered to be homogeneous and isotropic.

Mathematically operational methods were used and cylindrical symmetry was assumed. If a medium surrounds a pipe it has been assumed to be infinite in extent, the effect of the surface thus being neglected. This does not introduce appreciable error if the depths of burial are large compared to the radius of the pipe. The effect of the steady flow of the liquid in the pipes has been neglected.

Plan of Attack

In order to set up the specific problems considered for solution the general equation for small motions in liquids and solids has been solved by operational mathematical methods. When applied to a liquid, this equation was derived by Webster for a conservative system. This means that no wave energy is lost due to dissipative forces. Since cylindrical bodies were considered in this study, cylindrical coordinates have been used throughout. In order to simplify the problem still further, cylindrical symmetry has been assumed. This means that for a buried pipe no account has been taken of the effect of the surface. The solutions of the general equation are in terms of Bessel and Hankel functions. For the specific problems considered, the appropriate solutions which make the solutions finite along the axis and at infinity have been used. Also where there are no reflections of the wave energy the solutions which hold for outgoing waves are used. The applications of the mathematical results to each more specific case contain only the steady state guided waves in the liquid cylinder and the surrounding medium.

It has been assumed that the reader is familiar with vector notation. (2).

CHAPTER II

MATHEMATICAL FOUNDATIONS

In order to provide a mathematical foundation for this study the general equation for small motions in an elastic body has been used as a starting point. This equation is given by Webster (3) as

$$\rho \frac{\partial^2 \overline{q}}{\partial t^2} = (\lambda + \mu)\nabla(\nabla . \overline{q}) + \mu\nabla^2 \overline{q} \tag{2.1}$$

where $\overline{q}$ is the vibrational displacement, λ and μ are elastic constants due to Lame', ρ is the density, t is time and

$$\nabla^2 \overline{q} = \nabla(\nabla . \overline{q}) - \nabla x \ (\nabla x \ \overline{q}) . \tag{2.2}$$

In Equation (2.1) no body forces such as gravity are taken into consideration. In the following investigation all body forces have been neglected.

The Laplace Transformation (4) is now applied to Equation (2.1). This results in

$$\rho\left[s^2\overline{Q} - s\overline{q}\,(0+) - \overline{q}'\ (0+)\right] = (\lambda + \mu)\nabla(\nabla . \ \overline{Q}) + \mu\nabla^2\overline{Q} \tag{2.3}$$

where $\overline{Q}$ is the transform of variable $\overline{q}$, $\overline{q}\,(0+)$ is the displacement at the reference time ($t = 0$), and $\overline{q}'(0+)$ is the velocity at the reference time.

For transient solutions of vibrational problems it is frequently desirable that $\overline{q}$ (0+) and $\overline{q}'$(0+) have some specified non-zero value. For the solution of steady state problems, however, $\overline{q}$ (0+) and $\overline{q}'$ (0+) may both be taken to have zero values (5). Since steady state solutions are desired, it has been assumed that $\overline{q}$ (0+) and $\overline{q}'$(0+) are both zero. Equation (2.3) becomes

$$\rho s^2\overline{Q} = (\lambda + \mu)\nabla(\nabla . \overline{Q}) + \mu\nabla^2\overline{Q} \tag{2.4}$$

and Equation (2.2) becomes

$$\nabla^2\overline{Q} = \nabla(\nabla . \overline{Q}) - \nabla x (\nabla x\overline{Q}) . \tag{2.5}$$

In any elastic solid two forms of wave motion are possible. These are (1) waves of dilatation which are commonly known as compressional waves and (2) waves of shear which are also known as transverse waves. The vibrational motion in dilatational waves is in the direction of propagation while the motion in shear waves is perpendicular to the direction of propagation. In the interior of an ideal elastic solid these waves are propagated independently. These two forms of wave motion may be separated mathematically by the following Equation (6):

$$\overline{Q} = -\nabla\Phi + \nabla x\overline{A} \tag{2.6}$$

with the condition $\nabla . \overline{A} = 0$.

The function Φ is a scalar potential while the function $\overline{A}$ is a vector potential.

Since Φ and $\overline{A}$ are independent in the interior of an elastic solid, substitution of Equation (2.6) into Equation (2.4) leads to

$$\rho\, s^2 \Phi = (\lambda + 2\mu)\, \nabla^2\, \Phi \tag{2.7}$$

and

$$\rho\, s^2\, \overline{A} = \mu \nabla^2\, \overline{A} \quad . \tag{2.8}$$

In cylindrical coordinates

$$\nabla x \overline{A} = \begin{vmatrix} \frac{\hat{r}}{r} & \hat{\theta} & \frac{\hat{k}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix} \tag{2.9}$$

where $\hat{r}$, $\hat{\theta}$, and $\hat{k}$ are unit vectors.

For cylindrical symmetry

$$\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = 0. \tag{2.10}$$

For $\nabla . \overline{A} = 0$

$$\frac{\partial A_r}{\partial r} = \frac{\partial A_\theta}{\partial \theta} = \frac{\partial A_z}{\partial z} = 0. \tag{2.11}$$

These conditions are satisfied if

$$A_r = A_z = \frac{\partial A_\theta}{\partial \theta} = 0. \tag{2.12}$$

Then from the Equation (2.8)

$$s^2 A_\theta = \frac{\mu}{\rho}\left(\frac{\partial A_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r^2} + \frac{\partial^2 A_\theta}{\partial z^2}\right). \tag{2.13}$$

Equation (2.7) may be written

$$s^2 \Phi = \frac{\lambda + 2\mu}{\rho}\left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2}\right). \tag{2.14}$$

The compressional wave velocity V_c in an elastic solid is given by (7)

$$V_c^2 = \frac{\lambda + 2\mu}{\rho} \tag{2.15}$$

and the shear wave velocity V_s is given by (7)

$$V_s^2 = \frac{\mu}{\rho} . \tag{2.16}$$

Substitution of Equations (2.15) and (2.16) into Equations (2.13) and (2.14) leads to

$$s^2 A_\theta = V_s^2\left(\frac{\partial^2 A_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r^2} + \frac{\partial^2 A_\theta}{\partial z^2}\right) \tag{2.17}$$

and

$$s^2 \Phi = V_c^2\left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2}\right). \tag{2.18}$$

To solve Equation (2.17) in terms of Bessel functions, the following procedure was used.

Let

$$A_\theta = A_{\theta r} \; A_{\theta z} \tag{2.19}$$

where $A_{\theta r}$ is a function of r alone and $A_{\theta z}$ is a function of z alone. Then Equation (2.17) becomes

$$s^2 A_{\theta r} A_{\theta z} = V_s^2 \left(A_{\theta z} \frac{d^2 A_{\theta r}}{dr^2} + \frac{A_{\theta z}}{r} \frac{dA_{\theta r}}{dr} - \frac{A_{\theta z} \, A_{\theta r}}{r^2} + A_{\theta r} \frac{d^2 A_{\theta z}}{dz^2} \right). \tag{2.20}$$

Let

$$\frac{d^2 A_{\theta r}}{dr^2} + \frac{1}{r} \frac{d A_{\theta r}}{dr^2} - \frac{A_{\theta r}}{r^2} = - n^2 A_{\theta r} \tag{2.21}$$

where n can have real or pure imaginary values.

Substitution of Equation (2.21) into Equation (2.20) leads to

$$s^2 A_{\theta z} = V_s^2 \left[- n^2 A_{\theta z} + \frac{d^2 A_{\theta z}}{dz^2} \right]. \tag{2.22}$$

Equations (2.21) and (2.22) may be re-written

$$\frac{d^2 A_{\theta r}}{dr^2} + \frac{1}{r} \frac{d A_{\theta r}}{d r} + (n^2 - \frac{1}{r^2}) \; A_{\theta r} = 0 \tag{2.23}$$

and

$$\frac{d^2 A_{\theta z}}{dz^2} = \left(\frac{s^2}{V_s^2} + n^2\right) A_{\theta z} \quad . \tag{2.24}$$

Equation (2.23) is a Bessel equation with solutions (8)

$$A_{\theta r} = A_1 J_1 (nr) + B_1 Y_1 (nr) \quad . \tag{2.25}$$

In Equation (2.24) let

$$\frac{s^2}{V_s^2} + n^2 = \beta^2 \quad . \tag{2.26}$$

Then Equation (2.24) becomes

$$\frac{d^2 A_{\theta z}}{dz^2} = \beta^2 A_{\theta z} \tag{2.27}$$

with a solution (9)

$$A_{\theta z} = C_1 e^{\beta z} + D_1 e^{-\beta z} \quad . \tag{2.28}$$

From Equations (2.19), (2.25) and (2.28)

$$A_\theta = \left[A_1 J_1 (nr) + B_1 Y_1 (nr)\right]\left[C_1 e^{\beta z} + D_1 e^{-\beta z}\right] \quad . \tag{2.29}$$

Correspondingly, to solve Equation (2.18) in terms of Bessel functions let

$$\Phi = \Phi_r \Phi_z \tag{2.30}$$

where Φ_r is a function of r alone and Φ_z is a function of z alone.

Substitution of Equation (2.30) into Equation (2.18) yields

$$s^2 \Phi_r \Phi_z = V_s^2 \left(\Phi_z \frac{d^2\Phi_r}{dr^2} + \frac{\Phi_z}{r} \frac{d\Phi_r}{dr} + \Phi_r \frac{d^2\Phi_z}{dz^2} \right) \quad . \tag{2.31}$$

Let

$$\frac{d^2\Phi_r}{dr^2} + \frac{1}{r} \frac{d\Phi_r}{dr} = - m^2 \Phi_r \tag{2.32}$$

where m can be real or pure imaginary.

Substitution of Equation (2.32) into Equation (2.31) yields

$$s^2 \Phi_z = V_c^2 \left(- m^2 \Phi_z + \frac{d^2\Phi_z}{dz^2} \right) \quad . \tag{2.33}$$

Equations (2.32) and (2.33) may be re-written

$$\frac{d^2\Phi_r}{dr^2} + \frac{1}{r} \frac{d\Phi}{dr} + m^2 \Phi_r = 0 \quad , \tag{2.34}$$

and

$$\frac{d^2\Phi_z}{dz^2} = \left(\frac{s^2}{V_c^2} + m^2\right) \Phi_z \quad . \tag{2.35}$$

Equation (2.34) is a Bessel equation of order zero and has solutions

$$\Phi_r = A_0 J_0 (mr) + B_0 Y_0 (mr) \quad . \tag{2.36}$$

In Equation (2.35) let

$$\frac{s^2}{V_c^2} + m^2 = \alpha^2 \quad . \tag{2.37}$$

Equation (2.35) then becomes

$$\frac{d^2 \Phi_z}{dz^2} = \alpha^2 \Phi_z \tag{2.38}$$

with solutions

$$\Phi_z = C_0 e^{\alpha z} + C_1 e^{-\alpha z} \quad . \tag{2.39}$$

Combining Equations (2.30), (2.36) and (2.39) leads to

$$\Phi = \left(A_0 J_0 (mr) + B_0 Y_0 (mr)\right)\left(C_0 e^{\alpha z} + D_0 e^{-\alpha z}\right) \quad . \tag{2.40}$$

Equations (2.39) and (2.40) are the general solutions to the differential Equations (2.7) and (2.8). These solutions can be expressed in a different form by the introduction of Hankel functions $H_p^{(1)}(x)$ and $H_p^{(2)}(x)$. Thus (10)

$$H_p^{(1)}(mr) = J_p (mr) + i Y_p (mr) \tag{2.41}$$

and

$$H_p^{(2)}(mr) = J_p (mr) - i Y_p (mr) \quad . \tag{2.42}$$

Substitution of Equations (2.41) and (2.42) into Equations (2.29) and (2.40) results in

$$A_\theta = \left[A_3 H_1^{(1)}(nr) + B_3 H_1^{(2)}(nr)\right]\left[C_1 e^{\beta z} + D_1 e^{-\beta z}\right] \tag{2.43}$$

and

$$\Phi = \left[A_2 H_0^{(1)}(mr) + B_2 H_0^{(2)}(mr)\right]\left[C_0 e^{\alpha z} + D_0 e^{-\alpha z}\right] \tag{2.44}$$

where

$$A_2 = \frac{1}{2}\left(A_0 - i\,B_0\right) \tag{2.45}$$

$$A_3 = \frac{1}{2}\left(A_1 - i\,B_1\right) \tag{2.46}$$

$$B_2 = \frac{1}{2}\left(A_0 + i\,B_0\right) \tag{2.47}$$

$$B_3 = \frac{1}{2}\left(A_1 + i\,B_1\right) \quad . \tag{2.48}$$

In steady state solutions $e^{i\omega t} H_p^{(1)}(x)$ represents waves traveling inward to the origin and $e^{i\omega t} H_p^{(2)}(x)$ represents waves traveling outward from the origin (11). Use will be made later of these two properties to simplify the applications to vibrations in fluid tubes.

Other forms of solutions for Φ and A_θ are obtained when m and n are imaginary; that is

$$m = im_1 \tag{2.49}$$

$$n = in_1 \quad . \tag{2.50}$$

Thus (12)

$$J_p(ix) = i^p I_p(x) \tag{2.51}$$

$$J_p(ix) + i\,Y_p(ix) = \frac{2}{\pi}\, i^{-(p+1)} K_p(x) \quad . \tag{2.52}$$

Substitution of Equations (2.51) and (2.52) into Equations (2.21) and (2.32) leads to solutions

$$\Phi = \left(A_4 I_0(m_1 r) + B_4 K_0(m_1 r)\right)\left(C_2 e^{\alpha_1 z} + D_2 e^{-\alpha_1 z}\right) \tag{2.53}$$

$$A_\theta = \left(A_5 I_1(n_1 r) + B_5 K_1(n_1 r)\right)\left(C_3 e^{\beta_1 z} + D_3 e^{-\beta_1 z}\right) \tag{2.54}$$

where

$$A_4 = A_0 - B_0 \tag{2.55}$$

$$B_4 = \frac{-2B_0 i}{\pi} \tag{2.56}$$

$$A_5 = (A_1 - B_1)\, i \tag{2.57}$$

$$B_5 = \frac{-2B_1}{\pi} \tag{2.58}$$

and

$$\frac{s^2}{V_c^2} - m_1^2 = \alpha_1^2 \tag{2.59}$$

$$\frac{s^2}{V_s^2} - n_1^2 = \beta_1^2 \quad . \tag{2.60}$$

Equations (2.40), (2.44) and (2.53) are solutions to Equation (2.7) and Equations (2.29), (2.43) and (2.54) are solutions to Equation (2.8). In the application of these equations to specific problems, the choice of form depends on the boundary conditions to be satisfied. Thus, the particular equations used for a specific problem will determine whether m and n are to be real or imaginary.

CHAPTER III

BOUNDARY CONDITIONS FOR THE GENERAL SOLUTION

The exact boundary conditions for various configurations of liquid cylinders and liquid filled pipes will depend on the specific configuration of the individual problem. However, several conditions may be applied to the general solutions to make them more useful. In this chapter the general solutions of Equations (2.7) and (2.8) have had these conditions applied to them.

The Laplace transformed displacement $\overline{Q}$ is given in terms of the potentials Φ and $\overline{A}$ by Equation (2.6) This is

$$\overline{Q} = -\nabla\Phi + \nabla x\overline{A} \quad . \tag{3.1}$$

With the aid of Equations (2.9), (2.10), (2.11), and (2.12)

$$\nabla x A_\theta \hat{\theta} = -\hat{r}\frac{\partial A_\theta}{\partial z} + \frac{\hat{k}}{r}\left(A_\theta + r\frac{\partial A_\theta}{\partial r}\right) \tag{3.2}$$

and

$$\nabla\Phi = \hat{r}\frac{\partial\Phi}{\partial r} + \hat{k}\frac{\partial\Phi}{\partial z} \quad . \tag{3.3}$$

Now $\overline{Q}$ is a vector and may be split into two components, a component in the r direction Q_r and into a component in the z

direction Q_z . Then

$$Q_z = \frac{A_\theta}{r} + \frac{\partial A_\theta}{\partial r} - \frac{\partial \Phi}{\partial z} \tag{3.4}$$

and

$$Q_r = - \frac{\partial \Phi}{\partial r} - \frac{\partial A_\theta}{\partial z} \quad . \tag{3.5}$$

Solutions for guided waves in the z-direction are obtained by the choice of zero for the C's and non-zero values for the D's if α has an imaginary value. Thus $e^{i\omega t}$ multiplied by $e^{-i\alpha z}$ yields $e^{i\omega\left(t - \frac{\alpha z}{\omega}\right)}$. The reason for this result will be apparent when the solutions are reduced to steady state conditions. Thus Equations (2.40), (2.44), (2.53), (2.29), (2.43) and (2.54) become

$$\Phi = \left[A_0 J_0 (mr) + B_0 Y_0 (mr)\right] D_0 e^{-\alpha z} \tag{3.6}$$

$$\Phi = \left[A_2 H_0^{(1)} (mr) + B_2 H_1^{(2)} (mr)\right] D_0 e^{-\alpha z} \tag{3.7}$$

$$\Phi = \left[A_4 I_0 (m_1 r) + B_4 K_0 (m_1 r)\right] D_0 e^{-\alpha_1 z} \tag{3.8}$$

$$A_\theta = \left[A_1 J_1 (nr) + B_1 Y_1 (nr)\right] D_1 e^{-\beta z} \tag{3.9}$$

$$A_\theta = \left[A_3 H_1^{(1)} (nr) + B_3 H_1^{(2)} (nr)\right] D_1 e^{-\beta z} \tag{3.10}$$

$$A_\theta = \left[A_5 I_1 (n_1 r) + B_5 K_1 (n_1 r)\right] D_1 e^{-\beta_1 z} \tag{3.11}$$

where m, n, α and β for Equations (3.6), (3.7), (3.9), and (3.10) are given by Equations (2.26) and (2.37), while for Equations (3.8) and

(3.11) m, n, α, β are given by

$$\frac{s^2}{V_s^2} - n_1^2 = \beta_1^2 \tag{3.12}$$

$$\frac{s^2}{V_c^2} - m_1^2 = \alpha_1^2 \quad . \tag{3.13}$$

In Equations (3.6) through (3.11), D_0 and D_1 are multiplying constants and their values may be included in the A's and B's. Thus, in the following this has been done.

To evaluate Q_z find the components of Q_z as given by Equation (3.4):

$$\frac{A_\theta}{r} = \left(\frac{A_1}{r} J_1 (nr) + \frac{B_1}{r} Y_1 (nr) \right) e^{-\beta z} \tag{3.14}$$

$$\frac{\partial A_\theta}{\partial r} = \left(A_1 \left(nJ_0 (nr) - \frac{J_1 (nr)}{r} \right) + B_1 \left(nY_0 (nr) - \frac{Y_1 (nr)}{r} \right) \right) e^{-\beta z} \tag{3.15}$$

$$- \frac{\partial \Phi}{\partial z} = \alpha \left(A_0 J_0 (mr) + B_0 Y_0 (mr) \right) e^{-\alpha z} \quad . \tag{3.16}$$

Therefore

$$Q_z = \left(A_0 J_0 (mr) + B_0 Y_0 (mr) \right) \alpha \, e^{-\alpha z} + \left(A_1 J_0 (nr) + B_1 Y_0 (nr) \right) n \, e^{-\beta z} \quad . \tag{3.17}$$

Correspondingly one evaluates Q_r from the components given by

Equation (3.5). This leads to

$$Q_r = m\left[A_0 J_1\,(mr) + B_0 Y_1\,(mr)\right] e^{-\alpha z}$$

$$+ \beta\left[A_1 J_1\,(nr) + B_1 Y_1\,(nr)\right] e^{-\beta z} \quad . \tag{3.18}$$

Correspondingly from Equations (3.7) and (3.10) $\overline{Q}$ is given by

$$Q_r = m\left[A_2 H_1^{(2)}\,(mr) + B_2 H_1^{(2)}\,(mr)\right] e^{-\alpha z}$$

$$+ \beta\left[A_3 H_1^{(1)}\,(nr) + B_3 H_1^{(2)}\,(nr)\right] e^{-\beta z} \tag{3.19}$$

and

$$Q_z = \alpha\left[A_2 H_0^{(1)}\,(mr) + B_2 H_0^{(2)}\,(mr)\right] e^{-\alpha z}$$

$$+ n\left[A_3 H_0^{(1)}\,(nr) + B_3 H_0^{(2)}\,(nr)\right] e^{-\beta z} \quad . \tag{3.20}$$

From Equations (3. 9) and (3.11)

$$Q_r = m_1\left[-A_4 I_1\;(mr) + B_4 K_1\,(m_1 r)\right] e^{-\alpha_1 z}$$

$$+ \beta_1\left[A_5 I_1\,(n_1 r) + B_5 K_1\,(n_1 r)\right] e^{-\beta_1 z} \tag{3.21}$$

$$Q_z = \alpha_1\left[A_4 I_0\,(m_1 r) + B_4 K_0\,(m_1 r)\right] e^{-\alpha_1 z}$$

$$+ n_1\left[A_5 I_0\,(n_1 r) - B_5 K_0\,(n_1 r)\right] e^{-\beta_1 z} \quad . \tag{3.22}$$

In the applications of the solutions for Equation (2.1) to various physical conditions steady state solutions will be considered. For steady state solutions s may be replaced by $i\omega$. This justifies assuming that $\overline{q}$ (0+) and $\overline{q}'$ (0+) are zero (5). With this substitution Equations (3.17), (3.18), (3.19), (3.20), (3.21), and (3.22) may be written

$$q_r = m\left[A_0 J_1(mr) + B_1 Y_1(mr)\right] e^{i\omega t - \alpha z} + \beta\left[A_1 J_1(nr) + B_1 Y_1(nr)\right] e^{i\omega t - \beta z} \tag{3.23}$$

$$q_z = \alpha\left[A_0 J_0(mr) + B_0 Y_0(mr)\right] e^{i\omega t - \alpha z} + n\left[A_1 J_0(nr) + B_1 Y_0(nr)\right] e^{i\omega t - \beta z} \tag{3.24}$$

$$q_r = m\left[A_2 H_1^{(1)}(mr) + B_2 H_1^{(2)}(mr)\right] e^{i\omega t - \alpha z} + \beta\left[A_3 H_1^{(1)}(nr) + B_3 H_1^{(2)}(nr)\right] e^{i\omega t - \beta z} \tag{3.25}$$

$$q_z = \alpha\left[A_2 H_0^{(1)}(mr) + B_2 H_0^{(2)}(mr)\right] e^{i\omega t - \alpha z} + n\left[A_3 H_0^{(1)}(nr) + B_3 H_0^{(2)}(nr)\right] e^{i\omega t - \beta z} \tag{3.26}$$

$$q_r = m_1 \left\{ A_4 I_1 (m_1 r) + B_4 K_1 (mr) \right\} e^{i\omega t - \alpha_1 z}$$

$$+ \beta_1 \left\{ -A_5 I_1 (n_1 r) + B_5 K_1 (n_1 r) \right\} e^{i\omega t - \beta_1 z} \tag{3.27}$$

$$q_z = \alpha_1 \left\{ A_4 I_0 (m_1 r) + B_4 K_0 (m_1 r) \right\} e^{i\omega t - \alpha_1 z}$$

$$+ n_1 \left\{ A_5 I_0 (n_1 r) - B_5 K_0 (n_1 r) \right\} e^{i\omega t - \beta_1 z} \tag{3.28}$$

where

$$\alpha^2 = m^2 - \frac{\omega^2}{V_c^2} \tag{3.29}$$

$$\beta^2 = n^2 - \frac{\omega^2}{V_s^2} \tag{3.30}$$

$$m_1^2 + \alpha_1^2 = \frac{-\omega^2}{V_c^2} \tag{3.31}$$

$$n_1^2 + \beta_1^2 = \frac{-\omega^2}{V_s^2} . \tag{3.32}$$

In the following the factor $e^{i\omega t}$ will be omitted for convenience in writing the equations.

The stresses in a homogeneous solid are as follows (13):

$$s_{rr} = \lambda \Delta + 2\mu \frac{\partial q_r}{\partial r} \tag{3.33}$$

$$s_{rz} = \mu \left(\frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r} \right) \tag{3.34}$$

where

$$\triangle = \frac{\partial q_r}{\partial r} + \frac{q_r}{r} + \frac{\partial q_z}{\partial z} \quad . \tag{3.35}$$

Correspondingly the strains (13)

$$e_{rr} = \frac{\partial q_r}{\partial r} \tag{3.36}$$

$$e_{zz} = \frac{\partial q_z}{\partial z} \tag{3.37}$$

$$e_{rz} = \frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r} \tag{3.38}$$

where q is the displacement, e is the strain and s is the stress.

For fluids $\mu = 0$. This yields

$$s_{rr} = \lambda\left(\frac{\partial q_r}{\partial r} + \frac{q_r}{r} + \frac{\partial q_z}{\partial z}\right) \tag{3.39}$$

$$s_{rz} = 0. \tag{3.40}$$

This completes the discussion of the general conditions which apply to all steady state applications of the solutions of Equation (2.1). In the following chapters specific problems will be considered.

CHAPTER IV

APPLICATIONS TO LIQUID CYLINDERS

In order to check results previously obtained by Jacobi (1) and others the results of the previous chapter will be applied in this chapter to some previously solved problems.

Liquid Cylinder with Pressure Release Walls

A liquid cylinder is considered which has walls such that no pressure is exerted on the walls. This means that the radial stress at the surface of the cylinder is zero. An approximation to this condition in practice is a thin walled rubber tube. Since the liquid is ideal, the vector potential part of the solution is zero. Under these conditions Equations (3.23) and (3.24) become

$$q_r = m\left[A_0 J_1(mr) + B_0 Y_1(mr)\right] e^{-\alpha z} \tag{4.1}$$

and

$$q_z = \alpha\left[A_0 J_0(mr) + B_0 Y_0(mr)\right] e^{-\alpha z} \quad . \tag{4.2}$$

For Equations (4.1) and (4.2) to describe the motion along the axis of the tube q_r and q_z must be finite. This means that B_0 must be zero. This leads to

$$q_r = m A_0 J_1(mr)\, e^{-\alpha z} \tag{4.3}$$

$$q_z = \alpha A_0 J_0(mr)\, e^{-\alpha z} . \tag{4.4}$$

Since the radial stress is zero at the surface of the tube,

$$s_{aa} = 0 \tag{4.5}$$

where a is the radius of the tube.

From Equations (3.39), (4.3), (4.4) and (4.5)

$$s_{aa} = \rho \omega^2 J_0(ma)\, A_0 e^{-\alpha z} . \tag{4.6}$$

Since in general A_0 and ω^2 are not zero, $J_0(ma) = 0$ and the m's are zeros of $J_0(ma)$. In this case m does not have a zero value and there is a low frequency cutoff since from Equation (3.29)

$$\omega^2 = V_c^2 (m^2 - \alpha^2) . \tag{4.7}$$

This result has been reported by Jacobi (1).

Liquid Tube with Rigid Walls

Next a liquid filled pipe with rigid walls is considered. The condition that the walls be perfectly rigid is that the radial component of displacement in the liquid be zero at the inside surface of the pipe. This yields from Equation (4.3)

$$q_r = m A_0 J_1(ma)\, e^{-\alpha z} = 0 . \tag{4.8}$$

Since m and A_0 are not in general zero, $J_1(ma) = 0$. In this case, m = 0 is a zero of $J_1(ma)$, and the zero mode of vibration is transmitted. There is no low frequency cutoff. This result was also reported by Jacobi (1).

Liquid Cylinder Buried in Infinite Liquid

For a liquid cylinder buried in an infinite liquid body the boundary conditions may be taken that the radial stress and displacement are continuous across the boundary. This is expressed mathematically as

$$(s_{aa})_1 = (s_{aa})_2 \tag{4.9}$$

$$(q_a)_1 = (q_a)_2 \tag{4.10}$$

where the subscripts 1 and 2 refer to the liquid cylinder and the surrounding fluid body respectively.

The solutions given in terms of Bessel functions will be used for medium 1, while the solutions given in terms of the Hankel functions will be used for medium 2. This is to permit a finite solution along the axis and to permit outgoing waves in medium 2.

Using Equations (3.23), (3.25) and (3.39) results in

$$(s_{rr})_1 = \lambda_1 (m^2 - \alpha^2)\, J_0(mr)\, A_0 e^{-\alpha z} \tag{4.11}$$

$$(s_{rr})_2 = \lambda_2 (n^2 - \beta^2)\, H_0^{(2)}(nr)\, A_3 e^{-\beta z} \tag{4.12}$$

$$(q_r)_1 = m J_1(mr)\, A_0 e^{-\alpha z} \tag{4.13}$$

$$(q_r)_2 = n H_1^{(2)}(nr)\, A_3 e^{-\beta z} \tag{4.14}$$

Equations (4.9) and (4.10) yield

$$\lambda_1 (m^2 - \alpha^2)\, J_0(ma)\, A_0 e^{-\alpha z} = \lambda_2 (n^2 - \beta^2)\, H_0^{(2)}(na) A_3 e^{-\beta z} \tag{4.15}$$

and Equations (4.13) and (4.14) yield

$$mJ_1(ma)A_0 e^{-\alpha z} = nH_1^{(2)}(na)e^{-\beta z} \qquad (4.16)$$

along the tube at radius a.

Since Equations (4.15) and (4.16) hold for all values of z,

$$\alpha = \beta, \qquad (4.17)$$

$$\lambda_1(m^2 - \alpha^2) J_0(ma) A_0 - \lambda_2(n^2 - \alpha^2) H_0^{(2)}(na) A_3 = 0 \qquad (4.18)$$

$$mJ_1(ma) A_0 - nH_1^{(2)}(na) A_3 = 0 . \qquad (4.19)$$

If A_0 and A_3 are not identically zero, the determinant of the coefficients vanishes. Thus,

$$\begin{vmatrix} \lambda_1(m^2 - \alpha^2) J_0(ma) & \lambda_2(n^2 - \alpha^2) H_0^{(2)}(na) \\ m J_1(ma) & nH_1^{(2)}(na) \end{vmatrix} = 0 . \qquad (4.20)$$

Values of m, n and α which satisfy this equation determine the permitted frequencies and phase velocities. This result has been obtained by Jacobi (1).

This completes the comparison of the results obtained by previous workers with the results obtained by the operational solutions introduced in this paper. The following chapter introduces solutions which are extensions of previous work.

CHAPTER V

LIQUID CYLINDER IN AN INFINITE ELASTIC SOLID

A liquid cylinder in an infinite elastic solid approximates a liquid filled pipe buried in the ground if the walls of the pipe are quite thin.

Because shear waves are possible in the solid, two functions are needed to describe the wave motion. In the liquid cylinder only one function is needed as shear waves are not propagated.

However, there is a choice of two forms of solution in the liquid cylinder. These are the Bessel functions of the first kind, $J_p(mr)$, and the modified Bessel functions of the first kind, $I_p(mr)$. The $J_p(mr)$ functions are considered first. The solutions using the $I_p(mr)$ functions may be obtained by using Equations (3. 27) and (3. 28), or they may be obtained by replacing m by im in the equations resulting from the treatment of the case involving the $J_p(mr)$ functions.

Boundary conditions at the contact of the liquid and the solid can be set up by assuming continuity in the radial displacement and stress. If the liquid is considered to be ideal, it will not adhere to the solid and therefore there will be no shearing stress at the surface of the solid. The resulting boundary conditions will be

$$(q_a)_1 = (q_a)_2 \tag{5.1}$$

$$(s_{aa})_1 = (s_{aa})_2 \tag{5.2}$$

and

$$(s_{az})_2 = 0 \tag{5.3}$$

where the subscript 1 refers to the liquid cylinder and the subscript 2 refers to the elastic solid.

In the liquid cylinder the radial displacement is given from Equation (4. 3) by

$$(q_r)_1 = mJ_1(mr)\,A_0 e^{\alpha z} \quad , \tag{5.4}$$

and the radial stress is, from using Equations (3. 39), (4. 3), and (4. 4),

$$(s_{rr})_1 = \omega^2 \rho_1 J_0(mr)\,A_0 e^{-\alpha z} \quad . \tag{5.5}$$

In the elastic solid a form of solution for outgoing waves is used, i.e., $H_p^{(2)}(x)$. The compressional and shear components of radial displacement are obtained from Equations (3. 25), (3. 26) and (3. 33) as

$$(q_r)_2 = nH_1^{(2)}(nr)\,A_3 e^{-\beta z} + \gamma H_1^{(2)}(pr) \quad e^{-\gamma z} \quad , \tag{5.6}$$

and the radial stress is

$$(s_{rr})_2 = \left[\left(\lambda_2(n^2 - \beta^2) + 2\mu n^2\right) H_0^{(2)}(nr) - \frac{2\mu n}{r} H_1^{(2)}(nr)\right] A_1 e^{-\beta z}$$
$$+ 2\mu\gamma \left[pH_0^{(2)}(pr) - \frac{1}{r} H_1^{(2)}(pr)\right] A_2 e^{-\gamma z} \quad . \tag{5.7}$$

The tangential stress in the solid is given from Equations (3. 25), (3. 26) and (3. 34) by

$$(s_{rz})_2 = -\mu\left[2n\beta H_1^{(2)}(nr)\,A_1 e^{-\beta z} + (p^2 + r^2)H_1^{(2)}(pr)A_2 e^{-\gamma z}\right]. \tag{5.8}$$

At the boundary between the liquid and solid, using Equations (5.1), (5.4), and (5.6) results in

$$mJ_1(ma)\,A_0e^{-\gamma z} = nH_1^{(2)}(na)\,A_1e^{-\beta z} + \gamma H_1^{(2)}(pa)A_2e^{-\gamma z}, \qquad (5.9)$$

and using Equations (5.2), (5.5) and (5.7) results in

$$\omega^2\rho_1J_0(ma)\,A_0e^{-\alpha z} = \left[\left(\lambda_2(n^2-\beta^2) + 2\mu n^2\right)H_0^{(2)}(na) - \frac{2\mu n}{a}H_1^{(2)}(na)\right]A_1e^{-\beta z}$$

$$+\quad 2\mu\gamma\left[p\,H_0^{(2)}(pa) - \frac{1}{a}H_1^{(2)}(pa)\right]A_2e^{-\gamma z}. \qquad (5.10)$$

From Equations (5.3) and (5.8) results

$$-\mu\left[2n\beta H_1^{(2)}(na)\,A_1e^{-\beta z} + (p^2+\gamma^2)\,H_1^{(2)}(pa)\,A_2e^{-\gamma z}\right] = 0. \quad (5.11)$$

If Equations (5.9), (5.10), and (5.11) are to hold as z varies along the tube, they must be independent of z. This is true if

$$\alpha = \beta = \gamma. \qquad (5.12)$$

Thus, β and γ may be replaced by α.

If solutions of Equations (5.9), (5.10) and (5.11) exist for non-zero values of the A's, the determinant of the coefficients of the A's must vanish. That is

$$\begin{vmatrix} mJ_1(ma) & nH_1^{(2)}(na) & \gamma H_1^{(2)}(pa) \\ 0 & 2n\alpha H_1^{(2)}(na) & (p^2+\gamma^2)H_1^{(2)}(pa) \\ \omega^2\rho_1(ma) & \left[\lambda^2(n^2-\beta^2)+2\mu n^2\right]H_0^2(na) - \dfrac{2\mu n}{a}H_1^{(2)}(na) & 2\mu\gamma\left[p\,H_0^{(2)}(pa) - \dfrac{1}{a}H_1^{(2)}(pa)\right] \end{vmatrix} = 0 \qquad (5.13)$$

where

$$m^2 - \alpha^2 = \frac{\omega^2}{V_{c_1}^2} \qquad (5.14)$$

$$n^2 - \alpha^2 = \frac{\omega^2}{V_{c_2}^2} \tag{5.15}$$

$$p^2 - \alpha^2 = \frac{\omega^2}{V_{s_2}^2} \; . \tag{5.16}$$

Examination of Equations (2.21) and (2.32) reveals that m, n, and p are each real or imaginary. Complex values for m, n, and p do not satisfy the differential equations. Now $H_0^{(2)}(x)$ is complex if x is real but is imaginary if x is negative imaginary, and correspondingly $H_0^{(2)}(x)$ is real if x is negative imaginary but complex if x is real.

Therefore, if m is real and n and p are negative imaginary, Equation (5.13) can be solved for the existing frequencies of vibration and the phase velocities.

The following substitutions will put Equations (5.13), (5.14), (5.15) and (5.16) in dimensionless form.

$$ma = M \tag{5.17}$$

$$na = -iN \tag{5.18}$$

$$pa = -iP \tag{5.19}$$

$$a\alpha = iA \tag{5.20}$$

$$\frac{\omega a}{V_{c_2}} = W \tag{5.21}$$

$$\frac{V_{c_2}}{V_{c_1}} = g_1 \tag{5.22}$$

$$\frac{V_{c_2}}{V_{s_2}} = g_2 \tag{5.23}$$

$$\frac{\rho_1}{\rho_2} = h_1 \tag{5.24}$$

These substitutions provide equations in convenient form for calculation. Thus, Equations (5.14), (5.15) and (5.16) become

$$A^2 + M^2 = W^2 g_1^2 \tag{5.25}$$

$$A^2 - N^2 = W^2 \tag{5.26}$$

$$A^2 - P^2 = W^2 g_2^2 \ . \tag{5.27}$$

Equation (5.13) becomes

$$\begin{vmatrix} MJ_1(M) & iNH_1^{(2)}(-iN) & iAH_1^{(2)}(-iP) \\ 0 & 2ANiH_1^{(2)}(-iN) & -(P^2+A^2)H_1^{(2)}(-iP) \\ W^2h_1g_2^2J_0(M) & \begin{array}{l}-(A^2+P^2)H_0^{(2)}(-iN) \\ \quad + 2iNH_1^{(2)}(-iN)\end{array} & \begin{array}{l}2Ai\left[-iPH_0^{(2)}(-iP)\right. \\ \quad \left.-H_1^{(2)}(-iP)\right]\end{array} \end{vmatrix} = 0 \ . \tag{5.28}$$

Now

$$H_0^{(2)}(-ix) = \frac{2i}{\pi} K_0(x) \tag{5.29}$$

$$H_1^{(2)}(-ix) = \frac{-2}{\pi} K_1(x) \ . \tag{5.30}$$

Thus, Equation (5.28) becomes

$$\begin{vmatrix} MJ_1(M) & -NK_1(N) & -AK_1(P) \\ 0 & 2ANK_1(N) & (P^2 + A^2)K_1(P) \\ W^2 h_1 g_2^2 J_0(M) & (A^2 + P^2) J_0(N) + 2NK_1(N) & 2A\left[PK_0(P) + K_1(P)\right] \end{vmatrix} = 0 . \tag{5.31}$$

Equation (5.31) may be simplified to

$$A\left((1 + \frac{P^2}{A^2})^2 \frac{K_0(N)}{\frac{N}{A} K_1(N)} - \frac{4P}{A}\frac{K_0(P)}{K_1(P)} - h_1(1 - \frac{P^2}{A^2})^2 \frac{J_0(M)}{\frac{M}{A} J_1(M)}\right) = 2(1 - \frac{P^2}{A^2}) . \tag{5.32}$$

The other form of solution mentioned at the beginning of this chapter is obtained by replacing M by iM in Equations (5.17), (5.25) and (5.32). By using Equation (2.51) this results in

$$A^2 - M^2 = W^2 g_1^2 \tag{5.33}$$

$$A^2 - N^2 = W^2 \tag{5.34}$$

$$A^2 - P^2 = W^2 g_2^2 \tag{5.35}$$

$$A\left((1 + \frac{P^2}{A^2})^2 \frac{K_0(N)}{\frac{N}{A} K_1(N)} - \frac{4P}{A}\frac{K_0(P)}{K_1(P)} + h_1(1 - \frac{P^2}{A^2})^2 \frac{I_0(M)}{\frac{M}{A} I_1(M)}\right) = 2(1 - \frac{P^2}{A^2}) . \tag{5.36}$$

Equations (5.25), (5.26), (5.27) and (5.32) were solved for A, W, M, N, and P for specifically chosen values of the ratio W/A. These calculations were made on an IBM 650. The write-up of the IBM program is given in Appendix A.

From the values of W and A curves were plotted showing the relationship between wave length and phase velocity.

The phase velocity may be obtained from the factor $e^{i\omega t - \alpha z}$ in Equations (5.23) and (5.24). Any function of $t - \frac{z}{V_p}$ represents wave motion traveling in the positive z direction with phase velocity V_p (14). Comparison of these two expressions yields

$$V_p = \frac{i\omega}{\alpha} \quad . \tag{5.37}$$

In terms of the dimensionless quantities W and A , Equation (5.37) becomes

$$V_p = \frac{W}{A} V_{c_2} \quad . \tag{5.38}$$

Thus

$$\frac{V_p}{V_{c_2}} = \frac{W}{A} \tag{5.39}$$

and

$$\frac{V_p}{V_{c_1}} = \frac{W}{A} g_1 \quad . \tag{5.40}$$

This gives the ratio of the phase velocity and the velocity in the liquid cylinder.

With the wave length denoted by Λ

$$\omega = 2\pi \frac{V_p}{\Lambda} \quad . \tag{5.41}$$

Thus in terms of A

$$\Lambda = \frac{2\pi a}{A} \tag{5.42}$$

and

$$\frac{\Lambda}{2a} = \frac{\pi}{A} \quad . \tag{5.43}$$

This gives the ratio of the wave length and the diameter of the liquid cylinder.

Figures 1, 2, and 3 show the computed relationships between

$\frac{V_p}{V_{c_1}}$ and $\frac{\Lambda}{2a}$ for Equations (5.25), (5.26), (5.27) and (5.32).

These curves are for guided waves in the liquid cylinder. The following list gives the values of g_1, g_2, and h_1 in the Figures:

Figure	g_1	g_2	h_1
1	1.5	1.25	0.4
2	2.0	1.5	0.4
3	2.5	1.5	0.4

In each of these Figures two curves are shown. Due to the periodic nature of $J_0(M)$ and $J_1(M)$ there are more curves to the left of these, but none to the right. Search was made on the IBM 650 for more curves to the right but none were found.

The solid lines of the curves are from calculated points and the

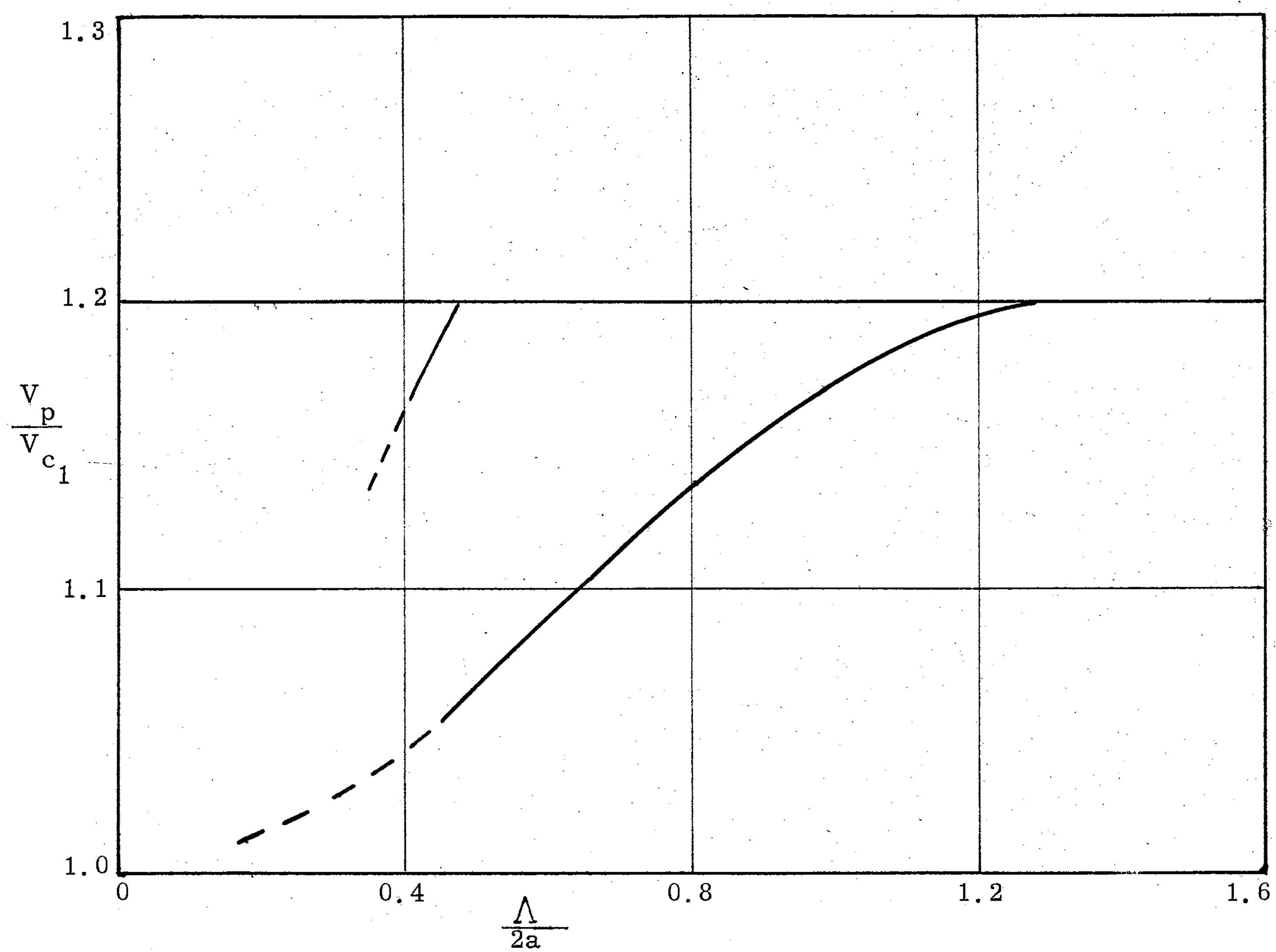

Fig. 1. Relationship of Phase Velocity and Wave Length When $V_{s_2} / V_{c_1} = 6/5$

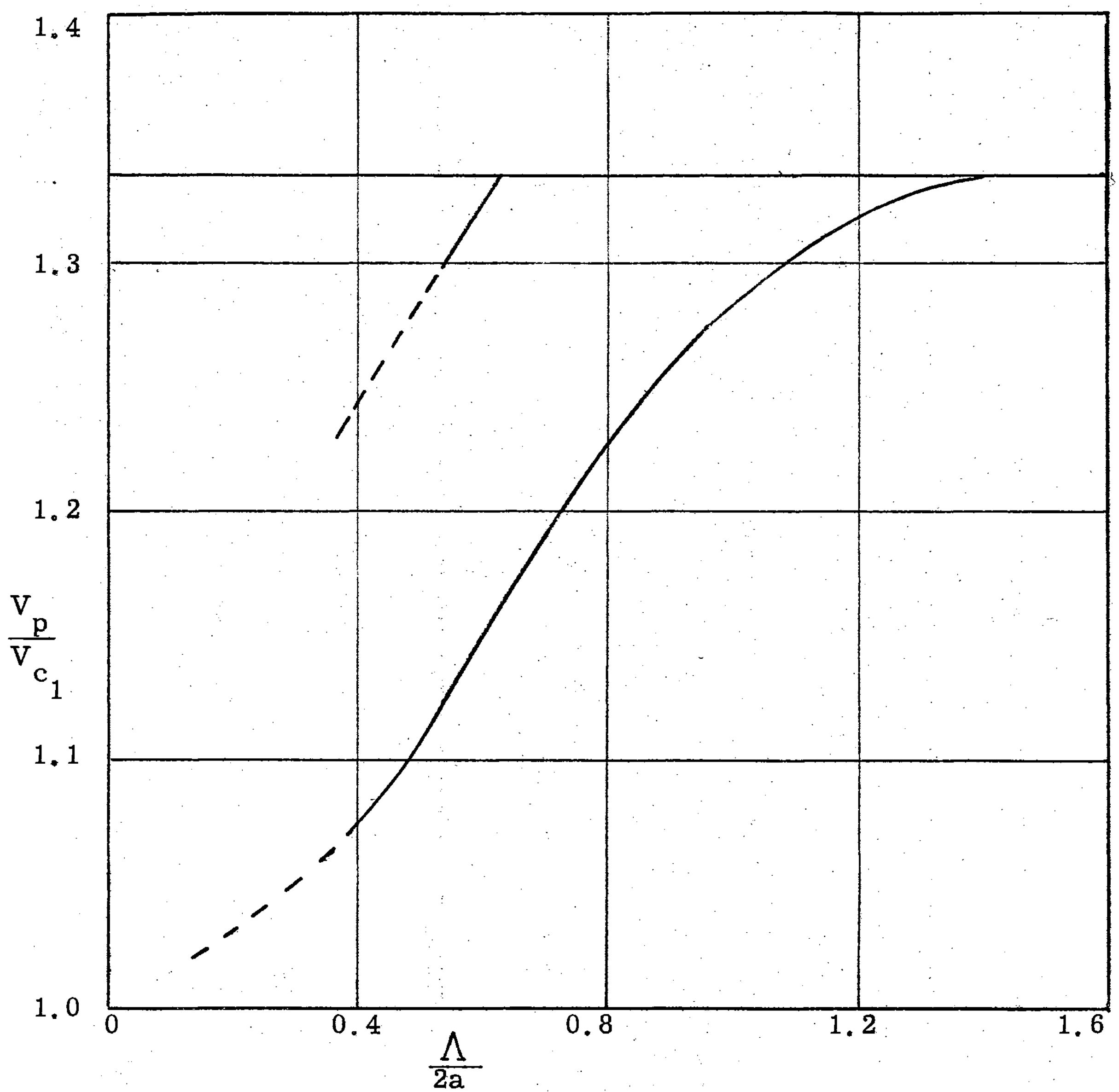

Fig. 2. Relationship of Phase Velocity and Wave Length When $V_{s_2} / V_{c_1} = 4/3$

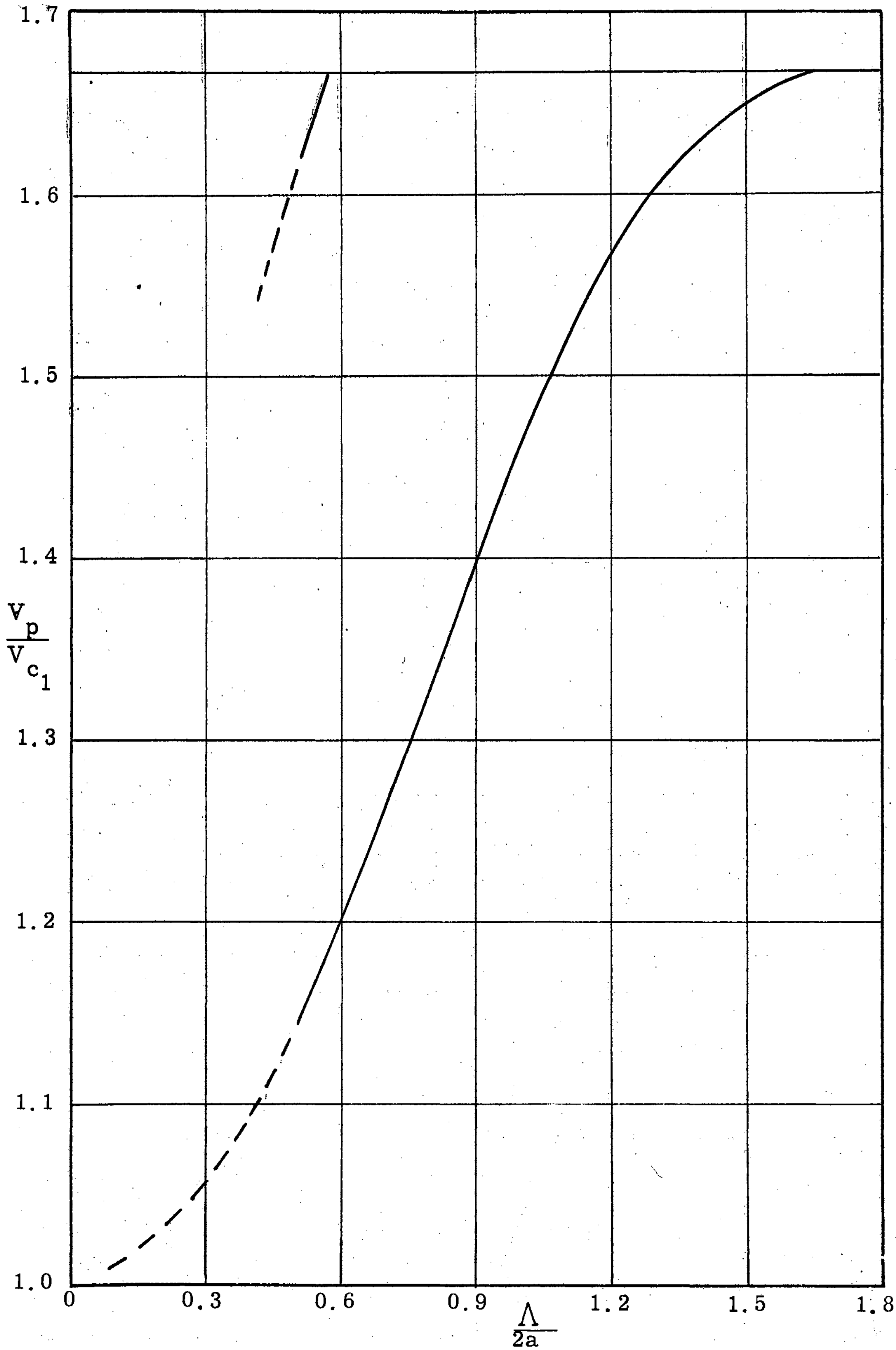

Fig. 3. Relationship of Phase Velocity and Wave Length When $V_{s_2} / V_{c_1} = 5/3$

dashed lines are extrapolated. In each Figure the upper limit of the ratio of V_p to V_{c_1} has been drawn in. Their limit is calculated from the ratio of g_1 and g_2 and is 6/5, 4/3, and 5/3 for Figures 1, 2, and 3 respectively. The lower limit for each curve is unity.

In Fig. 1 there is a low frequency cutoff at about

$$\frac{\Lambda}{2a} = 1.28 \tag{5.44}$$

for the first curve and a low frequency cutoff for the second curve at about

$$\frac{\Lambda}{2a} = 0.48 \; .$$

There is probably no high frequency cutoff.

Correspondingly the curves in Figs. 2 and 3 exhibit low frequency cutoff points.

It should be noted that at the low frequency cutoff point the wave length is less than two diameters of the liquid cylinder. Thus for a liquid cylinder two feet in diameter and a shear velocity in the solid of 2000 feet per second, the cutoff frequency is higher than 500 cycles per second.

Examination of Equations (5.25) and (5.27) reveals that g_1 must be greater than g_2 . This means that the shear wave velocity in the elastic solid must be greater than the compressional wave velocity in the liquid. If this velocity relationship does not hold, this solution to the problem does not exist.

Figure 4 shows the value of M plotted as a function of $\frac{\Lambda}{2a}$. Curves A, B, and C correspond respectively to the curves on the right in Figs. 1, 2, and 3. Thus it is possible to calculate a profile of the relative amplitude distribution of the guided waves at any given time in a plane perpendicular to the axis of the liquid cylinder. It should be noted that the maximum amplitude of these guided waves is at the axis of the liquid cylinder.

Equations (5.33), (5.34), (5.35) and (5.36) are for another type of guided wave which has its maximum amplitude at the contact of the liquid cylinder and the elastic solid.

From Equation (5.33) one may determine that the phase velocity of the guided wave is less than the compressional wave velocity in the liquid cylinder. From Equation (5.35) one may correspondingly determine that the phase velocity of the guided waves is less than the shear wave velocity in the elastic solid. Other than these conditions, there are no restrictions on the velocities which limit the existence of this solution.

The determination of the phase velocities and cutoff frequencies, if any, would require the solution of these equations for sets of curves. This could be done by a modification of the IBM program used for the other set of equations.

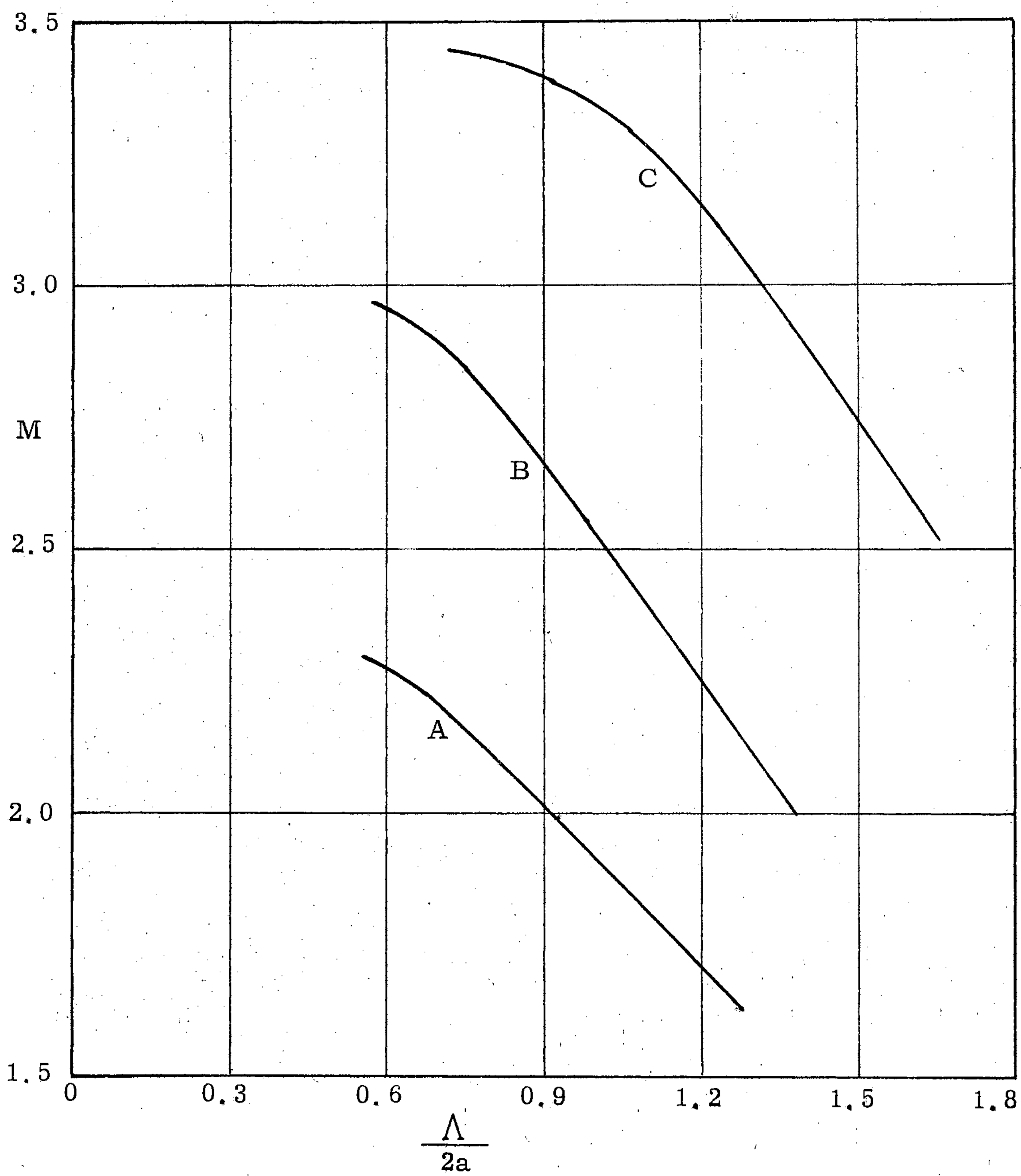

Fig. 4. Relationship of Eigenvalue M and Wave Length of the Guided Waves

CHAPTER VI

SUMMARY AND CONCLUSIONS

The purpose of this study was (1) to provide an operational mathematical basis for the study of steady state guided waves in liquid cylinders and surrounding elastic solids, (2) to check this operational method by re-solving typical problems that had been solved by classical mathematical methods, (3) to provide a procedure for solving more complex problems in guided wave propagation in liquid cylinder and elastic solids, and (4) to show how other vibrational problems having cylindrical symmetry could be solved with these methods.

The general equation for small motions in liquids and elastic solids was solved by operational mathematical methods in terms of Bessel and Hankel functions. These solutions in terms of vibrational displacements were applied to some of the less complex problems in liquid cylinder vibrations. Classical mathematical solutions had previously been published for these problems. The solutions developed in this study checked the published results.

The solutions of the general equation for small motions was applied to the problem of a liquid cylinder in an elastic solid. Two sets of equations were derived for two types of guided wave vibrations along the liquid cylinder and the elastic solid. One of these sets of equations was solved for phase velocities and wave lengths of the guided waves.

The phase velocities for the first set of guided waves are bounded above by the shear velocity in the elastic solid and bounded below by the compressional wave velocity in the liquid cylinder. Thus, unless the shear wave velocity in the elastic solid is greater than the compressional wave velocity in the liquid, this solution does not exist.

Inspection of the equations shows that the phase velocities for the second set of waves are bounded above by the shear wave velocity in the elastic solid and by the compressional wave velocity in the liquid. Thus, there are no restrictions on the existence of these guided waves.

These two sets of guided waves differ in that the maximum amplitude of vibration in the first set is in the liquid tube while for the second set of guided waves the maximum amplitude is at the contact of the liquid cylinder and the elastic solid.

The IBM program used to obtain the calculations mentioned above has been included in Appendix A.

In Appendix B equations are given resulting from the application of the general solutions to the problems of

(1) Elastic cylinder in infinite liquid,

(2) Liquid filled pipe in space, and

(3) Liquid filled pipe buried in an elastic solid.

Examination of these equations leads to some general statements about the limits on the phase velocities possible in the various cases.

Thus, in the case of the liquid filled pipe in an elastic solid, phase velocities of the guided waves are bounded above by the compressional wave velocity in the liquid, the shear wave velocity in the pipe, or the shear wave velocity in the elastic solid.

Further work can be done on these problems by investigating the effect of changes in the boundary conditions. The liquids considered in this study were assumed to be ideal so that at the contact of the liquid and elastic solid the shear stress in the solid was taken to be zero. However, many liquids wet solids and a thin layer adheres to the surface of the solid. Thus, it would be interesting to investigate the effect of replacing the boundary conditions expressing the shear stress as zero at the surface of the elastic solid with one which provides for continuity of axial displacement across the contact. This would probably yield solutions more nearly fitting the effects observed in practice.

Although in this study the general solutions have been applied only to cases for steady state guided waves, the equations developed in Chapter II can be applied to transient vibrational problems. For these transient problems the Laplace transformed equations would be used with the appropriate initial conditions. Use of the Inversion Integral would yield solutions in the time domain.

Conclusions are:

(1) A set of equations was set up by operational mathematical methods for application to problems of guided waves in liquid cylinders and elastic solids.

(2) These methods checked published results, and

(3) Applications were extended to the solution of more complex problems concerned with liquid cylinders and elastic solids.

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APPENDIX A

IBM FORTRAN PROGRAM

The IBM program which was used for the computation of curves determined by Equations (5.25), (5.26), (5.27) and (5.32) is given in the following.

The equations are re-written here as follows:

$$1 + \frac{M^2}{A^2} = \frac{W^2}{A^2} g_1^2 \tag{A.1}$$

$$1 - \frac{N^2}{A^2} = \frac{W^2}{A^2} \tag{A.2}$$

$$1 - \frac{P^2}{A^2} = \frac{W^2}{A^2} g_2^2 \tag{A.3}$$

$$f(A) = A \left((1 + \frac{P^2}{A^2})^2 \frac{K_0(N)}{\frac{N}{A} K_1(N)} - \frac{4P}{A} \frac{K_0(P)}{K_1(P)} - h_1 (1 - \frac{P^2}{A^2})^2 \frac{J_0(M)}{\frac{M}{A} J_1(M)} \right) - 2(1 - \frac{P^2}{A^2}) \tag{A.4}$$

where f(A) is less in absolute value than a given δ .

In writing the program in FORTRAN, Latin characters were used. The following list gives the program equivalent for the characters in the equations.

In Equations	In the Program
δ	DELTA
h_1	G3
A	ALPHA
W	OMEGA
M	EM
N	EN
P	P
$\frac{W}{A}$	WA
$\frac{M}{A}$	EMA
$\frac{N}{A}$	ENA
$\frac{P}{A}$	PA
$J_0(M)$	EMJ0
$J_1(M)$	EMJ1
$K_0(N)$	ZERKN
$K_1(N)$	ONEKN
$K_0(P)$	ZERKP
$K_1(P)$	ONEKP

The following is the IBM FORTRAN Program:

```
1 0 READ,G1,G2,G3,WA,ALPHA,DELTA,
1 1 ESZ,ABMIN,ABDER
    TMIN=1.0/G2
    TMAX=1.0/G1
    NCNT=0
2 0 IF(WA-1.0)3,1,1
3 0 ENA=SQRTF(1.0-WA*WA)
    CPHI=WA*G2
    IF(CPHI-1.0)4,1,1
4 0 PA=SQRTF(1.0-CPHI*CPHI)
    CMU=WA*G1
    IF(1.0-CMU)5,1,1
5 0 EMA=SQRTF(CMU*CMU-1.0)
7 0 EN=ENA*ALPHA
    P =PA*ALPHA
    EM=EMA*ALPHA
    ZERKN=BEK0F(EN)
    ONEKN=BEK1F(EN)
    ZERKP=BEK0F(P)
    ONEKP=BEK1F(P)
    EMJ0=BEJ0F(EM)
    EMJ1=BEJ1F(EM)
8 0 IF(ABSF(EMJ1)-ABMIN)22,22,9
9 0 RAT1=ZERKN/ONEKN
    RAT2=ZERKP/ONEKP
    RAT3=EMJ0/EMJ1
    RAT4=((PA*PA+2.)*PA*PA+1.)/ENA
    RAT5=-4.0*PA
10 0 RAT6=G3*CPHI*CPHI*CPHI*CPHI/
```

```
10 1 EMA
       TERM1=RAT1*RAT4
       TERM2=RAT2*RAT5
       TERM3=RAT3*RAT6
       VALU=ALPHA*(TERM1+TERM2-TERM3)
       VALU=VALU-2.0*CPHI*CPHI
       DUMMY=PRNTF(NCNT,ALPHA,VALU,
     1 ZERKN,ONEKN,ZERKP,ONEKP,EMJ0,
     2 EMJ1,CORR)
       IF(ESZ-ABSF(VALU))32,25,25
32 0 IF(NCNT-1)33,36,36
33 0 IF(VALU)34,35,35
34 0 NSRT=-1
       ALP1=ALPHA
       VAL1=VALU
       GO TO 11
35 0 NSRT=1
       ALP2=ALPHA
       VAL2=VALU
       GO TO 11
36 0 IF(VALU)37,38,38
37 0 ALP1=ALPHA
       VAL1=VALU
       IF(NSRT)11,39,40
40 0 NSRT=0
       GO TO 39
38 0 ALP2=ALPHA
       VAL2=VALU
       IF(NSRT)40,39,11
```

```
39 0 CORR=VAL2*(ALP2-ALP1)/(VAL2-
39 1 VAL1)
     ALPHA=ALP2-CORR
     GO TO 50
11 0 TERM4=((EN*RAT1+2.0)*RAT1-EN)*
11 1 RAT4
12 0 TERM5=((P*RAT2+2.0)*RAT2-P)*
12 1 RAT5
13 0 TERM6=((EM*RAT3-2.0)*RAT3+EM)*
13 1 RAT6
     DERIV=TERM4+TERM5+TERM6
24 0 IF(ABDER-ABSF(DERIV))14,23,23
14 0 CORR=VALU/DERIV
     IF(CORR)15,30,17
15 0 IF(5.0+CORR)16,16,19
16 0 CORR=-2.0
     GO TO 19
17 0 IF(CORR-0.5)19,18,18
18 0 CORR=0.5
19 0 IF(ALPHA-CORR-0.05)21,20,20
20 0 ALPHA=ALPHA-CORR
     IF(NSRT)53,50,53
21 0 CORR=0.2*CORR
     GO TO 19
22 0 ALPHA=ALPHA+0.05
     GO TO 7
23 0 IF(DERIV)28,29,29
25 0 OMEGA=WA*ALPHA
     PUNCH,WA,ALPHA,OMEGA,EM,EN,P
```

```
26 0 DUMMY=PRNTF(WA,ALPHA,OMEGA,
26 1 EM,EN,P,EMA,ENA,PA,NCNT)
27 0 WA=WA+DELTA
     NCNT=0
     IF(WA-TMIN)2,1,1
28 0 DERIV=-0.1
     GO TO 14
29 0 DERIV=0.1
     GO TO 14
30 0 CORR=0.1
     GO TO 19
50 0 IF(ALPHA-ALP1)51,55,51
51 0 IF(ALPHA-ALP2)53,55,53
55 0 DUMMY=PRNTF(ALPHA,VALU,Z,ALP1,
55 1 VAL1,Z,ALP2,VAL2,Z,Z)
     NCNT=9999999999
     GO TO 25
52 0 ALPHA=0.5*(ALP1+ALP2)
53 0 NCNT=NCNT+1
     GO TO 7
     END
```

APPENDIX B

OTHER PROBLEMS CONCERNING VIBRATIONS IN LIQUIDS AND SOLIDS

The solution to Equation (2.1) developed in Chapters II and III may be applied to a variety of problems concerned with vibrations in liquids and solids having cylindrical symmetry. Equations are given in the following pages for: (1) elastic cylinder in a liquid, (2) liquid filled pipe in space, and (3) liquid filled pipe in an elastic medium.

The boundary conditions for each case are listed. The equations giving the displacements and stresses are also given. Finally the equations resulting from applying these boundary conditions are given.

Elastic Cylinder in a Liquid

Two of the solutions for the problem of a solid elastic cylinder in a liquid of infinite extent are presented here. There are three possible solutions to this problem. However, one is restricted by the ratios among the velocities. The solution restricted by the velocities and one of the unrestricted ones is given below. At the contact of the liquid and solid the displacement and radial stress are continuous and the shear stress is zero. These conditions are expressed as

$$(q_a)_1 = (q_a)_2 \tag{B.1}$$

$$(s_{aa})_1 = (s_{aa})_2 \tag{B.2}$$

and

$$(s_{az})_1 = (s_{az})_2 . \tag{B.3}$$

In the elastic cylinder

$$(q_r)_1 = mA_0J_1(mr)e^{-\alpha z} + B A_1J_1 (mr) e^{-\beta z} \tag{B.4}$$

$$(s_{rr})_1 = \left\{ \left[\lambda_2(m^2-\alpha^2) + 2\mu m^2 \right] J_0(mr) - \frac{2\mu m}{r} J_1(mr) \right\} A_0 e^{-\alpha z} + \frac{2\mu\beta}{r} \left[nJ_0(nr) - J_1(nr) \right] A_1 e^{-\beta z} \tag{B.5}$$

$$(s_{rz})_1 = \mu \left[2m\alpha J_1(mr) A_0 e^{-\alpha z} + (n^2 + \beta^2) A_1J_1(nr)e^{-\beta z} \right]. \tag{B.6}$$

In the liquid medium

$$(s_{rr})_2 = \omega^2\rho_1A_2H_0^{(2)} (pr) e^{-\gamma z} \tag{B.7}$$

$$(q_r)_2 = pH_1^{(2)}(pr) e^{-\gamma z} \tag{B.8}$$

$$(s_{rz}) = 0 . \tag{B.9}$$

With these equations and the procedure used in Chapter V the dimensionless equations result:

$$A^2 + M^2 = W^2g_1^2 \tag{B.10}$$

$$A^2 + N^2 = W^2g_2^2 \tag{B.11}$$

$$A^2 - P^2 = W^2 . \tag{B.12}$$

$$\begin{vmatrix} MJ_1(M) & AJ_1(N) & PK_1(P) \\ h_1(N^2-A^2)J_0(M) - h_1MJ_1(M) & 2h_1A\left[NJ_0(N) - J_1(N)\right] & W^2g_2^2K_0(P) \\ -2MAJ_1(M) & (N^2-A^2)J_1(N) & 0 \end{vmatrix} = 0 \tag{B.13}$$

In Equation (B.13)

$$g_1 = \frac{V_{c_2}}{V_{c_1}} \tag{B.14}$$

$$g_2 = \frac{V_{c_2}}{V_{s_1}} \tag{B.15}$$

$$h_1 = \frac{\rho_1}{\rho_2} \quad . \tag{B.16}$$

Replacing M by iM and N by iN leads to another form of solution.

Thus

$$A^2 - M^2 = W^2g_1^2 \tag{B.17}$$

$$A^2 - N^2 = W^2g_2^2 \tag{B.18}$$

$$A^2 - P^2 = W^2 \quad . \tag{B.19}$$

$$\begin{vmatrix} -MI_1(M) & AI_1(N) & PK_1(P) \\ h_1\left[MI_1(M) - (N^2+A^2)I_0(M)\right] & 2h_1A\left[NI_0(N) - I_1(N)\right] & W^2g_2^2K_0(P) \\ 2MAI_1(M) & -(N^2+A^2)I_1(N) & 0 \end{vmatrix} = 0 \tag{B.20}$$

Solutions of Equations (B.10), (B.11), (B.12), and (B.13) yield phase velocities for guided waves in the elastic cylinder. For high frequencies these waves tend to concentrate in the cylinder. There are some restrictions on the velocities for the existence of this solution. They are

$$V_{c_2} > V_{c_1} \tag{B.21}$$

and

$$V_{c_2} > V_P > V_{c_1} \quad . \tag{B.22}$$

Equation (B.22) means that the phase velocity V_P is bounded above by the velocity in the liquid and below by the compressional wave velocity in the elastic cylinder.

For a steel cylinder in water this solution does <u>not</u> exist since the velocity in steel is greater than in water.

Solutions of Equations (B.17), (B.18), (B.19), and (B.20) yield the phase velocities and frequencies of waves which are guided along the elastic cylinder but tend to concentrate at the contact of the liquid and solid.

The ratios of the velocities do not restrict the existence of these solutions. Thus these guided waves should be expected in, for example, a steel cylinder in water. The phase velocity V_P is less than the shear velocity V_{s_1} in the elastic cylinder or the compressional wave velocity in the liquid, depending on which is smaller.

Liquid Filled Pipe in Space

A problem of theoretical interest is the guided waves in a liquid filled metal pipe in space.

There are as many as six possible solutions to this case but the existence of only two of them is not restricted by the relationships of the velocities. Equations for these two solutions are given here.

The boundary conditions are: at the contact of the liquid and pipe the radial displacement and stress are continuous, and the shear stress is zero; on the outside of the pipe the radial and shear stresses are zero.

These conditions are expressed at $r = a$ (inside radius of pipe) as

$$(q_a)_1 = (q_a)_2 \tag{B. 23}$$

$$(s_{aa})_1 = (s_{aa})_2 \tag{B. 24}$$

$$(s_{az})_2 = 0 \tag{B. 25}$$

and at $r = b$ (outside radius of pipe)

$$(s_{bb})_2 = 0 \tag{B. 26}$$

$$(s_{bz})_2 = 0 . \tag{B. 27}$$

In the liquid cylinder

$$(q_r)_1 = mJ_1(mr)A_0e^{-\alpha z} \tag{B. 28}$$

$$(s_{rr})_1 = \omega^2 P_1 J_0(mr)A_0e^{-\alpha z} . \tag{B. 29}$$

In the pipe

$$(q_r)_2 = n\left[A_1 J_1(nr) + B_1 Y_1(nr)\right] e^{-\beta z} \qquad \text{(B. 30)}$$

$$+ \gamma\left[A_2 J_1(pr) + B_2 Y_1(pr)\right] e^{-\gamma z} \quad .$$

$$(s_{rr})_2 = \lambda_2 (n^2 - \beta^2)\left[A_1 J_0(nr) + B_1 Y_0(nr)\right] e^{-\beta z}$$

$$+ 2\mu_2 n^2 \left[A_1 J_0(nr) + B_1 Y_0(nr)\right] e^{-\beta z}$$

$$\frac{-2\mu_2 n}{r} \left[A_1 J_1(nr) + B_1 Y_1(nr)\right] e^{-\beta z} \qquad \text{(B. 31)}$$

$$+ 2\mu_2 \, p\gamma \left[A_2 J_0(pr) + B_2 Y_0(pr)\right] e^{-\gamma z}$$

$$\frac{-2\mu_2 \gamma}{r} \left[A_2 J_1(pr) + B_2 Y_1(pr)\right] e^{-\gamma z}$$

$$(s_{rz})_2 = - 2\mu_2 \beta n \left[A_1 J_1(nr) + B_1 Y_1(nr)\right] e^{-\beta z} \qquad \text{(B. 32)}$$

$$- \mu_2 (p^2 + \gamma^2)\left[A_2 J_1(pr) + B_2 Y_1(pr)\right] e^{-\gamma z} .$$

With these equations and the aid of the procedure used in Chapter V, the following dimensionless equations result:

$$A^2 + M^2 = W^2 g_1^2 \qquad \text{(B. 33)}$$

$$A^2 + N^2 = W^2 \qquad \text{(B. 34)}$$

$$A^2 + P^2 = W^2 g_2^2 \qquad \text{(B. 35)}$$

$$\begin{vmatrix} MJ_1(M) & NJ_1(N) & NY_1(N) & AJ_1(P) & AY_1(P) \\ h_1W^2g_1^2J_0(M) & (P^2-A^2)J_0(N) & (P^2-A^2)Y_0(N) & 2APJ_0(P) & 2APY_0(P) \\ & -2NJ_1(N) & -2NY_1(N) & -2AJ_1(P) & -2AY_1(P) \\ 0 & 2ANJ_1(N) & 2ANY_1(N) & (A^2-P^2)J_1(P) & (A^2-P^2)Y_1(P) \\ 0 & (P^2-A^2)J_0(\frac{bN}{a}) & (P^2-A^2)Y_0(\frac{bN}{a}) & 2APJ_0(\frac{bP}{a}) & 2APY_0(\frac{bP}{a}) \\ & \frac{-2NaJ_1(\frac{bN}{a})}{b} & \frac{-2NaY_1(\frac{bN}{a})}{b} & -2AJ_1(\frac{bP}{a}) & -2AY_1(\frac{bP}{a}) \\ 0 & 2ANJ_1(\frac{bN}{a}) & 2ANY_1(\frac{bN}{a}) & (A^2-P^2)J_1(\frac{bP}{a}) & (A^2-P^2)Y_1(\frac{bP}{a}) \end{vmatrix} = 0 \tag{B.36}$$

where

$$g_1 = \frac{V_{c_2}}{V_{c_1}} \tag{B.37}$$

$$g_2 = \frac{V_{c_2}}{V_{s_2}} \tag{B.38}$$

$$h_1 = \frac{\rho_1}{\rho_2} \quad . \tag{B.39}$$

The other solution mentioned above is derived by using Equations (3.27) and (3.28).

In the liquid cylinder

$$(q_r)_1 = -mI_1(mr)A_0e^{-\alpha z} \tag{B.40}$$

$$(s_{rr})_1 = \rho_1 \omega^2 I_0(mr) A_0 e^{-\alpha z} . \tag{B.41}$$

In the pipe

$$\begin{aligned}(q_r)_2 &= n \left[-A_1 I_1(nr) + B_1 K_1(nr)\right] e^{-\beta z} \\ &\quad + \gamma \left[A_2 I_1(pr) + B_2 K_1(pr)\right] e^{-\gamma z}\end{aligned} \tag{B.42}$$

$$\begin{aligned}(q_z)_2 &= \beta \left[A_1 I_0(nr) + B_1 K_0(nr)\right] e^{-\beta z} \\ &\quad + p \left[A_2 I_0(pr) - B_2 K_0(pr)\right] e^{-\gamma z}\end{aligned} \tag{B.43}$$

$$\begin{aligned}(s_{rr})_2 &= -\lambda_2 (n^2 + \beta^2) \left[A_1 I_0(nr) + B_1 K_0(nr)\right] e^{-\beta z} \\ &\quad -2\mu_2 n^2 \left[A_1 I_0(nr) + B_1 K_0(nr)\right] e^{-\beta z} \\ &\quad + \frac{2\mu_2 n}{r} \left[A_1 I_1(nr) - B_1 K_1(nr)\right] e^{-\beta z} \\ &\quad + 2\mu_2 \gamma p \left[A_2 I_0(pr) + B_2 K_0(pr)\right] e^{-\gamma z} \\ &\quad - \frac{2\mu_2 \gamma}{r} \left[A_2 I_1(pr) + B_2 K_1(pr)\right] e^{-\gamma z}\end{aligned} \tag{B.44}$$

$$\begin{aligned}(s_{rz})_2 &= 2\mu_2 n \beta \left[A_1 I_1(nr) - B_1 K_1(nr)\right] e^{-\beta z} \\ &\quad + \mu_2 (p^2 - \gamma^2) \left[A_2 I_1(pr) + B_2 K_1(pr)\right] e^{-\gamma z} .\end{aligned} \tag{B.45}$$

With these equations, the boundary conditions, and the methods used above, the following dimensionless equations result:

$$A^2 - M^2 = W^2 g_1^2 \tag{B.46}$$

$$A^2 - N^2 = W^2 \tag{B.47}$$

$$A^2 - P^2 = W^2 g_2^2 \; . \tag{B.48}$$

$$\begin{vmatrix}
-MI_1(M) & -NI_1(N) & NK_1(N) & AI_1(P) & AK_1(P) \\[2ex]
h_1 W^2 g_2^2\, I_0(M) & \begin{array}{l} -(A^2+P^2)I_0(N) \\ +\,2NI_1(N) \end{array} & \begin{array}{l} -(A^2+P^2)K_0(N) \\ -\,2NK_1(N) \end{array} & \begin{array}{l} 2API_0(P) \\ -2AI_1(P) \end{array} & \begin{array}{l} -2APK_0(P) \\ -2AK_1(P) \end{array} \\[2ex]
0 & 2NAI_1(N) & -2NAK_1(N) & -(P^2+A^2)I_1(P) & -(P^2+A^2)K_1(P) \\[2ex]
0 & \begin{array}{l} -(A^2+P^2)I_0(\frac{Nb}{a}) \\ +\,2NI_1(\frac{Nb}{a}) \end{array} & \begin{array}{l} -(A^2+P^2)K_0(N) \\ -\,2NK_1(\frac{Nb}{a}) \end{array} & \begin{array}{l} 2API_0(\frac{Pb}{a}) \\ -2AI_1(\frac{Pb}{a}) \end{array} & \begin{array}{l} -2APK_0(\frac{Pb}{a}) \\ -2AK_1(\frac{Pb}{a}) \end{array} \\[2ex]
0 & 2NAI_1(\frac{Nb}{a}) & -\,2NAK_1(\frac{Nb}{a}) & -(P^2+A^2)I_1(\frac{Pb}{a}) & -(P^2+A^2)K_1(\frac{Pb}{a})
\end{vmatrix} = 0 \tag{B.49}$$

In the solution determined by Equations (B.33), (B.34), (B.35) and (B.36) the waves tend to concentrate in the liquid. Examination of the equations reveals that the phase velocities have no upper limit but are bounded below by the compressional wave velocity in the liquid or the shear velocity in the pipe depending on which is larger. That is,

$$V_p > V_{c_1} \tag{B.50}$$

and

$$V_p > V_{s_2} \quad . \tag{B.51}$$

In the solution determined by Equations (B.46), (B.47), (B.48), and (B.49) the waves tend to concentrate in the pipe. The phase velocities have no lower bound but are bounded above by the compressional wave velocity in the liquid or the shear wave velocity in the pipe. That is,

$$V_p < V_{c_1} \tag{B.52}$$

and

$$V_p < V_{s_2} \quad . \tag{B.53}$$

Liquid Filled Pipe in an Elastic Medium

A problem of considerable practical importance is the guided vibrations in a liquid filled metal pipe buried in an elastic solid. The equations developed in Chapters II and III may be applied to this problem. When the pipe is buried several diameters below the surface, the elastic solid may be considered to be infinite in extent. Thus, circular symmetry may be assumed.

The boundary conditions are: at the contact of the liquid and pipe the radial displacement and stress are continuous and the shear stress is zero; at the contact of the pipe and elastic medium the radial and tangential displacements and shear stresses are continuous.

These conditions are expressed at $r = a$ (inside radius of pipe) by

$$(q_a)_1 = (q_a)_2 \tag{B.54}$$

$$(s_{aa})_1 = (s_{aa})_2 \tag{B.55}$$

$$(s_{az})_2 = 0 \tag{B.56}$$

and at $r = b$ (outside radius of pipe) by

$$(q_b)_2 = (q_b)_3 \tag{B.57}$$

$$(s_{bb})_2 = (s_{bb})_3 \tag{B.58}$$

$$(q_z)_2 = (q_z)_3 \tag{B.59}$$

$$(s_{bz})_2 = (s_{bz})_3 \quad , \tag{B.60}$$

The 1, 2, and 3 subscripts refer to the liquid tube, the pipe, and the elastic solid, respectively.

In the liquid tube two forms of solutions are possible. The first gives the radial displacement and stress as

$$(q_r)_1 = kJ_1\,(kr)\,A_0 e^{-\alpha z} \tag{B.61}$$

and

$$(s_{rr})_1 = \omega^2 \rho_1 J_0\,(kr)\,A_1 e^{-\alpha z} \tag{B.62}$$

where

$$k^2 - \alpha^2 = \frac{\omega^2}{V_{c_1}^2} \quad . \tag{B.63}$$

The second solution gives the radial displacement and stress as

$$(q_r)_1 = -\,kI_1\,(kr)\,A_0 e^{-\alpha z} \tag{B.64}$$

and

$$(s_{rr})_1 = \omega^2 \rho_1 I_0\,(kr)\,A_0 e^{-\alpha z} \tag{B.65}$$

where

$$k^2 + \alpha^2 + \frac{\omega^2}{V_{c_1}^2} = 0 \; . \tag{B.66}$$

Correspondingly, in the pipe two forms of solution are possible. The first solution gives the radial displacement and stress, and tangential displacement and shear stress as

$$\begin{aligned}(q_r)_2 = \; & \ell \left[A_1 J_1(\ell r) + B_1 Y_1(\ell r)\right] e^{-\beta z} \\ & + \gamma \left[A_2 J_1(\ell r) + B_2 Y_1(mr)\right] e^{-\beta z}\end{aligned} \tag{B.67}$$

$$\begin{aligned}(q_z)_2 = \; & \beta \left[A_1 J_0(\ell r) + B_1 Y_0(\ell r)\right] e^{-\beta z} \\ & + m \left[A_2 J_0(mr) + B_2 Y_0(mr)\right] e^{-\gamma z}\end{aligned} \tag{B.68}$$

$$\begin{aligned}(s_{rr})_2 = \; & \lambda_2(\ell^2 - \beta^2) \left[A_1 J_0(\ell r) + B_1 Y_0(\ell r)\right] e^{-\beta z} \\ & + 2\mu_2 \ell^2 \left[A_1 J_0(\ell r) + B_1 Y_0(\ell r)\right] e^{-\beta z} \\ & - \frac{2\mu_2 \ell}{r} \left[A_1 J_1(\ell r) + B_1 Y_1(\ell r)\right] e^{-\beta z} \\ & + 2\mu_2 m \gamma \left[A_2 J_0(mr) + B_2 Y_0(mr)\right] e^{-\gamma z} \\ & - \frac{2\mu_2 \gamma}{r} \left[A_2 J_1(mr) + B_2 Y_1(mr)\right] e^{-\gamma z}\end{aligned} \tag{B.69}$$

$$\begin{aligned}(s_{rz})_2 = \; & -2\mu_2 \beta \ell \left[A_1 J_1(\ell r) + B_1 Y_1(\ell r)\right] e^{-\beta z} \\ & -\mu (m^2 + \gamma^2) \left[A_2 J_1(mr) + B_2 Y_1(mr)\right] e^{-\gamma z}\end{aligned} \tag{B.70}$$

where

$$\ell^2 - \beta^2 = \frac{\omega^2}{V_{p_2}^2} \tag{B.71}$$

$$m^2 - \gamma^2 = \frac{\omega^2}{V_{s_2}^2} \ . \tag{B.72}$$

The second form of solution in the pipe gives the components of displacement and stress as

$$\begin{aligned}(q_r)_2 = \; & \ell \left[-A_1 I_1(\ell r) + B_1 K_1(\ell r)\right] e^{-\beta z} \\ & + \gamma \left[A_2 I_1(mr) + B_2 K_1(mr)\right] e^{-\beta z}\end{aligned} \tag{B.73}$$

$$\begin{aligned}(q_z)_2 = \; & \beta \left[A_1 I_0(\ell r) + B_1 K_0(\ell r)\right] e^{-\beta z} \\ & + m \left[A_2 I_0(mr) - B_2 K_0(mr)\right] e^{-\gamma z}\end{aligned} \tag{B.74}$$

$$\begin{aligned}(s_{rr})_2 = \; & -\lambda_2(\ell^2 + \beta^2) \left[A_1 I_0(\ell r) + B_1 K_0(\ell r)\right] e^{-\beta z} \\ & - 2\mu_2 \ell^2 \left[A_1 I_0(\ell r) + B_1 K_0(\ell r)\right] e^{-\beta z} \\ & + \frac{2\mu_2 \ell}{r} \left[A_1 I_1(\ell r) - B_1 K_1(\ell r)\right] e^{-\beta z} \\ & + 2\mu_2 \gamma m \left[A_2 I_0(mr) - B_2 K_0(mr)\right] e^{-\gamma z} \\ & - \frac{2\mu_2 \gamma}{r} \left[A_2 I_1(mr) + B_2 K_1(mr)\right] e^{-\gamma z}\end{aligned} \tag{B.75}$$

$$\begin{aligned}(s_{rz})_2 = \; & 2\mu_2 \ell \beta \left[A_1 I_1(\ell r) - B_1 K_1(\ell r)\right] e^{-\beta z} \\ & + \mu_2(m^2 - \gamma^2) \left[A_2 I_1(mr) + B_2 K_1(mr)\right] e^{-\gamma z}\end{aligned} \tag{B.76}$$

where

$$\ell^2 + \beta^2 + \frac{\omega^2}{V_{c_2}^2} = 0 \tag{B.77}$$

$$m^2 + \gamma^2 + \frac{\omega^2}{V_{c_2}^2} = 0 \; . \tag{B.78}$$

In the elastic solid, the form of wave motion for outward wave travel is used. Thus

$$(q_r)_3 = nH_1^{(2)}(nr)\,A_3 e^{-\delta z} + \epsilon\, H_1^{(2)}(pr)\,A_4 e^{-\epsilon z} \tag{B.79}$$

$$(q_z)_3 = \delta\, H_1^{(2)}(nr) A_3 e^{-\delta z} + pH_0^{(2)}(pr)\,A_4 e^{-\epsilon z} \tag{B.80}$$

$$(s_{rr})_3 = \left[\left(\lambda_3(n^2 - \delta^2) + 2\mu_3 n^2\right) H_0^{(2)}(nr) - \frac{2\mu_3 n}{r} H_1^{(2)}(nr)\right] A_3 e^{-\delta z} + 2\mu_3\,\epsilon \left(pH_0^{(2)}(pr) - \frac{1}{r}\, H_1^{(2)}(pr)\right) A_4 e^{-\epsilon z} \tag{B.81}$$

$$(s_{rz})_3 = -\mu_3 \left(2n\delta H_1^{(2)}(nr)\,A_3 e^{-\delta z} + (p^2 + \epsilon^2)\, H_1^{(2)}(pr)\, e^{-\epsilon z}\right) \tag{B.82}$$

When the boundary conditions are used it may be noted that there are eight ways in which these boundary conditions may be satisfied. Complete equations are given here for only two of these cases.

Use of the following equations puts the final equations in dimensionless form:

$$ak = K \tag{B.83}$$

$$a\ell = L \tag{B.84}$$

$$am = M \tag{B.85}$$

$$an = -iN \tag{B.86}$$

$$a p = -iP \tag{B.87}$$

$$a\alpha = iA \tag{B.88}$$

$$\frac{V_{c_2}}{V_{c_1}} = g_1 \tag{B.89}$$

$$\frac{V_{c_2}}{V_{s_2}} = g_2 \tag{B.90}$$

$$\frac{V_{c_2}}{V_{c_3}} = g_3 \tag{B.91}$$

$$\frac{V_{c_2}}{V_{s_3}} = g_4 \tag{B.92}$$

$$\frac{\rho_1}{\rho_2} = h_1 \tag{B.93}$$

$$\frac{\rho_3}{\rho_2} = h_2 \quad . \tag{B.94}$$

Using these equations, the boundary conditions, and the first set of equations for the displacements and stresses results in

$$A^2 + K^2 = W^2 g_1^2 \tag{B.95}$$

$$A^2 + L^2 = W^2 \tag{B.96}$$

$$A^2 + M^2 = W g_2^2 \tag{B.97}$$

$$A - N^2 = W^2 g_3^2 \tag{B.98}$$

$$A^2 - P^2 = W^2 g_4^2 \tag{B.99}$$

$$\begin{vmatrix}
KJ_1(K) & LJ_1(L) & LY_1(L) & AJ_1(M) & AY_1(M) & 0 & 0 \\
W^2g_2^2h_1J_0(K) & \begin{array}{l}(M^2-A^2)J_0(L)\\ -2L_aJ_1(L)\end{array} & \begin{array}{l}(M^2-A^2)Y_0(L)\\ -2L_aY_1(L)\end{array} & \begin{array}{l}2MAJ_0(M)\\ -2AJ_1(M)\end{array} & \begin{array}{l}2MAY_0(M)\\ -2AY_1(M)\end{array} & 0 & 0 \\
0 & -2ALJ_1(L) & -2LY_1(L) & (M^2-A^2)J_1(M) & (M^2-A^2)Y_1(M) & 0 & 0 \\
0 & LJ_1(\frac{Lb}{a}) & LY_1(\frac{Lb}{a}) & AJ_1(\frac{Mb}{a}) & AY_1(\frac{Mb}{a}) & NK_1(\frac{Nb}{a}) & AK_1(\frac{Pb}{a}) \\
0 & \begin{array}{l}(M^2-A^2)J_0(\frac{b}{a}L)\\ -2LJ_1(\frac{b}{a}L)\end{array} & \begin{array}{l}(M^2-A^2)Y_0(\frac{b}{a}L)\\ -2LY_1(\frac{b}{a}L)\end{array} & \begin{array}{l}2MAJ_0(\frac{b}{a}M)\\ \frac{-2AM}{b}J_1(\frac{b}{a}M)\end{array} & \begin{array}{l}2MAY_0(\frac{b}{a}M)\\ \frac{-2aA}{b}Y_1(\frac{b}{a}M)\end{array} & \begin{array}{l}-h_2\frac{(P^2+A^2)g_2^2}{g_4^2}K_0(\frac{Nb}{a})\\ \frac{-2h_2g_2^2}{g_4^2}\frac{Na}{b}K_1(\frac{Nb}{a})\end{array} & \begin{array}{l}\frac{-2h_2}{g_4^2}APK_0(\frac{Pb}{a})\\ \frac{-2h_2}{g_4^2}\frac{Aa}{b}K_1(\frac{Pb}{a})\end{array} \\
0 & -AJ_0(\frac{b}{a}L) & -AY_0(\frac{b}{a}L) & MJ_0(\frac{b}{a}M) & MY_0(\frac{bM}{a}) & -AK_0(\frac{Nb}{a}) & -PK_0(\frac{Pb}{a}) \\
0 & 2ALJ_1(\frac{bL}{a}) & -2ALY_1(\frac{bL}{a}) & (A^2-M^2)J_1(\frac{bM}{a}) & (A^2-M^2)Y_1(\frac{Mb}{a}) & \frac{2NAg_2^2h_2}{g_4^2}K_1(\frac{Nb}{a}) & (P^2+A^2)\frac{h_2g_2^2}{g_4^2}K_1(\frac{Pb}{a})
\end{vmatrix} = 0 . \qquad \text{(B.100)}$$

Using the second set of equations for the displacements and stress and the same procedure results in

$$A^2 - K^2 = W^2g_1^2 \qquad \text{(B.101)}$$

$$A^2 - L^2 = W^2 \qquad \text{(B.102)}$$

$$A^2 - M^2 = W^2g_2^2 \qquad \text{(B.103)}$$

$$A^2 - N^2 = W^2g_3^2 \qquad \text{(B.104)}$$

$$A^2 - P^2 = W^2g_4^2 \qquad \text{(B.105)}$$

$$\begin{vmatrix}
-KI_1(K) & -LI_1(L) & LK_1(L) & AI_1(M) & AK_1(M) & 0 & 0 \\
h_1g_2^2W^2I_0(K) & \begin{matrix}-(M^2+A^2)I_0(L)\\ +2LI_1(L)\end{matrix} & \begin{matrix}-(M^2+A^2)K_0(L)\\ -2LK_1(L)\end{matrix} & \begin{matrix}2AMI_0(M)\\ -2AI_1(M)\end{matrix} & \begin{matrix}-2AMK_0(M)\\ -2AK_1(M)\end{matrix} & 0 & 0 \\
0 & -2ALI_1(L) & 2ALK_1(L) & (M^2+A^2)I_1(M) & (M^2+A^2)K_1(M) & 0 & 0 \\
0 & -LI_1(\frac{Lb}{a}) & LK_1(\frac{Lb}{a}) & AI_1(\frac{Mb}{a}) & AK_1(\frac{Mb}{a}) & NK_1(\frac{Nb}{a}) & AK_1(\frac{Pb}{a}) \\
0 & -AI_0(\frac{Lb}{a}) & -AK_0(\frac{Lb}{a}) & MI_0(\frac{Mb}{a}) & MK_0(\frac{Mb}{a}) & -AK_0(\frac{Nb}{a}) & -PK_0(\frac{Pb}{a}) \\
0 & \begin{matrix}-(M^2+A^2)I_0(\frac{Lb}{a})\\ \frac{+2La}{b}I_1(\frac{Lb}{a})\end{matrix} & \begin{matrix}-(M^2+A^2)K_0(\frac{Lb}{a})\\ \frac{-2La}{b}K_1(\frac{Lb}{a})\end{matrix} & \begin{matrix}2AMI_0(\frac{Mb}{a})\\ \frac{-2Aa}{b}I_1(\frac{Mb}{a})\end{matrix} & \begin{matrix}-2AMK_0(\frac{Mb}{a})\\ \frac{-2Aa}{b}K_1(\frac{Mb}{a})\end{matrix} & \begin{matrix}\frac{-h_2g_2^2}{g_4^2}(A^2+P^2)K_0(\frac{Nb}{a})\\ \frac{-2h_2g_2^2}{g_4^2}\frac{Na}{b}K_1(\frac{Nb}{a})\end{matrix} & \begin{matrix}\frac{-2h_2}{g_4^2}APK_0(\frac{Pb}{a})\\ \frac{-2h_2}{g_4^2}\frac{Aa}{b}K_1(\frac{Pb}{a})\end{matrix} \\
0 & -2ALI_1(\frac{Lb}{a}) & +2ALK_1(\frac{Lb}{a}) & (M^2+A^2)I_1(\frac{Mb}{a}) & (M^2+A^2)K_1(\frac{Mb}{a}) & \frac{+2h_2g_2^2}{g_4^2}ANK_1(\frac{Nb}{a}) & \frac{h_2g_2^2}{g_4^2}(P^2+A^2)K_1(\frac{Pb}{a})
\end{vmatrix} = 0. \qquad \text{(B.106)}$$

The phase velocities in the solutions for Equations (B.95) through (B.100) will be bounded above by the shear wave velocity in the elastic solid (medium 3). The phase velocity also is bounded below by the shear wave velocity in the pipe. Thus, this set of equations has no solution for a liquid filled steel pipe in the ground because steel has a higher shear wave velocity than earth materials close to the surface.

The phase velocities in the solutions for Equations (B.101) through (B.106) will be bounded above by the following:

$$V_P < V_{c_1} \qquad \text{(B.107)}$$

$$V_P < V_{s_2} \tag{B.108}$$

$$V_P < V_{s_3} \quad . \tag{B.109}$$

For a water or oil filled steel pipe in the earth the phase velocity of these guided waves will probably be limited by the shear velocity of the earth material.

In any of the problems that may be solved by the procedure used here the limitations on the phase velocities may be determined by examination of the equations applying to that case.

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