REPRESENTATION THEOREMS
FOR SUMMABILITY OPERATORS AND LINEAR FUNCTIONALS
ON BOUNDED SEQUENCES

By

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Submitted to the Faculty of the Graduate School of
the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
DOCTOR OF PHILOSOPHY
May, 1960

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PREFACE

During the past few years Professor R. B. Deal and his students have been concerned with a theory of abstract summability methods. As a part of this approach, the present paper studies each class of summability methods as a class of operators on a space of sequences. As a consequence, the space of bounded sequences, the space of convergent sequences, the space of sequences convergent to zero and their conjugate spaces are investigated in detail. Several theorems concerning characterization of some classes of summability methods and a few theorems on decompositions are established from structural analysis of these spaces.

My thanks are due to Professor R. B. Deal and O. H. Hamilton who served as my advisors during preparation of this paper and to Professor L. Wayne Johnson and O. H. Hamilton for their sound counsel and kind interest given me in all matters pertaining to my work.

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I

INTRODUCTION

This paper is primarily concerned with the general theory of summability methods; in particular, with the analysis of structures in each class of summability methods from a functional analysis viewpoint.

The summability methods generalize the notion of the usual limit process, which was explicitly defined by Cauchy, and may attach a limit to a divergent series. The use of divergent series goes back to Euler. In those days, they asked only for the "limit" of a given series. They believed that the limit must exist a priori. They also had some techniques available and obtained many useful results in analysis. On the other hand they were really perplexed when different techniques produced different values for the same series and when these useful techniques lead to many absurdities. In the last century with the revival of the rigorous approach, divergent series gradually disappeared from analysis until Cesaro gave a formal definition for the (C, 1) summability method in 1890. Once definitions were given, all the mysteries were gone, and the theory of summability was well founded as a part of analysis. With the development of functional analysis in the Lwow school, centered around Banach, a general theory of summability methods was approached as an application of functional analysis. Among the contributions to this area, S. Mazur's paper in 1930 [1] is distinguished. A fairly comprehensive knowledge about the general theory of summability methods can be found in S. Mazur and W. Orlicz's paper in 1955 [2].

In part II, two theorems on summability operators are given from which some of the important classical theorems follow directly as collorarics. One theorem proves that any summability operator is a continuous linear operator, and the other theorem establishes a necessary and sufficient condition for a continuous linear operator to belong to the class of conservative operators in terms of a basis in the space of convergent sequences. These two theorems are used to characterize various classes of summability operators.

Part III is algebraic in nature. The theorems here establish the fact that any conservative summability operator can be, in a sense, decomposed uniquely into a regular summability operator and a summability operator which sums all bounded sequences. For this purpose, inclusion relations among various classes of summability operators are established in the beginning. Since the class $\mathcal{T}_r$ of regular summability operators is not a subspace of the space $\mathcal{T}$ of all summability operators, it is necessary to consider the subspace of $\mathcal{T}$ which is spanned by $\mathcal{T}_r$. This turns out to be the subspace, $\mathcal{T}_z$, of conservative operators which map the space of sequences converging to zero into itself. Also fortunately, $\mathcal{T}_z \cap \mathcal{T}_b$ turn out to be the class, $\mathcal{T}_0$ of operators which map all bounded sequence into sequences converging to zero. Now a decomposition of the conservative operators, $\mathcal{T}_c$, has the form

$$\mathcal{T}_c / \mathcal{T}_o = \mathcal{T}_b / \mathcal{T}_o \oplus \mathcal{T}_z / \mathcal{T}_o$$

$$= \mathcal{T}_b / \mathcal{T}_o \oplus \mathcal{T}_\theta / \mathcal{T}_o \oplus \mathcal{T}_A,$$

where $\mathcal{T}_A = \{\delta A ; \delta \epsilon R\}$ for some fixed regular summability operator A. $\mathcal{T}_b$, is the class of operators which take bounded sequences into convergent sequences, and $\mathcal{T}_\theta$ is the class of operators which take convergent sequences into sequences converging to zero.

In part IV, a well-known representation of a continuous linear functional on C of the form,

$$f(\xi) = (a_o - \sum_{i=1}^{\infty} a_i) \lim_{i\to\infty} \xi^i + \sum_{i=0}^{\infty} a_i \xi^i, \text{ where } \sum_{i=0}^{\infty} |a_i| < \infty, \text{ [3]},$$

is generalized to a continuous linear functional on B. First, a representation of B* is established in an integral form with respect to a finitely additive set function on all subsets of the positive integers, I. A Banach limit, as a continuous linear functional on B, is completely characterized in terms of the corresponding set function, which can be considered as a probability function on all subsets of I. Then with respect to a fixed Banach limit L, a decomposition of B* is given as follows:

$$B^* = S \oplus (H) \oplus \{\delta L ; \delta \epsilon R\},$$

where S is the subspace of functionals having a representation

$f(\xi) = \sum_{i=1}^{\infty} a_i \xi^i$ with $\sum_{i=1}^{\infty} |a_i| < \infty$ and (H) is the subspace whose functionals vanish on the subspace of convergent sequences C. This decomposition theorem parallels the decomposition theorem given in part III.

In part V, four different characterizations of the class $\mathcal{T}_b$ are given and their equivalence is established. The first characterization is a direct application of the decomposition theorem in part IV, which in turn gives natural conditions on the matrix, (a_{ij}) of an operator $T \in \mathcal{T}_b$. The operator is characterized by the following conditions on (a_{ij}):

1. $\sum_{i=1}^{\infty} |a_{ij}| < M$, $i = 1, 2, \ldots$

2. $\lim_{i \to \infty} \sum_{j \in E} a_{ij}$ exists for all subsets E of I.

It is also interesting that the above conditions are equivalent to the following conditions given by Schur [4],

1. $\lim_{i \to \infty} a_{ij}$ exists for $j = 1, 2, \ldots$

2. $\sum_{j=1}^{\infty} |a_{ij}|$ converges uniformly with respect to i.

II

DEFINITIONS AND FUNDAMENTAL THEOREMS

ξ or $(\xi^1, \xi^2, \xi^3, \cdots)$ or $\{\xi^i\}$

A bounded sequence of real numbers.

B The set of bounded sequences of real numbers with the following operations and the norm,

(1) $\xi + \gamma = \{\xi^i + \gamma^i\}$; $\xi, \gamma \in B$

(2) $\alpha \xi = \{\alpha \xi^i\}$; $\xi \in B$ and α is any real number

(3) $\|\xi\| = \sup_i |\xi^i|$

Then B is a complete normed vector space.

C The set of convergent sequences with same operations and the same norm as B. Then C is also complete normed vector space and a subspace of B.

Z The set of sequences converging to zero. Z is a subspace of C.

R The field of real numbers.

Summability transformation:

A linear transformation T which is given by a matrix (a_{ij}), or,

$T(\xi) = \{t_i(\xi)\}$, where $t_i(\xi) = \sum_{j=1}^{\infty} a_{ij} \xi^j$; $(i=1, 2, \cdots)$ which

transforms a manifold of B into a manifold of B.

Summability operator:

A summability transformation which maps a subspace of B into itself.

$\mathcal{T}$ The class of all summability operators on B.

$\mathcal{T}_b$ A class of summability operators on B to C.

$\mathcal{T}_c$ A class of summability operators on C to C. If an operator $T \in \mathcal{T}_c$, then T is called a conservative summability operator.

$\mathcal{T}_z$ A subclass of $\mathcal{T}_c$ whose members map Z to Z.

$\mathcal{T}_o$ A class of summability operators which map B to Z.

$\mathcal{T}_r$ A class of summability operators which preserves limits. If an operator $T \in \mathcal{T}_r$, then T is called a regular summability operator.

In the space C, a linear functional $\ell(\xi) = \lim_{i \to \infty} \xi^i$ is continuous. The subspace Z of C is the null space of the linear functional $\ell(\xi)$. Let e_o be a vector in C such that $e_o^i = 1$ for all i. Since $\ell(e_o) = 1$, e_o does not belong to Z, and C can be decomposed into a direct sum of two subspace

$$C = \left\{ \alpha e_o \right\} \oplus Z \qquad \alpha \in R \tag{2.1}$$

Let $e_k = \delta_k^i$, $(k = 1, 2, 3, \ldots)$. Then the set of vectors $\{e_k\}$ forms a basis for Z.

For, if $\xi \in Z$, then $\lim_{i \to \infty} \xi^i = 0$ and consequently

$$\lim_{n \to \infty} \left\| \xi - \sum_{i=1}^{n} \xi^i e_i \right\| = 0,$$

that is,
$$\xi = \sum_{i=1}^{\infty} \xi^i e_i \tag{2.2}$$

Therefore any vector ξ in C has a representation of the form

$$\xi = \alpha^o e_o + \sum_{i=1}^{\infty} \alpha^i e_i .$$

Since $\quad \ell(\xi) = \alpha^o\, \ell(e_o) = \alpha^o$ and $\xi^i = \alpha^o + \alpha^i$

$$\alpha^o = \lim_{i \to \infty} \xi^i = \xi^o$$

$$\alpha^i = \xi^i - \xi^o$$

and
$$\xi = \xi^o e_o + \sum_{i=1}^{\infty} (\xi^i - \xi^o) e_i \qquad (2.3)$$

The representation (2.3) is unique and the set of vectors $\{ e_k ;\ k = 0, 1, 2, \cdots \}$ forms a basis for C.

<u>Lemma 1.</u> Any linear functional of the form $f(\xi) = \sum_{i=1}^{\infty} \alpha_i \xi^i$ defined on Z is continuous and $\| f \| = \sum_{i=1}^{\infty} |\alpha_i|$. Consequently $f(\xi)$ can be extended to the space B without changing the representation $f(\xi) = \sum_{i=1}^{\infty} \alpha_i \xi_i$.

Proof:

Consider a sequence of continuous linear functionals $f_n(\xi) = \sum_{i=1}^{n} \alpha_i \xi^i$; $n = 1, 2, 3, \cdots$. Since $f(\xi)$ is defined on Z, $f_n(\xi)$ converges for all $\xi \in Z$ and $\lim_{n \to \infty} f_n(\xi) = \sum_{i=1}^{\infty} \alpha_i \xi^i$. That is, the sequence f_n converges weakly to f, and f is a continuous linear functional [5].

Let $\quad \xi^i = \operatorname{sgn} \alpha_i \qquad$ for $i = 1, 2, \cdots n$

$\qquad\quad = 0 \qquad$ for $i > n$,

then $\quad f(\xi) = \sum_{i=1}^{n} |a_i| \quad$ and $\quad \|\xi\| = 1.$

Since $\quad f(\xi) \leqslant \|f\|, \sum_{i=1}^{n} |a_i| \leqslant \|f\|$, for all n.

Taking the limit with respect to n;

$$\sum_{i=1}^{\infty} |a_i| \leqslant \|f\|.$$

On the other hand, $\; |f(\xi)| \leqslant \sum_{i=1}^{\infty} |a_i| \, |\xi^i| \leqslant \sum_{i=1}^{\infty} |a_i| \, \|\xi\|.$

Letting $\|\xi\| \leqslant 1$, $\qquad \|f\| \leqslant \sum_{i=1}^{\infty} |a_i|.$

Consequently $\qquad \|f\| = \sum_{i=1}^{\infty} |a_i|. \qquad$ (2.4)

Define $\overline{f}$ by $\overline{f}(\xi) = \sum_{i=1}^{\infty} a_i \xi^i$ for all $\xi \in B$, then $\overline{f}$ coincides with f and it is a continuous extension of f to B. Obviously $\|\overline{f}\| = \|f\|$ because the derivation of (2.4) is valid with ξ in B. This completes the proof.

If $f(\xi)$ is a continuous linear function on C and ξ is given by the representation (3.3), then

$$f(\xi) = \xi^o f(e_o) + f\left(\sum_{i=1}^{\infty} (\xi^i - \xi^o)\, e_i\right)$$

$$= \xi^o f(e_o) + \sum_{i=1}^{\infty} f(e_i)(\xi^i - \xi^o). \qquad (2.5)$$

From the lemma 1, $\sum_{i=1}^{\infty} |f(e_i)| < \infty$. The formula (2.5) can be written

$$f(\xi) = \left[f(e_o) - \sum_{i=1}^{\infty} f(e_i)\right] \xi^o + \sum_{i=1}^{\infty} f(e_i)\xi^i$$

$$= a_o \xi^o + \sum_{i=1}^{\infty} a_i \xi^i,$$

where $a_o = f(e_o) - \sum_{i=1}^{\infty} f(e_i), \quad a_i = f(e_i).$ (2.6)

It is easily seen that $\|f\| \leqslant \sum_{i=0}^{\infty} |a_i|$.

Since $\sum_{i=1}^{\infty} |a_i|$ exists, given $\epsilon > 0$, there exists an N such that

$\sum_{i=N+1}^{\infty} |a_i| < \epsilon$.

Define a vector ξ such that $\xi^i = \operatorname{sgn} a_i$ for $1 \leqslant i \leqslant N$

$= \operatorname{sgn} a_o$ for $i > N$,

then $$f(\xi) = |a_o| + \sum_{i=1}^{N} |a_i| + \operatorname{sgn} a_o \sum_{i=N+1}^{\infty} a_i$$

$$\geqslant \sum_{i=0}^{N} |a_i| - \epsilon \geqslant \sum_{i=0}^{\infty} |a_i| - 2\epsilon.$$

Since ϵ is arbitrary

$$\|f\| \geqslant \sum_{i=0}^{\infty} |a_i|,$$

hence $$\|f\| = \sum_{i=0}^{\infty} |a_i|.$$

<u>Theorem 1.</u> Any summability operator T with matrix (a_{ij}) whose domain is a subspace of B is continuous and

$$\|T\| = \sup_i \sum_{j=1}^{\infty} |a_{ij}|.$$

Proof:

Since each $t_i(\xi)$ is a linear functional defined on Z, it is continuous and $\|t_i\| = \sum_{j=1}^{\infty} |a_{ij}|$ by lemma 1. $T(\xi) \in B$ implies the sequence $\{|t_i(\xi)|\}$ is a bounded sequence for each $\xi \in Z$. By the resonance theorem [6], the sequence of the norms $\{\|t_i\|\}$ is a bounded sequence. Let M be a bound for this sequence. For $\|\xi\| \leqslant 1$, $\|T(\xi)\| = \sup_i |t_i(\xi)| \leqslant \sup_i \|t_i\| < M$.

$$\|T\| = \sup_{\|\xi\| \leq 1} \|T(\xi)\| \leqslant \sup_i \|t_i\| < M.$$

Therefore T is a continuous summability transformation. For $\epsilon > 0$, there exists an N and $\xi \in Z$ such that

$$|t_N(\xi)| > \sup_i \|t_i\| - \epsilon, \text{ where } \|\xi\| \leqslant 1$$

$$\|T\| \geqslant |t_N(\xi)| > \sup_i \|t_i\| - \epsilon.$$

Since ϵ is arbitrary

$$\|T\| \geqslant \sup_i \|t_i\|,$$

hence $\|T\| = \sup_i \|t_i\|$.

<u>Theorem 2</u>. Any summability operator given by $T(\xi) = \{t_i(\xi)\} = \left\{\sum_{j=1}^{\infty} a_{ij}\xi^i\right\}$, on B into B, belongs to the class $\mathcal{T}_c$ if and only if the sequence $\{t_i(\xi)\}$ converges for each vector e_k in the basis for C. The limit function $t(\xi) = \lim_{i \to \infty} t_i(\xi)$ is a continuous linear functional and is given by

$$\lim_{i\to\infty} t_i(\xi) = \delta_o \xi^o + \sum_{j=1}^{\infty} \delta_j(\xi^j - \xi^o),$$

where $\delta_o = \lim_{i\to\infty} t_i(e_o) = \lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij}$

and $\delta_j = \lim_{i\to\infty} t_i(e_j) = \lim_{i\to\infty} a_{ij}$.

Proof:

Since the operator T is on B into B, T is a continuous operator by Theorem 1. Consequently each $t_i(\xi)$ is a continuous linear functional.

Suppose $T \in \mathcal{T}_c$. The sequence $\{t_i(\xi)\}$ of continuous linear functionals converges for all ξ in C. That is, $\{t_i\}$ converges weakly to t. But $\{t_i\}$ converges weakly if and only if $\{t_i\}$ converges on an everywhere dense set in C [5]. The set of finite linear combinations of base vectors is everywhere dense in C. Therefore weak convergence of a sequence $\{t_i\}$ is equivalent to the convergence of $\{t_i(\xi)\}$ for each base vector, and the limit function t is a continuous linear functional on C.

Since t is a continuous linear functional on C, t has a representation of the form given by (2.6). That is

$$t(\xi) = \delta_o \xi^o + \sum_{j=1}^{\infty} \delta_j(\xi^j - \xi^o).$$

For $\xi = e_k$, $\quad t(e_k) = \lim_{i\to\infty} t_i(e_k) = \lim_{i\to\infty} a_{ik} = \delta_k$.

For $\xi = e_o$, $\quad t(e_o) = \lim_{i\to\infty} t_i(e_o) = \lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} = \delta_o$.

Corollary 1. (Kojima-Schur) An operator given by $T(\xi) = \left\{t_i(\xi)\right\} = \left\{\sum_{j=1}^{\infty} a_{ij}\xi^j\right\}$ belongs to $\mathcal{T}_c$ if and only if the following conditions are satisfied.

(1) $\sum_{j=1}^{\infty} |a_{ij}| < M$

(2) $\lim_{i\to\infty} a_{ij}$ exists for all j

(3) $\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij}$ exists.

Proof:

If $T \in \mathcal{T}_c$, T is an operator on C into C. By Theorem 1, T is a continuous operator and

$$\|T\| = \sup_i \|t_i\| = \sup_i \sum_{j=1}^{\infty} |a_{ij}|$$

Therefore there exists a number M such that $\sum_{j=1}^{\infty}|a_{ij}| < M$. By Theorem 2, conditions (2) and (3) are satisfied.

Conversely suppose conditions (1), (2), and (3) are satisfied. From the condition (1) T is a continuous operator on B. Conditions (2) and (3) imply the convergence of $\left\{t_i(\xi)\right\}$ for each base vector. By Theorem 2, $T \in \mathcal{T}_c$.

Corollary 2. (Silverman-Toeplitz) An operator given by $T(\xi) = \left\{t_i(\xi)\right\} = \sum_{j=1}^{\infty} a_{ij}\xi^j$ belongs to $\mathcal{T}_r$ if and only if the following conditions are satisfied.

(1) $\sum_{j=1}^{\infty} |a_{ij}| < M$

(2) $\lim_{i\to\infty} a_{ij} = 0 \quad$ for $i = 1, 2, \ldots$

(3) $\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} = 1$

Proof:

$\mathcal{T}_r \subset \mathcal{T}_c$ and for $T \in \mathcal{T}_r$, $\delta_o = 1$ and $\delta_j = 0$. Therefore conditions (1), (2), and (3) are satisfied. If conditions (1), (2), and (3) are satisfied, $T \in \mathcal{T}_c$ and

$$t(\xi) = \xi^o = \lim_{i\to\infty} \xi^i$$

Hence T is a regular operator.

Corollary 3. An operator given by $T(\xi) = \left\{t_i(\xi)\right\} = \left\{\sum_{j=1}^{\infty} a_{ij}\xi^j\right\}$ belongs to $\mathcal{T}_z$ if and only if the following conditions are satisfied.

(1) $\sum_{j=1}^{\infty} |a_{ij}| < M$

(2) $\lim_{i\to\infty} a_{ij} = 0$

(3) $\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij}$ exists.

Proof:

Since $\mathcal{T}_z \subset \mathcal{T}_c$, $T \in \mathcal{T}_z$ implies that conditions (1) and (3) are satisfied. The mapping of Z into Z implies condition (2). Conversely, if conditions (1), (2), (3) are satisfied, $T \in \mathcal{T}_c$. Moreover the condition (2) implies T is a mapping of Z into Z. Consequently T belongs to $\mathcal{T}_z$.

III

SOME DECOMPOSITION THEOREMS

Consider the set $\mathcal{T}$ of all summability operators on B.

If $T \in \mathcal{T}$, $T(\xi) = \sum_{j=1}^{\infty} a_{ij}\xi^j$ and $\sum_{j=1}^{\infty} |a_{ij}| < M(T)$, $i = 1, 2, 3, \dots$

If one defines an addition operation and a scalar multiplication as follows:

$$T(\xi) + T'(\xi) = \sum_{i=j}^{\infty} (a_{ij} + a'_{ij})\xi^j \quad , \qquad (i = 1, 2, 3, \dots)$$

$$c\, T(\xi) = \sum_{j=1}^{\infty} (c\, a_{ij})\xi^j \quad , \qquad (i = 1, 2, 3, \dots)$$

where $T(\xi) = \sum_{j=1}^{\infty} a_{ij}\xi^j$

and $T'(\xi) = \sum_{j=1}^{\infty} a_{ij}\xi^j$,

then $\mathcal{T}$ becomes a vector space. Now, some classes of transformation are distinguished from $\mathcal{T}$ and are characterized in the following table.

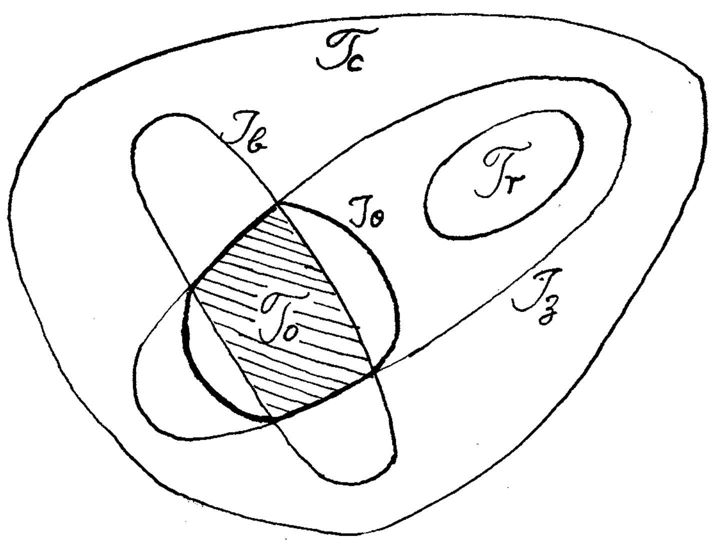
Class of Transformation	$\mathcal{T}_c$	$\mathcal{T}_r$	$\mathcal{T}_b$	$\mathcal{T}_z$	$\mathcal{T}_\theta$	$\mathcal{T}_0$
$\delta_j = \lim_{i\to\infty} a_{ij}$	δ_j	$\delta_j = 0$	δ_j	$\delta_j = 0$	$\delta_j = 0$	$\delta_j = 0$
$\delta = \lim_{i\to\infty} \sum_{j=1} a_{ij}$	δ	$\delta = 1$	$\delta = \sum_{j=1}^{\infty} \delta_j$	δ	$\delta = 0$	$\delta = 0$
Subspace Mapping Relations	C→C	C→C	B→C	C→C Z→Z	C→Z	B→Z

(1) In all cases $\sum_{j=1}^{\infty} |a_{ij}| < M$.

(2) For $\mathcal{T}_b$, $\mathcal{T}_\theta$, the uniform convergence of $\sum_{j=1}^{\infty} |a_{ij}|$ is required.

(3) For $\mathcal{T}_r$, $\lim_{i \to \infty} t_i(\xi) = \lim_{i \to \infty} \xi^i$.

The inclusion relations are shown by the following diagram:



For any classes except $\mathcal{T}_r$ in the above list, it is easily seen that they form subspaces of the space of summability operators. It is interesting to notice that the class $\mathcal{T}_r$ forms a convex set in $\mathcal{T}_z$, (actually a flat or coset of $\mathcal{T}_z$ in $\mathcal{T}_c$), even though $\mathcal{T}_r$ is not a subspace. It is well known that $\mathcal{T}_b \cap \mathcal{T}_r = \phi$ (null set) [7], and obviously $\mathcal{T}_\theta \cap \mathcal{T}_r = \phi$.

<u>Lemma 2.</u> $\mathcal{T}_b \cap \mathcal{T}_z = \mathcal{T}_0$.

Proof:

If $T \in \mathcal{T}_b \cap \mathcal{T}_z$, then $\delta_j = 0$.

Moreover, it is well known that $T \in \mathcal{T}_b$ implies that

$\lim_{n \to \infty} t_n(\xi) = \sum_{i=1}^{\infty} \delta_i \xi^i$. Hence $T \in \mathcal{T}_0$.

Therefore $\mathcal{T}_b \cap \mathcal{T}_z \subset \mathcal{T}_0$.

Conversely if $T \in \mathcal{T}_0$, $T(B) \subset Z$. Hence $T(C) \subset Z$ and $T \in \mathcal{T}_z$.

Also $T(B) \subset Z$ implies $T \in \mathcal{T}_b$. Thus $\mathcal{T}_0 \subset \mathcal{T}_b \cap \mathcal{T}_z$.

<u>Theorem 3.</u> $\mathcal{T}_c/\mathcal{T}_0 = \mathcal{T}_b/\mathcal{T}_0 \oplus \mathcal{T}_z/\mathcal{T}_0$

Proof:

Let $T(\xi) = \left\{ t_i(\xi) \right\} = \left\{ \sum_{j=1}^{\infty} a_{ij} \xi^j \right\}$ for $T \in \mathcal{T}_c$.

Since $\sum_{j=1}^{\infty} |a_{ij}| < M$, $\sum_{j=1}^{n} |a_{ij}| < M$ for all n.

Taking the limit with respect to i yields

$$\sum_{j=1}^{n} |\delta_j| \leq M \qquad \text{for all } n.$$

Then $\displaystyle \sum_{j=1}^{\infty} |\delta_j| = \lim_{n \to \infty} \sum_{j=1}^{n} |\delta_j| \leq M.$

Now
$$t_i(\xi) = \sum_{j=1}^{\infty} a_{ij} \xi^j$$

$$= \sum_{j=1}^{\infty} (a_{ij} - \delta_j) \xi^j + \sum_{j=1}^{\infty} \delta_j \xi^j.$$

Therefore T is a sum of two summability operators, one from $\mathcal{T}_z$ and the other from $\mathcal{T}_b$.

$$\mathcal{T}_c = \mathcal{T}_b + \mathcal{T}_z$$

Now to show the uniqueness of the decomposition suppose $\left\{t_i(\xi)\right\}$ has another decomposition of the same form,

$$t_i(\xi) = \sum_{j=1}^{\infty} \beta_{ij}\xi^j + \sum_{j=1}^{\infty} \gamma_{ij}\xi^j$$

where $\left\{\sum_{j=1}^{\infty} \beta_{ij}\xi^j\right\} \in \mathcal{T}_z$

and $\left\{\sum_{j=1}^{\infty} \gamma_{ij}\xi^j\right\} \in \mathcal{T}_b$

Then $\sum_{j=1}^{\infty} (a_{ij} - \delta_j)\xi^j + \sum_{j=1}^{\infty} \delta_j\xi^j = \sum_{j=1}^{\infty} \beta_{ij}\xi^j + \sum_{j=1}^{\infty} \gamma_{ij}\xi^j$

or $\sum_{j=1}^{\infty} (a_{ij} - \delta_j)\xi^j - \sum_{j=1}^{\infty} \beta_{ij}\xi^j = \sum_{j=1}^{\infty} \gamma_{ij}\xi^j - \sum_{j=1}^{\infty} \delta_j\xi^j.$

Since $\sum_{j=1}^{\infty} (a_{ij} - \delta_j)\xi^j - \sum_{j=1}^{\infty} \beta_{ij}\xi^j \in \mathcal{T}_z$

and $\sum_{j=1}^{\infty} \gamma_{ij}\xi^j - \sum_{j=1}^{\infty} \delta_j\xi^j \in \mathcal{T}_b,$

they belong to $\mathcal{T}_0 = \mathcal{T}_b \cap \mathcal{T}_z$. By taking the coset decomposition one obtains a direct sum decomposition

$$\mathcal{T}_c/\mathcal{T}_0 = \mathcal{T}_b/\mathcal{T}_0 \oplus \mathcal{T}_z/\mathcal{T}_0 .$$

Now consider a decomposition of $\mathcal{T}_z$. Let $T_z \in \mathcal{T}_z$ and A be any regular summability operator, then if $\delta \neq 0$

$$T_z = \delta(1/\delta\ T_z - A) + \delta A$$
$$= T\theta + \delta A, \text{ where } T_\theta = \delta(1/\delta\ T_z - A) \in \mathcal{T}_\theta \qquad (3.1)$$

If $\delta = 0$, then $T_z \in \mathcal{T}_\theta$ and the equation (3.1) is still true with $T_\theta = T_z$ and $\delta = 0$.

Now suppose T_z has another decomposition of the form

$$T_z = T'_\theta + \delta' A \qquad \text{where } T'_\theta \in \mathcal{T}_\theta .$$

Then $\quad T_\theta + \delta A = T'_\theta + \delta'_A$

and $\quad T_\theta - T'_\theta = (\delta' - \delta)A.$

Since $\quad T_\theta - T'_\theta \in \mathcal{T}_\theta$, $(\delta' - \delta)A \in \mathcal{T}_\theta$, which implies $\delta = \delta'$ and consequently $T_\theta = T'_\theta$. Therefore the decomposition given by (3.1) is unique. The above argument gives the following direct sum decomposition.

$$\mathcal{T}_z = \mathcal{T}_\theta \oplus \left\{\delta A;\ \delta \in R\right\}$$

To consider the decomposition with respect to $\mathcal{T}_0$, first notice that $\mathcal{T}_0 \subset \mathcal{T}_\theta$.

If $\mathcal{T}_A = \left\{\delta A;\ \delta \in R \quad + \mathcal{T}_0\right\}$ the coset decomposition of $\mathcal{T}_z$ with respect to $\mathcal{T}_0$ is given by

$$\mathcal{T}_z/\mathcal{T}_0 = \mathcal{T}_\theta/\mathcal{T}_0 \oplus \mathcal{T}_A. \qquad (3.2)$$

Combining (3.2) with Theorem 3, yields the following theorem:

<u>Theorem 4</u>. $\mathcal{T}_c$ can be decomposed into a direct sum of the form

$$\mathcal{T}_c/\mathcal{T}_0 = \mathcal{T}_b/\mathcal{T}_0 \oplus \mathcal{T}_\theta/\mathcal{T}_0 \oplus \mathcal{T}_A.$$

IV

STRUCTURE OF B*

Let I be the set of all positive integers and Σ be the family of all subsets of I. For each set $E \in \Sigma$, let $e_E \in B$ be defined as follows:

$$e_E = \{e_E^i\} \text{ where } \quad e_E^i = 1 \text{ if } i \in E \qquad (4.1)$$
$$e_E^i = 0 \text{ if } i \notin E.$$

That is, e_E is the characteristic function of E. Consider the linear manifold D generated by $\{e_E | E \in \Sigma\}$, that is, $\xi \in D$ if and only if

$\xi = \sum_{i=1}^{k} a_i e_{E_i}$, where $a_i \in R$ and $E_i \in \Sigma$. It is asserted that the linear manifold D is everywhere dense in the Banach space B. To see this, take any vector ξ in B. Since ξ is bounded, there exists an M such that $|\xi^i| < M$ for $i = 1, 2, 3, \ldots$

Let $E_m^{(n)} = \left\{ i \,\middle|\, -M + \frac{m}{n} M \leqslant \xi^i < -M + \frac{m+1}{n} M \right\}$,

$m = 0, 1, 2, \ldots 2n-1$ and define a sequence $\{\xi_{(n)}\}$ of vectors in D as follows

$$\xi_{(n)} = \sum_{m=0}^{2n-1} \left(\frac{m}{n} - 1\right) M \, e_{E_m^{(n)}} \qquad n = 1, 2, 3, \ldots \qquad (4.2)$$

For any vector ξ in B, define

$$\xi_E^i = \xi^i \qquad \text{if } i \in E$$
$$= 0 \qquad \text{if } i \notin E,$$

then $$\xi = \sum_{m=0}^{2n-1} \xi_{E_m}(n). \tag{4.3}$$

using (4.2) and (4.3) yields

$$\xi - \xi_{(n)} = \sum_{m=0}^{2n-1} \xi_{E_m}(n) - (\frac{m}{n} - 1)M\, e_{E_m}(n) \ ,$$

and $$\left\| \xi_{E_m}(n) - (\frac{m}{n} - 1)M\, e_{E_m}(n) \right\| < \frac{M}{n},$$

hence $$\left\| \xi - \xi_{(n)} \right\| = \max_m \left\| \xi_{E_m}(n) - (\frac{m}{n} - 1)M\, e_{E_m}(n) \right\| < \frac{M}{n}.$$

Therefore, any vector ξ in B is the limit of a sequence from D. This proves:

Lemma 3. In the space B, the linear manifold D is everywhere dense.

Let f be a continuous linear functional on B and ξ be the limit of a sequence $\xi_{(n)}$ from D. Since $f(\xi)$ is continuous

$$f(\xi) = \lim_{n\to\infty} f(\xi_{(n)})$$

As a matter of fact, one can use any sequence $\{\xi_{(n)}\}$ which converges to ξ. Therefore, one can use some specific sequence in D. Suppose the sequence defined in (4.2) is used. Then

$$f(\xi_{(n)}) = \sum_{m=0}^{2n-1} (\frac{m}{n} - 1)M\, f(e_{E_m}(n))$$

$$= \sum_{m=0}^{2n-1} (\frac{m}{n} - 1)M\, \mu(E_m^{(n)}), \qquad \text{where } \mu(E) = f(e_E),$$

and hence $$f(\xi) = \lim_{n\to\infty} \sum_{m=0}^{2n-1} (\frac{m}{n} - 1)M\, \mu(E_m^{(n)}). \tag{4.5}$$

If $E_1 \cap E_2 = \phi$, $f(e_{E_1}) + f(e_{E_2}) = f(e_{E_1} + e_{E_2}) = f(e_{E_1} \cup E_2)$

or $\quad \mu(E_1) + \mu(E_2) = \mu(E_1 \cup E_2)$

Therefore μ is a finitely additive set function defined on the field of subsets Σ. Now $f(\xi)$ can be written in the integral form

$$f(\xi) = \int \xi \, d\mu, \tag{4.6}$$

where $\int \xi \, d\mu$ is a compact notation for (4.5) and μ is a finitely additive set function.

For every set E, the total variation of μ on E is given by the definition

$$\mathcal{V}_\mu(E) = \sup \sum_{i=1}^{n} |\mu(E_i)|, \text{ where } E_i \text{ are disjoint subsets of } E.$$

If for any disjoint sets $E_1, E_2, \cdots, E_n$, a vector ξ is defined by

$$\xi = \sum_{i=1}^{n} \operatorname{sgn} f(e_{E_i}) e_{E_i},$$

then $\quad f(\xi) = \operatorname{sgn} \mu(E_i) \mu(E_i) = \sum_{i=1}^{n} |\mu(E_i)|, \quad$ and $\|\xi\| = 1$.

Hence $\quad \sum_{i=1}^{n} |\mu(E_i)| \leq \|f\| \quad$ for all disjoint sets E_{ij} $i = 1, \cdots n$.

Thus $\quad \mathcal{V}_\mu(E) \leq \|f\| \quad$ for all $E \in \Sigma$. $\quad$ (4.7)

The inequality $\|f\| \leq \mathcal{V}_\mu(I)$ is proved as follows.

Given $\epsilon > 0$, there exists $\xi \in D$ such that

$$f(\xi) + \epsilon > \|f\| \quad \text{and} \quad \|\xi\| \leq 1. \tag{4.8}$$

Now ξ has a representation of the form

$$\xi = \sum_{i=1}^{n} a_i e_{E_i}, \text{ where } |a_i| \leq 1, \ \{E_i\} \text{ are disjoint and } \bigcup_{i=1}^{n} E_i = I.$$

Therefore (4.8) becomes

$$\|f\| < \sum_{i=1}^{n} a_i \mu(E_i) + \epsilon \leq \sum_{i=1}^{n} |\mu(E_i)| + \epsilon \leq \mathcal{V}_\mu(I) + \epsilon,$$

and since ϵ is arbitrary

$$\|f\| \leq \mathcal{V}_\mu(I), \tag{4.9}$$

so that (4.7) and (4.9) give

$$\|f\| = \mathcal{V}_\mu(I).$$

That is, μ is a bounded set function and the total variation of μ on I is equal to the norm of the continuous linear functional f.

Conversely, if μ is a bounded finitely additive set function on Σ, then $\int \xi \, d\mu$ can be defined as follows and is a continuous linear functional on B.

If $\xi \in D$, and $\xi = \sum_{i=1}^{n} a_i e_{E_i}$, define $\int \xi \, d\mu = \sum_{i=1}^{n} a_i \mu(E_i)$.

If $\|\xi\| \leq 1$, $|a_i| \leq 1$ for $i = 1, 2, \ldots, n$, and

$$\left| \int \xi \, d\mu \right| \leq \sum_{i=1}^{n} |a_i| \, |\mu(E_i)| \leq \sum_{i=1}^{n} |\mu(E_i)| \leq \mathcal{V}_\mu(I)$$

Obviously $\xi \, d\mu$ is a bounded linear functional on D. Therefore it is a continuous linear functional on D. Since $\overline{D} = B$, it follows from the Hahn-Banch extension theorem that $\int \xi \, d\mu$ can be extended to the whole space B with preservation of the norm and this extension is unique [9]. Therefore, (4.5) gives a representation for continuous linear functionals on B. This proves the following representation theorem:

Theorem 5. Any continuous linear functional $f(\xi)$ on B has a unique representation of the form $\int \xi \, d\mu$, where μ is a bounded finitely additive set function on Σ and $f(e_E) = \mu(E)$ for all $E \in \Sigma$. Conversely, any finitely additive bounded set function on Σ defines uniquely a continuous linear functional $\int \xi \, d\mu$.

A Banach limit Lim ξ is a linear functional on B which has the following properties

(1) $\operatorname{Lim} \xi = \lim_{n \to \infty} \xi^n \qquad \text{if } \xi \in C$

(2) $\underline{\lim}_{n \to \infty} \xi^n \leq \operatorname{Lim} \xi \leq \overline{\lim}_{n \to \infty} \xi^n.$

From the condition (1)

$$\operatorname{Lim} e_o = 1,$$

$$\operatorname{Lim} e_E = 0, \qquad \text{if } E \text{ is a finite set,}$$

and from (2)

$$0 \leq \operatorname{Lim} e_E \leq 1 \qquad \text{for all } E.$$

Therefore the bounded finitely additive set function on Σ corresponding to the Banach limit has the following corresponding properties.

$$\begin{aligned} &(1) \quad \mu(I) = 1, \\ &(2) \quad \mu(E) = 0, \quad \text{if } E \text{ is finite,} \\ &(3) \quad 0 \leq \mu(E) \leq 1 \quad \text{for all } E. \end{aligned} \tag{4.10}$$

Conversely, suppose a finitely additive set function μ has the properties (4.10), and let $f(\xi)$ be the continuous linear functional defined by μ. If $\xi \in C$, then $f(\xi) = \int \xi \, d\mu = \xi^o$, where $\xi^o = \lim_{i \to \infty} \xi^i$.

Let $\xi \in B$ and $\overline{\xi}^o = \overline{\lim}_{i \to \infty} \xi^i$, $\underline{\xi}^o = \underline{\lim}_{i \to \infty} \xi^i$. Define three sequences $\{\xi_u^i\}$, $\{\xi_v^i\}$, $\{\xi_w^i\}$ as follows:

if $\xi^i > \overline{\xi}^o$, then $\xi_u^i = \xi^i - \overline{\xi}^o$, $\xi_v^i = \overline{\xi}^o$, $\xi_w^i = 0$,

if $\xi^i < \underline{\xi}^o$, then $\xi_u^i = 0$, $\xi_v^i = \underline{\xi}^o$, $\xi_w^i = \xi^i - \underline{\xi}^o$ and

if $\underline{\xi}^o \leq \xi^i \leq \overline{\xi}^o$, then $\xi_u^i = 0$, $\xi_v^i = \xi^i$, $\xi_w^i = 0$.

From this definition

$$\left\{\xi^i\right\} = \left\{\xi_u^i\right\} + \left\{\xi_v^i\right\} + \left\{\xi_w^i\right\},$$

$$\lim_{n\to\infty} \xi_u^i = \lim_{n\to\infty} \xi_w^i = 0,$$

$$\underline{\xi}^o \leq \xi_v^i \leq \overline{\xi}^o.$$

Now $f(\xi) = f(\xi_u) + f(\xi_v) + f(\xi_w) = f(\xi_v)$.

For ξ_v, one can obtain a sequence $\left\{\xi_{(n)}\right\}$ in D converging to ξ_v, as in (4.2), by taking

$$E_m^{(n)} = \left\{ i \,\middle|\, \underline{\xi}^o + \frac{m}{n}(\overline{\xi}^o - \overline{\xi}_o) \leq \xi_v^i < \underline{\xi}^o + \frac{m+1}{n}(\overline{\xi}^o - \underline{\xi}^o) \right\}$$

$$m = 0, 1, 2, \ldots, n-1$$

and
$$\xi_{(n)} = \sum_{m=0}^{n-1} \left[(1 - \frac{m}{n}) \underline{\xi}^o + \frac{m}{n} \overline{\xi}^o \right] e_m^{(n)}.$$

Here, $\sum_{m=0}^{n-1} E_m^{(n)} = I$ and $\left\{ E_m^{(n)},\ m = 0, 1, \cdots n-1 \right\}$ are mutually disjoint.

Hence,
$$f(\xi_{(n)}) = \sum_{m=0}^{n-1} \left[(1 - \frac{m}{n}) \underline{\xi}^o + \frac{m}{n} \overline{\xi}^o \right] \mu(E_m^{(n)}).$$

Since $\sum_{m=0}^{n-1} \mu(E_m^{(n)}) = 1$ and $\mu(E_m^{(n)}) \geq 0$

$$\underline{\xi}^o \leq f(\xi_n) \leq \overline{\xi}^o - \frac{(\overline{\xi}^o - \underline{\xi}^o}{n}.$$

Taking the limit as $n \to \infty$,

$$\underline{\xi}^{o} \leqslant f(\xi) \leqslant \overline{\xi}^{o}.$$

That is, $f(\xi)$ defined by μ and satisfying (4.10) is a Banach limit. This proves the following theorem.

Theorem 6. A continuous linear functional $\int \xi \, d\mu$ defined by a finitely additive set function μ on Σ is a Banach limit if and only if μ has the following properties:

(1) $\mu(I) = 1$,

(2) $\mu(E) = 0$, if E is a finite set,

(3) $\mu(E) \geqslant 0$.

Suppose μ is a bounded completely additive set function on Σ. Then once μ is defined at each point of I, μ is defined on Σ.

Let $\mu(\, i \,) = a_i$, then $\mu(E) = \sum_{i \in E} a_i$. (4.11)

From the definition of $\mathcal{V}_\mu(I)$,

$$\mathcal{V}_\mu(I) \leqslant \sum_{i=1}^{\infty} |a_i|$$

Also $|\mu(E^+)| + |\mu(E^-)| \leqslant \mathcal{V}_\mu(I)$ where $E^+ = \{i \,|\, a_i \geqslant 0\}$, $E^- = \{i \,|\, a_i < 0\}$,

or $\sum_{i=1}^{\infty} |a_i| \leqslant \mathcal{V}_\mu(I)$.

Consequently $\mathcal{V}_\mu(I) = \sum_{i=1}^{\infty} |a_i|$. (4.12)

Consider a linear functional $f(\xi) = \sum_{i=1}^{\infty} a_i \xi^i$, then $|f(\xi)| \leqslant \|\xi\| \sum_{i=1}^{\infty} |a_i|$.

Therefore $f(\xi)$ is continuous, and it is easy to see $\|f\| = \sum_{i=1}^{\infty} |a_i|$.

Now $f(e_E) = \sum_{i \in E} \alpha_i$ and the set function defined by $f(\xi) = \sum_{i=1}^{\infty} \alpha_i \xi^i$

coincides with μ given by (4.11).

This proves the following theorem:

Theorem 7. A continuous linear functional $f(\xi) = \int \xi \, d\mu$ has a series representation $\sum_{i=1}^{\infty} \alpha_i \xi^i$ if and only if the bounded finitely additive set function, μ, is completely additive, where $f(e_i) = \alpha_i$ and $\|f\| = \sum_{i=1}^{\infty} |\alpha_i|$.

Consider now a decomposition of a continuous linear functional $f(\xi)$.

Let $f(e_i) = \alpha_i$. Since $\sum_{i=1}^{n} |\alpha_i| \leqslant \|f\|$, it follows that $\sum_{i=1}^{\infty} |\alpha_i| \leqslant \|f\|$.

Then $f(\xi) = \sum_{i=1}^{\infty} \alpha_i \xi^i$ is a continuous linear functional and

$L_f(\xi) = f(\xi) - S_f(\xi)$ is also a continuous linear functional. By Theorem 5 let $L_f(\xi)$ be represented by $\int \xi \, d\mu$, where μ is a bounded finitely additive set function. From the definitions of L_f, S_f, and $\int \xi \, d\mu$, it follows that the value of μ on all finite sets is zero.

Now $$f(\xi) = L_f(\xi) + S_f(\xi) = L_f(\xi) + \sum_{i=1}^{\infty} f(e_i)\xi^i$$

where $\sum_{i=1}^{\infty} |f(e_i)| \leqslant \|f\|$ and $L_f(e_E) = 0$ on any finite set E.

Suppose $f(\xi) = L_f'(\xi) + \sum_{i=1}^{\infty} \alpha_i' \xi^i$ where $\sum_{i=1}^{\infty} |\alpha_i'|$ exists and

$L_f'(e_E) = 0$ on any finite set E.

Then $$L_f'(\xi) + \sum_{i=1}^{\infty} \alpha_i' \xi^i = L_f(\xi) + \sum_{i=1}^{\infty} f(e_i)\xi^i.$$

For $\xi = e_i$, $\alpha_i' = f(e_i) = \alpha_i$, hence $L_f'(\xi) = L_f(\xi)$ for all $\xi \in B$.

Since B* is a normed vector space

$$\|f\| \leqslant \|L_f\| + \sum_{i=1}^{\infty}|\alpha_i|.$$

For any $\|\xi\| = 1$, consider $\eta = \eta^i$ such that $\eta^i = \operatorname{sgn} \alpha_i$, $i \leqslant n$

$$\eta^i = (\operatorname{sgn} L_f(\xi))\xi^i \text{ for } i > n.$$

Then
$$\begin{aligned} f(\eta) &= L_f(\eta) + \Sigma\, \alpha_i \eta_i \\ &= [\operatorname{sgn} L_f(\xi)]\, L_f(\xi) + \sum_{i=1}^{n} |\alpha_i| + [\operatorname{sgn} L_f(\xi)] \sum_{i=n+1}^{\infty} \alpha_i \xi^i, \\ &= |L_f(\xi)| + \sum_{i=1}^{n} |\alpha_i| + [\operatorname{sgn} L_f(\xi)] \sum_{i=n+1}^{\infty} \alpha_i \xi^i. \end{aligned}$$

Since $\|\eta\| = 1$,

$$\|f\| \geqslant f(\eta) = |L_f(\xi)| + \sum_{i=1}^{n}|\alpha_i| + \epsilon_n.$$

Since ξ and n are arbitrary and $\lim_{n\to\infty} \epsilon_n = 0$,

$$\|f\| \geqslant \|L_f\| + \sum_{i=1}^{\infty} |\alpha_i|.$$

Therefore $\|f\| = \|L_f\| + \|S_f\|$.

This gives the following theorem.

Theorem 8. Any continuous linear functional on B has a unique expression of the form $f(\xi) = L_f(\xi) + \sum_{i=1}^{\infty} \alpha_i \xi^i$, where $L_f(e_E) = 0$ for a finite set E and $\sum_{i=1}^{\infty} |\alpha_i| < \infty$. Moreover $\|f\| = \|L_f\| + \sum_{i=1}^{\infty}|\alpha_i|$.

Theorem 9. The set S of continuous linear functionals of the form

$$f(\xi) = \sum_{i=1}^{\infty} a_i \xi^i, \quad \sum_{i=1}^{\infty} |a_i| < \infty,$$

forms a subspace of B*.

Proof:

Let $\quad f_i(\xi) = \sum_{j=1}^{\infty} a_{ij} \xi^j, \quad \sum_{j=1}^{\infty} |a_{ij}| < \infty$ be a Cauchy sequence in B*.

Then, given $\epsilon > 0$, there exists an N such that for $m, n > N$

$$\|f_m - f_n\| < \epsilon$$

or

$$\sum_{j=1}^{\infty} |a_{mj} - a_{nj}| < \epsilon$$

Taking partial sum,

$$\sum_{j=1}^{k} |a_{mj} - a_{nj}| < \epsilon, \text{ for arbitrary } k$$

Letting $n \to \infty$ and putting $\lim_{n \to \infty} a_{nj} = \delta_j$, (which always exists), yields

$$\sum_{j=1}^{k} |a_{mj} - \delta_j| < \epsilon, \text{ for all } k.$$

Since k is arbitrary, $\sum_{j=1}^{\infty} |a_{mj} - \delta_j| < \epsilon$, and

$$\sum_{j=1}^{\infty} |\delta_j| = \sum_{j=1}^{\infty} |a_{mj} - \delta_j - a_{mj}| \leqslant \sum_{j=1}^{\infty} |a_{mj} - \delta_j| + \sum_{j=1}^{\infty} |a_{mj}| < \sum_{j=1}^{\infty} |a_{mj}| + \epsilon,$$

or $\quad \sum_{j=1}^{\infty} |\delta_j| - \epsilon < \sum_{j=1}^{\infty} |a_{mj}|$, for $m > N$.

Hence $\quad \sum_{j=1}^{\infty} |\delta_j| \leqslant \underline{\lim}_{m \to \infty} \sum_{j=1}^{\infty} |a_{mj}|$.

Also $\sum_{j=1}^{\infty}|a_{mj}| \leqslant \sum_{j=1}^{\infty}|a_{mj} - \delta_j| + \sum_{j=1}^{\infty}|\delta_j| < \epsilon + \sum_{j=1}^{\infty}|\delta_j|$, for all $m > N$.

Then $\overline{\lim}_{m\to\infty} \sum_{j=1}^{\infty}|a_{mj}| \leqslant \sum_{j=1}^{\infty}|\delta_j|$.

Then it follows that

$$\overline{\lim}_{m\to\infty} \sum_{j=1}^{\infty}|a_{mj}| \leqslant \sum_{j=1}^{\infty}|\delta_j| \leqslant \underline{\lim}_{m\to\infty} \sum_{j=1}^{\infty}|a_{mj}|,$$

or

$$\lim_{m\to\infty} \sum_{j=1}^{\infty}|a_{mj}| = \sum_{j=1}^{\infty}|\delta_j|. \tag{4.13}$$

Now define a continuous linear functional $f(\xi)$ by

$$f(\xi) = \sum_{j=1}^{\infty} \delta_j \xi^j. \tag{4.14}$$

Then $\|f_m - f\| = \sum_{j=1}^{\infty}|a_{mj} - \delta_j| \longrightarrow 0$, as $m \longrightarrow \infty$

Therefore the limit of a Cauchy sequence in S belongs to S. It is easily seen that S is a linear manifold, hence S is a subspace.

Theorem 10. The set K of all continuous linear functionals $L(\xi)$ such that $L(e_E) = 0$ for all finite sets E forms a subspace and $L(\xi)$ has a representation of the form

$$L(\xi) = a_o L_o(\xi) + \theta(\xi),$$

where L_o is a fixed Banach limit and θ is a continuous linear function such that $\theta(\xi) = 0$ for all $\xi \in C$.

Proof:

The set of all $L(\xi)$ obviously forms a linear manifold. Let $\{L_i\}$ be a Cauchy sequence in this manifold K. From the completeness of B*,

$\lim_{i\to\infty} L_i = L$ is a continuous linear functional, and $\lim_{i\to\infty} L_i(\xi) = L(\xi)$ for all $\xi \in B$. In particular, if $\xi = e_E$ and E is finite, then $L_i(e_E) = 0$. Therefore $L(e_E) = \lim_{i\to\infty} L_i(e_E) = 0$ for all finite set E, that is, L belongs to the manifold K, which implies the manifold K is a subspace of B*.

Consider a subset Ⓗ of K such that $L(e_o) = 0$ if $L \in$ Ⓗ. By the same argument as above, one can easily show that Ⓗ is a subspace of K. Let $L \in K$ and $L \notin$ Ⓗ, then $L(e_o) \neq 0$.

Now $$L(\xi) = L(e_o)\left[\frac{1}{L(e_o)} L(\xi)\right] - L(e_o)L_o(\xi) + L(e_o)L_o(\xi)$$

$$= L(e_o)L_o(\xi) + L(e_o)\left[\frac{1}{L(e_o)} L(\xi) - L_o(\xi)\right],$$

where L_o is a Banach limit.

Since $$L(e_o)\left[\frac{1}{L(e_o)} L(\xi) - L_o(\xi)\right] \in Ⓗ,$$

$$L(\xi) = \alpha_o L_o(\xi) + \theta(\xi), \text{ where } \alpha_o = L(e_o),$$

$$\theta(\xi) = L(e_o)\left[\frac{1}{L(e_o)} L(\xi) - L_o(\xi)\right].$$

Theorem 9 and Theorem 10 give the following theorem.

Theorem 11. The space B* of all continuous linear functionals on B is the direct sum of three subspaces, $\{\alpha L_o;\ \alpha \in R\}$, Ⓗ and S;

$$B^* = Ⓗ \oplus \{\alpha L_o\} \oplus S.$$

One can show that the subspace Ⓗ is not a trivial one, that is, there exists a continuous linear functional $\theta \in$ Ⓗ and $\theta \not\equiv 0$.

To prove this, let $\xi_o \in B$ but $\xi_o \notin C$ and consider a manifold given by $\{\alpha\xi_o + \eta;\ \alpha \in R,\ \eta \in C\}$.

Define $\theta(\xi_o) = 1$ and $\theta(\eta) = 0$ for all $\eta \in C$, and $\theta(\alpha\xi_o + \eta) = \alpha$. Then θ is a linear functional on the manifold, and one needs only to show that θ is continuous on the manifold.

If $a \neq 0$ then $\left|\theta\left(\frac{a\xi_o + \eta}{\|a\xi_o + \eta\|}\right)\right| = \frac{|a|}{|a|\,\|\xi_o + \eta'\|} = \frac{1}{\|\xi_o + \eta'\|}$,

where $\eta' = \frac{\eta}{a}$. Therefore θ is continuous if $\inf_{\eta \in C} \|\xi_o + \eta\| \neq 0$.

Let $\eta = \eta^i$ and $\eta^o = \lim_{i\to\infty} \eta^i$. Then by (3.3),

$$\xi_o + \eta = \xi_o + \eta^o e_o + \sum_{i=1}^{\infty} (\eta^i - \eta^o) e_i$$

or $$(\xi_o + \eta)^i = \xi_o^i + \eta^o + (\eta^i - \eta^o)$$

and $$\|\xi_o + \eta\| \geqslant \max\left[\, \left|\overline{\lim_{i\to\infty}}\, \xi_o^i + \eta^o\right|, \left|\underline{\lim_{i\to\infty}}\, \xi_o^i + \eta^o\right| \,\right]$$

$$\geqslant 1/2\left[\, \overline{\lim_{i\to\infty}}\ \xi_o^i - \underline{\lim_{i\to\infty}}\ \xi_o^i \,\right].$$

Hence $$\inf_{\eta \in C} \|\xi_o + \eta\| \geqslant 1/2\left[\, \overline{\lim_{i\to\infty}}\ \xi_o^i - \underline{\lim_{i\to\infty}}\ \xi_o^i \,\right].$$

Actually the equality holds because for any $\epsilon > 0$, there exists an N such that for all $i > N$,

$$\underline{\lim}\ \xi_o^i - \epsilon < \xi_o^i < \overline{\lim}\ \xi_o^i + \epsilon.$$

If for $i \leqslant N$, $\eta^i = \xi_o^i$, and for $i > N$, $\eta^i = -1/2\left(\overline{\lim_{i\to\infty}}\ \xi_o^i + \underline{\lim_{i\to\infty}}\ \xi_o^i \right)$,

then $$(\xi_o + \eta)^i = 0 \qquad \text{if } i \leqslant N$$

and $$-1/2\left(\overline{\lim_{i\to\infty}}\ \xi_o^i - \underline{\lim_{i\to\infty}}\, \xi_o^i \right) - \epsilon < (\xi_o + \eta)^i < 1/2\left(\overline{\lim_{i\to\infty}}\ \xi_o^i - \underline{\lim_{i\to\infty}}\, \xi_o^i \right) + \epsilon \text{ if } i > N,$$

or $$\|\xi_o + \eta\| \leqslant 1/2\left(\overline{\lim_{i\to\infty}}\ \xi_o^i - \underline{\lim_{i\to\infty}}\ \xi_o^i \right) + \epsilon.$$

Since ϵ is arbitrary,

$$\inf_{\eta \in C} \|\xi_o + \eta\| \leqslant 1/2\left[\, \overline{\lim_{i\to\infty}}\ \xi_o^i - \underline{\lim_{i\to\infty}}\ \xi_o^i \,\right].$$

Therefore $\inf \| \xi_o + \eta \| = 1/2 (\overline{\lim\limits_{i\to\infty}} \xi_o^i - \underline{\lim\limits_{i\to\infty}} \xi_o^i) > 0$, from which

it follows that
$$\| \theta \| = \frac{2}{\overline{\lim\limits_{i\to\infty}} \xi_o^i - \underline{\lim\limits_{i\to\infty}} \xi_o^i} \; . \tag{4.15}$$

By using the Hahn-Banach extension theorem θ can be extended to the whole space B with preservation of the norm, where the norm is given by (4.15).

Therefore θ is a linear continuous functional on B and, by definition $\theta(\xi) = 0$ for all ξ in C, which proves $\theta \in ⒽH$ and $\theta \not\equiv 0$.

In connection with Theorem 11, the following theorem gives some more information about the structure of the space B and its conjugate space B^*.

<u>Theorem 12</u>.
$$Z^* \cong S$$
$$S^* \cong B,$$

Consequently, $B^* = Ⓗ \oplus \{ a_o L_o \} \oplus S \cong Z^{***}$.

Proof:

If $f \in Z^*$, then since Z is a subspace of B, f can be extended continuously to the whole space B and f has the form

$$f(\xi) = \theta(\xi) + a_o L_o(\xi) + \sum_{i=1}^{\infty} a_i \xi^i \text{ where } \theta \in Ⓗ \text{ and } \sum_{i=1}^{\infty} |a_i| < \infty.$$

Since $(\theta + a_o L_o)(\xi) = 0$ for $\xi \in Z$, f has the form $f(\xi) = \sum_{i=1}^{\infty} a_i \xi^i$ for all $\xi \in Z$, and $\sum_{i=1}^{\infty} |a_i| < \infty$. Therefore $f \in S$. Conversely if $f \in S$,

that is, if $f(\xi) = \sum_{i=1}^{\infty} a_i \xi^i$ and $\sum_{i=1}^{\infty} |a_i| < \infty$,

then
$$|f(\xi)| \leqq \sum_{i=1}^{\infty} |a_i| |\xi^i| \leq \| \xi \| \sum_{i=1}^{\infty} |a_i|$$

and $f(\xi)$ is continuous on Z, which proves $Z^* = S$.

Since S is a subspace of B^*, any ξ in B can be considered as a continuous linear functional on S by $f(\xi) = \xi(f) = \sum_{i=1}^{\infty} a_i \xi^i$

and $\quad |f(\xi)| \leqslant \|\xi\| \cdot \|f\|$.

In the space S, define $\sigma^i(e_j) = \delta_j^i$; $j = 1, 2, 3, \ldots$, then it is easily seen that $\{\sigma^i\}$ forms a basis and any f can be represented in the form

$$f = \sum_{i=1}^{\infty} a_i \sigma^i \text{ and } \left\| f - \sum_{i=1}^{n} a_i \sigma^i \right\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Now let ξ be any continuous linear functional in S, then

$$\xi(f) = \xi\left(\sum_{i=1}^{\infty} a_i \sigma^i\right) = \lim_{n\to\infty} \xi\left(\sum_{i=1}^{n} a_i \sigma^i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} a_i \xi(\sigma^i)$$

$$= \sum_{i=1}^{\infty} a_i \xi(\sigma^i).$$

Since $\|\xi(f)\| \leqslant \|\xi\|$ for all $\|f\| \leqslant 1$, $\|\xi(\sigma^i)\| \leqslant \|\xi\|$,

therefore $\{\xi^i\} = \{\xi(\sigma^i)\}$ is a bounded sequence and

$$\xi(f) = \sum_{i=1}^{\infty} a_i \xi^i.$$

That is, any continuous linear functional on S can be considered as an element in B. Therefore, $B \cong S^*$.

Now by substitution,

$$B^* = \textcircled{H} \oplus \{a_o L_o\} \oplus S \cong S^{**} \cong Z^{***}.$$

V

OPERATORS IN B

Let T be an operator on B and $T(\xi) = \left(t_1(\xi),\ t_2(\xi), \ldots\right) = \left\{t_i(\xi)\right\}$, where $T(\xi + \eta) = T(\xi) + T(\eta) = \left\{t_i(\xi)\right\} + \left\{t_i(\eta)\right\} = \left\{t_i(\xi + \eta)\right\}$ hence $t_i(\xi + \eta) = t_i(\xi) + t_i(\eta)$.

Also $T(\alpha\xi) = \alpha T(\xi)$ implies $t_i(\alpha\xi) = \alpha t_i(\xi)$. Therefore, T induces a sequence of linear functionals on B. If T is a summability operator on B, T is always continuous and the norm is given by Theorem 1.

Theorem 1 serves to give a one to one correspondence between the summability operators on B and the sequences of continuous linear functionals on B. If T is not a summability operator, one needs to restrict T to be a continuous operator in order to obtain a similar theorem.

Theorem 13. Necessary and sufficient conditions for an operator $T(\xi) = \left\{t_i(\xi)\right\}$ to be a continuous operator on B are

(1) $t_i(\xi)$ is a continuous linear functional on B, $i = 1, 2, 3, \ldots$.

(2) $\|t_i\| < M$ for all $i = 1, 2, \ldots$.

Under such conditions, $\|T\| = \sup_i \|t_i\|$.

Proof:

Let T be a continuous operator. Then

$$\|T\| = \sup_{\|\xi\|=1} \|T(\xi)\| = \sup_{\|\xi\|=1} \left[\sup |t_i(\xi)|\right], \qquad (5.1)$$

so $\quad \sup_i |t_i(\xi)| \leqslant \|T\| \quad$ for all $\|\xi\| = 1$,

or $\quad |t_i(\xi)| \leqslant \|T\| \quad$ for all $\|\xi\| = 1$,

hence $\quad \|t_i\| \leqslant \|T\| \quad$ for all i. $\qquad (5.2)$

That is, each $t_i(\xi)$ is a continuous linear functional and the sequence of norms $\{\|t_i\|\}$ is uniformly bounded.

Conversely if $T(\xi) = \{t_i(\xi)\}$ is an operator on B and the sequence $\{t_i\}$ satisfies conditions (1) and (2) of the theorem, $\|t_i\|$ exists for each i and $\|t_i\| < M$ for all i. Then

$$|t_i(\xi)| < M \text{ for all } i \text{ and all } \xi \text{ such that } \|\xi\| = 1.$$

Then, $\quad \sup_i |t_i(\xi)| \leqslant M \quad$ for all $\|\xi\| = 1$,

or $$\|T\| = \sup_{\|\xi\|=1} \|T(\xi)\| = \sup_{\|\xi\|=1} \left[\sup_i |t_i(\xi)|\right] \leqslant M, \cdots \qquad (5.3)$$

and T is a continuous operator.

From (5.2), $\sup_i \|t_i\| \leqslant \|T\|$ and from (5.3), one can obtain $\|T\| \leqq \sup \|t_i\|$ by setting $M = \sup \|t_i\|$, which proves $\|T\| = \sup \|t_i\|$.

Remark 1:

In Theorem 13, operators are defined on B, but no use is made of the fact that the domain of an operator is B. All that is required in the above theorem is that the domain of an operator is a subspace of B. Therefore, "a continuous operator on B' can be replaced by "a continuous linear transformation on a subspace of B into B," and Theorem 13 is still valid. But such generality is not usually required.

The linear functionals $t_i(\xi)$ for an operator T are usually given by a matrix. In that case, Theorem 1 becomes very effective.

Now consider a Cauchy sequence $\{t_i\}$ in the subspace S of B^*, given by $t_i(\xi) = \sum_{j=1}^{\infty} a_{ij}\xi^j$. Since S is a subspace of B^*,

$$t = \lim_{i\to\infty} t_i \in S \quad \text{and} \quad \lim_{i\to\infty} \|t_i - t\| = 0 \tag{5.5}$$

Putting $\delta_j = \lim a_{ij}$, t has a representation of the form

$$t(\xi) = \sum_{j=1}^{\infty} \delta_j \xi^j.$$

Since $\sum_{j=1}^{\infty} |\delta_j| < \infty$, given $\epsilon > 0$, there exists an N such that

$$\sum_{j=N}^{\infty} |\delta_j| < \epsilon/2$$

From (5.5), given $\epsilon > 0$, there exists an $N_1 > N$ such that $s > N_1$ implies $\sum_{j=1}^{\infty} |a_{sj} - \delta_j| < \epsilon/2$,

and so $\sum_{j=N}^{\infty} |a_{sj} - \delta_j| < \epsilon/2.$

Hence, $\sum_{j=N}^{\infty} |a_{sj}| \leq \sum_{j=N}^{\infty} |a_{sj} - \delta_j| + \sum_{j=N}^{\infty} |\delta_j| < \epsilon$, for $s > N_1$.

Since the set $\{s | s \leq N_1\}$ is finite, given $\epsilon > 0$ there exists a positive integer $N_2 > N_1$ such that

$$\sum_{j=N_2}^{\infty} |a_{sj}| < \epsilon \qquad \text{for } s = 1, 2, \ldots N_1.$$

Now $\sum_{j=N_2}^{\infty} |a_{sj}| < \epsilon$ for $s = 1, 2, 3, \ldots,$

That is, $\sum_{j=1}^{\infty} |a_{sj}|$ converges uniformly with respect to s.

Conversely, suppose $\lim_{s \to \infty} a_{sj} = \delta_j$ exists for each j and $\sum_{j=1}^{\infty} |a_{sj}|$ converges uniformly with respect to s. Since $\sum_{j=1}^{\infty} |a_{sj}|$ converges uniformly, given $\epsilon > 0$, there exists an N such that $\sum_{j=N+1}^{\infty} |a_{sj}| < \epsilon$ for all s. Since $\{a_{sj} | s = 1, 2, 3, \cdots\}$ is a Cauchy sequence, given $\epsilon > 0$, there exists N_1 such that for $p, q > N_1$, $|a_{pj} - a_{qj}| < \frac{\epsilon}{N}$, $j = 1, 2, 3, \ldots N$.

Then $\sum_{j=1}^{\infty} |a_{pj} - a_{qj}| \leqslant \sum_{j=1}^{N} |a_{pj} - a_{qj}| + \sum_{j=N+1}^{\infty} |a_{pj}| + \sum_{j=N+1}^{\infty} |a_{qj}| < 3\epsilon$

for $p, q > N_1$.

Since $\|t_p - t_q\| = \sum_{j=1}^{\infty} |a_{pj} - a_{qj}| < 3\epsilon$ for $p, q > N$, t_i is a Cauchy sequence in B^*.

This proves the following theorem:

<u>Theorem 14</u>. Necessary and sufficient conditions for a sequence $\{t_i(\xi)\} = \left\{ \sum_{j=1}^{\infty} a_{ij} \xi^i ;\ i = 1, 2, 3, \cdots \right\}$ to be a Cauchy sequence in B^* are

(1) $\lim_{i \to \infty} a_{ij}$ exists for each j,

(2) $\sum_{j=1}^{\infty} |a_{ij}|$ converge uniformly with respect to i.

Remark 2:

Theorem 14 can be stated in a stronger form. The existence of $\lim_{i\to\infty} a_{ij}$ for each j implies the existence of $\lim_{i\to\infty} t_i(e_k)$ and so the domain of summability of each continuous linear functional must contain the subspace Z of B. Also, for $t_i(\xi) = \sum_{j=1}^{\infty} a_{ij}\xi^i$, the norm $\|t_i\|$ must be defined by $\|t_i\| = \sum_{j=1}^{\infty} |a_{ij}|$, which is also true if the domain of summability of $t_i(\xi)$ contains the subspace Z of B. Hence in Theorem 14, the phrase "a Cauchy sequence in B^*" can be replaced by "a Cauchy sequence in C^*" or "a Cauchy sequence in Z^*." In another words, Z^* has a minimal property with respect to the validity of the Theorem 14.

The next theorem is concerned with the so called weak convergence of a sequence of continuous linear functionals in S, or the pointwise convergence of a sequence of continuous linear functions.

Theorem 15. A summability operator $T = \{t_i(\xi)\} = \left\{\sum_{j=1}^{\infty} a_{ij}\xi^i\right\}$ belongs to the class of transformations $\mathcal{T}_b$ if and only if the following two conditions are satisfied:

(1) $\sum_{j=1}^{\infty} |a_{ij}| < M$ for $i = 1, 2, 3, \ldots$,

(2) $\lim_{i\to\infty} \sum_{j\in E}^{\infty} a_{ij}$ exists for all $E \in \Sigma$.

Proof:

$T \in \mathcal{T}_b$ is equivalent to the condition that the sequence of continuous linear functionals $\{t_i\}$ converges weakly. A sequence of continuous linear functionals $\{t_i\}$ converges weakly if and only if the following two conditions are satisfied:

(1)' $\|t_i\| < M$

(2)' $\lim_{i\to\infty} t_i(\xi)$ exists for all ξ in an everywhere dense set in B,

and $\lim_{i\to\infty} t_i(\xi) = t(\xi)$ is a continuous linear functional on B [5].

The condition (1)' is equivalent to condition (1) $\sum_{j=1}^{\infty} |a_{ij}| < M$.

In the space B, D is an everywhere dense set in B, where D is the set of finite linear combinations of characteristic vectors $\{e_E\}$. Therefore the convergence on D is equivalent to the convergence on all characteristic vectors, and the condition (2)' reduces to the condition (2).

$$\lim_{i\to\infty} t_i(e_E) = \lim_{i\to\infty} \sum_{j\in E} a_{ij} \quad \text{exists for all } E \in \Sigma .$$

In the conjugate space B^*, the weak convergence of a sequence of continuous linear functionals may not imply strong convergence (Cauchy convergence in the norm). But, if a sequence of continuous linear functionals is restricted to the form $t_i(\xi) = \sum_{j=1}^{\infty} a_{ij}\xi^j$, then strong and weak convergence do imply each other. This fact is proved in the following theorem.

<u>Theorem 16</u>. In a double sequence $\{a_{ij};\ i = 1, 2, 3, \ldots,\ j = 1, 2, 3, \ldots\}$ the conditions

(1) $\sum_{j=1}^{\infty} |a_{ij}| < M$

A

(2) $\lim_{i\to\infty} \sum_{j\in E} a_{ij}$ exists for all $E \in \Sigma$

are equivalent to the conditions

B

(1)' $\lim_{i \to \infty} a_{ij} = \delta_j$ exists for each j

(2)' $\sum_{j=1}^{\infty} |a_{ij}|$ converge uniformly with respect to i.

Proof:

First, conditions B imply A. Given $\epsilon > 0$, there exists an N such that $\sum_{j=N+1}^{\infty} |a_{ij}| < \epsilon$ for all i. Let E be any set and I_N be a set of integers $1, 2, 3, \ldots N$.

Then $\sum_{j \in E} a_{ij} = \sum_{j \in I_N \cap E} a_{ij} + \sum_{j \in (I - I_N) \cap E} a_{ij}$.

Given $\epsilon > 0$, there exists an N_1 such that for $p, q > N_1$, $|a_{pj} - a_{qj}| < \frac{\epsilon}{N}$.

Now $\left| \sum_{j \in E} a_{pj} - \sum_{j \in E} a_{qj} \right| = \left| \sum_{j \in I_N \cap E} (a_{pj} - a_{qj}) + \sum_{j \in (I - I_N) \cap E} (a_{pj} - a_{qj}) \right|$,

$$\leq \sum_{j \in I_N \, E} |a_{pj} - a_{qj}| + \sum_{j \in I - I_N} |a_{pj}| + \sum_{j \in I - I_N} |a_{qj}|$$

$$\leq \epsilon + \epsilon + \epsilon = 3\epsilon.$$

Therefore, $\left\{ \sum_{j \in E} a_{ij} \right\}$ is a Cauchy sequence and $\lim \sum_{j \in E} a_{ij}$ exists for all $E \in \Sigma$. Moreover, since $\lim_{i \to \infty} \sum_{j=1}^{\infty} |a_{ij}| = \sum_{j=1} |\delta_j|$, the sequence $\sum_{j=1}^{\infty} |a_{ij}|$ is bounded, or

$$\sum_{j=1}^{\infty} |a_{ij}| < M, \qquad i = 1, 2, \ldots .$$

Now it will be shown that conditions A imply B. From the condition (2) $\delta_j = \lim_{i\to\infty} a_{ij}$ is defined for all $j = 1, 2, 3, \cdots$. From the condition (1), $\sum_{j=1} |a_{ij}| < M$ for all n. Now letting i increase without bound yields

$$\lim_{i\to\infty} \sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} |\delta_j| < M \qquad \text{for all } n.$$

Hence $\sum_{j=1}^{\infty} |\delta_i| < M$,

that is, the series $\sum_{j=1}^{\infty} \delta_j$ converges absolutely. Also from (2), $\sum_{j=1}^{\infty} a_{ij}$ converges absolutely for each i. Now put $\beta_{ij} = a_{ij} - \delta_j$.

Then
$$\sum_{j\in E} \beta_{ij} = \sum_{j\in E} (a_{ij} - \delta_j) = \sum_{j\in E} a_{ij} - \sum_{j\in E} \delta_j \tag{5.6}$$

Since conditions (1) and (2) imply $\lim_{i\to\infty} \sum_{j\in E} a_{ij} = \sum_{J\in E} \delta_j$ for all $E \in \Sigma$ [8].

$$\lim_{i\to\infty} \sum_{j\in E} \beta_{ij} = 0 \qquad \text{for all } E \in \Sigma, \tag{5.7}$$

and
$$\sum_{j=1}^{\infty} |\beta_{ij}| < 2M \qquad i = 1, 2, 3, \cdots. \tag{5.8}$$

If one can show that $\lim_{i\to\infty} \sum_{j=1}^{\infty} |\beta_{ij}| = 0$ under the conditions (5.7) and (5.8) then it is easily seen that $\sum_{j=1}^{\infty} |a_{ij}|$ converges uniformly with respect to i. Suppose the sequence $\sum_{j=1}^{\infty} |\beta_{ij}|$ does not converge to zero, then there exists a subsequence converging to a number $\gamma > 0$.

Therefore, assume
$$\lim_{i\to\infty} \sum_{j=1}^{\infty} |\beta_{ij}| = \gamma > 0. \tag{5.9}$$

First choose m_1 and $n_1 < n_2$ such that

$$\sum_{j=1}^{n_1} |\beta_{m_1 j}| < \frac{1}{2} \quad \text{and} \quad \sum_{j=n_2+1}^{\infty} |\beta_{m_1 j}| < \frac{1}{2}.$$

Suppose two sequences $m_1, m_2, \ldots m_{k-1}$ and $n_1, n_2, \ldots, n_k$ are chosen. Now choose m_k so that

$$\sum_{j=1}^{n_k} |\beta_{m_k j}| < \frac{1}{2^k}$$

and then choose $n_{k+1} > n_k$ such that

$$\sum_{j=n_{k+1}+1}^{\infty} |\beta_{m_{kj}}| < \frac{1}{2^k}.$$

By induction two sequences $m_1, m_2, \ldots m_k, \cdots$ and $n_1, n_2, \ldots, n_k \ldots$ are defined.

Since $\displaystyle\sum_{j=n_k+1}^{n_{k+1}} |\beta_{m_{kj}}| = \sum_{j=1}^{\infty} |\beta_{m_{kj}}| - \sum_{j=1}^{n_k} |\beta_{m_{kj}}| - \sum_{j=n_{k+1}+1}^{\infty} |\beta_{m_{kj}}|,$

$$\lim_{k \to \infty} \sum_{j=n_k+1}^{n_{k+1}} |\beta_{m_{kj}}| = \lim \sum_{j=1}^{\infty} |\beta_{m_{kj}}| = \gamma.$$

Now put $J_k = n_k + 1, n_k + 2, \ldots, n_{k+1}$

$= J_k^+ + J_k^-$

where $J_k^+ = \{n_{k+i} | \beta_{m_k n_k + i} \geqslant 0\}$

$J_k^- = \{n_{k+i} | \beta_{m_k n_k + i} < 0\},$

and let

$$H_k = J_k^+ \text{ if } \sum_{j \in J^+} \beta_{m_{kj}} \geqslant - \sum_{j \in J^-} \beta_{m_{kj}},$$

and $\quad H_k = J_k^- \quad \text{if} \quad -\sum_{j\in J^-} \beta_{m_{kj}} > \sum_{j\in J^+} \beta_{m_{kj}}.$

Let $H = \sum_{k=1}^{\infty} H_k.$

Then $\quad \left|\sum_{\substack{j\in H_i \\ i\neq k}} \beta_{m_{kj}}\right| \leq \sum_{\substack{j\in H_i \\ i\neq k}} \left|\beta_{m_{kj}}\right| \leq \sum_{j=1}^{n_k} \left|\beta_{m_{kj}}\right| + \sum_{j=n_{k+1}+1} \left|\beta_{m_{kj}}\right| < \frac{1}{2^{k-1}},$

and hence
$$\lim_{k\to\infty} \left[\sum_{\substack{j\in H_i \\ i\neq k}} \beta_{m_{kj}}\right] = 0 \tag{5.10}$$

By hypothesis, $\lim_{k\to\infty} \sum_{j\in H} \beta_{m_{kj}} = 0,$

and $\quad \sum_{j\in H} \beta_{m_{kj}} = \sum_{j\in H_k} \beta_{m_{kj}} + \sum_{\substack{j\in H_i \\ i\neq k}} \beta_{m_{kj}}$

therefore, $\quad \lim_{k\to\infty} \sum_{j\in H_k} \beta_{m_{kj}} = 0.$

But $\quad \left|\sum_{j\in H_k} \beta_{m_{kj}}\right| \geqslant \frac{1}{2} \sum_{j\in J_k} \left|\beta_{m_{kj}}\right|,$

and since $\frac{1}{2}\lim_{k\to\infty} \sum_{j\in J_k} \left|\beta_{m_{kj}}\right| = \gamma/2,$ the sequence $\left\{\sum_{j\in H_k} \beta_{m_{kj}}\right\}$ can not converge to zero, which contradicts (5.10) and so

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} \left|\beta_{ij}\right| = 0.$$

Since $a_{ij} = \beta_{ij} + \delta_j$, and $\sum_{j=1}^{\infty} \left|\beta_{ij}\right|$ converges uniformly and

$\sum_{j=1}^{\infty}|\delta_j| < M, \quad \sum_{j=1}^{\infty}|a_{ij}|$ converges uniformly with respect to i.

This completes the proof of Theorem 16.

If the conditions B in Theorem 16 are satisfied, then

$$\lim_{i\to\infty} \sum_{j\in E}|a_{ij}| = \sum_{j\in E}|\delta_j|, \quad \text{for all } E \in \Sigma. \tag{5.11}$$

Conversely, if $\sum_{j=1}^{\infty}|a_{ij}| < M$ and (5.11) is satisfied. Since

$\sum_{j=1}^{\infty}|\delta_j| < \infty$, then for every $\epsilon > 0$, there exists an N such that

$$\sum_{j=N}^{\infty}|\delta_j| < \epsilon/2.$$

Also, there exists an N_1 such that if $p > N_1$,

$$\left| \sum_{j=N}^{\infty}|a_{pj}| - \sum_{j=N}^{\infty}|\delta_j| \right| < \epsilon/2.$$

Then, $\sum_{j=N}^{\infty}|a_{pj}| \leq \left| \sum_{j=N}^{\infty}|a_{pj}| - \sum_{j=N}^{\infty}|\delta_j| \right| + \sum_{j=N}^{\infty}|\delta_j| < \epsilon$, for $p > N_1$,

or $\sum_{j=1}^{\infty}|a_{pj}|$ converges uniformly with respect to p. Remembering,

also, that if $\sum_{j=1}^{\infty}|a_{ij}| < M$ and $\lim_{i\to\infty} \sum_{j\in E}|a_{ij}|$ exists for all $E \in \Sigma$,

then $\lim_{i\to\infty} \sum_{j\in E}|a_{ij}| = \sum_{j=1}^{\infty}|\delta_j|$, one obtains the following corollary.

Corollary 4. In a double sequence $\{a_{ij}\ ;\ i = 1, 2, \ldots,\ j = 1, 2, 3, \ldots\}$ the following three conditions (A), (B), and (C) are equivalent,

$$(A)\quad \begin{cases} (1) & \sum_{j=1}^{\infty} |a_{ij}| < M. \\ (2) & \lim_{i\to\infty} \sum_{j\in E} a_{ij} \text{ exists for all } E \in \Sigma, \end{cases}$$

$$(B)\quad \begin{cases} (1) & \lim_{i\to\infty} a_{ij} = \delta_j \text{ exists for all } j. \\ (2) & \sum_{j=1}^{\infty} |a_{ij}| \text{ converge uniformly with respect to } i. \end{cases}$$

$$(C)\quad \begin{cases} (1) & \sum_{j=1}^{\infty} |a_{ij}| < M, \\ (2) & \lim_{i\to\infty} \sum_{j\in E} |a_{ij}| \text{ exists for all } E \in \Sigma, \end{cases}$$

Combining Theorems 14, 15, and 16 yields the following theorem.

Theorem 17. A summability operator $T(\xi) = \{t_i(\xi)\} = \left\{\sum_{j=1}^{\infty} a_{ij}\xi^i\right\}$ belongs to the class of operators $\mathcal{T}_b$ if and only if the sequence $\{t_i\}$ in B^* is a Cauchy sequence.

Remark 3:

In the space B^*, a sequence $\{t_i\}$ in which each t_i is a continuous linear function on B converges weakly, that is the $\lim_{i\to\infty} t_i(\xi)$ exists for all ξ in B, but $\{t_i\}$ may not converge strongly (convergence in norm). But if each t_i is restricted to the form $t_i(\xi) = \sum_{j=1}^{\infty} a_{ij}\xi^i$, weak convergence and strong convergence are equivalent as a consequence of Theorem 16.

A SELECTED BIBLIOGRAPHY

1. S. Mazur, Eine Anwendung der Theorie der Operationen bei der Untersuchung der Toepolitzschen Limitierungs Verfahren. pp. 40-50, Tom. II, Vol. 1-3, Studia Matematika, 1930.

2. S. Mazur, W. Orlicz, On Linear Methods of Summability, pp. 129-160, Tom. XIV, Vol. 14-15, Studia Matematika, 1954-56.

3. S. Banach, pp. 67, Theorie des Operations Lineaires, 1932.

4. J. Schur, Über Lineares Transformationen in der Theorie der Unendlichen Reihen, pp. 79-111, Vol. 151, Journal fur die Reine und Angewante Mathematik, 1921.

5. K. Yoshida, pp. 62, Functional Analysis, Iwanami Press, 1956.

6. Ibid., pp. 58.

7. G. Hardy, pp. 52, Divergent Series, Oxford, Clarendon Press, 1949.

8. Ibid., pp. 44.

9. W. Sierpinski, pp. 80, General Topology, University of Toronto, 1952.

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