TRANSIENT ANALYSIS OF A SINGLY EXCITED ELECTROMECHANICAL TRANSDUCER

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PREFACE

Conversion of electrical energy into or from mechanical energy by the use of magnetic energy is the basis of operation of all electromechanical transducers. The method of accomplishing this conversion is dependent upon the particular transducer. The conversion may be achieved by utilizing one of the following types of phenomena: (a) the force reaction between two current-carrying conductors, (b) the force reaction between a current-carrying conductor and a magnetic field, or (c) the force reaction resulting from that property of a magnetic field which causes it to always tend to conform itself so that the maximum amount of flux is attained. The definition of the singly excited electromechanical transducer requires that the principle of operation involve the phenomenon listed in either (b) or (c).

Devices utilizing the phenomenon listed in (b) include the dynamic loudspeaker, the velocity and dynamic microphones and vibration pick-up transducers. The transient response of these devices has been investigated, and the results are available in the books and journals in the field of electroacoustics.

Devices using the phenomenon listed in (c) include solenoids, magnetic brakes, magnetic clutches and relays. The transient response of these devices has not been investigated to any great extent because of the complexity of solving the equations which must be used to describe their response. In addition, the final condition of operation of the device was usually considered to be the only important thing. Recently, however, additional requirements have been placed on these transducers. Therefore, it is important that additional knowledge be obtained about the transient response of this type of electromechanical transducer.

This thesis contains the following: the procedure of setting up the nonlinear differential equations which best describe the system under study, an explanation of the necessity of selecting certain independent variables, the development of the particular form of the phase-plane method necessary to solve the two simultaneous nonlinear differential equations, the process of setting up the logical steps in programming the solution on the IBM 650 computer, and the comparison of the computed and experimental results. The particular form of the transducer used to evaluate the technique presented was the electromagnetic relay. This technique is applicable to other types of transducers in this category.

An expression of appreciation is extended to: Dr. Clark A. Dunn, Dr. L. Wayne Johnson, Dr. Herbert L. Jones and Professor H. G. Thuesen, members of my advisory committee, for their encouragement, help and prompt consideration. I am especially indebted to Professor Charles F. Cameron, chairman of my committee, for his supervision, guidance, encouragement and help throughout my doctoral program. It has been through his efforts that research work on electromagnetic relays was possible.

My special thanks go to my wife, who typed the first draft of this thesis and who had the job of setting up the equations on the typewriter.

To the secretaries: Mrs. Walker, who typed the final form of this thesis, and Mrs. Giddens, who proof-read the first draft, I express my appreciation.

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CHAPTER I

INTRODUCTION

Conversion of electrical energy to mechanical energy by the medium of a magnetic field has existed for at least a century. One of the first electromechanical energy converters was the telegraph repeater. This device is better classified as a singly excited electromechanical transducer. The singly excited electromechanical transducer consists of a single excitation coil, an associated magnetic circuit and the mechanical system.

Considerable progress has been made in singly excited transducers since the first telegraph repeater. One of the present day counterparts of the telegraph repeater is the electromechanical relay (generally called relay). Outwardly, the relay is a simple device consisting of a coil of wire on a magnetic circuit which closes when the coil is energized and opens when deenergized. The relay has played an important part in this period of automation, and as a consequence has been used in many new and different applications. Before the new and different applications, the relay performs its function reasonably well. Consequently, at that time, little needed to be known about the basic behavior of the singly excited electromechanical transducer in order to produce a relay that served its purpose. In recent years relays have been used in applications where certain environmental, weight and space requirements had to be satisfied. Since relay manufacturing has been more of an art than a science, considerable difficulty has been encountered in producing relays which will satisfy the requirements now being placed upon them. Part of the difficulty encountered has been caused by inaccurate or insufficient information about the environment in which the relay is expected to perform. In addition the weight, space, power and environmental requirements produce conflicting constraints on the relay design.

More basic knowledge must be determined about the behavior of the singly excited electromechanical transducer if these new requirements on relays are to be satisfied. Even though the relay appears simple in its operation, the prediction of its response to certain conditions of excitation is extremely difficult, if not impossible. In general, electromechanical energy conversion devices are described by nonlinear differential equations. At the present time, "no general analytical methods exist for solving nonlinear equations; consequently, a separate solution is required for each set of excitation conditions of interest."[1]

Some information is known about the steady state condition of the relay in its energized or deenergized state. However, the transient condition of the relay is not as well understood even though it is this condition that primarily determines the relay's performance. The transient condition exists during that period of time from the instant the coil is energized, or deenergized, to the time the current reaches its steady state value. During this interval the mechanical system should have changed from a condition of armature open or closed to a condition of armature closed or open. The solutions of the nonlinear differential equations describing the response of the single excited electromechanical transducer should result in being able to predict the performance of this

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[1] White, David C. and Herbert H. Woodson, Electromechanical Energy Conversion, (New York, 1959), p. 158.

device. The effect of certain parameters of each particular part of the device can be determined from a study of the solutions of the nonlinear differential equations.

In this type of problem, where the principle of superposition does not apply because of the nonlinear form of the differential equations, it was necessary to use the experimental information along with the theoretical or analytical information. From experimental observations it was possible to arrive at certain facts needed to determine the analytical relations. By a combined approach, using both experimental knowledge and analytical results, the solution to a particular set of describing relations was obtained. Even though the analytical or mathematical approach was separated from the experimental observation in this paper for purposes of clarity, the actual process of arriving at a solution to this problem was an integrated effort using experimental and analytical knowledge. At every step in the process, experimental observations were used to verify or to disprove the results arrived at by reasoning. In this manner, misleading concepts could be corrected or at least re-evaluated in the light of this nonlinear device.

Some information is available about the design of electromechanical transducers but there is only a limited amount of information about the transient response. Probably the first comprehensive coverage of design was published in The Bell System Technical Journal of January 1954, later published in book form. (R. L. Peek and H. N. Wagar, 1955). However, little was presented directly about the transient response of the transducer except for the contact system. The contact system affects the transducer response only in the fact that additional spring forces are applied to the mechanical system. Some information about the dynamic response of the electromechanical transducer is given in the text by Peek and Wagar.

One of the first articles published which gave some treatment of the transient response of the transducer was "Relay Characteristics and Uses", by Professor Charles F. Cameron, 1955. This article showed the results of recording the transient coil current as a function of time. It was indicated that the transient coil current could be used to determine some information about the response of the mechanical system. The transient response of solenoids and other magnetic coils had been investigated,[2] but the article by C. F. Cameron was one of the first to suggest using the transient coil current as a means of determining the response of the mechanical system of the transducer.

In 1953, in connection with a contract with the Bureau of Ships, U. S. Navy, Professor C. F. Cameron initiated and held the first Symposium on Electro-magnetic Relays at Oklahoma State University (then Oklahoma A & M College). A number of papers where presented dealing with the problems involved with electromagnetic relays or electromechanical transducers. However, the transient response of the transducer received little attention the first several years the Symposium was held. In 1956 Professor C. F. Cameron and the author presented a paper, "Relay Characteristics," that dealt specifically with the transient response of electromechanical transducers. That paper presented experimental records of the transient coil current and the transient displacement of the armature simultaneously. The simultaneous presentation of the transient coil current and armature displacement has been used very effectively

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[2]Rudenburg, Reinhold, Transient Performance of Electric Power Systems, (New York 1950).

in the experimental analysis to obtain insight and knowledge about the response of the transducer. Additional papers have been presented by Professor C. F. Cameron and the author, showing experimental evidence of the fact that the transient coil current can be used to analyze the response of the mechanical system of the transducer.[3, 4, 5]

The equations of motion of the transducer have been developed by other authors,[6] but the dependent variables selected usually have been the coil current and armature displacement. In addition, the coupling term usually has involved the inductance as a function of the coil current and armature displacement. It is shown in this thesis that, by selecting the magnetic flux instead of the coil current as the dependent variable, much simpler relations result. Also the need to use the inductance was eliminated. The coil current can be computed after the flux and armature displacement relations are determined, thereby giving the same final variables as the other methods.

These simplified but simultaneous nonlinear differential equations were then solved by using the phase-plane or phase-space method. The phase-space method is essentially a graphical procedure but the solution was obtained by numerical means. In order to obtain sufficient data by this method, a computer program was written for the IBM 650 using the

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[3]Cameron, C. F. and D. D. Lingelbach, "Evaluation of Relay Transient", 6th Symposium on Electromagnetic Relays, (1958), p. 51-52.

[4]Cameron, C. F. and D. D. Lingelbach, "Transient Characteristics of Electromagnetic Relays", 5th Symposium on Electromagnetic Relays, (1957) p. 67-78.

[5]Cameron, C. F. and D. D. Lingelbach, "Transient Coil Current as a Means of Relay Evaluation", Proceedings of the 1958 Electronic Component Conference, p. 129-137.

[6]White, D. C. and H. H. Woodson, Electromechanical Energy Conversion, (New York, 1959), p. 64-69, 90-100, 159-157.

Fortran language. Selected variables were changed, and the corresponding calculated response was obtained by the use of the computer program. These computed transient response curves were then compared with the experimental transient response curves taken from similar transducers.

The theoretical analysis was confined mainly to the energization of the transducer to a step function of voltage. Other types of voltage driving functions can be handled, with slight modification, by the method explained in this thesis. Since the experimental observations were confined to a step voltage because of power source restrictions, it was not considered in the scope of this thesis to solve for the response to other voltage driving functions.

The response of the transducer to the removal of the voltage driving function is not too complicated and can be solved by making certain assumptions which do not significantly reduce the accuracy of the results.[7, 8] Some information about the performance of the transducer can be determined by studying the coil current decay. This thesis will show some experimental observations of the response of the transducer to the removal of the voltage driving function. However, since the theoretical analysis of the release case can be treated essentially the same way as the operate case, no attempt will be made in this thesis to show the complete procedure.

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[7]Cameron, C. F. and Allen, E. F., "Analysis of Armature Motion During Release", Symposium on Electromagnetic Relays, (1956), pp. 39-41.

[8]Cameron, C. F. and Lingelbach, D. D., "Transient Characteristics of Electro-magnetic Relays", 5th Symposium on Electromagnetic Relays, (1957), pp. 67-78.

CHAPTER II

THEORETICAL ANALYSIS

Introduction

There are essentially two approaches that may be used to determine the dynamic equations of motion of electromechanical transducers when represented by lumped parameters. One method employs the known force laws such as D'Alembert's principle for the mechanical system and Kirchhoff's laws for the electrical system. The second method is obtained from variational principles applied to certain energy functions.

In determining the coupling terms by this first method, one of two ways may be used. One way uses the concept of an arbitrary displacement and conservation of energy to obtain the mechanical forces of electrical origin, and Faraday's law and Coulomb's law to obtain the electrical terms of mechanical origin. The other way to obtain the coupling terms is by integrating the force densities obtained from electromagnetic field theory. This way is not necessary in the case of the lumped parameter system. The application of this first method requires a considerable amount of judgment and insight in determining the relative actions of the terms, especially in complicated systems containing many variables.

The second method is obtained from variational principles applied to certain energy functions. By the application of Hamilton's principle, the dynamic equations of motion of the system, including the coupling terms, can be obtained. This procedure is more sophisticated mathematically, but insight into the physical system can be lost. This method is considered to be one of the most powerful techniques in dynamics.[1] However, there is one major weakness in the variational system and this is the difficulty of determining the set of generalized variables or coordinates.[2] The application of Hamilton's principle to the selected energy functions results in the Euler-Lagrange equation (often called Lagrange's equation).

A brief discussion of the force laws mentioned previously will be given here since some of these will be applied later in the chapter to develop the dynamic equations of motion of the transducer. Kirchhoff's laws include two relations involving electrical circuits. Actually they are based upon the conservation of energy and mass but are stated in electrical terms that are generally used to describe an electrical circuit.

Kirchhoff's emf law may be stated in several ways. One way is that the sum of the voltage drops taken in a given direction around a loop (j th loop) equals zero. This is given as

$$\sum_{q=1}^{s} e_{j_q} = 0 \tag{2.1}$$

where e_{j_q} = the qth voltage drop in the jth loop.

The other Kirchhoff's law states that no charge can accumulate at a point in a circuit or that the algebraic sum of the charge flow at a point must be zero. From the definition of current, this is expressed by stating that the algebraic sum of the currents at a node (jth node) must be zero.

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[1] White, D. C. and H. H. Woodson, Electromechanical Energy Conversion, Chapter I, (New York, 1959).

[2] Koenig, H. E. and W. A. Blackwell, "On the Codification of Lagrangian Formulation," Proc. IRE, (New York, 1958), p. 1428-29.

This is given as:

$$\sum_{q=1}^{p} i_{j_q} = 0 \qquad (2.2)$$

where i_{j_q} = the qth current in the j th node.

Because of the analogous nature of systems, whether they be composed of electrical, mechanical, hydraulic or other components, the types of force laws must involves loops and nodes. In the case of the mechanical parts D'Alembert's principle gives the relation existing at a mechanical node in a system. This principle states that the sum of all the forces at a node (j th node) must be zero. These forces must include the inertial, applied and constraint forces. One way to represent D'Alembert's principle is given as:

$$\sum_{q=1}^{r} (a_{j_q} - f_{j_q}) = 0 \qquad (2.3)$$

where: $a_{jq} = d/dt\ (m_{jq}\ \dot{x}_{j_q})$ the qth inertial force at the j th node.

$m_{jq}\ \dot{x}_{jq}$ = the momentum of the qth mass at the jth node.

f_{jq} = the q th applied or constraint forces at the j th node.

The other law similar to Kirchhoff's voltage around the loop states that the sum of the displacements around a loop (j th loop) must be zero. This is given as

$$\sum_{q=1}^{u} x_{j_q} = 0 \qquad (2.4)$$

where x_{j_q} = the q th displacement in the j th loop.

The coupling terms, as indicated previously, must be determined by the concept of an arbitrary displacement and the conservation of energy.

The first step in the determination of the coupling terms is to simplify the procedure by extracting all of the purely electrical terms and the purely mechanical terms, including the loss terms. This separation leaves only the terms of one set that are also functions of the other set. This procedure also makes the remaining system a conservative electromechanical network. Therefore, the total stored energy (W) in the coupling network is given by

$$W = W_m + W_e \tag{2.5}$$

where W_m is the stored energy in the magnetic fields,

W_e is the stored energy in the electric fields.

The energy functions are defined to be state functions. This means the energy is a function of the instantaneous configuration of the system and not dependent upon the past history or the dyanmic state. Therefore, such energy losses caused by hysteresis or eddy currents must be treated in some manner outside of the coupling network. Defining the energy functions as state functions causes the energy to be a single valued function of the system variables, independent of the derivatives and integrals of the variables. This procedure is used later in the chapter to develop the electromechanical coupling term.

Development of the Electrical Equation of the Transducer Based on a Traditional Approach

The application of Kirchhoff's Voltage Law to the coil of the transducer when supplied with a step input of voltage (E), gives equation (2.6).

$$E = iR + N \frac{d\emptyset}{dt} \tag{2.6}$$

where: $d\emptyset/dt$ = time rate of change of magnetic flux linking N turns

E = supply voltage

R = coil circuit resistance

N = turns on the coil.

Since the flux (ϕ) is a function of the coil current (i) and the armature position (x), and both i and x are functions of time (t), then N $d\phi/dt$ must be expressed as follows:

$$N \frac{d\phi}{dt} = N \left[\frac{\partial\phi}{\partial i} \frac{di}{dt} + \frac{\partial\phi}{\partial x} \frac{dx}{dt}\right] . \qquad (2.7)$$

A relation which expresses the flux (ϕ)as a function of the coil current (i) and the armature position (x) may be developed as follows:

$$\phi = \frac{\mathcal{F}}{\mathcal{R}} = \frac{Ni}{\mathcal{R}_a + \mathcal{R}_i} \qquad (2.8)$$

where: $\mathcal{F}$ = magnetomotive force

$\mathcal{R}$ = reluctance of total magnetic circuit

N = turns on coil carrying current (i)

i = coil current

$\mathcal{R}_a$ = reluctance of the air gap

$\mathcal{R}_i$ = reluctance of the ferromagnetic or iron portion of the magnetic circuit.

The reluctance $\mathcal{R}_a$ and $\mathcal{R}_i$ may be represented as follows:

$$\mathcal{R}_a = x/\mu A = \sigma x \qquad (2.9)$$

$$\mathcal{R}_i = \frac{s}{\mu c} = \frac{s}{r + ai} , \qquad (2.10)$$

where: A = cross sectional area of magnetic circuit at the working air gap

x = armature position

μ = permeability of free space

$\sigma = 1/\mu A$

s = ratio of the effective length of the iron portion of the magnetic circuit to its effective cross sectional area

μc = permeability of the iron portion of the magnetic circuit

$r + ai$ = approximation of μc over a given range of i.

Substituting the relations given in equations (2.9) and (2.10) into equation (2.8) results in the following form:

$$\phi = \frac{Ni}{\sigma x + ai + r} \quad . \tag{2.11}$$

Equation (2.7) requires the first partial derivative of the flux (ϕ) with respect to the coil current (i) and the first partial derivative of the flux (ϕ) with respect to the armature displacement (x). These partial derivatives are obtained by performing the indicated operations upon equation (2.11) as follows:

$$\frac{\partial \phi}{\partial i} = \frac{(\sigma x + ai + r)\ N - Ni\ (a)}{(\sigma x + ai + r)^2} = \frac{N\ (\sigma x + r)}{(\sigma x + ai + r)^2} \tag{2.12}$$

$$\frac{\partial \phi}{\partial x} = \frac{-Ni\ (\sigma)}{(\sigma x + ai + r)^2} \quad . \tag{2.13}$$

Substituting equations (2.12) and (2.13) into equation (2.7) and then substituting equation (2.7) into equation (2.6) gives the electrical equation of the transducer in terms of the two dependent variables. These dependent variables being the coil current (i) and the armature displacement (x). The complete electrical equation of the transducer is shown by equation (2.14).

$$E = iR + \left[\frac{N^2\ (\sigma x + r)}{(\sigma x + ai + r)^2}\right] \frac{di}{dt} + \left[\frac{-N^2\ i\sigma}{(\sigma x + ai + r)^2}\right] \frac{dx}{dt} \tag{2.14}$$

Equation (2.14) is a non-linear differential equation in terms of two dependent variables, the coil current (i) and the armature displacement (x) and the independent variable, time (t). The symbols in equation (2.14) have been defined in several previous equations, but will be repeated here for convenience.

Symbols used in equation (2.14):

E = step voltage applied to the transducer coil

i = coil current

R = coil circuit resistance

N = coil turns

$\sigma = 1/\mu A$

μ = magnetic permeability of free space

A = cross sectional area of the magnetic circuit at the working air gap

$ai + r$ = approximation of the permeability of the ferromagnetic portion of the magnetic circuit, over a given range of the current (i)

x = armature position or displacement

Development of the Equation of the Mechanical System

The development of the mechanical equation of the transducer may be approached by using rectangular or cylindrical coordinates. Possibly the use of cylindrical coordinates would be more representative of the system usually encountered in the electromechanical transducer, but because of the methods of measurement used in mechanical systems, it is more convenient to solve the system in rectangular coordinates. The general form of the equation in both systems will be developed. Also the transformation from one to the other will be shown to show that the solution could have been attempted in either set of coordinates. The general form of the mechanical equation in cylindrical coordinates is based on the schematic diagram of the transducer shown in Figure 2.1.

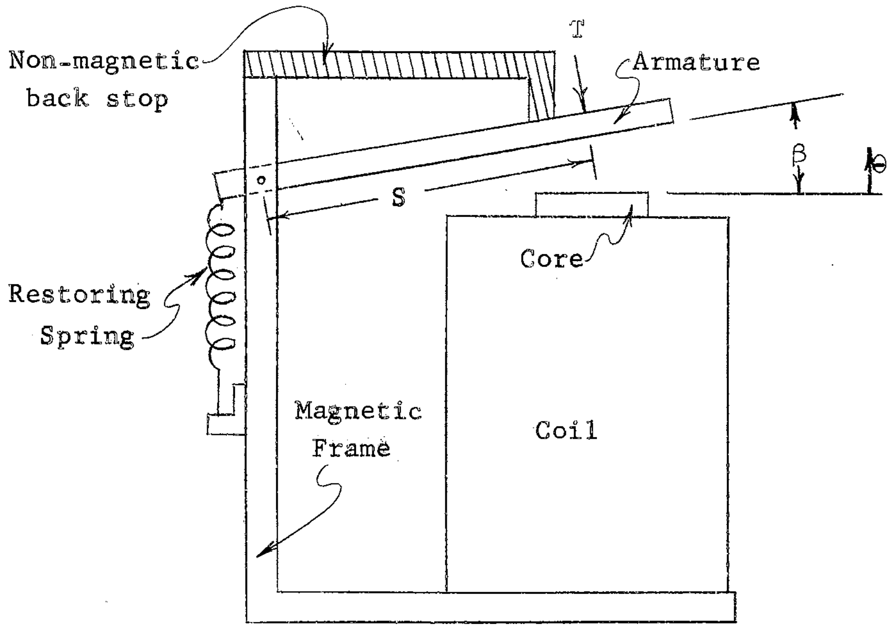

Figure 2.1

Since (θ) is the angular displacement, the $d\theta/dt$ is the angular velocity, and $d^2\theta/dt^2$ is the angular acceleration. The sum of the forces causing the armature to accelerate are shown as follows and are set equal to the angular inertial force.

$$T - T_o - \gamma\,(\beta - \theta) - \lambda(-\,d\theta/dt) = J\,(-\,d^2\theta/dt^2) \qquad (2.15)$$

where: T = the magnetic torque

T_o = initial torque caused by the restoring spring

$\gamma\,(B - \theta)$ = torque resulting from extending the restoring spring

λ = angular damping coefficient

J = polar mass moment of inertia of the armature about the pivot point

β = value of θ when armature is against the back stop.

Note: The minus signs involved with Θ are caused by the motion being opposite to the indicated positive direction of Θ

The conversion to rectangular coördinates is based upon the fact that any torque, by definition, is equal to some force times a moment arm. As shown in Figure 2.1, the length of the moment arm is (S), which is the distance from the center of the pivot to the center of the core when the armature is closed. This means the driving torque (T) is equal to the magnetic pull (F) times the length (S). Likewise the torque (T_o) caused by the restoring spring can be referred to an initial pull (P_o) times the length (S). All torque values can be represented by some effective force times the length (S). The polar mass moment of inertia (J) also can be represented by the square of the length (S) times an effective mass (M). With these conversions, the mechanical system shown in Figure 2.1 can be converted to the one shown in Figure 2.2. In Figure 2.2 the coordinate system is rectangular where the components of the transducer are the effective values derived from the model of the transducer in cylindrical coordinates.

Since Θ will be small, the total opening shown by the angle β in Figure 2.1 can be written as $G = S\beta$. Likewise x can be written as $x = S\Theta$ in which the transformation from the cylindrical coordinate system to the rectangular coordinate system is S. In other words, in order to obtain the quantity in rectangular coordinates, the quantity in cylindrical coordinates is operated upon by the length (S).

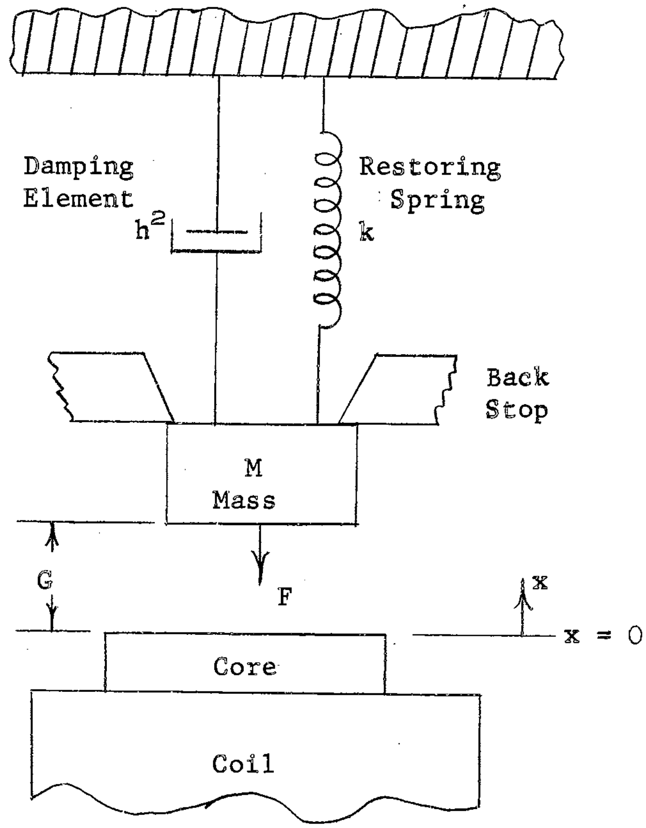

Figure 2.2

This procedure will give the following transformations:

$$F = T/S$$

$$P_o = T_o/S$$

$$G = S\beta$$

$$x = S\Theta$$

$$dx = Sd\Theta$$

$$dx^2 = Sd\Theta^2$$

$$k = \gamma/S^2$$

$$h^2 = \lambda/S^2$$

$$M = J/S^2.$$

If these transformations are substituted into equation (2.15) and some manipulations made, the following relation in rectangular coordinates results:

$$F - P_o - k\ (G - x) - h^2\ (- dx/dt) = M\ (- d^2x/dt^2). \tag{2.16}$$

Possibly a more standard form of equation (2.16) is given by equation (2.17).

$$F = -Md^2x/dt^2 - h^2dx/dt + k\ (G - x) + P_o \tag{2.17}$$

where: F = pull caused by the magnetic flux

P_o = effective value of the restoring spring force referred to the center of the core (pull center)

M = effective mass determined by dividing the polar mass moment of inertia (J) by the moment arm length squared

h^2 = effective damping coefficient referred to the pull center (center of core)

k = effective spring constant referred to the pull center

G = total distance between the center of the core and the armature at that point where the armature is against the back stop

x = distance measured from the surface of the pole face (called displacement).

To use equation (2.17) in attempting a solution to the problem, the pull (F) must be expressed as a function of the variable (x) and the coil current (i). To develop this pull (F) as functions of i and x, it is necessary to go back to the definition of work. By using the concept of virtual displacement on the function giving the magnetic energy in the air gap, a relationship for the pull (F) in terms of x and i can be obtained. With reference to Figure 2.2, the magnetic pull (F) tends to shorten the air gap. Consider the air gap to be shortened by a differential amount, $-dx$. The mechanical work involved is -Fdx. If -dx is considered to be the virtual displacement, under the constraint of constant flux (ϕ), then there is no energy exchanged with the electric circuit exciting the coil. Therefore the mechanical work - Fdx comes from a change in the stored magnetic energy (W_m). This is expressed by equation (2.18).

$$-F\,dx = -\,d\,W_m \quad \text{Constant } \phi \tag{2.18}$$

Therefore:

$$F = +\frac{d\,W_m}{dx} \qquad \phi = \text{constant}. \tag{2.19}$$

Since W_m is a function of several variables, equation (2.19) should be written in the form shown in equation (2.20)

$$F = +\frac{\partial W_m}{\partial x} \tag{2.20}$$

The energy stored in the magnetic field is determined by the time integration of the power supplied to the magnetic field. The power supplied to the magnetic field is given by equation (2.21).

$$p = ei = i\,N\,\frac{d\phi}{dt} = \mathcal{F}\,\frac{d\phi}{dt} \tag{2.21}$$

where: $\mathcal{F}$ = magnetomotive force or Ni.

Since energy is the time integration of power, the following expression may be written:

$$W_m = \int_0^t p\,dt = \int_0^{\phi} \mathcal{F}\,d\phi' = \mathcal{R}\int_0^{\phi} \phi'\,d\phi' \tag{2.22}$$

$$W_m = \phi^2\,\left(\frac{\mathcal{R}}{2}\right) \tag{2.23}$$

where: primes are used to indicate the variable of integration,

$\mathcal{R}$ = magnetic reluctance and is equal to $\mathcal{F}/\phi$.

The magnetic reluctance ($\mathcal{R}$) is a function of the armature displacement (x) and the coil current (i). This relation was developed previously and is given by equations (2.9) and (2.10) which is repeated as equation (2.24).

$$\mathcal{R} = \sigma x + r + ai \tag{2.24}$$

where: $\sigma = 1/\mu A$

μ = permeability of free space

A = cross sectional area of the magnetic circuit in the air gap

$r + ai$ = relation which approximates the relationship between the flux and mmf of the iron portion of the magnetic circuit.

If the operation indicated by equation (2.20) is performed on equation (2.23), the following results:

$$F = \frac{\partial W_m}{\partial x} = \frac{\phi^2}{2}\left(\frac{\partial \mathcal{R}}{\partial x}\right) \tag{2.25}$$

The flux squared (ϕ^2) can also be written as shown in equation (2.26) by using equation (2.24)

$$\phi^2 = \frac{\mathcal{F}^2}{\mathcal{R}^2} = \frac{(Ni)^2}{\mathcal{R}^2} = \frac{(Ni)^2}{(\sigma x + r + ai)^2} \quad . \tag{2.26}$$

The $\partial\mathcal{R}/\partial x$ can be obtained from equation (2.24) and gives equation (2.27).

$$\frac{\partial \mathcal{R}}{\partial x} = \sigma . \tag{2.27}$$

Substituting equations (2.26) and (2.27) into equation (2.25) gives:

$$F = \frac{N^2 i^2 \sigma}{2\,(\sigma x + r + ai)^2} \tag{2.28}$$

The relation given by equation (2.28) can be substituted into equation (2.17) giving:

$$\frac{N^2 i^2 \sigma}{2\,(\sigma x + r + ai)^2} = - M \frac{d^2x}{dt^2} - \frac{h^2 dx}{dt} + k\,(G - x) + P_o \qquad (2.29)$$

Rearranging equation (2.29) into a more standard form gives the following:

$$M \frac{d^2x}{dt^2} + h^2 \frac{dx}{dt} - k\,(G - x) - P_o + \frac{N^2 i^2 \sigma}{2\,(\sigma x + ai + r)^2} = 0 \qquad (2.30)$$

where: N = coil turns carrying the current (i)

M = effective mass of movable part and is determined by dividing the polar mass moment of inertia (J) by the moment arm length squared

h^2 = effective damping coefficient of the movable part referred to the pull center

k = effective spring constant referred to the pull center

G = length of open air gap which is the maximum value of the variable (x)

x = distance the movable part or armature is from the surface of the pole or pole face

i = the coil current

r + ai = the relation which approximates the relationship between the flux and the magnetomotive force of the iron portion of the magnetic circuit

$\sigma = 1/\mu A$

μ = permeability of free space

A = cross sectional area of the magnetic circuit in the region of the air gap

P_o = back tension or force acting on the back stop because of the restoring spring.

The previous equation is nonlinear in the two dependent variables x and i and would have to be solved simultaneously with equation (2.14) to obtain the coil current (i) and the armature displacement (x) as functions of time (t).

$$E - iR - \left[\frac{N^2 (\sigma x + r)}{(\sigma x + ai + r)^2}\right] \frac{di}{dt} + \left[\frac{N^2 \sigma i}{(\sigma x + ai + r)^2}\right] \frac{dx}{dt} = 0 \qquad (2.14)$$

Equation (2.14) is repeated here for clarity.

Development of the Electrical Equation Ultimately Used to Determine the Transducer Response

The application of Kirchhoff's Voltage Law to the coil of the transducer, when the coil is energized by a step voltage of E, gives equation (2.31).

$$E = iR + N\frac{d\phi}{dt} \qquad (2.31)$$

The coil current (i) and the magnetic flux (ϕ) are related in a fashion determined by the magnetic circuit (including air gaps) of the transducer. The relationship between i and ϕ is given by equation (2.32).

$$\phi = \frac{\mathcal{F}}{\mathcal{R}} = \frac{Ni}{\mathcal{R}i + \mathcal{R}a} \qquad (2.32)$$

where: $\mathcal{F}$ = magnetomotive force

$\mathcal{R}$ = total magnetic reluctance

$\mathcal{R}i$ = reluctance of ferromagnetic portion of the magnetic circuit

$\mathcal{R}a$ = reluctance of the air portion of the magnetic circuit.

The reluctance ($\mathcal{R}a$) of the air gap is shown by equation (2.33).

$$\mathcal{R}a = \frac{x}{\mu A} = \sigma x \tag{2.33}$$

where: x = length of the air gap

$\sigma = 1/\mu A$

μ = permeability of free space

A = cross sectional area of magnetic circuit in this region.

The reluctance ($\mathcal{R}i$) of the ferromagnetic or iron portion of the magnetic circuit must be represented in the form shown by equation (2.34) in order to reduce the complexity of the final electrical equation, and still include the effect of magnetic saturation.

$$\mathcal{R}_i = \frac{\mathcal{F}_i}{\phi_i} = r + ai \tag{2.34}$$

or

$$\mathcal{R} = \sigma x + ai + r \tag{2.35}$$

where: $\mathcal{F}_i$ = magnetomotive force of the iron portion

ϕ_i = magnetic flux in the iron portion resulting from $\mathcal{F}_i$

r = a parameter determined such that $r + ai$ approximates the non-linear relationship between $\mathcal{F}_i$ and ϕ_i over some desired section

a = (see definition of r above).

Substituting equation (2.35) into (2.32) results in equation (2.36):

$$\phi = \frac{Ni}{r + ai + \sigma x} \tag{2.36}$$

At this point in the development of the electrical equation, instead of substituting for $d\phi/dt$ as was done in the previous developments, i will be written in terms of ϕ and x.

$$i = \frac{r + \sigma x}{N} \left(\frac{\phi}{1 - b\phi} \right) \tag{2.37}$$

where: $b = a/N$.

From equation (2.31) and equation (2.37) it follows that

$$E = (r + \sigma x) \left(\frac{\phi}{1 - b\phi} \right) \frac{R}{N} + N \frac{d\phi}{dt}. \tag{2.38}$$

Equation (2.39) is a rearrangement of equation (2.38) into the standard differential equation form.

$$\frac{d\phi}{dt} + \frac{R}{N^2} (r + \sigma x) \left(\frac{\phi}{1 - b\phi} \right) = \frac{E}{N}. \tag{2.39}$$

Equation (2.39) is one of the two non-linear differential equations that will be solved simultaneously to obtain ϕ and x as functions of time.

Mechanical Equation Ultimately Used to Determine the Transducer Response

By the use of D'Alembert's Principle or, on a general basis, Euler-Lagrange Equations, the following equation results.

$$M \frac{d^2x}{dt^2} + h^2 \frac{dx}{dt} + k (x - G) - P_o + K\phi^2 = 0 \tag{2.40}$$

where: M = effective mass of armature

h^2 = viscous damping coefficient

k = spring constant of restoring spring

P_o = initial pull caused by restoring spring opposite to the magnetic force

$K\phi^2$ = magnetic force[3]

$K = \epsilon/A$

ϵ = pull constant determined by units used for ϕ and A

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[3]See any standard text dealing with static magnetic fields such as "Introductory Electrical Engineering", Reed and Corcoran, (New York 1957) p.304.

A = cross sectional area of the magnetic circuit at the air gap.

Equation (2.40) is the second of the two simultaneous differential equations that will be used to determine ϕ and x as functions of time. Once the values of ϕ and x are determined, equation (2.37) will be used to determine the corresponding values of i.

Inspection of equations (2.39) and (2.40) in comparison to equations (2.14) and (2.30) in the previous section, indicate the simplicity obtained by selecting ϕ as a dependent variable instead of i. Actually, equation (2.40) is not non-linear in x any more, although it is non-linear in the variable ϕ. One possible method of attack would be to combine the two equations into a third-order non-linear differential equation in ϕ. This could be accomplished by solving for x from equation (2.39) and substituting x and the first and second derivatives of x into equation (2.40). However, in this particular case, it is considered desirable to solve the two equations (2.39) and (2.40) simultaneously.

This decision resulted from the fact that the solution of the response is divided into three intervals. These three intervals are distinctive because of the values of x, dx/dt and dx^2/dt^2 that exist.

In the use of one combined equation in the variable ϕ, the identity of x would be lost thereby considerably complicating the calculating procedure.

The simultaneous solution of equations (2.39) and (2.40) is based on a method that is more fully developed in the book Analysis and Control of Nonlinear Systems by Y. H. Ku than in most other sources. The background for this method is discussed in Chapter III.

Equations of Motion Developed Using the Euler-Lagrange Equation

The second method mentioned in the introduction to this chapter is commonly referred to as Lagrange's equations of motion. Because of the general nature of this method a brief discussion will be given, and then this method will be used to develop the equations of motion of the transducer.

The simplest form of Lagrange's equation involves only conservative systems with fixed constraints. However, an extension can be made to nonconservative systems to obtain an alternate form of Lagrange's equations.

To account for the nonconservative dissipative forces, a velocity dependent function D, called Rayleigh's dissipation function, is used.[4] This results in the modified form of Lagrange's equation as is shown in equation (2.41).[5]

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_j}\right] - \frac{\partial D}{\partial \dot{q}_j} + F_j = 0 \tag{2.41}$$

where: q_j and $\dot{q}_j$ are the generalized corrdinates and generalized velocities, respectively, of the system at the j th terminal pair.

$(j = 1, 2, \ldots, H)$

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[4] White, D. C., and H. H. Woodson, Electromechanical Energy Conversion, (New York 1959), p. 61.

[5] The notation of using L for Lagrange's function and q_j, $\dot{q}_j$ as the as the j th generalized coordinate and velocity is fairly standard, so it will be used in this treatment. A number of texts have used this notation covering a long period of time of which the following two are examples. Higher Mathematics, Burington, R. S., and C. C. Torrance, (New York, 1939), Electromechanical Energy Conversion, White, D. C., and H. H. Woodson, (New York, 1959).

L = the conservative Lagrangian and is equal to the difference between the kinetic coenergy T_c and the potential energy V

D = Rayleigh's dissipation function and is defined as:

$$\sum_{j=1}^{H} \frac{1}{2} r_j (\dot{q}_j)^2 \qquad (2.42)$$

r_j = j th dissipative element

F_j = nonconservative force acting on the j th coordinate q_j.

The kinetic coenergy T_c instead of the energy T is used in the Lagrangian in order for it to be valid for nonlinear cases. One of the common nonlinear cases encountered in electromechanical transducers is the relation between flux linkages (λ) and the magnetomotive force (i). The magnetic stored energy (W_m) is determined as follows:

$$W_m (\lambda_h, x_g) = \int_0^{\lambda_h} \sum_{j=1}^{H} i_j' (\lambda_h, x_g) \, d\lambda_j' \, .$$

(h = 1,2,...,H and g = 1,2,...,G)

Where the primes denote variables of integration and where W_m is evaluated as the integral of $id\lambda$ for any fixed spacing (x_g's are constant). Figure 2.3 shows a graphical representation of the magnetic stored energy for k = 1.

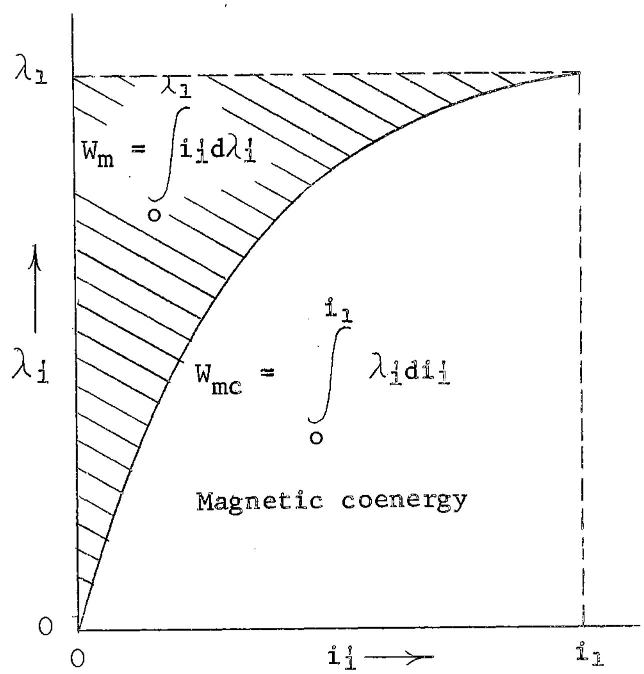

Figure 2.3

From Figure 2.3 or by integrating by parts the integral of $id\lambda$, the magnetic coenergy W_{mc} (unshaded area), is related to the magnetic energy W_m (shaded area,) by the following:

$$W_m = i_1 \lambda_1 - W_{mc}. \tag{2.44}$$

$$W_{mc} = i_1 \lambda_1 - \int_0^{\lambda_1} i_i d\lambda_i = \int_0^{i_1} \lambda_i di_i . \tag{2.45}$$

To obtain the Lagrangian L which is also valid for nonlinear systems the Lagrangian defined as follows must be studied.

$$L = L (q_j, \dot{q}_j, t). \tag{2.46}$$

$(j = 1, 2, \ldots, H)$

The differential of equation (2.46) is:

$$dL = \sum_{j=1}^{H} \frac{\partial L}{\partial q_j} dq_j + \sum_{j=1}^{H} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt . \quad (2.47)$$

The path of integration is arbitrary since L is a state function. A state function by definition is only a function of the instantaneous (or final) values independent of the dynamic state or past history. Therefore the path of integration is so selected that all the $\dot{q}$'s are constant when integrating with respect to the q's, and correspondingly the $\dot{q}$'s are constant when integrating with respect to the q's. Furthermore, these integrations are performed for a specific value of time t. Therefore, L is determined from the integral of equation (2.47) as:

$$L = \int_{0;0;t}^{q_j;0;t} dL(q_j', \dot{q}_j, t) + \int_{q_j;0;t}^{q_j;\dot{q}_j;t} dL\,(q_j, \dot{q}_j', t). \quad (2.48)$$

$(j = 1, 2, \ldots, H)$

By substituting the value of $dL(q_j, \dot{q}_j, t)$ from equation (2.47) into equation (2.48) the following results are obtained.[6]

$$L\,(q_j, \dot{q}_j, t) = \int_0^{q_j} \sum_{j=1}^{H} \frac{\partial L\,(q_j', 0, t)}{\partial q_j'} dq_j' +$$

$$\int_0^{\dot{q}_j} \sum_{j=1}^{H} \frac{\partial L\,(q_j, \dot{q}_j', t)}{\partial \dot{q}_j} d\dot{q}_j' \quad (2.49)$$

$(j = 1, 2, \ldots, H)$

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[6] The primed quantities are the variables of integration.

This particular procedure results in separating the Lagrangian into two functions. The first of these functions is:

$$\int_{0}^{q_j} \sum_{j=1}^{H} \frac{\partial L\,(q_j',0,t)}{\partial q_j'}\, dq_j' \qquad (2.50)$$

$(j = 1,2,\ldots,H)$

and is only a function of the coordinates q_j and t, and is independent of the velocities $\dot{q}_j$. No restrictions of linearity were made in the development of equation (2.49), and therefore equation (2.50) holds for the nonlinear case. Since the Lagrangian (L) is an energy function, then the partial of (L) with respect to the j th coordinate (q_j) is the force function (f_j). Then f_j is given as:

$$f_j\,(q_j,t) = \frac{\partial L(q_j,0,t)}{\partial q_j}. \qquad (2.51)$$

This makes the potential energy V as follows:

$$V = \int_{0}^{q_j} \sum_{j=1}^{H} -f_j\,(q_j',t)\, dq_j'. \qquad (2.52)$$

The second term of equation (2.49) is a function of the final values of the q_j coordinates but is a function of the velocities $\dot{q}_j$ and time t. This term is as follows:

$$\int_{0}^{\dot{q}_j} \sum_{j=1}^{H} \frac{\partial L\,(q_j,\, \dot{q}_j',\, t)}{\partial \dot{q}_j'}\, d\dot{q}_j'\,. \qquad (2.53)$$

$(j = 1,2,\ldots,H)$

In mechanics this is the kinetic energy which, for the linear case, is also the kinetic coenergy, since the curve in Figure 2.3 would be a straight diagonal line between (0,0) and (λ_1, i_1). This is evident by letting $i_1 = \dot{q}_1$. With this substitution in equation (2.53) the following results.

$$\int_0^{i_1} \frac{\partial L\,(q, i_1', t)}{\partial i_1'}\, di_1' \quad . \tag{2.54}$$

Magnetic coenergy was defined by equation (2.45) and is given as

$$W_{mc} = \int_0^{i_1} \lambda_1\,(i_1', x_1)\, di_1' \quad . \tag{2.55}$$

Therefore it follows that equation (2.54) has the form of coenergy. In mechanics this is the kinetic coenergy. The partial of the Lagrangian (L) with respect to the velocity ($\dot{q}_j$) is a momentum function (p_j) because the Lagrangian is an energy function. Therefore the kinetic coenergy (T_c) is given as

$$T_c = \int_0^{\dot{q}_j} \sum_{j=1}^{H} \frac{\partial L\,(q_j, \dot{q}_j', t)}{\partial \dot{q}_j}\, d\dot{q}_j' \;, \tag{2.56}$$

$(j = 1, 2, \ldots, H)$

or

$$T_c = \int_0^{\dot{q}_j} \sum_{j=1}^{H} p_j\, d\dot{q}_j' \quad . \tag{2.57}$$

$(j = 1, 2, \ldots, H)$

With the two right-hand terms of equation (2.49) now defined, then the Lagrangian for linear or nonlinear systems becomes:

$$L\ (q_j,\ \dot{q}_j,\ t) = \text{potential energy } V - \text{kinetic coenergy } T_c. \qquad (2.58)$$

The procedure for using Lagrange's equation to determine the equations of motion becomes that of determining the functions V, T_c, D and F in terms of q_j and $\dot{q}_j$. In the mechanical system the variables q_j and $\dot{q}_j$ are generally the displacement (x) and the velocity (v) respectively. However, in the electrical system, q_j may be the charge q or the flux linkages (λ). If q_j is charge q, then $\dot{q}_j$ is current (i), but if q_j is flux linkages (λ) then $\dot{q}_j$ is voltage (e). The selection of q_j then determines whether the system is considered to have a current driving source or a voltage driving source.

The equations of motion of the system represented by Figure 2.1 and 2.2 will now be obtained by using Lagrange's equation of motion. For the electrical system with j = 1, let $q_1 = q$ and $\dot{q}_1 = i$. With $q_1 = q$ (charge), then potential energy is the energy associated with a charge and therefore has dimensions of $q^2/2C$. This makes $-f_1$ equal to the voltage (e). The kinetic electrical energy is the energy associated with i and therefore has dimensions of $L\ i^2/2$, making the momentum function p_1 become the flux linkages (λ).

For the electrical system the potential energy (V) is

$$V_e = \int_0^q -\, e dq = \int_0^q -\frac{q}{C}\, dq = -\frac{q^2}{2C} \quad . \qquad (2.59)$$

Since there is no capacitance in the electrical circuit of the transducer, then $V_e = 0$.

The kinetic coenergy for the electrical system is

$$T_{ce} = \int_0^i \lambda' di' = \int_0^i N\phi' di'. \qquad (2.60)$$

In order to show that the use of Lagrange's equations will give the same equations of motion of the transducer as were obtained by force laws, the coupling term of mechanical origin in the electrical equation must be examined. This term was developed in final form in equation (2.25). The operation involved to obtain equation (2.25) was based upon maintaining the flux (ϕ) or flux linkages (λ) not a function of x. This condition will also have to be imposed in this development to obtain a result that can be compared with that obtained by the traditional approach.

For the mechanical system, let $j = 2$, $q_2 = x$, $\dot{q}_2 = \dot{x}$, $-f_2 = P_o + k(G - x)$, and $p_2 = -M\dot{x}$.[7] Therefore the potential energy (V_m) of the mechanical system is

$$V_m = \int_0^x \left[P_o + (G - x')\right] dx' = P_o x + kGx - \frac{kx^2}{2}. \qquad (2.61)$$

The kinetic coenergy of the mechanical system is

$$T_{cm} = \int_0^{\dot{x}} -M\dot{x}' d\dot{x}' = \frac{M\dot{x}^2}{2}. \qquad (2.62)$$

The dissipative function D is

$$D = +\frac{i^2 R}{2} - \frac{h^2 \dot{x}^2}{2}. \qquad (2.63)$$

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[7] See Figure 2.2. The minus sign associated with x and $\dot{x}$ is caused by x being opposite to the displacement from the equilibrium position.

The function F_1 for the electrical system is E while for the mechanical system it is zero, since no external driving function is considered. Combining the state functions for the electrical and mechanical system gives the Lagrangian L as follows:

$$L = T_c - V = \int_0^i N\phi di' - \frac{M\dot{x}^2}{2} - P_o x - kGx + \frac{kx^2}{2} . \qquad (2.64)$$

The equation of motion of the electrical system is determined from equation (2.41) with j = 1, or

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial i} \right) - \frac{\partial D}{\partial i} + F_1 = 0. \qquad (2.65)$$

Since L is not a function of q, then $\partial L/ \partial q = 0$. The $\partial L/ \partial i$ from equation (2.64) is

$$\frac{\partial L}{\partial i} = N\phi \qquad (2.66)$$

where the flux is assumed constant.

The $\partial D/ \partial i$ from equation (2.63) is

$$\frac{\partial D}{\partial i} = + iR. \qquad (2.67)$$

Substituting equations (2.66) and (2.67) into (2.65) and with $F_1 = E$ gives

$$\frac{-d}{dt} \left[N\phi \right] - iR + E = 0. \qquad (2.68)$$

Since N is constant, equation (2.68) becomes

$$N \frac{d\phi}{dt} + iR = E. \qquad (2.69)$$

Equation (2.69) is identical to equation (2.6); therefore, this procedures gives the same result for the electrical system.

The equation of motion for the mechanical system is determined from equation (2.41) with j = 2, or

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial D}{\partial \dot{x}} + F_2 = 0. \qquad (2.70)$$

The $\partial L/\partial x$ from equation (2.64) becomes

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x} \int_0^i N\phi di' - P_o - kG + kx. \qquad (2.71)$$

The flux (ϕ) is equal to $\mathcal{F}/\mathcal{R}$ where $\mathcal{F} = Ni$. Substituting these relations in the first term on the right hand side of equation (2.71) gives

$$\frac{\partial}{\partial x} \int_0^i \frac{N^2 i'}{\mathcal{R}} di' \quad . \qquad (2.72)$$

The constraints used in developing the magnetic pull as given by equation (2.28) were; the reluctance ($\mathcal{R}$) is constant when the current (i) or flux (ϕ) increases from zero, and the flux (ϕ) is constant when x is given a virtual change. Performing the operation indicated in equation (2.72) gives

$$\frac{\partial}{\partial x} \frac{N^2}{\mathcal{R}} \int_0^i i'di' = \frac{\partial}{\partial x} \left[\frac{N^2}{\mathcal{R}} \frac{i^2}{2}\right] . \qquad (2.73)$$

Since flux (ϕ) is equal to $Ni/\mathcal{R}$ then equation (2.73) may be written as

$$\frac{\partial}{\partial x} \left[\left(\frac{Ni}{\mathcal{R}}\right)^2 \frac{\mathcal{R}}{2}\right] \qquad (2.74)$$

Performing the operation indicated in equation (2.74) gives

$$\frac{\phi^2}{2}\frac{\partial \mathcal{R}}{\partial x} = \frac{\phi^2}{2}(\sigma) \tag{2.75}$$

Reluctance $\mathcal{R}$ is given by equation (2.24) as

$$\mathcal{R} = \sigma x + ai + r \tag{2.24}$$

Therefore the partial of $\mathcal{R}$ with respect to x is σ, and equation (2.71) becomes

$$\frac{\partial L}{\partial x} = \frac{N^2 i^2 \sigma}{2(\sigma x + ai + r)^2} - P_o - kG + kx. \tag{2.76}$$

The second term of equation (2.70) is determined from equation (2.64) as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left[-M\dot{x}\right] = -\frac{Md\dot{x}}{dt} = -\frac{Md^2x}{dt^2}\ . \tag{2.77}$$

The third term of equation (2.70) is determined from equation (2.63) as

$$\frac{\partial D}{\partial \dot{x}} = -h^2\dot{x} = -\frac{h^2 dx}{dt} \tag{2.78}$$

Since $F_2 = 0$, then the equation of motion of the mechanical system is determined by substituting equations (2.76), (2.77), and (2.78) into equation (2.70), giving

$$\frac{N^2 i^2 \sigma}{2(\sigma x + ai + r)^2} - P_o - kG + kx + \frac{Md^2x}{dt^2} + \frac{h^2 dx}{dt} = 0. \tag{2.79}$$

Rearrangement of equation (2.79) will show that it is identical to equation (2.30). Equation (2.30) was developed using D'Alembert's principle while equation (2.79) was developed using the Euler-Lagrange equation for nonconservative systems.

Either approach appears to involve a certain amount of insight or judgment when dealing with coupled systems. The Euler-Lagrange approach is more general, being based upon the determination of energy or state functions in terms of generalized coordinates. However, this approach has the disadvantage that insight into the physical phenomena may be lost by the more formal mathematical procedures. Also, in applying this energy approach to nonconservative systems, a dissipation function must be defined. In order for the dissipation term to be consistent with the force relationships, only one-half of the power loss in that element must be used. Physically this is not understood, but it is required in order to extend this Euler-Lagrange equation to nonconservative systems. H. E. Koenig and W. A. Blackwell have indicated that one major weakness of this approach is the lack of a general procedure for establishing a relationship between the variables of the energy function and the generalized coordinate.[8]

The approach using D'Alembert's principle is very satisfactory even though it may require more attention to the details of the directions in the which the forces act. It appears that, with moderate care in selecting reference points, the amount of work or the accuracy of the force method is comparable with that of the energy method even for complicated coupled systems.

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[8] Koenig, H. E., and W. A. Blackwell, "On the Codification of Lagrangian Formulation", IRE Proceedings, July 1958, pp. 1428-1429.

CHAPTER III

DISCUSSION OF A METHOD FOR NUMERICALLY SOLVING SIMULTANEOUS NONLINEAR DIFFERENTIAL EQUATIONS

The method under consideration is called the phase-plane or phase or space trajectory method. It is probably more fully developed for nonlinear systems in the book <u>Analysis and Control of Nonlinear Systems</u> by Y. H. Ku than in any other source. Apparently no one individual was responsible for the development of the method as no reference is given as to the origin. It appears to be an adaptation of the isoline method, but this is listed separately in Ku's book, thus suggesting the two methods are different. The original basis for this phase or space trajectory method seems to be the direction field method discussed formerly in Ford's <u>Differential Equations</u> and again more recently in <u>Differential Equations with Applications</u> by Betz, Burcham and Ewing. However, there are other recent texts that also disucss the direction field procedure as a graphical or numerical method. Since the direction field method seems to be a logical beginning of the phase-plane and space trajectory methods, this will be used as the starting point in the development of this space trajectory method.

Direction Field Method

The first-order differential equation, whether it be linear or nonlinear of the form given in equation (3.1), can be written in the form shown in equation (3.2).

$$A\ (x,t)\ \frac{dx}{dt} + B(x,t) = C(x,t). \tag{3.1}$$

$$\frac{dx}{dt} = f\ (x,t) = \dot{x} \tag{3.2}$$

where: $\dot{x} = dx/dt$.

It is assumed that $f(x,t)$ is a single valued function in the region of the x,t plane under consideration. This means that for each point $(x_o,\ t_o)$ of that region there is associated with it a value of dx/dt defined by equation (3.2) that is unique. Let the value of dx/dt when $x = x_o$ and $t = t_o$ be represented by $\dot{x}_o$. Thus the three numbers x_o, t_o, $\dot{x}_o$ define a line element in the x,t plane which is located by the values of $(x_o,\ t_o)$ and whose slope is determined by the differential equation (3.2) and represented by $\dot{x}_o$. The differential equation then defines a field of line elements called a direction field. A solution of the differential equation is any function $x = f(t)$ whose graphical representation matches this direction field. This means a curve which has a slope which fits the direction field at each of the points. Such a curve is called an integral curve. Since $f\ (x,t)$ is single-valued, then there passes a unique integral curve along these line elements which is a solution to the differential equation. This concept is probably best illustrated by considering a numerical example.

A simple example is the solution to the R-L circuit with a unit step as the driving emf. Consider the following numerical case.

$$10 = 10i + 1\ \frac{di}{dt} \tag{3.3}$$

Rearranging:

$$\frac{di}{dt} = 10 - 10i. \qquad (3.4)$$

The direction field is obtained by determining the value of di/dt at a number of points in the i-t plane. Since in this case, the value of di/dt is not a function of t then the direction of the line elements depends only on i as is shown in Figure 3.1. Therefore, when i = 0, di/dt = 10; when i = 0.1, di/dt = 9, etc. until i = 1 then di/dt = 0. The line elements shown in Figure 3.1 determine the direction field of the relation given in equation (3.4). The shape of the integral curve may be approximated by constructing a polygonal line. This polygonal line is obtained by commencing at some point (x_o, y_o) and drawing a line segment that coincides with the line element at that point. Then at the upper extremity of that line segment, say at point (x_1, y_1), draw another line segment that coincides with the line element at the point (x_1, y_1). By proceeding in this manner to more and more points, all inside the region defined, a series of connected line segments is obtained which form a polygonal line. If the line segments are made shorter and shorter by using more points, than a better approximation is obtained of the integral curve. To illustrate this procedure, suppose that the initial values of i and t are zero. This causes the integral curve to start at the point (a) in Figure 3.1. If only ten line segments are taken, then line a-b is obtained. It is noticed that the line a-b approximates the integral curve reasonably well until point (b) is approached. This was caused by the fact that the slope of the line elements changed rapidly as the value of i approached one. If at the point where i = 0.8, smaller increments were taken, such as i = 0.04, then the line a-c would result. It is seen that this gives a better approximation. The greatest error occured when the

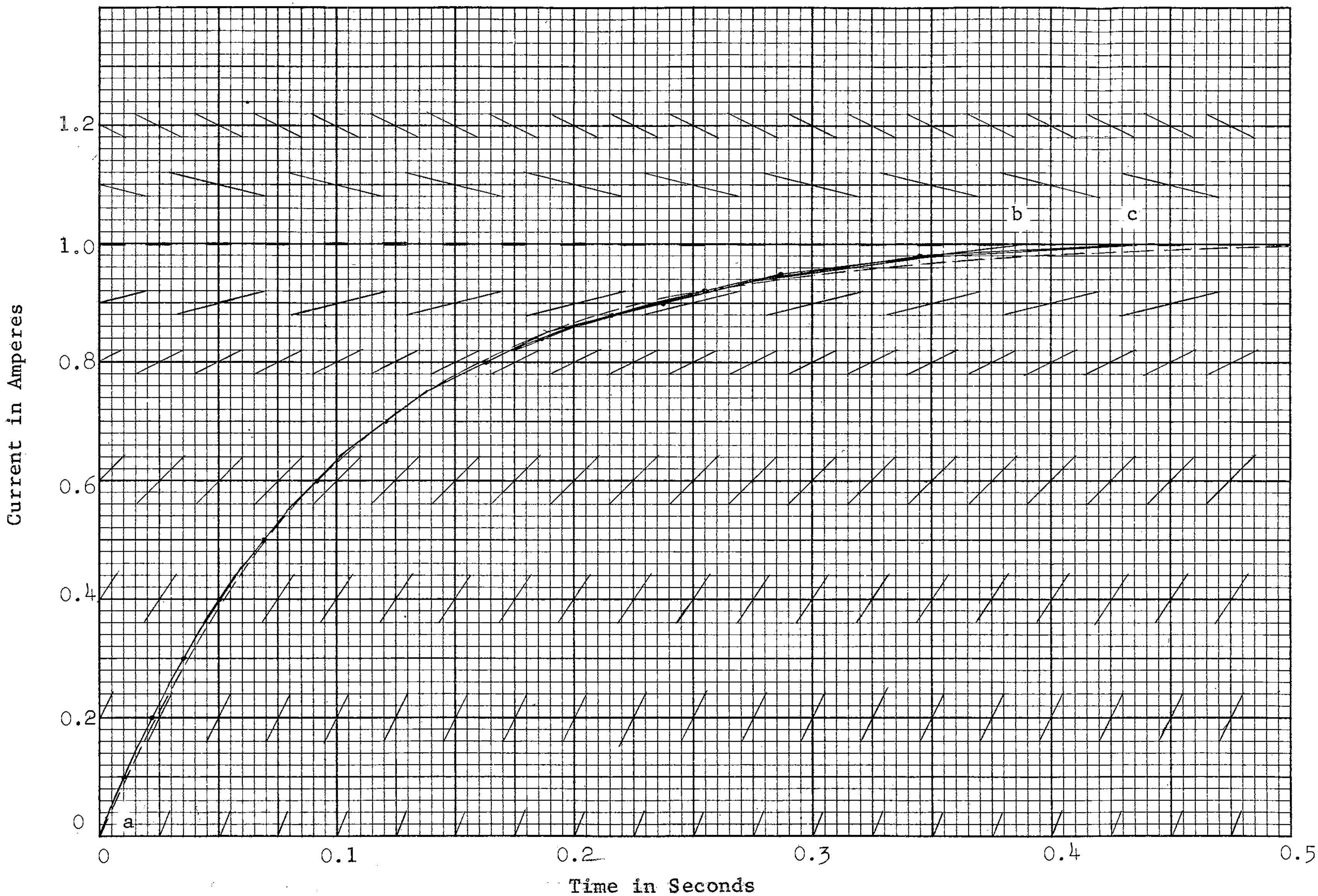


Figure 3.1. Example of a Solution by the Direction Field Method Compared to the Exact Solution

the slope of the line element changes from 1 to 0, since the ratio of these two values is infinite. The exact solution to equation (3.4) is given by equation (3.5) and is shown by the dashed line in Figure 3.1.

$$i = 1\ (1 - \epsilon^{-10t}). \tag{3.5}$$

The polygonal line is a fair approximation of the solution up to the value of i = 0.6. Smaller increments have to be taken to keep the same accuracy, especially as the value of i approaches 1.0. The initial conditions determine the starting point of the integral curve. If the initial value of i at t = 0 had been 0.4, then the integral curve would have started at that point with di/dt = 6. This simple example illustrates the procedure and some of the limitations that must be considered.

Phase-Plane Method Applied to a Second-Order Differential Equation

The discussion up to this point has been confined to a first-order differential equation, linear or nonlinear. This method is not restricted to first-order equations. It may be applied to a second-order nonlinear differential equation of the form shown in equation (3.6):

$$M\ (\dot{x},\ x,\ t)\ \frac{d^2x}{dt^2} + A(x,\ t)\ \frac{dx}{dt} + B(x,\ t) = f(t) \tag{3.6}$$

where: $\dot{x} = dx/dt$.

Equation (3.6) is much more general than is usually required but it is shown as an example to show the generality of the phase-plane method. Rearranging equation (3.6) in the form of equation (3.7) gives the starting point of this method.

$$\frac{d^2x}{dt^2} + C\frac{dx}{dt} + Dx = E \tag{3.7}$$

where: $C = A\ (x,t)/M\ (\dot{x},x,t)$

$D = B\ (x,t)/M\ (\dot{x},x,t)$

$E = f(t)/M\ (\dot{x},x,t)$.

Let $d^2x/dt^2 = w$ and $dx/dt = v$, then equation (3.7) becomes:

$$w = E - Cv - Dx. \tag{3.8}$$

Since $w = dv/dt = (dv/dx)\ (dx/dt) = v\ (dv/dx)$, then equation (3.8) becomes:

$$v\frac{dv}{dx} = w = E - Cv - Dx, \tag{3.9}$$

or

$$\frac{dv}{dx} = \frac{w}{v} = \frac{E - Cv - Dx}{v} \tag{3.10}$$

Equation (3.10) gives the slope of the line element in the v-x plane. This means that the change in v with a change in x is determined by the values of v and x. Usually the solution desired is x as a function of t. If the direction field is obtained for equation (3.10) then the integral curve can be approximated or a phase-plane plot made of equation (3.10). Since $dv/dt = v\ dv/dx$, then $dt = dx/v$.

therefore:

$$t = \int_{x_1}^{x_2} \frac{dx}{v} \tag{3.11}$$

Equation (3.11) is used to determine the time (t) as a function of x. Therefore a plot of time (t) versus displacement (x) is obtained or the displacement (x) may be determined as a function of time (t).

To help clarify the details of the phase-plane method the following numerical example will be considered.

$$\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 20x = 100. \tag{3.12}$$

Putting equation (3.12) into the standard phase-plane form gives:

$$w + 12v + 20\,x = 100, \tag{3.13}$$

or

$$\frac{dv}{dx} = \frac{w}{v} = \frac{100 - 12v - 20x}{v} \tag{3.14}$$

The initial values at $t = 0$ are $x = 0$, $v = 6$, and from equation (3.13), $w = 28$. Equation (3.14) gives the initial value of dv/dx as 4.67 or the initial slope of the v-x curve is + 4.67 as shown in Figure 3.2

Let $x = 0.1$, then from equation (3.14) Δv may be obtained as:

$$\frac{\Delta v}{.1} = \frac{100 - 12\,(6 + \Delta v) - 20\,(.1)}{6 + \Delta v}\,, \tag{3.15}$$

giving $\Delta v^2 + 7.2\Delta v - 2.6 = 0$ or $\Delta v = .345$.

The negative root makes no physical sense in this case. The value of Δv can also be determined from equation (3.14) by using the initial value of dv/dx and solving for Δv as follows:

$$\frac{\Delta v}{\Delta x} = 4.67$$

when $\Delta x = 0.1$, $\Delta v = .467$.

In this case the difference is about 30% since the value of Δx is large. If the value of Δx had been 0.01, then the value of Δv by the quadradic method would be 0.4125, compared to 0.467 determined from the slope. The difference has been reduced to about 11%. If the initial

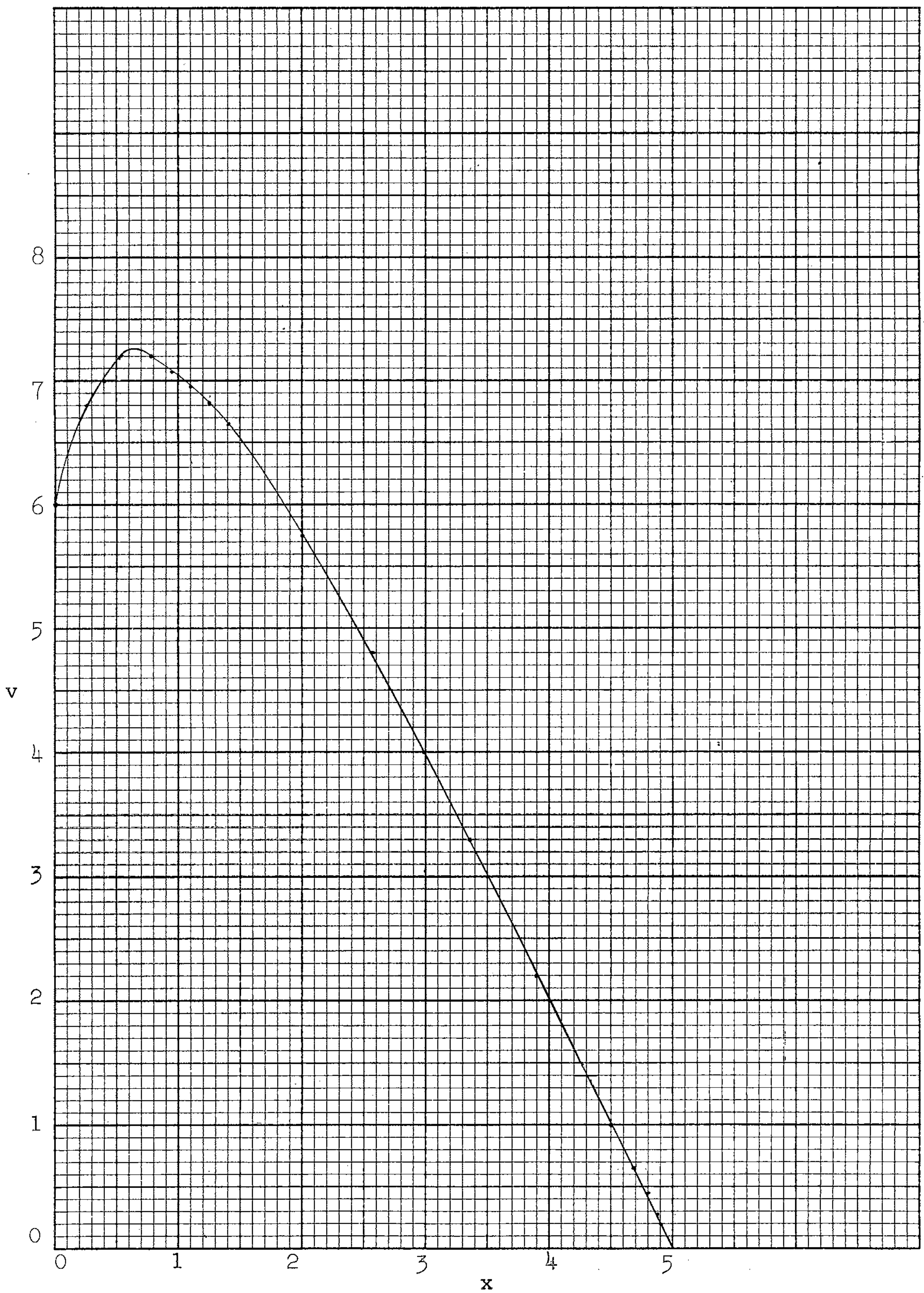


Figure 3.2. Phase-Plane Plot of Equation (3.12)

values of v and x are denoted as v_0, x_0 and the values at the next point as v_1, x_1, then the values at the new points are defined as $v_1 = v_0 + \Delta v_0$ and $x_1 = x_0 + \Delta x_0$. The value of Δv_0 is determined from equation (3.14) from the values of v_0, x_0 and Δx_0. The value of Δx_0 is selected by considering the accuracy desired. At the point v_1, x_1 a new slope is computed which is used to determine Δv_1 in the next interval. It is not necessary that $\Delta x_1 = \Delta x_0$, since the value of increments Δx is not dependent upon the values of v or x. In some cases it is desirable to change the value of Δx depending upon the accuracy desired. If the value of dv/dx is large, then small values of Δx are desirable. For the general case the value of v and x at any point is determined as:

$$v_{n+1} = v_n + \Delta v_n$$

and

$$x_{n+1} = x_n + \Delta x_n$$

where x_n and v_n are determined as follows:

$$v_n = v_0 + \sum_{k=0}^{n-1} \Delta v_k$$

$$x_n = x_0 + \sum_{k=0}^{n-1} \Delta x_k.$$

The values of Δx_k are selected, while the value of Δv_k is determined from equation (3.16).

$$\frac{\Delta v_k}{\Delta x_k} = \frac{100 - 12\,(v_k + \Delta v_k) - 20\,(x_k + \Delta x_k).}{v_k} \qquad (3.16)$$

The phase-plane plot of equation (3.12) in the v-x plane, determined by the procedure outlined, will give the curve as shown in Figure 3.2. The final values of v and x are 0 and 5 respectively. Equation (3.11) gives the value of t in terms of the values of x and v. The value of t can be determined graphically as the area under a curve of 1/v versus x. This relation is shown in Figure 3.3. Since the value of v approaches zero, then the value of 1/v approaches infinity which makes the value of t approach infinity as x approaches 5.

A plot of t versus x, determined from measuring the area under the curve of Figure 3.3 by counting squares, is shown in Figure 3.4 by the triangular points. The circled points are a plot of t versus x computed from an exact solution of equation (3.12). The close comparison of the plot of the exact solution and the numerical solution by the phase-plane method indicates the validity of such a procedure.

Phase-Plane Method Applied to Higher-Order Differential Equations

This procedure may be extended to higher-order differential equations. If a third-order differential equation were assumed, then a three dimensional phase space with coördinates of w, v, and x could be used. However, because of the difficulty in drawing and computing in three dimensions, it is desirable to use the projection of the curve on the three planes w - v, w - x and v - x. This interpretation follows from an examination of the third-order differential equation. Let the relation given in equation (3.17) be used for discussion.

$$\frac{d^3x}{dt^3} + a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + dx = F. \tag{3.17}$$

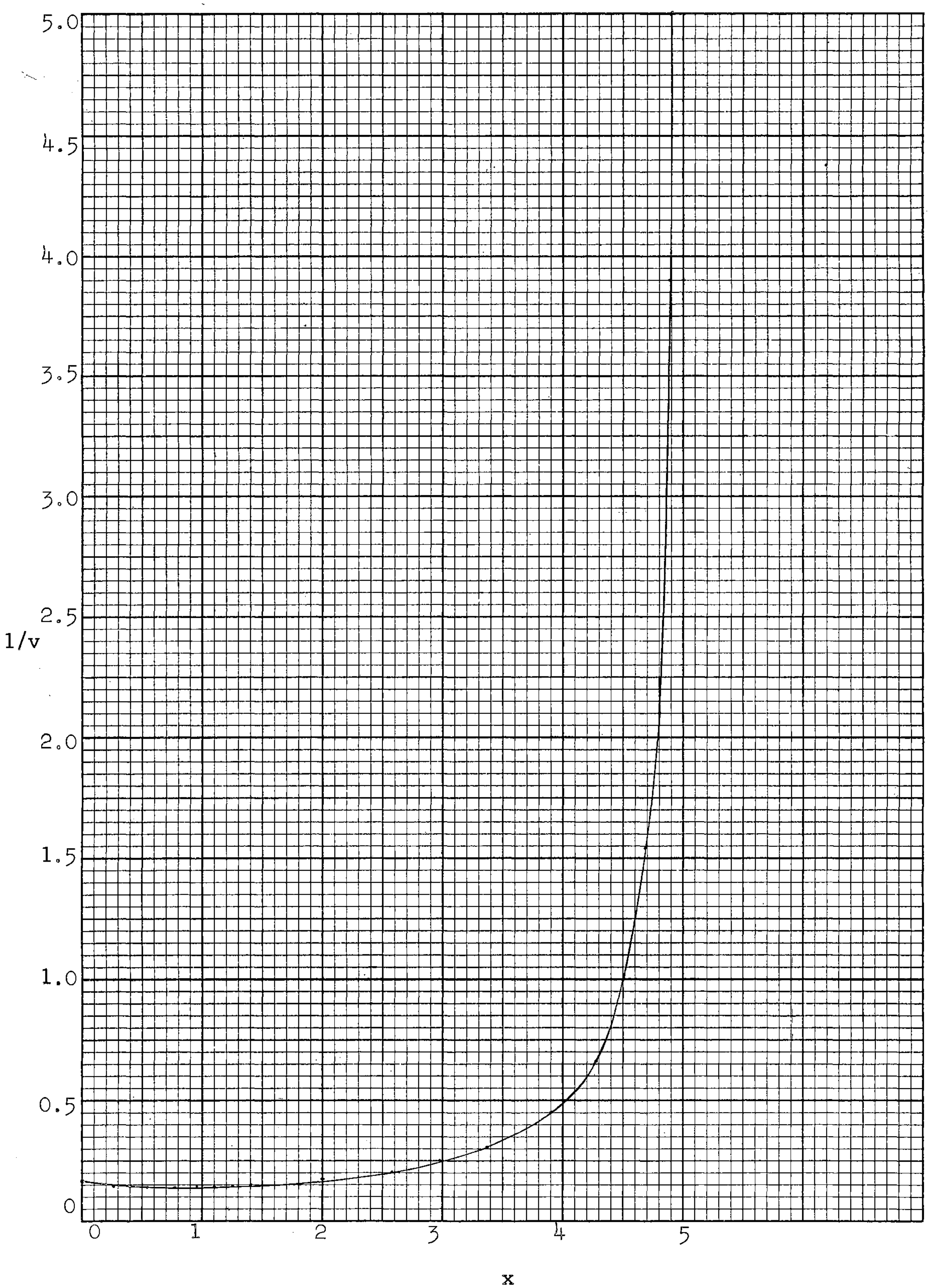


Figure 3.3. Example of the Use of Equation (3.11)

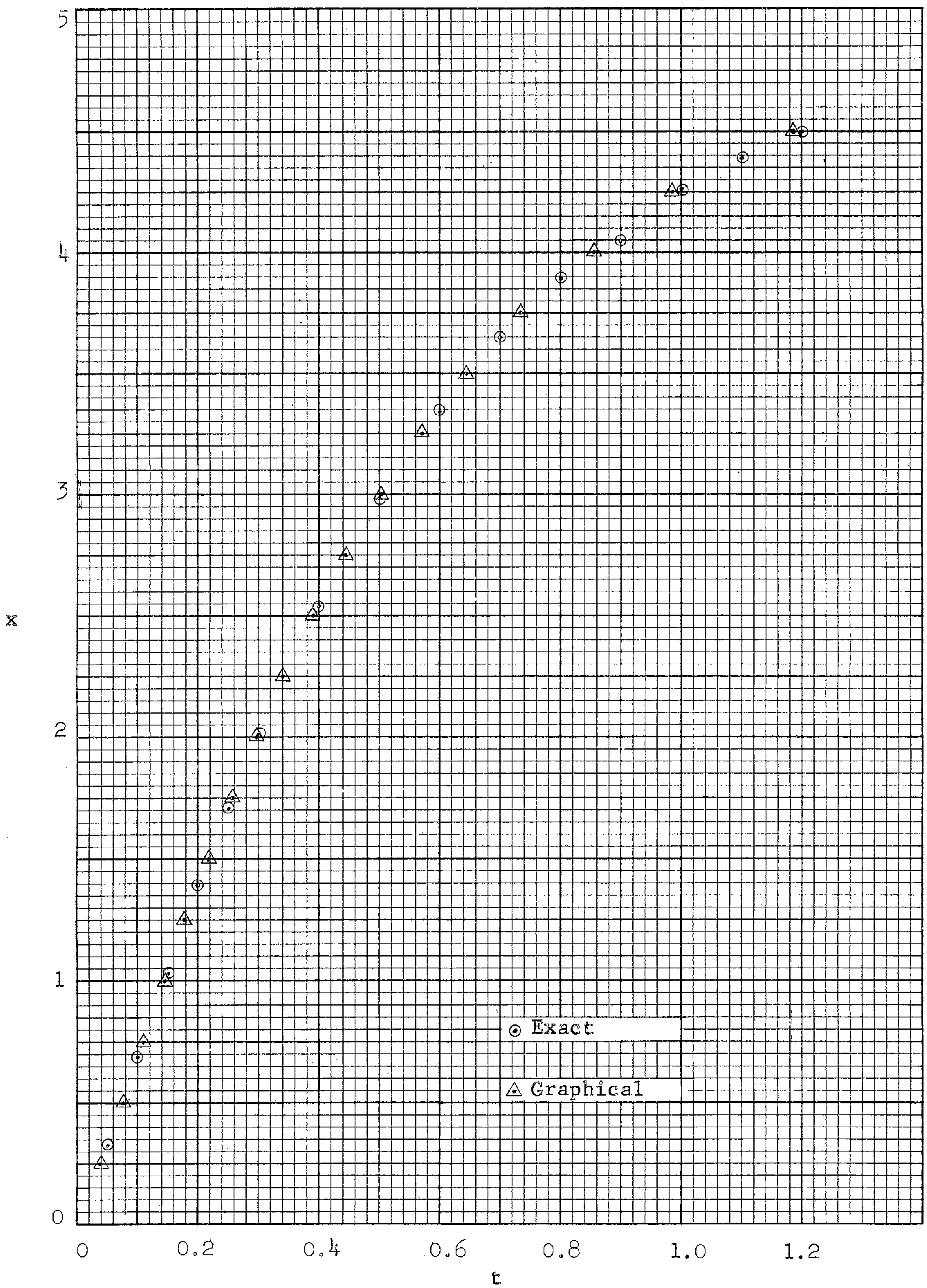


Figure 3.4. Comparison of the Graphical and Exact Solution of Equation (3.12)

Let $dx/dt = v$, $d^2x/dt^2 = w$ and $d^3x/dt^3 = \dot{w}$. From the definition then:

$$\dot{w} = \frac{dw}{dt} = \frac{dw}{dv}\frac{dv}{dt} = w\frac{dw}{dv}, \tag{3.18}$$

or

$$\frac{dw}{dv} = \frac{\dot{w}}{w}, \tag{3.19}$$

$$w = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}, \tag{3.20}$$

or

$$\frac{dv}{dx} = \frac{w}{v}, \tag{3.21}$$

$$\dot{w} = \frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt} = v\frac{dw}{dx}, \tag{3.22}$$

or

$$\frac{dw}{dx} = \frac{\dot{w}}{v}. \tag{3.23}$$

The relations lead to the statement that the slope of the projection of the space curve in the w - x plane is equal to the ratio of $\dot{w}$ to v. Also the slope of the projection of the space curve in the v - x plane is equal to the ratio of w to v, and the slope of the projection of the space curve in the w - v plane is equal to the ratio of $\dot{w}$ to w.

Note that equation (3.23) can be obtained from equations (3.19) and (3.21). In general any two of the equations (3.19), (3.21), or (3.23) will suffice to determine the slope of the space curve.

Equation (3.17) can be rearranged to give equation (3.24).

$$\dot{w} = F - aw - bv - cx, \tag{3.24}$$

or

$$\frac{dw}{dv} = \frac{\dot{w}}{w} = \frac{F - aw - bv - cx}{w}. \tag{3.25}$$

The reciprocal of equation (3.21) gives:

$$\frac{dx}{dv} = \frac{v}{w} \quad . \tag{3.26}$$

Equations (3.25) and (3.26) can be considered as two simultaneous phase-space equations where v becomes the independent variable and x and w the two dependent variables. Thus, a third-order differential equation can be replaced by two phase-space equations. In the case of a single phase-space equation a small change in x gives a change in v. For the case of two phase-space equations (given by equation (3.25) and (3.26)) a small change Δv, will simultaneously give a change, Δx and Δw. The procedure can be continued here as was explained in the case of the one phase-space equation. The new values of w, v, and x such as w_1, v_1 and x_1 are determined from the initial values of w_o, v_o and x_o as follows:

$$w_1 = w_o + \Delta w_o$$

$$v_1 = v_o + \Delta v_o$$

$$x_1 = x_o + \Delta x_o \quad .$$

The value of Δx_o is determined from equation (3.26) with $v = v_o$, $w = w_o$ and $dv = \Delta v_o$, while the value of Δw_o is determined from equation (3.25) with the additional relations of $x = x_o$ and $dw = \Delta w_o$. This type of calculation requires that Δv_o be small. A larger value of Δv_o may be used if the relation in equation (3.26) is substituted in equation (3.25), reducing it to two variables either w and v or x and v. This results in equation (3.25) being a quadradic in Δw_o or Δx_o, and complicates the process. It appears that several computations could be made by the simpler procedure in the same time required to solve the quadradic form. Even though the quadradic form is more accurate, it appears that by using a smaller value of Δv in the simpler procedure, the same accuracy may be obtained in the same computation time. In other words, it takes about the same time to obtain a given accuracy regardless of the two forms used. If the quadradic form is used, computing the slope at the middle of the interval results in greater accuracy especially when the initial slope is infinite.

Solution of Simultaneous Differential Equations by the Phase-Plane Method

The extension of this phase-plane method to simultaneous differential equations is similar to the procedure used to handle differential equations above the second-order. In fact, it is theoretically possible to reduce the simultaneous differential equations to a single differential equation, whose order is less than or equal to the sum of the orders of the original set. However, the resulting single equation is sometimes so complicated that it is practically impossible to solve. Likewise the significance of the initial conditions is lost, complicating the solution.

Consider the case of two simultaneous second-order differential equations given as follows:

$$\frac{d^2x_1}{dt^2} + a_1 \frac{dx_1}{dt} + b_1x_1 + c_1 \frac{d^2x_2}{dt^2} + d_1 \frac{dx_2}{dt} + e_1x_2 = F_1, \tag{3.27}$$

$$\frac{d^2x_2}{dt^2} + a_2 \frac{dx_2}{dt} + b_2x_2 + c_2 \frac{d^2x_1}{dt^2} + d_2 \frac{dx_1}{dt} + e_2x_1 = F_2. \tag{3.28}$$

Let $w_1 = d^2x_1/dt^2$, $v_1 = dx_1/dt$, $w_2 = d^2x_2/dt^2$ and $v_2 = dx_2/dt$. Then equations (3.27) and (3.28) become:

$$w_1 + a_1v_1 + b_1x_1 + c_1w_2 + d_1v_2 + e_2x_2 = F_1 \tag{3.29}$$

$$w_2 + a_2v_2 + b_2x_2 + c_2w_1 + d_2v_1 + e_2x_1 = F_2 \tag{3.30}$$

In a number of physical systems some of the values of a_1, b_1, ... a_2, b_2, ... will be zero, simplifying the equations. This procedure applies to nonlinear systems. Also the coefficients of each order of the derivative of the variable can be a combination of any lower order of the derivative of that variable, or of the other variables in the case of simultaneous differential equations.

Equation (3.30) should be rearranged so that the variable with the 1 subscript is first as in equation (3.29).

$$w_1 + a_1v_1 + b_1x_1 + c_1w_2 + d_1v_2 + e_1x_2 = F_1, \tag{3.29}$$

$$c_2w_1 + d_2v_1 + e_2x_1 + w_2 + a_2v_2 + b_2x_2 = F_2. \tag{3.20}$$

The procedure for handling simultaneous differential equations will be demonstrated by the use of equations (3.29) and (3.30). From the definitations $w_1 = dv_1/dt$ and $w_2 = dv_2/dt$, the following results can be obtained.

$$\frac{w_1}{w_2} = \frac{dv_1/dt}{dv_2/dt} = \frac{dv_1}{dv_2} \quad . \tag{3.31}$$

Likewise, since $v_1 = dx_1/dt$ and $v_2 = dx_2/dt$, then

$$\frac{v_1}{v_2} = \frac{dx_1/dt}{dx_2/dt} = \frac{dx_1}{dx_2} \quad . \tag{3.32}$$

Equations (3.31) and (3.32) give the changes in the one subscripted variable in terms of the two subscripted variable, or vice versa. Two additional equations are needed to give the relations between time derivatives of the same variable, as was shown in the case of a single differential equation. These two additional equations are:

$$\frac{dv_2}{dx_2} = \frac{w_2}{v_2} \quad , \tag{3.33}$$

and

$$\frac{dv_1}{dx_1} = \frac{w_1}{v_1} . \tag{3.34}$$

To illustrate the procedure, a numerical example will be considered. Let the two simultaneous differential equations be as follows:

$$w_1 + 2v_1 + 4x_1 + 3v_2 = 6, \tag{3.35}$$

$$w_2 + 3v_2 + 5x_2 + 2v_1 = 0. \tag{3.36}$$

Therefore:

$$\frac{dv_1}{dv_2} = \frac{w_1}{w_2} = \frac{6 - 2v_1 - 4x_1 - 3v_2}{- 3v_2 - 5x_2 - 2v_1} ; \tag{3.37}$$

$$\frac{dx_1}{dx_2} = \frac{v_1}{v_2} ; \tag{3.38}$$

$$\frac{dv_1}{dx_1} = \frac{w_1}{v_1} = \frac{6 - 2v_1 - 4x_1 - 3v_2}{v_1} ; \tag{3.39}$$

$$\frac{dv_2}{dx_2} = \frac{w_2}{v_2} = \frac{- 3v_2 - 5x_2 - 2v_1}{v_2} . \tag{3.40}$$

The regularly required initial conditions for a second-order differential equation are all that are needed to determine the initial values of equations (3.37) through (3.40). In this example, let $v_1 = 2$, $x_1 = 1$, $v_2 = 4$ and $x_2 = 0$ when $t = 0$ then the initial slopes are:

$$\frac{dv_1}{dv_2} = \frac{w_1}{w_2} = \frac{6 - 4 - 4 - 12}{- 12 - 0 - 4} = \frac{- 14}{- 16} = \frac{7}{8} ; \tag{3.41}$$

$$\frac{dx_1}{dx_2} = \frac{2}{4} = \frac{1}{2} ; \tag{3.42}$$

$$\frac{dv_1}{dx_1} = \frac{w_1}{v_1} = \frac{-14}{2} = -7 ; \tag{3.42}$$

$$\frac{dv_2}{dx_2} = \frac{w_2}{v_2} = \frac{-16}{4} = -4 \quad . \tag{3.44}$$

In this case all the initial slopes are well defined; that is, the values are finite. Starting with a small change in v_1, as Δv_1, then from equation (3.43) the change in x_1 as Δx_1 is determined from the initial slope of -7. For a given Δv_1, there is a corresponding Δv_2 from equation (3.41). This Δv_2 can then be used to determine Δx_2 by equation (3.44). The changes in all four coordinates have been determined by the use of only three of the four equations. This leaves the fourth, or equation (3.42) in this case, as a check, since $\Delta x_1/\Delta x_2$ is now known. As a check of the initial conditions, the following relation can be obtained:

$$\frac{dx_1}{dx_2} = \frac{dx_1/dv_1}{dx_2/dv_2} \left(\frac{dv_1}{dv_2}\right) . \tag{3.45}$$

For this example, this becomes:

$$\frac{dx_1}{dx_2} = \frac{-1/7}{-1/4} \left(\frac{7}{8}\right) = \frac{1}{2} \tag{3.46}$$

which checks equation (3.42). It should be pointed out that there are changes in all the variables with this change of Δv_1, and that the slopes should be taken at the middle of the interval involving Δv_1 if small increments are not possible. The complication of this type of procedure may make it desirable to use small values of Δv_1.

The procedure then becomes a process of selecting values of v_{1o} such as Δv_{1o}, and computing the changes in the other variables as Δx_{1o}, Δv_{2o} and Δx_{2o}. These changes added to the original values of v_{1o}, x_{1o}, v_{2o} and x_{2o} determine new values of the variables as v_{11}, x_{11}, v_{21} and x_{21}. These new values are used to determine the slopes at the new points; then another change in v_1 is assumed as Δv_{11}, and the process is repeated. The result is eight phase-plane plots of which only four or two are usually desired. These two plots desired are usually the $v_2 - x_2$ and $v_1 - x_1$ plot. From these the $v_2 - t$, $x_2 - t$, $v_1 - t$ and $x_1 - t$ are obtained, which is generally the solution desired.

This section has shown how the phase-plane method may be applied to solve differential equations whether they are linear or nonlinear. Also the application of this method to solve simultaneous nonlinear differential equations has been shown. Because of the large number of computations involved in the phase-plane method, it has been practical only recently. The use of the digital computer is almost a necessity in the solution of a problem using the phase-plane method.

Solution of the Transducer Equation by the Phase-Space Method

The two particular nonlinear differential equations that are used to describe the response of the electromagnetic transducer were developed in Chapter II. These two nonlinear differential equations are repeated here for convenience. Equation (3.47) was derived from the electrical system, and equation (3.48) was derived from the mechanical system.

$$\frac{d\phi}{dt} + \frac{R\ (r + \sigma x)}{N^2} \left[\frac{\phi}{1 - b\phi}\right] - \frac{E}{N} = 0\ . \tag{3.47}$$

$$\frac{d^2x}{dt^2} + \left[\frac{h^2}{M}\right] \frac{dx}{dt} + \frac{k\ (x - G)}{M} - \frac{P_o}{M} + \frac{K\phi^2}{M} = 0. \tag{3.48}$$

The desired solutions are the coil current (i) and the armature position (x) as functions of time (t). To determine the coil current (i) as a function of time (t), a third equation is necessary and is shown in equation (3.49).

$$i = \frac{r + \sigma x}{N} \left[\frac{\emptyset}{1 - b\emptyset}\right] . \tag{3.49}$$

To simplify the writing of the equations the following notations will be used:

$$\ddot{\emptyset} = d^2\emptyset/dt^2$$

$$\dot{\emptyset} = d\emptyset/dt$$

$$\ddot{x} = d^2x/dt^2$$

$$\dot{x} = dx/dt$$

$$h_1 = h^2/M$$

$$k_1 = k/M$$

$$K_1 = K/M$$

$$H_1 = (P_o + kG)/M.$$

With these notations equations (3.47) and (3.48) become

$$\dot{\emptyset} = \frac{E}{N} - \frac{R\,(r + \sigma x)}{N^2} \left[\frac{\emptyset}{1 - b\emptyset}\right] \tag{3.50}$$

and

$$\ddot{x} = - h_1\dot{x} - k_1 x - K_1\emptyset^2 + H_1 . \tag{3.51}$$

In the use of the phase-plane method for simultaneous differential equations it is necessary that the equations be of the same order. Since equation (3.50) is of the first-order and equation (3.51) is of the second-order, then the time derivative must be taken of equation (3.50) giving equation (3.52).

$$\ddot{\emptyset} = - \frac{R}{N^2} \left[\frac{(r + \sigma x)\,\dot{\emptyset}}{(1 - b\emptyset)^2} + \frac{\sigma\emptyset}{1 - b\emptyset}\,\dot{x}\right] . \tag{3.52}$$

There are four phase-plane equations needed and these are:

$$\frac{d\dot{\phi}}{d\phi} = \frac{\ddot{\phi}}{\dot{\phi}} = -\frac{R}{N^2}\left[\frac{(r+\sigma x)}{(1-b\phi)^2} + \left(\frac{\sigma\phi}{1-b\phi}\right)\frac{\dot{x}}{\dot{\phi}}\right], \tag{3.53}$$

$$\frac{d\dot{\phi}}{d\dot{x}} = \frac{\ddot{\phi}}{\ddot{x}} = \frac{-\frac{R}{N^2}\left[\frac{(r+\sigma x)}{(1-b\phi)^2}\dot{\phi} + \left(\frac{\sigma\phi}{1-b\phi}\right)\dot{x}\right]}{-h_1\dot{x} - k_1 x - K_1\phi^2 + H_1}, \tag{3.54}$$

$$\frac{d\dot{x}}{dx} = \frac{\ddot{x}}{\dot{x}} = -h_1 - \frac{k_1 x}{\dot{x}} - \frac{K_1\phi^2}{\dot{x}} + \frac{H_1}{\dot{x}}, \tag{3.55}$$

$$\frac{d\phi}{dx} = \frac{\dot{\phi}}{\dot{x}} \quad . \tag{3.56}$$

The initial conditions for this set of equations are $\ddot{x} = 0$, $\dot{x} = 0$, $x = G$, $\ddot{\phi} = RE\,(r+\sigma G)/N^3$, $\dot{\phi} = E/N$ and ϕ = residual flux (W). These initial conditions determine the initial slopes of the phase-plane curves. Equation (3.53) determines the slope of the curve in the $\dot{\phi} - \phi$ plane, equation (3.54) determines the slope of the curve in the $\dot{\phi} - \dot{x}$ plane, equation (3.55) determines the slope of the curve in the $\dot{x} - x$ plane, and equation (3.56) determines the slope of the curve in the $\phi - x$ plane. The flux (ϕ) is assumed to be the independent variable, and increments of ϕ such as $\Delta\phi$ will be assumed. Since the ultimate result is to determine the variables as functions of time (t), then the time (t) must be computed. In this case the value of t may be computed from the changes in flux (ϕ) or the changes in armature displacement (x). The appropriate equations to use are given as follows:

$$t = \int_{x_1}^{x_2} \frac{1}{\dot{x}}\, dx \tag{3.58}$$

or

$$t = \int_{\phi_1}^{\phi_2} \frac{1}{\dot{\phi}} \, d\phi \quad . \tag{3.59}$$

For every increment of flux ($\Delta\phi$) there is a change of Δx in x, $\Delta\dot{x}$ in $\dot{x}$, $\Delta\dot{\phi}$ in $\dot{\phi}$, and Δt in t, so usually the change Δt is computed for each increment and then the value of time (t) becomes the sum of the Δt's.

In the solution of these simultaneous nonlinear differential equations by this method the time interval must be divided into three parts. This division is necessary because of the types of retraints existing on the mechanical system. The restraints are in the form of limited travel of the armature and an initial force holding the armature against a stop. These three intervals consist of the time the armature remains stationary after the step voltage has been applied to the coil, the time required for the armature to move from the backstop to the residual stop, and the time after the armature closes until the current reaches its final value.

The first interval exists until the magnetic pull becomes equal to the back tension. During this interval the following conditions exist: $x = G$, $\dot{x} = 0$, and $\ddot{x} = 0$. The end of the second interval occurs when the value of x has decreased to a value equal to the residual gap. During this second interval both x and ϕ are changing. The third interval exists until the flux has reached the final value. This is characterized by the following conditions: $x = D$, $\dot{x} = 0$, and $\ddot{x} = 0$.

The solutions of the four phase-plane equations (3.53) through (3.56) results in obtaining numerically x and ϕ as functions of t. The numerical solution was obtained by writing a program for the IBM 650 digital computer. The program was written in the Fortran language because of the close similiarity to the language of mathematics. Figure 3.5 shows a flow chart which gives the basic steps used in setting up the program. A listing of the program in Fortran language is given in Appendix A. The resulting machine language program compiled from the Fortran program was used with numerical data to check the accuracy of the program and to obtain some data showing the effect of changing some of the parameters. The details of setting up equation (3.53) through (3.56) in the proper form, with the correct numerical coefficients for the set of units selected and the special notation that was used in the Fortran program, is given next. The validity of the mathematical solution is shown by comparing the numerical solution with the experimental results. This is given in the following chapters.

Development of the Equations to Program the Solution on the IBM 650 Digital Computer

Measurements made in the use of the electromechanical transducer, and especially the relay, involve a mixed set of units. Distances are measured in inches or thousandths of an inch, current is measured in amperes; force in grams and emf in volts. Therefore, it was necessary to select a consistent set of units. The set that most nearly seemed to represent the majority of the units used is the English or foot-pound second system. Therefore equations (3.53) through (3.56) must be modified to take into account the units. The equations in final form are the following:

$$\frac{d\dot{\phi}}{d\phi} = \frac{\ddot{\phi}}{\dot{\phi}} = -\frac{10^8 R}{N^2}\left[\frac{(r+\sigma x)}{[1-b(\phi-w)]^2} + \frac{\sigma(\phi-w)\dot{x}}{[1-b(\phi-w)]\dot{\phi}}\right] \qquad (3.60)$$

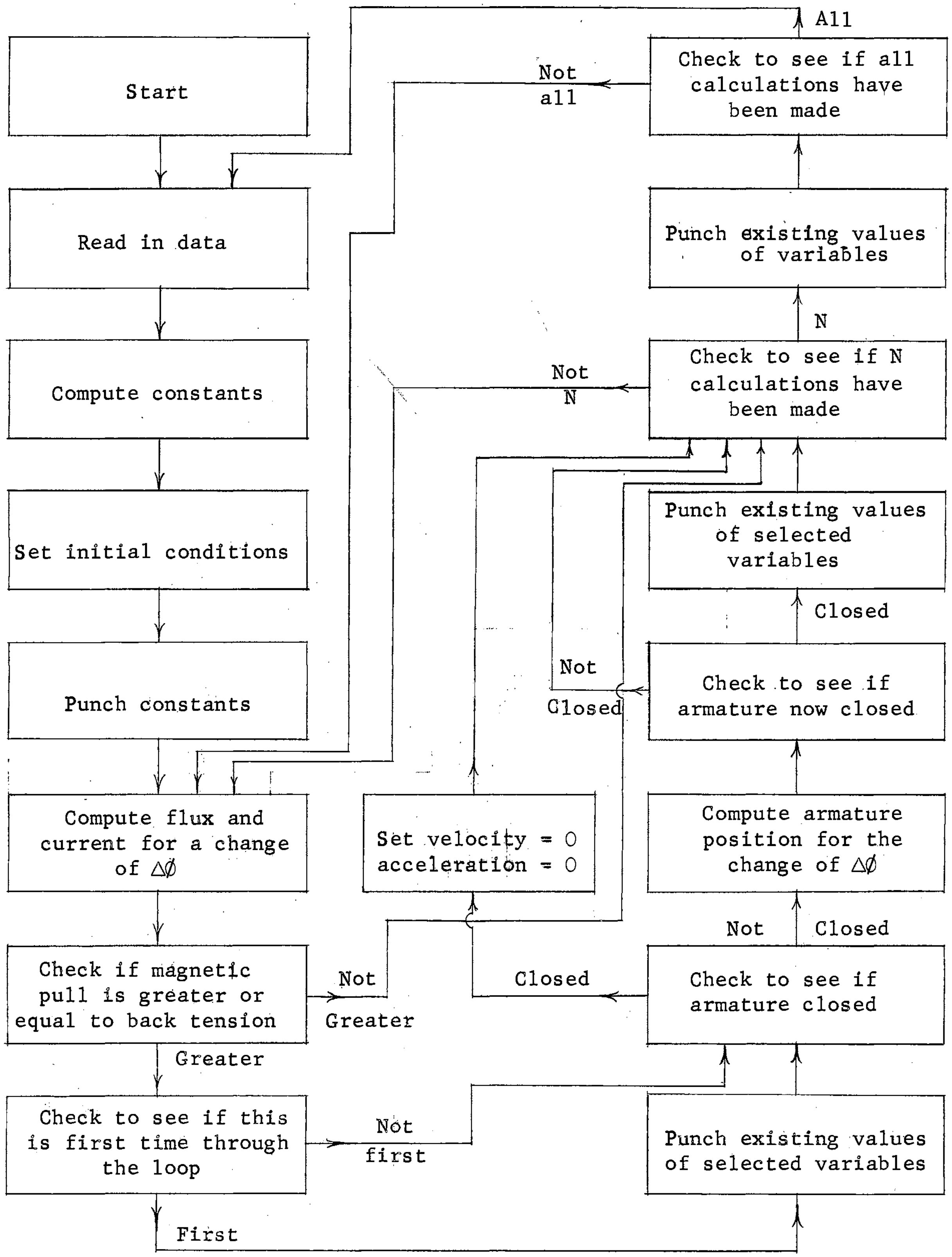


Figure 3.5. Logic Chart of Computer Program

$$\frac{d\dot{\phi}}{d\dot{x}} = \frac{\ddot{\phi}}{\ddot{x}} = \frac{\frac{-10^8 R}{N^2}\left[\frac{(r+\sigma x)\dot{\phi}}{[1-b(\phi-w)]^2} + \frac{\sigma(\phi-w)\dot{x}}{1-b(\phi-w)}\right]}{-h_1\dot{x} - k_1 x - K_1\phi^2 + H_1}, \qquad (3.61)$$

$$\frac{d\dot{x}}{dx} = \frac{\ddot{x}}{\dot{x}} = -h_1 - k_1\frac{x}{\dot{x}} - \frac{K_1\phi^2}{\dot{x}} + \frac{H_1}{\dot{x}}, \qquad (3.62)$$

$$\frac{d\phi}{dx} = \frac{\dot{\phi}}{\dot{x}}, \qquad (3.63)$$

$$\dot{\phi} = \frac{10^8 E}{N} - \frac{10^8 R\,(r+\sigma x)}{N^2}\left[\frac{\phi - w}{1-b(\phi-w)}\right], \qquad (3.64)$$

$$i = \frac{r+\sigma x}{N}\left[\frac{\phi - w}{1-b(\phi-w)}\right]. \qquad (3.65)$$

Where: x = armature displacement in feet

ϕ = air gap flux in maxwells

R = coil circuit resistance in ohms

N = coil turns

$h_1 = h^2/M$

h^2 = effective damping coefficient in pounds per foot per second

M = effective armature mass in slugs

$k_1 = k/M$

k = effective spring constant in pounds per foot

$K_1 = K/M$

K = coefficient of magnetic pull equation and is equal to $8.86 \times 10^{-8}/(\mu A)$ in pounds per square root maxwells

A = cross sectional area of the air gap in square inches

$H_1 = (P_o + kG)/M$

P_o = initial back tension in pounds

G = initial value of armature displacement or open air gap in feet

$r + ai$ = approximation of the reluctance of the iron part of the magnetic circuit over a limited range of values of i

i = coil current through the N turns in amperes

E = supply voltage in volts

$b = a/N$

$\sigma = 12/\mu A$ (12 caused by x being in feet)

μ = permeability of free space and equals 3.19 for English system of units

$\dot{x} = dx/dt$

$\ddot{x} = d^2x/dt^2$

$\dot{\phi} = d\phi dt$

$\ddot{\phi} = d^2\phi/dt^2$

W = value of residual flux in maxwells.

Equations (3.60) through (3.65) are used in the Fortram program to obtain x, ϕ and i as functions of t. To obtain the value of t the following equations were used:

$$\Delta t_j = \Delta\phi_j/\dot{\phi}_j, \quad (j = 0, 1, 2...n) \tag{3.66}$$

$$t = \sum_{j=0}^{N} \Delta t_j \tag{3.67}$$

when $t = 0$, $j = 0$.

As a check during the second interval, the values of Δt_j were also computed as $\Delta x_j/\dot{x}_j$ and printed along with the values computed from equation (3.66). Because of the restrictions in the symbols used in the Fortran program, some of the symbols used in equations (3.60) through

(3.65) had to be changed. There are several constants that are computed in the program and that are used to determine the value of the increment $\Delta\phi$. This required that the maximum value of the flux (ϕ) must be known since flux is selected as the independent variable. This is determined from equation (3.64) by setting $\dot{\phi} = 0$ and $x = D$ and solving of ϕ. The value of the maximum flux (ϕ_m) is given by the following equation:

$$\phi_m = \frac{EN}{R\ (r + \sigma D) + ENb} \qquad (3.68)$$

where all the symbols were defined at the beginning of this section except D. The symbol D is the residual gap or the value of the closed air gap. To keep the armature from sticking to the core when the coil is deenergized, a small non-magnetic shim (called a residual pin) is used resulting in a small air gap when the armature is closed.

The input data necessary for the program to compute the solution consists of eighteen variables. Most of the variables are involved with the transducer parameters except three variables which are used to change the program. The symbols used will be listed with the Fortran first, then the symbols used in the equation, then a brief definition. The eighteen input variables are listed in the order required by the program.

A = A = cross sectional area of the air gap in square inches

HI = h^2 = damping coefficient in lbs/ft/sec.

GR = M = effective armature mass in slugs

SI = k = effective spring constant in lbs/ft

PO = P_o = initial back tension in lbs.

R = R = coil circuit resistance in ohms

TN = N = coil turns

E = E = supply voltage in volts

G = G = initial value of armature displacement or open air gap in ft.

D = D = residual gap in feet

B = b = coefficient in magnetic reluctance equation

Q = r = coefficient in magnetic reluctance equation

W = W = initial value of residual flux in maxwells

V = 10^8 = units constant

UO = μ = permeability of free space. Equals 3.19 in English units

ERR = the number used to keep a check on the calculating error

INC = total number of increments of the flux

NO = number of calculation loops made before punching values.

Those listed below are the symbols used in the program for computation purposes.

CUR = i = coil current

Y = ϕ = flux

X = x = armature position or displacement

C = $\sigma = 12/(\mu A)$

H = $h_1 = h/M$

S = $k_1 = k/M$

P = $K_1 = K/M$

RN = R/N^2

EN = E/N

XD = $\dot{x} = dx/dt$

XDD = $\ddot{x} = d^2x/dt^2$

U = $H_1 = (P_o + kG)/M$

BM = $\phi_m = EN/\left[R(r + \sigma D) + ENb\right]$

DLY = $\Delta\phi = (\phi_m - W)/$ number of increments (INC)

LP = total number of times calculations printed to complete program

YDA = $10^8 E/N$

DI = $r + \sigma x$

DE = $1 - b(\phi - w)$

$$YDB = \frac{10^8 R (r + \sigma x)(\phi - w)}{N^2 [1 - b(\phi - w)]}$$

$$YD = \dot{\phi} = \frac{d\phi}{dt} = \frac{10^8 E}{N} - \frac{10^8 R (r + \sigma x)(\phi - w)}{N^2 [1 - b(\phi - w)]}$$

$$YDDA = \frac{-10^8 R (r + \sigma x)\dot{\phi}}{N^2 [1 - b(\phi - w)]^2}$$

$$YDDB = \frac{-10^8 R\sigma(\phi - w)\dot{x}}{N^2 [1 - b(\phi - w)]}$$

$$YDD = \ddot{\phi} = d^2\phi/dt^2 = -\frac{10^8}{N^2}\left[\frac{(r + \sigma x)\dot{\phi}}{[1 - b(\phi - w)]^2} + \frac{\sigma(\phi - w)\dot{x}}{[1 - b(\phi - w)]}\right]$$

YRA = $\ddot{\phi}/\dot{\phi}$

DLYD = $\Delta\dot{\phi} = (\Delta\phi)\ \ddot{\phi}/\dot{\phi}$

$$YD2 = \dot{\phi} = \sum_{n=0}^{INC} \Delta\dot{\phi}_n + \dot{\phi}_o$$

$$TI = t = \sum_{n=0}^{INC} \Delta t_n$$

$$CUR = i = \frac{(r + \sigma x)(\phi - w)}{N [1 - b(\phi - w)]}$$

BIAS = $P_o - \Delta P$ = value of pull at which armature starts to move

PYY = $K\phi^2/M$ = magnetic pull

$$XDD = \ddot{x} = -h_1\dot{x} - k_1 x - K_1\phi^2 + H_1$$

$$XYRA = \ddot{x}/\ddot{\phi}$$

$$DLXD = \Delta\dot{x} = (\Delta\dot{\phi})\ \ddot{x}/\ddot{\phi}$$

$$XD = \dot{x} = \sum_{n=0}^{INC} \Delta\dot{x}_n$$

$$XRA = \dot{x}/\ddot{x}$$

$$DLX = \Delta x = (\Delta\dot{x})\ \dot{x}/\ddot{x}$$

$$X = x = \sum_{n=0}^{INC} \Delta x_n + G.$$

The program was arranged to punch out the values of the coil current (i), time (t), flux (ϕ), pull, d^2x/dt^2, $d\phi/dt$ and $d^2\phi/dt^2$ when the armature started to move, and the values of coil current (i), time (t), armature position (x), flux (ϕ), pull, dx/dt, d^2x/dt^2, $d\phi/dt$, $d^2\phi/dt^2$, t_1 and t_2 after n loop calculations. Also built into the program were three error-checking procedures that could be used to check the magnitude of an error. These errors were determined by computing a given variable two ways. The first error-checking calculation is based upon computing $\dot{\phi}$ two ways. Because of the nature of this problem, $\dot{\phi}$ was available directly from equation (3.64). Also, $\dot{\phi}$ could be computed by adding the successive value of $\Delta\dot{\phi}$ to the initial value ($\dot{\phi}_o$) of $\dot{\phi}$. A ratio of these two values of $\dot{\phi}$ was then computed and compared to some value such as 1.01 which represents a 1% error. If the computed ratio exceeded the allowable deviation the excess was punched out. The value of $\dot{\phi}$ used in computing the other variables was always computed from the equation, so that there was no error directly caused by summing.

The second error-checking scheme was to use equation (3.63). In this equation the ratio of ϕ to x existing in all intervals was compared to the ratio of $\dot{\phi}$ to $\dot{x}$. Actually the total product $(\Delta\phi)\ \dot{x}/(\Delta x)\ \dot{\phi}$ derived from equation (3.63) should be equal to 1.0 if there is no error. This product then was compared to some allowable deviation. If the product exceeded the allowable value then the excess was punched out. If there was no need to sense these errors, then a large value of ERR could be read in, and no error values would be punched out. The third error-sensing arrangement was to compute Δt two ways. The value of a Δt_1 was computed from $\phi/\dot{\phi}$, while the value of a Δt_2 was computed from $x/\dot{x}$. In this case both values of Δt were punched each time a punch operation occured in the normal loop. Since Δt_2 only existed during the interval when the armature was moving, the comparison is only valid during that interval. The time (t) was computed for the values of Δt_1 since the flux was changing all of the time. A sample of the Frotran program is given in Appendix A.

The actual data obtained from this program is presented in Chapters IV and V which show a comparison of the results obtained by calculation and by experiment. It took the computer, on the average, about ten minutes to compute until the value of the armature displacement became equal to the closed or residual value. To make a complete calculation, of 56 points in this case, the time required was approximately 15 minutes for each set of input data. The program is optimized because of the built-in SOAP program used to obtain the final output program. High speed storage is not used by the program but the three index registers and the floating point unit are.

CHAPTER IV

QUALITATIVE DISCUSSION OF THE EXPERIMENTAL AND COMPUTED DATA

Experimental information was used in two ways in the solution of the problem of determining the transient response of the singly excited electromechanical transducer. A certain amount of experimental data had to be obtained in order to gain insight into and familiarity with the details of the problem. This information was used to determine the method of approach and the conditions involved in the analytical solution. The results of the analytical solution were then studied in relation to the experimental results to evaluate the accuracy of the mathematical model and the procedure used to arrive at the solution.

In addition, in this case, considerable experimental results will be presented to show the feasibility of using the response of one system in evaluating the response of a coupled second system.[1] In most electrical measuring devices this is the basic principle used to obtain some type of results which can be perceived by an individual. For example, the body is incapable of directly sensing the presence of a magnetic field. However, considerable information is obtained about magnetic fields by observing the response of devices that do respond to magnetic fields. In fact, the transient response of a magnetic field is somewhat unknown because of

[1]Cameron, C. F. and D. D. Lingelbach, "Oscilloscopic Analysis of Relay Performance", Automatic Control, (New York, 1958), pp. 10-11.

the inability to make reasonable direct measurements of the field under transient conditions. Just recently new devices have been made which have possibilities of studying transient magnetic fields by using a different principle. As indicated in Chapter I, the scope of this thesis has been limited to the response of a singly excited electromechanical transducer to a step input of voltage. However, the technique presented in this thesis can be applied to other conditions by using certain modifications.

Transient Coil Current Build-up

The transient coil current build-up is the instantaneous value of the coil current from the instant the coil is energized until the current reaches it steady state value, or until another switching action takes place in the coil circuit.

Inspection of equations (2.39) and (2.40) indicates that a change in any of the terms in these two equations would give some change in the instantaneous value of the coil current. It is possible that a change in some of the terms or variables would give less change than others in the coil current. Likewise, the condition of any one variable to cause a change in the coil current is based upon the values of the other variables existing at any given time. This condition is caused by the nonlinearity of the transducer. Experimentally, there are some variables that may be changed more conveniently than others. These will be varied, and the effect on the coil current will be observed. The details of the experimental setup are presented in Appendix B. In general, the coil current was obtained by a photographic record being made of the trace presented by the electron beam on the face of a cathode ray tube. Since only transient conditions were being recorded, the shutter of the camera was held open while the electron beam was swept across the face of the tube. In the case of simultaneous traces, the two beams of the dual oscilloscope were used. The use of simultaneous traces are a necessity for comparison purposes because of the difficulty of obtaining consistency in the transient operation of a mechanical device. With simultaneous traces there is no question as to whether each one existed under identical conditions.

The variables which are convenient to change are the initial value of the spring force called back tension, the open value of the armature called air gap, the closed value of the armature called residual gap, applied voltage, the coil circuit resistance, the constant of the spring, and a separate circuit electromagnetically coupled to the coil circuit called a slug or sleeve.[2]

The influence of a change in the back tension on the transient coil current build-up is shown by the traces in Figures 4.1 and 4.2. In order to explain the changes that take place in the transient coil current build-up with a change in the variables, it is desirable to take the general shape of the traces in Figure 4.1 and define some particular prints. The points to be primarily used in future discussions are located on a sketch in Figure 4.3 showing the general shape of the coil current.

[2]Cameron, C. F., D. D. Lingelbach and Douglas Jeng, "Transient Analysis of Relays with Slugs and Sleeves," Sixth Symposium on Electromagnetic Relays, (Princeton, Indiana, 1958). pp. 90-93.

Traces:

(a, b, c, d) Coil Current build-up

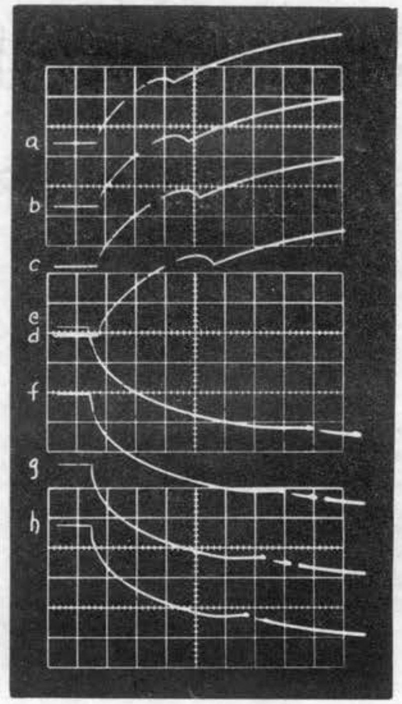
(a) Back tension 40 grams

(b) Back tension 60 grams

(c) Back tension 90 grams

(d) Back tension 140 grams

(e, f, g, h) Coil current decay



Oscillogram Data

Time scale: 5 milliseconds per division (horizontal)

Current scale: 19.75 milliamperes per division (vertical)

Turns: 5840

Air gap: 0.027 inches

Coil circuit resistance: 467 ohms

Voltage: 36 volts dc

Figure 4.1. Coil Current Build-up and Decay for Variable Back Tension

Traces:

(a, b, c, d) Coil current build-up

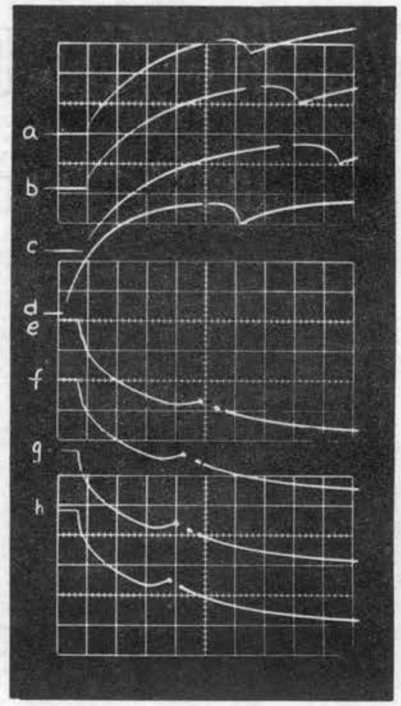
(a) Back tension 195 grams

(b) Back tension 230 grams

(c) Back tension 255 grams

(d) Back tension 320 grams

(e, f, g, h) Coil current decay



Oscillogram Data

Time scale: 5 milliseconds per division except trace (d) which is 10 milliseconds (horizontal)

Current scale: 19.75 milliamperes per division (vertical)

Turns: 5840

Air gap: 0.027 inches

Coil circuit resistance: 467 ohms

Voltage: 36 volts dc

Figure 4.2. Coil Current Build-up and Decay for Variable Back Tension

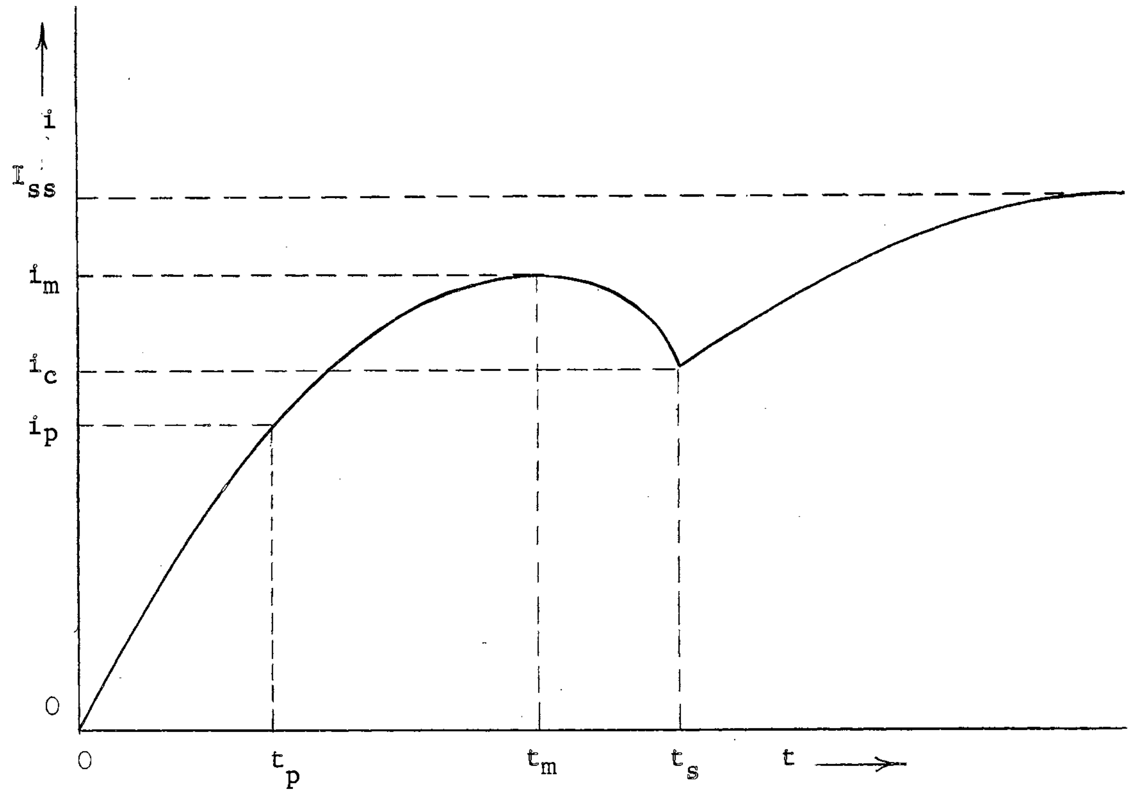


Figure 4.3

There are usually four values of the coil current (i) that are of use in describing the transient coil current build-up trace. Associated with these four current values are three finite time values. Commencing with the lowest current value marked is the value called i_p. This is defined as the pick-up value of current and is that value of current at which the magnetic pull becomes just equal to the total force holding the armature open. The next current point is i_c and is the value of the current at the lowest point on what is called the cusp in the current trace. The point i_m is next and is the value of the current at which the time rate of change of the current is zero. The final current value marked is the steady state value of the current and is noted as I_{ss}.

The three time values generally used are: first, t_p which is called the pick-up time and is defined as the time interval from the instant the coil is energized to the instant the magnetic pull becomes equal to the restoring force on the armature; the second time value is t_m and is the time at which the time rate of change of the current is zero; the last time value is t_s and is called the seating time. It is at this point (t_s) that the armature reaches the end of its travel as it closes. This point is determined experimentally be observing simultaneously the transient coil current and the armature motion.

Careful examination of the traces in Figure 4.1 and 4.2 will reveal that certain conditions exist in the trace for a change in the value of the back tension. At the point where $t = 0$, and $i = 0$, it will be found that the time rate of change of the coil current is the same for all values of back tension. In addition, the shape of the instantaneous value of the coil current is the same for all values of back tension for values of the coil current less than the smallest value of pick-up current in the group.

Increases in the back tension increase the value of i_p and the corresponding t_p. The values of t_p or i_p have been indicated by causing a blanking pulse to blank the electron beam at the instant the armature starts to move. The beginning of the blank in the coil current trace is caused by the breaking of a contact mounted on the armature so that movement of the armature breaks an electrical circuit, causing a blanking pulse. Therefore, the values of i_p and t_p exist at the beginning of the blank.

Increasing back tension also increases the time interval of t_s minus t_p. This time interval is referred to as the armature travel time or transit time. The time interval t_s minus t_m also increases with an increase in the back tension.

Associated with the increases in the time intervals are the increases in the current. Increasing the back tension increases the current increment of i_m minus i_c. The reason for presenting time intervals and current increments in describing the changes caused by changing a variable is to help differentiate between the types of changes. For example, if only the change in the magnitude of i_c were used to describe the effect of a change in back tension, one would not know whether the change was only a reflection of the change in i_p or a change directly related to back tension. Actually, not only should the magnitudes of the current and time be examined, but the time rate of change of the current should be checked. However, if small enough time increments were involved, then the data would give essentially the same results as examining the slope of the trace.

Figure 4.4 shows a plot of the calculated coil current build-up for a transducer in which the back tension was varied. Only the plots of coil current for three values of back tension are shown. Comparison of Figure 4.4 with Figure 4.1 shows that the computed results give the same changes as those exhibited experimentally. The relays used in obtaining the data for Figures 4.1 and 4.4 are not the same. In selecting the values of the parameters to compute the response, representative values were used whenever they were known. In some cases, such as the magnetic circuit parameters, the values had to be calculated since the exact relation between the flux and the magnetomotive force was not known. Also, a simple series magnetic circuit was used in the mathematical model. This kept the computer program from being more

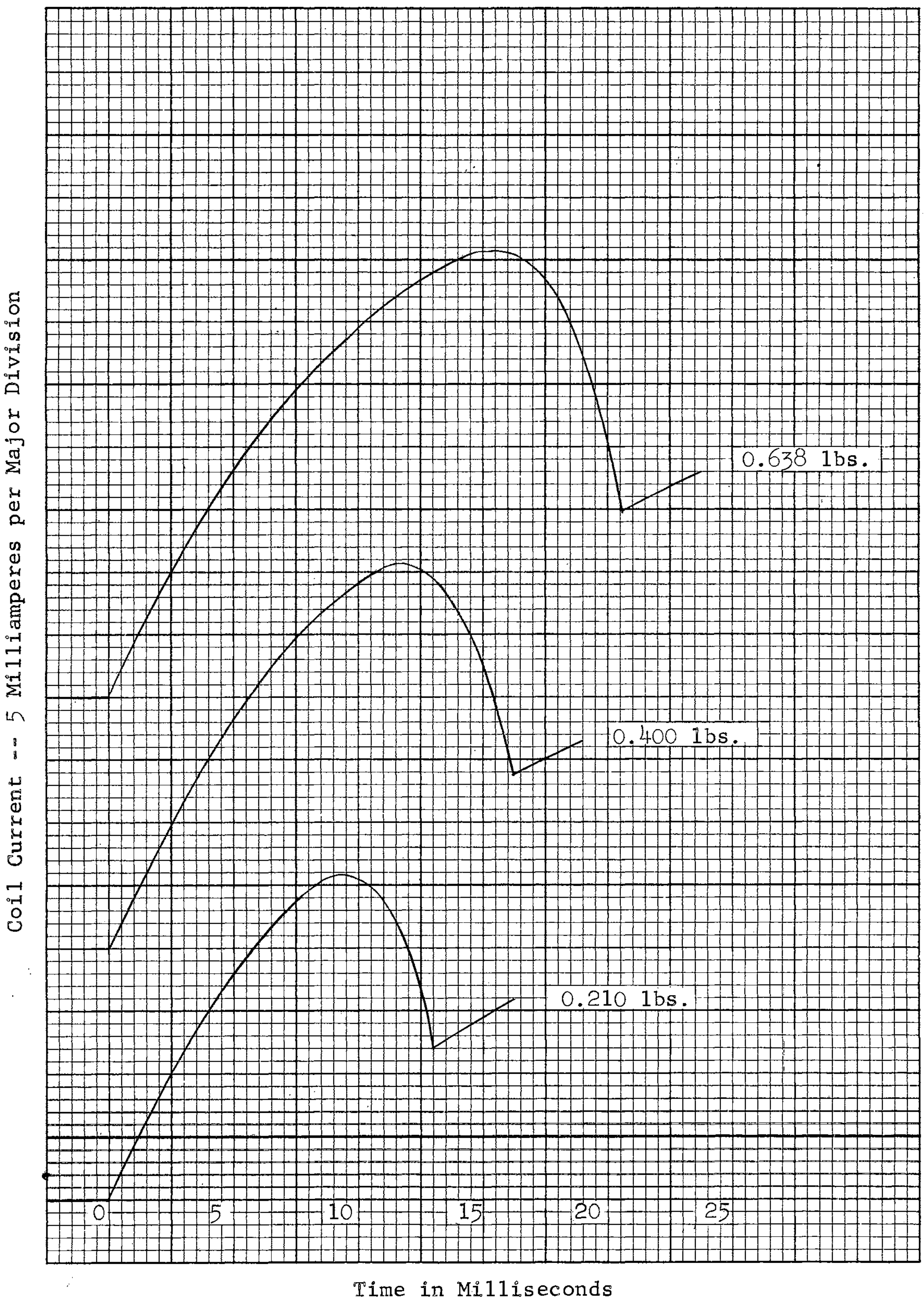
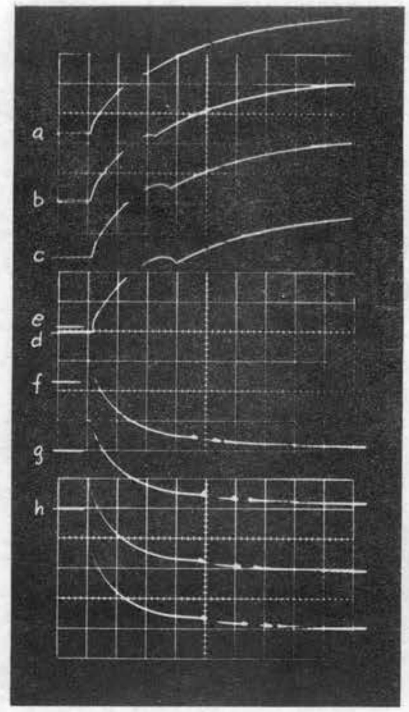


Figure 4.4. Computed Coil Current Build-up for Three Values of Back Tension

complicated than necessary to obtain the desired results in the time available. There are a number of places where additional refinements in the computer program would be necessary in order to be able to obtain a given response. The objective here was to show that a reasonable solution could be obtained by this method. If the solution obtained showed the same trends as the experimental results, then it was considered to be entirely satisfactory since definite limitations existed in the mathematical model used. One definite advantage of this type of solution is that added refinements can be made without having to completely revise the procedure.

To observe the effect of air gap on the coil current build-up, the traces presented in Figure 4.5 and 4.6 were obtained. Each coil current build-up trace was taken for a different value of the air gap and with constant back tension. Careful inspection will show that the slope of the trace at $t = 0$ is different for each of the values of air gap. The slope at $t = 0$ should increase with an increase in air gap. Because of the switching problem and coupling factors, it is difficult experimentally to obtain a well defined trace at $t = 0$. Therefore it is desirable to observe the change caused by changing the air gap by selecting some value of time between the values of zero and t_p and determining the current. By using the same value of time for all the traces under consideration, a comparison can be made from the values of coil current (existing at that value of time) to detect the change in air gap. If the slope is greater, then the current at a fixed time would also be greater. This difference in slope is about the only significant thing that can be used to distinguish between the changes caused by changing back tension, and those caused by changing air gap.



Traces:

(a, b, c, d) Coil current build-up

(a) Air gap: 0.008 inches

(b) Air gap: 0.013 inches

(c) Air gap: 0.018 inches

(d) Air gap: 0.023 inches

(e, f, g, h) Coil current decay

Oscillogram Data

Time scale: 5 milliseconds per division (horizontal)

Current scale: 16 milliamperes per division (vertical)

Back tension: 65 grams

Turns: 5840

Voltage: 36 volts dc

Coil circuit resistance: 562 ohms

Figure 4.5. Coil Current Build-up and Decay for Variable Air Gap

Traces:

(a, b, c, d) Coil current build-up

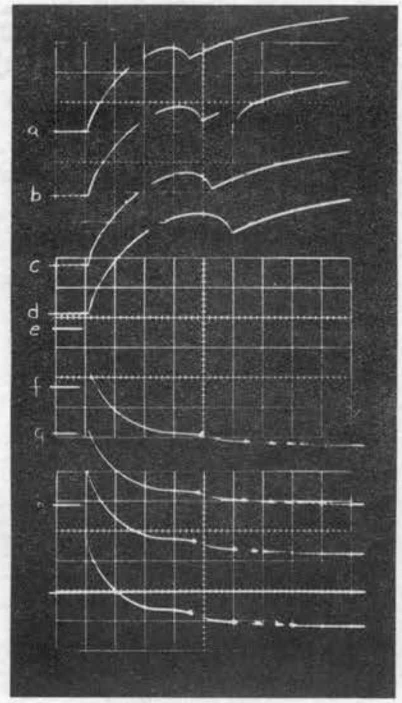
(a) Air gap: 0.028 inches

(b) Air gap: 0.033 inches

(c) Air gap: 0.038 inches

(d) Air gap: 0.043 inches

(e, f, g, h) Coil current decay



Oscillogram Data

Time scale: 5 milliseconds per division (horizontal)

Current scale: 16 milliamperes per division (vertical)

Back tension: 65 grams

Turns: 5840

Voltage: 36 volts dc

Coil circuit resistance: 562 ohms

Figure 4.6. Coil Current Build-up and Decay for Variable Air Gap

The data taken to determine whether the change is one of air gap or of back tension is obtained during the time interval that the armature is not moving. This means that the value of x is G and that $\dot{x}$ and $\ddot{x}$ are zero. If magnetic saturation can be neglected, then it is possible to obtain approximate relations that can be used to predict the pick-up time if certain parameters are known. The parameters which determine the pick-up time are the supply voltage (E), the coil circuit resistance (R), the coil turns (N), the pick-up flux (ϕ_p) and the magnetic circuit reluctance ($\mathcal{R}_p$). It was shown in a previous chapter that the magnetic pull is proportional to the square of the magnetic flux in the air gap. Also the reluctance ($\mathcal{R}_p$) is determined, in part, by the length of the air gap. With these conditions existing, then the following voltage equation may be written for the transducer coil circuit.

$$E = iR + J\,\frac{di}{dt};\ i \leq i_p\ . \qquad (4.1)$$

The solution to equation (4.1) for the coil current (i) is

$$i = \frac{E}{R}\ (1 - \epsilon^{-Rt/J});\quad i \leq i_p\ . \qquad (4.2)$$

Solving equation (4.2) for the time (t) gives

$$t = \frac{J}{R}\ \ln\ \frac{I_{ss}}{I_{ss} - i}\ ;\quad t \leq t_p\ , \qquad (4.3)$$

where $I_{ss} = E/R$ = steady state current

J = effective inductance of the coil when the armature is open.

The effective inductance J can be shown from its definition to be equal to $N^2/\mathcal{R}$. Substituting this relation into equation (4.3) gives

$$t = \frac{N^2}{\mathcal{R}R}\ \ln\ \frac{I_{ss}}{I_{ss} - i}\ ;\qquad t \leq t_p\ . \qquad (4.4)$$

Equation (4.4) shows that for a given value of i, the time (t) varies inversely with the reluctance, or conversely that for a fixed value of t the current i varies directly with the $\mathcal{R}$. Therefore, in the examination of the coil current build-up traces it is possible to detect the changes in the air gap by comparing the instantaneous values of current at the same value of time (t). Attention should be called to the fact that equations (4.1) through (4.4) are accurate for $t \leq t_p$ and $i \leq i_p$.

Equation (4.4) can be rearranged so that the magnitude of the reluctance $\mathcal{R}$ may be calculated by obtaining a pair of values of time (t) and the corresponding value of current (i) from a transient coil current build-up trace. If this pair of values is (t_1, i_1) then the reluctance $\mathcal{R}_1$ can be calculated from equation (4.5) as

$$\mathcal{R}_1 = \frac{N^2}{Rt_1} \ln \frac{I_{ss}}{I_{ss} - i_1} \quad ; \quad i_1 \leq i_p \quad . \tag{4.5}$$

When the value of $\mathcal{R}_1$ has been determined, the value of flux existing at this time can be obtained from equation (4.6).

$$\phi_1 = i_1 \frac{N}{\mathcal{R}_1} \; ; \quad i_1 \leq i_p \; . \tag{4.6}$$

If the pair of values of current and time are selected as the pick-up values then the pick-up flux ϕ_p and the reluctance $\mathcal{R}_p$ are determined when using equations (4.5) and (4.6)

By substituting equation (4.6) into equation (4.4) a relation is obtained which shows the effect of varying the back tension upon the pick-up time t_p. This relation is given in equation (4.7).

$$t_p = \frac{N^2}{R\mathcal{R}_p} \ln \frac{NI_{ss}}{NI_{ss} - \phi_p\mathcal{R}_p} \tag{4.7}$$

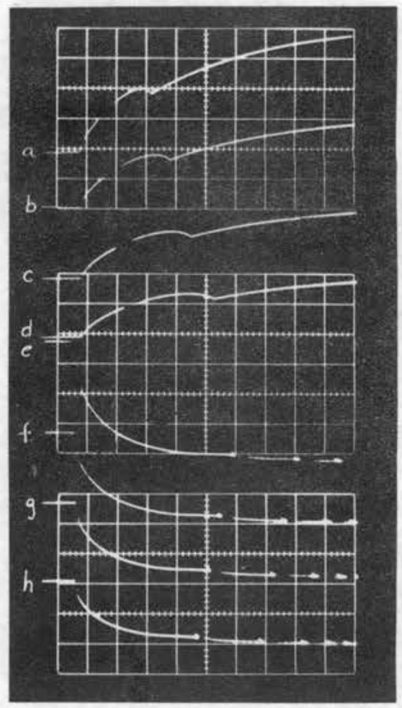
Since $\emptyset_p$ varies as the square root of the back tension, increasing back tension will increase the value of $\emptyset_p$. From equation (4.7) it is seen that an increase in the value of $\emptyset_p$ will increase the pick-up time (t_p) in a logarithmic manner.

With fixed values of the internal parameters such as back tension and air gap, it is still possible to change the transient coil current build-up by changing the external conditions. The magnitude of the supply voltage may be changed with constant coil circuit resistance, or both the resistance and voltage may be changed or only the resistance changed. These three types of input conditions influence the transient response differently.

The influence of different values of supply voltage on the transient coil current build-up is shown by the trace in Figures 4.7 and 4.8. Some definite changes in the shape of the trace may be noted. The slope of the trace at t = 0 increases with an increase in the voltage. This result can be predicted by taking the derivative of equation (4.2) with respect to time. This results in the following equation.

$$\frac{di}{dt} = \frac{E}{J}\epsilon^{\frac{-Rt}{J}} \quad ; \quad t \leq t_p \quad . \tag{4.8}$$

At t = 0 the slope of the transient coil current build-up trace is E/J. This shows that the slope varies directly with E. Since di/dt at t = 0 is greater with a larger E, then the time required for the coil current to reach a given value decreases. This is shown by the time interval from zero to the beginning of the blanking pulse. In addition, the time required for the armature to move from the open to the closed position increases with a decrease in the applied voltage.



Traces:

(a, b, c, d) Coil current build-up

(a) Voltage: 36.0 volts dc

(b) Voltage: 26.7 volts dc

(c) Voltage: 21.2 volts dc

(d) Voltage: 17.7 volts dc

(e, f, g, h) Coil current decay

Oscillogram Data

Time scale: 5 milliseconds per division (horizontal)

Current scale: 19.5 milliamperes per division (vertical)

Turns: 5840

Back tension: 40 grams

Coil circuit resistance: 467 ohms

Air gap: 0.027 inches

Figure 4.7. Coil Current Build-up and Decay for Variable Voltage

Traces:

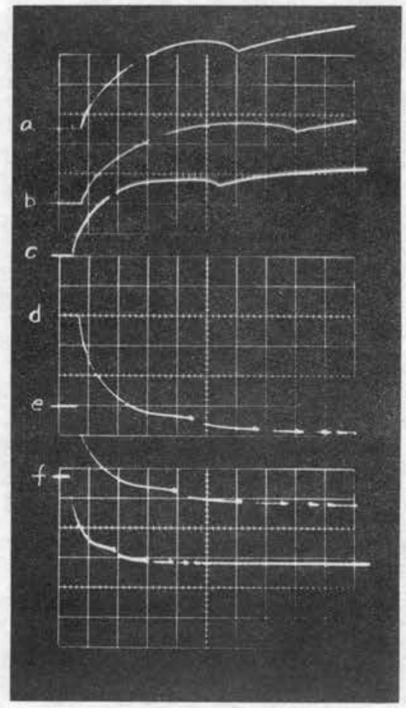
(a, b, c) Coil current build-up

(a) Voltage: 15.0 volts dc

(b) Voltage: 12.5 volts dc

(c) Voltage: 10.8 volts dc

(d, e, f) Coil current decay



Oscillogram Data

Time scale: 5 milliseconds per division (horizontal) except trace (c) which is 10 milliseconds

Current scale: 8.25 milliamperes per division (vertical)

Turns: 5840

Back tension: 40 grams

Coil circuit resistance: 467 ohms

Air gap: 0.027 inches

Figure 4.8. Coil Current Build-up and Decay for Variable Voltage

The effect of changing the coil circuit resistance on the transient coil current build-up is shown by the traces in Figures 4.9 and 4.10. In this case there is not change in the initial slope of the coil current build-up. This is indicated by equation (4.8) since at $t = 0$, R does not appear. Inspection of the traces will show that the pick-up time (t_p) increases with an increase in the coil circuit resistance. Rearrangement of equation (4.4) by substituting $E/R = I_{ss}$ will give equation (4.9):

$$t = \frac{N^2}{R\mathcal{R}} \ln \frac{E}{E - iR} \quad ; \quad i \leq i_p. \tag{4.9}$$

Since the variable R occurs in both factors of equation (4.9) it is awkward to see the effect of changing R. In one factor, increasing R decreases the value of the factor while in the other factor the opposite change occurs. At first thought it might appear that there is a finite value of R that results in the pick-up time (t_p) being a minimum. However, taking the derivative of equation (4.9) with respect to R, and setting that equal to zero, shows this is not true.

The armature travel time also increases with an increase in the coil circuit resistance. Inspection of the traces in Figures 4.9 and 4.10 shows that the increase in the pick-up time is smaller than the increase in the armature travel time.

If the supply voltage (E) and the coil circuit resistance (R) are changed together such that the ratio remains constant, certain unique things result. By definition the steady state current (I_{ss}) is fixed. The traces in Figure 4.11 show the result of changing E and R together but keeping their ratio constant. Examination of the traces shows that the slope of the trace at $t = 0$ increases with an increase in the supply voltage. A check with equation (4.8) shows that at $t = 0$ the value of di/dt depends

Traces:

(a, b, c, d) Coil current build-up

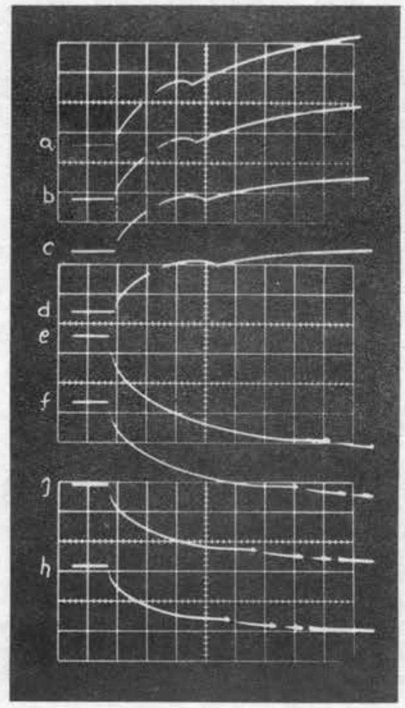
(a) Coil circuit resistance: 467 ohms

(b) Coil circuit resistance: 564 ohms

(c) Coil circuit resistance: 735 ohms

(d) Coil circuit resistance: 880 ohms

(e, f, g, h) Coil current decay



Oscillogram Data

Time scale: 5 milliseconds per division (horizontal)

Current scale: 19.75 milliamperes per division (vertical)

Air gap: 0.027 inches

Back tension: 40 grams

Voltage: 36 volts dc

Turns 5840

Figure 4.9. Coil Current Build-up and Decay for Variable Coil Circuit Resistance

Traces:

(a, b, c, d) Coil current build-up

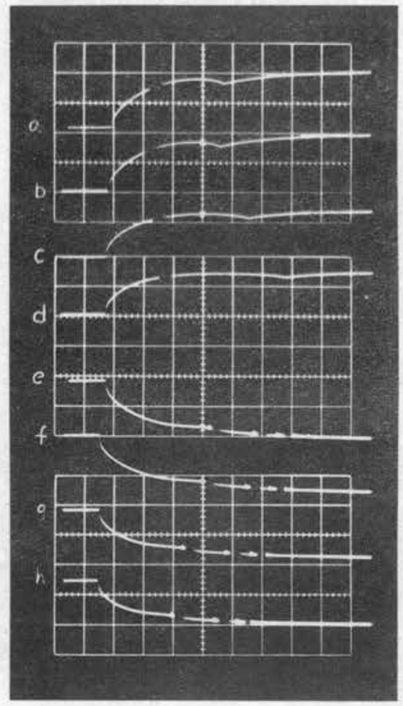
(a) Coil circuit resistance: 1000 ohms

(b) Coil circuit resistance: 1000 ohms

(c) Coil circuit resistance: 1220 ohms

(d) Coil circuit resistance: 1364 ohms

(e, f, g, h) Coil current decay



Oscillogram Data

Time scale: 5 milliseconds per division (horizontal)

Current scale: 19.75 milliamperes per division (vertical)

Air gap: 0.027 inches

Back tension: 40 grams

Voltage: 36 volts dc

Turns: 5840

Figure 4.10. Coil Current Build-up and Decay for Variable Coil Circuit Resistance

Traces: Coil current build-up

(a) Voltage: 24.6 volts

Coil circuit resistance: 214 ohms

(b) Voltage: 49.2 volts

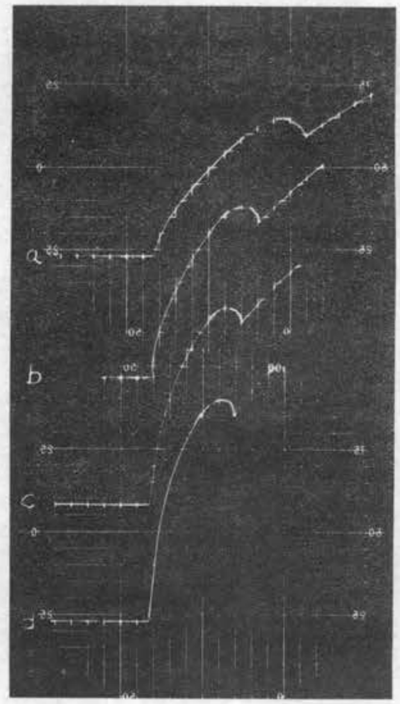
Coil circuit resistance: 428 ohms

(c) Voltage: 70.0 volts

Coil circuit resistance: 610 ohms

(d) Voltage: 95 volts

Coil circuit resistance: 825 ohms



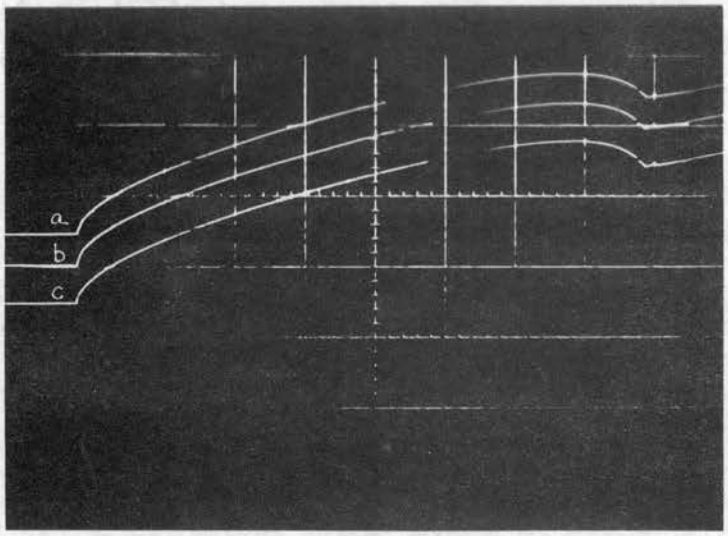
Time scale: 1 millisecond per small division (horizontal)

Current scale: 5.7 milliamperes per small division (vertical)

Figure 4.11. Coil Current Build-up for Constant Steady State Current but Variable Voltage and Resistance

only on E and J and not R. Therefore, increasing E increases the slope. The rate of change of i with time increases with an increase in E; therefore, the pick-up time (t_p) decreases with an increase in E. Since increasing E decreases the pick-up time, and increasing R increases the pick-up time, the result of increasing both E and R together reduces the effect of each. Examination of equation (4.4) shows that the pick-up time varies inversely with the coil circuit resistance since I_{ss} is constant. The traces in Figure 4.11 also show that the armature travel time decreases with an increase in the coil circuit resistance when both E and R are increased together. However, the change in the armature travel time is less than the change in the pick-up time when both E and R are changed.

The parameters discussed previously have been those that are either part of the electrical system or part of the mechanical system that affects the electrical system before the armature moves. There are two parameters, the mass and the spring constant, which influence the response only when the armature is moving. Actually there is a third parameter, the damping, which is involved with the mechanical system, but this one is difficult to change. Experimentally these two parameters are hard to change without changing another parameter. The traces shown in Figure 4.12 show the effect of changing only the spring constant. Because the spring constant does not become effective until the armature moves, the pick-up time does not change with a change in the spring constant. Experimentally, it is difficult to change springs without changing the back tension even when the back tension is adjusted to be constant. Measurement of the back tension by the standard hand type gram gauge is not very accurate.



Traces: Coil current build-up

(a) Spring constant 55.8 lbs/ft.

(b) Spring constant 25.1 lbs/ft.

(c) Spring constant 10.0 lbs/ft.

Oscillogram Data:

Time scale: 2 milliseconds per division (horizontal)

Pick-up current: 10.4 milliamperes

Steady state current: 29 milliamperes

Supply voltage: 36 volts dc

Figure 4.12. Coil Current Build-up for Variable Spring Constant

The armature travel time does change with the spring constant. An increase in the spring constant increases the armature travel time. However, the spring constant times the total armature displacement must be of the same order of magnitude as the back tension before the spring constant gives much change. It will be shown in the next chapter that, even with a spring constant of zero, the armature travel time is not zero. The effect of a change in the spring constant on the transient coil current build-up is shown by the traces in Figure 4.12. The shape of the cusp is the primary change associated with a change in the spring constants. The computed effect of a change in the spring constant, from the computer program, is shown by the curves in Figure 4.13. Figure 4.13 shows the computed transient coil current build-up and the computed armature motion for two values of the spring constant. It was considered desirable in this case to show both the coil current build-up and the armature motion, because of the peculiar shape of the coil current when the spring constant was large. The curves shown by the solid lines are for the case of a large spring constant. Comparison of both the coil current and the armature motion shows why the coil current had a slight dip and why the cusp is so long. The armature velocity increased at first, then decreased before reaching its final closure value. This is one example of the use of the transient coil current to detect unusual changes in the armature motion. The dashed curves represent the response when the spring constant is smaller and more normal.

The second mechanical variable that influences the armature travel time is the armature mass. If the back tension is kept constant when the mass is changed, then the mass does not directly influence the pick-up time. Whether the armature mass can influence the pick-up time

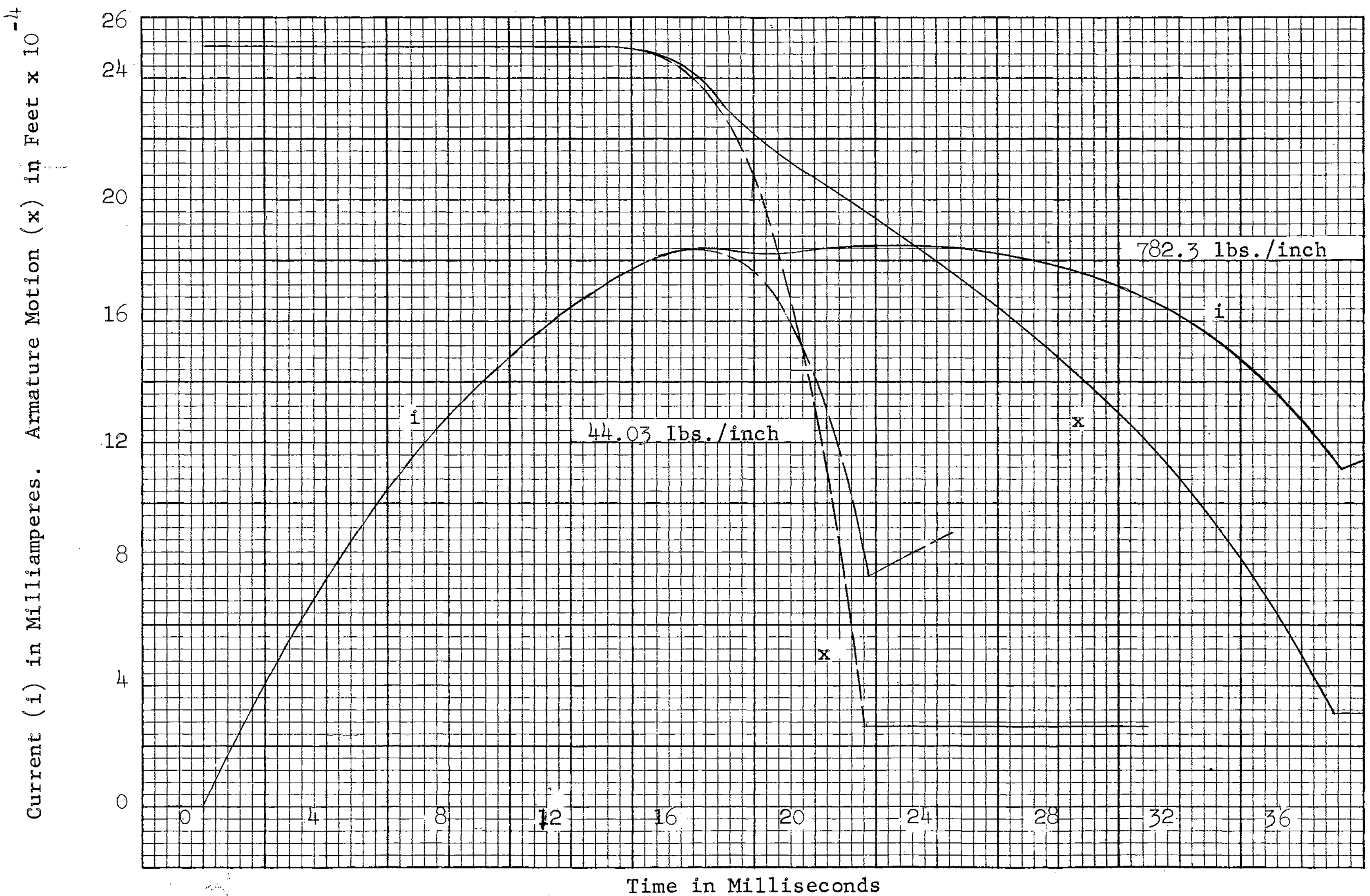
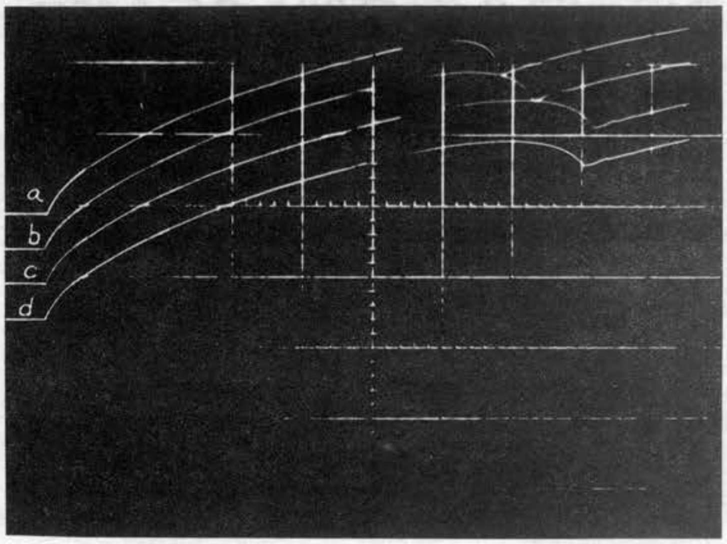


Figure 4.13. Computed Coil Current Build-up and Armature Motion for Two Values of the Spring Constant

at all depends on how the mass is changed. If the reluctance of the magnetic circuit is changed when the mass is changed, then the pick-up time is changed. If the mass is changed without changing the magnetic reluctance, then the only change is in the armature travel time. The assumption made in this discussion is that the mass does not change the reluctance of the magnetic circuit; therefore, pick-up time is not changed. The effect of changing armature mass on the transient coil current build-up is shown by the traces in Figure 4.14. The armature travel time increases with an increase in armature mass. The largest change occurs when the total mass is small, and the change approaches a fairly constant value when the mass becomes larger. If the damping term in the mechanical system equation is negligible, then the travel time approaches zero as the mass approaches zero. Figures 4.15, 4.16 and 4.17 show the computed results of a change in mass on the transient coil current build-up and the armature motion. The smallest mass was used to obtain the data used to plot the curves in Figure 4.15, and the succeeding Figures 4.16 and 4.17 are the results obtained by increasing the mass. The increased mass resulted in a smaller velocity, therefore increasing the length of the coil current cusp.

A somewhat comprehensive discussion was given of the influence of various parameters on the transient coil current build-up because this is the only quantity that can be recorded on all types of transducers, including the hermetically sealed types. In hermetically sealed types, the amount and kind of information that can be obtained about the response is very limited. Therefore, if it can be shown that a particular quantity can be used to study the response of the transducer, then this information is of considerable value in evaluating a particular



Traces: Coil current build-up

(a) Armature mass 10.5 grams total

(b) Armature mass 34.5 grams total

(c) Armature mass 47.5 grams total

(d) Armature mass 62.5 grams total

Oscillogram Data:

Time scale: 5 milliseconds per division (horizontal)

Pick-up current: 21.8 milliamperes

Steady state current: 33 milliamperes

Supply voltage: 36 volts dc

Figure 4.14. Coil Current Build-up for Variable Armature Mass

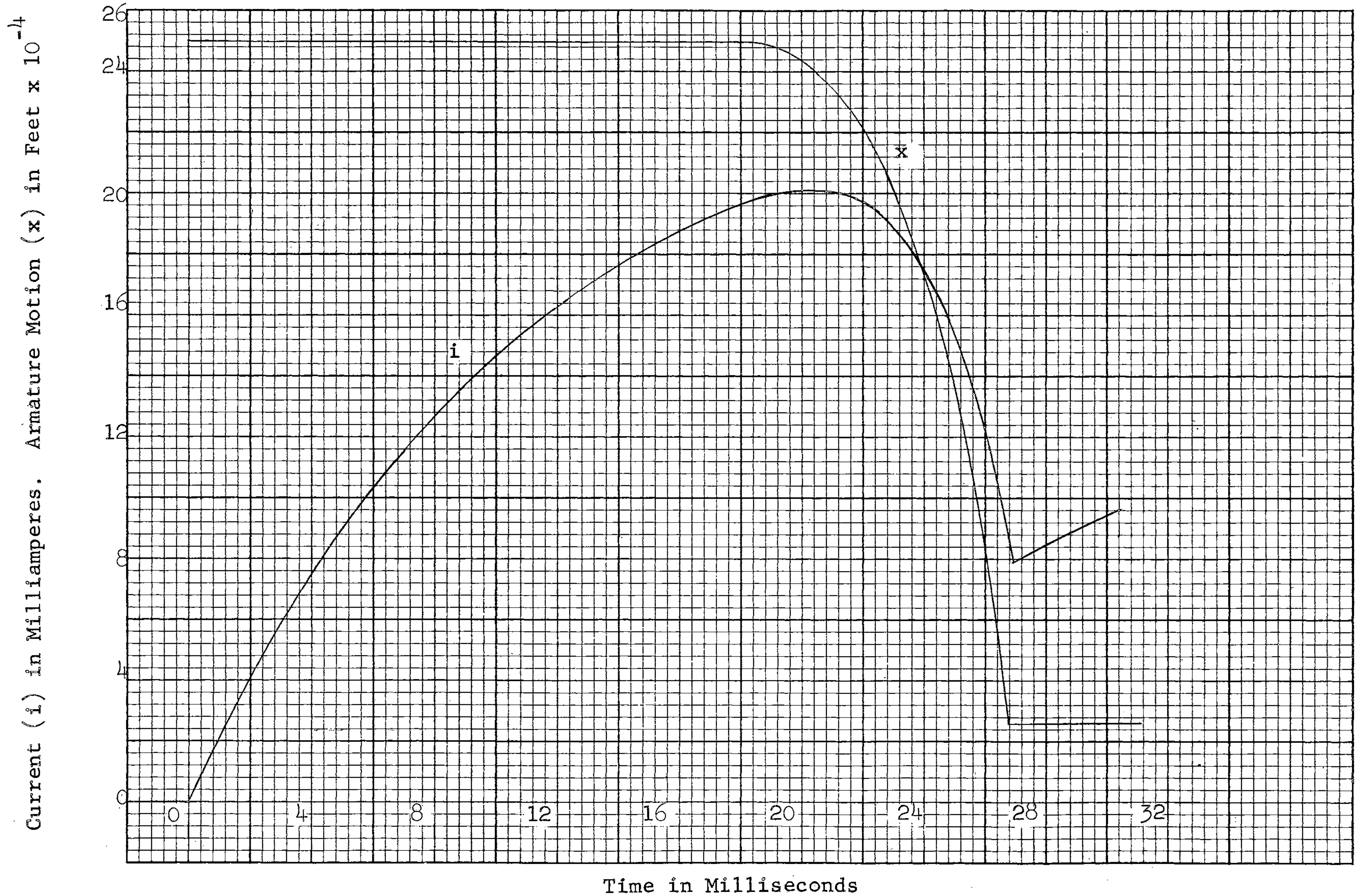


Figure 4.15. Computed Curves of Coil Current Build-up and Armature Motion for a Mass of 0.00205 Slugs

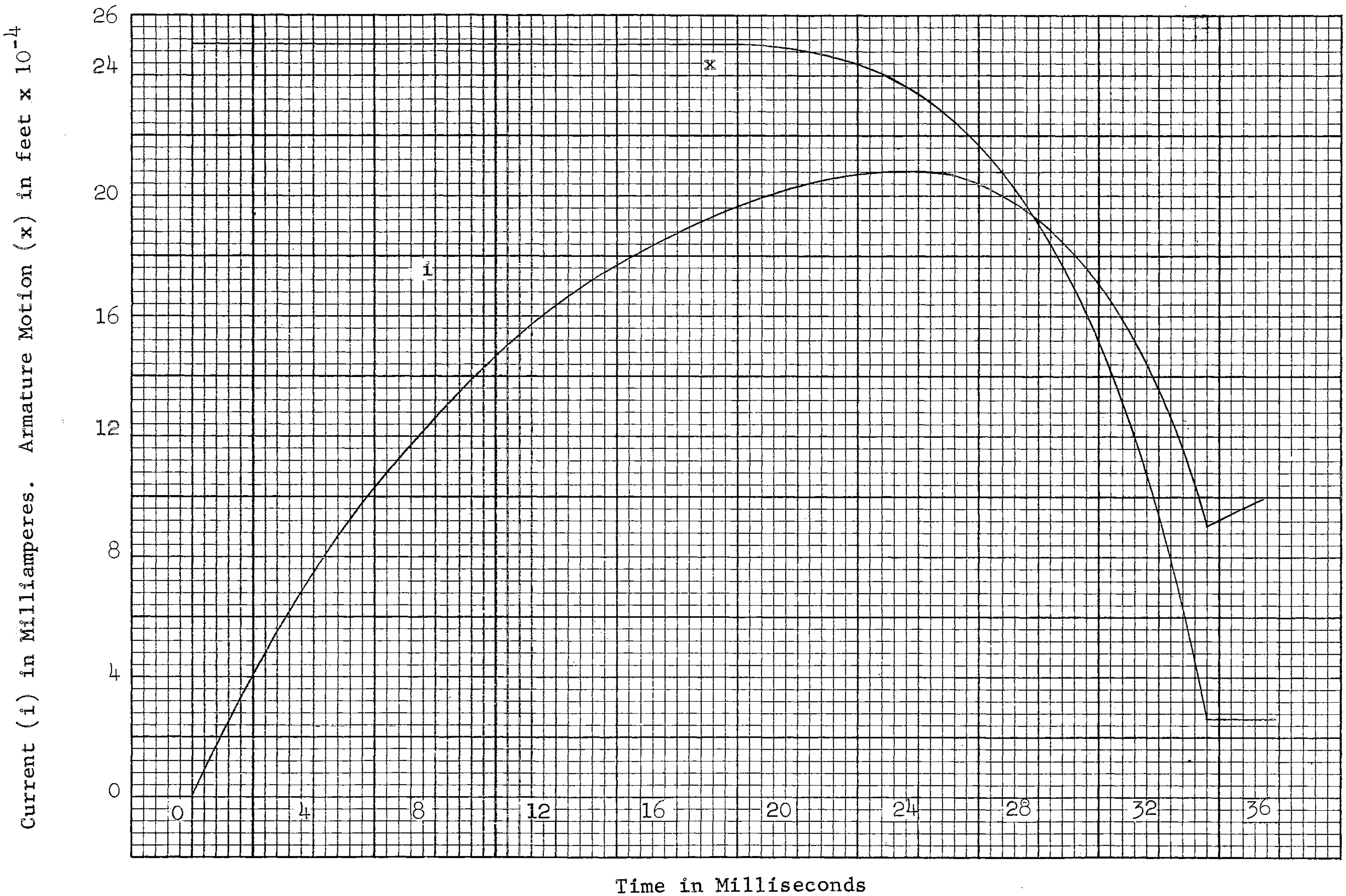


Figure 4.16. Computed Curves of Coil Current Build-up and Armature Motion for a Mass of 0.01025 Slugs

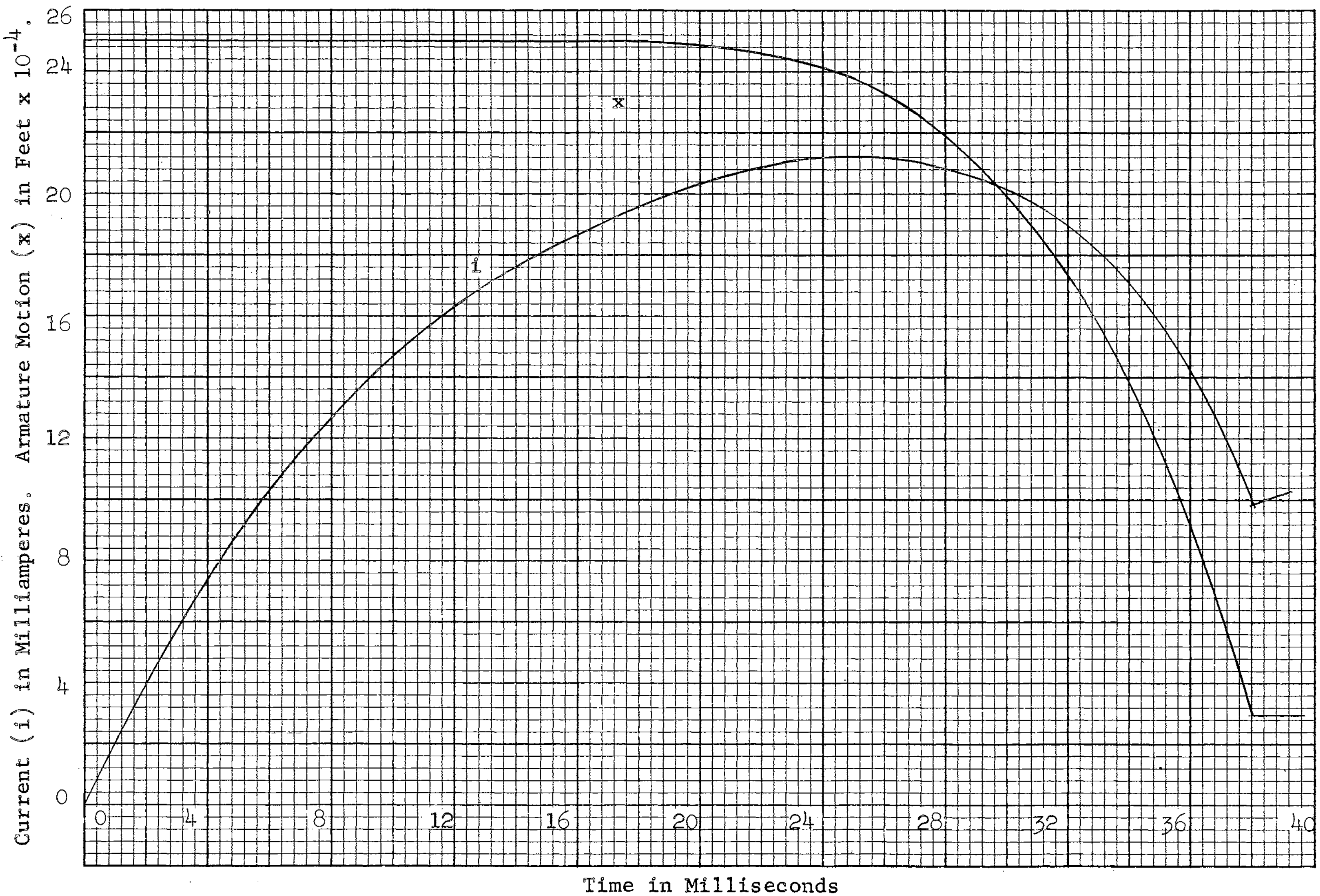


Figure 4.17. Computed Curves of Coil Current Build-up and Armature Motion for a Mass of 0.0205 Slugs

transducer as to its reliability in a certain application.[3] In modern applications of transducers, even though the unit itself is not sealed, it may be a part of a sealed package. Then the transient coil current build-up becomes important in determining whether the response of the transducer has changed during its assembly or packaging.

Transient Armature Motion

The transient armature motion shows the response of the mechanical system to the electrical driving function. There are several ways to obtain an electrical signal that is proportional to the armature position. A number of the methods require some attachment of a sensing device to detect the position. This attachment of another device to the armature creates the problem of changing the effective mass of the armature, especially when the size of the transducer under study is small. The method that was used in obtaining data for this thesis involved the use of a light beam and photosensitive pick-up. By this procedure no mechanical attachments were made to the armature, so no changes could occur in the effective mass. (Some additional details of the experimental setup are given in Appendix B.) Since the armature motion can be obtained only on open type transducers, this discussion will not be as comprehensive as that for the transient coil current build-up.

The effect of changes in back tension on the armature motion is shown in Figure 4.18. Increasing back tension, when the pick-up value of current is small, results in little change in the armature travel time. Therefore,

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[3] Cameron, C. F. and D. D. Lingelbach, "The Dynamics of Relays", <u>Electronic Industries</u>, (1959), Part I, Sept., pp. 70-76; Part II, Oct., pp. 86-90; Part III, Nov., pp. 96-101.

Traces:

(a, c, e) Armature motion

(a) Back tension 75 grams

(c) Back tension 50 grams

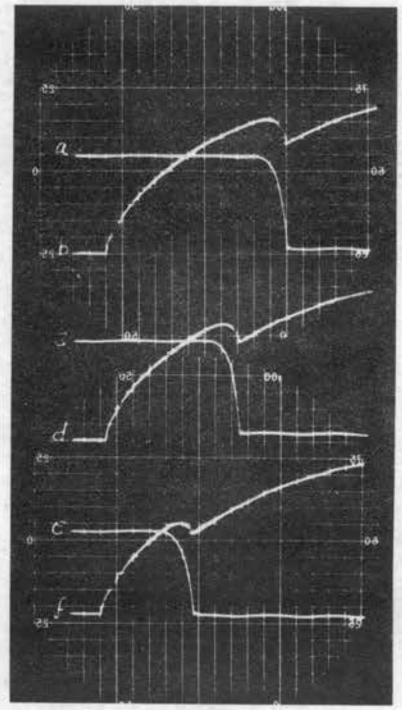
(e) Back tension 25 grams

(b, d, f) Coil current build-up

(b) Back tension 75 grams

(d) Back tension 50 grams

(f) Back tension 25 grams



Oscillogram Data:

Time scale: 4 milliseconds per small division (horizontal)

Current scale: 2.4 milliamperes per small division (vertical)

Air gap: 0.021 inches

Figure 4.18. Armature Motion and Coil Current Build-up for Variable Back Tension

the armature motion curve has the same shape once the armature starts moving. However, when the back tension becomes large enough to cause the pick-up current to approach the steady state value of the current, then a change in back tension gives a change in the armature travel time. As stated in the discussion of transient coil current, increasing the back tension increases the pick-up time. The computed results of the change in the armature motion with a change in back tension are shown by Figures 4.19, 4.20 and 4.21. Figures 4.19 and 4.20 show the results when the back tension is small and is changed. Figure 4.21 shows the results when the back tension is changed to a large value. Comparison of Figures 4.20 and 4.21 show the changes that exist when the value of the back tension used causes the value of pick-up current to approach the steady state value of current.

The effect of the value of the air gap on the armature motion is shown by the traces in Figure 4.22. The armature travel time increases with an increase in air gap for two reasons. One reason is that the armature has to travel further, and second, the reluctance of the magnetic circuit increases. Therefore, increasing the air gap causes the armature travel time to increase rapidly, especially at the larger values of air gap. The increase in the armature travel time results in a decrease in the impact velocity of armature.

The value of the supply voltage affects the armature motion as shown by the traces in Figure 4.23. The armature travel time increases with a decrease in the value of the supply voltage. Associated with this increase in armature travel time is a decrease in the armature impact velocity.

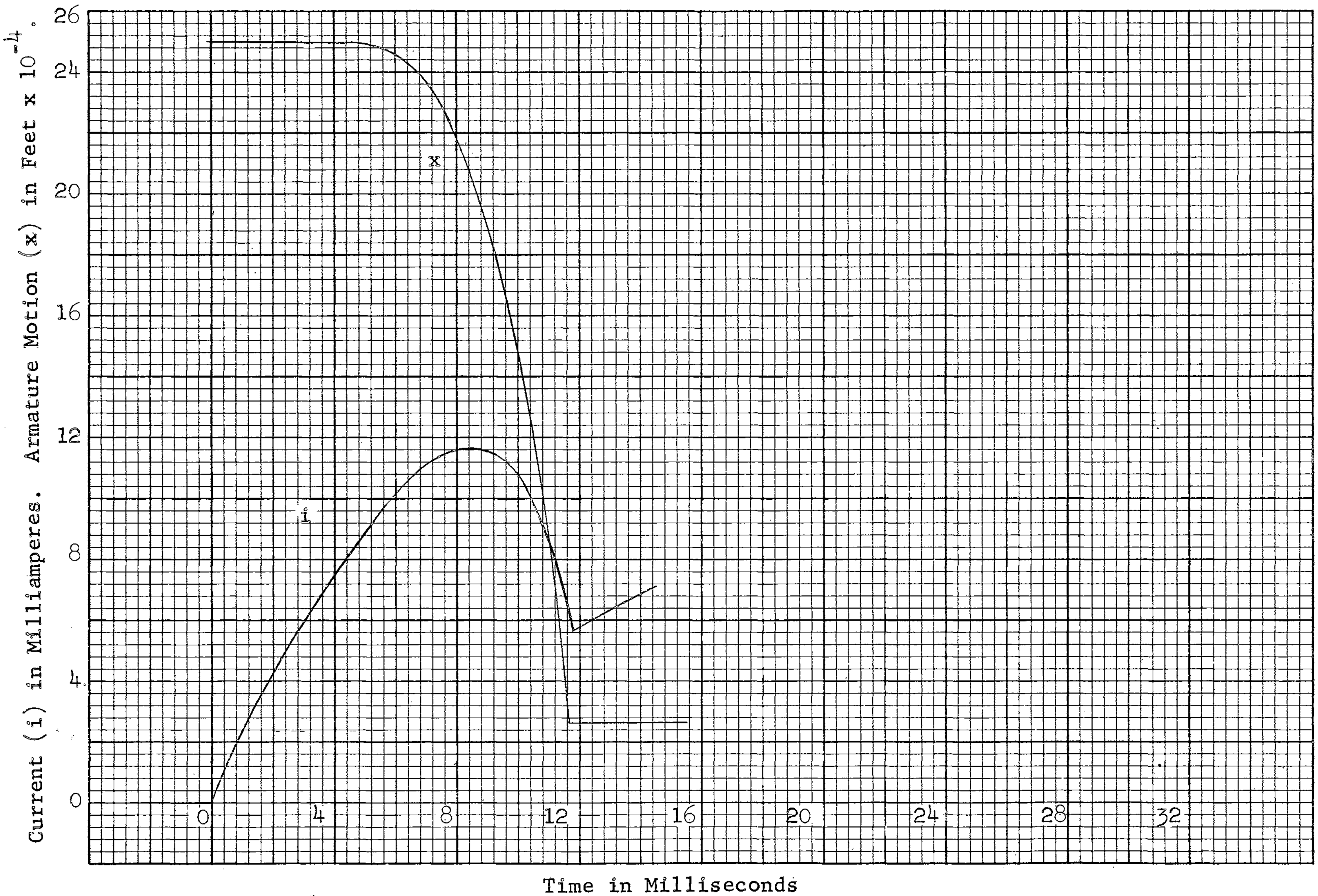


Figure 4.19. Computed Curves of Coil Current Build-up and Armature Motion for a Back Tension of 0.132 pounds

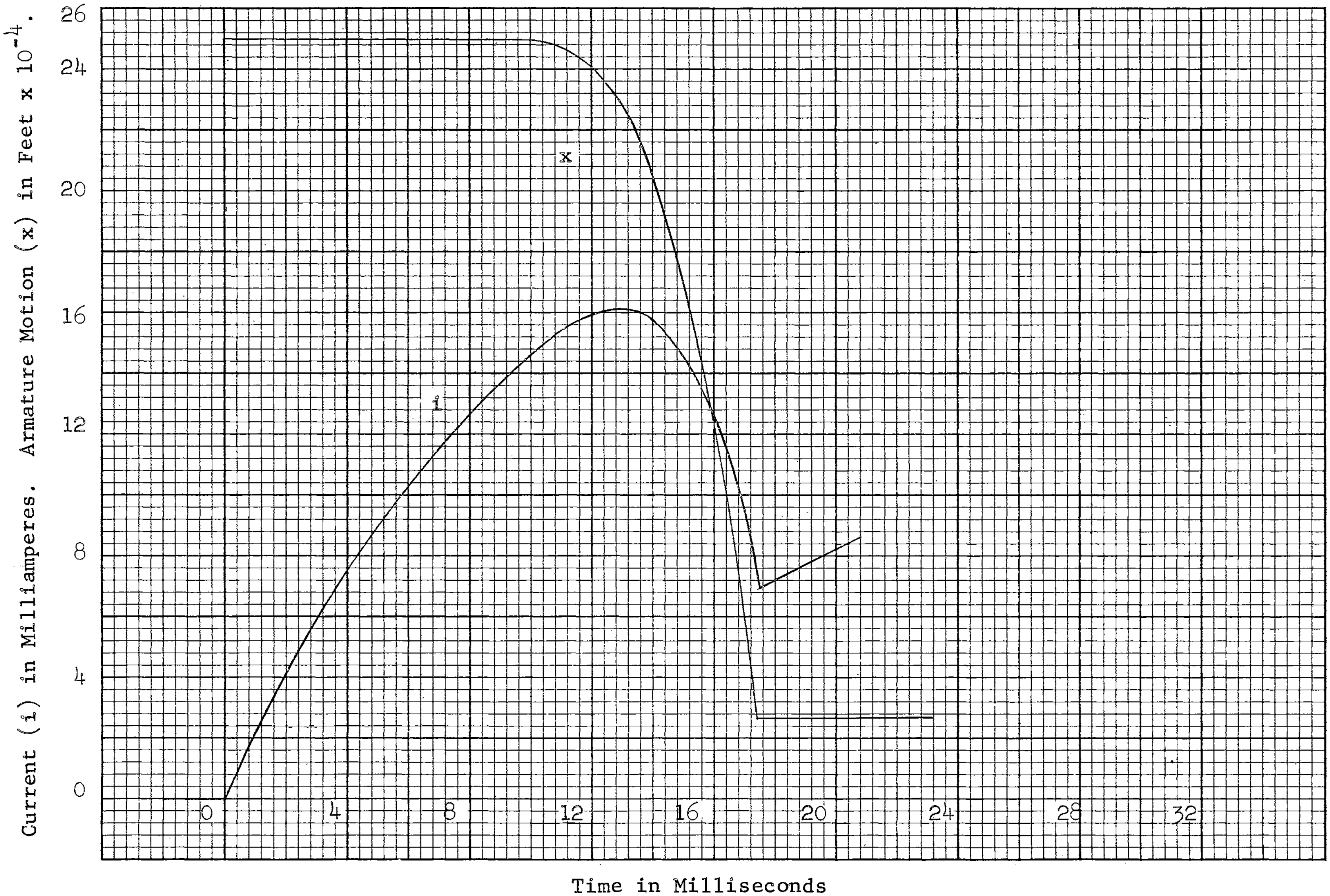


Figure 4.20. Computed Curves of Coil Current Build-up and Armature Motion for a Back Tension of 0.474 pounds

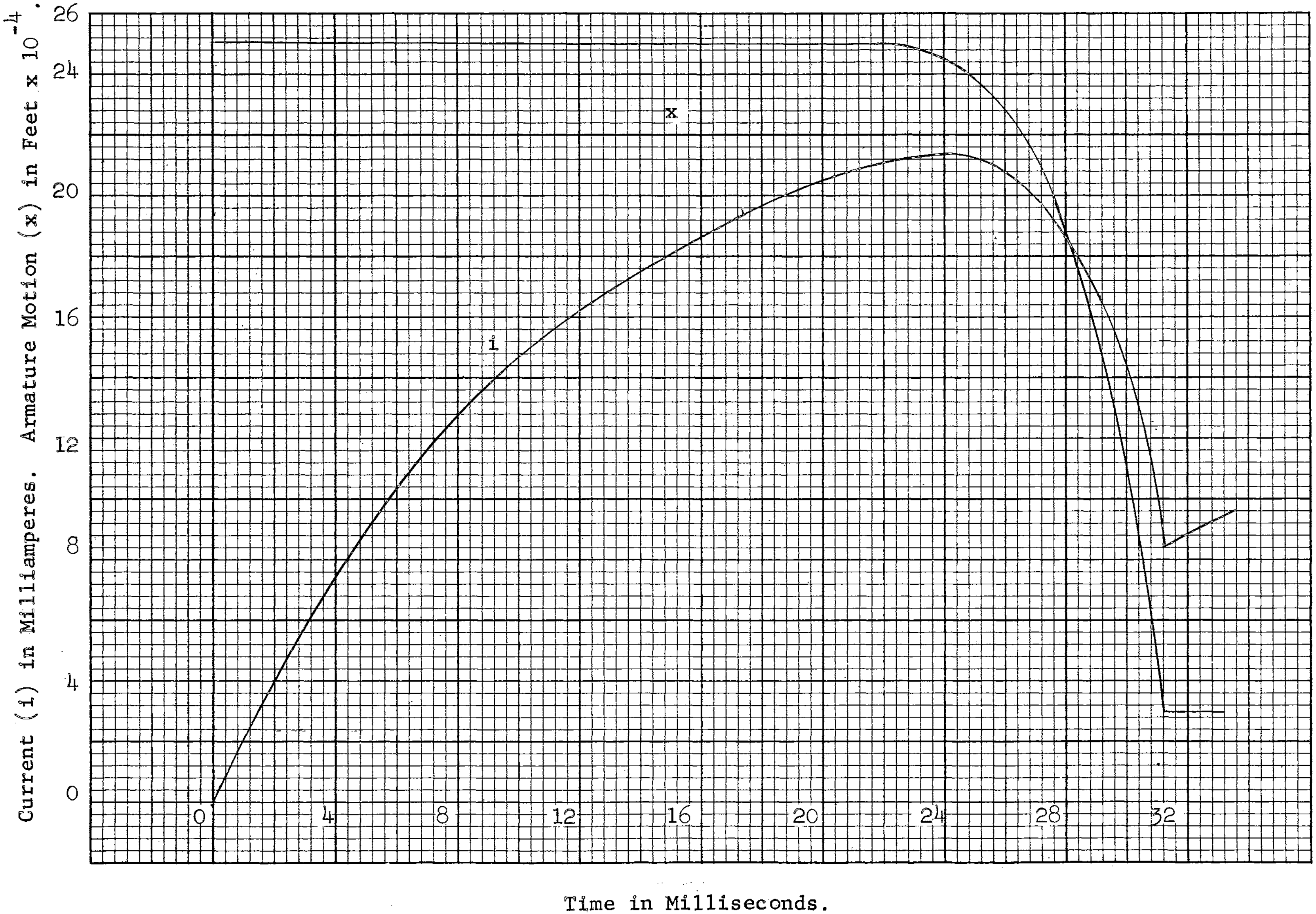
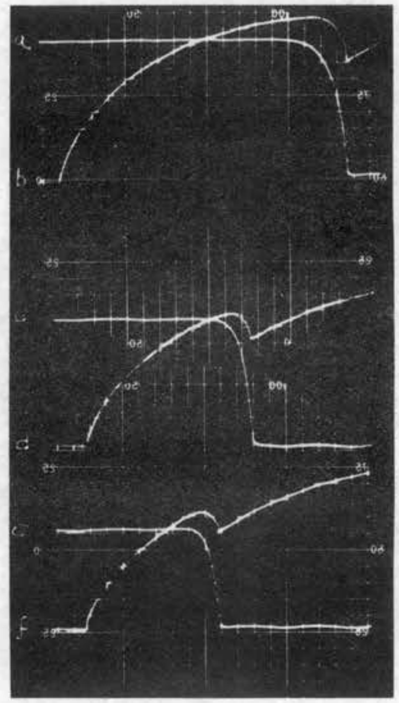


Figure 4.21. Computed Curves of Coil Current Build-up and Armature Motion for a Back Tension of 1.00 pound



Traces:

(a, c, e) Armature motion

(a) Air gap 0.052 inches

(c) Air gap 0.031 inches

(e) Air gap 0.021 inches

(b, d, f) Coil current build-up

(b) Air gap 0.052 inches

(d) Air gap 0.031 inches

(f) Air gap 0.021 inches

Oscillogram Data:

Time scale: 4 milliseconds per small division (horizontal)

Current scale: 2.4 milliamperes per small division (vertical)

Back tension: 50 grams

Figure 4.22. Armature Motion and Coil Current Build-up for Variable Air Gap

Traces:

(a, c, e) Coil current build-up

(a) Voltage: 24 volts

(c) Voltage: 28 volts

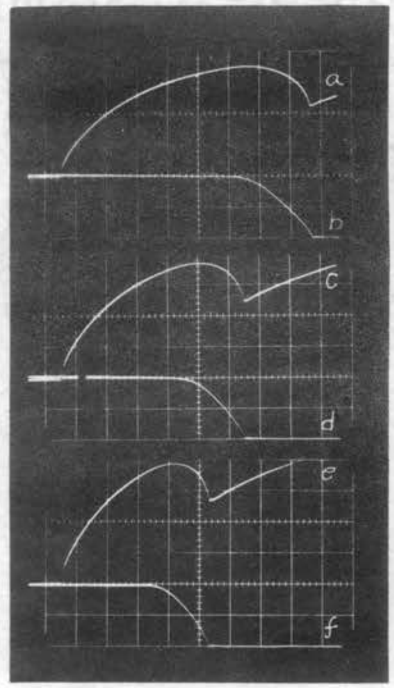
(e) Voltage: 32 volts

(b, d, f) Armature motion

(b) Voltage: 24 volts

(d) Voltage: 28 volts

(f) Voltage: 32 volts



Oscillogram Data

Time scale: 5 milliseconds per division (horizontal)

Current scale: 12.8 milliamperes per division (vertical)

Air gap: 0.046 inches

Residual gap: 0.002 inches

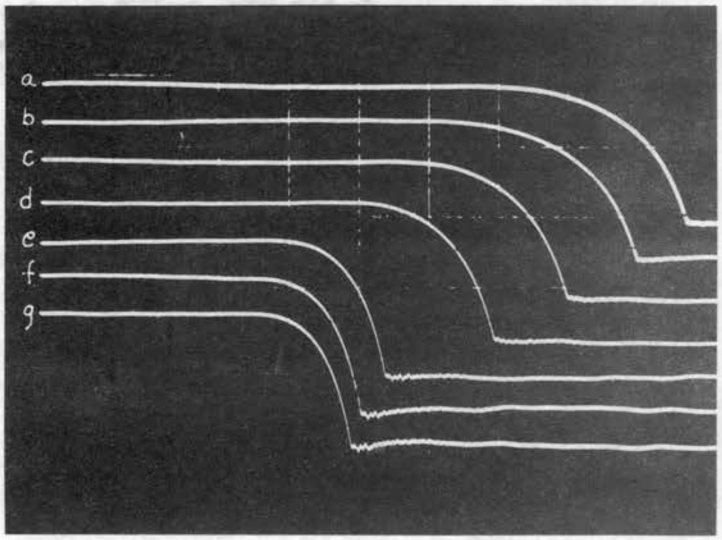
Figure 4.23. Armature Motion and Coil Current Build-up for Variable Voltage

Increasing the coil circuit resistance changes the shape of the armature motion trace in a manner similar to decreasing the supply voltage. This effect is shown by the traces in Figure 4.24. Increasing the coil circuit resistance increases the armature travel time and decreases the armature impact velocity.

If both the supply voltage (E) and the coil circuit resistance (R) are increased such that the ratio E/R is constant, then only a slight change in the armature travel time results with a change in the coil resistance when the value of R is less than twice the value of the coil resistance. As the coil circuit resistance (R) becomes more than twice the coil resistance, then a noticeable increase in armature travel time occurs when the coil resistance is changed. This effect is shown by the traces in Figure 4.25.

Normally the armature mass or the spring constant of a given relay does not change a great deal. However, it is of interest to know the particular effect that each of the two variables, mass and the spring constant, have on the transducer response. Figures 4.15, 4.16 and 4.17 show the computed response of the armature motion and the coil current. Increasing the mass decreases the armature impact velocity and increases the armature travel time.

The effect of a stiff spring is shown by the computed armature motion and coil current curves in Figure 4.26. The effect of a stiff spring, for the case computed in Figure 4.26, was that of causing the armature to slow down during its closure. This is shown by the decrease in the slope of the armature motion curve in Figure 4.26. Figure 4.13 also shows the effect of a stiff spring and a weaker spring on the armature motion and coil current.



Traces: Armature Motion

(a) Coil circuit resistance 3790 ohms

(b) Coil circuit resistance 3680 ohms

(c) Coil circuit resistance 3460 ohms

(d) Coil circuit resistance 3080 ohms

(e) Coil circuit resistance 1930 ohms

(f) Coil circuit resistance 1465 ohms

(g) Coil circuit resistance 1250 ohms

Oscillogram Data:

Time scale: 5 milliseconds per division (horizontal)

Supply voltage: 36 volts dc

Figure 4.24. Armature Motion for Variable Coil Circuit Resistance

Traces:

(a, c) Armature motion

(a) Voltage: 49.2 volts

Coil circuit resistance: 470 ohms

(c) Voltage: 95 volts

Coil circuit resistance: 825 ohms

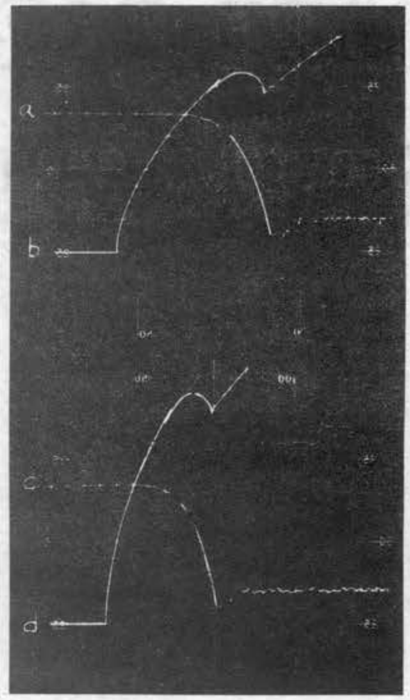
(b, d) Coil current build-up

(b) Voltage: 49.2 volts

Coil circuit resistance: 470 ohms

(d) Voltage: 95 volts

Coil circuit resistance: 825 ohms



Oscillogram Data

Time scale: 1 millisecond per small division (horizontal)

Current scale: 5.7 milliamperes per small division (vertical)

Figure 4.25. Armature Motion and Coil Current Build-up for Constant Steady State Current but Variable Voltage and Resistance

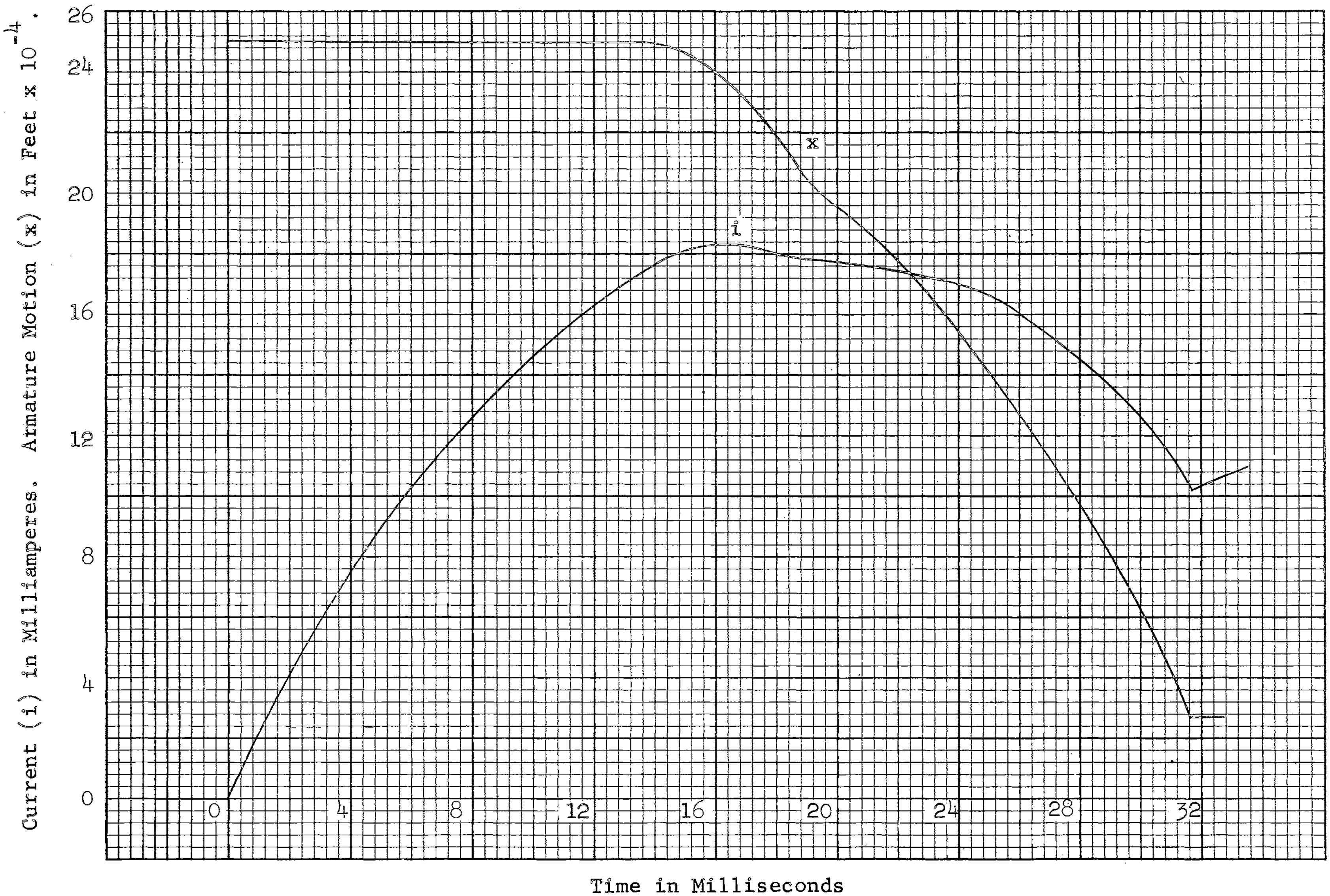


Figure 4.26. Computed Coil Current Build-up and Armature Motion for a Spring Constant of 616.3 Pounds Per Foot

As explained in Chapter III, computations were made of several other variables in addition to the armature motion and coil current. Figure 4.28 shows a plot of some of the other quantities associated with the mechanical system. The variables plotted in Figure 4.28 are armature motion, armature velocity, armature acceleration and magnetic pull, all as functions of time. Figure 4.27 shows the variables associated with the electrical system. These variables are coil current, flux and time rate of change of flux, all as functions of time.

The magnitude of the rate of change of flux is seen to be essentially the mirror image of the transient coil current to a different scale. The mirror image effect is caused by the fact that rate of change of flux obtained from the emf equation of the coil circuit can be expressed as follows:

$$\frac{d\phi}{dt} = (E - iR)/N \qquad (4.10)$$

This equation shows that, for constant values of N, E and R, the value of $d\phi/dt$ is proportional to $(1 - i)$, thereby causing the mirror image result as shown by the $\dot{\phi}$ curve in Figure 4.27. The flux curve follows the current curve, as it should, until the armature moves, then it becomes somewhat quadradic in form until the armature closes. The flux then again follows the current curve with a different scale factor.

Figure 4.28 indicates that the velocity is continually increasing as the armature closes. Also, the acceleration is continually increasing as the armature closes. The pull curve could almost be represented as two straight lines. One straight line could be drawn from near the origin to a time where the armature has completed about half of its

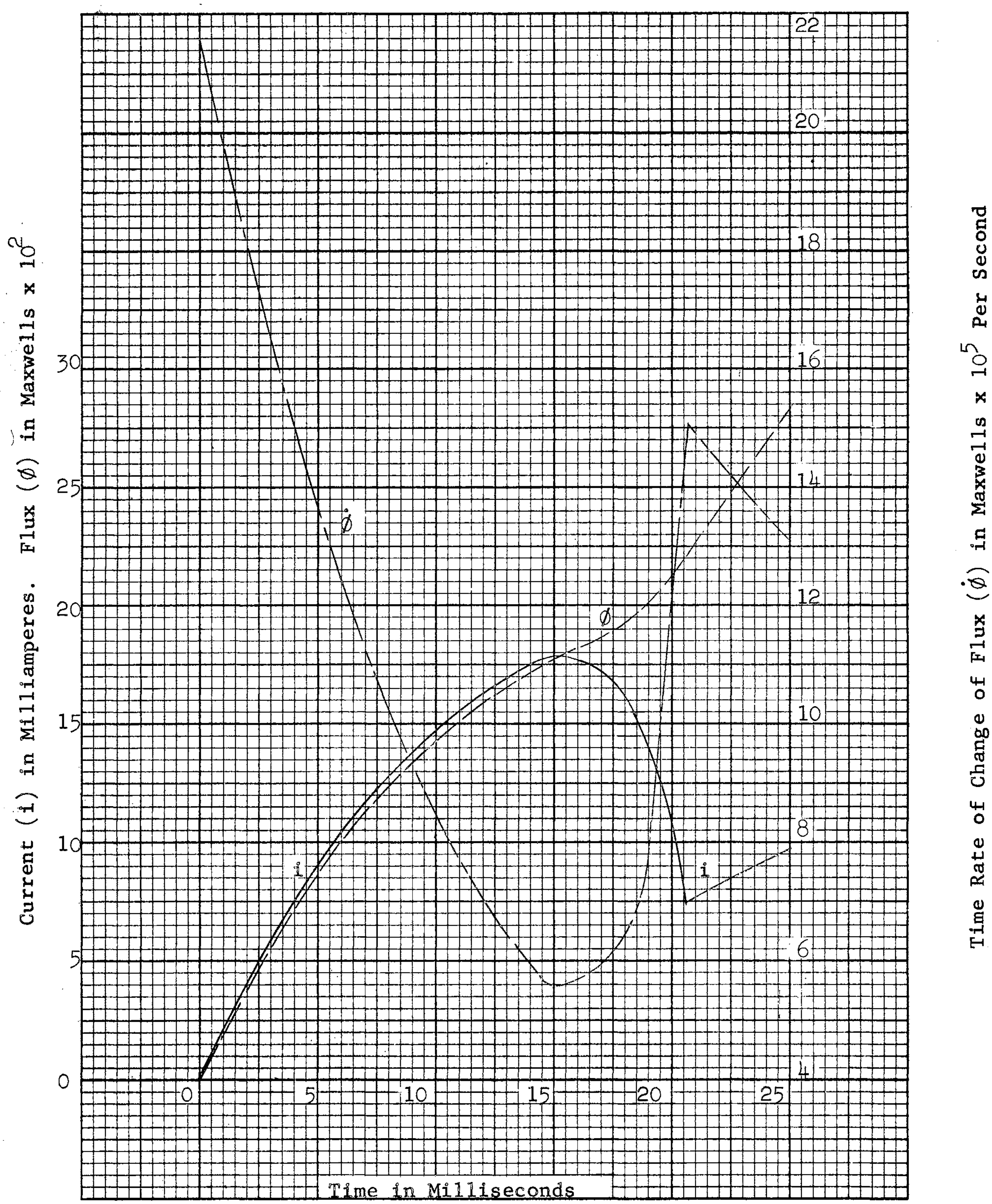


Figure 4.27. Computed Curves of Coil Current (i), Flux (Ø), and the Time Rate of Change of Flux ($\dot{Ø}$) as Functions of Time

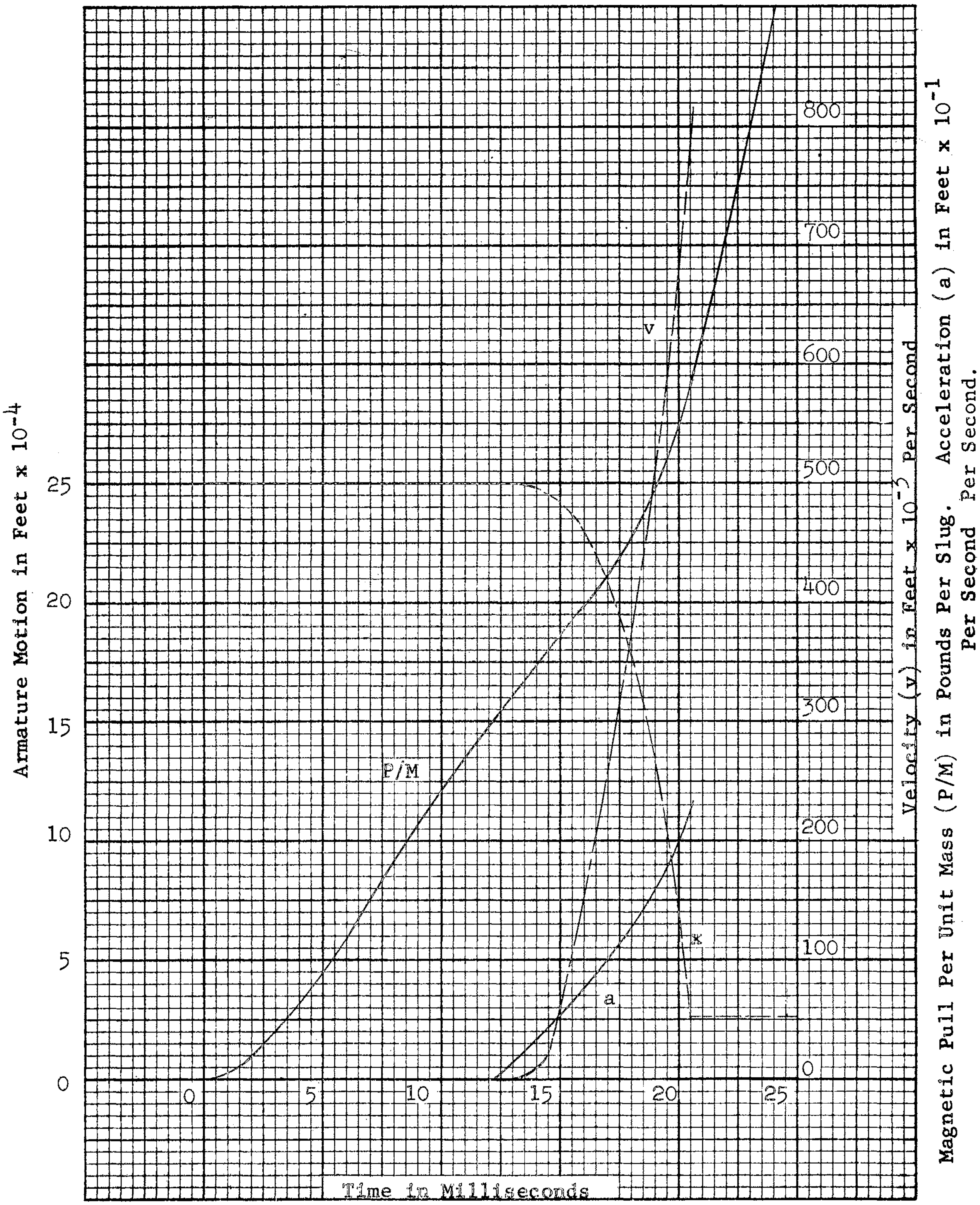


Figure 4.28. Computed Curves of Armature Motion (x), Armature Velocity (v), Armature Acceleration (a) and Magnetic Pull Per Unit Mass (P/M) as Functions of Time

travel and the other straight line of approximately twice the slope from that point on.

A qualitative discussion has been given in this chapter which presents the changes that take place primarily in the transient coil current build-up and the armature motion. It was brought out that a change in any variable resulted in some change existing in the transient coil current. In some cases two variables gave some of the same general changes, but in most cases at lease one phase of the effect was unique.[4] Not all of the parameters affecting the transient coil current were changed because of the difficulty in physically making a change. It is possible, but highly improbable, that any two variables would result in giving identical changes in the transient coil current because of the nonlinear characteristic of the transducer. This behavior is desirable from an evaluation standpoint but is undesirable when a general solution to the response to other types of driving function is attempted, since the superposition theorem can not be used.[5]

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[4]Cameron, C. F. and D. D. Lingelbach, "Relay Chatacteristics," Symposium on Electromagnetic Relays, (1956), pp. 41-48.

[5]Cameron, C. F., D. D. Lingelbach and C. C. Freeny, "Armature Overtravel in Relays," Seventh Symposium on Electromagnetic Relays," (1959), pp. 67-70.

CHAPTER V

QUANTITATIVE DISCUSSION OF THE EXPERIMENTAL AND COMPUTED RESULTS

The previous chapter presented a qualitative discussion of the comparison of the experimental and computed results. Only general trends or changes were presented as they affected the transient coil current build-up and armature motion. Additional data was obtained from the experimental and computed results which show quantitatively the effect of the variables on the performance of the transducer. Instead of expressing the change in pick-up time or transit time as a function of the magnitude of the variable itself, a more general non-dimensional quantity will be used. This non-dimensional quantity, defined as the ratio of the pick-up current to the steady state current and called Γ, has certain desirable properties. For one thing, it expresses the changes of five of the variables in a common form. Also, the values of Γ by definition can vary only between zero and one. This means that the use of such a parameter gives more uniformity to the use of curves to express the results. Since the mass (M) and the spring constant (K) do not directly affect the pick-up current, then the quantity Γ can not be used to show the effect of these two variables.

The mathematical model of the transducer used in the computational procedure is not identical to a known transducer; however, the values used are representative of the 10-cubic inch size. A set of values was selected which was used to represent an average or normal operating point. Whenever the influence of a variable upon the transient response was desired, only the variable under study was allowed to vary. Therefore, unless otherwise specified, the following values of the variables were used to obtain the response of the transducer.

Voltage: 24 volts (step input)

Turns: 11,000

Coil circuit resistance: 1000 ohms

Back tension: 0.22 pounds or 100 grams

Armature mass: 0.00205 slugs or 30 grams

Spring constant: 10.53 lbs/in. or 400 gms/in.

Damping constant: 0.1 lbs/ft/sec.

Cross sectional area of air gap: 0.1104 sq. inches

Open air gap: 0.0025 feet or 0.030 inches

Closed air gap: 0.00031 feet or 0.00375 inches.

Probably the most important characteristics of the performance that can be shown on a quantitative basis are the pick-up time and the armature travel time. Since these quantities are plotted versus the per unit pick-up current (Γ), the effect of a change of the variable upon the pick-up current is indirectly shown. The variables which determine the value of Γ can be obtained from the definition of the pick-up flux. Let ϕ_p be the pick-up flux. Then the following relation exists.

$$\phi_p = i_p \frac{N}{\mathcal{R}_p} \quad . \tag{5.1}$$

By definition $\Gamma = i_p/I_{ss}$ and $I_{ss} = E/R$. With these relations available, equation (5.2) can be written.

$$\Gamma = \frac{R\mathcal{Q}_p \phi_p}{EN} \, . \qquad (5.2)$$

The effect of Γ upon the pick-up time (t_p) and the armature travel time (k) when Γ is changed by changing back tension is shown in Figures 5.1 and 5.2. Figure 5.1 shows the computed values, and Figure 5.2 shows the measured values from a relay of the same class. The close comparison of the curves in Figures 5.1 and 5.2 leaves no doubt that the procedure used to compute the response is correct. In fact, the existence of a minimum value of k, as the back tension is varied, was first shown by the computed results. It was later that experimental data was obtained which verified that such a minimum existed. The reason that such a minimum was not suspected is that it is difficult or almost impossible to visualize that decreasing the back tension would result in the travel time being longer. Since the experimental data in Figure 5.2 only went to a value of Γ = 0.3, additional experimental data was taken to prove that the travel time had a minimum when the back tension was varied. To obtain experimental data for $\Gamma < 0.3$ requires that some of the other variables be changed; therefore, the absolute comparison of the two sets of experimental data can not be made. The additional experimental data is shown in Figure 5.3. The minimum seems to occur for a Γ of approximately 0.5; however, this appears to be a function of the values of the other variables existing when this data was obtained.

Considerable other experimental data is available from the figures in Chapter IV showing the effect of the change in supply voltage, coil resistance, air gap and other variables. Because of a lack of time, and because of the desire to obtain additional information about the response of the relay which is difficult to obtain experimentally,

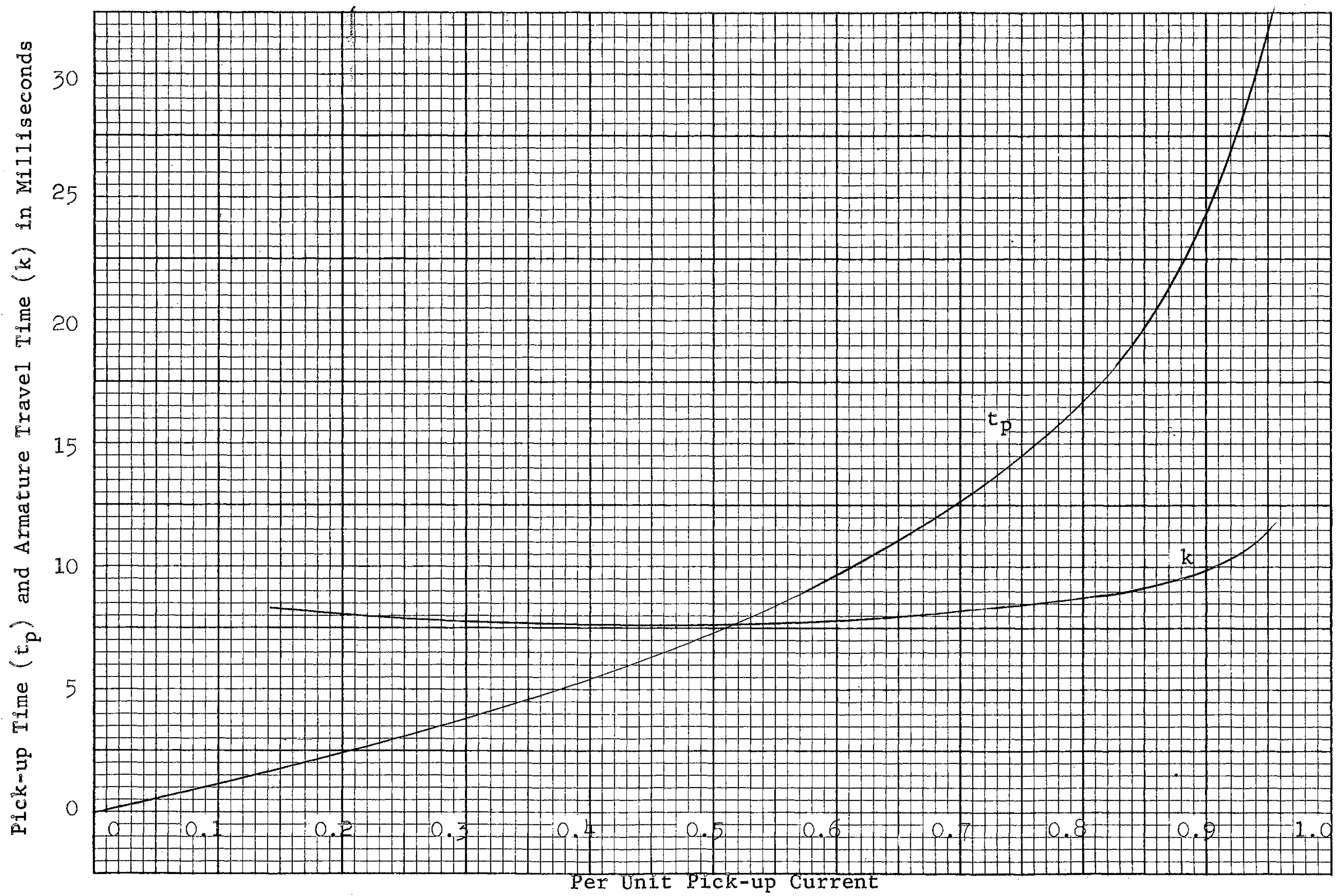


Figure 5.1. Computed Values of Pick-up Time and Armature Travel Time With Back Tension as a Variable

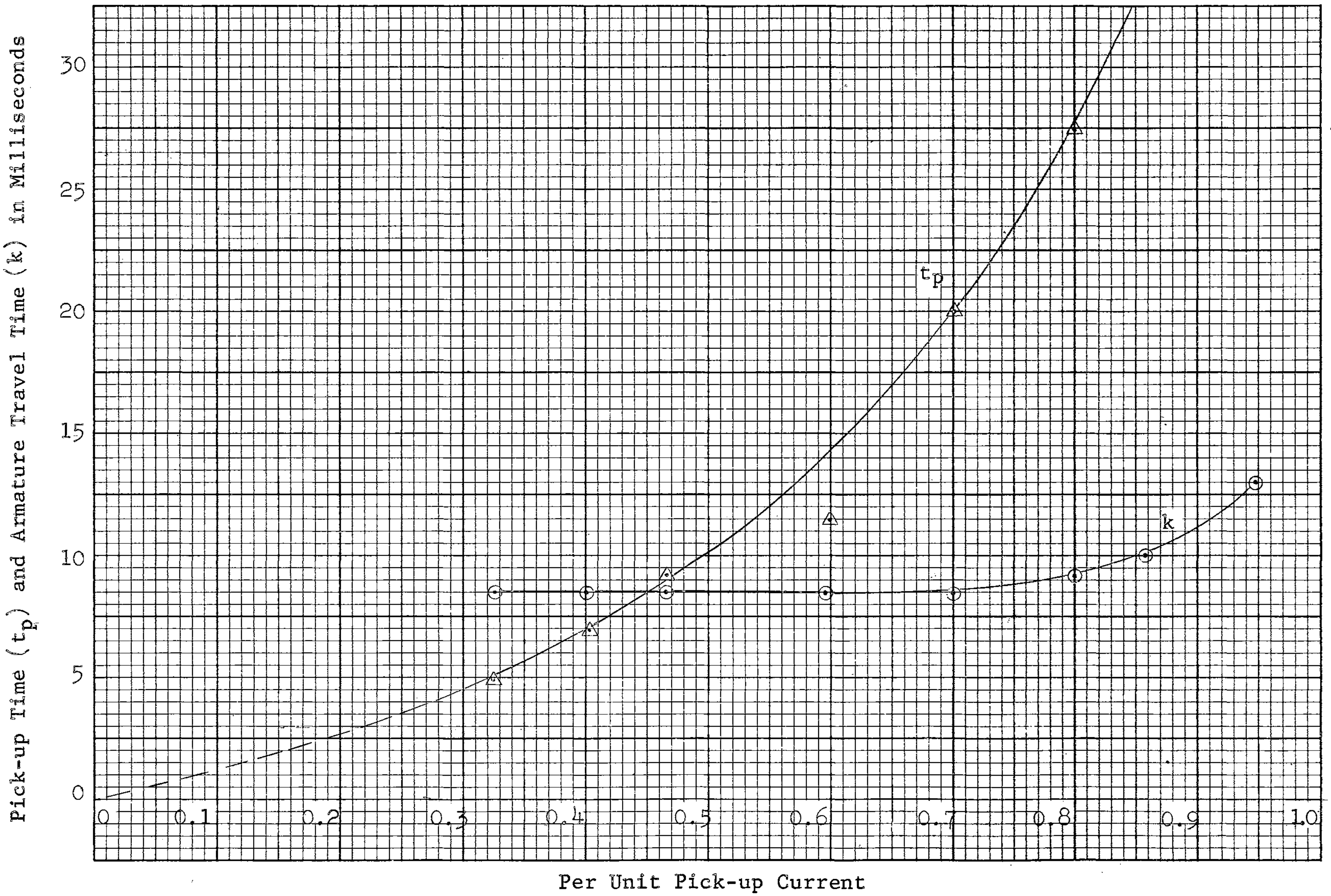


Figure 5.2. Measured Values of Pick-up Time and Armature Travel Time With Back Tension as a Variable

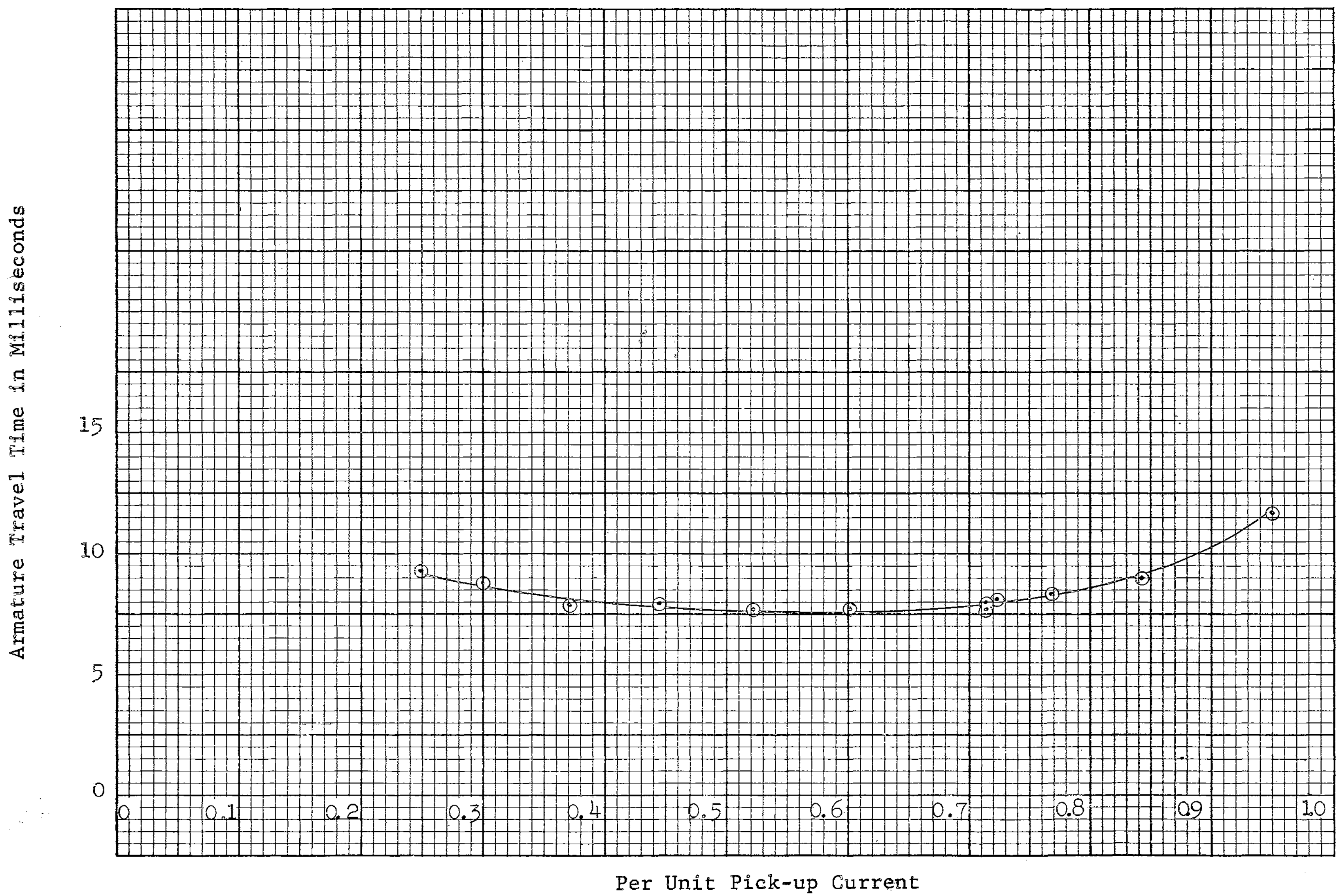


Figure 5.3. Measured Values of Armature Travel Time with Back Tension as a Variable

only computed information showing the effect of the spring constant and the armature mass was obtained. Experimental information showing the effect of the armature mass and spring constant was obtained for some limited values, but obtaining such information is difficult. It is difficult to change the spring constant of a spring, and therefore experimental information can only be obtained by using a set of different springs. The problem arises in resetting the back tension to the same value each time, especially if the armature is held in place by the spring. Also the range of the values of the spring constant in springs is limited so that the range of the computed results and experimental results is different.

The computed effect on the armature travel time of a change in the spring constant is shown by the curve in Figure 5.4. The range of values of the spring constant is larger than usual in order to show the effect outside the normal range. This is the advantage of an analytical approach, in that it allows for investigation in the so-called "fringe" areas.

Figure 5.5 shows the measured or experimental results on the armature travel time for a change in the spring constant. Because of the limited range, the scale on the abscissa is ten times larger in Figure 5.5 than in Figure 5.4. The scale on the ordinate is two and one-half times larger in Figure 5.5 than in Figure 5.4. The results indicated by each figure compare favorably.

Another variable which is difficult to accurately vary experimentally is the armature mass. Therefore, two sets of computed results were obtained in order to gain additional insight into the response of the transducer to a change in armature mass. Figure 5.6 shows the

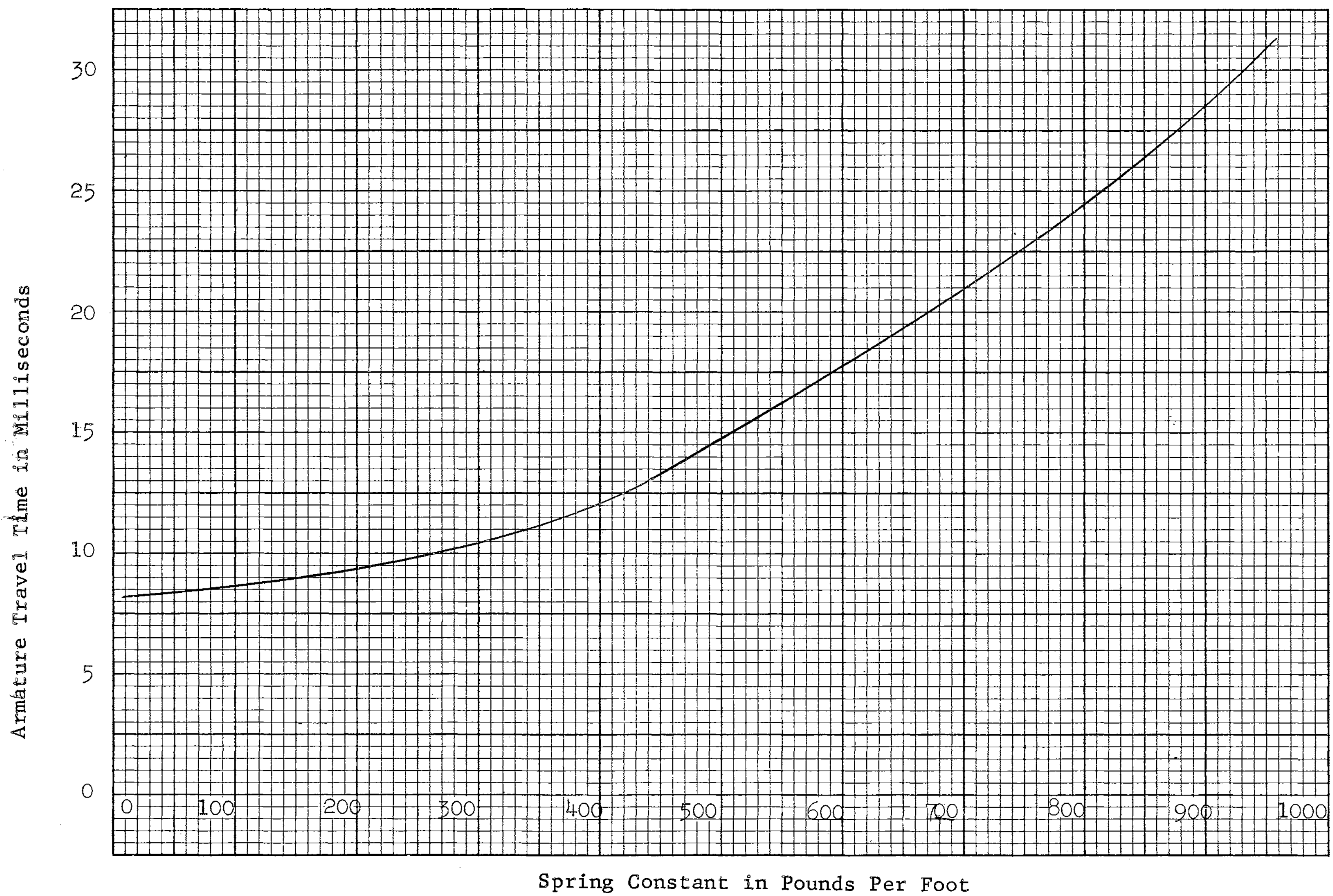


Figure 5.4. Computed Values of Armature Travel Time With Spring Constant as a Variable

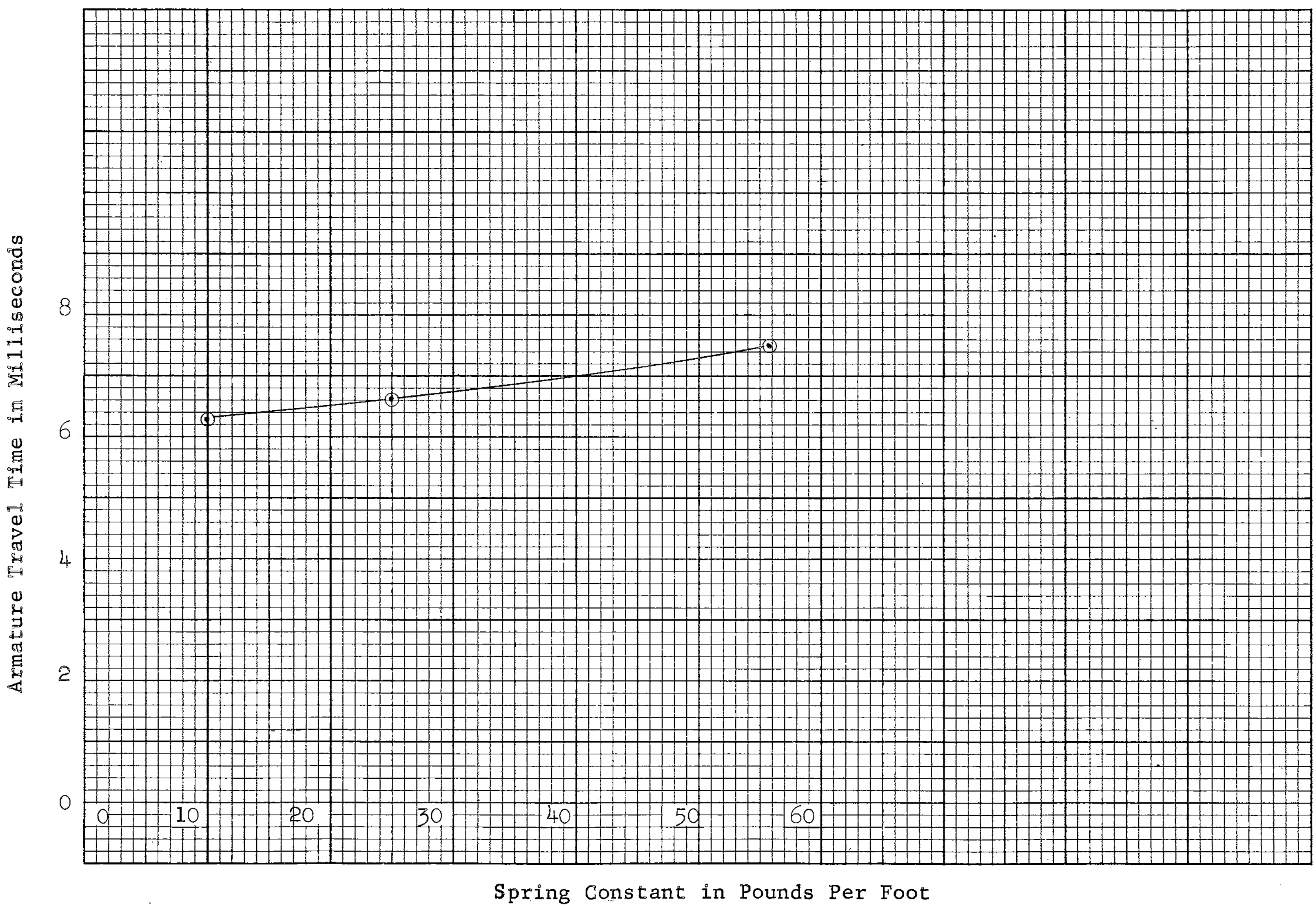


Figure 5.5. Measured Values of Armature Travel With Spring Constant as a Variable

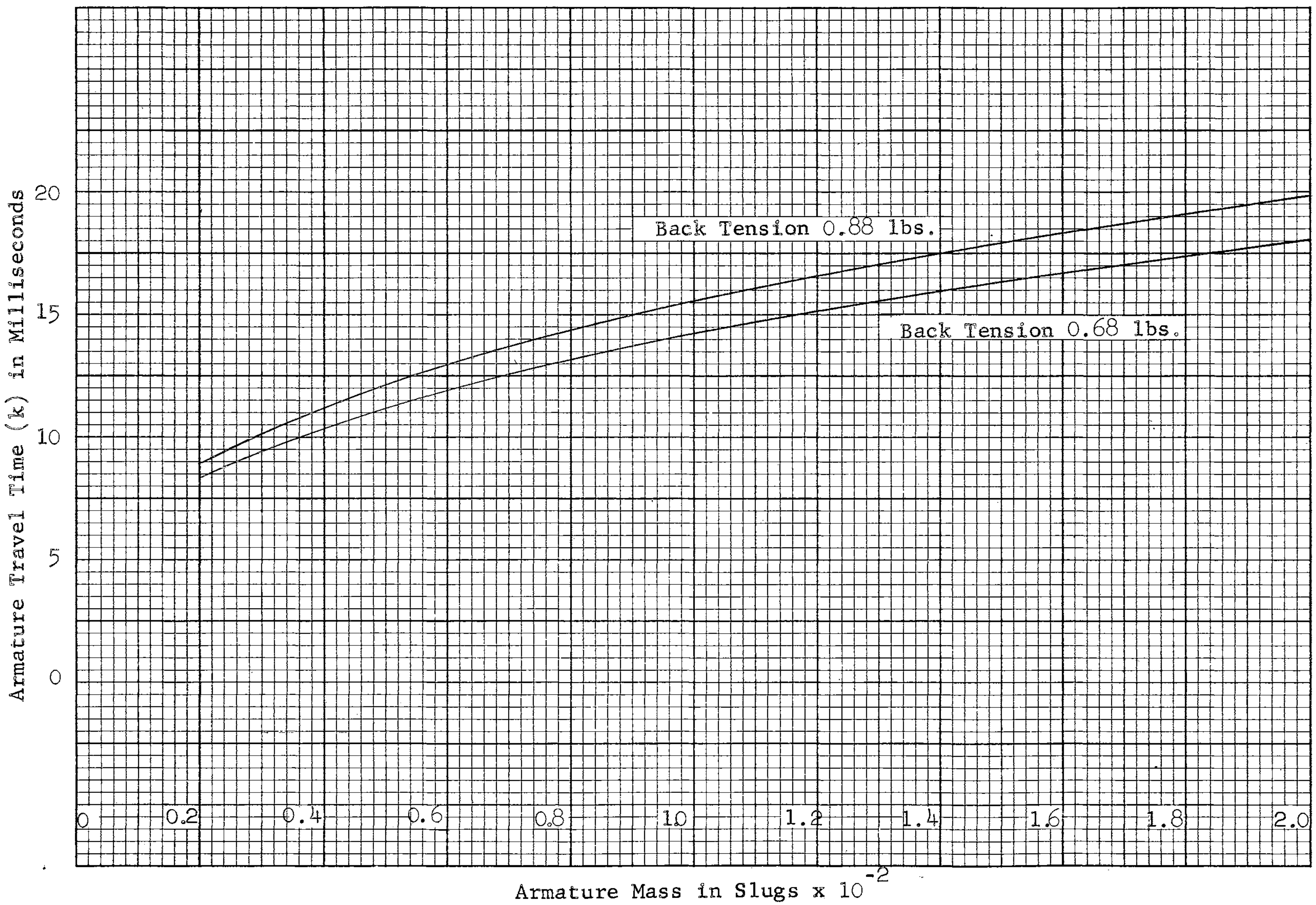


Figure 5.6. Computed Values of Armature Travel Time With Mass as a Variable

computed effect on the armature travel time of a change in armature mass for two values of back tension. The effect of back tension seems to be primarily that of shifting the time inversely with a change in back tension. Also, as the mass approaches zero, the armature travel time approaches approximately the same value for all values of back tension. It appears that theoretically at same negative value of mass the armature travel time is zero for all values of back tension. At first, it might appear that with zero mass, the armature travel time would be zero. This is not true when mechanical damping is present. This condition is realized by examining equation (5.3).

$$M\ddot{x} + h^2\dot{x} + Kx = F \tag{5.3}$$

where $\ddot{x} = d^2x/dt^2$

$\dot{x} = dx/dt$.

If the mass approaches zero in equation (5.3) it will still take the armature some finite time to close because of the $\dot{x}$ term. If the damping term (h^2) is negligible, then as the mass approaches zero so will the armature travel time.

The experimental results of the effect of a change in armature mass on the armature travel time is shown by the curve in Figure 5.7. The range of mass values covered in the experimental data is about one-nineth of the range covered in the computed results. The comparison between the computed and experimental results in this case is also favorable.

The quantitative comparison between the computed and experimental values was very good in all the cases shown. The computed procedure has the advantage that controlled variations may be made in any of the variables, and also that the fringe areas can be investigated that otherwise are difficult to do experimentally.

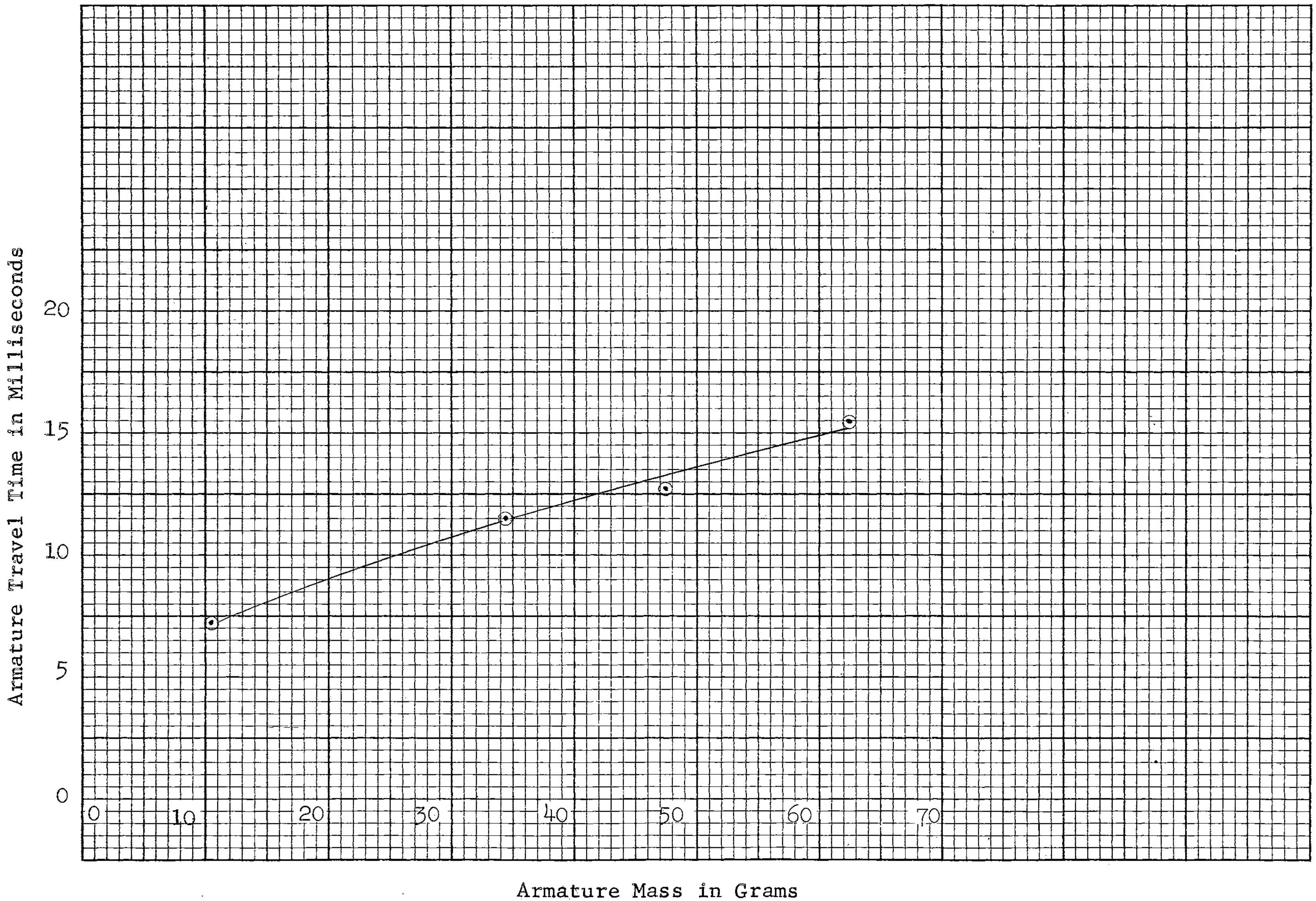


Figure 5.7. Measured Values of Armature Travel Time With Mass as a Variable

CHAPTER VI

SUMMARY AND CONCLUSIONS

The equations which are needed to describe the singly excited electromechanical transducer are referred to as nonlinear differential equations. There are several ways the equations may be written, depending upon the variables which are selected as independent. Also the equations may be written using the force laws (such as D'Alembert's principle) or the variational approach (using Lagrange's equations). The first part of the thesis described the procedure and the approximations used to set up the equations representing the electrical system and the mechanical system. The main difficulty was involved in determining the particular form of the coupling term of mechanical origin in the electrical equation and the coupling term of electrical origin in the mechanical equation. Naturally, the form of the coupling term depended upon the variables selected as independent. It was shown that by selecting the flux, instead of the current, as an independent variable simpler equations result.

The solution of the nonlinear differential equations was obtained by a numerical procedure using the phase-plane method. Increments of flux were selected, and the corresponding increments of armature displacement and time were computed. The current was computed from the flux and the computed values of the armature displacement. In the computational procedure the values of a number of other variables had to be calculated. These included the first and second time rates of change of the flux, the velocity and acceleration of the armature, and the magnetic pull.

A computer program was written for the IBM 650 Computer using the Fortran language. This was necessary in order to obtain a sufficient amount of accurate results to compare with the experimental results. Because of the nature of the original differential equations and the type of solution used, several error sensing calculations were incorporated into the program. The overall computational error was less than one-half of one percent for the increments used in the first and third computational intervals. The computations had to be divided into three intervals because of the conditions existing for the armature. In the first and third intervals the armature velocity and acceleration was zero. However, the armature displacement was the open value in the first interval and was the closed value in the third interval. The second interval was characterized by the fact that the armature was in a transient condition. During the second interval the second time rate of change of the flux took on large positive and negative values in consecutive increments. This caused the one error check to show a deviation of as much as six percent for certain modes of operation. However, the error on the time calculations was zero up to one hundredth of a millisecond or about a one percent error.

Conclusions

The close comparison of the computed transient response with the experimental transient response indicated that the method of solution used was accurate. The model used to represent the electromechanical transducer was accurate in that the effect of mechanical damping and magnetic saturation was included. These two effects are not considered in most solutions because of the increased complication resulting from their consideration. In the particular mathematical model used, the effect of magnetic leakage flux was not considered because of the extra time required to obtain the desired amount of computed values. The effect of the leakage flux can be considered by adding some extra terms in the term involving the resistance in the electrical equation.

Additional refinements could be made in the computer program such as changing the flux increment size for different computational intervals. Also the variables could be stepped a certain percentage over some fixed range. It appears that it would be possible to write a program to solve two or more general second order nonlinear differential equations. However, if the system described by the equations contained spring loaded components with initial values, then special arrangements would have to be made to detect the particular intervals.

The program that was written was used to determine the effect of some of the variables on the transient response of the electromechanical transducer. Provisions were even made to determine the effect of different values of residual flux existing in the magnetic circuit when the coil was energized. The result from the computer program that was especially valuable was the instantaneous value of the magnetic flux. This allowed the calculation of the instantaneous magnetic pull which can not be measured on an experimental model. The magnetic flux is practically impossible to measure directly, especially on transducers with small air gaps.

The results accomplished by the procedure given in this thesis make it possible to accurately predict, for the first time, the response of a singly excited electromechanical transducer without actually having to construct a model.

SELECTED BIBLIOGRAPHY

1. "A Guide to Relay Literature." Automatic Control, (Nov. 1958), pp. 31-42. A good bibliography of relay literature published in periodicals.

2. American Standard for Relays. New York: American Standards Association, Inc., 1957.

3. Betz, H., P. B. Burcham, and G. M. Ewing. Differential Equations with Applications. New York: Harper and Brothers, 1954.

4. Bewley, L. V. Two Dimensional Fields in Electrical Engineering. New York: The Macmillan Co., 1948.

5. Cameron, C. F. "Relay Characteristics and Uses." Tele-Tech and Electronic Industries, Vol. 14, (April 1955), pp. 97, 173-174.

6. ______, and D. D. Lingelbach. "Relay Characteristics." Symposium on Electromagnetic Relays. Princeton, Indiana: Potter and Brumfield, Inc., 1956, pp. 41-48.

7. Fitzgerald, E., and C. Kingsley. Electric Machines. New York: McGraw-Hill Book Co., 1952.

8. Ford, Lester R. Differential Equations. New York: McGraw-Hill Book Co., Inc., 1933.

9. Gardner, M. F., and J. L. Barnes. Transients in Linear Systems, Vol. I. New York: John Wiley and Sons, Inc., 1942.

10. Goldman, S. Transformation Calculus and Electrical Transients. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1949.

11. Investigation of Dynamic Characteristics of Relays. Research Reports SC-4336(TR), SC-4337(TR), SC-4345(TR), SC-4422(RR), and SC-4422(RR). Washington 25, D.C.: Office of Technical Services, Department of Commerce.

12. Ku, Y. H. Analysis and Control of Nonlinear Systems. New York: Ronald Press Co., 1959. Contains a very extensive chronological bibliography dealing with nonlinear systems.

13. ______. Electric Energy Conversion. New York: Ronald Press Co., 1959.

14. Lago, G. V., and D. L. Waidelich. Transient in Electrical Circuits. New York: Ronald Press Co., 1958.

15. Olson, H. F. Dynamical Analogies. 2nd Ed. Princeton, New Jersey: D. Van Nostrand Co., Inc., 1958.

16. Peek, R. L., and H. N. Wagar. Switching Relay Design. Princeton, New Jersey: D. Van Nostrand Co., Inc., 1955.

17. Rudenburg, Reinhold. Transient Performance of Electric Power Systems. New York: McGraw-Hill Book Co., 1950.

18. Symposium on Electromagnetic Relays. Princeton, Indiana: Potter and Brumfield, Inc., 1953, 1954, 1955, 1956, 1957, 1958 and 1959.

19. Synge, J. L., and B. A. Griffith. Principles of Mechanics. New York: McGraw-Hill Book Co., Inc., 1959. Good development of Lagrange's equations and Hamilton's principle.

20. White, D. C., and H. H. Woodson. Electromechanical Energy Conversion. New York: John Wiley and Sons, Inc., 1959.

APPENDIX A

LISTING OF COMPUTER PROGRAM IN FORTRAN LANGUAGE

```
C 0000 0  D. D. LINGELBACH, SOLUTION OF
C 0000 0 ELECTROMECHANICAL TRANSDUCER
C 0000 0  EQUATIONS. TWO SIMULTANEOUS
C 0000 0 NONLINEAR DIFFERENTIAL EQS.
     1 0 READ,A,HI,GR,SI,PO,R,TN,E,G,D,
     1 1 B,Q,W,V,UO,ERR,INC,NO
C 0000 0  CUR= COIL CURRENT, Y= FLUX
C 0000 0  X = ARMATURE POSITION
C 0000 0  CALCULATION OF EQ. CONSTANTS
C 0000 0  AND SET DELTA Y.
         C=12.0/(3.19*A)
         H=HI/GR
         S=SI/GR
         P=8.86E-8/(3.19*A*GR)
         RN=R/(TN*TN)
         EN=E/TN
         X=G
         XD=0.0
         XDD=0.0
         T1=0.0
         DLT2=0.0
         MOVE =1
         U=(PO+SI*G)/GR
         BM=E*TN/(R*(Q+C*D)+E*TN*B)
         Y=W
         DLY=(BM-W)/FLOTF(INC)
         PUNCH,C,H,S,P,U,BM,DLY
         LP=INC/NO
C 0000 0  TWO LOOPS, K LOOP MAKES N
C 0000 0  CALCULATIONS, J LOOP PUNCHES
C 0000 0   EVERY N-TH CALC.
         DO11J=1,LP
         DO 10 K=1,NO
C 0000 0  TIME AND CURRENT CALCULATIONS
C 0000 0  BEFORE ARM. STARTS TO MOVE
         YDA=V*EN
         DI=Q+C*X
         DE=1.0-B*(Y-W)
         YDB=V*RN*DI*(Y-W)/DE
         YD=YDA-YDB
         YDDA=-V*RN*DI*YD/(DE*DE)
         YDDB=-V*RN*C*(Y-W)*XD/DE
         YDD=YDDA+YDDB
         YRA=YDD/YD
```

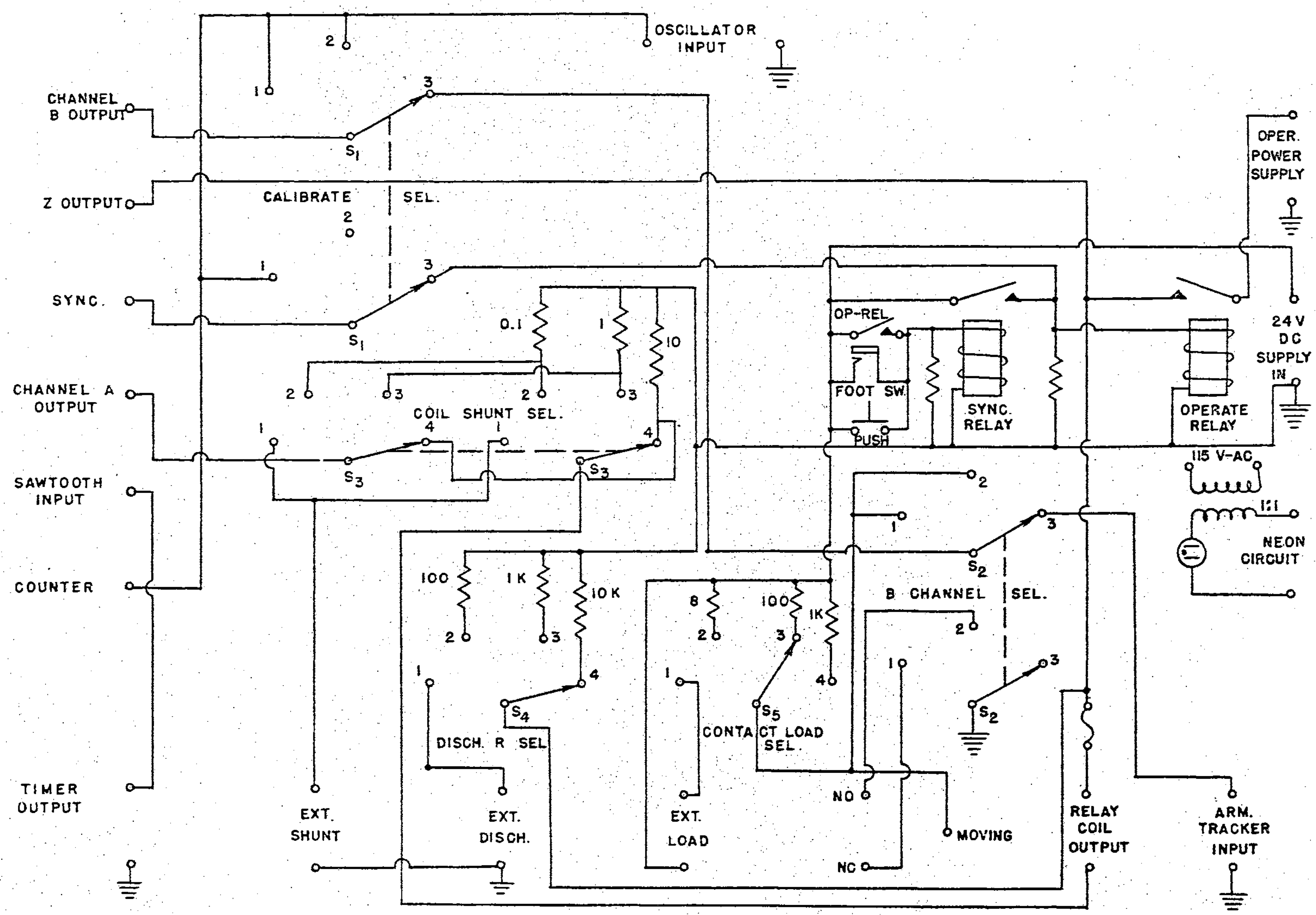
```
              DLYD=YRA*DLY
              YD2=YD+DLYD
              Y=Y+DLY
              DE=1.0-B*(Y-W)
              YDB=V*RN*DI*(Y-W)/DE
              YD=YDA-YDB
              TST1=YD2/YD
              IF(TST1-ERR)4,4,5
         5 0  OV=TST1-ERR
              PUNCH,OV
         4 0  DLT1=DLY/YD
              T1=T1+DLT1
              CUR=DI*(Y-W)/(TN*DE)
              BIAS=PO/GR-P*DLY*DLY
              PYY=P*Y*Y
              IF(PYY-BIAS)10,10,13
C 0000 0        PULL = BACK TENSION.  ARM.
C 0000 0        STARTS TO MOVE. ARM. MOTION
C 0000 0        CALCULATIONS.
        13 0  IF(MOVE-1)3,14,3
        14 0  PUNCH,CUR,T1,Y,PYY,XDD,YD,YDD
              MOVE =MOVE  + NO
         3 0  IF(D-X)6,9,9
         6 0  YDDA=-V*RN*DI*YD/(DE*DE)
              YDDB=-V*RN*C*(Y-W)*XD/DE
              YDD=YDDA+YDDB
              YRA=YDD/YD
              DLYD=YRA*DLY
              XDD=-H*XD-S*X-PYY+U
              XYRA=XDD/YDD
              DLXD=XYRA*DLYD
              XD=XD+DLXD
              IF(XDD)2,7,7
         2 0  XRA=XD/XDD
              DLX=XRA*DLXD
              X=X+DLX
              DLT2=DLX/XD
              CK1=DLY/DLX
              CK2=YD/XD
              TST2=CK1/CK2
              IF(TST2-ERR)7,7,8
         8 0  OV2=TST2-ERR
              PUNCH,OV2
         7 0  IF(D-X)10,12,12
        12 0  PUNCH,CUR,T1,X,Y,PYY,XD,XDD,YD
C 0000 0        ARM. CLOSED. DX AND DV = 0
C 0000 0        TIME AND CURRENT CALC. CONT.
         9 0  XD=0.0
              XDD=0.0
        10 0  CONTINUE
        11 0  PUNCH,CUR,T1,X,Y,PYY,XD,XDD,YD
        11 1  ,YDD,DLT1,DLT2
              GO TO 1
              END
```

APPENDIX B

TEST PROCEDURES AND EQUIPMENT

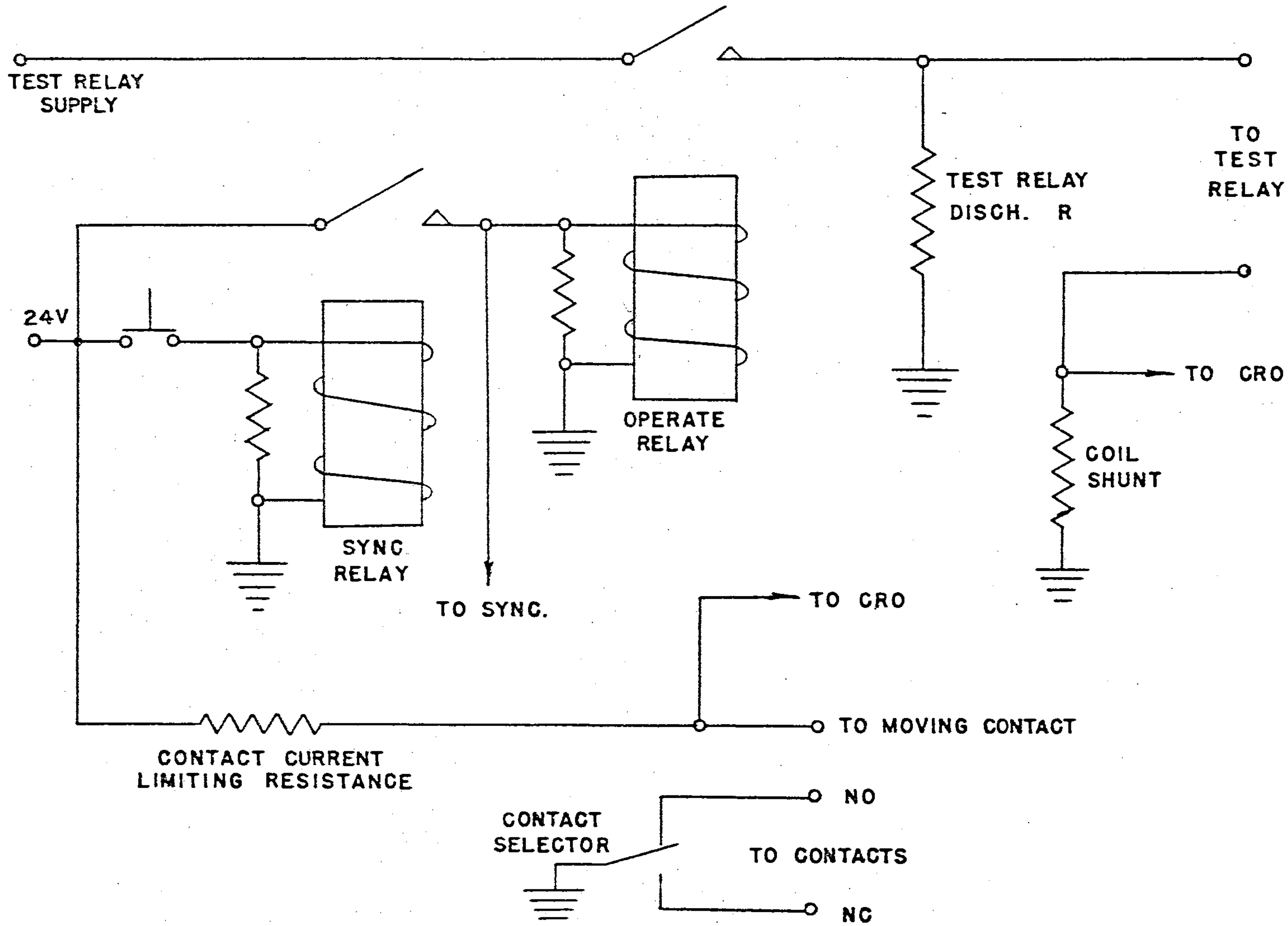
In the study of transients, the instantaneous variation of a parameter is observed. The most desirable piece of equipment for this kind of measurement is the cathode-ray oscillograph or oscilloscope. The next most important consideration is a means of obtaining a record of the event in a minimum time. The advent of the "Polaroid" camera has made this possible. In a minute or so the record of the transient response in the form of traces on a photographic print can be obtained by the use of a "Polaroid" camera attached to the front of the oscilloscope. This short time interval allows another chance to obtain the required data if the previous trys are not successful.

In order to simplify the procedure and to shorten the time required to set up the necessary circuit conditions to obtain the transient coil current, the switching circuit shown in Figure B.1 was used. This shows a complete circuit diagram of the switching scheme used. The main things accomplished by this panel can be more clearly shown by referring to the simplified diagram in Figure B.2. One requirement was that a voltage be obtained from the coil current of sufficient magnitude to drive the oscilloscope Y input. This was obtained by providing a small resistance in series with the coil of the test relay. This resistance is called a shunt. One side of the resistance was connected to the oscilloscope Y input and the other side was grounded. Since coil resistances vary in



CIRCUIT DIAGRAM OF SWITCHING PANEL

Figure B.1



SIMPLIFIED DIAGRAM OF SWITCHING PANEL

Figure B.2.

magnitude some arrangement was provided to keep the shunt resistance approximately one percent of the coil resistance. Another requirement was some means of starting the horizontal motion of the cathode-ray beam before the voltage signal is applied to the Y input of the oscilloscope. This delay of the voltage signal was accomplished by the use of a relay called a sync relay. The sync relay supplied a signal to the trigger input of the oscilloscope, causing the beam to move horizontally. At some time later (approximately 3 milliseconds) the relay under test was energized. This procedure provided the clean break in the change in the shape of the trace at the instant the test relay was energized. The definite location of this point on the trace was necessary in order to obtain accurate values of the pick-up and seating time.

When the relay is de-energized, then some means must be provided for obtaining a fixed resistance as the current decays to zero. This was accomplished by placing a resistance in parallel with the coil of the test relay. This resistance, called the discharge resistance, was arranged to be variable in value to accomodate the various values of coil resistance.

Since the switching panel was designed to test a particular form of the electromechanical transducer, the relay, a circuit was provided for obtaining data about the operation of the contacts.

To obtain the response of the mechanical system a device developed by Professor Cameron's research team was used. This device operates on the sensing of a change in the intensity of a light beam and therefore makes no mechanical connections with the transducer. Since the device has no moving parts and is an open loop type electronic system, its response to high speed changes, such as armature bounce, is very accurate.

There are some commercial devices on the market which perform the function of converting motion to an electrical signal. However, most of these are closed loop electronic systems with feedback which may be subject to hunting at high frequencies of mechanical vibration or require some attachment of the device to the armature. The results obtained by Professor Cameron's light beam device are as accurate as the oscilldscope. The accuracy of the oscilloscope is about 3% when careful measurements are made using the photographic records.

Possibly the most inaccurate measurements were involved with the mechanical measurements such as force and distance. The mechanical force was measured using a hand operated gram gauge. An accuracy of five percent was considered good in these measurements. The air gap values were measured using thickness gauges. Because of the cramped conditions involved in measuring the air gap, accuracies of five percent were considered good for values below twenty thousandths and above ten thousandths. Below ten thousandths an accuracy of $\pm$ one thousandths was feasible. Current and voltage measurements had accuracies of about one percent, depending upon the magnitude involved.

VITA

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Doctor of Philosophy

Thesis: TRANSIENT ANALYSIS OF A SINGLY EXCITED ELECTROMECHANICAL TRANSDUCER

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