JUMP RESONANCE IN A THIRD ORDER NONLINEAR CONTROL SYSTEM

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CHAPTER I

INTRODUCTION

In the last few years much effort has been applied to the study of nonlinear systems. Perhaps much of this has been due to the present space race. Prior to that time efforts in this country have, to a great extent, avoided analyses of systems nonlinear in nature or have linearized those systems if an analysis was absolutely necessary. There has been good reason for this generally, since there are no exact solutions to nonlinear differential equations except in very few cases. One of the most powerful propositions for the solution of linear systems, the superposition principal, is of no value in nonlinear systems. These properties and others such as entrainment, frequency demultiplication and multiplication, and discontinuous jump resonance, peculiar to nonlinear systems alone, have to a certain extent hindered the growth of the field.

The particular property known as jump resonance, also referred to as ferroresonance by some writers, is the subject of this thesis. The phenomena of jump resonance can occur in a physical system whenever the resonance curve for the system can be made to be skewed to the right or the left by the addition of a nonlinearity into the system. The skewing of the resonance curve will produce multiple-valued amplitudes for a single frequency. An example of this type of multi-valuedness is shown in Figure 1. Duffing (1), an early writer in the field, first published his work in 1918. In his

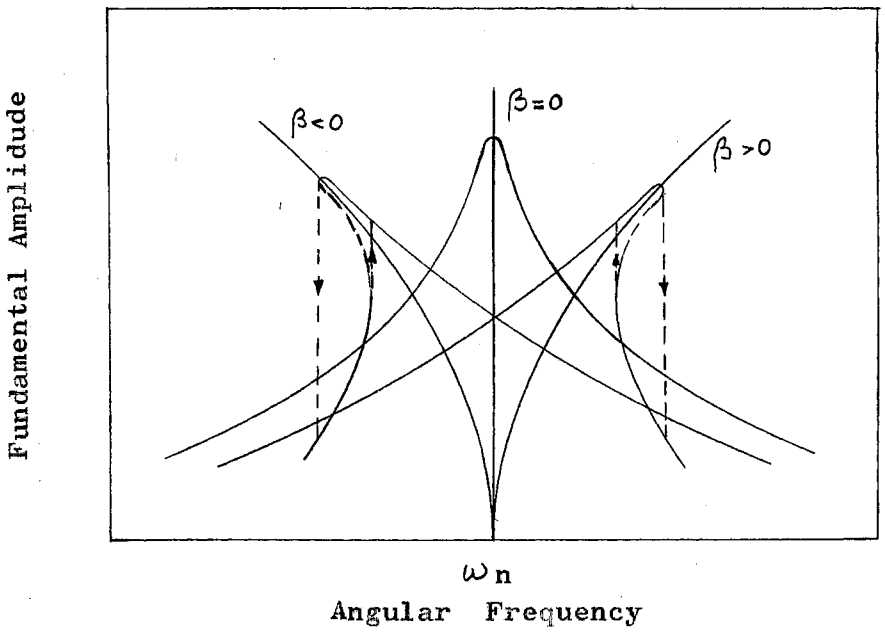

Figure 1. Resonance in Linear and Nonlinear Systems

disclosure he discusses the various ramifications of the equation

$$y'' + \omega_n^2 y + ay^3 = G \sin \omega t. \qquad \text{I-1}$$

Solutions to this equation exhibit jump resonance.

A later report by Appleton (2) gives a detailed analysis of the phenomena in a second order dissipative system. Appleton observed the occurrence of jumps in a vibration galvanometer. In a more recent work, Klotter (3) makes a general analysis of second order vibration systems.

Discontinuous jump resonance has also been studied in second order control systems (4,5). In a paper by Hopkin and Ogata (6) a third order system is studied in which it is assumed that linear modes of operation exist at low frequencies and nonlinear modes at high frequencies. Many other higher ordered systems have been studied and reported, however, very little mention is made of

discontinuous jumps.

In this paper an investigation is reported that has been made to determine conditions required to produce jump resonance in a typical third order control system having only real negative poles in the forward transfer function. The conditions, input amplitude and frequency, have been found to be related to system parameters by simple relationships. Restriction of the study to control systems whose forward transfer function has only real negative poles (the possibility of one at the origin is not excluded) is a weak restriction and should be no deterrent of application to systems with complex poles. A novel method of graphical solution is presented; though laborious in its construction, it gives an insight into solutions in the neighborhood of phase angles of 90 degrees.

CHAPTER II

SYSTEM ANALYSIS

The system under investigation will be analyzed by assuming that the solution to the system equation is a single harmonic term having the same frequency as the forcing function. That this is only an approximation is recognized, and it will be shown that such an approximation is sufficiently accurate for many engineering applications. It must be remembered that many harmonics are present; however, the higher ordered terms are relatively small.

A graphical method of analysis is utilized in this chapter and its relation with the fundamental frequency response curve presented.

System Equation

The system investigated in this research is an elementary control system of the type shown diagrammatically in Figure 2. It is a system with a control element, h, in the feedback path. This element, h, is a nonlinear device that converts y, the output or response into f(y).

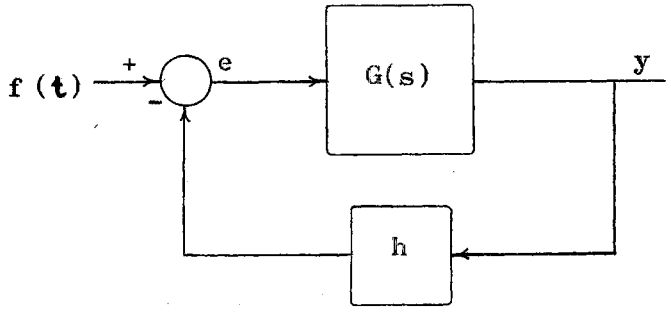

Figure 2. Nonlinear Control System

The forward transfer function, G(s), for the system is defined in the following way:

$$\frac{Y(s)}{E(s)} = G(s) = \frac{K}{s(T_1 s+1)(T_2 s+1)} \qquad \text{II-1}$$

Y(s) is the Laplace transform of the time variable output or response y; E(s) is the Laplace transform of the time variable error signal e; and K and T_1 and T_2 are the usual gain and time delay constants, respectively, for the system. The block labeled h in Figure 2 is not representable as a transfer function since it is the nonlinear element. It is a time invariant element whose output and input is related as follows:

$$f(y) = y + cy^3 \qquad \text{II-2}$$

Since Equation II-1 relates the output to the input for the block G(s), it may be written

$$Y(s)\ s(T_1 s + 1)(T_2 s + 1) = KE(s). \qquad \text{II-3}$$

This equation becomes, after transformation into the time domain,

$$T_1 T_2 y''' + (T_1 + T_2)y'' + y' = Ke. \qquad \text{II-4}$$

In this equation the primes are used to denote time derivatives. With reference to Figure 2, the following may also be written

$$e = f(t) - f(y). \qquad \text{II-5}$$

It must be remembered that y and e are both functions of time although they are not so explicitly written. Substitution of II-2 and II-5 into II-4 yields

$$T_1 T_2 y''' + (T_1 + T_2)y'' + y' + Ky + Kcy^3 = Kf(t). \qquad \text{II-6}$$

After division by $T_1 T_2$, there is obtained

$$y''' + \left[(T_1 + T_2)/(T_1 T_2)\right] y'' + (1/T_1 T_2)\, y' + (K/T_1 T_2)y + (Kc/T_1 T_2)y^3 = (K/T_1 T_2)\, f(t). \qquad \text{II-7}$$

For ease in writing, Equation II-7 will be written as

$$\mu y''' + \gamma y'' + \delta y' + \alpha y + \beta y^3 = F \cos(\omega t - \varphi) \qquad \text{II-8}$$

in which

$$\mu = 1, \qquad \text{II-9}$$

$$\gamma = (T_1 + T_2)/T_1T_2, \qquad \text{II-10}$$

$$\delta = 1/T_1T_2, \qquad \text{II-11}$$

$$\alpha = K/T_1T_2, \qquad \text{II-12}$$

$$\beta = Kc/T_1T_2, \qquad \text{II-13}$$

and

$$F \cos(\omega t - \varphi) = (K/T_1T_2)\, f(t). \qquad \text{II-14}$$

The constants defined above are all real positive quantities. The input to the system, $f(t)$, is a time variable cosine function of angular frequency ω and amplitude F. It is the forcing function for the system. The phase angle φ is the angle between the forcing function and the fundamental amplitude response function y of the system. So that the solution will be in its simplest form, the forcing function $F \cos(\omega t - \varphi)$ is written

$$F \cos(\omega t - \varphi) = v \cos \omega t + u \sin \omega t, \qquad \text{II-15}$$

in which

$$v = F \cos \varphi, \qquad \text{II-16}$$

$$u = F \sin \varphi, \qquad \text{II-17}$$

$$\varphi = \arctan u/v, \qquad \text{II-18}$$

and

$$F^2 = v^2 + u^2. \qquad \text{II-19}$$

In the subsequent analysis it is required that $|F|$ be a fixed value for any given range of frequency ω. The angle φ, as previously defined above, will not be fixed and it constitutes an unknown quantity. The fundamental amplitude response is unknown in magnitude; however, its phase position is taken as the reference phasor.

First Approximation Solution

Equation II-8 has no known exact solutions; however, a number of approximate solutions may be obtained. Klotter (3) uses the Ritz Averaging Method for obtaining a solution to a second order forced vibration equation and it is the method to be used in this thesis to obtain a solution which includes the third harmonic components. For the first approximation a simpler technique is more useful (7). The solution desired is the steady state value and is assumed in this approximation to be

$$y = A_1 \cos \omega t \qquad \text{II-20}$$

in which $A_1 > 0$ is the fundamental amplitude that is to be evaluated. Upon substitution of II-20 into Equation II-8 and if the forcing function is written as given by the right hand term of Equation II-15, Equation II-8 becomes

$$A_1\omega(\mu\omega^2-\delta) \sin \omega t + (\alpha-\gamma\omega^2 + 3/4\beta A_1^2) A_1 \cos \omega t + 1/4\beta A_1^3 \cos 3\omega t = v \cos \omega t + u \sin \omega t. \qquad \text{II-21}$$

In this approximation the third harmonic term is assumed to be small. Thus for equality it is necessary that

$$v = (\alpha - \gamma\omega^2 + 3/4\beta A_1^2)A_1 \qquad \text{II-22}$$

and

$$u = A_1\omega(\mu\omega^2 - \delta). \qquad \text{II-23}$$

Upon substitution of Equations II-22 and II-23 into Equation II-19, there is obtained

$$(\alpha - \gamma\omega^2 + 3/4\,\beta A_1^2)^2 A_1^2 + A_1^2\,\omega^2(\mu\omega^2 - \delta)^2 = F^2. \qquad \text{II-24}$$

A graphical analysis will be used to study this equation more fully.

1. The u - v Plane

A_1 of Equation II-24 may be determined for variable ω through the use of a digital computer, but more of an insight may be obtained by plotting u and v in a u - v plane. u and v are then coordinates of a point for some value of A_1 and ω. For ω fixed and variable A_1, a locus of constant ω is obtained. This is illustrated in Figure 3.

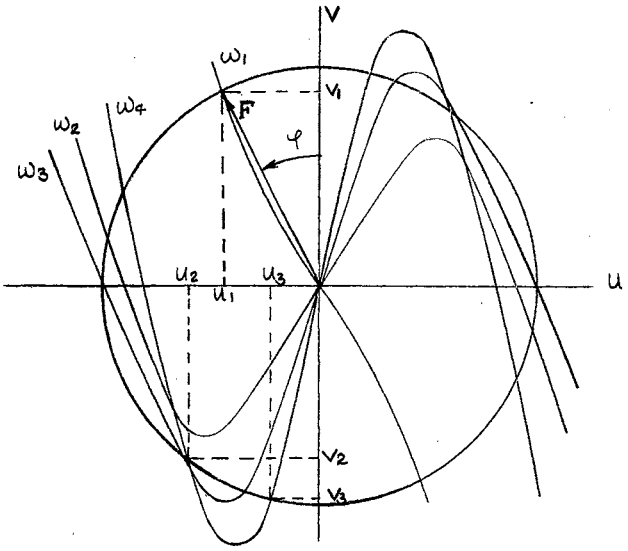

Figure 3. The u - v Plane and Loci of Constant ω
$\omega_1 < \omega_2 < \omega_3 < \omega_4$

Illustrated on this figure also is a circle of radius F. Equation II-19 is the equation of this circle. Values of u and v prescribed by the intersection of the circle of radius F with the loci of constant ω are solutions for the forcing function of amplitude F. The angle φ is measured in a counterclockwise direction from the vertical to the radius vector F. A_1 is used as the reference phasor. It lies along v and is directed in the positive v direction. The magnitude of A_1 is calculated using Equation II-23.

$$A_1 = \frac{u}{\omega(\mu\omega^2 - \delta)} \qquad \text{II-25}$$

The solution thus obtained represents the amplitude of the fundamental response for the applied forcing function of magnitude F at the frequency ω. The forcing function leads the response function by an angle φ. If the frequency is now changed, new values of A_1 and φ will be obtained. A set of data for a fundamental amplitude frequency response curve may thus be determined. This curve will resemble the $\beta > 0$ curve of Figure 1. It may be observed from Figure 3 that, for values of F sufficiently large, three intersections of a constant ω locus are possible in the left half plane. Intersections in the right half plane for the curves drawn yield values of $A_1 < 0$. These values of A_1 are not defined for it is required that $A_1 > 0$. There are solutions possible for $A_1 > 0$ in the fourth quadrant; however, the amplitudes are very small and are for the most part out of the range of interest. Solutions for the assumed conditions are not physically realizable in the first quadrant. (The phase angle for a third order system approaches 270 degrees as ω approaches infinity.)

The shape of the constant ω loci suggests writing an equation for these curves. This may be accomplished by eliminating A_1 in Equations II-22 and II-23. The result is

$$v = au + bu^3 \qquad \text{II-26}$$

in which

$$a = \frac{\alpha - \gamma\omega^2}{\omega(\mu\omega^2 - \delta)} \qquad \text{II-27}$$

and

$$b = \frac{3/4\,\beta}{\omega^3(\mu\omega^2 - \delta)^3} \; . \qquad \text{II-28}$$

In Equation II-27, a is positive for $\frac{\alpha}{\gamma} < \omega^2 < \frac{\delta}{\mu}$. This is the range

of interest in this study. For this range of ω, b is negative. In Equation II-26, if $|u|$ is small and u is negative, the cubic term is small relative to the linear term for ω near the value of $\sqrt{\alpha/\gamma}$. For larger values of $|u|$, the cubic term is more prominent and for some negative value of u, $v = 0$. As ω is increased, $|u|$ for $v = 0$ becomes larger and approaches a limiting value. This maximum value of $|u|$ may be established by differentiating u of Equation II-23 with respect to ω and equating the result to zero. The details are performed in Appendix A and the results are given here. The angular frequency for maximum $|u|$ is given by Equation A-39 and is

$$\omega_u^2 = \frac{1}{8\mu\gamma}\left[2\gamma\delta + 3\alpha\mu \pm \sqrt{(2\gamma\delta + 3\alpha\mu)^2 - 16\alpha\delta\gamma\mu}\right]. \qquad \text{II-29}$$

The amplitude for maximum $|u|$ may be found by using Equation A-46.

$$A_{1u} = 2\omega_u \sqrt{\frac{\gamma(\delta - \mu\omega_u^2)}{3\beta(3\mu\omega^2 - \delta)}} \qquad \text{II-30}$$

The value of maximum $|u|$ is

$$|u|_{max} = 2\omega_u^2 (\mu\omega_u^2 - \delta)\sqrt{\frac{\gamma(\delta - \mu\omega_u^2)}{3\beta(3\mu\omega_u^2 - \delta)}}\,. \qquad \text{II-31}$$

The constant ω locus, for the ω_u as determined by II-29, intersects the u-axis at $-|u|_{max}$. The value of $|u|_{max}$ is given by II-31.

The constant ω locus for $\omega > \omega_u$ will intersect the u-axis at a value of $|u| < |u|_{max}$. For this ω, a is greater than a_u. (a_u is the a corresponding to the ω_u angular frequency.) The constant ω locus lies below the ω_u locus for values of u between the origin and the point at which the two loci intersect. This intersection occurs at negative values of v.

The constant ω locus for $\omega < \omega_u$ will intersect the u-axis at

$|u| < |u|_{max}$. For this ω, a is less than a_u and its locus lies above the constant ω_u locus for values of u between the origin and the point of intersection. This intersection occurs at a positive value of v.

A circle of radius $F = |u|_{max}$ drawn on the u - v plane will pass through the point $(-|u|_{max}, 0)$. This F circle can intersect constant ω loci only in the second quadrant, or if φ is sufficiently larger than 90 degrees, it can intersect constant ω loci in the third quadrant. (Sufficiently larger means in the general neighborhood of 150 degrees.) For this forcing function, $F = |u|_{max}$, the angle φ increases as the frequency increases. The angle φ is 90 degrees at the value of $\omega = \omega_u$. As the frequency is increased ($\omega > \omega_u$) the angle φ may become smaller or a sudden jump to a much larger value of φ may occur. The first harmonic approximation does not predict this jump in phase.

For values of $F > |u|_{max}$, the angle φ is always less than 90 degrees or much larger than 90 degrees. No real solutions exist in the neighborhood of 90 degrees. The solutions which exist are illustrated by (u_2, v_2) and (u_3, v_3) of Figure 3. The point (u_2, v_2) is not physically realizable.

No general formula can be given specifying this region of no real solutions; but numerical solutions to Equation II-24, in the neighborhood of $\varphi = 90$ degrees for F sufficiently large, will yield complex values for A_1. These are unacceptable for the physical problem. The real solutions for A_1, such as determined by (u_2, v_2), are not physically realizable and are within regions of instability. Such a region is defined in the A_1 -ω plane and has its counterpart

in the u - v plane. Appendix C describes it in the u - v plane. The point (u_3, v_3) in Figure 3 is both stable and realizable.

2. The $A_1 - \omega$ Plane

The $A_1 - \omega$ plane, Figure 1, is partially reproduced as Figure 4. The skewing to the right of vertical is a consequence of $\beta > 0$. ($\beta < 0$ causes a skewing to the left.) The various components of importance in this figure are the backbone curve, locus of vertical tangents, and the locus of horizontal tangents. The backbone curve is the curve defined for $v = 0$. It is the condition for which a

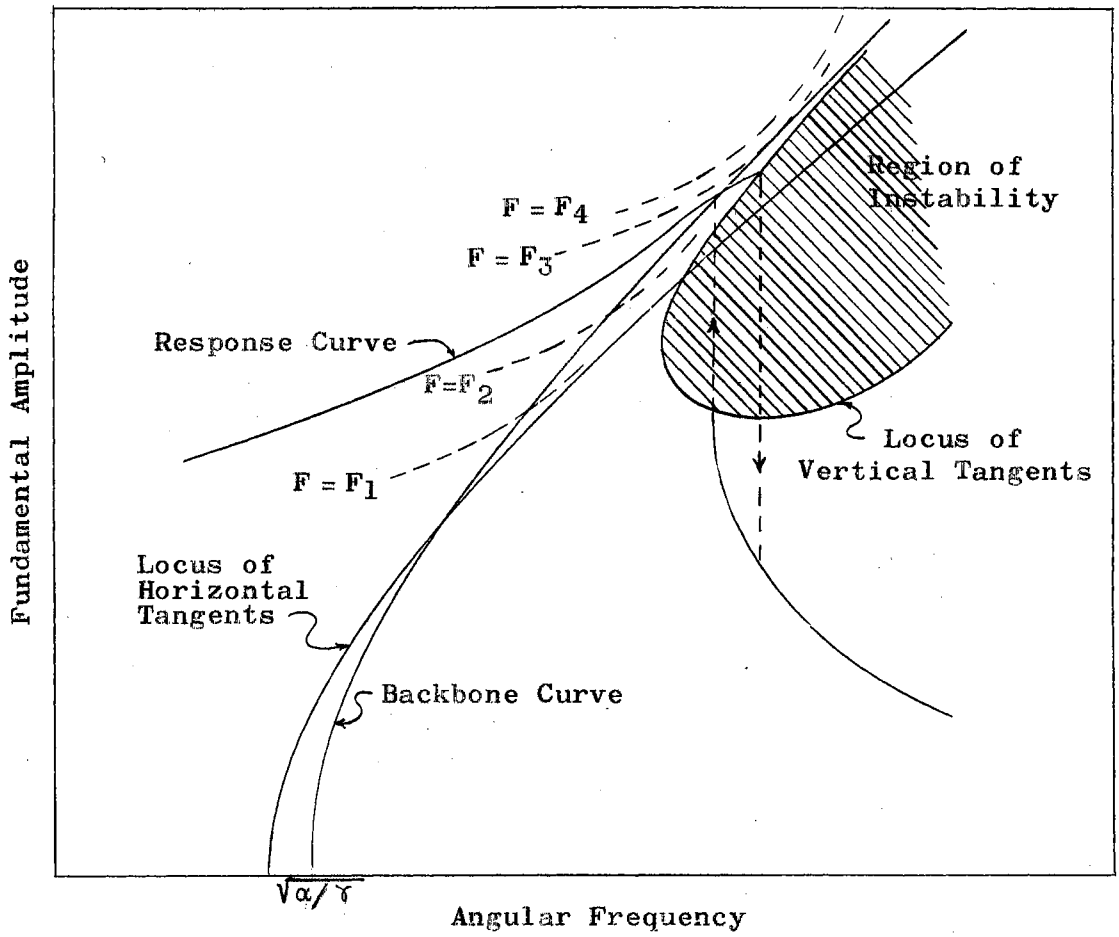

Figure 4. $A_1 - \omega$ Plane Showing the General Region of Instability

90 degree phase angle exists between the forcing function F and the fundamental response A_1. Equation II-22 equated to zero yields the backbone curve.

$$\alpha - \gamma\omega^2 + 3/4\beta A_1^2 = 0 \qquad \text{II-32}$$

This expression may be solved for A_1. Its value is

$$A_1 = + \sqrt{\frac{\gamma\omega^2 - \alpha}{3/4\,\beta}} \qquad \text{II-33}$$

and it is shown plotted on Figure 4. The ω intercept is $\omega = +\sqrt{\alpha/\gamma}$ The condition $v = 0$ yields for u,

$$u = F \qquad \text{II-34}$$

or

$$A_1 \omega (\mu\omega^2 - \delta) = F. \qquad \text{II-35}$$

If A_1 from Equation II-33 is substituted in Equation II-35, the real value of frequency satisfying this expression is the frequency at which intersection occurs. Thus

$$\omega (\mu\omega^2 - \delta) \sqrt{\frac{(\gamma\omega^2 - \alpha)}{3/4\,\beta}} = F \qquad \text{II-36}$$

If both sides of II-36 are squared there is obtained

$$\omega^2 (\mu\omega^2 - \delta)^2 \left(\frac{\gamma\omega^2 - \alpha}{3/4\,\beta}\right) - F^2 = 0. \qquad \text{II-37}$$

This is a fourth degree equation in ω^2 and has no general solution. An upper bound on F, however, may be determined. Such a limit on F was established in the preceding section in the discussion of the u - v plane. The Equation II-35 is shown plotted on Figure 4 for several values of F. The intersection of the backbone curve and F defines a particular response curve. The value $F = F_3$ corresponds

to the $|u|_{max}$ value. It is quite apparent that real positive solutions are not possible for $F > F_3$ in a region in the vicinity of 90 degrees. F_3 is shown tangent to the backbone curve.

The region of instability indicated on Figure 4 is defined by the locus of vertical tangents (7). The locus may be determined by taking the derivative of Equation II-24 with respect to A_1 and equating the derivative of ω^2 to zero. The result is

$$(\alpha - \gamma\omega^2 + 3/4\beta A_1^2)(\alpha - \gamma\omega^2 + 9/4\beta A_1^2) + \omega^2(\mu\omega^2 - \delta)^2 = 0. \qquad \text{II-38}$$

This equation defines the region of instability in the $A_1 - \omega$ plane. The expression may be solved for A_1^2 and its solution is

$$A_1^2 = \frac{8}{9\beta}\left[-(\alpha - \gamma\omega^2) \pm \sqrt{a^2 - 3}\right]. \qquad \text{II-39}$$

In this expression a is the quantity previously defined in Equation II-27.

For $a^2 = 3$,

$$A_1^2 = \frac{8}{9\beta}(\gamma\omega^2 - \alpha) \qquad \text{II-40}$$

the double root on the extreme left is obtained. Since a is given by

$$a = \frac{\alpha - \gamma\omega^2}{\omega(\mu\omega^2 - \delta)} \qquad \text{II-27}$$

then, the expression $a\mu\omega^3 + \gamma\omega^2 - a\delta\omega - \alpha = 0.$ II-41

A solution for the real positive value of ω in this cubic yields the frequency at which A_1 has a single value. Substitution of the numerical value in expression II-40 then leads to the numerical value of A_1.

The locus of horizontal tangents is found by taking the derivative of II-24 with respect to the frequency ω, and equating the

derivatives of A_1 to zero. The details of this derivation are performed in the appendix and the final result is given by Equation A-32. The equation is repeated here.

$$A_1 = \pm \left\{ \frac{2}{3\gamma\beta} \left[3\mu^2\omega^4 + 2(\gamma^2 - 2\mu\delta)\omega^2 + (\delta^2 - 2\alpha\gamma) \right] \right\}^{1/2} \qquad \text{II-42}$$

The equation is shown plotted on Figure 4. It is of interest to note the effect of a change in parameters on this curve relative to the backbone curve.

A form of the equation for the locus of horizontal tangents is

$$2\gamma(\alpha - \gamma\omega^2 + 3/4\beta A_1^2) = (\mu\omega^2 - \delta)(3\mu\omega^2 - \delta). \qquad \text{II-43}$$

The left side of the equality is zero for $v = 0$. This defines the backbone curve. The zeros on the right are $\omega^2 = \delta/\mu$ and $\omega^2 = \delta/3\mu$. Thus at $\omega^2 = \delta/\mu$ or $\omega^2 = \delta/3\mu$ the locus of horizontal tangents intersects the backbone curve. The intersection may be found by substituting these values of ω in the equation

$$\alpha - \gamma\omega^2 + 3/4\beta A_1^2 = 0. \qquad \text{II-32}$$

The results are

$$A_1\Big|_{\omega^2 = \delta/\mu} = \left[\frac{4}{3} \frac{\alpha}{\beta} \left(\frac{\gamma\delta}{\mu\alpha} - 1 \right) \right]^{1/2} \qquad \text{II-44}$$

and

$$A_1\Big|_{\omega^2 = \delta/3\mu} = \left[\frac{4}{9} \frac{\alpha}{\beta} \left(\frac{\gamma\delta}{\mu\alpha} - 3 \right) \right]^{1/2} \qquad \text{II-45}$$

The latter equation is the smaller of the two amplitudes. If $\gamma\delta/\mu\alpha = 3$, A_1 is zero. The locus of horizontal tangents lies to the right of the backbone curve. The backbone curve and the locus of horizontal tangents intersect at

$$(\omega = \sqrt{\delta/3\mu}\ ,\ A_1 = 0)\ \text{and}\ \left(\omega = \sqrt{\delta/\mu}\ ,\ A_1 = 2\left[\frac{2}{3}\ \frac{\alpha}{\beta}\right]^{1/2}\right) \qquad \text{II-46}$$

The locus crosses the ω axis for

$$3\mu^2\omega^4 + 2(\gamma^2 - 2\mu\delta)\omega^2 + (\delta^2 - 2\alpha\gamma) = 0. \qquad \text{II-47}$$

The solution for ω^2 is

$$\omega^2 = \frac{1}{6\mu^2}\left[-2(\gamma^2 - 2\mu\delta) \pm \sqrt{4(\gamma^2 - 2\mu\delta)^2\mu^2 12(\delta^2 - 2\alpha\gamma)}\right]. \qquad \text{II-48}$$

For a real solution to this expression, it is required that $\delta^2 < 2\alpha\gamma$. This is true in a system in which jump resonance can occur.

The solution for A_1 for $\omega^2 = \alpha/\gamma$ is

$$A_1\Big|_{\omega^2 = \alpha/\gamma} = + \frac{\mu\alpha}{\gamma}\left\{\frac{2}{3\gamma\beta}\left(\frac{\gamma\delta}{\mu\alpha} - 3\right)\left(\frac{\gamma\delta}{\mu\alpha} - 1\right)\right\}^{1/2}. \qquad \text{II-49}$$

CHAPTER III

FORCING FUNCTION FOR JUMP RESONANCE

In the preceding chapter certain fundamental concepts directly related to the solution of the system equation were developed. These concepts will now be used to prescribe conditions necessary for jump resonance to occur. This chapter will establish the following: (1) the forcing function amplitude required to produce jump resonance; (2) maximum forcing function for real amplitudes in the vicinity of 90 degrees phase displacement between forcing function and response function; and (3) the frequency at which lower jump resonance occurs.

Threshold Forcing Function Amplitude

The maximum amplitude of the forcing function, for which no jump can occur, will be defined to be the threshold forcing function amplitude F_T. This F_T may be found by writing $F = f(u)$. Such a representation is given by Equation A-16 in the appendix.

$$b^2u^6 + 2abu^4 + (1 + a^2)u^2 = F^2 \qquad \text{III-1}$$

The maximum and minimum values of F as a function of u are then given by

$$\frac{dF}{du} = 3b^2u^4 + 4abu^2 + 1 + a^2 = 0. \qquad \text{III-2}$$

Comparison of this equation with Equation A-26 reveals the two to

be identical. The roots of Equation III-2 are

$$u^2 = \frac{1}{3b}\left(-2a \pm \sqrt{a^2 - 3}\right). \qquad \text{III-3}$$

If $a^2 = 3$, then the value of u^2 is

$$u^2 = -\frac{2a}{3b}. \qquad \text{III-4}$$

Neither a maximum nor a minimum occurs at this point. This is also the point at which the locus of vertical tangents has a single value. Thus the fundamental response curve is tangent to the locus of vertical tangents for this value of forcing function amplitude. In Equation III-4, $a = \pm\sqrt{3}$. The value $a = +\sqrt{3}$ is to be used for $a > 0$ in the range $\alpha/\gamma < \omega^2 < \delta/\mu$. Upon substitution of u from Equation III-4 into Equation III-1, the value of F_T is

$$F_T = \sqrt{-\frac{8}{9}\frac{\sqrt{3}}{b}}. \qquad \text{III-5}$$

Thus for values of $F \leqq F_T$ jump resonance can not occur. This limit on F is important for it establishes the maximum amplitude of the input signal that a system may receive. If a given system is subject to jump resonance, jump can be prevented from occurring by placing a limit on its input.

A frequency limit is also established for $a^2 = 3$. Roots to Equation II-41,

$$a\mu\omega^3 + \gamma\omega^2 - a\delta\omega - \alpha = 0 \qquad \text{II-41}$$

will yield the frequency limit. The equation has only one real positive root for ω. This may be found by using the general formula for the cubic.

Maximum Forcing Function Amplitude For Solution Near $\varphi = 90$ Degrees

As previously established, values of F exceeding $|u|_{max}$ as given by Equation II-31,

$$|u|_{max} = \left| 2\omega^2 (\mu\omega^2 - \delta) \sqrt{\frac{\gamma(\delta - \mu\omega^2)}{3\beta(3\mu\omega^2 - \delta)}} \right| \qquad \text{II-31}$$

yield non-real values of A_1 in the neighborhood of $\varphi = 90$ degrees. Real solutions do exist for $|F| = |u|_{max}$ for $\varphi \leqq 90$ degrees. For solutions to be continuous in the region, $|F|$ must be less than $|u|_{max}$; however, no criteria have been established setting a limit on $|F|$ for continuous real solutions.

Frequency of Lower Jump Resonance

The frequency at which a sudden increase in amplitude occurs for decreasing frequency is the frequency of lower jump resonance. This frequency may be found by means of a graphical solution in the u - v plane. The locus of vertical tangents plotted in the u - v plane, intersected by the circle of magnitude F, determines the values of u and v at which jump occurs. The procedure used is presented in Appendix C.

CHAPTER IV

EXPERIMENTAL ANALYSIS

In Chapter II an analysis was made to determine the conditions under which jump resonance occurred in a third order nonlinear control system. In order to verify the analysis a simulation study was made of three representative systems governed by the equation previously analyzed. The systems were simulated on a Donner electronic analog computer and fundamental amplitude response curves for three different forcing function amplitudes were obtained. Values of 3, 4, and 5 were used for the ratio $\delta\gamma/\mu\alpha$. The value of $(\beta T_1 T_2/K) = c$ (Equation II-13) was maintained fixed.

This chapter will discuss the system simulation and the experimental procedures used to verify the analysis.

System Simulation

The equation programmed on the analog computer was Equation II-6.

$$T_1T_2y''' + (T_1 + T_2)y'' + y' + Ky + Kcy^3 = Kf(y) \qquad \text{II-6}$$

The equation was written in this form in order to experience less difficulty in the programming procedure. A simplified block diagram of the system is shown in Figure 5.

The box labeled G, the forward transfer function, was simulated in accordance with the usual transfer function techniques (9). A detailed sketch of G that was used is shown in Figure 6. The two

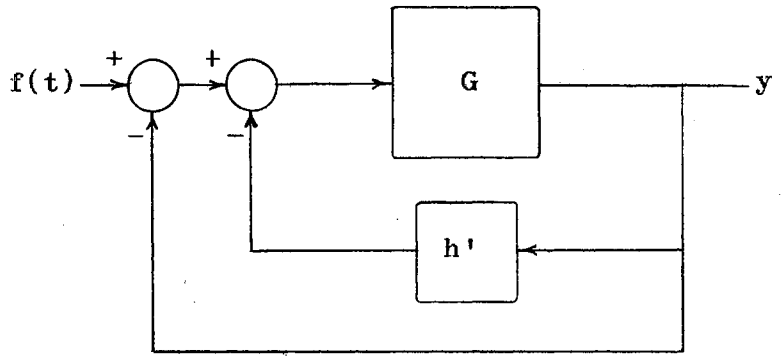

Figure 5. Simplified Block Diagram of the System

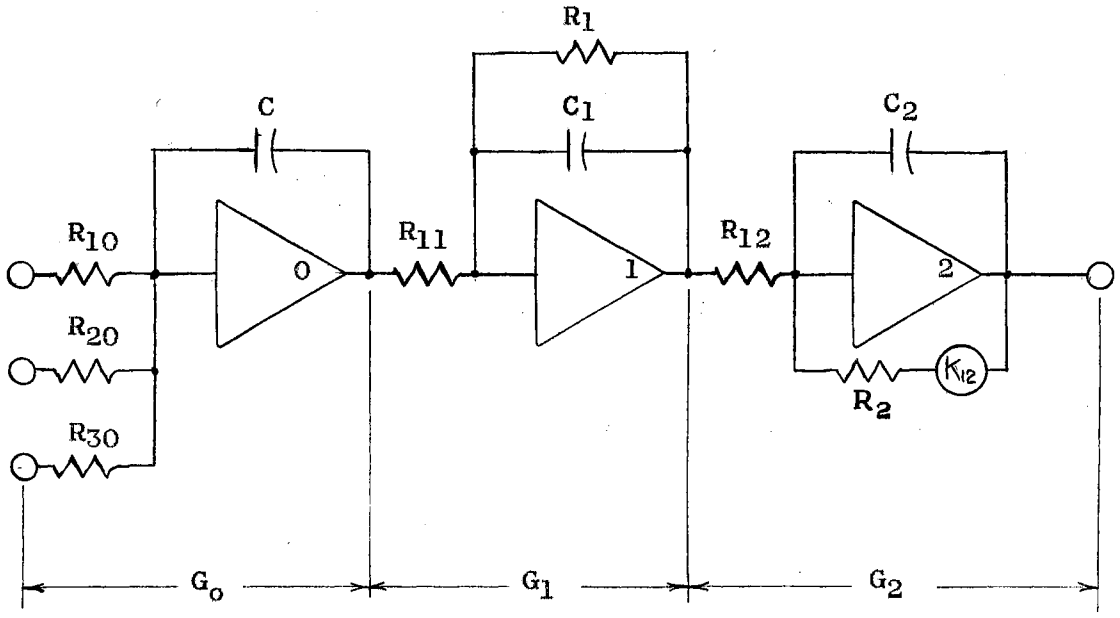

Figure 6. Analog Computer Representation for the Forward Transfer Function

summing elements in Figure 5 have been replaced by the summing-integrating amplifier 0. The transfer function for each individual amplifier was defined as follows:

$$G_o = \frac{1}{R_{io}Cs} \quad , \quad i = 1,2,3 \qquad \text{IV-1}$$

$$G_1 = \frac{R_1/R_{11}}{(T_1 s + 1)} \quad , \quad T_1 = R_1 C_1 \qquad \text{IV-2}$$

$$G_2 = \frac{R_2/R_{22}}{(T_2 s + 1)} \quad , \quad T_2 = R_2 C_2 \qquad \text{IV-3}$$

The over-all transfer function is the product of the individual transfer functions

$$G = G_o G_1 G_2 \qquad \text{IV-4}$$

or

$$G = \frac{\frac{1}{R_{io}C}\ \frac{R_1}{R_{11}}\ \frac{R_2}{R_{22}}}{s(T_1 s + 1)(T_2 s + 1)} \quad . \qquad \text{IV-5}$$

The purpose of the three resistors at the input to amplifier 1 was to provide for the three inputs f(t), y, and cy^3.

The nonlinearity was produced by means of a ten-segment diode function generator. The complete schematic of the computer circuit used in the simulation study is shown in Figure 7.

The system scale factors used were 2 and 4. The component values for the computer are given in Table I. These values were determined using the usual analog computer techniques (10).

Experimental Procedure

The systems studied were chosen to be representative of systems normally encountered in practice. Because these systems generally have time constants T_1 and T_2 in the order of 1 second,

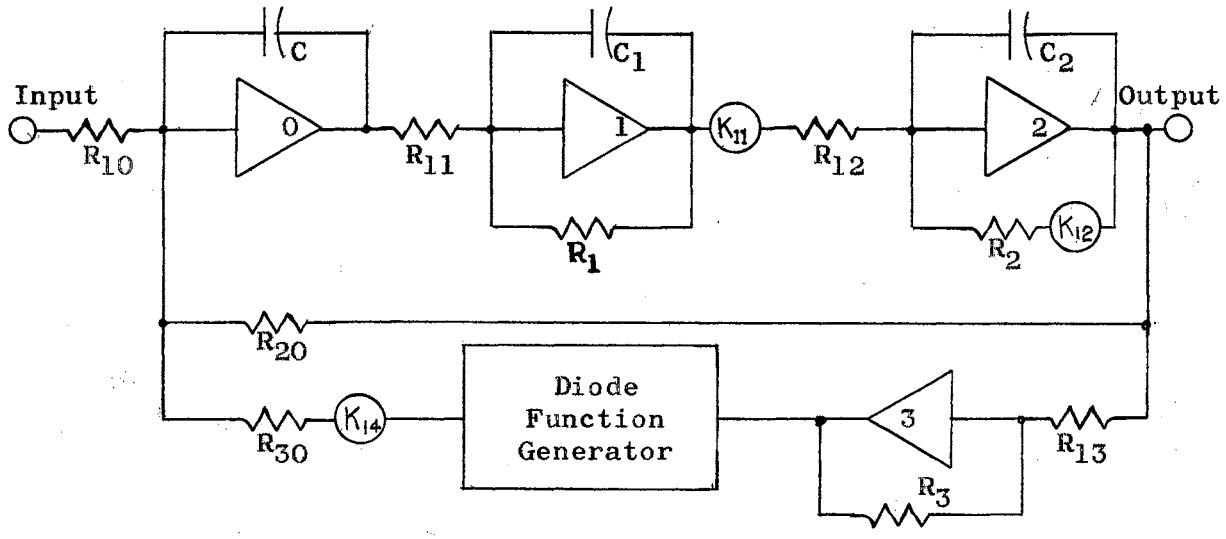

Figure 7. Computer Circuit for Simulation of System Equation

a time scale transformation was made so that no unnecessary delays would be experienced in waiting for transients to disappear. The time constant T_1 was fixed at 0.001 seconds while the constant T_2 was given the three values 0.05, 0.0167, and 0.001 seconds. It would have been desirable to use a fourth value of $T_2 = \infty$; however, the system is unstable for this value. The gain K was fixed at 400, a value that gave a stable closed loop control system for all values of T_2.

The system shown in Figure 7 was simulated on a Donner electronic analog computer. A Hewlett-Packard sine wave oscillator connected to the input served to produce the forcing function for the system. A DuMont cathode ray oscilloscope and Hewlett-Packard a-c vacuum tube voltmeter were connected to the output. A Hewlett-Packard a-c vacuum tube voltmeter was also connected to the input. A Deltron Company phase meter was connected to the input and the output and the computer components adjusted to values selected from

TABLE I

COMPUTER COMPONENT VALUES

Component	$\delta\gamma/\mu\alpha$		
	3	4	5
T_1 seconds	0.001	0.001	0.001
T_2 seconds	0.05	0.0167	0.001
c	0.35×10^{-3}	0.35×10^{-3}	0.35×10^{-3}
K	400	400	400
K_{11}	0.2	0.6	1.0
K_{12}	0.2	0.6	1.0
K_{14}	0.7	0.7	0.7
R_{10}	0.250	0.250	0.250
R_{20} MΩ	0.250	0.250	0.250
R_{30} MΩ	0.1	0.1	0.1
R_{11} MΩ	0.1	0.1	0.1
R_1 MΩ	0.1	0.1	0.1
C_1 μfd	0.01	0.01	0.01
R_{12} MΩ	0.1	0.1	0.1
R_2 MΩ	0.1	0.1	0.1
C_2 μfd	0.01	0.01	0.01
R_{13} MΩ	0.5	0.5	0.5
R_3 MΩ	1.0	1.0	1.0
μ	1	1	1
α	8×10^7	24×10^7	40×10^7
β	2.8×10^4	8.4×10^4	14×10^4
δ	2×10^5	6×10^5	10×10^5
γ	1.2×10^3	1.6×10^3	2×10^3
$\sqrt{\alpha/\gamma}$	258	387	447
$\sqrt{\delta/\mu}$	447	774	1000

Table I. With the computer operating, data for fundamental amplitude frequency response curves were taken for three forcing function amplitudes. The amplitudes used correspond to the three curves:

1. Response curve for threshold forcing function.
2. Response curve for forcing function equal to $|u|_{max}$ for $v = 0$.
3. Response curve for solutions that are discontinuous in the neighborhood of $\varphi = 90$ degrees.

Experimental Observations

No great difficulty was experienced in obtaining data for the three systems examined. Data were taken for the systems for frequencies below the value of $\omega = \sqrt{\alpha/\gamma}$ to $\omega = \sqrt{\delta/\mu}$. The wave form was observed throughout a particular run to note any harmonic distortion that was present. The wave shapes were very nearly sinusoidal. However, some distortion could be observed particularly at values of $\omega < \sqrt{\alpha/\gamma}$ and in the region of resonance. No attempt was made to determine the third harmonic components in the system. For large values of forcing function amplitude a phenomena of some consequence was observed. It was found that forcing amplitudes, for which no solutions existed in the vicinity of $\varphi = 90$ degrees, produced a type of low frequency oscillation. The oscillation appeared to be of a relaxation type and modulated the fundamental frequency at a frequency of 1/9 to 1/12 of the fundamental. Because of the nature of the wave shape, no data could be obtained when the oscillation occurred.

The Hewlett-Packard Company vacuum tube voltmeters used in the observations were checked against a 1/2% General Electric Company

voltmeter and found to be within 1% of full scale on all scales used in the study. The Hewlett-Packard Company oscillator was checked against several similar models by the same company and it was found to track closer than could be read. The computer components are 1% tolerance components and it was felt that greater precision voltmeters or frequency checking the oscillators was not justifiable.

CHAPTER V

EXPERIMENTAL AND ANALYTICAL RESULTS

Experimental data were obtained from the computer by using the procedure described in Chapter IV. Three systems were analyzed for values of $\gamma\delta/\mu\alpha$ equal to 3, 4, and 5. These systems, subsequently, will be referred to as systems 3, 4, and 5. Data were obtained for three particular forcing function amplitudes: (1) threshold forcing function amplitude $F = F_T$; (2) forcing function amplitude $F = |u|_{max}$; and (3) forcing function amplitude $F > |u|_{max}$. These particular forcing amplitudes were chosen to illustrate the analysis.

Threshold Forcing Function Amplitude $F = F_T$

For a forcing function amplitude F_T, data are presented in Figures 8, 9, and 10. The experimental and analytical data are shown on the curves. Good agreement is obtained between the experimental results and analytical data considering the fact that only a first harmonic term is used in the analysis. The region in which variation exists is in the region of 90 degrees phase lag, that is, near resonance. In each of the three curves, points are labeled indicating the values of 90 degrees phase lag and maximum amplitudes. It may be observed that these do not fall on their respective loci, the backbone curve and the locus of horizontal tangents. These curves do, however, have the same general trend. It should also be

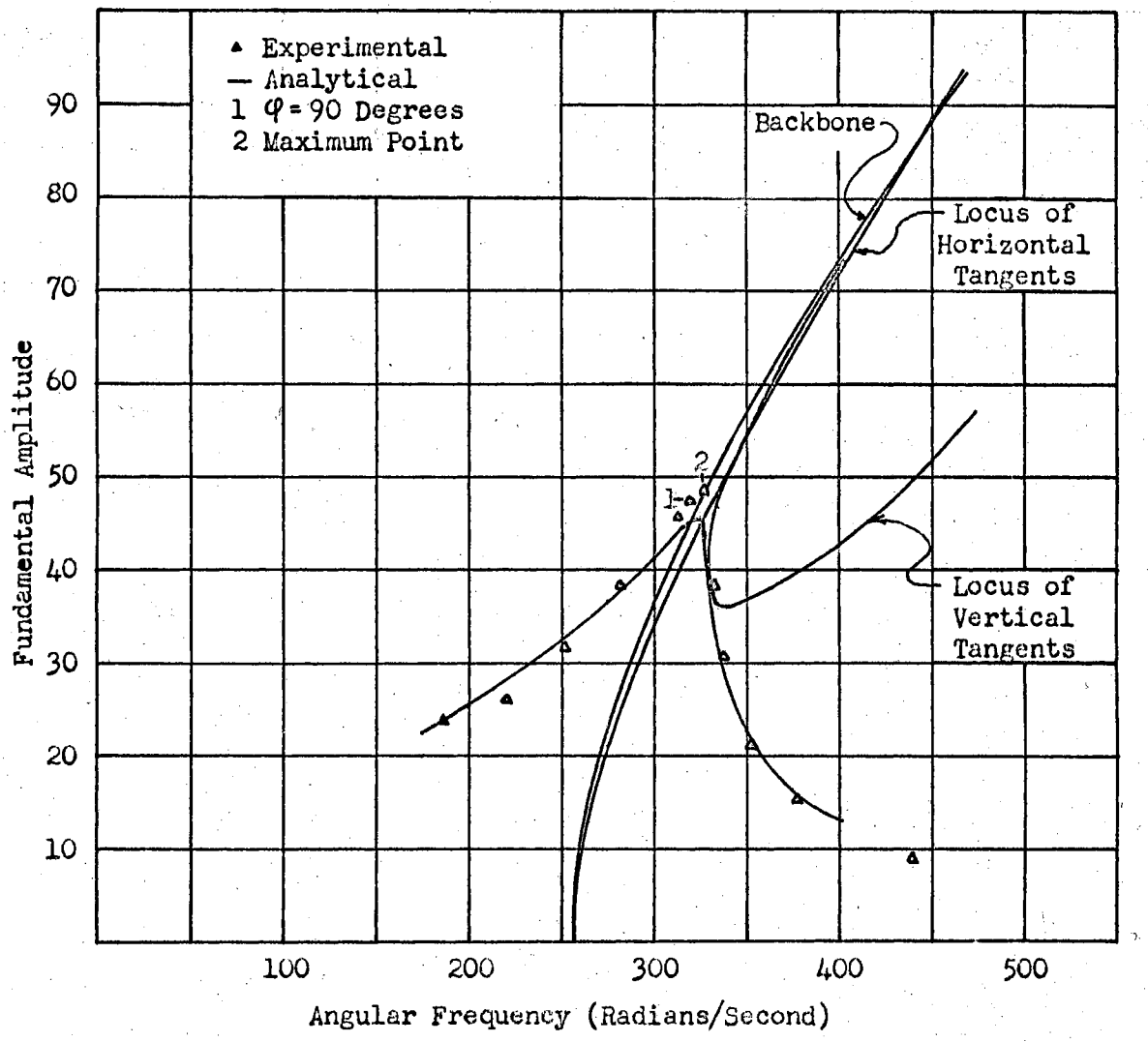

Figure 8. Fundamental Amplitude Response Curve, $\delta\gamma/\mu\alpha = 3$, $F = 140.6 \times 10^7 = F_T$

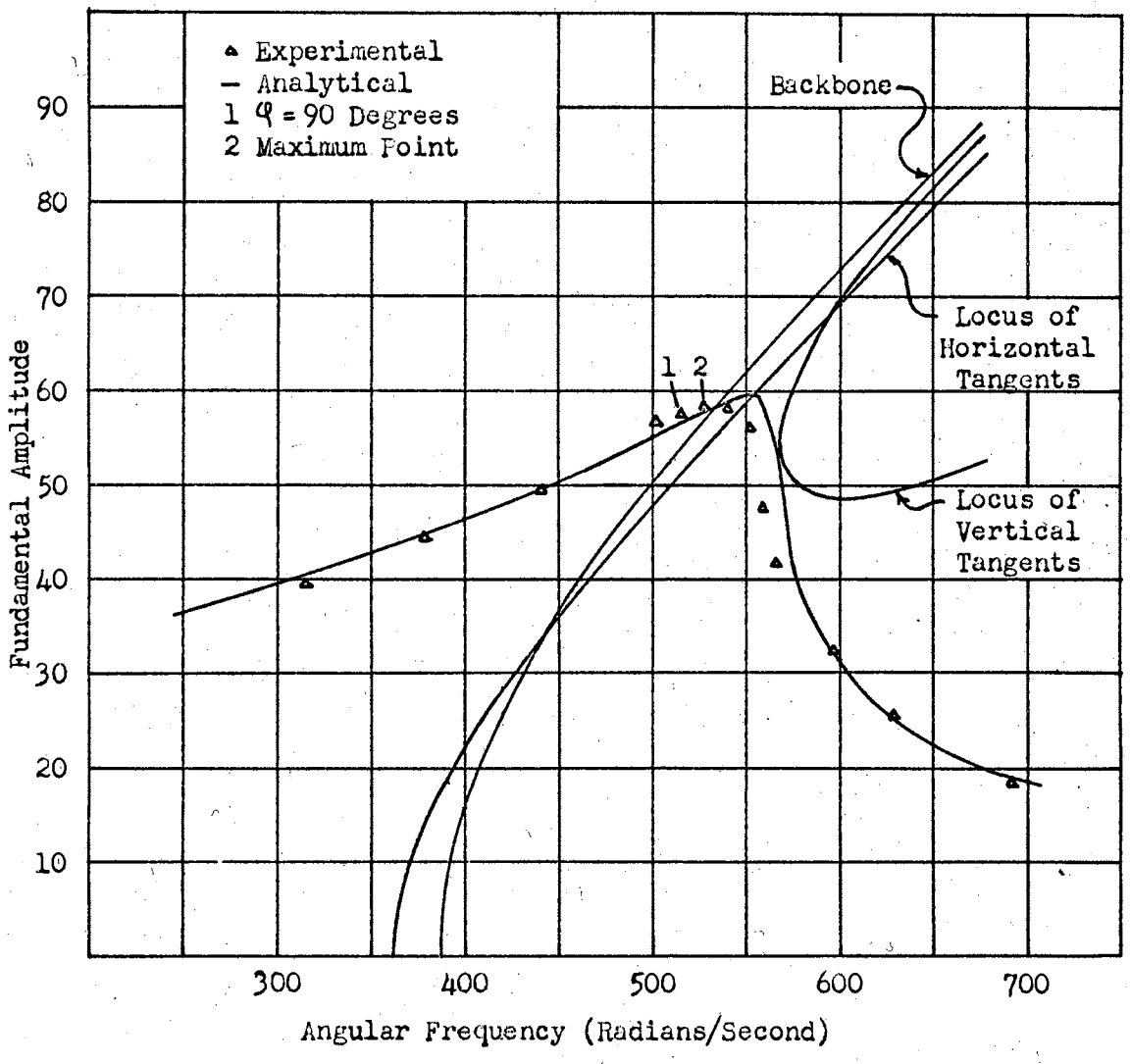

Figure 9. Fundamental Amplitude Response Curve, $\delta\gamma/\mu\alpha = 4$, $F = 984 \times 10^7 = F_T$

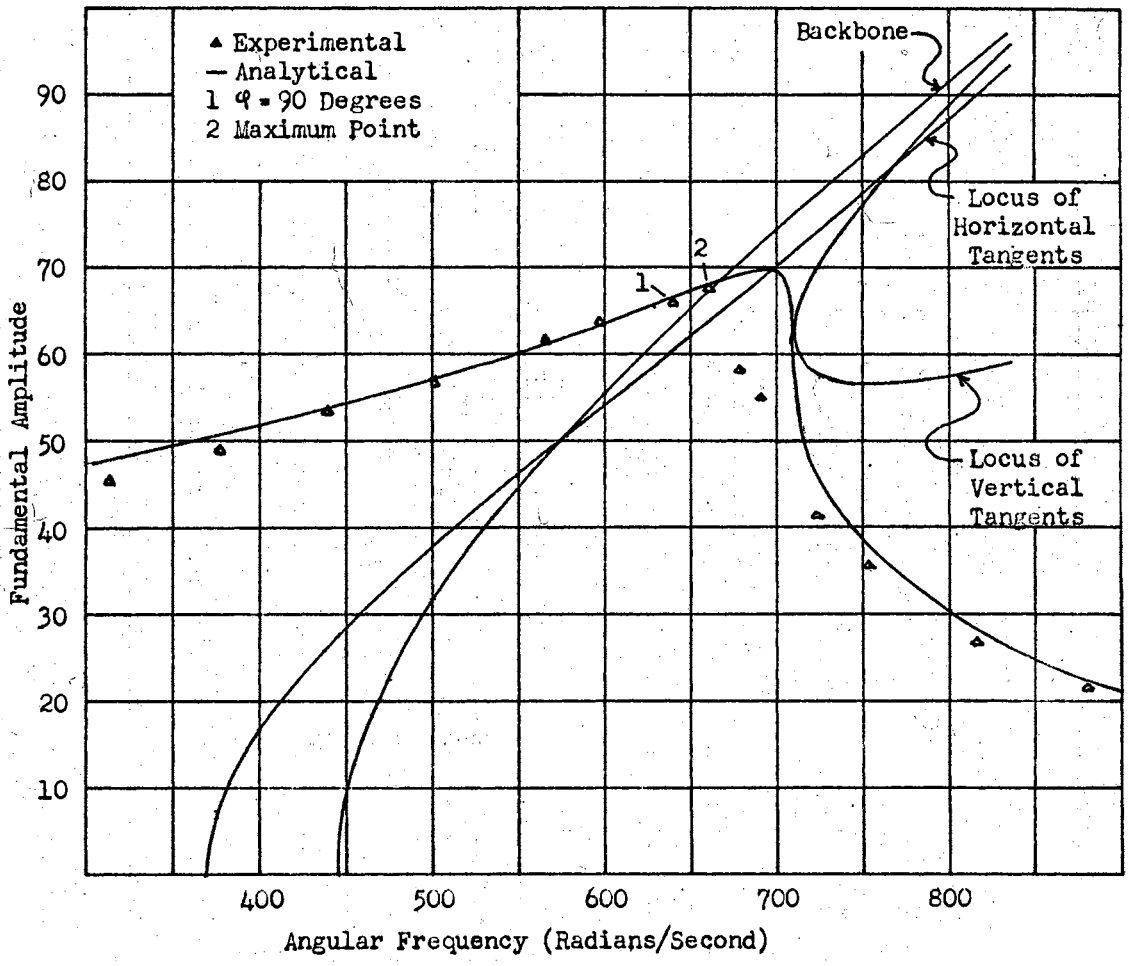

Figure 10. Fundamental Amplitude Response Curve, $\delta\gamma/\mu\alpha = 5$, $F = 2530 \times 10^7 = F_T$

noted that better agreement is obtained for system 3 than for system 5. This is due primarily to the greater harmonic content for the latter system. The per cent harmonic content for the systems at maximum amplitude is 0.88% for 3, 1.5% for 4, and 1.6% for 5. These values were calculated from the analysis in Appendix B where consideration is given to third harmonic components. The third harmonic components change the shape of the constant ω loci in the u - v plane by causing a displacement of the loci to the right. This affects the actual value of F_T for a given system. A smaller value of F_T would be required if the third harmonic components were included in the analysis.

Very good agreement between simulation data and experimental data is obtained in the region of $\omega = \sqrt{\alpha/\gamma}$. This is a region in which very low observable distortion occurred. Slight discrepancies between the two curves is attributable to instrumentation.

Forcing Function Amplitude $F = |u|_{max}$

Figures 11, 12, and 13 illustrate the response curves obtained for the three systems for a forcing function of amplitude $|u|_{max}$. The experimental points agree with the analytical curves to a fair extent everywhere, except at resonance. Figures 12 and 13 particularly indicate the lack of a real solution in the vicinity of 90 degrees. The upward curving away from the backbone illustrates this point. The analytical response is tangent to the backbone whereas the experimental curve bends upward away from it. The lack of tangency between the experimental curve and the backbone is attributable to the third harmonic.

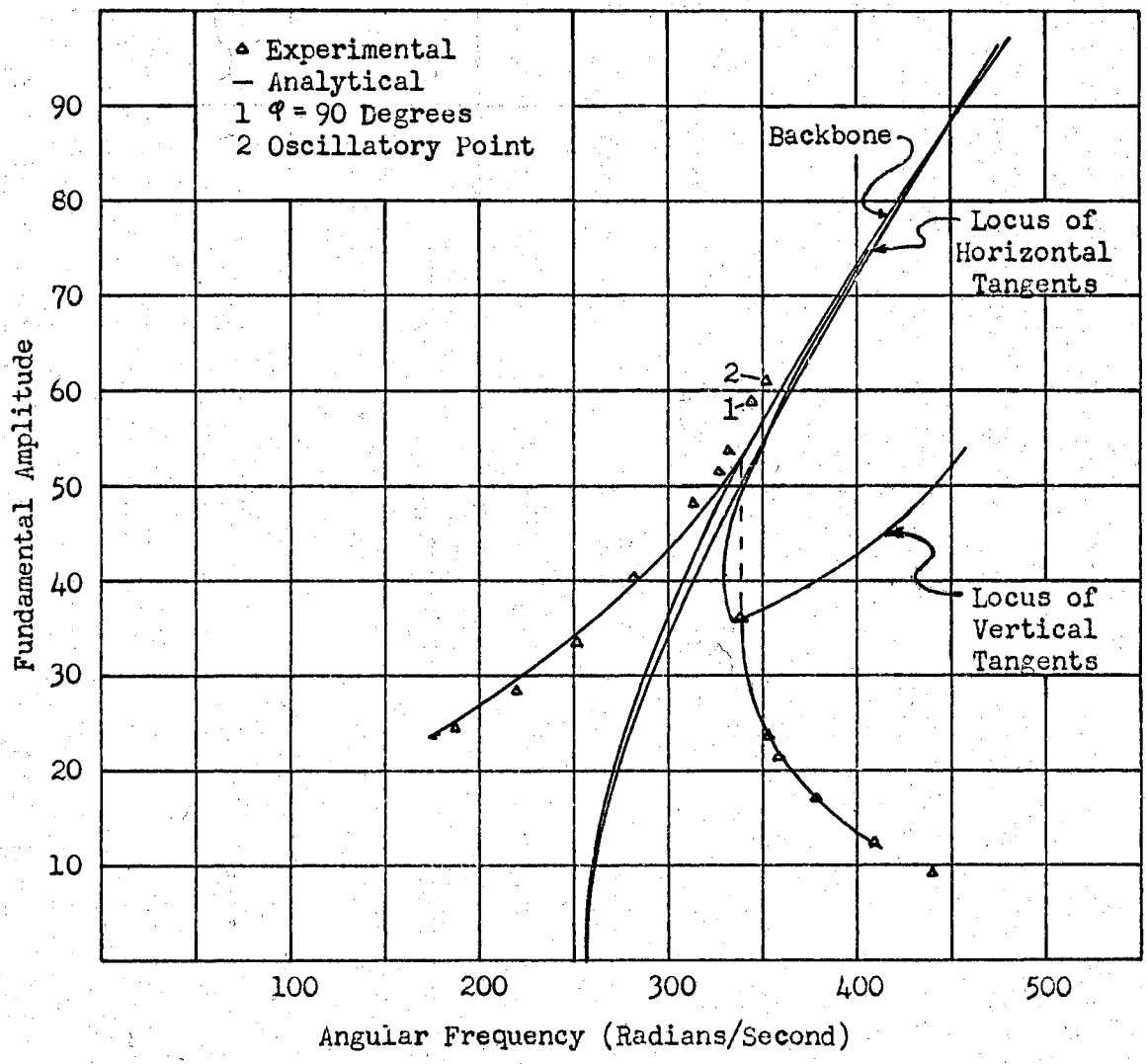

Figure 11. Fundamental Amplitude Response Curve, $\delta\gamma/\mu\alpha = 3$, $F = 153 \times 10^7 = |u|_{max}$

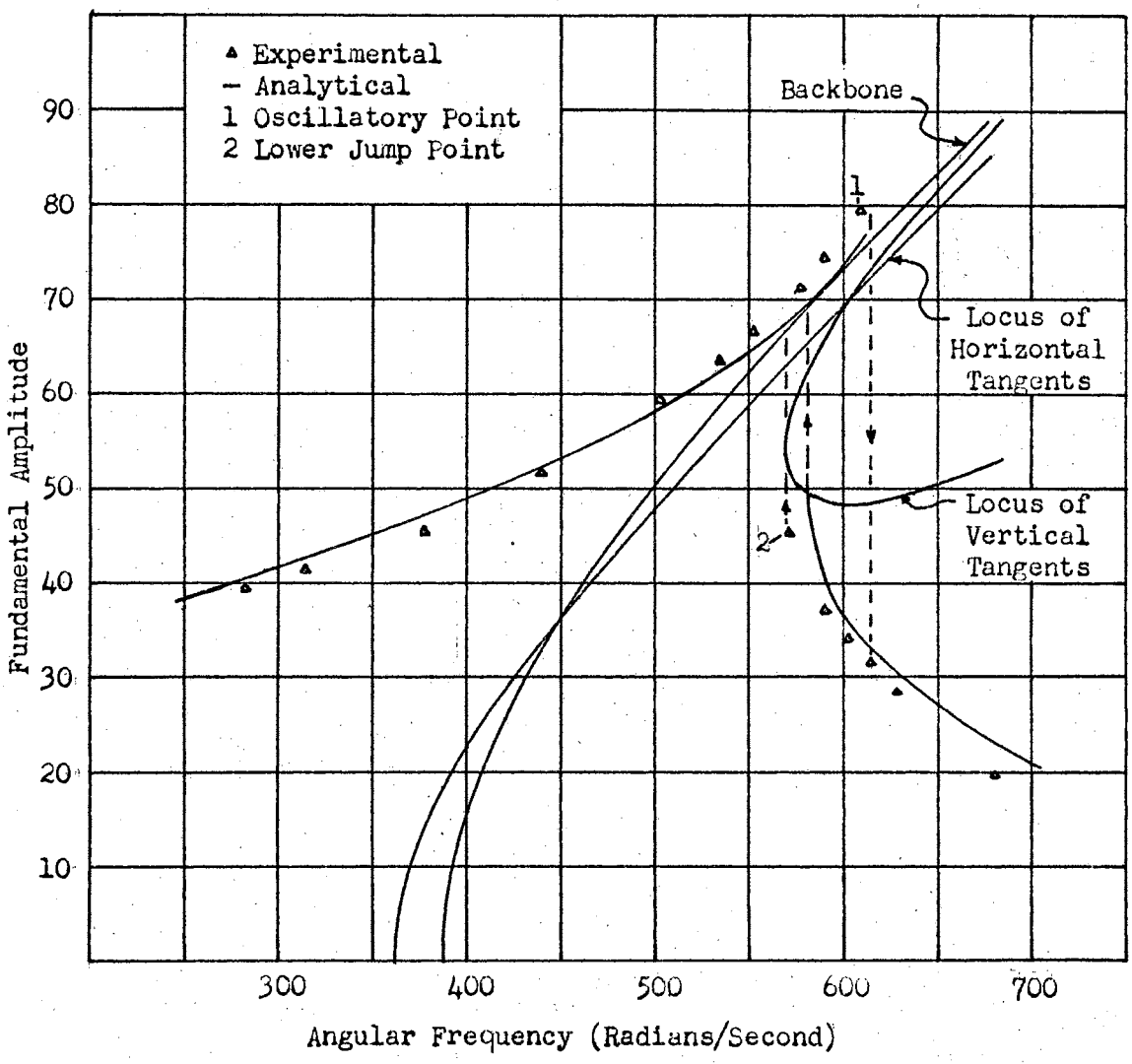

Figure 12. Fundamental Amplitude Response Curve, $\delta\gamma/\mu\alpha = 4$, $F = 1058 \times 10^7 = |u|_{max}$

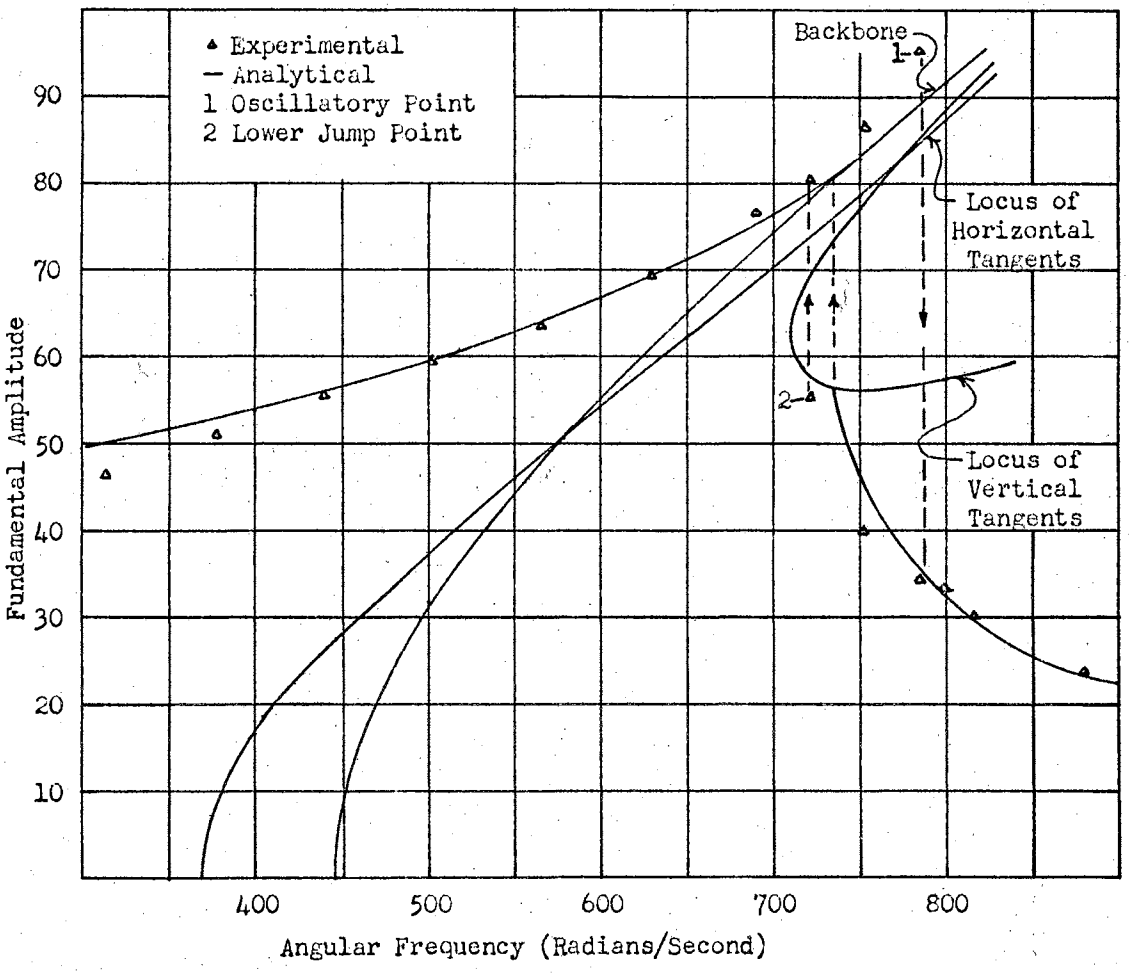

Figure 13. Fundamental Amplitude Response Curve, $\delta\gamma/\mu\alpha = 5$, $F = 2730 \times 10^7 = |u|_{max}$

Figures 14, 15, and 16 show the variation of phase angle with frequency for the three systems. These figures indicate the approach of the phase angle to 90 degrees; however, the phase angle is always less than 90 degrees. The phase angle's approach to 90 degrees is indicative of the response curve's approach to the backbone as discussed above. The curves indicate the frequency at which jump occurs for increasing frequency, but the analytical curves of the first harmonic approximation cannot predict this point. The jump point for decreasing frequency can, however, be determined and the method used is presented in Appendix C. Figures 11, 12, and 13 show the intersection of the locus of vertical tangents and the response curve. This is the point at which lower jump resonance occurs. These frequencies for the three systems are:

$\delta\gamma/\mu\alpha$	Calculated Lower Jump Resonance Frequency	Experimental Lower Jump Resonance Frequency
3	338	332
4	582	571
5	735	722

Forcing Function Amplitude $F > |u|_{max}$

Figures 17, 18, and 19 show the response curves for the three systems for $F > |u|_{max}$. These curves only serve to show that for this amplitude of forcing function the response curve is near the backbone; however, it is not tangent to it. The lower jump resonance frequencies are as follows:

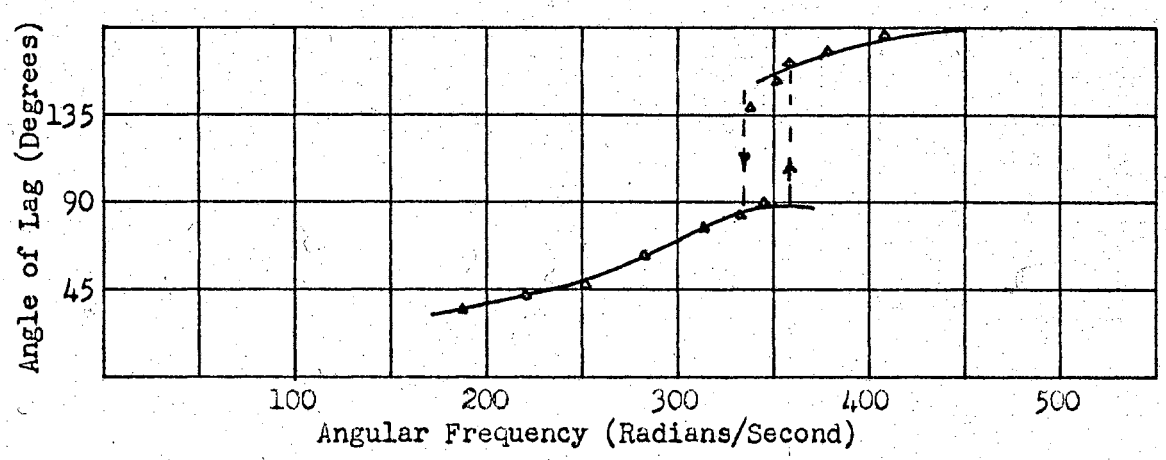

Figure 14. Phase Angle for System 3 for $F = 153 \times 10^7 = |u|_{max}$

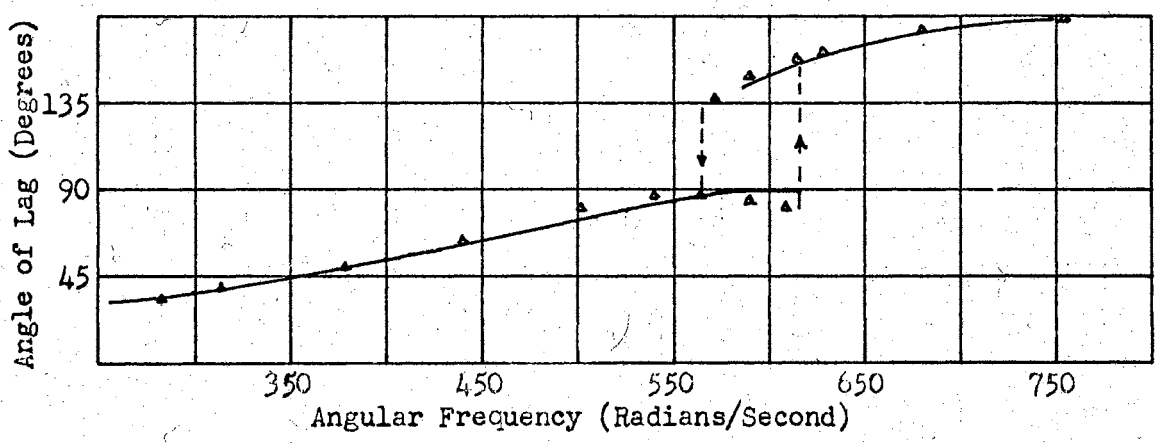

Figure 15. Phase Angle for System 4 for $F = 1058 \times 10^7 = |u|_{max}$

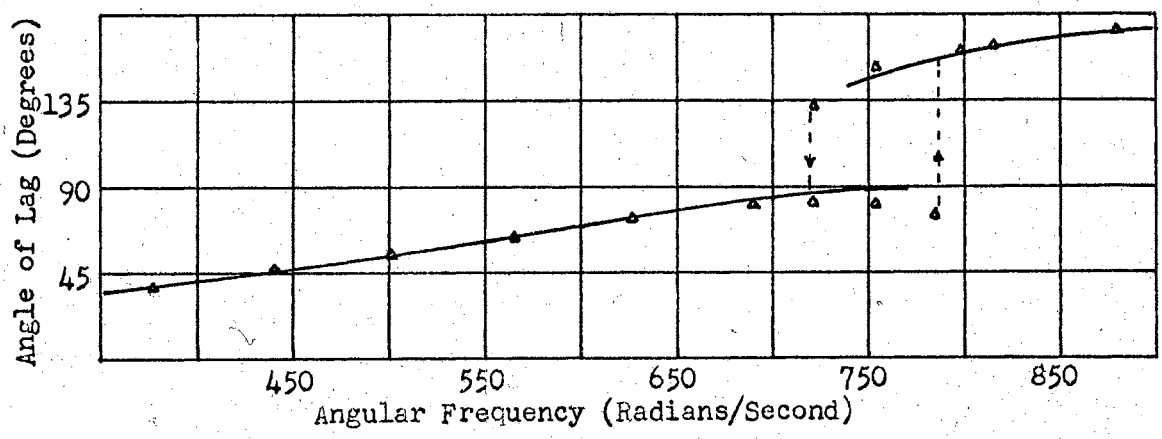

Figure 16. Phase Angle for System 5 for $F = 2730 \times 10^7 = |u|_{max}$

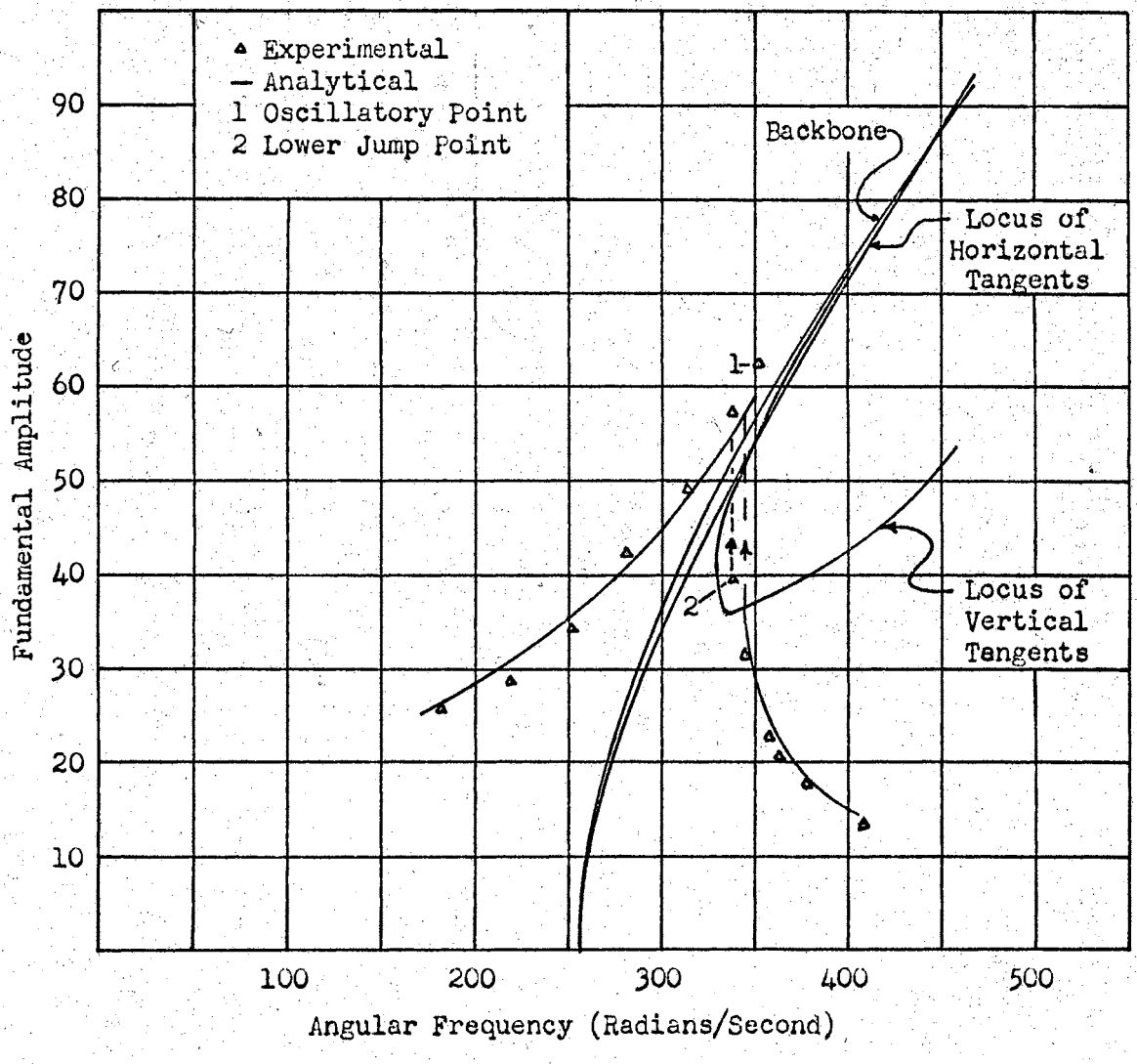

Figure 17. Fundamental Amplitude Response Curve, $\delta\gamma/\mu\alpha = 3$, $F = 163 \times 10^7 > |u|_{max}$

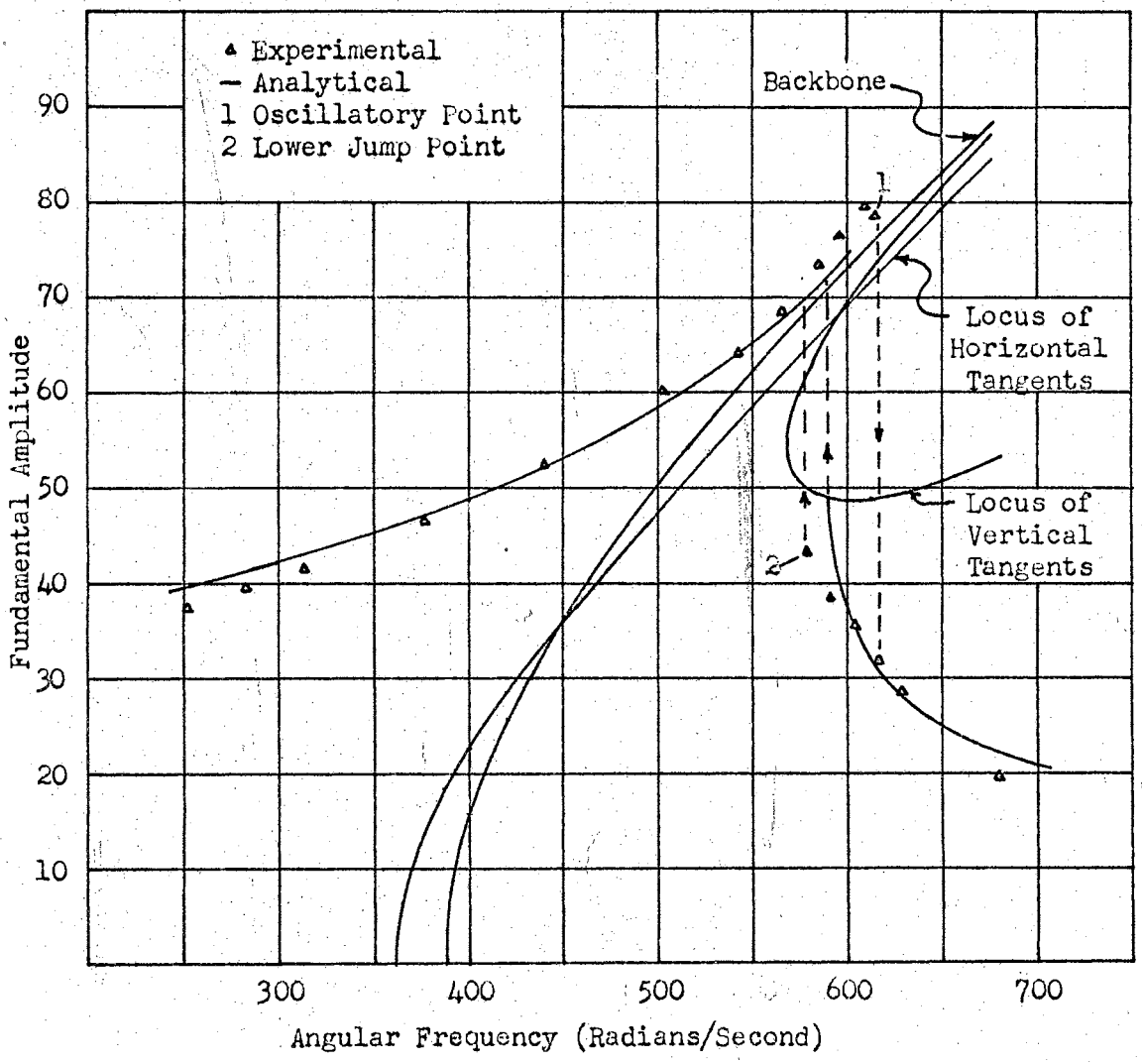

Figure 18. Fundamental Amplitude Response Curve, $\delta\,\gamma/\mu\alpha = 4$, $F = 1080 \times 10^7 > |u|_{max}$

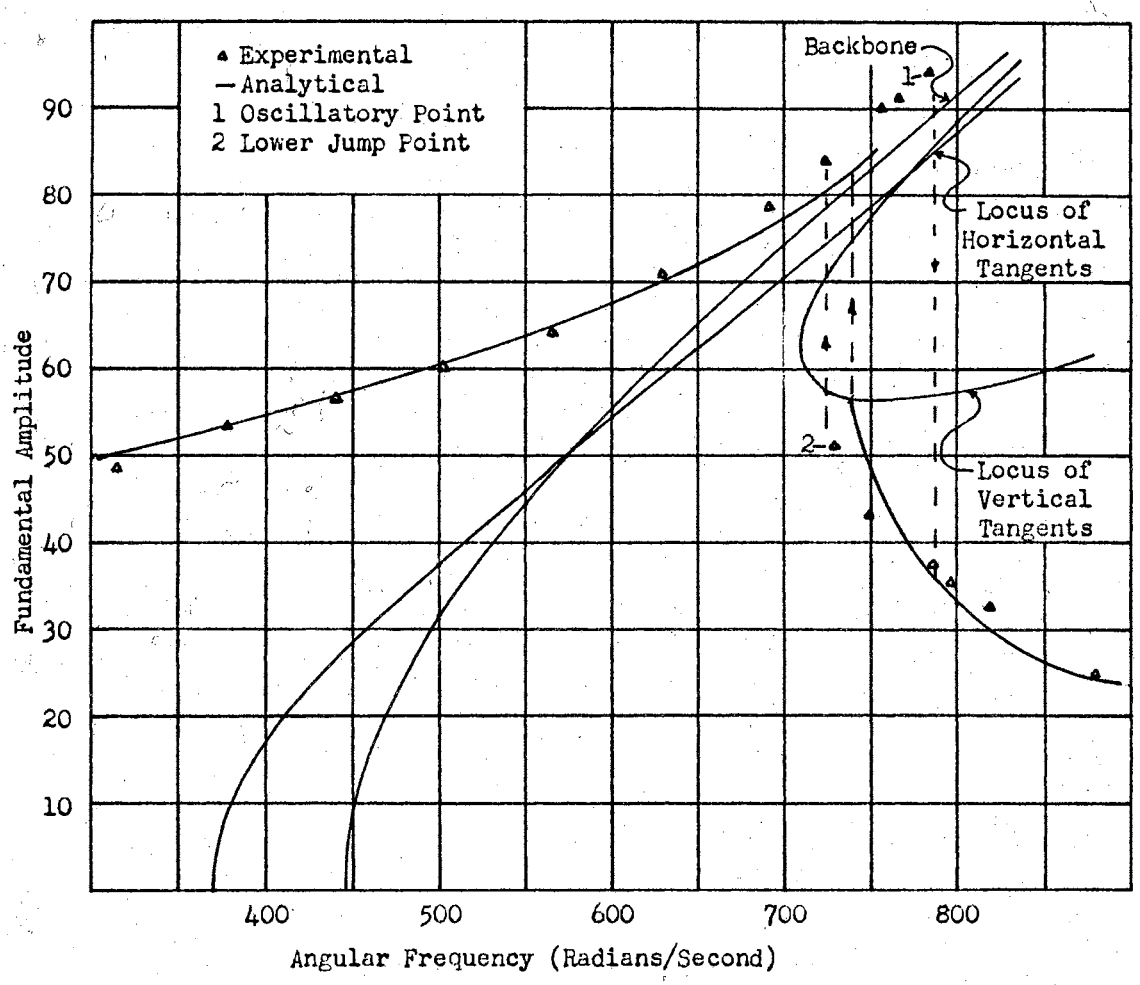

Figure 19. Fundamental Amplitude Response Curve, $6\gamma/\mu\alpha = 5$, $F = 2800 \times 10^7 > |u|_{max}$

$\delta\gamma/\mu\alpha$	Calculated Lower Jump Resonance Frequency	Experimental Lower Jump Resonance Frequency
3	345	339
4	590	578
5	740	728

Before jump resonance occurs, for increasing frequency, pronounced oscillations were observed for amplitudes of forcing function greater than $|u|_{max}$. The angular frequency at which these oscillations occurred are labeled on the curves.

CHAPTER VI

SUMMARY AND CONCLUSIONS

An approximate harmonic solution to a sinusoidally forced third order nonlinear differential equation representing a control system has been obtained and, in view of the approximation, the following findings have resulted:

1. Solutions to the differential equation may be obtained by utilizing a u - v plane in a graphical solution. The u - v plane is a plane in which components of the forcing function are plotted for constant angular frequency.

2. Jump resonance may be prevented from occurring by limiting the input signals to the system to values less than the threshold forcing function amplitude. This threshold forcing function amplitude is a function of the system parameters and may be calculated in terms of these parameters. Jump resonance will be prevented for values of

$$F \leqq F_T$$

in which

$$F_T = \sqrt{-\frac{8}{9}\frac{\sqrt{3}}{b}} \qquad \text{III-5}$$

with

$$b = \frac{3/4\beta}{\omega^3(\mu\omega^2 - \delta)^3} \, . \qquad \text{II-28}$$

In this equation ω is the positive real solution to the cubic

$$\sqrt{3}\mu\omega^3 + \gamma\omega^2 - \sqrt{3}\delta\omega - \alpha = 0 \qquad \text{II-41}$$

The positive real value of ω satisfying the equation is a frequency limitation on the input to the system.

3. Real solutions cannot be obtained in the immediate vicinity of values of $\varphi = 90$ degrees for values of forcing function greater than $|u|_{max}$. $|u|_{max}$ is a function of the system parameters and it may be calculated in terms of these parameters. $|u|_{max}$ is defined to be

$$|u|_{max} = \left| 2\omega_u^2(\mu\omega_u^2 - \delta)\sqrt{\frac{\gamma(\delta - \mu\omega_u^2)}{3\beta(3\mu\omega_u^2 - \delta)}} \right| \qquad \text{II-31}$$

in which ω_u^2 is

$$\omega_u^2 = \frac{1}{8\mu\gamma}\left[2\gamma\delta + 3\alpha\mu + \sqrt{(2\gamma\delta + 3\alpha\mu)^2 - 16\alpha\delta\gamma\mu} \right]. \qquad \text{II-29}$$

4. The frequency at which upper jump resonance occurs for increasing frequency cannot be determined with a first harmonic approximation; however, the lower jump resonance frequency for decreasing frequency can be determined with limited accuracy.

With these findings in mind, future research must be directed to solutions which consider subharmonics and superharmonics in the analysis. An analog computer study in which strip chart recordings are made of the amplitudes throughout the frequency range of interest would be required in such an investigation. From these recordings harmonic content could be determined using a Fourier series analysis.

It is believed by the writer that an analysis considering not only the third harmonic but also a subharmonic in the solution

would lead to quantitative data concerning the upper jump resonance frequency.

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APPENDIX A

DERIVATION OF EQUATIONS FOR THE FIRST HARMONIC ANALYSIS

General First Order Approximation

The system equation to be solved is

$$\mu y''' + \gamma y'' + \delta y' + \alpha y + \beta y^3 = v \cos \omega t + u \sin \omega t \qquad \text{A-1}$$

for

$$v^2 + u^2 = F^2 \qquad \text{A-2}$$

and

$$\varphi = \arctan \frac{v}{u} . \qquad \text{A-3}$$

For the first approximation $y = A_1 \cos \omega t$ is assumed to be a steady state solution, where ω is the angular velocity of the forcing function and A_1 is the fundamental amplitude response. Upon substitution of the assumed solution into Equation A-1, there results

$$\mu \omega^3 A_1 \sin \omega t - \gamma \omega^2 A_1 \cos \omega t - \delta \omega A_1 \sin \omega t + \alpha A_1 \cos \omega t + \frac{\beta A_1^3}{4} (\cos 3\omega t + 3 \cos \omega t) = v \cos \omega t + u \sin \omega t. \qquad \text{A-4}$$

By collecting terms and forming coefficients of the A_1's there is obtained

$$A_1 \omega (\mu \omega^2 - \delta) \sin \omega t + (\alpha - \gamma \omega^2 + \frac{3}{4} \beta A_1^2) A_1 \cos \omega t + \frac{\beta A_1^3}{4} \cos 3\omega t = v \cos \omega t + u \sin \omega t. \qquad \text{A-5}$$

If the third harmonic is considered negligible and coefficients of like terms on either side of the equality are equated, there results

$$v = (\alpha - \gamma \omega^2 + \frac{3}{4} \beta A_1^2) A_1 , \qquad \text{A-6}$$

$$u = \omega(\mu\omega^2 - \delta)A_1 . \qquad \text{A-7}$$

In accordance with Equation A-2, the response function for the system may then be written as

$$(\alpha - \gamma\omega^2 + \frac{3}{4}\beta A_1^2)^2 A_1^2 + \omega^2(\mu\omega^2 - \delta)^2 A_1^2 = F^2. \qquad \text{A-8}$$

The parametric representation , A-6 and A-7, may be written in the functional form $v = f(u)$ by solving A-7 for A_1 and substituting into A-6. A_1 is given by the expression

$$A_1 = \frac{u}{\omega(\mu\omega^2 - \delta)} \qquad \text{A-9}$$

and A_1^3 is

$$A_1^3 = \frac{u^3}{\omega^3(\mu\omega^2 - \delta)^3} . \qquad \text{A-10}$$

The expression for v is

$$v = (\alpha - \gamma\omega^2)\frac{u}{\omega(\mu\omega^2 - \delta)} + \frac{3}{4}\beta\frac{u^3}{\omega^3(\mu\omega^2 - \delta)^3}, \qquad \text{A-11}$$

or

$$v = au + bu^3. \qquad \text{A-12}$$

In this equation the constants a and b are defined as

$$a = \frac{\alpha - \gamma\omega^2}{\omega(\mu\omega^2 - \delta)} \qquad \text{A-13}$$

and

$$b = \frac{3}{4}\frac{\beta}{\omega^3(\mu\omega^2 - \delta)^3} . \qquad \text{A-14}$$

By using Equation A-12, Equation A-2 becomes

$$(au + bu^3)^2 + u^2 = F^2 \qquad \text{A-15}$$

or

$$b^2u^6 + 2abu^4 + (1 + a^2)u^2 = F^2. \qquad \text{A-16}$$

$$u = \omega(\mu\omega^2 - \delta)A_1 . \qquad \text{A-7}$$

In accordance with Equation A-2, the response function for the system may then be written as

$$(\alpha - \gamma\omega^2 + \frac{3}{4}\beta A_1^2)^2 A_1^2 + \omega^2(\mu\omega^2 - \delta)^2 A_1^2 = F^2. \qquad \text{A-8}$$

The parametric representation , A-6 and A-7, may be written in the functional form $v = f(u)$ by solving A-7 for A_1 and substituting into A-6。 A_1 is given by the expression

$$A_1 = \frac{u}{\omega(\mu\omega^2 - \delta)} \qquad \text{A-9}$$

and A_1^3 is

$$A_1^3 = \frac{u^3}{\omega^3(\mu\omega^2 - \delta)^3} \text{。} \qquad \text{A-10}$$

The expression for v is

$$v = (\alpha - \gamma\omega^2)\frac{u}{\omega(\mu\omega^2 - \delta)} + \frac{3}{4}\beta\frac{u^3}{\omega^3(\mu\omega^2 - \delta)^3}, \qquad \text{A-11}$$

or

$$v = au + bu^3. \qquad \text{A-12}$$

In this equation the constants a and b are defined as

$$a = \frac{\alpha - \gamma\omega^2}{\omega(\mu\omega^2 - \delta)} \qquad \text{A-13}$$

and

$$b = \frac{3}{4}\frac{\beta}{\omega^3(\mu\omega^2 - \delta)^3} . \qquad \text{A-14}$$

By using Equation A-12, Equation A-2 becomes

$$(au + bu^3)^2 + u^2 = F^2 \qquad \text{A-15}$$

or

$$b^2u^6 + 2abu^4 + (1 + a^2)u^2 = F^2\text{。} \qquad \text{A-16}$$

Locus of Vertical Tangents

The locus of vertical tangents may be found by taking the derivative of Equation A-8 with respect to A_1 and equating $d\omega^2/dA_1$ to zero. The derivative of expression A-8 with respect to A_1 is

$$(\alpha - \gamma\omega^2 + \tfrac{3}{4}\beta A_1^2)2A_1 + 2A_1^2(\alpha - \gamma\omega^2 + \tfrac{3}{4}\beta A_1^2)(-\gamma\omega^2 + \tfrac{6}{4}\beta A_1) + 2A_1\omega^2(\mu\omega^2 - \delta) + A_1^2\left[2\omega^2(\mu\omega^2 - \delta)(\omega^2)' + (\mu\omega^2 - \delta)(\omega^2)'\right] = 0. \quad \text{A-17}$$

The derivative of ω^2 with respect to A_1, $\frac{d\omega^2}{dA_1} = (\omega^2)'$, is zero. This equation may be written in the form

$$(\alpha - \gamma\omega^2 + \tfrac{3}{4}\beta A_1^2)(\alpha - \gamma\omega^2 + \tfrac{9}{4}\beta A_1^2) + \omega^2(\mu\omega^2 - \delta)^2 = 0. \quad \text{A-18}$$

The solution for A_1^2 may then be found by first writing the equation

$$A_1^4 + \frac{16}{9}\frac{(\alpha - \gamma\omega^2)}{\beta}A_1^2 + \frac{16}{17}\frac{1}{\beta^2}\left[(\alpha - \gamma\omega^2)^2 + \omega^2(\mu\omega^2 - \delta)^2\right] = 0. \quad \text{A-19}$$

The solution for A_1^2 is

$$A_1^2 = -\frac{16}{18}\frac{(\alpha - \gamma\omega^2)}{\beta} \pm \sqrt{\left|\frac{16}{9}\right|^2\left|\frac{\alpha - \gamma\omega^2}{\beta}\right|^2 - \frac{64}{27}\frac{1}{\beta^2}\left[(\alpha - \gamma\omega^2)^2 + \omega^2(\mu\omega^2 - \delta)^2\right]}. \quad \text{A-20}$$

This expression becomes, after algebraic simplification,

$$A_1^2 = \frac{8}{9\beta}\left[-(\alpha - \gamma\omega^2) \pm \frac{1}{2}\omega(\mu\omega^2 - \delta)\sqrt{a^2 - 3}\right]. \quad \text{A-21}$$

The single valued point on the locus of vertical tangents is the point at which the radical vanishes. This point is

$$A_1^2 = \frac{8}{9\beta}(\gamma\omega^2 - \alpha). \quad \text{A-22}$$

Equation A-18 may also be written in terms of u and v. If Equation

A-18 is divided by $\omega^2(\mu\omega^2 - \delta)^2$ the result is

$$\frac{(\alpha - \gamma\omega^2 + \frac{3}{4}\beta A_1^2)}{\omega(\mu\omega^2 - \delta)} \quad \frac{(\alpha - \gamma\omega^2 + \frac{9}{4}\beta A_1^2)}{\omega(\mu\omega^2 - \delta)} + 1 = 0 \qquad \text{A-23}$$

or

$$\left[\frac{\alpha - \gamma\omega^2}{\omega(\mu\omega^2 - \delta)} + \frac{3}{4}\frac{\beta A_1^2}{\omega(\mu\omega^2 - \delta)}\right]\left[\frac{\alpha - \gamma\omega^2}{\omega(\mu\omega^2 - \delta)} + \frac{9}{4}\frac{\beta A_1^2}{\omega(\mu\omega^2 - \delta)}\right] + 1 = 0. \qquad \text{A-24}$$

The values for a and A_1^2 are

$$a = \frac{\alpha - \gamma\omega^2}{\omega(\mu\omega^2 - \delta)} \qquad \text{and} \qquad A_1^2 = \frac{u^2}{\omega^2(\mu\omega^2 - \delta)^2}.$$

Therefore Equation A-24 is

$$\left[a + bu^2\right]\left[a + 3bu^2\right] + 1 = 0 \qquad \text{A-25}$$

or

$$3b^2u^4 + 4abu^2 + 1 + a^2 = 0. \qquad \text{A-26}$$

Equation A-18 represents the locus of vertical tangents in the A_1 - ω plane and Equation A-26 is its representation in the u - v plane. By writing $S = u^2$, Equation A-26 is written

$$3b^2S^2 + 4abS + a^2 + 1 = 0 \qquad \text{A-27}$$

and it has the following solution:

$$S = \frac{1}{3b}\left[-2a \pm \sqrt{a^2 - 3}\right] \qquad \text{A-28}$$

or

$$u = \pm\sqrt{\frac{1}{3b}\left[-2a \pm \sqrt{a^2 - 3}\right]} \qquad \text{A-29}$$

Locus of Horizontal Tangents

The locus of horizontal tangents may be found by taking the derivative of Equation A-8 with respect to ω and equating $dA_1/d\omega$

to zero. The derivative of A-8 is

$$2A_1^2(\alpha - \gamma\omega^2 + \frac{3}{4}\beta A_1^2)(-2\gamma\omega + \frac{6}{4}\beta A_1 \frac{dA_1}{d\omega}) + (\alpha - \gamma\omega^2 + \frac{3}{4}\beta A_1^2)^2 2\frac{dA_1}{d\omega}$$

$$+ A_1^2\left[2\omega^2(\mu\omega^2 - \delta)(2\mu\omega) + (\mu\omega^2 - \delta)^2 2\omega\right] + \omega^2(\mu\omega^2 - \delta)^2 2\frac{dA_1}{d\omega} = 0 \qquad \text{A-30}$$

or

$$-2\gamma(\alpha - \gamma\omega^2 + \frac{3}{4}\beta A_1^2) + (\mu\omega^2 - \delta)(3\mu\omega^2 - \delta) = 0. \qquad \text{A-31}$$

This equation may be solved for A_1 and its solution is

$$A_1 = \pm\left\{\frac{2}{3\gamma\beta}\left[3\mu^2\omega^4 + 2(\gamma^2 - 2\mu\delta)\omega^2 + (\delta^2 - 2\alpha\gamma)\right]\right\}^{1/2}. \qquad \text{A-32}$$

Maximum Value of u for Variable Frequency and v = 0

From Equations A-6 and A-7, u may be written as

$$u = \pm\left[\frac{4}{3}\left(\frac{\gamma\omega^2 - \alpha}{\beta}\right)\right]^{1/2}\omega(\mu\omega^2 - \delta) \qquad \text{A-33}$$

by solving Equation A-6 for A_1 and substituting into Equation A-7. The positive value must be used to yield a positive value for A_1. It is required that $\frac{du}{d\omega} = 0$. The derivative of u is

$$\frac{du}{d\omega} = \frac{\omega}{2}(\mu\omega^2 - \delta)\left\{\left[\frac{4}{3}\left(\frac{\gamma\omega^2 - \alpha}{\beta}\right)\right]^{-1/2}\frac{8}{3}\frac{\gamma\omega}{\beta}\right\} + \left[\frac{4}{3}\left(\frac{\gamma\omega^2 - \alpha}{\beta}\right)\right]^{1/2}(3\mu\omega^2 - \delta). \qquad \text{A-34}$$

Upon equating the derivative of u with respect to ω to zero, there is obtained

$$\frac{4\gamma\omega^2(\mu\omega^2 - \delta)}{3\beta\left[\frac{4}{3}\left(\frac{\gamma\omega^2 - \alpha}{\beta}\right)\right]^{1/2}} + \left[\frac{4}{3}\left(\frac{\gamma\omega^2 - \alpha}{\beta}\right)\right]^{1/2}(3\mu\omega^2 - \delta) = 0 \qquad \text{A-35}$$

or

$$\frac{4\gamma\omega^2(\mu\omega^2 - \delta) + 4(\gamma\omega^2 - \alpha)(3\mu\omega^2 - \delta)}{3\beta\left[\frac{4}{3}\left(\frac{\gamma\omega^2 - \alpha}{\beta}\right)\right]^{1/2}} = 0. \qquad \text{A-36}$$

The final result is

$$\gamma\omega^2(\mu\omega^2 - \delta) + (\gamma\omega^2 - \alpha)(3\mu\omega^2 - \delta) = 0 \qquad \text{A-37}$$

or

$$4\mu\gamma\omega^4 - (2\gamma\delta + 3\alpha\mu)\omega^2 + \alpha\delta = 0. \qquad \text{A-38}$$

The solution for ω^2 is

$$\omega^2 = \frac{1}{8\mu\gamma}\left[2\gamma\delta + 3\alpha\mu \pm\sqrt{(2\gamma\delta + 3\alpha\mu)^2 - 16\alpha\delta\gamma\mu}\right]. \qquad \text{A-39}$$

ω is the angular frequency at which a maximum value of u is obtained for $v = 0$.

To find the maximum value of u, the value of ω from Equation A-39 is substituted into Equation A-33. A_1 for maximum u may be calculated from Equation A-9. A_1 may also be determined by using the following procedure:

$$u = \omega(\mu\omega^2 - \delta)A_1 \qquad \text{A-7}$$

$$\frac{du}{d\omega} = A_1(3\mu\omega^2 - \delta) + (\mu\omega^3 - \delta\omega)\frac{dA_1}{d\omega} = 0 \qquad \text{A-40}$$

It is required that

$$\alpha - \gamma\omega^2 + \frac{3}{4}\beta A_1^2 = 0 \qquad \text{A-41}$$

and

$$-2\gamma\omega + \frac{3}{4}\beta A_1\frac{dA_1}{d\omega} = 0 \qquad \text{A-42}$$

or

$$\frac{dA_1}{d\omega} = \frac{4}{3}\frac{\gamma\omega}{\beta A_1}. \qquad \text{A-43}$$

Upon substituting Equation A-43 into Equation A-40, there is obtained

$$\frac{3\beta A_1^2(3\mu\omega^2 - \delta) + (\mu\omega^3 - \delta\omega)4\gamma\omega}{3\beta A_1} = 0 \qquad \text{A-44}$$

or

$$3\beta A_1^2(3\mu\omega^2 - \delta) = 4\gamma\omega^2(\delta - \mu\omega^2). \qquad \text{A-45}$$

The solution for A_1 is

$$A_1 = \pm 2\omega\sqrt{\frac{\gamma(\delta - \mu\omega^2)}{3\beta(3\mu\omega^2 - \delta)}}\ . \qquad \text{A-46}$$

APPENDIX B

AN APPROXIMATE SOLUTION FOR FIRST AND THIRD HARMONIC COMPONENTS

The differential equation to be solved is Equation II-8.

$$\mu y''' + \gamma y'' + \delta y' + \alpha y + \beta y^3 = F \cos(\omega t - \varphi) \qquad \text{II-8}$$

This equation is rewritten as

$$H \equiv \mu y''' + \gamma y'' + \delta y' + \alpha y + \beta y^3 - v \cos \omega t - u \sin \omega t = 0. \qquad \text{B-1}$$

A periodic function of the form

$$y = A_1 \cos \omega t + A_3 \cos 3\omega t + B_3 \sin 3\omega t \qquad \text{B-2}$$

is assumed to be a steady-state solution.

The Ritz Averaging Method (3) requires that the following integrals be zero:

$$\int_0^{2\pi} H[y] \cos \omega t \, d\omega t = 0 \qquad \text{B-3}$$

$$\int_0^{2\pi} H[y] \sin \omega t \, d\omega t = 0 \qquad \text{B-4}$$

$$\int_0^{2\pi} H[y] \cos 3\omega t \, d\omega t = 0 \qquad \text{B-5}$$

$$\int_0^{2\pi} H[y] \sin 3\omega t \, d\omega t = 0 \qquad \text{B-6}$$

These expressions upon evaluation yield the following:

from B-3,

$$(\alpha - \gamma\omega^2 + \frac{3}{4}\beta A_1^2) A_1 + 3\beta \left[\frac{1}{4} A_1^2 A_3 + \frac{1}{2} A_3^2 A_1 + \frac{1}{2} B_3^2 A_1\right] = v; \qquad \text{B-7}$$

from B-4, $$\omega(\mu\omega^2 - \delta)A_1 + \frac{3}{4}\beta A_1^2 B_3 = u \ ;$$

from B-5,

$$(\alpha - 9\gamma\omega^2 + \frac{3}{4}\beta A_3^2)\, A_3 - (27\mu\omega^2 - 3\delta)\omega B_3 + \frac{\beta A_1^3}{4} + 3\beta\left[\frac{1}{2}A_1^2 A_3 + \frac{1}{4}B_3^2 A_3\right] = 0 \ ; \qquad \text{B-9}$$

and from B-6,

$$(\alpha - 9\gamma\omega^2 + \frac{3}{4}\beta B_3^2)\, B_3 + (27\mu\omega^2 - 3\delta)\omega A_3 + \frac{3}{4}\beta\left[A_3^2 B_3 + 2A_1^2 B_3\right] = 0. \qquad \text{B-10}$$

Equation B-9 may be written as

$$(\alpha - 9\gamma\omega^2)\, A_3 + (3\delta - 27\mu\omega^2)\omega B_3 + \frac{1}{4}\beta A_1^3 + \frac{3}{2}\beta A_1^2 A_3 = 0 \qquad \text{B-11}$$

if terms such as A_3^3 and $B_3^2 A_3$ are assumed small compared to the other terms in the equation. Equation B-10 may then be written as

$$(\alpha - 9\gamma\omega^2)\, B_3 - (3\delta - 27\mu\omega^2)\omega A_3 + \frac{3}{2}\beta A_1^2 B_3 = 0. \qquad \text{B-12}$$

For ease in manipulating the equations, the frequency coefficients of A_3 and B_3 will be written as

$$C_1 = \alpha - 9\gamma\omega^2 \qquad \text{B-13}$$

and

$$C_2 = 3\delta\omega - 27\mu\omega^3 . \qquad \text{B-14}$$

Then the equations become

$$C_1 A_3 + C_2 B_3 + \frac{1}{4}\beta A_1^3 + \frac{3}{2}\beta A_1^2 A_3 = 0 \qquad \text{B-15}$$

and

$$C_1 B_3 - C_2 A_3 + \frac{3}{2}\beta A_1^2 B_3 = 0. \qquad \text{B-16}$$

Equation B-16 may be solved for A_3,

$$A_3 = \frac{1}{C_2}\left[C_1 + \frac{3}{2}\beta A_1^2\right]B_3 . \qquad \text{B-17}$$

Upon substitution into Equation B-15, the solution for B_3 is

$$B_3 = \frac{-\frac{1}{4}\beta A_1^3}{\frac{1}{C_2}(C_1 + \frac{3}{2}\beta A_1^2)^2 + C_2} . \qquad \text{B-18}$$

An approximate solution may now be obtained by iteration. In Equation B-7 and B-8, A_3 and B_3 are assumed zero. As a first approximation, A_1 may be found using the u - v plane as was done in Chapter II. The value of A_1 so found is substituted in Equation B-18 and the resulting value for B_3 substituted into Equation B-17. These third harmonic components may be used to produce new constant ω loci in a u - v plane. It may be observed that in Equation B-8 for positive values of B_3 the value of u becomes more positive. The same is true for values of v in Equation B-7. Thus, in consideration of the third harmonic components, a point on a locus is moved up and to the right. In the vicinity of first harmonic resonance the value of $|u|_{max}$ is reduced.

In order to illustrate the significance of the third harmonic components, values of A_1 were chosen, and A_3 and B_3 determined for three values of angular frequency for the three systems of this study. Figures 20 through 28 show the variation of the third harmonic components for the systems. The values of angular frequency chosen were at $\omega = \sqrt{\alpha/\gamma}$, $\omega = \frac{1}{3}\sqrt{\alpha/\gamma}$ and near resonance. The value of F used was $F = |u|_{max}$. The value of $\omega = \frac{1}{3}\sqrt{\alpha/\gamma}$ was a very deliberate choice. For this value of ω the third harmonic cosine coefficient is zero and the value of B_3 is negative. It is, however, to be noted that its magnitude is sufficiently large to

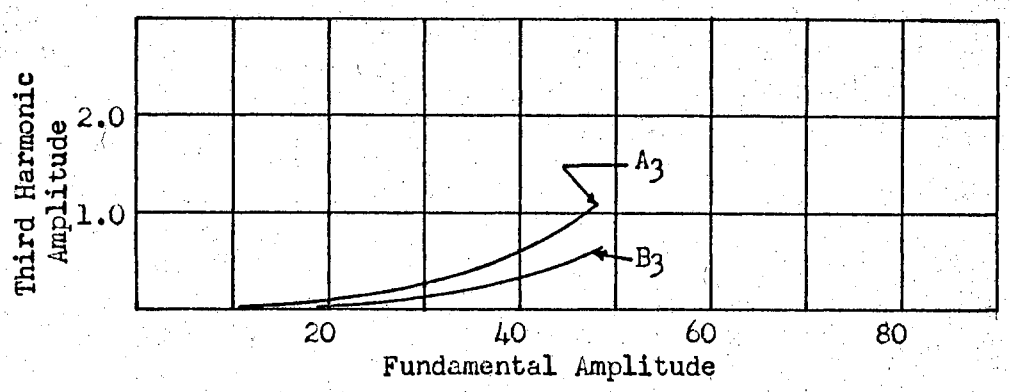

Figure 20. Third Harmonic Components for System 3, $\omega = \sqrt{\alpha/\gamma} = 258$

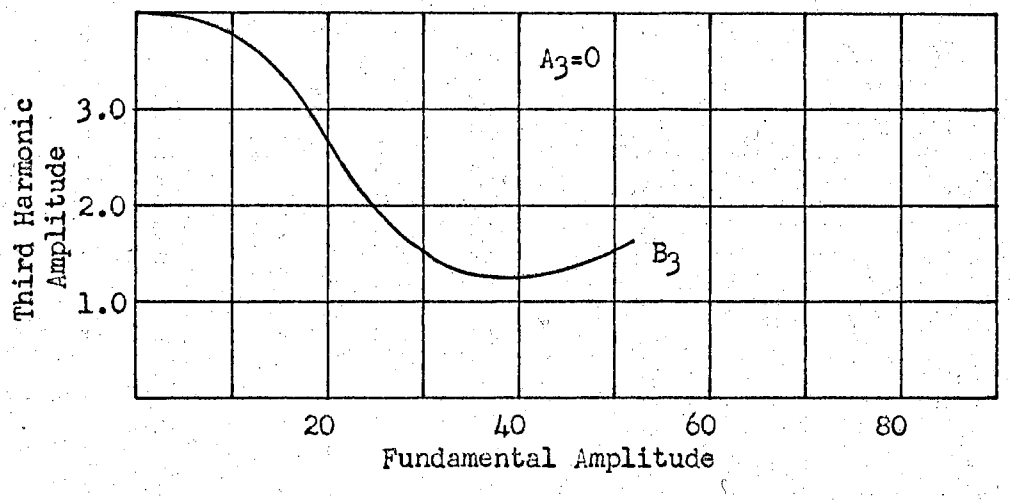

Figure 21. Third Harmonic Components for System 3, $\omega = \frac{1}{3}\sqrt{\alpha/\gamma} = 86$

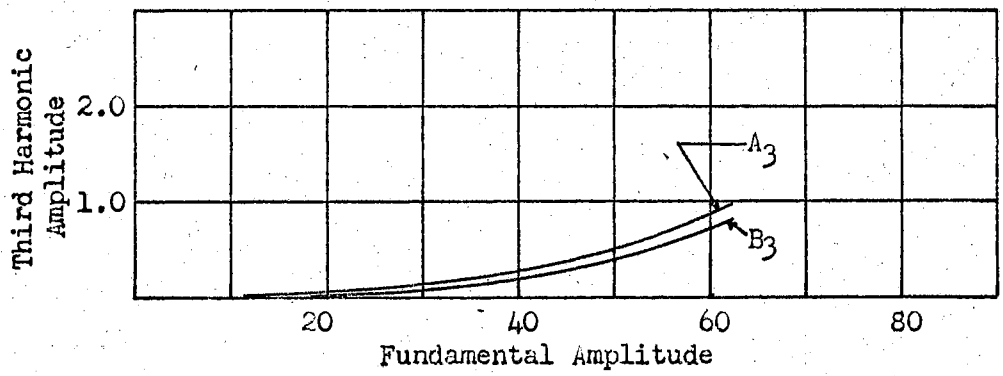

Figure 22. Third Harmonic Components for System 3, $\omega = 340$

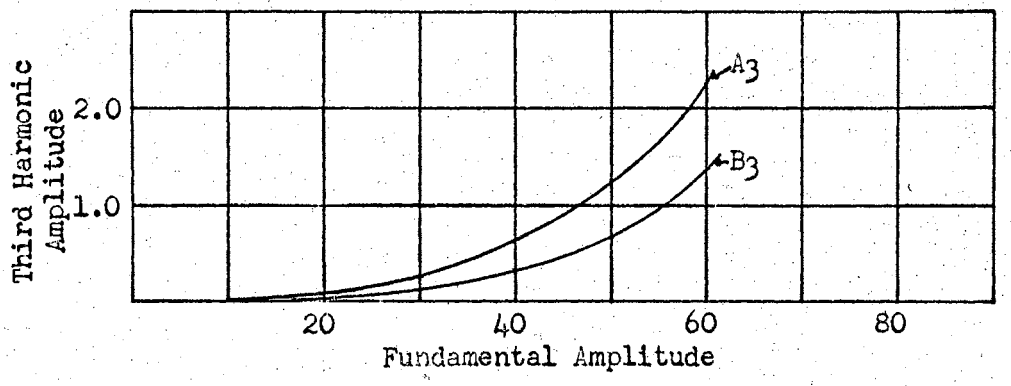

Figure 23. Third Harmonic Components for System 4, $\omega = \sqrt{\alpha/\gamma} = 387$

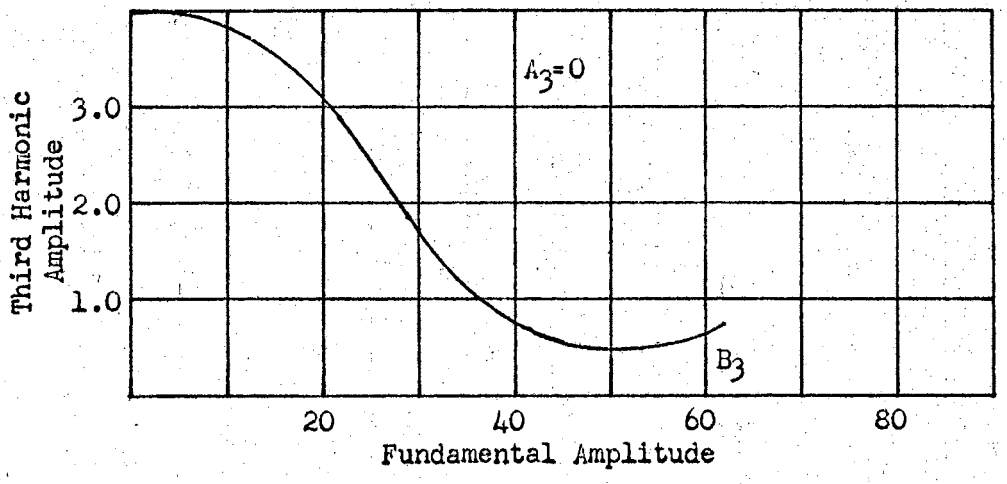

Figure 24. Third Harmonic Components for System 4, $\omega = \frac{1}{3}\sqrt{\alpha/\gamma} = 129$

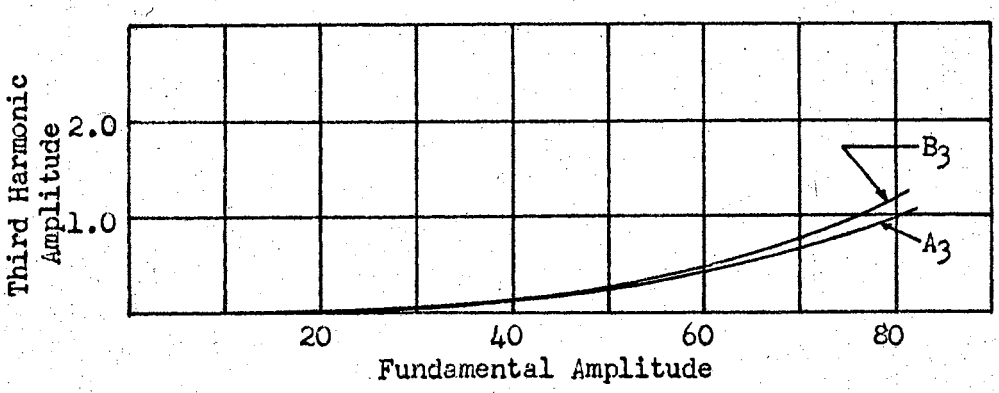

Figure 25. Third Harmonic Components for System 4, $\omega = 615$

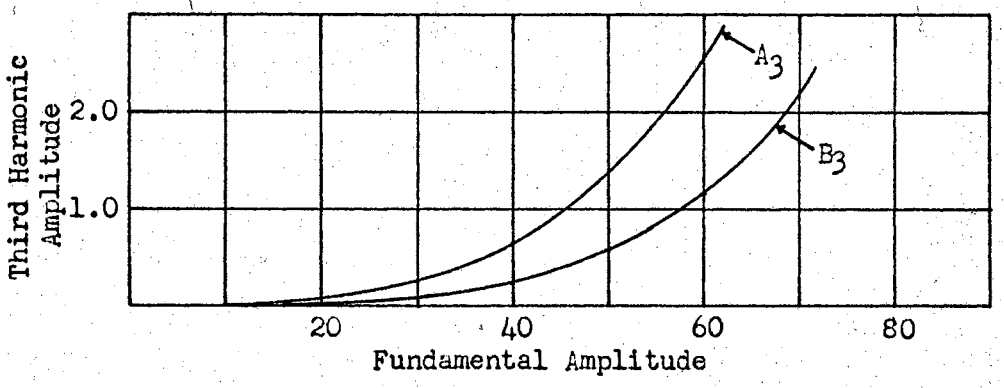

Figure 26. Third Harmonic Components for System 5, $\omega = \sqrt{\alpha/\gamma} = 447$

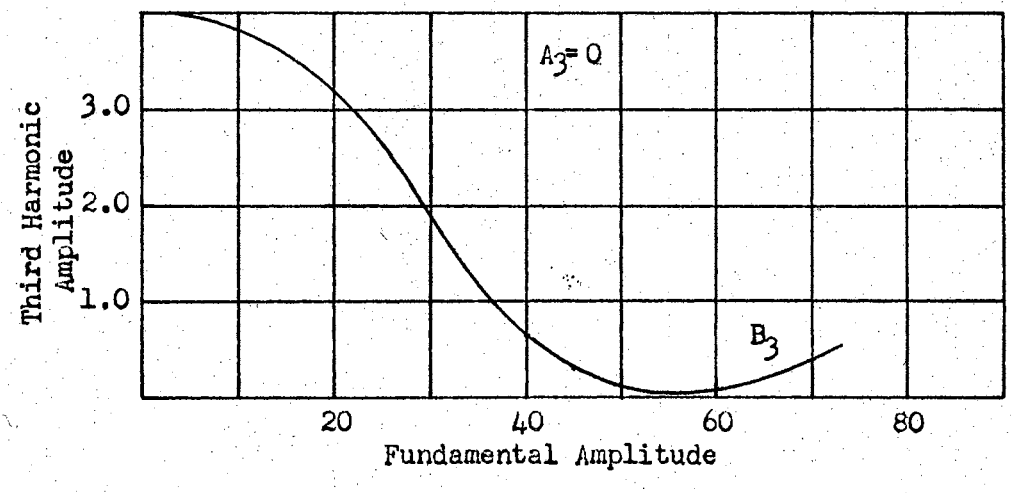

Figure 27. Third Harmonic Components for System 5, $\omega = \frac{1}{3}\sqrt{\alpha/\gamma} = 149$

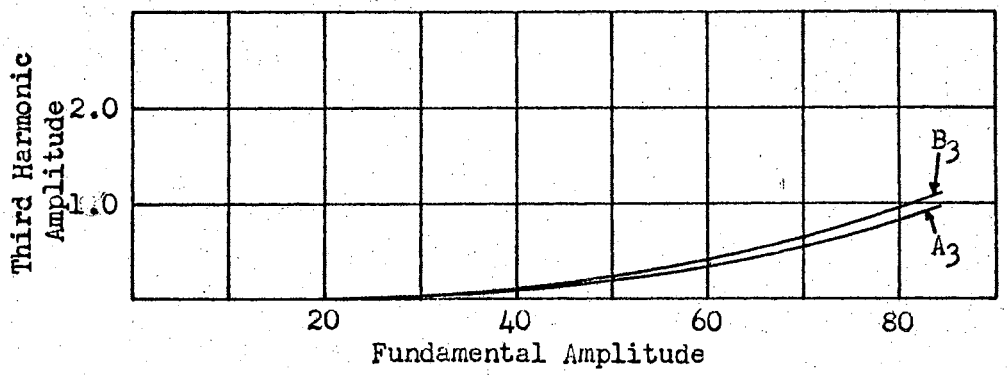

Figure 28. Third Harmonic Components for System 5, $\omega = 775$

invalidate one of the assumptions above. (The third harmonic components' products are small.)

The approximate third harmonic percentages for $F = |u|_{max}$ are as follows:

Angular Frequency	3	4	5
$\omega = \frac{1}{3}\sqrt{\alpha/\gamma}$	4	4.5	6.0
$\omega = \sqrt{\alpha/\gamma}$	1.2	3.0	3.6
Near Resonance	2.0	2.0	2.0

APPENDIX C

LOCUS OF VERTICAL TANGENTS IN THE u - v PLANE

In Figures 29, 30, and 31 are shown the u - v plane constant loci for the three systems examined in this research. A part of the locus of vertical tangents is also plotted on these curves. The locus of vertical tangents is used to determine the point at which lower jump resonance occurs. The three circles drawn on the figures, which represent the forcing function amplitudes $F = F_T$, $F = |u|_{max}$, and $F > |u|_{max}$, are used to determine the response curves. The intersection of a constant ω locus with a circle determines a point (u_1, v_1). The amplitude A_1 is found by using the equation

$$A_1 = \frac{u_1}{\omega(\mu\omega^2 - \delta)} .$$

The intersection of a circle with the locus of vertical tangents determines the point at which lower jump resonance occurs. If this point of intersection is (u_2, v_2), then the point u_2^2 is given by Equation III-3 and is

$$u_2^2 = \frac{1}{3b}\left[-2a + \sqrt{a^2 - 3}\right]. \qquad \text{C-1}$$

In this expression a and b determine a particular locus on which the point (u_2, v_2) lies. The point v_2 must satisfy the expression

$$v_2 = au_2 + bu_2^3 .$$

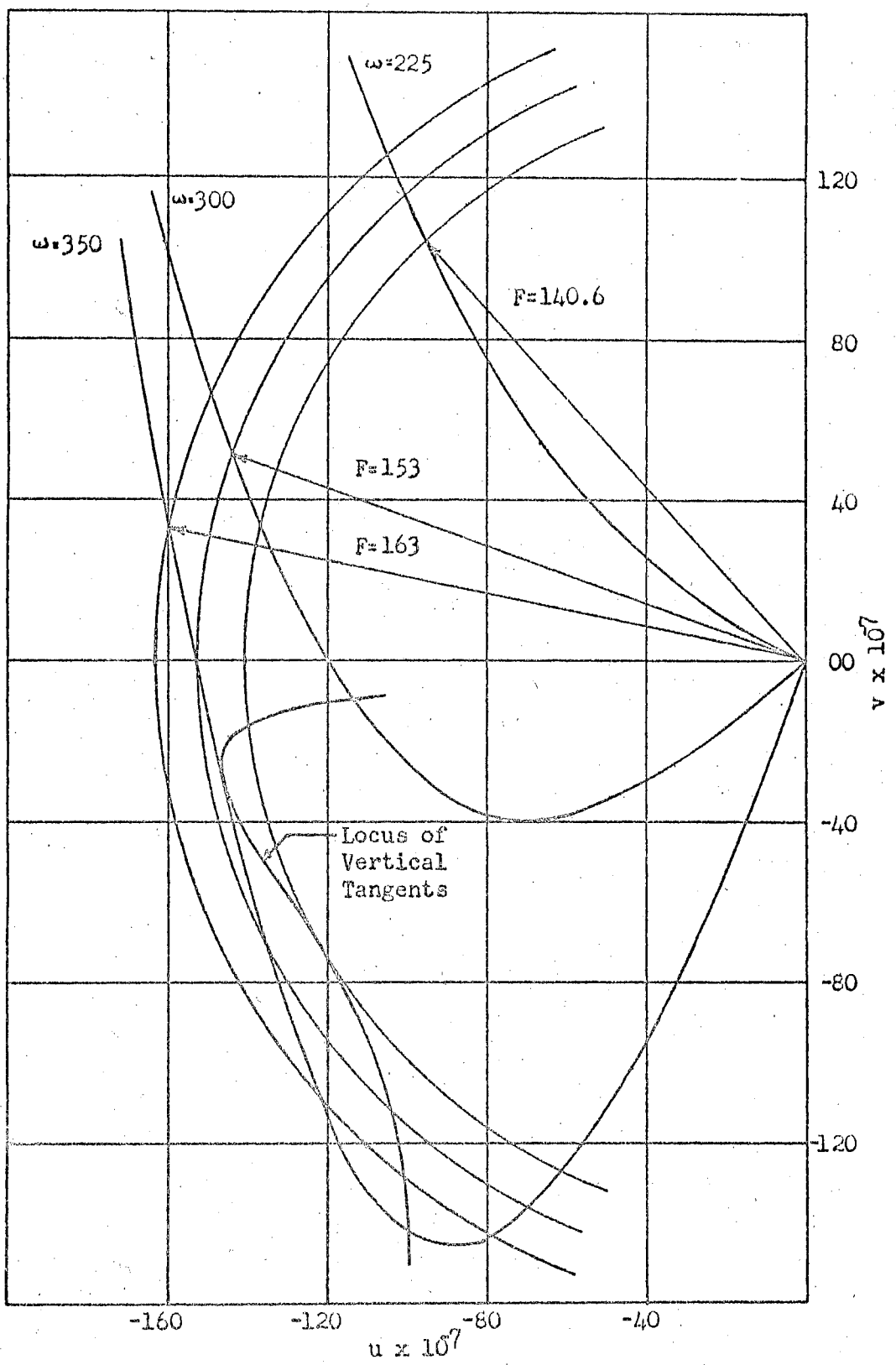

Figure 29. Constant ω Loci in the u - v Plane for System 3

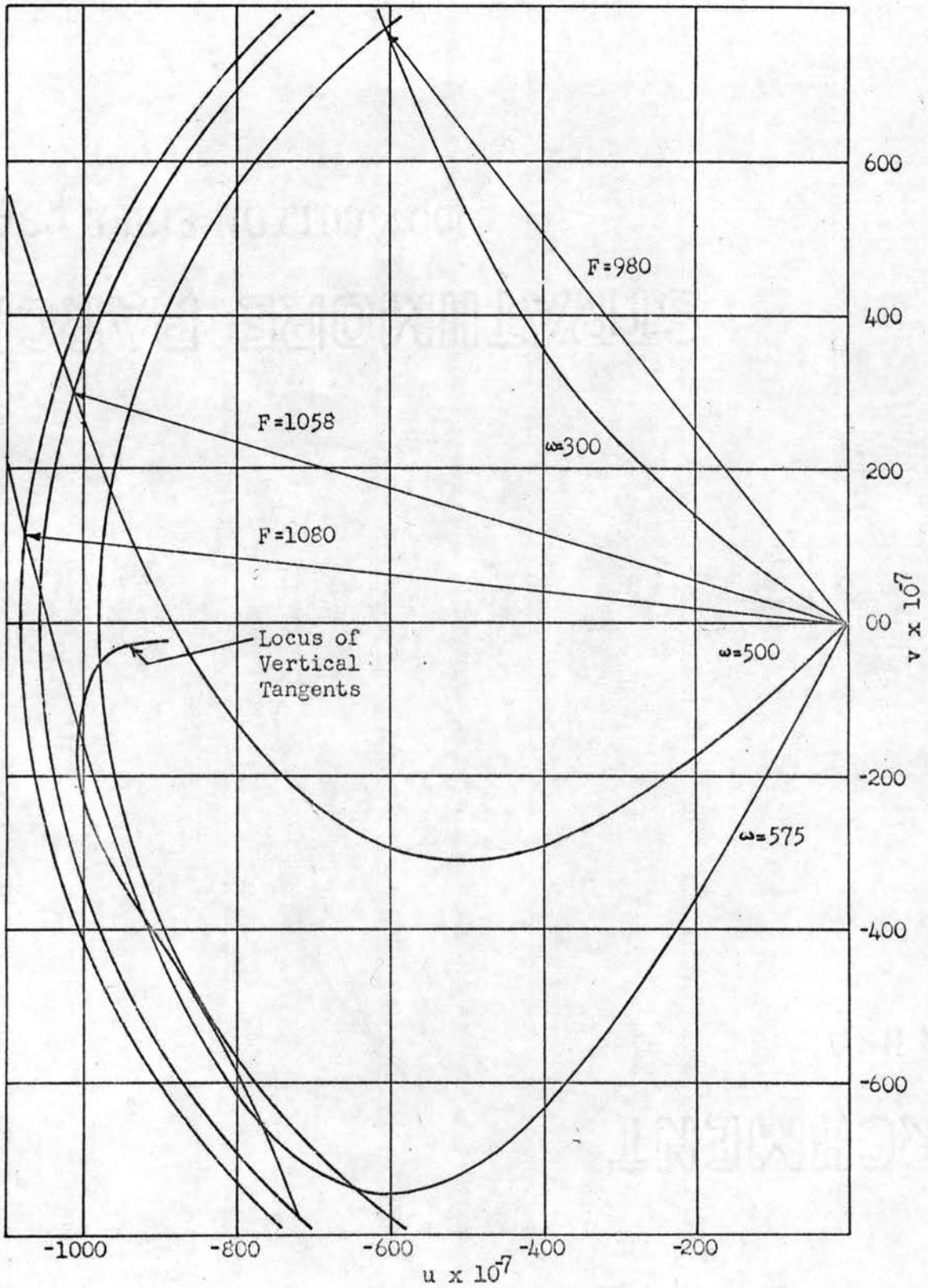

Figure 30. Constant ω Loci in the u - v Plane for System 4

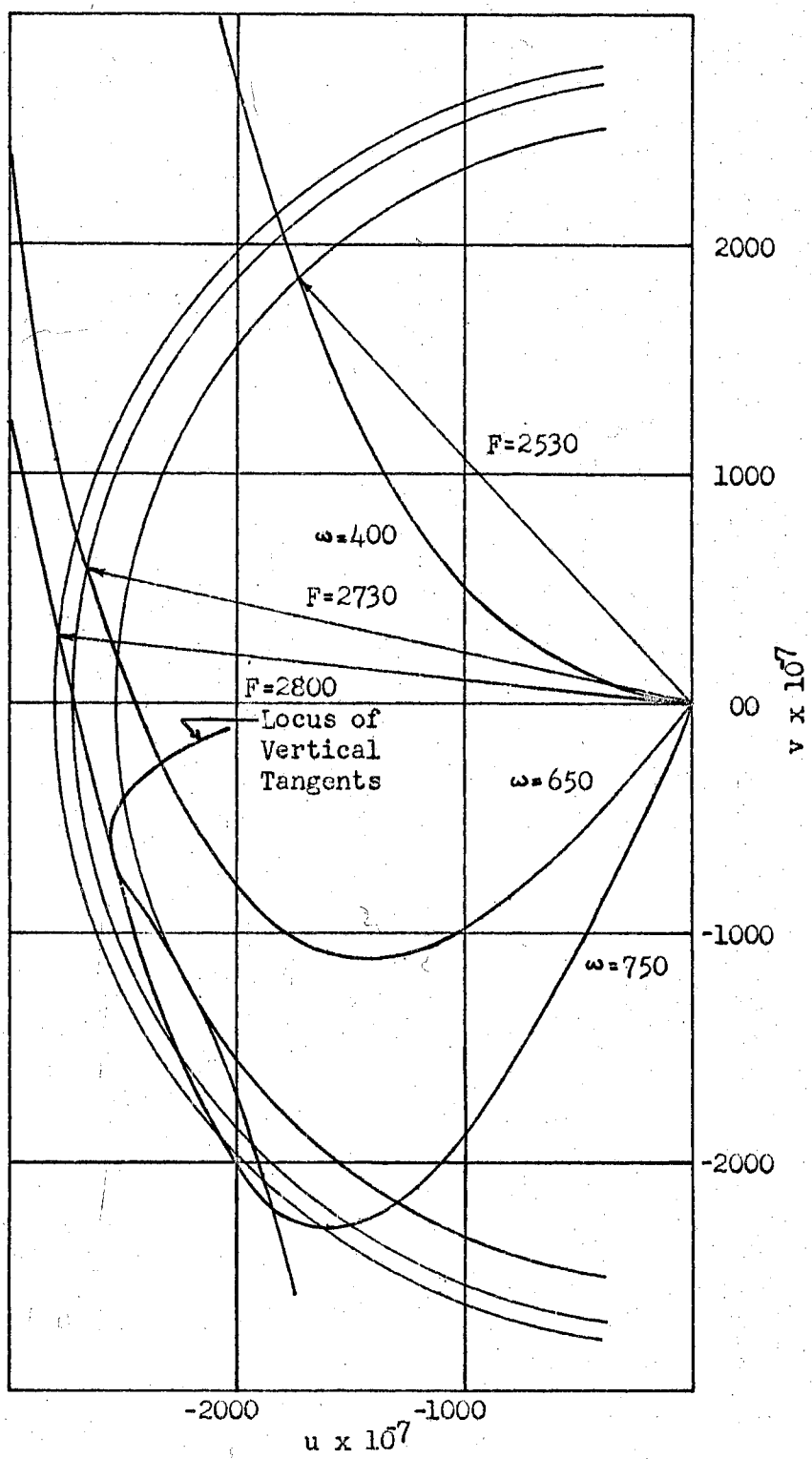

Figure 31. Constant ω Loci in the u - v Plane for System 5

If Equation C-1 is solved for b, the result is

$$b = \frac{1}{3u_2^2}\left[-2a + \sqrt{a^2 - 3}\right]. \qquad \text{C-3}$$

From Equation C-2 the solution for a is

$$a = \frac{v_2 - bu_2^3}{u_2} \qquad \text{C-4}$$

or

$$a = \frac{v_2}{u_2} - bu_2^2. \qquad \text{C-5}$$

By substituting b from Equation C-3 into Equation C-5, the result is

$$a = \frac{v_2}{u_2} + \frac{2a}{3} - \frac{1}{3}\sqrt{a^2 - 3}\ . \qquad \text{C-6}$$

Upon simplification of the expression, Equation C-6 becomes

$$a = \frac{3\left(\frac{v_2}{u_2}\right)^2 + 1}{2\frac{v_2}{u_2}} \qquad \text{C-7}$$

or

$$a = \frac{3}{2}\ \frac{v_2}{u_2}\ + \frac{1}{2}\ \frac{u_2}{v_2}. \qquad \text{C-8}$$

The positive real root of the cubic

$$a\mu\omega^3 + \gamma\omega^2 - a\delta\omega - \alpha = 0 \qquad \text{II-41}$$

is the lower jump resonance angular frequency.

APPENDIX D

SYMBOLS

A_1	Fundamental amplitude response, cosine coefficient
A_3	Third harmonic amplitude response, cosine coefficient
A_{1u}	Fundamental amplitude at maximum u for v = 0
C	Capacitance, farads
c	Nonlinear coefficient
E(s)	Laplace transform of system error
e	Instantaneous system error
F	Forcing function, $Kf(t)/T_1T_2$
F_T	Threshold forcing function
f(t)	Instantaneous system input
f(y)	Instantaneous output of nonlinear device
G(s)	Transfer function
H [y]	Differential equation of Ritz Averaging Method
K	Gain
R	Resistance, ohms
T_1	Time constant, seconds
T_2	Time constant, seconds
u	Horizontal component of forcing function
$\|u\|_{max}$	Maximum value of u for variable frequency and v = 0
v	Vertical component of forcing function
Y(s)	Laplace transform of system response

y	Instantaneous system response
α	Coefficient of response function, K/T_1T_2
β	Coefficient of nonlinear function, Kc/T_1T_2
γ	Coefficient of second derivative of response function, $(T_1 + T_2)/T_1T_2$
δ	Coefficient of first derivative of response function, $(1/T_1T_2)$
μ	Coefficient of third derivative of response function
φ	Angle of lag of system response
ω	Angular frequency, radians per second
ω_u	Angular frequency for maximum u, radians per second

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