A GENERALIZED DERIVATIVE

By

THOMAS WILLIAM CAIRNS

Bachelor of Science
Oklahoma State University
Stillwater, Oklahoma
1953

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1955

Submitted to the Faculty of the Graduate School of
The Oklahoma State University
in partial fulfillment of the requirements
for the degree of
DOCTOR OF PHILOSOPHY
May, 1960

A GENERALIZED DERIVATIVE

Thesis Approved:

R. B. Deal
Thesis Adviser

O. H. Hamilton

John E. Hoffman

H. Harrington

Franklin A. Graybill

Robert MacVicar
Dean of the Graduate School

PREFACE

My thanks are expressed to Professor R. B. Deal for his guidance and assistance in the writing of this thesis, to Professor L. Wayne Johnson for his counsel and encouragement throughout my graduate program and to the remainder of those who constitute my graduate committee.

TABLE OF CONTENTS

LIST OF TABLES

I INTRODUCTION

This work on a generalized derivative was motivated by the absence of strong techniques for numerical differentiation. It is clear that the process of differentiation is one which inherently magnifies errors whether they are errors in empirical data or interpolation errors. One technique for numerical differentiation of a function, f, is to fit f with a polynomial and then differentiate the polynomial. However, a small fitting error may not at all reflect a larger error in a derivative. Hence, it seems effective to define a derivative in terms of an integral, which is fundamentally a smoothing process and which, in the case of a finite approximation, give an approximate derivative of f which depends on the nature of f on an entire interval instead of just at the end points. Such an integral might well be put in a form to which one could apply a standard numerical integration technique. It also seems likely that in specific applications one could establish criteria, perhaps statistical, to determine a "best" interval for the approximation. A study of such criteria seems to be a logical and possibly fruitful sequel to this paper.

Some work has been done on generalized derivatives and the term has been used to describe several concepts in addition to the one described here. For example, Kassimatis [5] defines the n'the generalized Riemann derivative of a measurable function, f, at x by

$$D_h^n f(x) = \lim_{h \to 0} (2h)^{-n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(x+2jh-nh) \text{ for } h > 0,\ n = 1, 2, \ldots$$

In the case of

$$D^2 f(x) = \lim_{h \to 0} (2h)^{-2}[f(x-2h) - 2f(x) + f(x+2h)]$$

this is the same as Hobson's [4] definition of the generalized second derivative. These definitions provide generalized derivatives for some functions which do not have derivatives in the ordinary sense.

The n'th Peano derivative [3] of a continuous function, f, is a_n if f can be expressed by $f(c+h) = a_o + a_1 h + \ldots + \frac{a_n h^n}{n!} + \epsilon(h)h^n$ where $\lim_{h \to 0} \epsilon(h)h^n = 0$.

Laurent Schwartz has proposed the idea of a distribution as a generalization on the idea of a point function. His distribution is a functional defined on a set of testing functions which have the properties that each has derivatives of all orders which, as well as the testing function itself, vanishes at the ends of the interval [a,b]. Then each function, f, determines the functional

$$F(\phi) = \int_a^b f(x)\phi(x)dx$$

where ϕ ranges over the set of testing functions. Integration by parts shows that

$$-F(\phi') = \int_a^b f'(x)\phi(x)dx = -\int_a^b f(x)\phi'(x)dx$$

and we can define $F'(\phi) = -F(\phi')$. This is a generalized derivative in the sense that it is the derivative of a distribution which is a generalization on the idea of a point function. However, the approach of Schwartz is not amenable to smoothing noisy data by numerical approximation.

In this paper we use the following procedure for defining a generalized derivative. If a function, f, is fit on some interval [x-h,x+h] by a straight line in the sense that

$$\int_{x-h}^{x+h} [f(\xi) - a\xi - b]^2 d\xi$$

is a minimum, we obtain the line

$$y' = \frac{3}{2h^3}\int_{-h}^{h}\xi f(x+\xi)d\xi(x'-x) + \frac{1}{2h}\int_{-h}^{h}f(x+\xi)d\xi.$$

We define the generalized derivative

$$Df(x) = \lim_{h\to 0}\frac{3}{2h^3}\int_{-h}^{h}\xi f(x+\xi)d\xi$$

if the limit exists and if the integral mean, $\lim_{h\to 0}\frac{1}{2h}\int_{-h}^{h}f(x+\xi)d\xi = \bar{f}(x)$ also exists. Conditions necessary for the convergence of Df are much weaker than for the existence of the ordinary derivative and, in fact, $Df(x) = f'(x)$ if the latter exists.

One can state some interesting theorems which have analogies in the theory of ordinary derivatives. For example, we show that if $\bar{f}(x)$ exists, then $D\int_{a}^{x}f(\xi)d\xi = \bar{f}(x)$. We can determine some conditions under which $D[f(x)g(x)] = \bar{f}(x)Dg(x) + \bar{g}(x)Df(x)$. In addition, there are mean value theorems similar to the familiar ones. We can also discuss convergence of Df in terms of right and left-hand generalized derivatives and integral means which are defined to be, as the name implies, the parameters of the limiting lines of best fit on the intervals to the right and left of any point. These are designated by D^+f, D^-f, $\bar{f}^+$ and $\bar{f}^-$. We show, for example, that under some conditions $Df(x) = \lim_{h\to 0}\frac{1}{h}[\bar{f}_h^+(x) - \bar{f}_h^-(x)]$ where the subscripts indicate integral means over a finite interval.

Finally, in a manner similar to that used to develop the generalized derivative, we may define a generalized n'th derivative. To facilitate this development we generalize the Legendre polynomials to the interval [x-h,x+h] and examine the best fit n'th degree polynomial in the limit as h approaches zero. We are able to state, in particular, some relationships between this generalized n'th derivative and the generalized first derivative iterated n times.

II. A FIRST ORDER GENERALIZED DERIVATIVE

Consider a function $f(x)$ defined on the real numbers and Lebesgue square integrable on the interval $[x_o-h, x_o+h]$. We may associate with the function at the point x_o a straight line which, over the interval $[x_o-h, x_o+h]$ is the line of best fit by the least square criterion. That is, we know that it is possible to find a unique pair a, b such that

$$[E^2] = \int_{x-h}^{x+h} [f(x) - a(x-x_o) - b]^2 dx$$

is a minimum. If we make a change of variables $x = x_o + \xi$, we obtain the simpler expression

$$[E^2] = \int_{-h}^{h} [f(x_o+\xi) - a\xi - b]^2 \, d\xi.$$

We take partial derivatives with respect to a and b and set them equal to zero.

$$\int_{-h}^{h} [f(x_o+\xi) - a\xi - b] d\xi = 0$$

$$\int_{-h}^{h} [f(x_o+\xi) - a\xi - b]\xi d\xi = 0.$$

These equations are linear in a and b and can be written

$$a \int_{-h}^{h} \xi^2 \, d\xi + b \int_{-h}^{h} \xi \, d\xi = \int_{-h}^{h} \xi f(x_o+\xi) d\xi$$

$$a \int_{-h}^{h} \xi \, d\xi + b \int_{-h}^{h} d\xi = \int_{-h}^{h} f(x_o+\xi) d\xi.$$

integrating and solving for a and b, we obtain

$$\frac{2h^3}{3}a = \int_{-h}^{h} \xi f(x_o+\xi)d\xi$$

or

$$a = \frac{3}{2h^3}\int_{-h}^{h} \xi f(x_o+\xi)d\xi.$$

$$2hb = \int_{-h}^{h} f(x_o+\xi)d\xi$$

or

$$b = \frac{1}{2h}\int_{-h}^{h} f(x_o+\xi)d\xi.$$

We had a familiar alternative approach available to us for finding the best fit line. We can consider the functions 1, $(x-x_o)$, $(x-x_o)^2$, . . . , $(x-x_o)^n$, . . . to be the basis elements of the space of all functions Lebesgue square integrable on $[x_o+h, x_o-h]$. If we define an inner product in the following manner

$$[f,g] = \int_{x-h}^{x+h} f(x)g(x)dx$$

$$= \int_{-h}^{h} f(x_o+\xi)g(x_o+\xi)d\xi,$$

the Hilbert axioms are satisfied and the space is a Hilbert space. Consider now an arbitrary element, f, of the space and the subspace spanned by the vectors 1, and $x-x_o$. It is well known that the projection of f onto the subspace is precisely that linear combination of the base vectors, $a(x-x_o) + b$, which minimizes the distance to f. The distance in terms of the Hilbert norms is given by

$$[f(x) - a(x-x_o) - b,\ f(x) - a(x-x_o) - b]^{\frac{1}{2}} = \left[\int_{-h}^{h} [f(x_o+\xi) - a\xi - b]^2 d\xi\right]^{\frac{1}{2}}.$$

Thus, the distance from f to the subspace is the least square error expression. The criterion by which we determine a and b follows as a result of the fact that the difference between f and its projection on the subspace is orthogonal to every element in the subspace, in particular the base vectors. That is,

$$\int_{-h}^{h} [f(x_o+\xi) - a\xi - b]\xi d\xi = 0$$

$$\int_{-h}^{h} [f(x_o+\xi) - a\xi - b] d\xi = 0.$$

Thus we arrive at the same set of linear equations in a and b.

Several things are immediately clear. Our space is really L_2 defined on a general interval. Furthermore, we have a different space for each x_o and h, and as a result, a different best fit line. Our objective is to treat

$$a = \frac{3}{2h^3}\int_{-h}^{h} \xi f(x_o+\xi) d\xi$$

and

$$b = \frac{1}{2h}\int_{-h}^{h} f(x_o+\xi) d\xi$$

as operators on a space of functions and so we will change our designation of the variables by dropping the subscript. It is still necessary to distinguish between the independent variable in the coefficients, a and b, and in the base elements of the subspace. We will prime the latter. Hence, for a given x and h the best fit line is

$$y = \frac{3}{2h^3}\int_{-h}^{h} \xi f(x+\xi) d\xi (x'-x) + \frac{1}{2h}\int_{-h}^{h} f(x+\xi) d\xi.$$

We will also use the following notation:

$$D_h f(x) = \frac{3}{2h^3}\int_{-h}^{h} \xi f(x+\xi) d\xi = \frac{3}{2h^3}\int_{x-h}^{x+h} (x-\xi) f(\xi) d\xi$$

$$\overline{f}_h(x) = \frac{1}{2h}\int_{-h}^{h} f(x+\xi) d\xi = \frac{1}{2h}\int_{x-h}^{x+h} f(\xi) d\xi$$

and when we wish to emphasize the operator aspect of the latter, i.e., the operator which maps f into $\bar{f}_h$, we will use T_h; hence, $T_h f = \bar{f}_h$. Whenever we wish to use the derivative of f(x), we will write f'(x) or $\frac{df(x)}{dx}$.

Clearly, $\bar{f}_h(x)$ is the integral mean of f on an interval symmetric about x. A great deal has been published on the subject of integral means [6] and a few of the theorems below differ from known work only in notation or point of view.

From the second form of the relation defining $\bar{f}_h$ it is clear that $\bar{f}_h$ is absolutely continuous. If, for all ρ such that $-h \leq \rho \leq h$, $f(x+\rho)$ is bounded, that is, there exist m and M such that $m \leq f(x+\rho) \leq M$, then by the first mean value theorem

$$\bar{f}_h(x) = \frac{1}{2h}\int_{-h}^{h} f(x+\rho)d\rho = K, \text{ where } m \leq K \leq M.$$

If f is continuous on this interval, then there exists a ρ, such that $-h \leq \rho \leq h$ and $\bar{f}_h(x) = f(x+\rho)$. From these properties we obtain the following.

Theorem I: If $f(x+\rho) > 0$ for $-h \leq \rho \leq h$ and f is integrable on this interval, then $\bar{f}_h(x) > 0$.

Corollary: If $f(x+\rho) > g(x+\rho)$ for $-h \leq \rho \leq h$ and f and g are integrable on this interval, then $\bar{f}_h(x) > \bar{g}(x)$.

We may also see that T_h possesses properties of continuity and linearity associated with its operator character. Whenever we use T_h as an operator, we need require not only integrability, which will be assumed hereafter, but if the domain of the operator is a set of functions defined, for example, on a closed interval, [a,b], we must devise some method of taking care of what happens at the end of the interval. One possibility is to ask that the functions operated on by T_h be defined and integrable on the interval $[-h+a, b+h]$ in order that the result of the operation be defined on $[a,b]$, or we might assume that all the functions are identically zero outside the interval. In practice it might be more convenient to handle this difficulty in some other way, but at least in the above two cases we may show that the operator is linear and continuous in the sense that $\lim_{n\to\infty} f_n = g$ implies $\lim_{n\to\infty} T_h f_h = T_h g$ where convergence is in terms of a sup norm.

$D_h f$ possesses similar elementary properties. It is absolutely continuous and D_h is also a continuous, linear operator if defined on an appropriate set of functions. If, as before, $m \leq f(x+\xi) \leq M$ in the interval in question, we obtain by the mean value theorem

$$\frac{3}{4h}m \leq \frac{3}{2h^3}\int_0^h \xi f(x+\xi)d\xi \leq \frac{3}{4h}M$$

$$-\frac{3}{4h}M \leq \frac{3}{2h^3}\int_{-h}^0 \xi f(x+\xi)d\xi \leq -\frac{3}{4h}m$$

and adding

$$\left|\frac{3}{2h^3}\int_{-h}^h \xi f(x+\xi)d\xi\right| \leq \frac{3}{2}\,\frac{M-m}{2h} .$$

The following theorems establish an important relationship between the operator D_h and the ordinary derivative. We know that if $f'(x) \geq g'(x)$ for all x in some interval, then f-g is monotone increasing in the interval.

Lemma 1: If $f(x+\xi)$ is monotone increasing for $-h \leq \xi \leq h$, then $D_h f(x) \geq 0$.

Proof: Suppose that $f(x+\xi)$ is monotone increasing for $-h \leq \xi \leq h$. Then by the second mean value theorem there exists a ξ' such that $-h \leq \xi' \leq h$ and

$$D_h f(x) = \frac{3}{2h^3}\int_{-h}^h \xi f(x+\xi)d\xi$$

$$= \frac{3}{2h^3}\left[f(x-h)\int_{-h}^{\xi'}\xi d\xi + f(x+h)\int_{\xi'}^h \xi d\xi\right]$$

$$= \frac{3}{2h^3}\left[f(x-h)\,\frac{\xi^2}{2}\Big|_{-h}^{\xi'} + \left[f(x+h)\,\frac{\xi^2}{2}\right|_{\xi'}^{h}\right.$$

$$= \frac{3}{4h^3}\left[h^2[f(x+h) - f(x-h)] - \xi^2[f(x+h) - f(x-h)]\right]$$

$$= \frac{3}{4h^3}[f(x+h) - f(x-h)][h^2 - \xi'^2].$$

It is clear that (1) $f(x+h) > f(x-h)$ because of the monotonicity of f, and (2) $h^2 > \xi'^2$ since ξ' is interior to the interval and h is the end point. Hence $\bar{D}_h f(x) > 0$.

We have now established the following theorem.

Theorem II: If $f'(x+\xi) > g'(x+\xi)$ for all $-h \leq \xi \leq h$, then $\bar{D}_h f(x) > \bar{D}_h g(x)$.

Theorem III: If $f'(x+\xi)$ has a maximum and a minimum on $-h \leq \xi \leq h$, then $\min f'(x+\xi) \leq \bar{D}_h f(x) \leq \max f'(x+\xi)$.

Proof: Consider the line, $g(x) = [\max f'(x+\xi)]x + b$. Obviously $g'(x+\xi) \geq f'(x+\xi)$ in the interval. Then by the previous theorem $\bar{D}_h g(x) \geq \bar{D}_h f(x)$. But g is a straight line and operating on it with $\bar{D}_h$ yield simply its slope, $\bar{D}_h g(x) = \max f'(x+\xi)$. Hence, $\bar{D}_h g(x) \leq \max f'(x+\xi)$. The proof for the other inequality is similar.

Corollary: If $f'(x+\xi)$ is continuous for $-h \leq \xi \leq h$, then there exists a ρ such that $-h < \rho < h$ and $\bar{D}_h f(x) = f'(x+\rho)$.

It is clear that there is an intimate relationship between $f'(x)$ and $\bar{D}_h f(x)$ and also between $f(x)$ and $\bar{f}_h(x)$. In fact, if $f(x)$ has a continuous derivative, from the mean value theorem discussed above, it is clear that we can make $f'(x) - \bar{D}_h f(x)$ as small as we please by choosing a sufficiently small h. Let us define the notation $\lim_{h \to 0} \bar{D}_h f(x) = \bar{D}f(x)$, and $\lim_{h \to 0} \bar{f}_h(x) = \bar{f}(x)$. We see that under the conditions mentioned above $\bar{D}f(x) = f'(x)$. This is an important result, but the same result can be obtained under weaker conditions.

Theorem IV: Let $f'(x)$ exist and $f(x+\xi)$ be integrable on some interval containing $\xi = 0$. Then $\bar{D}f(x) = f'(x)$.

Proof: Differentiability implies that there exists a K such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|\xi'| < \delta$ implies

$$\left| \frac{f(x+\xi') - f(x-\xi')}{2} - K \right| < \epsilon .$$

This is also true on some interval on which f is integrable. Then

$$K - \epsilon < \frac{f(x+\xi') - f(x-\xi')}{2\xi'} < K + \epsilon ,$$

$$2\xi'^2 (K - \epsilon) < \xi'[f(x+\xi') - f(x-\xi')] < 2\xi'^2 (K + \epsilon)$$

By the mean value theorem, for h in the interval,

$$\frac{4h^3}{3}(K-\epsilon) < \int_{-h}^{h} \xi f(x+\xi)d\xi - \int_{-h}^{h} \xi f(x+\xi)d\xi < \frac{4h^3}{3}(K+\epsilon),$$

and finally

$$\left| \frac{3}{2h^3} \int_{-h}^{h} \xi f(x+\xi)d\xi - K \right| < \epsilon .$$

A proof similar to that of theorem IV would suffice to show that if f(x) is continuous, then $\bar{f}(x) = f(x)$.

Theorem V: If $\bar{f}(x)$ exists, then $D\int_a^x f(\xi)d\xi = \bar{f}(x)$.

Proof: By definition

$$D\int_a^x f(\xi)d\xi = \lim_{h\to 0} \frac{3}{2h^3} \int_{-h}^{h} \xi \int_a^{x+\xi} f(\mu)d\mu d\xi$$

$$= \lim_{h\to 0} \frac{3}{2h^3} \int_{-h}^{h} \xi^2 \left[\frac{1}{\xi}\int_x^{x+\xi} f(\mu)d\mu\right] d\xi$$

$$= \lim_{h\to 0} \frac{3}{2h^3} \left\{ \int_{-h}^{0} \xi^2 \left[\frac{1}{\xi}\int_x^{x+\xi} f(\mu)d\mu\right] d\xi + \int_0^{h} \xi^2 \left[\frac{1}{\xi}\int_x^{x+\xi} f(\mu)d\mu\right] d\xi \right\}$$

$$= \lim_{h\to 0} \frac{3}{2h^3} \left\{ \int_{0}^{h} \xi^2 \left[\frac{1}{\xi}\int_{x-\xi}^{x} f(\mu)d\mu\right] d\xi + \int_0^{h} \xi^2 \left[\frac{1}{\xi}\int_x^{x+\xi} f(\mu)d\mu\right] d\xi \right\}$$

$$= \lim_{h\to 0} \frac{3}{2h^3} \int_0^h \xi^2 \left[\frac{1}{2\xi}\int_{x-\xi}^{x+\xi} f(\mu)d\mu\right] d\xi .$$

By hypothesis, $\lim_{h\to 0} \frac{1}{2h}\int_{x-h}^{h+h} f(\mu)d\mu = \bar{f}(x)$. Hence, for every $\epsilon > 0$ there exists a $\delta>0$ such that $0<\xi\leq h<\delta$ implies

$$\left[\bar{f}(x)-\epsilon\right]\frac{3}{h^3}\int_0^h \xi^2 d\xi < \frac{3}{2h^3}\int_0^h \xi^2\left[\frac{1}{2\xi}\int_{x-\xi}^{x+\xi} f(\mu)d\mu\right]d\xi < \left[\bar{f}(x)+\epsilon\right]\frac{3}{h^3}\int_0^h \xi^2 d\xi,$$

and

$$\left|\frac{3}{h^3}\int_0^h \xi^2\left[\frac{1}{2\xi}\int_{x-\xi}^{x+\xi} f(\mu)d\mu\right]d\xi - \bar{f}(x)\right| < \epsilon.$$

The last two statements by the first mean value theorem.

Lemma 2: Suppose $\bar{f}$ exists on some interval containing x and for all points, x', in the interval $\lim_{h\to 0}\frac{1}{h}\int_0^h f(x'+\xi)d\xi = \lim_{h\to 0}\frac{1}{h}\int_{-h}^0 f'(x+\xi)d\xi = \bar{f}(x)$. Then $\bar{f}$ is continuous at x.

Proof: In the above interval there is a $\delta>0$ for every $\epsilon>0$ such that $0<h<\delta$ implies

$$\left|\frac{1}{h}\int_0^h f(x+\xi)d\xi - \bar{f}(x)\right| < \epsilon$$

and

$$\left|\frac{1}{h}\int_0^h f(x+k+\xi)d\xi - \bar{f}(x+k)\right| < \epsilon$$

where x+k+h is also in the interval. Now let $0<\delta'<\frac{\delta}{2}$.
Then the following are true for $0<h$, and $k<\delta'$.

$$\left|\frac{1}{k}\int_0^k f(x+\xi)d\xi - \bar{f}(x)\right| < \epsilon,$$

$$\left|\frac{1}{k}\int_0^k f(x+\xi)d\xi - \bar{f}(x+k)\right| < \epsilon,$$

and

$$\left|\bar{f}(x) - \bar{f}(x+k)\right| < \epsilon.$$

In particular, we may show that

$$\lim_{h\to 0}\frac{1}{2h}\int_{-h}^h [f(x+\xi) - \bar{f}(x)]d\xi = 0,$$

and

$$\lim_{h\to 0} \frac{1}{h^2}\int_{-h}^{h}[f(x+\xi) - \bar{f}(x)]^2 d\xi = 0$$

imply

$$\lim_{h\to 0} \frac{1}{h}\int_{0}^{h}[f(x+\xi) - \bar{f}(x)]d\xi = 0.$$

Lemma 3: If f is continuous in some interval and $Df(x) > 0$ for all x in the interval, then f is monotone increasing.

Proof: Since f is continuous, it is non-increasing on some interval if it is non-increasing at all. But if there were some interval in which f were non-increasing, then $D_h f(x) < 0$ on that interval and, hence, for some x in the limit.

Theorem VI: If $\bar{f}$ and Df exist in an interval, $[x-h, x+h]$, and $\lim_{h\to 0} \frac{1}{h^2}\int_{-h}^{h}[f(x'+\xi) - f(x')]d\xi = 0$ and $Df(x') > 0$ for all x' in the interval, then $D_h f(x) > 0$.

Proof: Clearly $Df(x) = D\bar{f}(x)$. But by lemma 2, $\bar{f}(x)$ is continuous. Then the conclusion follows by lemma 3 and lemma 1.

Theorem VII: If $\bar{f}$ and Df exist in an interval $[x-h, x+h]$ and if $\lim_{h\to 0} \frac{1}{h^2}\int_{-h}^{h}[f(x'+\xi) - \bar{f}(x')]^2 d\xi = 0$ and $m \leq Df(x') \leq M$ for all x' in the interval, then $m \leq D_h f(x) \leq M$.

Proof: This follows immediately from theorem VI.

Theorem VIII: If $\bar{f}$ and Df exist in an interval, $[x+h, x-h]$, and if $\lim_{h\to 0} \frac{1}{h^2}\int_{-h}^{h}[f(x'+\xi) - f(x')]^2 d\xi = 0$ and Df is continuous in the interval, then $D_h f(x) = Df(x+k)$ for some k such that $-h < k < h$.

Proof: This follows immediately from theorem VII.

Theorem IX: Let $\bar{f}(x)$, $Df(x)$, $\bar{g}(x)$ and $Dg(x)$ exist and let

$$\lim_{h\to 0} \frac{1}{h^2}\int_{-h}^{h}[f(x+\xi) - \bar{f}(x)]^2 d\xi = 0$$

and

$$\lim_{h\to 0} \frac{1}{h^2}\int_{-h}^{h}[g(x+\xi) - \bar{g}(x)]^2 d\xi = 0.$$

Then

$$D[f(x)g(x)] = \bar{f}(x)Dg(x) + \bar{g}(x)Df(x).$$

Proof:

$$D[f(x+\xi)g(x+\xi)] = \lim_{h\to 0} \frac{3}{2h^3}\int_{-h}^{h}\xi[f(x+\xi) - \bar{f}(x)][g(x+\xi) - \bar{g}(x)]d\xi$$

$$+ \bar{f}(x)Dg(x) + \bar{g}(x)Df(x).$$

Now $\left| \frac{3}{2h^3}\int_{-h}^{h}\xi[f(x+\xi) - \bar{f}(x)][g(x+\xi) - \bar{g}(x)]d\xi \right|$

$$\leq \frac{3}{2h^3}\left[\int_{-h}^{h}|\xi|[f(x+\xi) - \bar{f}(x)]^2 d\xi \int_{-h}^{h}|\xi|[g(x+\xi) - \bar{g}(x)]^2 d\xi\right]^{\frac{1}{2}}$$

$$= \frac{3}{2}\left[\frac{1}{h^3}\int_{-h}^{h}|\xi|[f(x+\xi) - \bar{f}(x)]^2 d\xi \; \frac{1}{h^3}\int_{-h}^{h}|\xi|[g(x+\xi) - \bar{g}(x)]^2 d\xi\right]^{\frac{1}{2}}.$$

However, $\left|\frac{1}{h^3}\int_{-h}^{h}|\xi|[f(x+\xi) - \bar{f}(x)]^2 d\xi\right| \leq \frac{1}{h^2}\int_{-h}^{h}[f(x+\xi) - \bar{f}(x)]^2 d\xi.$

Then

$$\lim_{h\to 0} \frac{1}{h^3}\int_{-h}^{h}\xi[f(x+\xi) - \bar{f}(x)]^2 = 0$$

and, hence,

$$\lim_{h\to 0} \frac{3}{2h^3}\int_{-h}^{h}\xi[f(x+\xi) - \bar{f}(x)][g(x+\xi) - \bar{g}(x)]d\xi = 0$$

and

$$D[f(x)g(x)] = \bar{f}(x)Dg(x) + \bar{g}(x)Df(x).$$

A final important property of these operators is that if $f(x+\rho) = g(x+\rho)$ almost everywhere for $-h \leq \rho \leq h$, then $T_h f(x) = T_h g(x)$ and $D_h f(x) = D_h g(x)$.

It is interesting to note that operating with T_h is a smoothing process. It maps arbitrary integrable functions into continuous ones and continuous functions into differentiable ones. Furthermore, T_h maps exponentials into exponentials, sines into sines, and n'th degree polynomials into n'th degree polynomials. For example,

(1) $T_h e^x = \frac{e^h - e^{-h}}{2h} e^x,$

(2) $T_h \sin x = \frac{1}{h} \sin h \sin x,$

(3) $T_h x^n = \sum_{j=0}^{\frac{n}{2}} \frac{n!}{(2j+1)!(n-2j)!} x^{n-2j} h^{2j}$ if n is even, and

$T_h x^n = \sum_{j=0}^{\frac{n-2}{2}} \frac{n!}{(2j+1)!(n-2j)!} x^{n-2j} h^{2j}$ if n is odd.

Note that the first term is x^n. The rest is a polynomial of degree n-2 in x and n or n-1 in h. Hence,

$$T_h \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} a_i x^i + p(x,h)$$

where p(x,h) is a polynomial of degree n-2 in x and n or n-1 in h.

Operating with D_h is similar to a differentiating process as might be deduced from the properties already described. D_h maps exponentials into exponentials, sines into cosines, and n'th degree polynomials into n-1'st degree polynomials. For example,

(1) $D_h e^x = \frac{3}{2h^3}[h(e^h+e^{-h}) - (e^h-e^{-h})]e^x,$

(2) $D_h \sin x = \frac{3}{h^3}[h\cos h + \sin h]\cos x,$

(3) $D_h x^n = \sum_{j=0}^{\frac{n-1}{2}} \frac{3}{2j+3} \frac{n!}{(n-2j-1)!(2j+1)!} x^{n-2j-1} h^{2j}$ if n is odd, and

$D_h x^n = \sum_{j=0}^{\frac{n-2}{2}} \frac{3}{2j+3} \frac{n!}{(n-2j-1)!(2j+1)!} x^{n-2j-1} h^{2j}$ if n is even.

The first term is nx^{n-1} and for $j=1$ we have $\frac{3}{5}\frac{n!}{(n-3)!3!}x^{n-3}h^2$. It follows that

$$D_h\sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} ia_i x^{i-1} + p(x,h)$$

where $p(x,h)$ is a polynomial of degree $n-3$ in x and n or $n-1$ in h.

Example 1: Let $f(x) = 1$ for $x \leq 0$

$= -1$ for $x > 0$, and let $h = 1$.

Then $D_1f(x) = 0$ for $x \leq -1$

$= \frac{3}{2}[x^2-1]$ for $-1 \leq x \leq 1$

$= 0$ for $x \geq 1$.

Furthermore, $\int D_1f(x)dx = \frac{x^3}{2} - \frac{3x}{2} + C$. This is the function of which $D_1f(x)$ is the ordinary derivative.

$T_1f(x) = -1$ for $x \leq -1$

$= -x$ for $-1 \leq x \leq 1$

$= 1$ for $x > 1$.

$\frac{d}{dx}T_hf(x) = 0$ for $x < -1$

$= -1$ for $-1 \leq x \leq 1$

$= 0$ for $x > 1$.

Example 2: Let $f(x) = x + 1$ for $x \leq 0$

$= -x + 1$ for $x > 0$, and let $h = 1$.

Then $D_1f(x) = \frac{3}{2}[\frac{x^3}{3} - x]$ for $-1 \leq x \leq 1$

$= 1$ for $x \leq -1$

$= -1$ for $x \geq 1$.

$T_1f(x) = 1 - \frac{x^2}{2}$ for $-1 \leq x \leq 1$

$= x + 1$ for $-1 \geq x$

$= -x + 1$ for $x \geq 1$.

$\frac{d}{dx}T_hf(x) = 1$ for $x \leq -1$

$$= -x \text{ for } -1 \leq x \leq 1$$

$$= -1 \text{ for } x \geq 1.$$

$$D_h f(x) = \frac{3}{4}\left[\frac{x^4}{6} - x^2\right] + \frac{5}{8} \text{ for } -1 \leq x \leq 1$$

$$= x \text{ for } x \leq -1$$

$$= -x \text{ for } x \geq 1.$$

Given in Table I are some additional properties of the operators T_h and D_h which might be useful. These fall into two categories, differentiation of $T_h f$ and $D_h f$ and iterative operations with T_h and D_h.

TABLE I

SOME ADDITIONAL PROPERTIES OF T_h and D_h

1. If $f(x+\rho)$ has a continuous derivative for $-h \leq \rho \leq h$, then $T_h f'(x) = \frac{d}{dx} T_h f(x)$, $T_h f'(x) = f'(x+\mu)$ for some μ such that $-h \leq \mu \leq h$, and $\frac{d}{dx} T_h f(x) = f'(x+\mu)$ for the μ above.

2. If $f(x+\rho)$ has a derivative for all ρ such that $-h \leq \rho \leq h$, and $m \leq f'(x+\rho) \leq M$ on this interval, then $T_h f'(x) = \frac{d}{dx} T_h f(x)$, $m \leq T_h f'(x) \leq M$, and $m \leq \frac{d}{dx} T_h f(x) \leq M$.

3. If $f(x+\rho)$ is continuous for $-h \leq \rho \leq h$, then $T_h f(x)$ is differentiable and, in fact, $\frac{d}{dx} T_h f(x) = \frac{1}{2h}[f(x+h) - f(x-h)]$.

4. If $f(x+\rho)$ has a continuous second derivative for all ρ such that $-h \leq \rho \leq h$, then $D_h f'(x) = f''(x+\mu)$ for some μ such that $-h \leq \mu \leq h$, and $\frac{d}{dx} D_h f(x) = f''(x+\mu)$ for the μ above.

5. If $f(x+\rho)$ has a second derivative for all ρ such that $-h \leq \rho \leq h$, and $m \leq f''(x+\mu) \leq M$ in this interval, then $D_h f'(x) = \frac{d}{dx} D_h f(x)$, $m \leq D_h f'(x) \leq M$, and $m \leq \frac{d}{dx} D_h f(x) \leq M$.

6. If $f(x+\rho)$ is continuous for all ρ such that $-h \leq \rho \leq h$, then $D_h f(x)$ is differentiable.

7. $T_j[T_h f(x)] = T_h[T_j f(x)]$ and $D_j[D_h f(x)] = D_h[D_j f(x)]$.

8. If $f(x+\rho)$ is continuous for all ρ such that $-h-j \leq \rho \leq h+j$, then $T_h[T_j f(x)] = f(x+\mu)$ for some μ such that $-h-j \leq \mu \leq h+j$. In particular, ${}_2T_h f(x) = T_h[T_h f(x)]$ has this property on the interval $[-2h, 2h]$.

9. If there exist m and M such that $m \leq f(x+\rho) \leq M$ for all ρ such that $-h-j \leq \rho \leq h+j$, then $m \leq T_h[T_j f(x)] \leq M$.

10. If $f(x)$ is such that $m \leq f''(x+\rho) \leq M$ for all ρ such that $-h-j \leq \rho \leq h+j$, then $m \leq D_h[D_j f(x)] \leq M$.

11. If $f(x+\rho)$ has a continuous second derivative for all ρ such that $-h-j \leq \rho \leq h+j$, then $D_h[D_j f(x)] = f''(x+\mu)$ for some μ such that $-h-j \leq \mu \leq h+j$.

12. If $f(x+\rho)$ has a continuous n'th derivative for all ρ such that $-nh \leq \rho \leq nh$, then ${}_nD_h f(x) = f^{(n)}(x+\mu)$ for some μ such that $-nh \leq \mu \leq nh$.

13. If f is such that $m \quad f^{(n)}(x+\rho) \quad M$ for all ρ such that $-nh \leq \rho \leq nh$, then $m \leq {}_nD_h f(x) \leq M$.

14. If $f(x+\rho)$ has n continuous derivatives for all ρ such that $-(m+n)h \leq \rho \leq (m+n)h$, then ${}_nD_h[{}_mT_h f(x)] = f^{(n)}(x+\mu)$ for some μ such that $-(m+n)h \leq \mu \leq (m+n)h$.

15. If f is such that $m \leq f^{(n)}(x+\rho) \leq M$ for all ρ such that $-(k+n)h \leq \rho \leq (k+n)h$, then $m \leq {}_nD_h[{}_kT_h f(x)] \leq M$.

III. RIGHT AND LEFT GENERALIZED DERIVATIVES

Hence we have defined a limiting process which yields the derivative when it exists and which converges under some circumstances when the derivative does not exist. An interesting aspect is that we have defined the derivative in terms of an integral. We will call this generalized derivative the "g-derivative" and $\overline{f}(x)$ the integral mean of $f(x)$. It is evident that continuity of $f(x)$ is not necessary for the existence of $Df(x)$. Hence, it is possible for $\lim_{h\to 0} D_h f(x)$ to exist without implying the existence of a unique limiting line. That is, even if the above expression exists, $\overline{f}(x)$ might not exist and there need not be a line analogous to the tangent line for the derivative. Therefore, we will alter the definition of the g-derivative to include the requirement that $\overline{f}(x)$ also exist to assure the existence of the limiting line $y' = Df(x)(x'-x) + \overline{f}(x)$.

We can also define right and left hand g-derivatives as well as right and left hand integral means. To define the g-derivative of f we first developed the expression for the best fit line by the least squares criterion to $f(x+\rho)$ over the interval $-h \le \rho \le h$. Let us now consider the best fit line to $f(x+\rho)$ over the interval on ρ, $[0,h]$. For a given value of hm and x this line is, of course, in general not the same line as before. We will call the slope of this line $D_h^+ f(x)$ and the ordinate of the point of intersection of this line and the line $x' = x + \frac{h}{2}$, $\overline{f}_h^+(x) = T_h^+ f(x)$. We may show that

$$D_h^+ f(x) = \frac{12}{h^3} \int_0^h (\xi - \frac{h}{2}) f(x+\xi)\, d\xi$$

and

$$\overline{f}_h^+(x) = \frac{1}{h}\int_0^h f(x+\xi)d\xi .$$

Now x is at the extreme left end of the interval and if we let h approach zero, the interval will shrink to the point x. It is also useful to define another integral mean which is the ordinate of the point of intersection of the best fit line and the line x' = x. We will call this $\hat{f}_h^+(x)$ and it is defined by

$$\hat{f}_h^+(x) = \frac{2}{h^2}\int_0^h (2h-3\xi)f(x+\xi)d\xi .$$

The best fit line is precisely the line

$$y = D_h^+f(x)(x'-x) + \hat{f}_h^+(x)$$

$$= D_h^+f(x)(x'-x-\frac{h}{2}) + \overline{f}_h^+(x).$$

We now define the limits which give us the right hand g-derivative

$$D^+f(x) = \lim_{h\to 0} D_h^+f(x)$$

and the right hand integral means

$$\overline{f}^+(x) = \lim_{h\to 0} \overline{f}_h^+(x)$$

and

$$\hat{f}^+(x) = \lim_{h\to 0} \hat{f}_h^+(x).$$

In the same way in which we defined the right hand g-derivative and integral means, we may obtain the left hand counterparts. We have

$$D_h^-f(x) = \frac{12}{h^3}\int_{-h}^0 (\xi+\frac{h}{2})f(x+\xi)d\xi ,$$

$$\overline{f}_h^-(x) = \frac{1}{h}\int_{-h}^{0} f(x+\xi)d\xi,$$

$$\hat{f}_h^-(x) = \frac{2}{h^2}\int_{-h}^{0} (2h+3\xi)f(x+\xi)d\xi,$$

$$D^- f(x) = \lim_{h\to 0} D_h^- f(x),$$

$$\overline{f}^-(x) = \lim_{h\to 0} \overline{f}_h^-(x),$$

$$\hat{f}^-(x) = \lim_{h\to 0} \hat{f}_h^-(x),$$

and the limiting line is

$$y' = D^- f(x)(x'-x) + \hat{f}^-(x)$$
$$= D^- f(x)(x'-x+\frac{h}{2}) + \overline{f}^-(x).$$

We can draw some analogies between the right and left-hand g-derivatives and the right and left-hand derivatives in ordinary differentiation. Also the right and left-hand integral means have analogies in the right and left-hand limits.

For a given function, $f(x+\rho)$, and given values of h, we have written expressions identifying the best fit lines for three different intervals on ρ: (a) $[-h,h]$, (b) $[0,h]$, and (c) $[-h,0]$. With respect to (a) we reviewed the fact that for a fixed h we were simply determining the projection of an arbitrary element of a Hilbert space on the subspace spanned by a constant and a straight line, 1 and $x'-x$. The same is true of the intervals (b) and (c). The following interesting relationship exists between the three: Let $g(x+\rho)$ be any function which is a straight line on $[0,h]$ and also on $[-h,0]$ (possibly discontinuous at 0). It is clear that $g(x+\rho)$ is an element of space (a) and that, in fact, all possible such functions (straight lines on each of $[0,h]$ and $[-h,0]$) comprise a subspace of this space (a). Suppose we call the L_2 space on $[-h,h]$, H, and the subspace spanned by a constant and a straight line, S_1, and the subspace comprised of all possible functions of the nature of $g(x+\rho)$ described above, S_2.

Then, as we stated, $S_2 \subset H$, $S_1 \subset H$, and, furthermore, $S_1 \subset S_2$. That is, any straight line on [-h,h] is also a straight line on both [-h,0] and [0,h].

We can show in such a case that the projection of an arbitrary vector, λ, on S_1 is the same as the projection on S_1 of the projection of λ on S_2. In this case the implication is that if $f(x+\rho)$ is fit for ρ on the interval [-h,h] by the function which is a straight line on [-h,0] and a straight line on [0,h] and which fits best by the least squares criterion, and if this best fit function is then fit for ρ on [-h,h] by a straight line, then the same straight line is obtained as would be if $f(x+\rho)$ were fit by a straight line for ρ on [-h,h]. This suggests that we can write D_h and $\bar{F}_h$ in terms of their right and left-hand counterparts simply by performing the above operations.

Let us consider a function such as comprises S_2 above.

Let
$$g(x') = a_r(x'-x) + b_r \text{ for } x' \geq x,$$
$$g(x') = a_l(x'-x) + b_l \text{ for } x' < x.$$

Then we have

$$D_h g(x') = \frac{3}{2h^3}\left[\int_{-h}^{0} \xi[a_l(x'+\xi-x) + b_l]d\xi + \int_{0}^{h} \xi[a_r(x'+\xi-x) + b_r]d\xi\right]$$
$$= \frac{3}{2h^3}\left[a_l\frac{h^3}{3} - [a_l(x'-x) + b_l]\frac{h^2}{2} + a_r\frac{h^3}{3} + [a_r(x'-x) + b_r]\frac{h^2}{2}\right]$$
$$= \frac{a_l + a_r}{2} + \frac{3}{4h}(b_r - b_l) + \frac{3}{4h}(x'-x)(a_r - a_l).$$

Suppose now that a_r, a_l, b_r and b_l are the best fit right and left-hand parameters for $x'=x$ for some function f, and some h. Then

$$D_h g(x) = \tfrac{1}{2}[D_h^+ f(x) + D_h^- f(x)] + \frac{3}{4h}[\hat{f}_h^+(x) - \hat{f}_h^-(x)].$$

But this must also be $D_h f(x)$. We may write a similar expression containing $\bar{F}_h^+(x)$ and $\bar{F}_h^-(x)$.

$$D_h f(x) = \tfrac{1}{2}[D_h^+ f(x) + D_h^- f(x)] + \frac{3}{4h}[\bar{F}_h^+(x) - \bar{F}_h^-(x)].$$

In the first expression $D_h f(x)$ is expressed in terms of the right and left hand parameters $D_h^+ f(x)$, $D_h^- f(x)$, $\hat{f}_h^+(x)$, and $\hat{f}_h^-(x)$ and in the second in terms of $D_h^+ f(x)$, $D_h^- f(x)$, $\bar{f}_h^+(x)$ and $\bar{f}_h^-(x)$.

We may deduce some properties of right and left hand means and g-derivatives immediately. We will consider the means first. Recall that if $f(x)$ is continuous, then $\bar{f}(x) = f(x)$. We can easily see that $\bar{f}(x) = \frac{1}{2}[\bar{f}^+(x) + \bar{f}^-(x)]$ if these exist. Furthermore, if any two of the means are equal, all three are equal.

It is useful to consider the relationships existing between $\hat{f}^+(x)$, $\bar{f}^+(x)$, and $D^+ f(x)$. We are interested in relationships which have analogous left hand counterparts. The graphical picture is clear: $\bar{f}_h^+(x)$ is the ordinate of the best fit line at the center of the interval $[x, x+h]$, that is, at $x+\frac{h}{2}$; $\hat{f}_h^+(x)$ is the ordinate of this line at x; $D_h f(x)$ is the slope of the line. Hence, the following relationship is true.

$$\hat{f}_h^+(x) = \bar{f}_h^+(x) - \frac{h}{2} D_h^+ f(x),$$

or

$$D_h^+ f(x) = \frac{2}{h}[\bar{f}_h^+(x) - \hat{f}_h^+(x)].$$

On the left side the expressions are

$$\hat{f}_h^-(x) = \bar{f}_h^-(x) + \frac{h}{2} D_h^- f(x)$$

or

$$D_h^- f(x) = \frac{2}{h}[\hat{f}_h^-(x) - \bar{f}_h^-(x)].$$

If we keep the graphical representation in mind, some of the results are easy to anticipate.

Theorem X: If $\bar{f}_h^+(x)$ exists in the limit as $h \quad 0$, then $\hat{f}_h^+(x)$ may not be unbounded.

Proof: Suppose $\lim_{h\to 0} \frac{1}{h}\int_0^h f(x+\xi)d\xi$ exists. We have

$$\hat{f}_h^+(x) = \frac{2}{h^2}\int_0^h (2h - 3\xi)f(x+\xi)d\xi.$$

Now $2h-3\xi$ is a monotone function. Hence by the second mean value theorem there exists a μ such that $0 \leq \mu \leq h$ and

$$\hat{f}_h^+(x) = \frac{4}{h}\int_0^{\mu} f(x+\xi)d\xi - \frac{6}{h}\int_{\mu}^{h} f(x+\xi)d\xi$$
$$= \frac{10}{h}\int_0^{\mu} f(x+\xi)d\xi - \frac{6}{h}\int_0^{h} f(x+\xi)d\xi .$$

We have given that there exists a K such that for every $\epsilon > 0$ there exists a δ 0 such that h δ implies

$$\left| \frac{1}{h}\int_0^{h} f(x+\xi)d\xi - K \right| < \epsilon.$$

Since 0 μ h,

$$\left| \frac{1}{\mu}\int_0^{\mu} f(x+\xi)d\xi - K \right| < \epsilon .$$

$$\left| \frac{10}{h}\int_0^{\mu} f(x+\xi)d\xi - \frac{10\mu}{h} K \right| < \frac{10\mu}{h} < 10\epsilon .$$

It follows that $\lim\limits_{h\to 0} \hat{f}_h^+(x) = -6K + \lim\limits_{h\to 0} 10\frac{\mu}{h}K$ if $\lim\limits_{h\to 0}\frac{\mu}{h}$ exists. In any case $0 \leq \frac{\mu}{h} \leq 1$ and so even if $\frac{\mu}{h}$ does not converge it is bounded by 0 and 1 and

$$-6K - \epsilon < \hat{f}_h^+(x) < 4K + \epsilon .$$

Corollary: If $\bar{f}_h^+(x)$ converges as $h \to 0$ and $\hat{f}_h^+(x)$ does not, then (1) $\hat{f}_h^+(x)$ has a bounded discontinuity and (2) $D_h^+f(x)$ has an unbounded discontinuity.

Proof: Since $\bar{f}_h^+(x)$ converges as h 0, $\hat{f}_h^+(x) + \frac{h}{2}D_h^+f(x)$ converges but by hypothesis and the last theorem $\hat{f}_h^+(x)$ has a bounded, oscillating discontinuity. Then $\frac{h}{2}D_h^+f(x)$ is bounded and oscillating as $h \to 0$ and must take on non-zero values for arbitrarily small h's, and $D_h^+f(x)$ is, therefore unbounded. It may not diverge properly because if it did, $\frac{h}{2}D_h^+f(x)$ would either diverge properly or have a limit and likewise $\hat{f}_h^+(x)$. This violates the hypothesis.

Theorem XI: If $\lim [\bar{f}_h^+(x) - \hat{f}_h^+(x)]$ converges, but not to zero, then $D_h^+f(x)$ becomes unbounded as $h \to 0$.

Proof: Since $\frac{h}{2} D_h^+f(x) = [\bar{f}_h^+(x) - \hat{f}_h^+(x)]$. Then $\lim_{h\to 0} \frac{h}{2} D_h^+f(x) = K$ It follows that $D_h^+f(x)$ must become unbounded.

Corollary: If $\bar{f}^+(x)$ and $\hat{f}^+(x)$ exist and are not equal, then $D_h^+f(x)$ becomes unbounded as $h \to 0$.

Theorem XII: If $\lim_{h\to 0} [\bar{f}_h^+(x) - \hat{f}_h^+(x)] = 0$ and hence if $\bar{f}^+(x) = \hat{f}^+(x)$, then $\lim_{h\to 0} \frac{h}{2} D_h^+f(x) = 0$.

Note that this does not imply that $D^+f(x)$ exists. For example at $x = 0$, $f(x) = (x)^{\frac{1}{3}}$ is such that $\bar{f}^+(x) = \hat{f}^+(x) = 0$ but $D_h^+f(x)$ becomes unbounded in the limit.

Theorem XIII: Suppose $\lim D_h^+f(x)$ exists. Then $\lim_{h\to 0} [\bar{f}_h^+(x) - \hat{f}_h^+(x) = 0$. and $\bar{f}_h^+(x)$ and $\hat{f}_h^+(x)$ converge or diverge together.

Theorem XIV: If $D_h^+f(x)$ diverges properly as $h \to 0$, and:

(a) $\lim_{h\to 0} \frac{h}{2} D_h^+f(x) = K$, then $\lim_{h\to 0} [\bar{f}_h^+(x) - \hat{f}_h^+(x)] = K$ and $\bar{f}_h^+(x)$ and $\hat{f}_h^+(x)$ converge or diverge together. If they converge $\bar{f}^+(x) - \hat{f}^+(x) = K$

(b) $\lim_{h\to 0} \frac{h}{2} D_h^+f(x)$ diverges, then $\lim_{h\to 0} [\bar{f}_h^+(x) - \hat{f}_h^+(x)]$ diverges and by a previous lemma $\bar{f}_h^+(x)$ and $\hat{f}_h^+(x)$ each diverge as $h \to 0$.

Theorem XV: Suppose $D^+f(x) = D^-f(x)$ and $\bar{f}^+(x) = \bar{f}^-(x)$. Then $D^+f(x) = D^-f(x) = Df(x)$.

Proof: The hypothesis states that the right and left hand best fit lines merge into one in the limit. The conclusion of the theorem follows immediately from the facts that

(1) all the parameters are continuous functions of h,

(2) the function composed of the best fit lines on $[-h,0]$ and $[0,h]$ is a better fit than any straight line on $[-h,h]$ unless they be the same

... former is at least as good a fit.

... two expressions

$$\ldots f(x) + D_h^- f(x)] + \frac{3}{4h}[\hat{f}_h^+(x) - \hat{f}_h^-(x)]$$

$$\ldots{}_h^+ f(x) + D_h^- f(x)] + \frac{3}{4h}[\bar{f}_h^+(x) - \bar{f}_h^-(x)].$$

... and $\bar{f}^-(x)$ exist, then $\bar{f}(x) = \frac{1}{2}[\bar{f}^+(x) + \bar{f}^-(x)]$. ... believe the same thing about g-derivatives, that ... xists, it is the average of the right and left- ... ever, the first expression above yields the information that $Df(x) = \frac{1}{2}[\ldots f(x) + D^- f(x)]$ assuming the existence of all the limits, if and only if $\lim_{h\to 0} \frac{3}{4h}[\hat{f}_h^+(x) - \hat{f}_h^-(x)] = 0$. Furthermore, if this is true, some manipulation shows that

$$Df(x) = \lim_{h\to 0} \frac{1}{h}[\bar{f}_h^+(x) - \bar{f}_h^-(x)].$$

It is interesting to note that even if $\bar{f}^+(x) = \hat{f}^+(x)$ and $\bar{f}^-(x) = \hat{f}^-(x)$, in general, $\lim_{h\to 0} \frac{1}{h}[\bar{f}_h^+(x) - \bar{f}_h^-(x)] = \lim_{h\to 0} \frac{1}{h}[\hat{f}_h^+(x) - \hat{f}_h^-(x)]$. Recall that the existence of $Df(x)$ implies the existence of $\bar{f}(x)$ by definition. It is possible to write $D_h f(x)$ in many forms of which we have shown two. Another is

$$D_h f(x) = \frac{1}{h}[\bar{f}_h^+(x) - \bar{f}_h^-(x)] + \frac{1}{4h}[f_h^-(x) - f_h^+(x)].$$

Theorem XVI: If $f(x+\mu)$ is right and left-hand differentiable at $\mu=0$, then $\lim_{h\to 0} \frac{1}{h}[\hat{f}_h^+(x) - \hat{f}_h^-(x)] = 0$.

Proof: There exists a K such that for every $\epsilon > 0$ there exists a $\delta_1 > 0$ such that $0 < \mu < \delta_1$ implies $\left|\frac{f(x+\mu) - f(x)}{\mu} - K\right| < \epsilon$. There exists an L such that for every $\epsilon > 0$ there exists a $\delta_2 > 0$ such that $\mu > 0$, $\mu < \delta_2$ implies $\left|\frac{-f(x) + f(x+\mu)}{\mu} - L\right| < \epsilon$. For every $\epsilon > 0$ there exists a $\delta > 0$ such that $|\mu| < \delta$ implies $|f(x+\mu) - f(x)| < \epsilon$. All these follow from the hypothesis. From these expressions we obtain by the mean value theorem for integrals

$$4f(x) + 2h(K-\epsilon) < \frac{4}{h}\int_0^h f(x+\xi)d\xi < 4f(x) + 2h(K+\epsilon),$$

$$3f(x) + 2h(K+\epsilon) < \frac{6}{h^2}\int_0^h \xi f(x+\xi)d\xi < 3f(x) + 2h(K+\epsilon).$$

Subtracting

$$f(x) - 4h\epsilon < \frac{4}{h}\int_0^h f(x+\xi)d\xi - \frac{6}{h^2}\int_0^h \xi f(x+\xi)d\xi < f(x) + 4h\epsilon.$$

Again by the mean value theorem

$$-2h(L+\epsilon) + 4f(x) < \frac{4}{h}\int_{-h}^0 f(x+\xi)d\xi < -2h(L-\epsilon) + 4f(x),$$

$$2h(L-\epsilon) - 3f(x) < \frac{6}{h^2}\int_{-h}^0 \xi f(x+\xi)d\xi < -3f(x) + 2h(L+\epsilon).$$

Adding

$$f(x) - 4h\epsilon < \frac{4}{h}\int_{-h}^0 f(x+\xi)d\xi + \frac{6}{h^2}\int_{-h}^0 \xi f(x+\xi)d\xi < f(x) + 4h\epsilon.$$

Combining these two results, we obtain

$$-8h\epsilon < f_h^+(x) - f_h^-(x) < 8h\epsilon$$

or

$$\frac{1}{h}\left|f_h^+(x) - f_h^-(x)\right| < 8\epsilon.$$

The last statement is true for $\delta = \min(\delta_1, \delta_2, \delta_3)$ and $0 < h \leq \delta$.

We should note that for a finite h the right and left-hand means and g-derivatives retain all the properties of $T_h f$ and $D_h f$ which are the results of their being the parameters of the best fit line. These are primarily the continuity and differentiability properties as well as the mean value theorems. The fact that T_h^+, T_h and T_h^- are smoothing operators carries over to the limiting case. It follows from what has already been said that T^+f, for example, is at least as smooth and differentiable as f.

We could show that if, for some h, -h ρ h implies that relative to a set of measure 2h, f(x+ρ) is continuous at ρ=0, then $\bar{f}(x+\rho)$ is continuous at ρ=0. The same holds for right and left-hand continuity with $\bar{f}^+(x+\rho)$ and $\bar{f}^-(x+\rho)$. It is also true that if f(x+ρ) is differentiable relative to such a set of measure 2h at ρ=0, then $\bar{f}(x+\rho)$ is also differentiable. It is, of course, not true that if on [-h,h] f(x+ρ) is continuous at ρ=0 relative to a set of measure 2h, that $\bar{f}(x) = f(x)$, necessarily. Under these conditions it may be precisely at ρ=0 at which f(x+ρ) is discontinuous. However, we may say that $\bar{f}(x)$ is equal to $\lim_{\rho\to 0} f(x+\rho)$ taking ρ's only from theset of measure 2h. The same is true of differentiation if we make it relative to the set of measure 2h.

$$\frac{d\bar{f}(x)}{dx} = \lim_{\rho\to 0} \frac{f(x+\rho) - \lim_{\mu\to 0} f(x+\mu)}{\rho}$$

where ρ and μ are chosen only from the set of measure 2h. This also holds for right and left-hand differentiation of $\bar{f}^+$ and $\bar{f}^-$ relative to such a set.

Table II gives some additional properties of left and right-hand g-derivatives.

TABLE II

SOME ADDITIONAL PROPERTIES OF RIGHT HAND G-DERIVATIVES

1. $D^{+}\overline{f}^{+}(x) = D^{+}f(x)$.
2. $T^{+}[T^{+}f(x)] = T^{+}[T^{-}f(x)] = T^{+}f(x)$.
3. If f is right differentiable at x, then $f'_{+}(x) = D^{+}f(x)$.
4. If f is right continuous at x, then $\overline{f}^{+}$ is also.
5. If f is right differentiable at x, then $\overline{f}^{+}$ is also.
6. If $\lim_{\rho \to 0} f(x+\rho)$ exists, then $\overline{f}^{+}(x)$ exists and they are equal.
7. If f is right continuous at x, then $f(x) = \overline{f}^{+}(x)$.
8. If $\lim_{\rho \to 0} \overline{f}^{+}(x+\rho)$ exists, it is equal to $\overline{f}^{+}(x)$.

IV HIGHER ORDER G-DERIVATIVES

The fact that a useful relationship has been established between the ordinary derivative and the slope of a line best fit over some interval by least squares criterion suggests that a similar relationship might exist between the second derivative and the best fit quadratic. Furthermore, since the criterion of best fit is least squares, one might anticipate that a simplification in the approach might be obtained if we were to fit with Legendre polynomials. Both of these are true. However, in the same way that we fit the line over a general interval [x-h, x+h] we must adapt Legendre polynomials to such an interval. What we have by making such an adaptation might well be called "generalized Legendre polynomials".

The elementary properties of Legendre polynomials are well known:

(1) They are orthogonal on the bi-unit interval.

(2) The expansion of an arbitrary Lebesque square integrable function in L_2 in terms of Legendre polynomials minimizes the integral of the square of error.

(3) The orthogonality conditions imply that such an expansion requires that increasing the degree of the polynomial fit by one necessitates calculating only one new coefficient.

Furthermore, the Legendre polynomials satisfy the familiar Legendre equation and can be obtained by orthogonalizing the functions 1, x, x^2, . . . using the Schmidt orthogonalization process and the L_2 inner product. The functions which we will define as generalized Legendre polynomials will be obtainable by a simple change of variable in Legendre polynomials or, more especially, by orthogonalization by the Schmidt process using the inner product previously defined in this paper,

$$[f,g] = \int_{-h}^{h} f(x+\xi)g(x+\xi)d\xi .$$

They will be polynomials in the primed variables $(x'-x)$ like the straight line discussed previously, and they will also be normalized. Let us denote by P_n the familiar nth degree Legendre polynomial, and p_n the nth degree generalized Legendre polynomial.

By definition, then,

$$p_n(x'-x) = \left[\frac{2n+1}{2h}\right]^{\frac{1}{2}} P_n\left(\frac{x'-x}{h}\right)$$

Fixing x and h gives us an L_2 space on which $[p_i,p_j] = \delta_{ij}$ where the p_i's form a basis for the space. That is,

$$[p_i,p_j] = \left[\frac{2i+1}{2h}\right]^{\frac{1}{2}}\left[\frac{2j+1}{2h}\right]^{\frac{1}{2}} \int_{-h}^{h} P_i\left(\frac{\xi}{h}\right)P_j\left(\frac{\xi}{h}\right)d\xi$$

$$= \left[\frac{2i+1}{2}\right]^{\frac{1}{2}}\left[\frac{2j+1}{2}\right]^{\frac{1}{2}} \int_{-1}^{1} P_i(\rho)P_j(\rho)d\rho = \delta_{ij} .$$

If we expand a function f in these polynomials, the n'th coefficient is given by $a_n = [f,p_n] = \int_{-h}^{h} f(x+\xi)p_n(\xi)d\xi$.

In addition to the properties mentioned above which carry over from the bi-unit interval to the general interval, we have several recursion relations in revised form and also a new form of Rodriguez' Formula. Given in Table III is a listing of the analogous relations, the form on the bi-unit interval followed by the form on the general interval.

TABLE III

RECURSION FORMULAS

1. Rodriguez formula: $P_n(\phi) = (2^n n!)^{-1} \frac{d^n}{d\phi'^n} (\phi^2-1)$

1'. $p_n(x'-x) = (\frac{2n+1}{h})^{\frac{1}{2}} (2^n n!)^{-1} h^n \frac{d^n}{dx'^n} [(\frac{x'-x}{h}) - 1]^n$

2. $(n+1)P_{n+1}(\phi) - (2n+1)P_n(\phi) + nP_{n-1}(\phi) = 0$

2'. $(n+1)(\frac{1}{2n+3})^{\frac{1}{2}} p_{n+1}(x'-x) - (2n+1)^{\frac{1}{2}}(\frac{x'-x}{h})p_n(x'-x) + n(\frac{1}{2n-1})^{\frac{1}{2}} p_{n-1}(x'-x) = 0$

3. $P'_{n+1}(\phi) - P'_n(\phi) = (n+1)P_n(\phi)$

3'. $h(\frac{1}{2n+3})^{\frac{1}{2}} p'_{n+1}(x'-x) - (x'-x)(\frac{1}{2n+1})^{\frac{1}{2}} p'_n(x'-x) = (n+1)(\frac{1}{2n+1})^{\frac{1}{2}} p_n(x'-x)$

4. $\phi P'_n(\phi) - P'_{n-1}(\phi) = nP_n(\phi)$

4'. $(x'-x)(\frac{1}{2n+1})^{\frac{1}{2}} p'_n(x'-x) - h(\frac{1}{2n-1})^{\frac{1}{2}} p'_{n+1}(x'-x) = (2n+1)^{\frac{1}{2}} p_n(x'-x)$

5. $P'_{n+1}(\phi) - P'_{n-1}(\phi) = (2n+1)P_n(\phi)$

5'. $h(\frac{1}{2n+3})^{\frac{1}{2}} p'_{n+1}(x'-x) - h(\frac{1}{2n-1})^{\frac{1}{2}} p'_{n-1}(x'-x) = (2n+1)^{\frac{1}{2}} p_n(x'-x)$

6. $(\phi^2-1)P'_n(\phi) = n\phi P_n(\phi) - nP_{n-1}(\phi)$

6'. $[(\frac{x'-x}{h})-1]h(\frac{1}{2n+1})^{\frac{1}{2}} p'_n(x'-x) = n(\frac{x'-x}{h})(\frac{1}{2n+1})^{\frac{1}{2}} p'_n(x'-x) - n(\frac{1}{2n-1})^{\frac{1}{2}} p_{n-1}(x'-x)$

Given below is a comparison of the first three Legendre polynomials and the first three such polynomials transformed to the general interval.

(1) $P_o(x) = 1,$ $\qquad$ $p_o(x'-x) = \left[\frac{1}{2h}\right]^{\frac{1}{2}};$

(2) $P_1(x) = x,$ $\qquad$ $p_1(x'-x) = \left[\frac{3}{2h}\right]^{\frac{1}{2}} \frac{x'-x}{h};$

(3) $P_2(x) = \frac{1}{2}(3x^2-1),$ $\qquad$ $p_2(x'-x) = \frac{1}{2}\left[\frac{5}{2h}\right]^{\frac{1}{2}}\left[3(\frac{x'-x}{h})^2 - 1\right].$

If, as before, the Legendre expansion of $f(x')$ is given by

$$f(x') = \sum a_i p_i(x'-x),$$

then

$$a_o = \left[\frac{1}{2h}\right] \int_{-h}^{h} f(x+\xi)d\xi,$$
$$a_1 = \frac{1}{h}\left[\frac{3}{2h}\right] \int_{-h}^{h} \xi f(x+\xi)d\xi,$$
$$a_2 = \frac{1}{2}\left[\frac{5}{2h}\right] \int_{-h}^{h} (3\frac{\xi^2}{h^2} - 1)f(x+\xi)d\xi.$$

We have used lower case a's here to represent the Legendre coefficients. We are more especially interested, however, in the coefficient of each power of $(x'-x)$ in this expansion. We will introduce additional notation as follows: let b_{ij} be the coefficient of $(x'-x)^j$ in the i'th generalized Legendre polynomial. That is

$$p_o(x'-x) = b_{oo},$$
$$p_1(x'-x) = \qquad b_{11}(x'-x),$$
$$p_2(x'-x) = b_{20} \qquad + b_{22}(x'-x)^2,$$
$$\vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$p_n(x'-x) = b_{no} \qquad + b_{n2}(x'-x)^2 + \dots + b_{nn}(x'-x)^n,$$

$$p_{n+1}(x'-x) = \quad b_{n+1,1}(x'-x) + \dots + b_{n+1,n+1}(x'-x)^{n+1} .$$

Now if f is expanded in Legendre polynomials, the contribution of p_n to the expansion is

$$a_n b_{no} + a_n b_{n2}(x'-x)^2 + \dots + a_n b_{nn}(x'-x)^n$$

and the coefficient of $(x'-x)^j$ in which we are particularly interested is $a_n b_{nj}$. This coefficient is a function of x and h. It is clear that the recursion relations listed above serve simply to establish relations between the various b_{ij}'s and could be written in terms of these.

We will now determine what happens to this generalized Legendre expansion in the limit as h goes to zero. The term containing the n'th degree polynomial may be written as

$$a_n p_n(x'-x) = C[\frac{1}{h}\int_{-h}^{h} P_n(\frac{\xi}{h}) f(x+\xi)d\xi] P_n(\frac{x'-x}{h}),$$

where C is the normalizing factor and independent of h. This can be written

$$a_n p_n(x'-x) = C[\frac{1}{h}\int_{-h}^{h} \sum_{i=0}^{\frac{n}{2}} K_{2i}(\frac{\xi}{h})^{2i} f(x+\xi)d\xi] \sum_{i=0}^{\frac{n}{2}} K_{2i}(x'-x)^{2i}(\frac{1}{h})^{2i}$$

if n is even. For n odd there is a similar expression. Here the K's are independent of x and h. The entire first factor is a function of x and h, say F(x,h). Then we have

$$a_n p_n(x'-x) = F(x,h) \sum_{i=0}^{\frac{n}{2}} K_{2i}(x'-x)^{2i}(\frac{1}{h})^{2i}$$

and

$$\lim_{h\to 0} a_n p_n(x'-x) = \lim_{h\to 0} F(x,h)\left[\frac{K_n}{h^n}(x'-x)^n + \frac{K_{n-2}}{h^{n-2}}(x'-x)^{n-2} + \dots + K_0\right].$$

Suppose now that $\lim_{h\to 0} F(x,h)\left[\frac{K_n}{h^n}\right]$ exists. But for $m = 0$

$$\lim_{h\to 0} F(x,h)K_{n-m}\left(\frac{1}{h}\right)^{n-m} = \frac{K_{n-m}}{K_n} \lim_{h\to 0} \frac{F(x,h)K_n}{h^n} \lim_{h\to 0} h^m = 0$$

since the first limit exists. We have, therefore, proved the following:

Theorem XVII: Suppose f can be expanded in a generalized Legendre series. Then, if $\lim_{h\to 0} a_n b_{n,n}$ exists, $\lim_{h\to 0} a_n b_{n,m} = 0$ for all $m < n$. That is, if in the limit as h approaches zero, the coefficient of $(x'-x)^n$ in $a_n p_n(x'-x)$ exists, then the coefficients of all lower degree terms in $a_n p_n(x'-x)$ approach zero since they are all higher degree in h.

We are thus brought to the interesting fact that if f is expanded in a finite Legendre series, and if limits are taken for the coefficient of $(x'-x)^m$ in such an expansion and they all exist, the only non-zero contribution to this coefficient is from $p_m(x'-x)$. If $n > m$, any coefficient of $(x'-x)^m$ in $p_n(x'-x)$ approaches zero with h. These facts suggest the relationship between the n'th derivative of f and the coefficient of $(x'-x)^n$ in $a_n p_n(x'-x)$.

Theorem XVIII: If $f^n(x+\rho) \geqslant 0$ for all ρ such that $-h \leq \rho \leq h$, then the coefficient of $p_n(x'-x)$ in the generalized Legendre expansion,

$$\int_{-h}^{h} p_n(\xi)f(x+\xi)d\xi \geqslant 0.$$

Proof: Rodrigues' formula gives us

$$p_n(x'-x) = \left(\frac{2n+1}{h}\right)^{\frac{1}{2}} \left(2^n n!\right)^{-1} h^n \frac{d^n}{dx'^n}\left[\left(\frac{x'-x}{h}\right)^2 - 1\right]^n .$$

Then

$$\int_{-h}^{h} f(x+\xi)p_n(\xi)d\xi = \left(\frac{2n+1}{h}\right)^{\frac{1}{2}} \left(2^n n!\right)^{-1} h^n \int_{-h}^{h} f(x+\xi)\frac{d^n}{d\xi^n}\left[\left(\frac{\xi}{h}\right)^2 - 1\right]^n d\xi$$

$$= K_{n,h}\int_{-h}^{h} f(x+\xi)\frac{d^n}{d\xi^n}\left[\left(\frac{\xi}{h}\right)^2 - 1\right]^n d\xi$$

$$= K_{n,h}\left[f(x+\xi)\frac{d^{n-1}}{d\xi^{n-1}}[(\tfrac{\xi}{h})^2 - 1]^n\Big|_{-h}^{h} - \int_{-h}^{h} f'(x+\xi)\frac{d^{n-1}}{d\xi^{n-1}}[(\tfrac{\xi}{h})^2 - 1]^n d\xi\right]$$

as a result of integrating by parts. It is clear that we may continue to integrate by parts until we arrive at

$$\int_{-h}^{h} f(x+\xi)p_n(\xi)d\xi = K_{n,h}\left[f(x+\xi)\frac{d^{n-1}}{d\xi^{n-1}}[(\tfrac{\xi}{h})^2-1]^n\Big|_{-h}^{h}\right.$$

$$- f'(x+\xi)\frac{d^{n-2}}{d\xi^{n-2}}[(\tfrac{\xi}{h})^2 - 1]^n\Big|_{-h}^{h} + \ldots \mp f^{n-1}(x+\xi)\frac{d}{d\xi}[(\tfrac{\xi}{h})^2-1]\Big|_{-h}^{h}$$

$$\left.\pm f^{n}(x+\xi)[(\tfrac{\xi}{h})^2 - 1]^n\Big|_{-h}^{h} \mp \int_{-h}^{h} f^{n}(x+\xi)[(\tfrac{\xi}{h})^2 - 1]^n d\xi\right],$$

where the sign of the last term is $\begin{bmatrix}+\\-\end{bmatrix}$ if n is $\begin{bmatrix}\text{even}\\\text{odd}\end{bmatrix}$. Consider the fact that

$$\frac{d}{d\xi}[(\tfrac{\xi}{h})^2 - 1]^n = \frac{2\xi n}{h^2}[(\tfrac{\xi}{h})^2 - 1]^{n-1}$$

and

$$\frac{d^2}{d\xi^2}[(\tfrac{\xi}{h})^2 - 1]^n = \frac{4n(n-1)}{h^4}\xi^2[(\tfrac{\xi}{h})^2 - 1]^{n-2} + \frac{2n}{h^2}[(\tfrac{\xi}{h})^2 - 1]^{n-1}.$$

If we continue taking derivatives, in every term appears a power of $[(\frac{\xi}{h})^2 - 1]$, and there one such term in each derivative which is one degree less than the term of minimum degree in the preceding derivative. In fact, after n-1 differentiations the term which is lowest degree in $[(\frac{\xi}{h})^2 - 1]$ will be precisely

$$n!\, h^{-2(n-1)}2^{n-1}\xi^{n-1}[(\tfrac{\xi}{h})^2 - 1].$$

This term and, indeed, every other term in the expansion of

$$\int_{-h}^{h} f(x+\xi)p_n(\xi)d\xi$$

except the one of least degree is zero when evaluated at h or -h. Hence

$$\int_{-h}^{h} f(x+\xi)p_n(\xi)d\xi = K_{n,h}(-1)^n \int_{-h}^{h} f^n(x+\xi)[(\frac{\xi}{h})^2 - 1]^n d\xi.$$

Now, on the interval $[-h,h]$, $[(\frac{\xi}{h})^2 - 1] \leqslant 0$. Hence $(-1)^n[(\frac{\xi}{h})^2 - 1]^n > 0$. Since we know that $K_{n,h} > 0$ and, by hypothesis, $f^n(x+\xi) > 0$, it follows that

$$\int_{-h}^{h} f(x+\xi)p_n(\xi)d\xi \geqslant 0.$$

Corollary: Given a function with the same properties as in the previous theorem, the coefficient of $(x'-x)^n$ in the Legendre expansion is greater than zero.

Proof: This follows immediately from the theorem and the fact that $b_{n,n} > 0$ for all n.

Theorem XIX: Let f and g be functions such that $f^n(x+\rho) \geqslant g^n(x+\rho)$ for $-h \leqslant \rho \leqslant h$. Then

$$\int_{-h}^{h} f(x+\xi)p_n(\xi)d\xi \geqslant \int_{-h}^{h} g(x+\xi)p_n(\xi)d\xi.$$

Proof: $f^n(x+\rho) \geqslant g^n(x+\rho)$ implies $\frac{d^n}{dx^n}[f(x+\rho) - g(x+\rho)] \geqslant 0$ and by the previous theorem

$$\int_{-h}^{h} [f(x+\xi) - g(x+\xi)]p_n(\xi)d\xi \geqslant 0,$$

and finally

$$\int_{-h}^{h} f(x+\xi)p_n(\xi)d\xi \geqslant \int_{-h}^{h} g(x+\xi)p_n(\xi)d\xi.$$

Theorem XX: Suppose $m \leqslant f^n(x+\rho) \leqslant M$ for all $-h \leqslant \rho \leqslant h$. Then $m \leqslant n!\, a_n b_{n,n} \leqslant M$, where $a_n b_{n,n}$ is the coefficient of $(x'-x)$ in $p_n(x'-x)$ in the Legendre expansion of $f(x)$.

Proof: Consider the auxiliary functions $g(x') = \frac{m}{n!}(x'-x)^n$ and $h(x') = \frac{M}{n!}(x'-x)^n$. It is clear that $g^n(x') = m$ and $h^n(x') = M$. Hence for $-h \leq \rho \leq h$

$$g^n(x'-\rho) \leq f^n(x'+\rho) \leq h^n(x'+\rho).$$

Now let

$$\sum_{i=0}^{n} \alpha_i (x'-x)^i, \sum_{i=0}^{n} \beta_i (x'-x)^i, \sum_{i=0}^{n} \gamma_i (x'-x)^i$$

be respectively the best fit n'th degree polynomials of g, f, and h. By the previous theorem $\alpha_n \leq \beta_n \leq \gamma_n$. But $\alpha_n = \frac{m}{n!}$, $\gamma_n = \frac{M}{n!}$. Then

$$\frac{m}{n!} \leq \beta_n \leq \frac{M}{n!} \text{ or } m \leq n!\ \beta_n \leq M.$$

Corollary 1: If $f^n(x+\rho)$ is continuous for all ρ such that $-h \quad \rho \quad h$, then there exists a number ρ', $-h \leq \rho' \leq h$, such that $f^n(x+\rho') = n!\beta_n$ where β_n is the coefficient of the n'th degree term in the best fit n'th degree polynomial.

Corollary 2: Suppose $f(x+\rho)$ is n times differentiable for all ρ in some neighborhood of zero. Then $\lim_{h \to 0} \beta_n = \frac{1}{n!} f^n(x)$ where β_n is as in Corollary 1.

The last corollary establishes the same kind of relationship between the n'the derivative of f and the best fit n'th degree polynomial as we have already discussed between the first derivative and the best fit straight line. In fact, it defines a generalized n'the derivative.

Suppose the best fit n'th degree polynomial to $f(x+\rho)$ for ρ such that $-h \leq \rho \leq h$ is

$$\sum_{i=0}^{n} \beta_i (x'-x)^i.$$

Then $D_h^n f(x) = n!\beta_n$ and $D^n f(x) = \lim_{h \to 0} n!\beta_n$. Recall that β_n is also the coefficient of $(x'-x)$ in

$$\left[\int_{-h}^{h} f(x+\xi)p_n(\xi)d\xi\right]p_n(x'-x).$$

We will conclude by demonstrating a relationship between $D_h^n f(x)$ and ${}_nD_h f(x)$ which was previously defined as the D_h operator iterated n times. We see finally that if $f(x+\rho)$ is bounded above and below by M and m respectively, then

(1) for every ρ such that $-nh \leq \rho \leq nh$, $m \leq {}_nD_h f(x) \leq M$.

(2) for every ρ such that $-h \leq \rho \leq h$, $m \leq D_h^n f(x) \leq M$.

Then, if the n'th derivative of f exists and is continuous on some interval containing x, $\lim_{h \to 0} D_h^n f(x) = \lim_{h \to 0} {}_nD_h f(x) = f^{(n)}(x)$. This is typical of the relationship between $D_h^n f(x)$ and ${}_nD_h f(x)$ for functions of various differentiability characteristics.

BIBLIOGRAPHY

1. Achieser, N. I. Theory of Approximation: Frederick Ungar Publishing Company: New York, 1956.

2. Halperin, Israel. Introduction to the Theory of Distributions: University of Toronto Press, 1952.

3. Haslam - Jones, U.S.: On a generalized derivative: Quarterly Journal of Mathematics, Oxford Series, (2)4, pp. 190-197, 1953.

4. Hobson, E. W. The Theory of the Functions of a Real Variable and the Theory of Fourier Series: Harren Press, Washington, D. C., 1950.

5. Kassimatis, C. Functions which have generalized Riemann derivatives: Canadian Journal of Mathematics, 10(1958), pp. 413-420.

6. McShane, E. J. Integration: Princeton University Press, 1947.

VITA

Thomas William Cairns

Candidate for the Degree of

Doctor of Philosophy

Thesis: A GENERALIZED DERIVATIVE

Major: Mathematics

Biographical:

Born: The writer was born at Hutchinson, Kansas, November 13, 1931, the son of Edmund A. and Gladys Cairns.

Undergraduate Study: He attended elementary and secondary school at Hutchinson and graduated from Hutchinson High School in 1949. From September, 1949 to May, 1953 he attended Oklahoma State University and was granted a Bachelor of Science degree with a major in physics.

Graduate Study: He attended Oklahoma State University from September, 1953 to August, 1954 and was granted a Master of Science degree in Mathematics in January of 1955. In September, 1956 he was discharged from the army and returned to pursue studies leading to the Doctor of Philosophy degree. He was in residence until August, 1959.

Experiences: The writer entered the United States Army in September, 1954 and was discharged in September, 1956. During his service career he spent eighteen months in research with the National Security Agency with the title of "mathematician". In September, 1959 he accepted an appointment to the position of Assistant professor with the mathematics department of the University of Tulsa.

Organizations: The writer is a member of Phi Kappa Phi, Pi Mu Epsilon, Omicron Delta Kappa and Sigma Pi Sigma honorary societies and is a member of the American Mathematical Society, Mathematical Association of America and the Society of Industrial and Applied Mathematics.