VARIANCE COMPONENTS IN TWO-WAY CLASSIFICATION MODELS WITH INTERACTION

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CHAPTER I

INTRODUCTION

Components of variance has been discussed in many papers and analysis of variance components has become one of the basic tools of research in several fields of scientific investigation. In the problem of estimation, the researcher always tries to ascertain whether an estimator, best suited to the problem under consideration, possesses the well known properties of being unbiased, efficient, consistent, sufficient, minimum variance, etc. In practice, an objective of an investigation will be to strive to obtain minimum variance (best) unbiased estimators.

Any estimator, whether biased or unbiased minimum variance, must be a function of observations. It is known that sufficient statistics contain all the information in the sample about the parameters of a density function which describes a given population. It would be further desirable to ascertain whether a set of sufficient statistics can be reduced to a minimal set by employing the scheme given by Lehmann and Scheffe [8]. Moreover, the Rao-Blackwell theorem says that minimum variance unbiased estimates of the function of parameters must be based on a set of minimal sufficient statistics; but it does not enable us to determine which estimator is best if two or more unbiased estimators exist for the same function and each is based on a set of minimal sufficient statistics. If the density function from which the minimal set was obtained has the property of being complete, the unbiased estimator of the function based on a set of minimal sufficient statistics is unique, and has minimum variance. Unfortunately, with regard to the problems under consideration in this thesis, the density functions are not complete when an Eisenhart Model II is assumed [4].

D. L. Weeks [9] has given a minimal set of sufficient statistics in case of BIB and GD-PBIB designs when there is no block treatment interaction. Unfortunately, in practice we do not always have such a nice situation.

Hence, the problem of this thesis is:

(i) To determine a minimal set of sufficient statistics for the parameters of the Balanced Incomplete Block Design when there is block-treatment interaction.

(ii) To find a minimal set of sufficient statistics for Group Divisible Partially Balanced Incomplete Block Designs with two associate classes when there is block-treatment interaction.

(iii) To find the distribution of each statistic in a minimal set of sufficient statistics for (i) and (ii).

(iv) To determine pairwise independence in each set.

CHAPTER II

NOTATIONS AND SYMBOLS

We shall introduce here the definitions of symbols which we shall use often in this thesis. They will be classified in three parts as follows:

(1) Abbreviations

(2) Scalars

(3) Matrices

(1) Abbreviations

(a) BIB is an abbreviation for Balanced Incomplete Block.

(b) PBIB is an abbreviation for Partially Balanced Incomplete Block.

(c) GD-PBIB is an abbreviation for Group Divisible, Partially Balanced Incomplete Block Design. If GD is prefixed by S, SR, or R, it will denote the Singular, Semi-Regular, or Regular Group Divisible, Partially Balanced Incomplete Block Design, respectively.

(d) E denotes Mathematical Expectation.

(e) MVN is an abbreviation for Multivariate Normal.

(f) $\mathfrak{I}$ denotes an operation on a density function which, when properly defined, reduces the dimension of the space of the sufficient statistics.

(g) $R[\mu, \beta, \tau, (\beta\tau)]$ = Reduction due to μ, β, τ, and $(\beta\tau)$.

(h) $R[(\beta\tau) | \mu, \beta, \tau]$ = Reduction due to $(\beta\tau)$ adjusted for μ, β, τ.

(2) Scalars

(a) b is equal to the number of blocks in a design.

(b) t is equal to the number of treatments in a design.

(c) r is equal to the number of replicates of each treatment.

(d) k is equal to the number of plots per block.

(e) m denotes the number of times any treatment is replicated in any block, if it appears in that block.

(f) λ denotes in a BIB, the number of times two different treatments occur together in all blocks.

(g) λ_i, $(i = 1, 2)$, denotes in a PBIB, the number of times two different treatments which are i-th associates occur together in all blocks.

(h) λ_j' is the non-centrality parameter of the non-central chi-square distribution.

(i) g is the number of groups in a GD-PBIB Design.

(j) n is the number of treatments per group in GD-PBIB Designs.

(k) $v = k^{-1}(rk - r + \lambda_1) = k^{-1}[\lambda_2 t + n(\lambda_1 - \lambda_2)]$.

(3) Matrices

(a) X is a Design Matrix of a two-way classification model.

(b) X_1 is a partition of X corresponding to blocks.

(c) X_2 is a partition of X corresponding to treatments.

(d) X_3 is a partition of X corresponding to interaction.

(e) Y is a vector of observable random variables.

(f) J_q^s is an s x q matrix of all one's. j_1^n will be used to denote an n x 1 vector of one's.

(g) $N = X_2'X_1$

(h) $M = X_1'X_3$

(i) $L = X_2'X_3$

(j) D is a diagonal matrix

(k) P is an orthogonal matrix. When partitioning a matrix, partitions will be denoted by the addition of a subscript. Further partitions of a partition will be denoted by an additional subscript. Thus $P = (P_1, P_2) = (P_{11}, P_{12}, P_{21}, P_{22}, P_{23})$.

(l) Σ is a covariance matrix

(m) ϕ_w represents a w x w matrix of all zeros.

(n) $A = [X_2 - X_1(X_1'X_1)^{-1}X_1'X_2]$

(o) I_w is the identity matrix of dimension w x w.

Additional symbols if needed, will be defined as the discussion develops.

We shall now prove two lemmas which will be needed for the proofs of the theorem in the ensuing chapters.

<u>Lemma 1</u>: Let X denote the design matrix of two way classification model $Y = X\beta + e$ where the rank of X is bk and where X is of the form $X = (j_1^{bkm}, X_1, X_2, X_3)$. Then there exists a set of bk(m - 1) orthogonal rows P such that $X_1'P = \phi$, $X_2' = \phi$, $X_3'P = \phi$, and $J_{bkm}^1 = \phi$.

Proof: Consider the matrix product

$$XX' = (J_1^{bkm}, X_1, X_2, X_3)\begin{bmatrix} J_{bkm}^{1} \\ X_1' \\ X_2' \\ X_3' \end{bmatrix} = J_{bkm}^{bkm} + X_1X_1' + X_2X_2' + X_3X_3'$$

Since XX' is symmetric, there exists an orthogonal matrix Q such that $Q'XX'Q = D$ where D is a diagonal matrix. The number of non-zero elements on the diagonal of D is bk since X is of rank bk. Partition Q into $Q = (C, P)$ where C and P are of dimensions bkm x bk and bkm x bk(m - 1) respectively, and such that

$$Q'XX'Q = \begin{bmatrix} C' \\ P' \end{bmatrix} XX'[C, P] = \begin{bmatrix} D^* & \phi \\ \phi & \phi \end{bmatrix}$$

where D is bk x bk. Therefore,

$$P'J_{bkm}^{bkm} P + P'X_1X_1'P + P'X_2X_2'P + P'X_3X_3'P = \phi$$

The matrices J_{bkm}^{bkm}, X_1X_1', X_2X_2', and X_3X_3' are each positive semi-definite, each being the product of a matrix and its transpose. The matrices $P'J_{bkm}^{bkm}P$, $P'X_1X_1'P$, $P'X_2X_2'P$, and $P'X_3X_3'P$ are also positive semi-definite for the same reason. Since each diagonal element of each of these matrices is the sum of squares of real numbers and the sum of these sum of squares is zero, the diagonal elements of each of the four afore mentioned matrices must be equal to zero. If any off diagonal element is non-zero, there would be at least one of the principal minors which would be negative, a contradiction of positive semi-definiteness. We therefore conclude that each of the matrices must be equal to the null matrix.

It is therefore obvious that

$$J^{1}_{bkm}P = \phi, \; X_i'P = \phi_i, \quad i = 1, 2, 3 .$$

Lemma 2: Let N be a t x b matrix of rank m. Let P be an orthogonal matrix such that $P'NN'P = D$ where D is diagonal with characteristic roots of NN' on the diagonal. If $s \leq m$ of the characteristic roots are equal to d_0 ($d_0 \neq 0$), then the matrix $d_0^{-1/2}P_0'N = C'$ (say) is a set of s orthogonal rows such that $C'N'NC = d_0I_s$ where P_0 is such that $P_0'NN'P_0 = d_0I_s$.

Proof: Since we are given that s characteristic roots of NN' are equal we can partition P into (P_0, P_1) such that

$$(1) \qquad \begin{bmatrix} P_0' \\ P_1' \end{bmatrix} NN'(P_0, P_1) = D = \begin{bmatrix} d_0I_s & \phi \\ \phi & D_1 \end{bmatrix}$$

where D_1 is diagonal. Hence $P_0'NN'P_0 = d_0I_s$, that is $(d_0^{-1/2}P_0'N)(N'P_0d_0^{-1/2}) = I_s$. Consider now $(d_0^{-1/2}P_0'N)N'N(N'P_0d_0^{-1/2}) = Z$ (say), then we may write $Z = (d_0^{-1/2}P_0'N)N'(P_0P_0' + P_1P_1')N(N'P_0d_0^{-1/2})$. From (1) above, $P_0'NN'P_1 = \phi$. Therefore,

$$\begin{aligned} Z &= d_0^{-1/2}(P_0'NN'P_0)(P_0'NN'P_0)d_0^{-1/2} \\ &= d_0^{-1/2}(d_0I_s)(d_0I_s)d_0^{-1/2} \\ &= d_0I_s \end{aligned}$$

Hence the lemma is proved.

CHAPTER III

THE BALANCED INCOMPLETE BLOCK

In this chapter we shall be concerned with finding a set of minimal sufficient statistics in a balanced incomplete block design when there is a component of variance corresponding to the block-treatment interaction and an Eisenhart Model II is assumed.

The Balanced Incomplete Block Design is defined as a design with the following properties:

(a) There are b blocks and t treatments.

(b) There are k experimental units per block ($k < t$).

(c) There is one and only one observation per cell.

(d) A treatment cannot appear more than once in a block.

(e) Each treatment is replicated exactly r times.

(f) The number of blocks in which a pair of treatments appear together is exactly λ.

We are going to discuss a case where there is block-treatment interaction and so we shall assume $m > 1$ in order to obtain an estimate of the error variance. We shall, therefore, replace (c), (d), (e), and (f) by (c'), (d'), (e'), and (f') respectively as given below, where a cell is a group of experimental units subjected to a particular block-treatment combination.

(c') There are exactly m observations per cell.

(d') A treatment cannot appear more than once in the cells of the same block but it can appear m times in the same cell as follows from (c').

(e') Each treatment appears exactly m times in each of r different blocks.

(f') The number of blocks in which a pair of treatments appears together is exactly λ. This can also be worded as: the number of times a pair of treatments appears together in all blocks is $m\lambda$.

Specifically,

(I) $$y_{ijk} = \mu + \beta_i + \tau_j + (\beta\tau)_{ij} + e_{ijk}$$

where $i = 1, 2, \ldots, b$; $j = 1, 2, \ldots, t$; $k = n_{ij}$,

$$n_{ij} = \begin{cases} 0 \text{ if treatment } j \text{ does not appear in block } i. \\ 1, 2, \ldots, m, \text{ if treatment } j \text{ appears in block } i. \end{cases}$$

The observations y_{ij0} do not exist.

Under model II the following assumptions are made:

(1) β_i, τ_j, $(\beta\tau)_{ij}$ and e_{ijk} are each distributed normally.

(2) $E(e_{ijk}) = 0$ for all i, j, k.

$$E(e_{ijk}e_{uvw}) = \begin{cases} \sigma^2 \text{ if } i = u,\ j = v,\ k = w \\ 0 \text{ otherwise} \end{cases}$$

(3) $E(\beta_i) = 0$ for all i.

$$E(\beta_i\beta_p) = \begin{cases} \sigma_1^2 \text{ if } i = p \\ 0 \text{ otherwise} \end{cases}$$

(4) $E(\tau_j) = 0$ for all j.

$$E(\tau_j\tau_u) = \begin{cases} \sigma_2^2 \text{ if } j = u \\ 0 \text{ otherwise} \end{cases}$$

(5) $E(\tau\beta)_{ij} = 0$ for all i and j.

$$E[(\tau\beta)_{ij}(\tau\beta)_{uv}] = \begin{cases} \sigma_3^2 & \text{if } i = u,\ j = v \\ 0 & \text{otherwise} \end{cases}$$

(6) $E(e_{ijk}\beta_s) = 0$ for all i, j, k, and s.

(7) $E(e_{ijk}\tau_p) = 0$ for all i, j, k, and p.

(8) $E[e_{ijk}(\beta\tau)_{uv}] = 0$ for all i, j, k, and u, v.

(9) $E(\beta_i\ \tau_j) = 0$ for all i, and j.

(10) $E[\beta_i(\beta\tau)_{uv}] = 0$ for all i and u, v.

(11) $E[\tau_j(\beta\tau)_{uv}] = 0$ for all j and u, v.

(12) μ is constant.

The following relationships hold in BIB design when under the assumptions given above there is a block-treatment interaction.

(1) $\sum_i n_{ij} = mk$

(2) $\sum_i n_{ij} = mr$

(3) $\sum_i n_{ij}n_{ij'} = m^2\lambda\ (j \neq j')$

(4) $bk = tr$

(5) $\lambda(t - 1) = r(k - 1)$

The matrix model which fulfills the conditions set forth above can be written as

$$\text{(II)} \qquad Y = \mu J_1^{bkm} + X_1\beta + X_2\tau + X_3(\beta\tau) + e$$

where Y is the vector of bkm observations and we shall consider elements ordered according to blocks, then treatments. X_1, X_2, and X_3 are of dimension bkm x b, bkm x t, and bkm x bk, respectively. β, τ, $(\beta\tau)$, and e are vectors of b, t, bk, and bkm random variables respectively. The distributional properties can be written in matrix form as follows:

(1) e is distributed as the MVN(ϕ, $\sigma^2 I_{bkm}$).

(2) β is distributed as the MVN(ϕ, $\sigma_1^2 I_b$).

(3) τ is distributed as the MVN(ϕ, $\sigma_2^2 I_t$).

(4) $(\tau\beta)$ is distributed as the MVN(ϕ, $\sigma_3^2 I_{bk}$).

(5) $E(e\beta') = \phi$, $E(e\tau') = \phi$, $E[e(\beta\tau)'] = \phi$, $E(\beta\tau') = 0$,
$E[\beta(\tau\beta)'] = \phi$, $E[\tau(\beta\tau)'] = \phi$.

The following relationships hold for the matrices of the model.

(1) $X_1'X_1 = mkI_b$

(2) $X_2'X_2 = mrI_t$

(3) $X_3'X_3 = mI_{bk}$

(4) $J_{bkm}^{bkm} X_1 = mkJ_b^{bkm}$

(5) $J_b^{bkm} X_1' = J_{bkm}^{bkm}$

(6) $J_{bkm}^{bkm} X_2 = rmJ_t^{bkm}$

(7) $J_t^{bkm} X_2 = J_{bkm}^{bkm}$

(8) $J_{bkm}^{bkm} X_3 = mJ_{bk}^{bkm}$

(9) $J_{bk}^{bkm} X_3' = J_{bkm}^{bkm}$

(10) If $X_2'X_1 = N$, $NN' = m^2[(r - \lambda)I_t + \lambda J_t^t]$

(11) If $X_3'X_1 = M'$, $MM' = m^2 k I_b$

(12) If $X_3'X_2 = L'$, $LL' = m^2 r I_t$

(13) $(X_2' - m^{-1}k^{-1}NX_1')X_2 = A'X_2 = \lambda k^{-1} m(tI_t - J_t^t)$

(14) $(X_2' - m^{-1}k^{-1}NX_1')X_1 = \phi$

(15) $ML' = mN'$

(16) $MNL' = m^3[(r - \lambda)I_t + \lambda J_t^t]$

(17) $J_t^t N = mkJ_b^t$

(18) $L'J_t^t N = m^2 k J_b^{bk}$

(19) $J_t^t L = mJ_{bk}^t$

(20) $L'J_t^t NM = m^3 k J_{bk}^{bk}$

(21) $L'J_t^t L = m^2 J_{bk}^{bk}$

(22) $M'N'J_t^t L = m^3 k J_{bk}^{bk}$

(23) $L'J_t^t = mJ_t^{bk}$

(24) $N'J_t^t N = m^2 k J_b^b$

(25) $N'J_t^t = mkJ_t^b$

(26) $M'N'J_t^t = m^2 k J_t^{bk}$

(27) $L'J_t^t = m^{-1}k^{-1}M'N'J_t^t$

(28) If $F' = X_3' - m^{-1}k^{-1}M'X_1' - m^{-1}\lambda^{-1}t^{-1}k(L' - m^{-1}k^{-1}M'N')$

$(X_2' - m^{-1}k^{-1}NX_1')$, then $F'J_1^{bkm} = 0$, $F'X_1 = \phi$, $F'X_2 = \phi$

and $m^{-1}F'F$ is an idempotent matrix of rank $bk - b - t + 1$.

(29) $X_1X_1'X_3X_3' = X_3X_3'X_1X_1' = mX_1X_1'$

(30) $X_2X_2'X_3X_3 = X_3X_3'X_2X_2' = mX_2X_2'$

We shall now define an operation, say $\mathcal{I}$, which when operated on the joint distribution of the elements of the vector Y, gives a set of sufficient statistics which is minimal. This has been explained in the latter part of this chapter where we have discussed the minimal set of sufficient statistics.

The vector Y is distributed as the multivariate normal with mean $\bar{\mu}$ and covariance matrix Σ where

$$\bar{\mu} = E(Y) = \mu J_1^{bkm}$$

and

$$\Sigma = E(Y - \bar{\mu})(Y - \bar{\mu})' = (X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I)$$

The joint density of the elements of Y is given by

(III) $$g(Y, \theta) = (2\pi)^{-\frac{bkm}{2}} \left|\Sigma\right|^{-1/2} \exp\left[-2^{-1}(Y - \bar{\mu})'\Sigma^{-1}(Y - \bar{\mu})\right]$$

Consider now the operation $\mathcal{I}$ on $g(Y, \theta)$ to be of the form

$$\mathcal{I}\, g(Y, \theta) = (2\pi)^{-\frac{bkm}{2}} \left|\Sigma\right|^{-1/2} \exp\left[-2^{-1}(Y - \bar{\mu})'PP'\Sigma^{-1}PP'(Y - \bar{\mu})\right]$$

where P is an orthogonal bkm x bkm matrix to be defined.

Let P be partitioned as follows: $P = (R_1, R_2, R_3, R_4, R_5)$ where the dimensions of R_i ($i = 1, 2, 3, 4, 5$) are $bkm \times 1$, $bkm \times b-1$, $bkm \times t-1$, $bkm \times bk - b - t + 1$, $bkm \times bk(m-1)$, respectively. We shall now define these five partitions of P so that the condition of orthogonality is satisfied.

Let $R_1' = (bkm)^{-1/2} J_{bkm}^1$ and R_5 be constructed in the same manner as the matrix P of Lemma 1. We then have $R_1'R_1 = 1$ and $R_5'R_5 = I_{bk(m-1)}$.

Consider now the matrix $NN' = m^2[(r - \lambda)I_t + \lambda J_t^t]$. We can get the characteristic roots of NN' by solving the determinantal equation $|NN' - \ell I| = 0$ for ℓ. The characteristic roots of NN' are then $m^2(r - \lambda)$ and $m^2[r + (t-1)\lambda] = m^2rk$ of multiplicities $(t-1)$ and 1, respectively. Let Q be an orthogonal matrix which diagonalizes NN', that is

$$Q'NN'Q = \begin{bmatrix} m^2rk & \phi \\ \phi & m^2(r-\lambda)I_{t-1} \end{bmatrix}$$

Partition Q into (P_1, P_3) where P_1 and P_3 are of dimension $t \times 1$ and $t \times (t-1)$, respectively. Then

$$\begin{bmatrix} P_1' \\ P_3' \end{bmatrix} NN'(P_1, P_3) = \begin{bmatrix} m^2rk & \phi \\ \phi & m^2(r-\lambda)I_{t-1} \end{bmatrix} = D_1 \text{ (say)}$$

By Lemma 2 the orthogonal set of rows which diagonalizes N'N and gives the non-zero characteristic roots of N'N is $D_1^{-1/2}Q'N$. Thus

$$(D_1^{-1/2}Q'N)N'N(N'QD_1^{-1/2}) = D_1$$

Since the rank of NN' is t, the rank of N'N is also t. Since N'N is $b \times b$, there will be $b - t$ zero characteristic roots of N'N. If by P_2

we denote the matrix which diagonalizes $N'N$, we may write

$$P_2'N'NP_2 = \begin{bmatrix} m^2rk & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & m^2(r-\lambda)I_{t-1} \end{bmatrix}$$

We can partition P_2 into (P_{20}, P_{21}, P_{22}) and have

$$P_2'N'NP_2 = \begin{bmatrix} P_{20}' \\ P_{21}' \\ P_{22}' \end{bmatrix} N'N(P_{20}, P_{21}, P_{22}) = \begin{bmatrix} m^2rk & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & m^2(r-\lambda)I_{t-1} \end{bmatrix}$$

We can write $P_{22}' = (r-\lambda)^{-1/2}m^{-1}P_3'N$.

Since $A' = (X_2' - m^{-1}k^{-1}NX_1')$, the orthogonal matrix which diagonalizes NN' will also diagonalize $A'A$, for

$$Q'(mrI - m^{-1}k^{-1}NN')Q = mrI - m^{-1}k^{-1}D_1$$

where

$$mrI - m^{-1}k^{-1}D_1 = \begin{bmatrix} 0 & \phi \\ \phi & mk^{-1}\lambda tI_{t-1} \end{bmatrix}$$

Consider now $F' = X_3' - m^{-1}k^{-1}M'X_1' - m^{-1}\lambda^{-1}t^{-1}k(X_3'AA')$. Since $m^{-1}F'F = m^{-1}F'X_3$ is an idempotent matrix of rank $bk - b - t + 1$, we can have P_4' as $bk - b - t + 1 \times bk$ orthogonal vectors from $bk \times bk$ orthogonal matrix which would diagonalize $m^{-1}F'F$. This can be done since we can always choose P_4 corresponding to non-zero characteristic roots of the idempotent matrix.

We now define the matrix P of which we spoke when the operation

Σ was discussed. Define P in the following manner.

$$P' = \begin{bmatrix} (bkm)^{-1/2} J^1_{bkm} \\ (km)^{-1} P'_{21} X'_1 \\ (km)^{-1} P'_{22} X'_1 \\ (\frac{k}{\lambda tm})^{1/2} P'_3 A' \\ m^{-1/2} P'_4 F' \\ P'_5 \end{bmatrix} = \begin{bmatrix} (bkm)^{-1/2} J^1_{bkm} \\ (km)^{-1/2} P'_{21} X'_1 \\ [km^3(r-\lambda)]^{-1/2} P'_3 N X'_1 \\ [\frac{k}{\lambda tm}]^{1/2} P'_3 A' \\ m^{-1/2} P'_4 F' \\ P'_5 \end{bmatrix}$$

where

$$R'_2 = \begin{bmatrix} (mk)^{-1/2} P'_{21} X'_1 \\ (mk)^{-1/2} P'_{22} X'_1 \end{bmatrix}$$

$$R'_3 = (\frac{k}{\lambda tm})^{1/2} P'_3 A'$$

$$R'_4 = m^{-1/2} P'_4 F'$$

and

$$R'_5 = P'_5$$

It can be verified that P is an orthogonal matrix. For proof, see Appendix I.

We shall first derive $P'\Sigma P$ and from Appendix II it follows that $P'\Sigma P$ assumes the form as given in Table I.

In order to find $P'\Sigma^{-1}P$ we shall make use of the fact that $(P'\Sigma P)^{-1} = P'\Sigma^{-1}P$. We also note that if we have a matrix of the form

TABLE I

$\underline{P'\Sigma P}$

$$
\begin{bmatrix}
\sigma^2+mk\sigma_1^2+mr\sigma_2^2+m\sigma_3^2 & \phi & \phi & \phi & \phi & \phi \\
\phi & [\sigma^2+mk\sigma_1^2+m\sigma_3^2]I_{b-t} & \phi & \phi & \phi & \phi \\
\phi & \phi & [\sigma^2+mk\sigma_1^2+mk^{-1}(r-\lambda)\sigma_2^2+m\sigma_3^2]I_{t-1} & [m^2k^{-2}\lambda t(r-\lambda)]^{1/2}\sigma_2^2 I_{t-1} & \phi & \phi \\
\phi & \phi & [m^2k^{-2}\lambda t(r-\lambda)]^{1/2}\sigma_2^2 I_{t-1} & [\sigma^2+\lambda k^{-1}mt\sigma_2^2+m\sigma_3^2]I_{t-1} & \phi & \phi \\
\phi & \phi & \phi & \phi & [\sigma^2+m\sigma_3^2]I_{bk-b-t+1} & \phi \\
\phi & \phi & \phi & \phi & \phi & \sigma^2 I_{bk(m-1)}
\end{bmatrix}
$$

$$C = \begin{bmatrix} c_1 I_s & c_3 I_s \\ c_3 I_s & c_2 I_s \end{bmatrix}$$

then

$$C^{-1} = (c_1 c_2 - c_3^2)^{-1} \begin{bmatrix} c_2 I_s & -c_3 I_s \\ -c_3 I_s & c_1 I_s \end{bmatrix}$$

With the help of this result $P'\Sigma^{-1}P$ is shown in Table II.

Let us examine the form $P'(Y - \bar{\mu})$. We then have

$$P'(Y-\bar{\mu}) = \begin{bmatrix} (bkm)^{-1/2} J^1_{bkm}(Y-\mu J_1^{bkm}) \\ (mk)^{-1/2} P'_{21} X'_1 (Y-\mu J_1^{bkm}) \\ (mk)^{-1/2} P'_{22} X'_1 (Y-\mu J_1^{bkm}) \\ (\frac{k}{\lambda t m})^{1/2} P'_3 A'(Y-\mu J_1^{bkm}) \\ m^{-1/2} P'_4 F'(Y-\mu J_1^{bkm}) \\ P'_5 (Y-\mu J_1^{bkm}) \end{bmatrix} = \begin{bmatrix} (bkm)^{1/2}(y_{\cdots} - \mu) \\ (km)^{-1/2} P'_{21} X'_1 Y \\ (km)^{-1/2} P'_{22} X'_1 Y \\ (\frac{k}{\lambda t m})^{1/2} P'_3 A' Y \\ m^{-1/2} P'_4 F' Y \\ P'_5 Y \end{bmatrix}$$

where $y_{\cdots} = (bkm)^{-1} J^1_{bkm} Y$.

Letting $q = (Y - \bar{\mu})'PP'\Sigma^{-1}PP'(Y - \bar{\mu})$, we have

$$\begin{aligned} q = {} & (\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)^{-1}(bkm)\ (y_{\cdots} - \mu)^2 \\ & + [km(\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)]^{-1} Y'X_1 P_{21} P'_{21} X'_1 Y \\ & + [kmd_1]^{-1}[\sigma^2 + \lambda k^{-1} mt\sigma_2^2 + m\sigma_3^2] Y'X_1 P_{22} P'_{22} X'_1 Y \\ & + [m\sigma_3^2 + \sigma^2]^{-1} Y'FP_4 P'_4 F'Ym^{-1} + \sigma^{-2} Y'P_5 P'_5 Y \end{aligned}$$

TABLE II

$\underline{P'\Sigma^{-1}P}$

$$\begin{bmatrix}
[\sigma^2+mk\sigma_1^2+mr\sigma_2^2+m\sigma_3^2]^{-1} & \phi & \phi & \phi & \phi & \phi \\
\phi & [\sigma^2+mk\sigma_1^2+m\sigma_3^2]^{-1}I_{b-t} & \phi & \phi & \phi & \phi \\
\phi & \phi & d_1^{-1}[\sigma^2+\lambda k^{-1}mt\sigma_2^2+m\sigma_3^2]I_{t-1} & -d_1^{-1}[m^2k^{-2}\lambda t(r-\lambda)]^{\frac{1}{2}}\sigma_2^2 I_{t-1} & \phi & \phi \\
\phi & \phi & -d_1^{-1}[m^2k^{-2}\lambda t(r-\lambda)]^{\frac{1}{2}}\sigma_2^2 I_{t-1} & d_1^{-1}[\sigma^2+mk\sigma_1^2+mk^{-1}(r-\lambda)\sigma_2^2+m\sigma_3^2]I_{t-1} & \phi & \phi \\
\phi & \phi & \phi & \phi & [\sigma^2+m\sigma_3^2]^{-1}I_{bk-b-t+1} & \phi \\
\phi & \phi & \phi & \phi & \phi & \sigma^{-2}I_{bk(m-1)}
\end{bmatrix}$$

$$d_1^{-1} = \sigma^4 + mk\sigma^2\sigma_1^2 + mr\sigma^2\sigma_2^2 + 2m\sigma^2\sigma_3^2 + m^2\lambda t\sigma_1^2\sigma_2^2 + m^2k\sigma_1^2\sigma_3^2 + m^2r\sigma_2^2\sigma_3^2 + m^2\sigma_3^4$$

$$+ \frac{k}{\lambda tm}[\sigma^2 + mk\sigma_1^2 + mk^{-1}(r-\lambda)\sigma_2^2 + m\sigma_3^2]d_1^{-1} Y'AP_3P_3'A'Y$$

$$-2d_1^{-1}[m^2k^{-2}(r-\lambda)]^{1/2}Y'X_1P_{22}P_3'A'Y\sigma_2^2.$$

Define the seven statistics s_i ($i = 1, 2, 3, \ldots 7$) as follows:

$$s_1 = y\ldots$$

$$s_2 = (km)^{-1}Y'X_1P_{21}P_{21}'X_1'Y \qquad \text{not defined if } b = t$$

$$s_3 = (km)^{-1}Y'X_1P_{22}P_{22}'X_1'Y$$

$$s_4 = k^{-1}(r-\lambda)^{1/2}Y'X_1P_{22}P_3'A'Y$$

(IV)

$$s_5 = \frac{k}{\lambda tm} Y'AP_3P_3'A'Y$$

$$s_6 = m^{-1}Y'FP_4P_4'F'Y$$

$$s_7 = Y'P_5P_5'Y$$

These seven statistics are sufficient for the parameters μ, σ^2, σ_1^2, σ_2^2, σ_3^2. This follows from [7].

We shall now prove that this set of sufficient statistics is minimal for $g(Y, \theta)$. In order to prove this we shall make use of the scheme given by Lehmann and Scheffe [8]. This consists of defining a function $K(Y, Y_0) = \mathfrak{T}\, g(Y, \theta)/\mathfrak{T}\, g(Y_0, \theta)$ and finding the condition under which $K(Y, Y_0)$ is independent of parameters. We shall define $\mathfrak{T}$ to consist of operating on the exponent of $g(Y, \theta)$ with the matrix **P** which we have already defined. A set of sufficient statistics is minimal sufficient when $K(Y, Y_0)$ being independent of parameters implies $s_i = s_{i0}$ where the s_i are a proposed set of minimal sufficient statistics and s_{i0} are obtained from $\mathfrak{L}\, g(Y_0, \theta)$ in the same manner as s_i were obtained from $\mathfrak{L}\, g(Y, \theta)$. We can write $K(Y, Y_0) = \exp -2^{-1}(q - q_0)$ with q defined above and q_0 the same as q except s_i $(i = 1, \ldots, 7)$ to be replaced by s_{i0}, $(i = 1, 2, \ldots, 7)$.

Let us write $K(Y, Y_0) = \exp -2^{-1} \sum_{i=1}^{7} v_i u_i$ where v_i $(i = 1, \ldots, 7)$ are defined below and $u_i = s_i - s_{i0}$ $(i = 2, \ldots, 7)$ and $u_1 = bkm(s_1 - \mu)^2 - bkm(s_{10} - \mu)^2$.

$g(Y, \theta)$ may be written in the form

$$g(Y, \theta) = P(\theta)Q(Y)\exp\left[-\frac{1}{2} \sum_{i=1}^{k} v_i(\theta)u_i(Y)\right]$$

A necessary and sufficient condition for the set of sufficient statistics $u_i(Y)$ to be minimal for $g(Y, \theta)$ is that there exist no non-zero constants $a_1, a_2, \ldots, a_k$, c such that

$$\text{(V)} \qquad \sum_{i=1}^{k} a_i v_i(\theta) = c\ .$$

Thus it is enough to prove that for the following eight functions,

$$\text{(VI)} \quad \begin{aligned} v_1 &= (\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)^{-1} \\ v_2 &= (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)^{-1} \\ v_3 &= [\sigma^2 + mk\sigma_1^2 + mk^{-1}(r - \lambda)\sigma_2^2 + m\sigma_3^2]d_1^{-1} \\ v_4 &= [\sigma^2 + \lambda k^{-1}mt\sigma_2^2 + m\sigma_3^2]d_1^{-1} \\ v_5 &= -2\sigma_2^2 d_1^{-1} \\ v_6 &= (\sigma^2 + m\sigma_3^2)^{-1} \end{aligned}$$

$$v_7 = \sigma^{-2}$$

$$v_8 = v_1\mu .$$

(V) is not true for any $a_1, a_2, \ldots, a_8$ and c except when all vanish.

In (VI) it is clear that μ appears only in v_8 since $v_1, v_2, \ldots, v_7$ are homogeneous functions of σ, σ_1, σ_2, and σ_3 of degrees -2, the constant c can only be zero.

Effect the linear transformation:

$$x = \sigma^2$$

$$y = \sigma^2 + mk\sigma_1^2 + m\sigma_3^2$$

$$z = \sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2$$

$$w = m\sigma_3^2 + \sigma^2$$

The functions in (6) become:

$$v_1 = xyw[zw + \frac{\lambda t}{rk}(z - y)(y - w)]D^{-1}$$

$$v_2 = xzw[zw + \frac{\lambda t}{rk}(z - y)(y - w)]D^{-1}$$

$$v_3 = xyzw[y - \frac{r - \lambda}{rk}(z - y)]D^{-1}$$

$$v_4 = xyzw[w + \frac{\lambda t}{rk}(z - y)]D^{-1}$$

$$v_5 = -2xyzw[\frac{z - y}{mr}]D^{-1}$$

$$v_6 = xyz[zw + \frac{\lambda t}{rk}(z - y)(y - w)]D^{-1}$$

$$v_7 = yzw[zw + \frac{\lambda t}{rk}(z - y)(y - w)]D^{-1}$$

where $D = xyzw\,[\,zw + \frac{\lambda t}{rk}(z - y)(y - w)\,]$.

Observe that the term xy^2w^2 appears only in v_1, xz^2w^2 appears only in v_2, xy^2z^2 appears only in v_6, and yz^2w^2 appears only in v_7. This implies v_1, v_2, v_6, v_7 are mutually linearly independent of v_3, v_4, v_5. Now observe that after removing the common factor xyzw in v_3, v_4, and v_5, these are also linearly independent, thereby proving that (V) is not true unless $a_1, a_2, \ldots, a_7$ and c vanish. This condition then implies the set of sufficient statistics defined in (IV) is minimal.

Summarizing the results of this chapter will be accomplished by means of the following theorems and corollaries.

<u>Theorem 1:</u> <u>If an Eisenhart Model II is assumed in a balanced incomplete block design with interaction, then there are seven statistics in a minimal set of sufficient statistics if $b > t$ and there are six statistics in a minimal set if $b = t$.</u>

<u>Corollary 1.1.</u> <u>The explicit form of the statistics in a minimal set are as follows:</u>

1. $s_1 = y_{...}$

2. $s_2 = (km)^{-1}Y'X_1P_{21}P_{21}'X_1'Y$ if $b > t$, not defined if $b = t$.

3. $s_3 = (km)^{-1}Y'X_1P_{22}P_{22}'X_1'Y$

4. $s_4 = k^{-1}(r - \lambda)^{1/2}Y'X_1P_{22}P_3'A'Y$

5. $s_5 = \frac{k}{\lambda tm}\,Y'AP_3P_3'A'Y$

6. $s_6 = m^{-1}Y'FP_4P_4'F'Y$

7. $s_7 = Y'P_5P_5'Y$

where $P_{21}'N'NP_{21} = \phi_{b-t}$, $P_3'NN'P_3 = m^2(r-\lambda)I_{t-1}$, $m^{-1}P_4'F'FP_4 = I_{bk-b-t+1}$.

Corollary 1.2. The expectations of each of the statistics as defined in Corollary 1.1 are as follows:

1. $E(s_1) = \mu$

2. $E(s_2) = (b - t)(\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)$

3. $E(s_3) = (t - 1)[\sigma^2 + mk\sigma_1^2 + mk^{-1}(r - \lambda)\sigma_2^2 + m\sigma_3^2]$

4. $E(s_4) = (t - 1)k^{-2}m^2(r - \lambda)\lambda t\sigma_2^2$

5. $E(s_5) = (t - 1)(\sigma^2 + \lambda k^{-1}mt\sigma_2^2 + m\sigma_3^2)$

6. $E(s_6) = (bk - b - t + 1)(\sigma^2 + m\sigma_3^2)$

7. $E(s_7) = bk(m - 1)\sigma^2$

For the proof of the corollary see Appendix III.

Corollary 1.3. The distribution of each of the statistics of the minimal set as defined in Corollary 1.1 is as follows:

1. $s_1 \sim N[\mu, (bkm)^{-1}(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)]$

2. $s_2 \sim (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)\ \chi^2_{b-t}$ if $b > t$; not defined if $b = t$.

3. $s_3 \sim [\sigma^2 + mk\sigma_1^2 + mk^{-1}(r-\lambda)\sigma_2^2 + m\sigma_3^2]\ \chi^2_{t-1}$

4. $s_5 \sim [\sigma^2 + \lambda k^{-1}mt\sigma_2^2 + m\sigma_3^2]\ \chi^2_{t-1}$

5. $s_6 \sim [\sigma^2 + m\sigma_3^2] \, \chi^2_{bk-b-t+1}$

6. $s_7 \sim \sigma^2 \, \chi^2_{bk(m-1)}$

7. s_4 is distributed as a linear combination of independent chi-square variables that is $s_4 \sim \Sigma \, p_i \, \chi^2_{(1)}$ where p_i are the non-zero characteristic roots of $2^{-1}(A_4 + A_4')\not\Sigma$ where $A_4 = k^{-1}m^{-1}X_1N'P_3P_3'A'$.

The proof of this corollary appears in Appendix III.

<u>Corollary 1.4. The statistics s_i (i = 1, 2, . . . , 7), are pairwise independent except for pairs</u> (s_3, s_4), (s_3, s_5), <u>and</u> (s_4, s_5).

The proof of this corollary is given in Appendix V.

<u>Corollary 1.5. The seven statistics as defined in Corollary 1.1 may be computed from the following Analysis of Variance Table</u> (Table III).

See Appendix V for proof.

Table III

Analysis of Variance, Balanced Incomplete Block

Source	Statistic
Mean	$bkm\bar{y}^2_{\cdot\cdot\cdot} = bkms_1^2$
Blocks (ignoring treatments)	$(mk)^{-1}\Sigma(B_i - B_\cdot)^2$
Block-treatment-interaction-error component	$[km^3(r-\lambda)]^{-1}\Sigma(T_j - T_\cdot)^2 = s_3$
Block-interaction-error component	By subtraction (s_2)
Treatment-interaction Error Component	$(\frac{k}{\lambda tm})\Sigma Q_j^2 = s_5$
Interaction-Error Component	$m^{-1}(\sum_{n=1}^{bk} c_n^2 - k^{-1}\sum_{i=1}^{b} B_i^2 - \frac{k}{\lambda t}\Sigma Q_j^2)$
Intra-block Error with $s_4 = m^{-1}k^{-1}\Sigma T_jQ_j$	By subtraction (s_7)

The notation used here is explained in Appendices III and V.

CHAPTER IV

GROUP DIVISIBLE, PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS (WITH TWO ASSOCIATE CLASSES)

In this chapter we shall be interested in finding sets of minimal sufficient statistics for each of the three types of group divisible designs when there is a component of variance corresponding to the block-treatment interaction and an Eisenhart Model II is assumed.

Definitions:

An incomplete block design is said to be partially balanced with two associate classes if:

(1) there are b blocks and each with k experimental units.

(2) there are $t > k$ treatments, each of which satisfies the following:

(a) A treatment cannot appear more than once in a block;

(b) Each treatment appears exactly r times in all blocks;

(c) Each treatment has exactly n_i i-th associates;

(d) Two treatments which are i-th associates occur together in exactly λ_i blocks;

(3) any pair of treatments satisfy the following:

(a) The pair are either first or second associates;

(b) Any pair of treatments which are i-th associates, the number of treatments common to the j-th associate of the first and the k-th associate of the second is p^i_{jk} and is independent of the pair of treatments.

From the above definitions, the following relationships hold:

(i) $bk = rt$

(ii) $n_1 + n_2 = t - 1$

(iii) $n_1\lambda_1 + n_2\lambda_2 = rk - r$.

A group divisible, partially balanced incomplete block design is defined as a design in which the treatments are arranged such that there are g groups of n treatments each, such that any two treatments of the same group occur in exactly λ_1 blocks, and any two treatments which are in different groups occur together in exactly λ_2 blocks.

For the group divisible designs, the following relationships hold:

(i) $t = gn$

(ii) $n_1 = n - 1$

(iii) $n_2 = n(g - 1)$

(iv) $r \geqslant \lambda_1$

(v) $rk - \lambda_2 t \geqslant 0$

(vi) $(n - 1)\lambda_1 + n(g - 1)\lambda_2 = r(k - 1)$

They are classified into three types by Bose, Clatworthy, and Shrikhande [2] as follows:

(i) Singular if $r = \lambda_1$

(ii) Semi-Regular if $rk - \lambda_2 t = 0$

(iii) Regular if $r > \lambda_1$ and $rk - \lambda_2 t > 0$.

We are going to discuss a case where there is block-treatment interaction and we shall assume we have more than one observation per cell. We shall therefore replace (2) in the definition of an incomplete block design by (2') as follows where a cell is a group of experimental units subjected to a particular block-treatment combination.

(a) There are exactly m observations per cell;

(b) A treatment cannot appear more than once in different cells in the same block but appears m times in the same cell as follows from (a).

(c) Each treatment appears exactly m times in each of r different blocks.

(d) Each treatment has exactly n_i i-th associates.

(e) The number of times a pair of treatments which are i-th associates appear together in all blocks is $m\lambda_i$.

In spite of the above change, all the relationships (i) to (vi) given above are true.

We shall now discuss some of the general properties of all three types of designs before we find a set of minimal sufficient statistics for each.

We shall assume here the same model as in the BIB design with the same distributional properties of the random variables. The matrix model will be:

$$\text{(I)} \qquad Y = \mu J_1^{bkm} + X_1\beta + X_2\tau + X_3(\beta\tau) + e$$

where Y is distributed as the multivariate normal, mean $\bar{\mu} = \mu J_1^{bkm}$ and covariance matrix

$$\not{\Sigma} = X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I$$

All the results (1) to (30) which are true for the BIB Designs will hold here except (10), (13), (16), and (28). We shall replace (10), (13), (16), and (28) by (10'), (13'), (16'), and (28'), respectively.

(10') $\quad NN' = m^2[rB_0 + \lambda_1 B_1 + \lambda_2 B_2]$ where $B_t = n^t_{i\alpha}$, $(t = 0, 1, 2)$.

B_t is a t x t symmetric matrix,

$$n^t_{i\alpha} = \begin{cases} 1 \text{ if the i-th and } \alpha\text{-th treatments are t associates} \\ 0 \text{ otherwise} \end{cases}$$

$i, \alpha = 1, 2, \ldots, t$; $t = 0, 1, 2$. If $t = 0$, $B_0 = I_t$. Moreover, $B_0 + B_1 + B_2 = J^t_t$.

(13') $$(X_2' - m^{-1}k^{-1}NX_1')X_2 = (mrI_t - m^{-1}k^{-1}NN')$$

$$= [mrI_t - mk^{-1}(rB_0 + \lambda_1 B_1 + \lambda_2 B_2)]$$

$$= \frac{m}{k}[r(k-1)B_0 - \lambda_1 B_1 - \lambda_2 B_2]$$

(16') $\quad B_t B_s = \sum_{\ell=0}^{2} p^\ell_{st} B_\ell$, where p^ℓ_{st} is as defined previously with

$$p^0_{st} = \begin{cases} 0 & \text{if } s \neq t \\ n_s = n_t & \text{if } s = t \end{cases}$$

In defining p^0_{st} we are making use of the convention that a treatment will be considered its own 0-th associate.

(28') If

$$F' = X_3' - m^{-1}k^{-1}M'X_1' - \frac{k}{(rk-r+\lambda_1)m}(X_3'AA') - \frac{k(\lambda_1 - \lambda_2)}{\lambda_2 t(rk-r+\lambda_1)m}[X_3'A][B_0 + B_1]A'$$

then $F'J_1^{bkm} = \phi$, $F'X_1 = \phi$, $F'X_2 = \phi$, and $m^{-1}F'F$ is an idempotent matrix of rank bk-b-t+1.

The joint density of the elements of Y is given by

$$g(Y, \theta) = (2\pi)^{-\frac{bkm}{2}} |\Sigma|^{-1/2} \exp[-2^{-1}(Y-\bar{\mu})'\Sigma^{-1}(Y-\bar{\mu})]$$

Before we define the operation $\mathfrak{I}$ on $g(Y, \theta)$, it may be stated here that the elements of the vector Y can be ordered in such a way that the matrix NN' assumes the form as given by (10') and hence we can find the characteristic roots of NN' [1] and they are shown in Table IV.

Table IV

Characteristic Roots of NN' in GD-PBIB Designs

Multiplicities	Roots
1	m^2rk
$g - 1$	$m^2(rk - \lambda_2 t)$
$g(n - 1)$	$m^2(r-\lambda_1)$

Imposing the restrictions on the roots for each of the three types of designs we have the results as given in Table V.

Table V

Characteristic Roots of NN' for S, SR and R-GD-PBIB Designs

Multiplicities	Roots	Roots	Roots
1	m^2rk	m^2rk	m^2rk
$g - 1$	$m^2(rk - \lambda_2 t)$	0	$m^2(rk - \lambda_2 t)$
$g(n - 1)$	0	$m^2(r - \lambda_1)$	$m^2(r - \lambda_1)$

Since NN' is symmetric there exists an orthogonal matrix Q_3 such that $Q_3'NN'Q_3 = D_3$ where D_3 is diagonal with the characteristic roots of NN' displayed on the main diagonal. Partition Q_3 into (P_{30}, P_{31}, P_{32}) where P_{30}, P_{31}, and P_{32} are of dimension $t \times 1$, $t \times (g-1)$, and $t \times g(n-1)$ respectively. We then have,

$$\begin{bmatrix} P_{30}' \\ P_{31}' \\ P_{32}' \end{bmatrix} NN'(P_{30}, P_{31}, P_{32}) = \begin{cases} \begin{bmatrix} m^2rk & \phi & \phi \\ \phi & m^2(rk-\lambda_2 t)I_{g-1} & \phi \\ \phi & \phi & \phi \end{bmatrix} & (S) \\ \begin{bmatrix} m^2rk & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & m^2(r-\lambda_1)I_{g(n-1)} \end{bmatrix} & (SR) \\ \begin{bmatrix} m^2rk & \phi & \phi \\ \phi & m^2(rk-\lambda_2 t)I_{g-1} & \phi \\ \phi & \phi & m^2(r-\lambda_1)I_{g(n-1)} \end{bmatrix} & (R) \end{cases}$$

Since the non-zero characteristic roots of N'N are equal to the non-zero characteristic of NN' and ar e of the same multiplicity, there exists an orthogonal matrix Q_2 such that

$$Q_2' N' N Q_2 = \begin{bmatrix} m^2rk & \phi & \phi \\ \phi & \phi_{c_0+c_1'} & \phi \\ \phi & \phi & D_3^* \end{bmatrix}$$

where

c_0 = multiplicity of zero characteristic roots of NN'

$c_1' = b - t$

D_3^* = Diagonal matrix of the non-zero characteristic roots of NN' excluding the root m^2rk.

Partition Q_2 into (P_{20}, P_{21}, Q_{22}) where the dimensions of P_{20}, P_{21}, and Q_{22} are b x 1, b x c_0+c_1', and b x $\sum_{i=1}^{2} c_i$, respectively, where c_i

denotes the multiplicity of the i-th non-zero characteristic roots of NN' other than m^2rk. We may write,

$$\begin{bmatrix} P'_{20} \\ P'_{21} \\ Q_{22} \end{bmatrix} N'N\,(P_{20}, P_{21}, Q_{22}) = \begin{bmatrix} rk & \phi & \phi \\ \phi & \phi_{c_0+c_1'} & \phi \\ \phi & \phi & D_3^* \end{bmatrix}$$

Then for,

(i) S-GD-PBIB designs $Q'_{22} = P'_{22}$ will be of dimension $(g-1) \times b$;

(ii) SR-GD-PBIB designs $Q'_{22} = P'_{23}$ will be of dimension $g(n-1) \times b$;

(iii) R-GD-PBIB designs $Q_{22} = (P_{22}, P_{23})$.

Now we shall exhibit the relations among the partitions of Q_3 and Q_2 as given in Lemma 2. Then for

(i) S-GD-PBIB designs $P'_{22} = [m^2(rk - \lambda_2 t)]^{-1/2} P'_{31} N$.

(ii) SR-GD-PBIB designs $P'_{23} = [m^2(r - \lambda_1)]^{-1/2} P'_{32} N$.

(iii) R-GD-PBIB designs, the above two realtionships hold.

We shall now consider the matrix $A'A$. The orthogonal matrix which diagonalizes NN' also diagonalizes $A'A$ for

$$\begin{aligned} Q_3'A'AQ_3 &= Q_3'[X_2' - m^{-1}k^{-1}NX_1'][X_2 - m^{-1}k^{-1}X_1N']Q_3 \\ &= Q_3'[rmI - m^{-1}k^{-1}NN']Q_3 \\ &= mrI - m^{-1}k^{-1}D_3 \ . \end{aligned}$$

The characteristic roots of $A'A$ are then as given in Table VI.

Table VI

Characteristic Roots of $A'A$ for GD-PBIB Designs

Multiplicities	Roots
1	0
$g - 1$	$mk^{-1}\lambda_2 t$
$g(n - 1)$	$mk^{-1}[\lambda_2 t + n(\lambda_1 - \lambda_2)]$

By making use of restrictions for each of the three types of GD-PBIB designs we have the characteristic roots of $A'A$ in Table VII.

Table VII

Characteristic Roots of $A'A$ for S, SR, and R-GD-PBIB Designs

Multiplicities	Roots (S)	Roots (SR)	Roots (R)
1	0	0	0
$g - 1$	$mk^{-1}\lambda_2 t$	mr	$mk^{-1}\lambda_2 t$
$g(n - 1)$	mr	mv	mv

Consider now a bkm x bkm orthogonal matrix P' defined in the following way:

$$P' = \begin{bmatrix} R_1' \\ R_2' \\ C_3 R_3' \\ R_4' \\ P_5' \end{bmatrix}$$

where R_1, R_2, R_3, and R_4 are defined as follows and P_5' be constructed in the same manner as the matrix P of Lemma 1.

$$R_1' = (bkm)^{-1/2} J^1_{bkm}$$

$$R_2' = \begin{cases} \begin{bmatrix} (mk)^{-1/2} P_{21}' X_1' \\ (mk)^{-1/2} P_{22}' X_1' \end{bmatrix} & \text{for S-GD-PBIB Designs} \\ \begin{bmatrix} (mk)^{-1/2} P_{21}' X_1' \\ (mk)^{-1/2} P_{23}' X_1' \end{bmatrix} & \text{for SR-GD-PBIB Designs} \\ \begin{bmatrix} (mk)^{-1/2} P_{21}' X_1' \\ (mk)^{-1/2} P_{22}' X_1' \\ (mk)^{-1/2} P_{23}' X_1' \end{bmatrix} & \text{for R-GD-PBIB Designs} \end{cases}$$

$$c_3 R_3' = \begin{cases} \begin{bmatrix} \left(\frac{k}{\lambda_2 tm}\right)^{1/2} P_{31}' A' \\ (mr)^{-1/2} P_{32}' A' \end{bmatrix} & \text{for S-GD-PBIB Designs} \\ \begin{bmatrix} (mr)^{-1/2} P_{31}' A' \\ (mv)^{-1/2} P_{32}' A' \end{bmatrix} & \text{for SR-GD-PBIB Designs} \\ \begin{bmatrix} \left(\frac{k}{\lambda_2 tm}\right)^{1/2} P_{31}' A' \\ (mv)^{-1/2} P_{32}' A' \end{bmatrix} & \text{for R-GD-PBIB Designs} \end{cases}$$

$R_4' = m^{-1/2}P_4'F'$ where F' is as given in (28') and P_4' is a set of bk-b-t+1 x bk orthogonal vectors from a bk x bk orthogonal matrix which diagonalizes $m^{-1}F'F$. Consider the operation $\mp g(Y, \theta)$ to be

$$\text{(II)} \qquad \mp g(Y, \theta) = (2\pi)^{-\frac{bkm}{2}} |\Sigma|^{-1/2} \exp[-2^{-1}(Y-\bar{\mu})'PP'\Sigma^{-1}PP'(Y-\bar{\mu})]$$

where P is an orthogonal matrix defined above.

We shall now consider each of the three type of group divisible designs separately using the results we have derived so far in general.

Singular Group Divisible Partially Balanced Incomplete Block Designs.

In Appendix II $P'\Sigma P$ is shown to be of the form as given in Table VIII.

Table VIII

$$\begin{bmatrix} U_{11} & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & U_{22} & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & U_{33} & U_{34} & \phi & \phi & \phi \\ \phi & \phi & U_{43} & U_{44} & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & U_{55} & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & U_{66} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & U_{77} \end{bmatrix}$$

$$U_{11} = (\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)$$

$$U_{22} = (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)I_{c_0+c_1'}$$

$$U_{33} = [\sigma^2 + mk\sigma_1^2 + mk^{-1}(rk - \lambda_2 t)\sigma_2^2 + m\sigma_3^2]I_{g-1}$$

$$U_{34} = U_{43} = mk^{-1}(rk - \lambda_2 t)^{1/2}(\lambda_2 t)^{1/2}\sigma_2^2 I_{g-1}$$

$$U_{44} = [mk^{-1}\lambda_2 t\sigma_2^2 + m\sigma_3^2 + \sigma^2]I_{g-1}$$

$$U_{55} = (mr\sigma_2^2 + m\sigma_3^2 + \sigma^2)I_{g(n-1)}$$

$$U_{66} = (\sigma^2 + m\sigma_3^2)I_{bk-b-t+1}$$

$$U_{77} = \sigma^2 I_{bk(m-1)}$$

We must now determine the form of $P'\Sigma^{-1}P$. To evaluate this we note that $(P'\Sigma P)^{-1} = P'\Sigma^{-1}P$. The form of $P'\Sigma^{-1}P$ is given in Table IX.

Form of $P'\Sigma^{-1}P$ for Singular GD-PBIB Designs:

Table IX

$$\begin{bmatrix} W_{11} & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & W_{22} & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & W_{33} & W_{34} & \phi & \phi & \phi \\ \phi & \phi & W_{43} & W_{44} & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & W_{55} & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & W_{66} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & W_{77} \end{bmatrix}$$

where

$$W_{11} = (\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)^{-1}$$

$$W_{22} = (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)^{-1}I_{c_0+c_1'}$$

$$W_{33} = d_1^{-1}(\sigma^2 + mk^{-1}\lambda_2 t\sigma_2^2 + m\sigma_3^2)^{-1}I_{g-1}$$

$$W_{34} = W_{43} = -d_1^{-1}[m^2k^{-2}(rk - \lambda_2 t)\lambda_2 t]^{1/2}\sigma_2^2 I_{g-1}$$

$$W_{44} = d_1^{-1}[\sigma^2 + mk\sigma_1^2 + mk^{-1}(rk - \lambda_2 t)\sigma_2^2 + m\sigma_3^2]I_{g-1}$$

$$W_{55} = (mr\sigma_2^2 + m\sigma_3^2 + \sigma^2)^{-1}I_{g(n-1)}$$

$$W_{66} = (\sigma^2 + m\sigma_3^2)^{-1}I_{bk-b-t+1}$$

$$W_{77} = [\sigma^2]^{-1}I_{bk(m-1)}$$

$$d_1 = \sigma^4 + mk\sigma^2\sigma_1^2 + mr\sigma^2\sigma_2^2 + 2m\sigma^2\sigma_3^2 + m^2\lambda_2 t\sigma_1^2\sigma_2^2 + m^2k\sigma_1^2\sigma_3^2$$
$$+ m^2r\sigma_2^2\sigma_3^2 + m^2\sigma_3^4.$$

Evaluating $P'(Y - \bar{\mu})$ we have

$$P'(Y - \bar{\mu}) = \begin{bmatrix} (bkm)^{1/2}(y_{...} - \mu) \\ (km)^{-1/2}P_{21}'X_1'Y \\ (km)^{-1/2}P_{22}'X_1'Y \\ \left(\frac{k}{\lambda_2 tm}\right)^{1/2}P_{31}'A'Y \\ (rm)^{-1/2}P_{32}'A'Y \\ m^{-1/2}P_4'F'Y \\ P_5'Y \end{bmatrix}$$

Performing the multiplication $(Y-\bar{\mu})'PP'\Sigma^{-1}PP'(Y-\bar{\mu}) = q$ (say), we have

$$q = (bkm)(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)^{-1}(y_{\cdots} - \mu)^2$$

$$+ [km(\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)]^{-1} Y'X_1P_{21}P_{21}'X_1'Y$$

$$+ [kmd_1]^{-1}[\sigma^2 + mk^{-1}\lambda_2 t\sigma_2^2 + m\sigma_3^2] Y'X_1P_{22}P_{22}'X_1'Y$$

$$+ [m(\sigma^2 + m\sigma_3^2)]^{-1} Y'FP_4P_4'F'Y + \sigma^{-2} Y'P_5P_5'Y$$

$$+ \left(\frac{k}{\lambda_2 tm}\right)[\sigma^2 + mk\sigma_1^2 + mk^{-1}(rk - \lambda_2 t)\sigma_2^2 + m\sigma_3^2] d_1^{-1} Y'AP_{31}P_{31}'A'Y$$

$$+ [rm(mr\sigma_2^2 + m\sigma_3^2 + \sigma^2)]^{-1} Y'AP_{32}P_{32}'A'Y$$

$$- 2d_1^{-1}[m^2k^{-2}(rk - \lambda_2 t)\lambda_2 t]^{1/2}\sigma_2^2 Y'X_1P_{22}P_{31}'A'Y\left(\frac{1}{\lambda_2 tm}\right)^{1/2}$$

Define the eight statistics s_i $(i = 1, 2, \ldots 8)$ as follows:

$$s_1 = y_{\cdots}$$

$$s_2 = (km)^{-1} Y'X_1P_{21}P_{21}'X_1'Y \qquad \text{if } b > g, \text{ not defined if } b = g.$$

$$s_3 = (km)^{-1} Y'X_1P_{22}P_{22}'X_1'Y$$

$$s_4 = \left(\frac{k}{\lambda_2 tm}\right) Y'AP_{31}P_{31}'A'Y$$

$$s_5 = (rm)^{-1} Y'AP_{32}P_{32}'A'Y$$

$$s_6 = m^{-1} Y'FP_4P_4'F'Y$$

$$s_7 = Y'P_5P_5'Y$$

$$s_8 = [k^{-2}(rk - \lambda_2 t)]^{1/2} Y'X_1P_{22}P_{31}'A'Y$$

These eight statistics are sufficient for the parameters μ, σ^2, σ_1^2, σ_2^2, and σ_3^2. This follows from [7] and we shall show that these eight statistics form a minimal set of sufficient statistics by following the same procedure as we had for the BIB designs.

$g(Y, \theta)$ may be written in the form,

$$g(Y, \theta) = P(Q)\, Q(Y) \exp\left[-\frac{1}{2} \sum_{i=1}^{k} v_i(\theta) u_i(Y)\right]$$

A necessary and sufficient condition for the set of sufficient statistics $u_i(Y)$ to be minimal for $g(Y, \theta)$ is that there exist no non-zero constants $a_1, a_2, \ldots, a_k, c$, such that

$$\text{(IV)} \qquad \sum_{i=1}^{k} a_i v_i(\theta) = c .$$

Thus it is enough to prove that for the following nine functions:

$$v_1 = [\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2]^{-1}$$

$$v_2 = [\sigma^2 + mk\sigma_1^2 + m\sigma_3^2]^{-1}$$

$$v_3 = [\sigma^2 + mk\sigma_1^2 + mk^{-1}(rk - \lambda_2 t)\sigma_2^2 + m\sigma_3^2]d_1^{-1}$$

$$v_4 = [\sigma^2 + mk^{-1}\lambda_2 t\sigma_2^2 + m\sigma_3^2]d_1^{-1}$$

$$\text{(V)} \quad v_5 = -2\sigma_2^2 d_1^{-1}$$

$$v_6 = (\sigma^2 + m\sigma_3^2)^{-1}$$

$$v_7 = \sigma^{-2}$$

$$v_8 = [mr\sigma_2^2 + m\sigma_3^2 + \sigma^2]^{-1}$$

$$v_9 = v_1\mu .$$

(IV) is not true for any a_1, a_2, . . . , a_9, and c except when all vanish.

In (V) it is clear that μ appears only in v_9. Since v_1, v_2, . . . , v_8 are homogeneous functions of σ, σ_1, σ_2, and σ_3 of degree -2, the constant c can only be zero.

Effect the linear transformation,

$$x = \sigma^2$$

$$y = \sigma^2 + mk\sigma_1^2 + m\sigma_3^2$$

$$z = \sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2$$

$$u = mr\sigma_2^2 + m\sigma_3^2 + \sigma^2$$

$$w = \sigma^2 + m\sigma_3^2 .$$

The functions in (V) become:

$$v_1 = xyuw[zw + \frac{\lambda_2 t}{rk}(z - y)(y - w)]D^{-1}$$

$$v_2 = xzuw[zw + \frac{\lambda_2 t}{rk}(z - y)(y - w)]D^{-1}$$

$$v_3 = xyzuw[y - \frac{(rk - \lambda_2 t)}{rk}(z - y)]D^{-1}$$

$$v_4 = xyzuw[w + \frac{\lambda_2 t}{rk}(z - y)]D^{-1}$$

$$v_5 = -2xyzuw[\frac{z - y}{mr}]D^{-1}$$

$$v_6 = xyzu[zw + \frac{\lambda_2 t}{rk}(z - y)(y - w)]D^{-1}$$

$$v_7 = yzuw[zw + \frac{\lambda_2 t}{rk}(z - y)(y - w)]D^{-1}$$

$$v_8 = xyzw[zw + \frac{\lambda_2 t}{rk}(z - y)(y - w)]D^{-1}$$

where $D = xyzuw[zw + \frac{\lambda_2 t}{rk}(z - y)(y - w)]$.

Observe that the term xy^2uw^2 appears only in v_1, xz^2uw^2 appears only in v_2, xy^2z^2w appears only in v_6, yz^2uw^2 appears only in v_7, and xyz^2w^2 appears only in v_8. This implies v_1, v_2, v_6, v_7, and v_8 are mutually linearly independent of v_3, v_4, and v_5. Now observe that after removing the common factor xyzuw in v_3, v_4, and v_5, these are also linearly independent, thereby proving that (IV) is not true unless $a_1, a_2, \ldots, a_7$, and c vanish. This condition then implies the set of sufficient statistics defined in (IV) are minimal.

Summarizing the results for singular GD-PBIB Designs, we have the following theorem and corollaries:

Theorem 2: If an Eisenhart Model II is assumed in a singular, group divisible, partially balanced incomplete block design with two associate classes, then there are eight statistics in a minimal set of sufficient statistics if $b > g$ and seven statistics if $b = g$.

Corollary 2.1. The explicit form of a set of minimal sufficient statistics for a singular GD-PBIB design are as follows:

$$s_1 = y_{\cdots}$$

$$s_2 = (mk)^{-1}Y'X_1P_{21}P_{21}'X_1'Y \quad \text{if } b > g \text{ and is not defined if } b = g.$$

$$s_3 = (mk)^{-1}Y'X_1P_{22}P_{22}'X_1'Y \quad \text{or} \quad [m^3k(rk - \lambda_2 t)]^{-1}Y'X_1N'P_{31}P_{31}'NX_1'Y$$

$$s_4 = \left(\frac{k}{\lambda_2 tm}\right) Y'AP_{31}P_{31}'A'Y$$

$$s_5 = (rm)^{-1}Y'AP_{32}P_{32}'A'Y$$

$$s_6 = m^{-1}Y'FP_4P_4'F'Y$$

$s_7 = Y'P_5P_5'Y$

$s_8 = [k^{-2}(rk - \lambda_2 t)]^{1/2} Y'X_1P_{22}P_{31}'A'Y$ or $k^{-1}m^{-1}Y'X_1N'P_{31}P_{31}'A'Y$

<u>Corollary 2.2. The distributions of eight statistics as given in Corollary 2.1 are as follows:</u>

$s_1 \sim N[\mu, (bkm)^{-1}(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + \sigma^2)]$

$s_2 \sim [\sigma^2 + mk\sigma_1^2 + m\sigma_3^2]\ \chi^2_{b-g}$ if $b > g$ and is not defined if $b = g$.

$s_3 \sim [\sigma^2 + mk\sigma_1^2 + m\sigma_3^2 + mk^{-1}(rk - \lambda_2 t)\sigma_2^2]\ \chi^2_{g-1}$

$s_4 \sim [\sigma^2 + mk^{-1}\lambda_2 t\sigma_2^2]\ \chi^2_{g-1}$

$s_5 \sim [\sigma^2 + mr\sigma_2^2]\ \chi^2_{g(n-1)}$

$s_6 \sim [\sigma^2 + m\sigma_3^2]\ \chi^2_{bk-b-t+1}$

$s_7 \sim [\sigma^2]\ \chi^2_{bk(m-1)}$

$s_8 \sim \Sigma a_i\ \chi^2_{(1)}$ where a_i are non-zero characteristic roots of $s^{-1}[A_7 + A_7']\Sigma$ where $A_7 = m^{-1}k^{-1}X_1N'P_{31}P_{31}'A'$.

For proof of this corollary, see Appendix III.

<u>Corollary 2.3. The statistics as defined in Corollary 2.1 are pairwise independent except for the pairs</u> (s_3, s_4), (s_3, s_8), <u>and</u> (s_4, s_8).

For proof of this corollary, see Appendix IV.

<u>Corollary 2.4. The expectations of the eight statistics as defined in Corollary 2.1 are as follows:</u>

$E(s_1) = \mu_1$

$$E(s_2) = (b - g)[\sigma^2 + mk\sigma_1^2 + m\sigma_3^2]$$

$$E(s_3) = (g - 1)[\sigma^2 + mk\sigma_1^2 + m\sigma_3^2 + mk^{-1}(rk - \lambda_2 t)\sigma_2^2]$$

$$E(s_4) = (g - 1)[\sigma^2 + mk^{-1}\lambda_2 t\sigma_2^2]$$

$$E(s_5) = g(n - 1)[\sigma^2 + mr\sigma_2^2]$$

$$E(s_6) = (bk - b - t + 1)[\sigma^2 + m\sigma_3^2]$$

$$E(s_7) = bk(m - 1)\sigma^2$$

$$E(s_8) = m^2k^{-2}(g - 1)(rk - \lambda_2 t)(\lambda_2 t)\sigma_2^2$$

For proof of this corollary see Appendix III.

Semi-Regular GD-PBIB Designs.

In Appendix II $P'\Sigma P$ is shown to be of the form as given in Table X.

Table X

$$\begin{bmatrix} U_{11} & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & U_{22} & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & U_{33} & \phi & U_{35} & \phi & \phi \\ \phi & \phi & \phi & U_{44} & \phi & \phi & \phi \\ \phi & \phi & U_{53} & \phi & U_{55} & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & U_{66} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & U_{77} \end{bmatrix}$$

where

$$U_{11} = (\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)$$

$$U_{22} = (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)I_{c_0+c_1'}$$

$$U_{33} = [\sigma^2 + mk\sigma_1^2 + mk^{-1}(r - \lambda_1)\sigma_2^2 + m\sigma_3^2]I_{g(n-1)}$$

$$U_{35} = U_{53} = mk^{-1}[(r - \lambda_1)v]^{1/2}\sigma_2^2 I_{g(n-1)}$$

$$U_{44} = (\sigma^2 + mr\sigma_2^2 + m\sigma_3^2)I_{g-1}$$

$$U_{55} = (mv\sigma_2^2 + m\sigma_3^2 + \sigma^2)I_{g(n-1)}$$

$$U_{66} = (\sigma^2 + m\sigma_3^2)I_{bk-b-t+1}$$

$$U_{77} = \sigma^2 I_{bk(m-1)}$$

In order to determine $P'\Sigma^{-1}P$, we shall use the relation $(P'\Sigma P)^{-1} = P'\Sigma^{-1}P$. The form of $P'\Sigma^{-1}P$ is given in Table XI.

Table XI

$$\begin{bmatrix} W_{11} & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & W_{22} & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & W_{33} & \phi & W_{35} & \phi & \phi \\ \phi & \phi & \phi & W_{44} & \phi & \phi & \phi \\ \phi & \phi & W_{53} & \phi & W_{55} & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & W_{66} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & W_{77} \end{bmatrix}$$

where

$$W_{11} = [\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2]^{-1}$$

$$W_{22} = [\sigma^2 + mk\sigma_1^2 + m\sigma_3^2]^{-1}I_{c_0+c_1'}$$

$$W_{33} = [mv\sigma_2^2 + m\sigma_3^2 + \sigma^2]d_2^{-1} I_{g(n-1)}$$

$$W_{35} = W_{53}' = -[mk^{-1}[(r - \lambda_1)v]^{1/2}]\sigma_2^2 \, d_2^{-1} I_{g(n-1)}$$

$$W_{44} = [\sigma^2 + mr\sigma_2^2 + m\sigma_3^2]^{-1} I_{g-1}$$

$$W_{55} = [\sigma^2 + mk\sigma_1^2 + mk^{-1}(r - \lambda_1)\sigma_2^2 + m\sigma_3^2] I_{g(n-1)} d_2^{-1}$$

$$W_{66} = [\sigma^2 + m\sigma_3^2]^{-1} I_{bk-b-t+1}$$

$$W_{77} = \sigma^{-2} I_{bk(m-1)}$$

$$d_2 = \sigma^4 + mk\sigma^2\sigma_1^2 + mr\sigma^2\sigma_2^2 + 2m\sigma^2\sigma_3^2 + m^2(rk - r + \lambda_1)\sigma_1^2\sigma_2^2$$
$$+ m^2k\sigma_1^2\sigma_3^2 + m^2r\sigma_2^2\sigma_3^2 + m^2\sigma_3^4 .$$

We shall now ascertain the form $P'(Y - \bar{\mu})$. This is equal to:

$$P'(Y - \bar{\mu}) = \begin{bmatrix} (bkm)^{-1/2}(y_{...} - \mu) \\ (km)^{-1/2} P_{21}' X_1' Y \\ (km)^{-1/2} P_{23}' X_1' Y \\ (mr)^{-1/2} P_{31}' A' Y \\ (mv)^{-1/2} P_{32}' A' Y \\ m^{-1/2} P_4' F' Y \\ P_5' Y \end{bmatrix}$$

Performing the multiplication we have for $(Y-\bar{\mu})PP'\Sigma^{-1}PP'(Y-\bar{\mu}) = q$, where

$$q = (bkm)(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)^{-1}(y_{\cdot\cdot\cdot} - \mu)^2$$

$$+ [km(\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)]^{-1} Y'X_1P_{21}P_{21}'X_1'Y$$

$$+ [kmd_2]^{-1}[\sigma^2 + mv\sigma_2^2 + m\sigma_3^2] Y'X_1P_{23}P_{23}'X_1'Y$$

$$+ [m(\sigma^2 + m\sigma_3^2)]^{-1} Y'FP_4P_4'F'Y + \sigma^{-2} Y'P_5P_5'Y$$

$$+ [mr(\sigma^2 + mr\sigma_2^2 + m\sigma_3^2)]^{-1} Y'AP_{31}P_{31}'A'Y$$

$$+ [mvd_2]^{-1}[\sigma^2 + mk\sigma_1^2 + mk^{-1}(r - \lambda_1)\sigma_2^2 + m\sigma_3^2] Y'AP_{32}P_{32}'A'Y$$

$$-2d_2^{-1}[k^{-2}(r - \lambda_1)]^{1/2}\sigma_2^2 Y'X_1P_{23}P_{23}'A'Y$$

Define the eight statistics $s_i = (1, 2, 3, \ldots, 8)$ as follows:

$$s_1 = y_{\cdot\cdot\cdot}$$

$$s_2 = (km)^{-1} Y'X_1P_{21}P_{21}'X_1'Y$$

$$s_3 = (km)^{-1} Y'X_1P_{23}P_{23}'X_1'Y$$

$$s_4 = (mr)^{-1} Y'AP_{31}P_{31}'A'Y$$

(III')

$$s_5 = (mv)^{-1} Y'AP_{32}P_{32}'A'Y$$

$$s_6 = m^{-1} Y'FP_4P_4'F'Y$$

$$s_7 = Y'P_5P_5'Y$$

$$s_8 = [mk^{-1/2}(r - \lambda_1)^{1/2}] Y'X_1P_{23}P_{32}'A'Y \quad .$$

These eight statistics are sufficient for the parameters μ, σ^2, σ_1^2, σ_2^2, σ_3^2. This follows from [7], and we shall show that these eight statistics form a minimal set of sufficient statistics by following the same procedure as we had for the BIB designs.

$g(Y, \theta)$ may be written in the form

$$\text{(IV')} \qquad g(Y, \theta) = P(\theta)\,Q(Y)\,\exp\left[-\frac{1}{2}\sum_{i=1}^{k} v_i(\theta)\,u_i(Y)\right]$$

A necessary and sufficient condition for the set of sufficient statistics $u_i(Y)$ to be minimal for $g(Y, \theta)$ is that there exists no non-zero constants $a_1, a_2, \ldots, a_k$, c such that

$$\sum_{i=1}^{k} a_i v_i(\theta_i) = c \; .$$

Thus it is enough to prove that for the following nine functions:

$$v_1 = [\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2]^{-1}$$

$$v_2 = [\sigma^2 + mk\sigma_1^2 + m\sigma_3^2]^{-1}$$

$$v_3 = [\sigma^2 + mk\sigma_1^2 + mk^{-1}(r - \lambda_1)\sigma_2^2 + m\sigma_3^2]d_2^{-1}$$

$$v_4 = [\sigma^2 + mv\sigma_2^2 + m\sigma_3^2]d_2^{-1}$$

$$\text{(V')} \quad v_5 = -2\sigma_2^2\, d_2^{-1}$$

$$v_6 = (\sigma^2 + m\sigma_3^2)^{-1}$$

$$v_7 = \sigma^{-2}$$

$$v_8 = (\sigma^2 + mr\sigma_2^2 + m\sigma_3^2)^{-1}$$

$$v_9 = v_1\mu \; .$$

(IV') is not true for any a_1, a_2, . . . , a_9, and c except when all vanish.

In (V') it is clear that μ appears only in v_9. Since v_1, v_2, ..., v_8 are homogeneous functions of σ, σ_1, σ_2, and σ_3 of degree -2, the constant c can only be zero.

Effect the linear transformation,

$$x = \sigma^2$$

$$y = \sigma^2 + mk\sigma_1^2 + m\sigma_3^2$$

$$z = \sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2$$

$$u = mr\sigma_2^2 + m\sigma_3^2 + \sigma^2$$

$$w = \sigma^2 + m\sigma_3^2$$

The functions in (V') become:

$$v_1 = (xyuw)[zw + \frac{v}{r}(z - y)(y - w)]D^{-1}$$

$$v_2 = (xzuw)[zw + \frac{v}{r}(z - y)(y - w)]D^{-1}$$

$$v_3 = (xyzuw)[y - \frac{(r - \lambda_1)}{rk}(z - y)]D^{-1}$$

$$v_4 = xyzuw[w + \frac{v}{r}(z - y)]D^{-1}$$

$$v_5 = -2xyzuw[\frac{z - y}{mr}]D^{-1}$$

$$v_6 = xyzu[zw + \frac{v}{r}(z - y)(y - w)]D^{-1}$$

$$v_7 = yzuw[zw + \frac{v}{r}(z - y)(y - w]D^{-1}$$

$$v_8 = xyzw[zw + \frac{v}{r}(z - y)(y - w)]D^{-1}$$

where

$$D^{--} = xyzuw[zw + \frac{v_2}{rk}(z - y)(y - w)]$$

By following the process exactly similar to that for S-GD-PBIB designs we can conclude the set of sufficient statistics defined in (IV') are minimal. Hence from the above discussions we have the following theorems and corollaries.

Theorem 3. In a semi-regular group divisible, partially balanced incomplete block design with two associate classes there are eight statistics in a minimal set of sufficient statistics if $b > t - g + 1$ and seven statistics in a minimal set if $b = t - g + 1$.

Corollary 3.1. The explicit form of the statistics in a minimal set of sufficient statistics in a SR-GD-PBIB design are as follows:

$$s_1 = y_{\cdots}$$

$$s_2 = (mk)^{-1}Y'X_1P_{21}P_{21}'X_1'Y \quad \text{if } b > t-g+1; \text{ not defined if } b = t-g+1$$

$$s_3 = (mk)^{-1}Y'X_1P_{23}P_{23}'X_1'Y \quad \text{or} \quad [m^2k(r - \lambda)]^{-1}Y'X_1P_{32}P_{32}'NX_1'Y$$

$$s_4 = (mr)^{-1}Y'AP_{31}P_{31}'A'Y$$

$$s_5 = (mv)^{-1}Y'AP_{32}P_{32}'A'Y$$

$$s_6 = (m)^{-1}Y'FP_4P_4'F'Y$$

$$s_7 = Y'P_5P_5'Y$$

$$s_8 = [m^2k^{-2}(r - \lambda_1)]^{1/2}Y'X_1P_{23}P_{32}'A'Y = k^{-1}Y'X_1N'P_{32}P_{32}'A'Y$$

Corollary 3.2. The distribution of each of the statistics as given in Corollary 3.1 is as follows:

$$s_1 \sim N[\mu, (bkm)^{-1}(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)]$$

$$s_2 \sim (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)\, \chi^2_{(b-t+g-1)}$$

$$s_3 \sim [\sigma^2 + mk\sigma_1^2 + mk^{-1}(r - \lambda_1)\sigma_2^2 + m\sigma_3^2]\, \chi^2_{g(n-1)}$$

$$s_4 \sim [\sigma^2 + m\sigma_2^2]\, \chi^2_{(g-1)}$$

$$s_5 \sim [\sigma^2 + mv\sigma_2^2]\, \chi^2_{g(n-1)}$$

$$s_6 \sim [\sigma^2 + m\sigma_3^2]\, \chi^2_{(bk-b-t+1)}$$

$$s_7 \sim \sigma^2\, \chi^2_{[bk(m-1)]}$$

$s_8 \sim \Sigma a_i\, \chi^2_{(1)}$ where the a_i are the non-zero characteristic roots of $2^{-1}(A_7 + A_7')\Sigma$ where $A_7 = k^{-1}X_1N'P_{32}P_{32}'A'$.

For proof of this corollary, see Appendix III.

Corollary 3.3. The eight statistics as given in Corollary 3.1 are pairwise independent except for the pairs (s_3, s_5), (s_3, s_8), and (s_5, s_8).

For proof of this corollary, see Appendix IV.

Corollary 3.4. The expectations of the eight statistics as given in Corollary 3.1 are as follows:

$$E(s_1) = \mu$$

$$E(s_2) = (b - t + g - 1)(\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)$$

$$E(s_3) = g(n-1)[\sigma^2 + mk\sigma_1^2 + mk^{-1}(r-\lambda_1)\sigma_2^2 + m\sigma_3^2]$$

$$E(s_4) = (g-1)[\sigma^2 + mr\sigma_2^2]$$

$$E(s_5) = g(n-1)[\sigma^2 + mv\sigma_2^2]$$

$$E(s_6) = [\sigma^2 + m\sigma_3^2][bk - b - t + 1]$$

$$E(s_7) = \sigma^2[bk(m-1)]$$

$$E(s_8) = g(n-1)m^3(r-\lambda_1)(rk - r + \lambda_1)k^{-2}\sigma_2^2$$

For proof of this corollary, see Appendix III.

Regular GD-PBIB Designs.

In order to derive the elements of $P'\Sigma P$, we shall make use of the results derived for S and SR-GD-PBIB designs. $P'\Sigma P$ will be of the form as given in Table XII.

Table XII

$$\begin{bmatrix} U_{11} & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & U_{22} & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & U_{33} & \phi & U_{35} & \phi & \phi & \phi \\ \phi & \phi & \phi & U_{44} & \phi & U_{46} & \phi & \phi \\ \phi & \phi & U_{53} & \phi & U_{55} & \phi & \phi & \phi \\ \phi & \phi & \phi & U_{64} & \phi & U_{66} & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & U_{77} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & \phi & U_{88} \end{bmatrix}$$

where

$$U_{11} = (\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)$$

$$U_{22} = (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)I_{b-t}$$

$$U_{33} = [\sigma^2 + mk\sigma_1^2 + mk^{-1}(rk - \lambda_2 t)\sigma_2^2 + m\sigma_3^2]I_{g-1}$$

$$U_{35} = U_{53} = mk^{-1}[(rk - \lambda_2 t)\lambda_2 t]^{1/2}\sigma_2^2 I_{g-1}$$

$$U_{44} = [mk\sigma_1^2 + mk^{-1}(r - \lambda_1)\sigma_2^2 + m\sigma_3^2 + \sigma^2]I_{g(n-1)}$$

$$U_{46} = U_{64} = mk^{-1/2}[(r - \lambda_1)v]^{1/2}\sigma_2^2 I_{g(n-1)}$$

$$U_{55} = [mk^{-1}\lambda_2 t\sigma_2^2 + m\sigma_3^2 + \sigma^2]I_{g-1}$$

$$U_{66} = [mv\sigma_2^2 + m\sigma_3^2 + \sigma^2]I_{g(n-1)}$$

$$U_{77} = [\sigma^2 + m\sigma_3^2]I_{bk-b-t+1}$$

$$U_{88} = \sigma^2 I_{bk(m - 1)}$$

The form of $P'\Sigma^{-1}P$ is given in Table XIII.

Table XIII

$$\begin{bmatrix} W_{11} & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & W_{22} & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & W_{33} & \phi & W_{35} & \phi & \phi & \phi \\ \phi & \phi & \phi & W_{44} & \phi & W_{46} & \phi & \phi \\ \phi & \phi & W_{53} & \phi & W_{55} & \phi & \phi & \phi \\ \phi & \phi & \phi & W_{64} & \phi & W_{66} & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & W_{77} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & \phi & W_{88} \end{bmatrix}$$

$$W_{11} = (\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)^{-1}$$

$$W_{22} = (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)^{-1} I_{b-t}$$

$$W_{33} = [mk^{-1}\lambda_2 t\sigma_2^2 + m\sigma_3^2 + \sigma^2] d_1^{-1} I_{g-1}$$

$$W_{35} = W_{53} = -mk^{-1}[(rk - \lambda_2 t)\lambda_2 t]^{1/2} d_1^{-1}\sigma_2^2 I_{g-1}$$

$$W_{44} = [mv\sigma_2^2 + m\sigma_3^2 + \sigma^2] d_2^{-1} I_{g(n-1)}$$

$$W_{46} = W_{64} = -[mk^{-1/2}[(r - \lambda_1)v]^{1/2}]\sigma_2^2 I_{g(n-1)}$$

$$W_{55} = [\sigma^2 + mk\sigma_1^2 + mk^{-1}(rk - \lambda_2 t)\sigma_2^2 + m\sigma_3^2] d_1^{-1} I_{g-1}$$

$$W_{66} = [mk\sigma_1^2 + mk^{-1}(r - \lambda_1)\sigma_2^2 + m\sigma_3^2 + \sigma^2] d_2^{-1} I_{g(n-1)}$$

$$W_{77} = [\sigma^2 + m\sigma_3^2]^{-1} I_{bk-b-t+1}$$

$$W_{88} = \sigma^{-2} I_{bk(m-1)}$$

d_1 and d_2 are the same as those given in Singular and Semi-Regular GD-PBIB Designs, respectively.

Evaluating $P'(Y - \bar{\mu})$, we have

$$P'(Y - \bar{\mu}) = \begin{bmatrix} (bkm)^{1/2}(y_{\cdots} - \mu) \\ (km)^{-1/2}P'_{21}X'_1Y \\ (km)^{-1/2}P'_{22}X'_1Y \\ (km)^{-1/2}P'_{23}X'_1Y \\ (\frac{k}{\lambda_2 tm})^{1/2}P'_{31}A'Y \\ (mv)^{-1/2}P'_{32}A'Y \\ m^{-1/2}P'_4F'Y \\ P'_5Y \end{bmatrix}$$

Performing the multiplication $(Y-\bar{\mu})'PP'\Sigma^{-1}PP'(Y-\bar{\mu}) = q$ (say), we have

$$\begin{aligned} q &= (bkm)(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)^{-1}(y_{\cdots} - \mu)^2 \\ &+ [km(\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)]^{-1}Y'X_1P_{21}P'_{21}X'_1Y \\ &+ [kmd_1]^{-1}[\sigma^2 + mk^{-1}\lambda_2t\sigma_2^2 + m\sigma_3^2]Y'X_1P_{22}P'_{22}X'_1Y \\ &+ d_2^{-1}(km)^{-1}[mv\sigma_2^2 + m\sigma_3^2 + \sigma^2]Y'X_1P_{23}P'_{23}X'_1Y \\ &+ [m(\sigma^2 + m\sigma_3^2)]^{-1}Y'FP_4P'_4F'Y + \sigma^{-2}Y'P_5P'_5Y \\ &+ \frac{k}{\lambda_2 tm}[\sigma^2 + mk\sigma_1^2 + mk^{-1}(rk - \lambda_2t)\sigma_2^2 + m\sigma_3^2]d_1^{-1}Y'AP_{31}P'_{31}A'Y \\ &+ [mvd_2]^{-1}[\sigma^2 + mk\sigma_1^2 + mk^{-1}(r - \lambda_1)\sigma_2^2 + m\sigma_3^2]Y'AP_{32}P'_{32}A'Y \\ &- 2d_1^{-1}[m^2k^{-2}(rk - \lambda_2t)\lambda_2t]^{1/2}]\sigma_2^2Y'X_1P_{23}P'_{32}A'Y(m^2kv)^{-1/2} \end{aligned}$$

$$- 2d_2^{-1}[mk^{-1/2}[(r - \lambda_1)v]^{1/2}]\sigma_2^2 Y'X_1P_{23}P_{32}'A'Y(m^2kv)^{-1/2}$$

Define the ten statistics as follows:

$$\text{(III'')} \quad \begin{aligned}
s_1 &= y_{\bullet\cdot\cdot} \\
s_2 &= (km)^{-1}Y'X_1P_{21}P_{21}'X_1'Y \qquad \text{(not defined for } b = t) \\
s_3 &= (km)^{-1}Y'X_1P_{22}P_{22}'X_1'Y \\
s_4 &= (km)^{-1}Y'X_1P_{23}P_{23}'X_1'Y \\
s_5 &= \frac{k}{\lambda_2 tm} Y'AP_{31}P_{31}'A'Y \\
s_6 &= (mv)^{-1}Y'AP_{32}P_{32}'A'Y \\
s_7 &= m^{-1}Y'FP_4P_4'F'Y \\
s_8 &= Y'P_5P_5'Y \\
s_9 &= [k^{-2}(rk - \lambda_2 t)]^{1/2}Y'X_1P_{22}P_{31}'A'Y \\
s_{10} &= [k^{-2}(r - \lambda_1)]^{1/2}Y'X_1P_{23}P_{32}'A'Y
\end{aligned}$$

These ten statistics are sufficient for the parameters μ, σ^2, σ_1^2, σ_2^2, σ_3^2. This follows from [7], and we shall show that these ten statistics form a minimal set of sufficient statistics by following the same procedure as we had for the BIB designs.

$g(Y, \theta)$ may be written in the form

$$\text{(IV'')} \qquad g(Y, \theta) = P(\theta)Q(Y) \exp[-2^{-1} \sum_{i=1}^{k} v_i(\theta) u_i(Y)]$$

A necessary and sufficient condition for the set of sufficient statistics $u_i(Y)$ to be minimal for $g(Y, \theta)$ is that there exist no non-zero constants $a_1, a_2, \ldots, a_k$, c such that

$$\sum_{i=1}^{k} a_i v_i(\theta_i) = c \quad .$$

Thus it is enough to prove that for the following eleven functions,

$$v_1 = [\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2]^{-1}$$

$$v_2 = [\sigma^2 + mk\sigma_1^2 + m\sigma_3^2]^{-1}$$

$$v_3 = [\sigma^2 + mk^{-1}\lambda_2 t\sigma_2^2 + m\sigma_3^2]d_1^{-1}$$

$$v_4 = [\sigma^2 + mv\sigma_2^2 + m\sigma_3^2]d_2^{-1}$$

$$v_5 = -2\sigma_2^2 d_1^{-1}$$

$$v_6 = -2\sigma_2^2 d_2^{-1}$$

$$v_7 = (\sigma^2 + m\sigma_3^2)^{-1}$$

$$v_8 = \sigma^{-2}$$

$$v_9 = [\sigma^2 + mk\sigma_1^2 + mk^{-1}(r - \lambda_1)\sigma_2^2 + m\sigma_3^2]d_2^{-1}$$

$$v_{10} = [\sigma^2 + mk\sigma_1^2 + mk^{-1}(rk - \lambda_2 t)\sigma_2^2 + m\sigma_3^2]d_1^{-1}$$

$$v_{11} = v_1\mu \; .$$

(IV'') is not true for any $a_1, a_2, \ldots, a_{11}$ and c except when all vanish.

In (V'') it is clear that μ appears only in v_{11}. Since $v_1, v_2, \ldots, v_{10}$ are homogeneous functions of σ, σ_1, σ_2, and σ_3 of degree -2, the constant c can only be zero.

Effect the linear transformation,

$$x = \sigma^2$$

$$y = \sigma^2 + mk\sigma_1^2 + m\sigma_3^2$$

$$z = \sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2$$

$$u = mk\sigma_2^2 + m\sigma_3^2 + \sigma^2$$

$$w = \sigma^2 + m\sigma_3^2$$

The functions in (V'') become:

$$v_1 = xyuw[zw + \frac{\delta}{rk}(z - y)(y - w)]D^{-1}$$

$$v_2 = xzuw[zw + \frac{\delta}{rk}(z - y)(y - w)]D^{-1}$$

$$v_3 = xyzuw[w + \frac{\lambda_2 t}{rk}(z - y)]D_1^{-1}$$

$$v_4 = xyzuw[w + v(z - y)]D_2^{-1}$$

$$v_5 = -2xyzuw[\frac{z - y}{mr}]D_1^{-1}$$

$$v_6 = -2xyzuw[\frac{z - y}{mr}]D_2^{-1}$$

$$v_7 = xyzu[zw + \frac{\delta}{rk}(z - y)(y - w)]D^{-1}$$

$$v_8 = yzuw[zw + \frac{\delta}{rk}(z - y)(y - w)]D^{-1}$$

$$v_9 = xyzuw[y - \frac{(r - \lambda_1)}{rk}(z - y)]D_2^{-1}$$

$$v_{10} = xyzuw[y - \frac{rk - \lambda_2 t}{rk}(z - y)]D_1^{-1}$$

where D_1 and D_2 are the same as D defined for singular and semi-regular GD-PBIB Designs, respectively. D in this section can take value D_1 or D_2 as δ takes the values $\lambda_2 t$ or kv, respectively.

Observe that the term xy^2uw^2 appears only in v_1, xz^2uw^2 appears only in v_2, xy^2z^2u appears only in v_7, and yz^2uw^2 appears only in v_8. This implies v_1, v_2, v_7, and v_8 are mutually linearly independent of v_3, v_4, v_5, v_6, v_9, v_{10}. Now observe that after removing the common factor xyzuw in v_3, v_4, v_5, v_6, v_9, and v_{10}, these are also linearly independent, thereby proving that (IV'') is not true unless $a_1, a_2, \ldots, a_{11}$ and c vanish. This condition then implies the set of sufficient statistics defined in (IV'') are minimal.

Hence from the above discussions we have the following theorem and corollaries.

Theorem 4: Under the assumption of an Eisenhart Model II in a regular group divisible, partially balanced incomplete block design with two associate classes, there are ten statistics in a minimal set of sufficient statistics if $b > t$ and nine statistics in a minimal set if $b = t$.

Corollary 4.1. A set of minimal sufficient statistics for a regular, group divisible, partially balanced incomplete block design is as follows:

$s_1 = y\ldots$

$s_2 = (mk)^{-1} Y'X_1P_{21}P'_{21}X_1'Y$ if $b > t$, not defined if $b = t$.

$s_3 = (mk)^{-1} Y'X_1P_{22}P'_{22}X_1'Y$ or $[m^2k(rk - \lambda_2 t)]^{-1} Y'X_1N'P_{31}P'_{31}NX_1'Y$

$s_4 = (mk)^{-1} Y' X_1 P_{23} P'_{23} X'_1 Y$ or $[m^2 k(r - \lambda_1)]^{-1} Y' X_1 N' P_{32} P'_{32} N X'_1 Y$

$s_5 = \frac{k}{\lambda_2 tm} Y' A P_{31} P'_{31} A' Y$

$s_6 = (mv)^{-1} Y' A P_{32} P'_{32} A' Y$

$s_7 = m^{-1} Y' F P_4 P'_4 F' Y$

$s_8 = Y' P_5 P'_5 Y$

$s_9 = [k^{-2}(rk - \lambda_2 t)]^{1/2} Y' X_1 P_{22} P'_{31} A' Y$

$s_{10} = [k^{-2}(r - \lambda_1)] Y X_1 P_{23} P'_{32} A' Y$

Corollary 4.2. The distributions of the ten statistics as defined in Corollary 4.1 are as follows:

$s_1 \sim N[\mu, (bkm)^{-1}(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)]$

$s_2 \sim [\sigma^2 + mk\sigma_1^2 + m\sigma_3^2]\ \chi^2_{(b-t)}$ if $b > t$, not defined if $b = t$

$s_3 \sim [\sigma^2 + mk\sigma_1^2 + mk^{-1}(rk - \lambda_2 t)\sigma_2^2 + m\sigma_3^2]\ \chi^2_{(g-1)}$

$s_4 \sim [\sigma^2 + mk\sigma_1^2 + mk^{-1}(r - \lambda_1)\sigma_2^2 + m\sigma_3^2]\ \chi^2_{[g(n-1)]}$

$s_5 \sim [\sigma^2 + mk^{-1}\lambda_2 t\sigma_2^2]\ \chi^2_{(g-1)}$

$s_6 \sim [\sigma^2 + mv\sigma_2^2]\ \chi^2_{[g(n-1)]}$

$s_7 \sim [\sigma^2 + m\sigma_3^2]\ \chi^2_{[bk-b-t+1]}$

$s_8 \sim \sigma^2\ \chi^2_{bk(m-1)}$

$s_9 \sim \Sigma a_i\ \chi^2_{(1)}$ where a_i are the non-zero characteristic roots of $2^{-1}(A_1 + A_1')\Sigma$ where $A_1 = k^{-1}X_1N'P_{31}P_{31}'A'$.

$s_{10} \sim \Sigma b_i \chi^2_{(1)}$ where b_i are the non-zero characteristic roots of $2^{-1}(B_1 + B_1')\Sigma$ where $B_1 = k^{-1}X_1N'P_{32}P_{32}'A'$.

For proof see Appendix III.

Corollary 4.3. The ten statistics as defined in Corollary 4.1 are pairwise independent except for the pairs (s_3, s_5), (s_3, s_9), (s_4, s_6), (s_4, s_{10}), (s_5, s_9), and (s_6, s_{10}).

For proof see Appendix IV.

Corollary 4.3. The expectations of the ten statistics as defined in Corollary 4.1 are as follows:

$E(s_1) = \mu$

$E(s_2) = (b - t)(\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)$ if $b > t$, not defined if $b = t$.

$E(s_3) = (g - 1)[\sigma^2 + mk\sigma_1^2 + mk^{-1}(rk - \lambda_2 t)\sigma_2^2 + m\sigma_3^2]$

$E(s_4) = g(n - 1)[\sigma^2 + mk\sigma_1^2 + mk^{-1}(r - \lambda_1)\sigma_2^2 + m\sigma_3^2]$

$E(s_5) = (g - 1)[\sigma^2 + mk^{-1}\lambda_2 t\sigma_2^2]$

$E(s_6) = g(n - 1)[\sigma^2 + mv\sigma_2^2]$

$E(s_7) = (bk - b - t + 1)[\sigma^2 + m\sigma_3^2]$

$E(s_8) = bk(m - 1)\sigma^2$

$E(s_9) = m^2k^{-2}\lambda_2 t(rk - \lambda_2 t)\sigma^2(g - 1)$

$E(s_{10}) = g(n - 1)m^3(r - \lambda_1)(rk - r + \lambda_1)k^{-2}\sigma_2^2$

For proof of this corollary see Appendix III.

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APPENDIX I

To show that $P'P = I$, let $P'P = (p_{ij})$ $i, j = 1, \ldots, 6$, (BIB).

<u>Diagonal Terms</u>

$$p_{11} = (bkm)^{-1/2} J^1_{bkm} J^{bkm}_1 (bkm)^{-1/2} = (bkm)^{-1}(bkm) = 1$$

$$p_{22} = (km)^{-1/2} P'_{21} X'_1 X_1 P_{21} (km)^{-1/2} = (km)^{-1} km P'_{21} P_{21} = I_{b-t}$$

$$p_{33} = (km)^{-1/2} P'_{22} X'_1 X_1 P_{22} (km)^{-1/2} = (km)^{-1} km P'_{22} P_{22} = I_{t-1}$$

$$p_{44} = \left(\frac{k}{\lambda tm}\right)^{1/2} P'_3 A' A P_3 \left(\frac{k}{\lambda tm}\right)^{1/2} = \frac{k}{\lambda tm} P'_3 [X'_2 X_2 - m^{-1} k^{-1} NN'] P_3$$

$$= \frac{k}{\lambda tm} \left[mr I_{t-1} - m \frac{(r-\lambda)}{k} I_{t-1} \right]$$

$$= \frac{k}{\lambda tm} \cdot \frac{\lambda tm}{k} I_{t-1} = I_{t-1}$$

$$p_{55} = m^{-1/2} P'_4 F' F P_4 m^{-1/2} = I_{bk-b-t+1}$$

$$p_{66} = P'_5 P_5 = I_{bk(m-1)}$$

<u>Off-Diagonal Terms</u>

$$p_{12} = (bkm)^{-1/2} J^1_{bkm} X_1 P_{21} (km)^{-1/2} = c_1 J^1_b P_{21} = \phi$$

$$p_{13} = (bkm)^{-1/2} J^1_{bkm} X_1 P_{22} (km)^{-1/2} = c_2 J^1_b P_{22} = \phi$$

$$p_{14} = (bkm)^{-1/2} J^1_{bkm} A P_3 \left[\frac{k}{\lambda tm}\right]^{1/2} = c_3 J^1_t P_3 = \phi$$

$$p_{15} = (bkm)^{-1/2} J^1_{bkm} F P_4 m^{-1/2}$$

$$= c_4 J^1_{bkm} [X_3 - m^{-1}k^{-1}X_1X_1'X_3 - m^{-1}\lambda^{-1}t^{-1}kA(L - m^{-1}k^{-1}NM)$$

$$= c_4 [J^1_{bkm}X_3 - J^1_{bkm}X_3 - J^1_{bkm}A(L - m^{-1}k^{-1}NM)m^{-1}\lambda^{-1}t^{-1}k]$$

$$= \phi$$

$$p_{16} = (bkm)^{-1/2} J^1_{bkm} P_5 = \phi$$

$$p_{23} = (km)^{-1/2} P_{21}' X_1' X_1 P_{22} (km)^{-1/2} = (km)^{-1}(km) P_{21}' P_{22} = \phi$$

$$p_{24} = (km)^{-1/2} P_{21}' X_1' F P_4 (m)^{-1/2} = \phi$$

$$p_{26} = (km)^{-1/2} P_{21}' X_1' P_5 = \phi$$

$$p_{34} = (km)^{-1/2} P_{22}' X_1' A P_3 \left(\frac{k}{\lambda tm}\right)^{1/2} = \phi$$

$$p_{35} = (km)^{-1/2} P_{22}' X_1' F P_4 (m)^{-1/2} = \phi$$

$$p_{36} = (km)^{-1/2} P_{22}' X_1' P_5 (m)^{-1/2} = \phi$$

$$p_{45} = \left(\frac{k}{\lambda tm}\right)^{1/2} P_3' A' F P_4 (m)^{-1/2}$$

$$= \left(\frac{k}{\lambda tm}\right)^{1/2} m^{-1/2} P_3' [X_2' - m^{-1}k^{-1}NX_1'][X_3 - m^{-1}k^{-1}X_1M - m^{-1}\lambda^{-1}t^{-1}k(X_2$$

$$- m^{-1}k^{-1}X_1N')(L - m^{-1}k^{-1}NM)]P_4$$

$$= c_5 P_3' [X_2'X_3 - m^{-1}k^{-1}X_2'X_1N') - m^{-1}\lambda^{-1}t^{-1}k(X_2'X_2 - m^{-1}k^{-1}X_2'X_1N')$$

$$(L - m^{-1}k^{-1}NM)P_4$$

$$= c_5 P_3' (L - m^{-1}k^{-1}NM - m^{-1}\lambda^{-1}t^{-1}k(mrI_t - m^{-1}k^{-1}NN')(L - m^{-1}k^{-1}NM)]P_4$$

$$= c_5 P_3'[L - m^{-1}k^{-1}NM - m^{-1}\lambda^{-1}t^{-1}k(mrI_t - m^{-1}k^{-1}m^2(r-\lambda)I_t$$

$$- m^{-1}k^{-1}m^2\lambda J_t^t(L - m^{-1}k^{-1}NM)]P_4$$

$$= c_5 P_3'[L - m^{-1}k^{-1}NM - \frac{k}{\lambda tm}(\frac{\lambda tm}{k}I_t - \frac{m\lambda}{k}J_t^t)(L - m^{-1}k^{-1}NM)]P_4$$

$$= c_5 P_3'[L - m^{-1}k^{-1}NM - L + \frac{k}{\lambda tm}\frac{m\lambda}{k}J_t^t L + m^{-1}k^{-1}NM$$

$$- \frac{k}{\lambda tm}\frac{m\lambda}{k}m^{-1}k^{-1}NM]P_4$$

$$= c_5 P_3'[\frac{1}{t}J_t^t L - \frac{1}{t}J_t^t L]P_4$$

$$= \phi$$

$$p_{46} = (\frac{k}{\lambda tm})^{1/2}P_3'A'P_5 = \phi$$

$$p_{56} = m^{-1/2}P_4'F'P_5 = \phi$$

Hence $PP' = I_{bkm}$ and therefore P' is an orthogonal matrix.

To show P' is an orthogonal matrix for each of the three types of GD-PBIB designs, let P' be transferred to the following form after combining the partitions of Q_2 and Q_3.

$$P = \begin{bmatrix} (bkm)^{-1/2}J_{bkm}^1 \\ (mk)^{-1/2}P_2'X_1' \\ c_3P_3'A' \\ m^{-1/2}P_4'F' \\ P_5' \end{bmatrix}$$

where,

(i) P_2' is b-1 x b set of orthogonal vectors from an orthogonal matrix Q_2 corresponding to the characteristic roots of N'N other than m^2rk.

(ii) P_3', t-1 x t set of orthogonal vectors from an orthogonal matrix Q_3 corresponding to the characteristic roots of NN' other than m^2rk.

(iii)

$$c_3 = \begin{cases} \begin{bmatrix} (\frac{k}{\lambda_2 tm})^{1/2} I_{g-1} & \phi \\ \phi & (mr)^{-1/2} I_{g(n-1)} \end{bmatrix} & \text{for S designs} \\ \begin{bmatrix} (mr)^{-1/2} I_{g-1} & \phi \\ \phi & (mv)^{-1/2} I_{g(n-1)} \end{bmatrix} & \text{for SR designs} \\ \begin{bmatrix} (\frac{k}{\lambda_2 tm})^{1/2} I_{g-1} & \phi \\ \phi & (mv)^{-1/2} I_{g(n-1)} \end{bmatrix} & \text{for R designs .} \end{cases}$$

Let $P'P = (p_{ij})$, i, j = 1, 2, . . . , 5.

$$p_{11} = (bkm)^{-1/2} J^1_{bkm} J_1^{bkm} (bkm)^{-1} = (bkm)^{-1}(bkm) = 1$$

$$p_{12} = (bkm)^{-1/2} J^1_{bkm} X_1 P_2 (mk)^{-1/2} = \text{const. } J^1_b P_2 = \phi$$

$$p_{13} = (bkm)^{-1/2} J^1_{bkm} A P_3 C_3 = \text{const. } J^1_t P_3 = \phi$$

$$p_{14} = (bkm)^{-1/2} J^1_{bkm} F P_4 m^{-1/2}$$

$$= \text{const. } J^1_{bkm}[X_3 - m^{-1}k^{-1} X_1 X_1' X_3 - \frac{k}{(rk-r+\lambda_1)m}(AA'X_3) - \frac{k[\lambda_1 - \lambda_2]}{\lambda_2 t(rk-r+\lambda_1)m} A[B_0 + B_1]'A'X_3]$$

$$= \text{const.}\,[J^1_{bkm}X_3 - J^1_{bkm}X_3 - \frac{k}{(rk-r+\lambda_1)m} J^1_{bkm}AA'X_3 - \frac{k[\lambda_1 - \lambda_2]}{\lambda_2 t(rk-r+\lambda_1)m} J^1_{bkm}A[B_0 + B_1]'A'X_3]$$

$$= \phi$$

$$p_{15} = (bkm)^{-1/2}J^1_{bkm}P_5 = \phi$$

$$p_{22} = (mk)^{-1/2}P_2'X_1'X_1P_2(mk)^{-1/2} = (mk)^{-1}(mk)P_2'P_2 = I_{b-1}$$

$$p_{23} = (mk)^{-1/2}P_2'X_1'AP_3C_3 = \phi$$

$$p_{24} = (mk)^{-1/2}P_2'X_1'FP_4m^{-1/2} = \phi$$

$$p_{25} = (mk)^{-1/2}P_2'X_1'P_5 = \phi$$

$$p_{33} = C_3P_3'A'AP_3C_3 = I_{t-1}$$

$p_{34} = C_3P_3'A'FP_4m^{-1/2} = \phi$ This follows from the fact $X_1'F = 0$, $X_2'F = 0$

$$p_{35} = C_3P_3'A'P_5 = \phi$$

$$p_{44} = m^{-1/2}P_4'F'FP_4m^{-1/2} = P_4m^{-1}F'FP_4 = I_{bk-b-t+1}$$

$$p_{45} = m^{-1/2}P_4'F'P_5 = \phi$$

$$p_{55} = P_5'P_5' = I_{bk(m-1)}$$

Hence $PP' = I_{bkm}$. Therefore P' is an orthogonal matrix.

APPENDIX II

<u>The derivation of $P'\not\Sigma P$:</u> Letting $P'\not\Sigma P = (A_{ij})$ $i, j = 1, 2, \ldots, 6$, we shall then have for each i and j the following.

(1) $$A_{11} = (bkm)^{-1/2} J^1_{bkm} \not\Sigma J^{bkm}_1 (bkm)^{-1/2}$$
$$= (bkm)^{-1} J^1_{bkm}(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I)J_1^{bkm}$$
$$= (bkm)^{-1}(bk^2m^2\sigma_1^2 + tr^2m^2\sigma_2^2 + bkm^2\sigma_3^2 + bkm\sigma^2)$$
$$= (\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)$$

(2) $$A_{12} = (km)^{-1/2}(bkm)^{-1/2} J^1_{bkm}\not\Sigma X_1 P_{21}$$
$$= c_0 J^1_{bkm}(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I)X_1 P_{21}$$
$$= c_0(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)J^1_b P_{21} = \phi$$

(3) $$A_{13} = (km)^{-1/2}(bkm)^{-1/2} J^1_{bkm}\not\Sigma X_1 P_{22}$$
$$= c_1(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)J^1_b P_{22} = \phi$$

(4) $$A_{14} = \left(\frac{k}{\lambda tm}\right)^{1/2}(bkm)^{-1/2} J^1_{bkm}\not\Sigma A P_3$$
$$= c_2(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)J^1_{bkm} A P_3 = \phi$$

(5) $$A_{15} = m^{-1/2}(bkm)^{-1/2} J^1_{bkm}\not\Sigma F P_4$$
$$= m^{-1/2}(bkm)^{-1/2} J^1_{bkm}\not\Sigma[X_3 - m^{-1}k^{-1}X_1M - m^{-1}\lambda^{-1}t^{-1}k(X_2$$
$$- m^{-1}k^{-1}X_1N')(L - m^{-1}k^{-1}NM)$$

$$= c_3[m^2 J_{bk}^1 - m^2 J_{bk}^1] - m^{-1}\lambda^{-1}t^{-1}k[mrJ_t^1 - mrJ_t^1][L - m^{-1}k^{-1}NM]$$

$$= \phi$$

(6) $A_{16} = (bkm)^{-1/2} J_{bkm}^1 P_5 = \phi$

(7) $A_{22} = (km)^{-1} P_{21}' X_1' \Sigma X_1 P_{21}$

$$= (mk)^{-1} P_{21}' X_1' [X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + X_3 X_3' \sigma_3^2 + \sigma^2 I] X_1 P_{21}$$

$$= (mk)^{-1} P_{21}' [m^2 k^2 \sigma_1^2 I_b + NN' \sigma_2^2 + MM' \sigma_3^2 + mk\sigma^2 I] P_{21}$$

$$= (mk)^{-1} P_{21}' [m^2 k^2 \sigma_1^2 I_b + NN' \sigma_2^2 + m^2 k \sigma_3^2 I_b + mk\sigma^2 I_b] P_{21}$$

$$= (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2) I_{b-t}$$

(8) $A_{23} = (mk)^{-1} P_{21}' X_1' \Sigma X_1 P_{22}$

$$= (mk)^{-1} P_{21}' X_1' [X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + X_3 X_3' \sigma_3^2 + \sigma^2 I] X_1 P_{22}$$

$$= (mk)^{-1} P_{21}' (m^2 k^2 \sigma_1^2 I_b + N'N \sigma_2^2 + m^2 k \sigma_3^2 I_b + mk\sigma^2 I_b] P_{22}$$

$$= \phi$$

(9) $A_{24} = (km)^{-1/2} \left(\frac{k}{\lambda t m}\right)^{1/2} P_{21}' X_1' \Sigma A P_3$

$$= c_4 P_{21}' X_1' [X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + X_3 X_3' \sigma_3^2 + \sigma^2 I] A P_3$$

$$= c_4 P_{21}' [N' X_2' \sigma_2^2 + M X_3' \sigma_3^2] A P_3$$

$$= c_4 P_{21}' N' [rmI_t - m^{-1}k^{-1} NN') P_3 \sigma_2^2 + c_4 P_{21}' M(L' - m^{-1}k^{-1} M'N') \sigma_3^2 P_3$$

$$= c_5 P_{21}' N'N P_{22} \sigma_2^2 + c_4 P_{21}' (ML' - m^{-1}k^{-1} MM'N') P_3 \sigma_3^2$$

$$= \phi$$

(10) $A_{25} = (km)^{-1/2}m^{-1/2}P_{21}'X_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]FP_4$

$= c_6P_{21}'X_1'X_3X_3'[X_3 - m^{-1}k^{-1}X_1M - m^{-1}\lambda^{-1}t^{-1}k(AA'X_3)]P_4$

$= \phi$

(11) $A_{26} = (km)^{-1/2}P_{21}'X_1'\Sigma P_5 = \phi$

(12) $A_{33} = [km^3(r-\lambda)]^{-1}P_3'NX_1'\Sigma X_1N'P_3$

$= [km^3(r-\lambda)]^{-1}[P_3'NX_1'X_1X_1'X_1N'P_3\sigma^2 + P_3'NX_1'X_2X_2'X_1N'P_3\sigma_2^2$

$+ P_3'NX_1'X_3X_3'X_1N'P_3\sigma_3^2 + P_3'NX_1'X_1NP_3'\sigma^2]$

$= [km^3(r-\lambda)]^{-1}[m^2k^2m^2(r-\lambda)I_{t-1}\sigma_1^2 + m^4(r-\lambda)^2I_{t-1}\sigma_2^2$

$+ m^2km^2(r-\lambda)I_{t-1}\sigma_3^2 + mkm^2(r-\lambda)I_{t-1}\sigma^2]$

$= [\sigma^2 + mk\sigma_1^2 + mk^{-1}(r-\lambda)\sigma_2^2 + m\sigma_3^2]I_{t-1}$

(13) $A_{34} = [km^3(r-\lambda)]^{-1/2}[\frac{k}{\lambda tm}]^{1/2}P_3'NX_1'\Sigma AP_3$

$= c_7P_3'NX_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2]AP_3$

$= c_7[P_3'NX_1'X_1X_1'\sigma_1^2AP_3\sigma_1^2 + P_3'NX_1'X_2X_2'AP_3\sigma_2^2 + P_3'NX_1'X_3X_3'AP_3\sigma_3^2$

$+ P_3'NX_1'AP_3\sigma^2]$

$= [m^2k^{-2}\lambda t(r-\lambda)]^{1/2}\sigma_2^2I_{t-1}$

(14) $A_{35} = [km^3(r-\lambda)]^{1/2}m^{-1/2}P_3'NX_1'\Sigma FP_4$

$= c_8P_3'NX_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2][X_3 - m^{-1}k^{-1}X_1M$

$- m^{-1}\lambda^{-1}t^{-1}k(X_2 - m^{-1}k^{-1}X_1N')(L - m^{-1}k^{-1}NM)]P_4$

$$= c_8 P_3' N X_1' X_3 X_3' [X_3 - m^{-1}k^{-1}X_1 M - m^{-1}\lambda^{-1}t^{-1}k(X_2 - m^{-1}k^{-1}X_1 N')$$

$$(L - m^{-1}k^{-1}NM)] P_4$$

$$= c_8 P_3' NM [mI_{bk} - m^{-1}k^{-1}M'M - m^{-1}\lambda^{-1}t^{-1}k(L' - m^{-1}k^{-1}M'N')$$

$$(L - m^{-1}k^{-1}NM)] P_4$$

$$= c_8 P_3' [mNM - mNM - m^{-1}\lambda^{-1}t^{-1}k(NML - NML)] P_4$$

$$= \phi$$

(15) $A_{36} = [km^3(r-\lambda)]^{1/2} P_3' N X_1' \Sigma P_5 = \phi$

(16) $A_{44} = [\frac{k}{\lambda tm}] P_3' A' \Sigma A P_3$

$$= \frac{k}{\lambda tm} P_3' A' [X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + X_3 X_3' \sigma_3^2 + \sigma^2 I] A P_3$$

$$= \frac{k}{\lambda tm} [P_3' A' X_2 X_2' A P_3] \sigma_2^2 + [P_3' A' X_3 X_3' A P_3] \sigma_3^2 + \sigma^2 I_{t-1}$$

$$= \frac{k}{\lambda tm} P_3' [\lambda^2 k^{-2} m^2 (tI_t - J_t^t)(tI_t - J_t^t)] P_3 \sigma_2^2$$

$$+ P_3' [L - m^{-1}k^{-1}NM][L' - m^{-1}k^{-1}M'N'] P_3 \sigma_3^2 + \sigma^2 I_{t-1}$$

$$= \frac{k}{\lambda tm} \Big\{ P_3' [\lambda^2 k^{-2} m^2 t (tI_t - J_t^t)] P_3 \sigma_2^2 + P_3' [LL' - m^{-1}k^{-1}LM'N'$$

$$- m^{-1}k^{-1}NML' + m^{-2}k^{-2}NMM'N'] \sigma_3^2 P_3 \Big\} + \sigma^2 I_{t-1}$$

$$= k^{-1}\lambda m t \sigma_2^2 I_{t-1} + P_3' [m^2 r I_t - k^{-1}NN' - k^{-1}NN' + k^{-1}NN'] P_3 \sigma_3^2 \frac{k}{\lambda tm}$$

$$+ \sigma^2 I_{t-1}$$

$$= [\sigma^2 + k^{-1}\lambda m t \sigma_2^2 + m\sigma_3^2] I_{t-1}$$

$$(17)\quad A_{45} = \left(\frac{k}{\lambda tm}\right)^{1/2} m^{-1/2} P_3' A' \Sigma F P_4$$

$$= c_9 P_3' A'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2)[X_3 - m^{-1}k^{-1}X_1M$$

$$- m^{-1}\lambda^{-1}t^{-1}k(X_2 - m^{-1}k^{-1}X_1N')(L - m^{-1}k^{-1}NM)]P_4$$

$$= c_9 P_3'[L - m^{-1}k^{-1}NM][mI_{bk} - m^{-1}k^{-1}M'M - m^{-1}\lambda^{-1}t^{-1}k(L'$$

$$- m^{-1}k^{-1}M'N')(L - m^{-1}k^{-1}NM)]P_4$$

$$= c_9 P_3'[mL - m^{-1}k^{-1}LM'M - m^{-1}\lambda^{-1}t^{-1}k(L'L - m^{-1}k^{-1}L'NM$$

$$- m^{-1}k^{-1}M'NL + m^{-2}k^{-2}M'N'NM) - m^{-1}k^{-1}NM\{mI_{bk}$$

$$- m^{-1}k^{-1}M'M - m^{-1}\lambda^{-1}t^{-1}k(L' - m^{-1}k^{-1}M'N')(L - m^{-1}k^{-1}NM)\}]P_4$$

$$= c_9 P_3'[mL - m^{-1}k^{-1}LM'M - m^{-1}\lambda^{-1}t^{-1}k\{m^2rL - m^{-1}k^{-1}m^2rNM$$

$$- m^{-1}k^{-1}LM'N'L + m^{-2}k^{-2}LM'N'NM\} - k^{-1}NM$$

$$+ m^{-2}k^{-2}NMM'M + m^{-1}k^{-1}\lambda^{-1}t^{-1}k(NML'L - m^{-1}k^{-1}NML'NM$$

$$- m^{-1}k^{-1}NMM'N'L + m^{-2}k^{-2}NMM'N'NM\ ]P_4$$

$$= c_9 P_3'[mL - k^{-1}NM - m^{-1}\lambda^{-1}t^{-1}k(m^2rL - mk^{-1}rNM - k^{-1}NN'L$$

$$+ m^{-1}k^{-2}NN'NM) - k^{-1}NM + k^{-1}NM + m^{-1}\lambda^{-1}t^{-1}k(mNN'L$$

$$- k^{-1}NN'NM - mNN'L + k^{-1}NN'NM)]P_4$$

$$= c_9 P_3'[mL - k^{-1}NM - m^{-1}\lambda^{-1}t^{-1}k(m^2rL - mk^{-1}rNM - k^{-1}m^2[(r-\lambda)I_t$$

$$+ \lambda J_t^t]L + m^{-1}k^{-2}m^2[(r-\lambda)I_t + \lambda J_t^t]NM]P_4$$

$$= c_9 P_3'[mL - k^{-1}NM - m^{-1}\lambda^{-1}t^{-1}k(m^2 rL - mk^{-1}rNM$$

$$- k^{-1}m^2(r-\lambda)L + mk^{-2}(r-\lambda)NM)]P_4$$

$$= c_9 P_3'[mL - k^{-1}NM - m^{-1}\lambda^{-1}t^{-1}km^2(r - \frac{r-\lambda}{k})L$$

$$+ m^{-1}\lambda^{-1}t^{-1}kmk^{-1}(r - \frac{r-\lambda}{k})NM]P_4$$

$$= c_9 P_3'[mL - k^{-1}NM - mL + k^{-1}NM]$$

$$= \phi$$

(18) $$A_{55} = m^{-1}P_4'F'\Sigma FP_4$$

$$= m^{-1}P_4'F'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I)FP_4$$

$$= m^{-1}P_4'F'X_3X_3'FP_4\sigma_3^2 + m^{-1}P_4'F'FP_4\sigma^2$$

$$= (\sigma^2 + m\sigma_3^2)I_{bk-b-t+1}$$

$$[P_4'm^{-1}F'X_3X_3'FP_4\sigma_3^2 = P_4'm^{-1}mF'X_3P_4\sigma_3^2$$

$$= P_4'F'X_3P_4\sigma_3^2$$

$$= mP_4'm^{-1}F'X_3P_4\sigma_3^2$$

$$= m\sigma_3^2 I_{bk-b-t+1}]$$

(19) $$A_{56} = m^{-1}rP_4'F'\Sigma P_5$$

$$= m^{-1/2}P_4'F'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I)P_5 = \phi$$

(20) $$A_{66} = P_5'\Sigma P_5 = P_5'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I)P_5$$

$$= \sigma^2 I_{bk(m-1)}$$

The derivation of $P'\Sigma P$ for S-GD-PBIB Designs: Letting $P'\Sigma P = (A_{ij})$ i, j = 1, 2, . . . , 7, we shall then have for each i and j the following.

(1) $$A_{11} = (bkm)^{-1/2} J^1_{bkm} \Sigma J^{bkm}_1 (bkm)^{-1/2}$$

$$= (bkm)^{-1} J^1_{bkm} [X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I] J^{bkm}_I$$

$$= (bkm)^{-1} [bkm^2\sigma_1^2 + tr^2m^2\sigma_2^2 + bkm^2\sigma_3^2 + bkm\sigma^2]$$

$$= (\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)$$

(2) $$A_{12} = (mk)^{-1/2}(bkm)^{-1/2} J^1_{bkm} \Sigma X_1 P_{21}$$

$$= c_0 J^1_{bkm}(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I) X_1 P_{21}$$

$$= c_0(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2) J^1_b P_{21}$$

$$= \phi$$

(3) $$A_{13} = (km)^{-1/2}(bkm)^{-1/2} J^1_{bkm} \Sigma X_1 P_{22}$$

$$= c_1(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2) J^1_b P_{22}$$

$$= \phi$$

(4) $$A_{14} = \left(\frac{k}{\lambda_2 tm}\right)^{1/2}(bkm)^{-1/2} J^1_{bkm} \Sigma A P_{31}$$

$$= c_2 J^1_{bkm} [X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I] A P_{31}$$

$$= c_2(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2) J^1_{bkm} A P_{31}$$

$$= \phi$$

(5) $A_{15} = (mr)^{-1/2}(bkm)^{-1/2}J^1_{bkm}\not\Sigma AP_{32}$

$= c_3[\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2]J^1_{bkm}AP_{32}$

$= \phi$

(6) $A_{16} = (bkm)^{-1/2}m^{-1/2}J^1_{bkm}\not\Sigma FP_4$

$= c_4J^1_{bkm}[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2I]FP_4$

$= c_4[\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2]J^1_{bkm}FP_4$

$= \phi$

(7) $A_{17} = (bkm)^{-1/2}J^1_{bkm}\not\Sigma P_5' = \phi$

(8) $A_{22} = (mk)^{-1}P_{21}'X_1'\not\Sigma X_1P_{21}$

$= (mk)^{-1}P_{21}'X_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2I]X_1P_{21}$

$= (mk)^{-1}[m^2k^2\sigma_1^2 + m^2k\sigma_3^2 + mk\sigma^2]I_{c_0 + c_1'}$

$= [mk\sigma_1^2 + m\sigma_3^2 + \sigma^2]I_{c_0 + c_1'}$

(9) $A_{23} = (mk)^{-1}P_{21}'X_1'\not\Sigma X_1P_{22}$

$= (mk)^{-1}P_{21}'X_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2I]X_1P_{22}$

$= (mk)^{-1}P_{21}'[m^2k^2\sigma_1^2I_b + NN'\sigma_2^2 + MM'\sigma_3^2 + mk\sigma^2I_b]P_{22}$

$= \phi$

(10) $A_{24} = (mk)^{-1/2}P_{21}'X_1'\not\Sigma AP_{31}(\frac{k}{\lambda_2 tm})^{1/2}$

$= c_0P_{21}'X_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2I]AP_{31}$

$$= c_0 P_{21}'[N'X_2'\sigma_2^2 + MX_3'\sigma_3^2]AP_{31}$$

$$= c_0 P_{21}'[N'X_2'\sigma_2^2 + MX_3'\sigma_3^2][X_2 - m^{-1}k^{-1}X_1N']P_{31}$$

$$= c_0 P_{21}'N'[rmI_t - m^{-1}k^{-1}NN']\sigma_2^2 P_{31} + c_0 P_{21}'M(L' - m^{-1}k^{-1}M'N')P_{31}$$

$$= \phi$$

(11) $$A_{25} = (mk)^{-1/2} P_{21}'X_1'\Sigma AP_{32}(mr)^{-1/2}$$

$$= c_1 P_{21}'X_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]AP_{32}$$

$$= \phi$$

(12) $$A_{26} = (mk)^{-1/2} P_{21}'X_1'\Sigma FP_4 m^{-1/2}$$

$$= c_2 P_{21}'X_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]FP_4$$

$$= c_2 P_{21}'M[X_3'X_4 - m^{-1}k^{-1}M'M - \frac{k}{(rk-r+\lambda_1)m}(L' - m^{-1}k^{-1}M'N')$$
$$(L - m^{-1}k^{-1}NM) - \frac{k(\lambda_1 - \lambda_2)}{(rk-r+\lambda_1)\lambda_2 tm}(L' - m^{-1}k^{-1}M'N')$$
$$(B_0 + B_1)'(L - m^{-1}k^{-1}NM)]P_4$$

$$= \phi$$

(13) $$A_{27} = (mk)^{-1/2} P_{21}'X_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]P_5$$

$$= \phi$$

(13) $$A_{33} = (mk)^{-1/2} P_{22}'X_1'\Sigma X_1 P_{22}(mk)^{-1/2}$$

$$= (mk)^{-1} P_{22}'X_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]X_1 P_{22}$$

$$= (mk)^{-1}P_{22}'[m^2k^2\sigma_1^2 I_b + N'N\sigma_2^2 + m^2k\sigma_3^2 I_b + mk\sigma^2 I_b]P_{22}$$

$$= [mk\sigma_1^2 + m\sigma_3^2 + \sigma^2]I_{g-1} + P_{22}'N'NP_{22}\sigma_2^2(mk)^{-1}$$

$$= [mk\sigma_1^2 + m\sigma_3^2 + \sigma^2]I_{g-1} + m^2(rk - \lambda_2 t)(mk)^{-1}I_{g-1}$$

$$= [\sigma^2 + mk\sigma_1^2 + mk^{-1}(rk - \lambda_2 t)\sigma_2^2 + m\sigma_3^2]I_{g-1}$$

(14) $$A_{34} = (mk)^{-1/2}P_{22}'X_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]AP_{31}\left(\frac{k}{\lambda_2 tm}\right)^{1/2}$$

$$= (mk)^{-1/2}\left(\frac{k}{\lambda_2 tm}\right)^{1/2}P_{22}'N'[rmI - m^{-1}k^{-1}NN']\sigma_2^2 P_{31}$$

$$= (mk)^{-1/2}\left(\frac{k}{\lambda_2 tm}\right)^{1/2}[(rk - \lambda_2 t)m^2]^{-1/2}P_{31}'NN'[rmI - m^{-1}k^{-1}$$

$$NN']\sigma_2^2 P_{31}$$

$$= (mk)^{-1/2}\left(\frac{k}{\lambda_2 tm}\right)^{1/2}[(rk - \lambda_2 t)m^2]^{-1/2}\sigma_2^2[rmm^2(rk - \lambda_2 t)$$

$$- m^{-1}k^{-1}m^4(rk - \lambda_2 t)^2]I_{g-1}$$

$$= \frac{k^{-1}}{\lambda_2 t}[rk - \lambda_2 t]^{1/2}m[rk - rk + \lambda_2 t]\sigma_2^2 I_{g-1}$$

$$= mk^{-1}[(rk - \lambda_2 t)(\lambda_2 t)]^{1/2}\sigma_2^2 I_{g-1}$$

(15) $$A_{35} = (mk)^{-1/2}P_{22}'X_1'\Sigma A'P_{32}(mr)^{-1/2}$$

$$= (mk)^{-1/2}(mr)^{-1/2}P_{22}'X_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]A'P_{32}$$

$$= (mk)^{-1/2}(mr)^{-1/2}P_{22}'N'[rmI - m^{-1}k^{-1}NN']\sigma_2^2 P_{32}$$

$$= (mk)^{-1/2}(mr)^{-1/2}[(rk - \lambda_2 t)m^2]^{-1/2}P_{31}'NN'[rmI - m^{-1}k^{-1}NN']$$

$$\sigma_2^2 P_{32} = \phi$$

$$(16)\quad A_{36} = (mk)^{-1/2}P_{22}'X_1'\Sigma F P_4(m)^{-1/2}$$

$$= c_3 P_{22}'X_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]FP_4$$

$$= \phi$$

$$(17)\quad A_{37} = (mk)^{-1/2}P_{22}'X_1'\Sigma P_5$$

$$= \phi$$

$$(18)\quad A_{43} = A_{34}$$

$$(19)\quad A_{44} = \left(\frac{k}{\lambda_2 tm}\right)^{1/2}P_{31}'A'\Sigma AP_{31}\left(\frac{k}{\lambda_2 tm}\right)^{1/2}$$

$$= \left(\frac{k}{\lambda_2 tm}\right)P_{31}'A'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]AP_{31}$$

$$= \left(\frac{k}{\lambda_2 tm}\right)P_{31}'[(mrI - m^{-1}k^{-1}NN')^2\sigma_2^2 + (L - m^{-1}k^{-1}NM)(L'$$

$$- m^{-1}k^{-1}M'N')\sigma_3^2 + (mrI - m^{-1}k^{-1}NN')\sigma^2]P_{31}$$

$$= \frac{k}{\lambda_2 tm}P_{31}'[m^2k^{-2}\lambda_2^2t^2\sigma_2^2 + m^2k^{-1}\lambda_2 t\sigma_3^2 + m\lambda_2 tk^{-1}\sigma^2]I_{g-1}$$

$$= [mk^{-1}\lambda_2 t\sigma_2^2 + m\sigma_3^2 + \sigma^2]I_{g-1}$$

$$(20)\quad A_{45} = \left(\frac{k}{\lambda_2 tm}\right)^{1/2}P_{31}'A'\Sigma AP_{32}(mr)^{-1/2}$$

$$= c_4 P_{31}'A'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]AP_{32}$$

$$= c_4 P_{31}'[(mrI - m^{-1}k^{-1}NN')^2\sigma_2^2 + m(mrI - m^{-1}k^{-1}NN')\sigma_3^2$$

$$+ (mrI - m^{-1}k^{-1}NN')\sigma^2]P_{32}$$

$$= \phi$$

$$(21)\quad A_{46} = \left(\frac{k}{\lambda_2 tm}\right)^{1/2} P_{31}' A' \not\Sigma F P_4 (m)^{-1/2}$$

$$= \left(\frac{k}{\lambda_2 tm}\right)^{1/2} P_{31}' A' [X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + X_3 X_3' \sigma_3^2 + \sigma^2 I] F P_4 m^{-1/2}$$

$$= \phi$$

$$(22)\quad A_{47} = \left(\frac{k}{\lambda_2 tm}\right)^{1/2} P_{31}' A' \not\Sigma P_5$$

$$= \phi$$

$$(23)\quad A_{55} = (mr)^{-1/2} P_{32}' A' \not\Sigma A P_{32} (mr)^{-1/2}$$

$$= (mr)^{-1} P_{32}' (mrI - NN')^2 \sigma_2^2 + m(mrI - m^{-1}k^{-1}NN')\sigma_3^2$$

$$+ (mrI - m^{-1}k^{-1}NN')\sigma^2] P_{32}$$

$$= (mr)^{-1} [m^2 r^2 \sigma_2^2 + m^2 r \sigma_3^2 + mr\sigma^2] I_{g(n-1)}$$

$$= (mr\sigma_2^2 + m\sigma_3^2 + \sigma^2) I_{g(n-1)}$$

$$(24)\quad A_{56} = (mr)^{-1/2} P_{32}' A' \not\Sigma F P_4 (m)^{-1/2}$$

$$= \phi$$

$$(25)\quad A_{57} = (mr)^{-1/2} P_{32}' A' \not\Sigma P_5$$

$$= \phi$$

$$(26)\quad A_{66} = m^{-1/2} P_4' F' \not\Sigma F P_4 m^{-1/2}$$

$$= m^{-1/2} P_4' F [X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + X_3 X_3' \sigma_3^2 + \sigma^2 I] F P_4 m^{-1/2}$$

$$= m^{-1} P_4' F' X_3 X_3' F P_4 \sigma_3^2 + m^{-1} P_4' F' F P_4 \sigma^2$$

$$= (\sigma^2 + m\sigma_3^2) I_{bk-b-t+1}$$

(27) $A_{67} = m^{-1/2}P_4'F'\not\Sigma P_5 = \phi$

(28) $A_{77} = P_5'\not\Sigma P_5 = \sigma^2 I_{bk(m-1)}$

<u>The derivation of $P'\not\Sigma P$ for SR-GD-PBIB Designs</u>: Letting $P'\not\Sigma P = (A_{ij})$, i, j = 1, 2, . . . , 7 we shall then have for each i and j the same results as for S-GD-PBIB Designs except the following.

$$A_{33} = (mk)^{-1}P_{23}'X_1'\not\Sigma X_1 P_{23}$$

$$= (mk)^{-1}P_{23}'[m^2k^2\sigma_1^2 I_b + N'N\sigma_2^2 + m^2k\sigma_3^2 I_b + mk\sigma^2 I_b]P_{23}$$

$$= [mk\sigma_1^2 + m\sigma_3^2 + \sigma^2]I_{g(n-1)} + (mk)^{-1}[m^2(r-\lambda_1)]\sigma_2^2 I_{g(n-1)}$$

$$= [mk\sigma_1^2 + mk^{-1}(r-\lambda_1)\sigma_2^2 + m\sigma_3^2 + \sigma^2]I_{g(n-1)}$$

$$A_{34} = (mk)^{-1/2}(mr)^{-1/2}P_{23}'X_1'\not\Sigma AP_{31}$$

$$= (mk)^{-1/2}(mr)^{-1/2}P_{23}'N'[rmI - m^{-1}k^{-1}NN']\sigma_2^2 P_{31}$$

$$= (mk)^{-1/2}(mr)^{-1/2}[m^2(r-\lambda_1)]^{-1/2}P_{32}'NN'[rmI - m^{-1}k^{-1}NN']P_{31}\sigma_2^2$$

$$= \phi$$

$$A_{35} = (mk)^{-1/2}(mv)^{-1/2}P_{23}'X_1'\not\Sigma AP_{32}$$

$$= m^{-1}k^{-1/2}v^{-1/2}[m^2(r-\lambda_1)]^{-1/2}P_{32}'NN'[rmI - m^{-1}k^{-1}N'N]P_{32}\sigma_2^2$$

$$= m^{-2}k^{-1/2}v^{-1/2}[r - \lambda_1]^{-1/2}[rm[m^2(r-\lambda_1)] - m^{-1}k^{-1}[m^2(r-\lambda_1]^2]$$

$$I_{g(n-1)}\sigma_2^2$$

$$= mk^{-1/2}v^{-1/2}[r-\lambda_1]^{-1/2}[r(r-\lambda_1) - k^{-1}(r-\lambda_1)^2]\sigma_2^2 I_{g(n-1)}$$

$$= mk^{-3/2}v^{-1/2}[r-\lambda_1]^{1/2}[rk - (r-\lambda_1)]\sigma_2^2 I_{g(n-1)}$$

$$= mk^{-1}[(r-\lambda_1)(rk-r+\lambda_1)]^{1/2}\sigma_2^2 I_{g(n-1)}$$

$$A_{44} = (mr)^{-1}P'_{31}A'\Sigma AP_{31}$$

$$= (mr)^{-1}[m^2r^2\sigma_2^2 + m^2r\sigma_3^2 + mr\sigma^2]I_{g-1}$$

$$= (mr\sigma_2^2 + m\sigma_3^2 + \sigma^2]I_{g-1}$$

$$A_{55} = (mv)^{-1}P'_{32}A'\Sigma AP_{32}$$

$$= (mv)^{-1}P'_{32}[(mrI - m^{-1}k^{-1}NN')\sigma_2^2 + m(mrI - m^{-1}k^{-1}NN')\sigma_3^2$$

$$+ (mrI - m^{-1}k^{-1}NN')\sigma^2]P_{32}$$

$$= (mv)^{-1}[m^2v^2\sigma_2^2 + m^2v\sigma_3^2 + mv\sigma^2]I_{g(n-1)}$$

$$= [mv\sigma_2^2 + m\sigma_3^2 + \sigma^2]I_{g(n-1)}.$$

<u>The derivation of $P'\Sigma P$ for R-GD-PBIB Designs:</u> This follows from the results derived for $P'\Sigma P$ in the case of BIB, S-GD-PBIB, and SR-GD-PBIB designs.

APPENDIX III

DISTRIBUTIONS AND EXPECTATIONS OF THE s_i

In this appendix we shall find the distributions and expectations of each of the statistics in the minimal sets of sufficient statistics that we have found for the BIB and GD-BIB designs.

We shall first state a well-known theorem which we shall use in deriving the distribution of each statistic.

Theorem: <u>If Y is distributed as the multivariate normal, mean $\bar{\mu}$ and covariance matrix Σ, then $Y'AY$ is distributed as the non-central χ^2 with degrees of freedom k and non-centrality parameter λ if $A\Sigma$ is idempotent and where k is the rank of A and $\lambda = 2^{-1}\bar{\mu}'A\bar{\mu}$</u> [3].

1. $s_1 = y_{...}$

Since $y_{...}$ is a linear combination of normal variables $y_{...}$ is distributed normally, mean μ and variance $(bkm)^{-1}(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)$ or $s_1 \sim N[\mu, (bkm)^{-1}(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)$.

2. $s_2 = (km)^{-1}Y'X_1P_{21}P_{21}'X_1'Y$

Distribution of s_2. Let $A_2 = (km)^{-1}X_2P_{21}P_{21}'X_1'$. Then $A_2A_2 = A_2$. In order to apply the theorem we must show that:

$$A_2\Sigma A_2\Sigma = A_2\Sigma$$

or equivalently $$A_2\Sigma A_2 = A_2 .$$

$$A_2 \Sigma A_2 = (mk)^{-2} X_1 P_{21} P_{21}' X_1' (X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + X_3 X_3' \sigma_3^2$$

$$+ \sigma^2 I) X_1 P_{21} P_{21}' X_1'$$

$$= (mk)^{-2} X_1 P_{21} P_{21}' (m^2 k^2 \sigma_1^2 + N'N\sigma_2^2 + m^2 k \sigma_3^2 + mk\sigma^2 I) P_{21} P_{21}' X_1'$$

$$= (mk)^{-1} X_1 P_{21} P_{21}' X_1' [\sigma_1^2 + mk\sigma_1^2 + m\sigma_3^2]$$

$$= (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2) A_2$$

Let $B_2 = (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)^{-1} A_2$. Then $Y'B_2Y \sim \chi'^2(k_2, \lambda_2)$, where $k_2 = \text{rank } B_2 = \text{rank } A_2 = \text{tr } A_2 = (km)^{-1} \text{Tr}, (X_1 P_{21} P_{21}' X_1') = \text{tr } P_{21} P_{21}' = b-t.$

$$\lambda_2 = \mu^2 J_{bkm}^1 X_1 P_{21} P_{21}' X_1' J_1^{bkm} (\sigma^2 + k\sigma_1^2)^{-1} = \phi$$

Therefore $s_2 \sim (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)\ \chi^2_{b-t}$. Therefore, $E(s_2) = (b-t)(\sigma^2 + mk\sigma_1^2 + m\sigma_3^2$.

3. $s_3 = (km)^{-1} Y' X_1 P_{22} P_{22}' X_1'$

Let $A_3 = (km)^{-1} X_1 P_{22} P_{22}' X_1'$. $A_3 A_3 = A_3$.

$$A_3 \Sigma A_3 = (km)^{-2} X_1 P_{22} P_{22}' X_1' [X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + X_3 X_3' \sigma_3^2$$

$$+ \sigma^2 I] X_1 P_{22} P_{22}' X_1'$$

$$= (km)^{-2} X_1 P_{22} P_{22}' [m^2 k^2 \sigma_1^2 + N'N\sigma_2^2 + m^2 k \sigma_3^2 + mk\sigma^2] P_{22} P_{22}' X_1'$$

$$= (km)^{-1} X_1 P_{22} P_{22}' [mk\sigma_1^2 + mk^{-1}(r-\lambda)\sigma_2^2 + m\sigma_3^2 + \sigma^2] P_{22} P_{22}' X_1'$$

$$= (km)^{-1} X_1 P_{22} P_{22}' X_1' [mk\sigma_1^2 + mk^{-1}(r-\lambda)\sigma_2^2 + m\sigma_3^2 + \sigma^2]$$

Let $B_3 = [mk\sigma_1^2 + mk^{-1}(r-\lambda)\sigma_2^2 + m\sigma_3^2 + \sigma^2]^{-1} A_3$.

$Y'B_3Y \sim \chi'^2(k_3, \lambda_3)$, where k_3 = rank of B_3 = rank of A_3 = tr A_3 = $(mk)^{-1}\mathrm{tr}\,(X_1P_{22}P_{22}'X_1') = t - 1$. $\lambda_3 = \mu^2 J_{bkm}^1 X_1P_{22}P_{22}'J_1^{bkm} = 0$. Therefore

$$s_3 \sim [mk\sigma_1^2 + mk^{-1}(r-\lambda)\sigma_2^2 + m\sigma_3^2 + \sigma^2]\ \chi_{t-1}^2$$

and

$$E(s_3) = [\sigma^2 + mk\sigma_1^2 + mk^{-1}(r-\lambda)\sigma_2^2 + m\sigma_3^2](t-1)$$

4. $A_5 = \dfrac{k}{\lambda tm} AP_3P_3'A'$

Then $A_5A_5 = A_5$.

$$A_5\Sigma A_5 = \frac{k^2}{\lambda^2t^2m^2} AP_3P_3'A'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2I)AP_3P_3'A'$$

$$= \frac{k^2}{\lambda^2t^2m^2} AP_3P_3'[(\lambda k^{-1}m)^2(tI - J)(tI - J)\sigma_2^2 + (m^2rI_t - k^{-1}NN')\sigma_3^2 + \lambda k^{-1}m(tI - J)\sigma^2]P_3P_3'A$$

$$= \frac{k}{\lambda tm} AP_3P_3'A'\ \ \sigma^2 + \lambda k^{-1}mt\sigma_2^2 + [\frac{mrt}{\lambda t} - \frac{m(r-\lambda)}{\lambda t}]\sigma_3^2$$

$$= \frac{k}{\lambda tm} AP_3P_3'A'[\sigma^2 + \lambda k^{-1}mt\sigma_2^2 + m\sigma_3^2]$$

Let $B_5 = (\sigma^2 + \lambda k^{-1}mt\sigma_2^2 + m\sigma_3^2)^{-1}A_5$. Then $Y'B_5Y \sim \chi^2(k_5, \lambda_5)$, where k_5 = rank of B_5 = rank of A_5 = tr A_5

$$= \frac{k}{\lambda tm}\,\mathrm{tr}\ AP_3P_3'A'$$

$$= \frac{k}{\lambda tm}\,\mathrm{tr}A'AP_3P_3'$$

$$= \frac{k}{\lambda tm}\,\mathrm{tr}[\lambda k^{-1}m(tI - J)P_3P_3'] = \mathrm{tr}P_3P_3' = t-1$$

$$\lambda_5 = \mu^2 J^1_{bkm} A P_3 P_3' A' J_1^{bkm} = 0$$

Therefore

$$s_5 \sim [\sigma^2 + \lambda k^{-1} m t \sigma_2^2 + m\sigma_3^2] \;\; \chi^2_{t-1}$$

and $$E(s_5) = (t-1)(\sigma^2 + \lambda k^{-1} m t \sigma_2^2 + m\sigma_3^2)$$

5. Distribution and expectation of $s_6 = m^{-1} Y' F P_4 P_4' F' Y$

Let $A_6 = m^{-1} F P_4 P_4' F'$. Then

$$A_6 A_6 = m^{-2} F P_4 P_4' F' F P_4 P_4' F'$$

$$= m^{-1} F P_4 P_4' m^{-1} F' F P_4 P_4' F'$$

$$= m^{-1} F P_4 P_4' F' \qquad [(P_4' m^{-1} F' F P_4 = I_{bk-b-t+1}]$$

$$= A_6$$

$$A_6 \Sigma A_6 = m^{-2} F P_4 P_4' F[X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + X_3 X_3' \sigma_3^2 + \sigma^2] F P_4 P_4' F'$$

$$= m^{-2} F P_4 [P_4' F' X_3 X_3' F P_4 \sigma_3^2 + P_4' F' F P_4 \sigma^2] P_4' F'$$

$$= m^{-1} F P_4 [P_4' m^{-1} E m^{-1} E P_4 m \sigma_3^2 + P_4' m^{-1} E P_4 \sigma^2] P_4' F'$$

$$(F' X_3 = E)$$

$$= m^{-1} F P_4 P_4' F' [m\sigma_3^2 + \sigma^2]$$

Therefore,

$$s_6 \sim (\sigma^2 + m\sigma_3^2) \;\; \chi^2_{bk-b-t+1}$$

$$\lambda_6 = \mu^2 J^1_{bkm} F P_4 P_4' F' J_1^{bkm} = 0$$

$$E(s_6) = (\sigma^2 + m\sigma_3^2)(bk - b - t + 1)$$

Let $A_7 = P_5P_5'$. Then $A_7A_7' = A_7$.

$$A_7 \Sigma A_7 = P_5P_5'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]P_5P_5' = \sigma^2 A_7$$

Let $B_7 = \sigma^{-2}A_7$. Then $Y'B_6Y \sim \chi'^2(k_7, \lambda_7)$, where k_7 = rank of B_7 = rank of A_7 = tr P_5P_5' = tr $I_{bkm-bk} = bk(m - 1)$.

$$\lambda_7 = \mu^2 J^1_{bkm} P_5 P_5' J_1^{bkm} = 0$$

Therefore

$$s_7 \sim \sigma^2 \chi^2_{bkm - bk}$$

$$E(s_7) = (bkm - bk)\sigma^2$$

Now $s_4 = k^{-1}(r-\lambda)^{1/2} Y'X_1P_{22}P_3'A'Y = k^{-1}m^{-1}Y'X_1N'P_3P_3'A'Y$. Let $A_4 = k^{-1}m^{-1}X_1N'P_3P_3'A'$. Since A_4 is not symmetric, we may write

$$Y'A_4Y = 2^{-1}Y'[A_4 + A_4']Y.$$

Then since $4^{-1}(A_4 + A_4')\Sigma(A_4 + A_4')$ is not equal to $2^{-1}(A_4 + A_4')$, s_4 is not distributed as χ^2 variate but as a linear combination of χ^2 variates. That is,

$$s_4 \sim \Sigma a_i \chi^2_{(1)}$$

where a_i are the non-zero characteristic roots of $2^{-1}(A_4 + A_4')$.

$$E(s_4) = E[k^{-1}m^{-1}Y'X_1NP_3P_3'A'Y]$$

$$= E\ tr[k^{-1}m^{-1}Y'X_1N'P_3P_3'A'Y\ ]$$

$$= k^{-1}m^{-1}\, tr\, E[YY'X_1N'P_3P_3'A']$$

$$= k^{-1}m^{-1}\, tr\, [X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2I]X_1N'P_3P_3'A'$$

$$= k^{-1}m^{-1}\, tr\, [A'X_2X_2'X_1N'P_3P_3'\sigma_2^2 + A'X_3X_3'X_1N'P_3P_3'\sigma_3^2]$$

$$= k^{-1}m^{-1}\, tr\, P_3'[mrI - m^{-1}k^{-1}NN']NN'P_3\sigma_2^2$$

$$= k^{-1}m^{-1}[mrm^2(r-\lambda) - m^{-1}k^{-1}m^4(r-\lambda)^2]\, tr\, I_{t-1}\sigma_2^2$$

$$= k^{-1}m^2(r-\lambda)[r - \frac{r-\lambda}{k}]\, tr\, I_{t-1}\sigma_2^2$$

$$= k^{-1}m^2(r-\lambda)[\frac{r(k-1)}{k} + \frac{\lambda}{k}]\, trI_{t-1}\sigma_2^2$$

$$= k^{-2}m^2(r-\lambda)\lambda t\sigma_2^2\, tr\, I_{t-1}$$

$$= k^{-2}m^2(r-\lambda)\lambda t\sigma_2^2(t-1)$$

6. $s_4 = m^{-1}k^{-1}Y'X_1N'P_3P_3'A'Y.$

Substituting $(I - t^{-1}J)$ for P_3P_3', we have $s_4 = m^{-1}k^{-1}Y'X_1N'(I - t^{-1}J)A'Y = m^{-1}k^{-1}Y'X_1N'A'Y$. ($P_3P_3' = I - t^{-1}J$ because corresponding to a unique characteristic root m^2rk of NN', we have a unique vector $(1/\sqrt{t}, 1/\sqrt{t}, \ldots, 1/\sqrt{t})$ from the orthogonal t x t matrix which diagonalizes NN'). Since the j-th element of $Y'X_1N'$ is T_j and the j-th element of $A'Y$ is Q_j, this statistic may be written as $m^{-1}k^{-1}\Sigma T_jQ_j$.

7. In order to determine s_2 in terms of the block and treatment totals, consider

$$m^{-1}k^{-1}X_1X_1'Y = m^{-1}k^{-1}Y'X_1(P_2P_2')X_1'Y$$

$$= m^{-1}k^{-1}Y'X_1(P_{20}, P_{21}, P_{22})\begin{bmatrix} P_{20}' \\ P_{21}' \\ P_{22}' \end{bmatrix} X_1'Y$$

We can write $P_{20}P_{20}' = b^{-1}J_b^b$. This follows from the reason given for P_3P_3' in 6. above. Since $b^{-1}J_b^1N'NJ_1^b = m^2r^2tb^{-1} = m^2rk$, which is a characteristic root of N'N of multiplicity 1, we therefore write:

$$m^{-1}k^{-1}YX_1X_1'Y - (mbk)^{-1}Y'X_1JX_1'Y - m^{-1}k^{-1}Y'X_1P_{22}P_{22}'X_1'Y$$

$$= m^{-1}k^{-1}Y'X_1P_{21}P_{21}'X_1'Y$$

or writing this in terms of block and treatment totals we have

$$m^{-1}k^{-1}\sum_{i=1}^{b}(B_i - B.)^2 - [km^3(r-\lambda)]^{-1}[\sum_j(T_i - T.)^2] = m^{-1}k^{-1}Y'X_1P_{21}P_{21}'X_1'Y$$

where B_i is the i-th element of $X_1'Y$ and $B. = b^{-1}\Sigma B_i$. The statistics s_2 may be obtained then by subtracting s_3 from the corrected sum of squares of blocks.

Singular, Group Divisible, PBIB Designs.

In this section we shall find the distributions and expectations of the statistics in a minimal set of sufficient statistics for singular GD-PBIB Designs.

1. Distribution of $s_1 = y...$

Since s_1 is a linear combination of normal variables, s_1 is normally distributed with mean $E(y...) = \mu$ and variance $E(y... - \mu^2) = (bkm)^{-1}(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)$. That is

$$s_1 \sim N[\mu,\ (bkm)^{-1}(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)]$$

2. Distribution of $s_2 = (mk)^{-1}Y'X_1P_{21}P_{21}'X_1'Y$.

Let

$$A_1 = (mk)^{-1}X_1P_{21}P_{21}'X_1'$$

then

$$A_1A_1 = (mk)^{-2}X_1P_{21}P_{21}'X_1'X_1'P_{21}P_{21}'X_1'$$

$$= (mk)^{-1}X_1P_{21}P_{21}'X_1' = A_1$$

and

$$A_1\Sigma A_1 = (mk)^{-2}X_1P_{21}P_{21}'X_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2$$

$$+ \sigma^2I]X_1P_{21}P_{21}'X_1'$$

$$= (mk)^{-2}X_1P_{21}P_{21}'[m^2k^2\sigma_1^2I_b + N'N\sigma_2^2 + MM'\sigma_3^2$$

$$+ mk\sigma^2I_b]P_{21}P_{21}'X_1'$$

$$= (mk)^{-2}X_1P_{21}P_{21}'[m^2k^2\sigma_1^2I_b + N'N\sigma_2^2 + m^2k\sigma_3^2I_b$$

$$+ mk\sigma^2I_b]P_{21}P_{21}'X_1'$$

$$= (mk)^{-1}X_1P_{21}P_{21}'X_1'[mk\sigma_1^2 + m\sigma_3^2 + \sigma^2]$$

$$= [mk\sigma_1^2 + m\sigma_3^2 + \sigma^2]A_1$$

Let $B_1 = [mk\sigma_1^2 + m\sigma_3^2 + \sigma^2]^{-1}A_1$. Therefore $Y'B_2Y \sim \chi'^2(k_1, \lambda_1)$, where k_1 = rank of B_1 = rank of A_1 = tr $A_1 = (mk)^{-1}\text{tr}\, X_1P_{21}P_{21}'X_1' = b - g$.

$$\lambda_1 = \mu^2 J_{bkm}^1 X_1P_{21}P_{21}'X_1'J_1^{bkm}C(\sigma) = 0$$

Hence

$$s_2 \sim [\sigma^2 + mk\sigma_1^2 + m\sigma_3^2] \ \chi^2_{b-g}$$

$$E(s_2) = (b - g)[\sigma^2 + mk\sigma_1^2 + m\sigma_3^2]$$

3. Distribution of $s_3 = (km)^{-1}Y'X_1P_{22}'P_{22}X_1'Y$

Let

$$A_2 = (km)^{-1}X_1P_{22}'P_{22}X_1$$

Then

$$A_2A_2 = A_2$$

and

$$A_2\Sigma A_2 = (mk)^{-2}X_1P_{22}P_{22}'X_1'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]X_1P_{22}P_{22}'X_1'$$

$$= (mk)^{-1}X_1P_{22}P_{22}'[(mk\sigma_1^2 + m\sigma_3^2 + \sigma^2)I_b + m^{-1}k^{-1}NN'\sigma_2^2]P_{22}P_{22}'X_1'$$

$$= (mk)^{-1}X_1P_{22}[\sigma^2 + mk\sigma_1^2 + m\sigma_3^2 + mk^{-1}(rk-\lambda_2 t)\sigma_2^2]P_{22}'X_1'$$

$$= [\sigma^2 + mk\sigma_1^2 + m\sigma_3^2 + mk^{-1}(rk-\lambda_2 t)\sigma_2^2]A_2$$

Let $B_2 = [\sigma^2 + mk\sigma_1^2 + m\sigma_3^2 + mk^{-1}(rk-\lambda_2 t)\sigma_2^2]^{-1}A_2$. Then $Y'B_2Y \sim \chi'^2(k_2, \lambda_2)$ where k_2 = rank of B_2 = rank of A_2 = tr A_2

$$= \mathrm{tr}\,(mk)^{-1}X_1P_{22}P_{22}'X_1' = g - 1$$

and

$$\lambda_3 = \mu^2 J_{bkm}^1 X_1 P_{22}P_{22}' J_1^{bkm} C(\sigma) = 0$$

Hence

$$s_3 \sim [\sigma^2 + mk\sigma_1^2 + m\sigma_3^2 + mk^{-1}(rk-\lambda_2 t)\sigma_2^2]\,\chi^2_{g-1}$$

$$E(s_3) = (g-1)[\sigma^2 + mk\sigma_1^2 + m\sigma_3^2 + mk^{-1}(rk-\lambda_2 t)\sigma_2^2]$$

4. Distribution of $s_4 = (\frac{k}{\lambda_2 tm})Y'AP_{31}P_{31}'A'Y$.

Let

$$A_3 = (\frac{k}{\lambda_2 tm})AP_{31}P_{31}'A'.$$

Then

$$A_3A_3 = A_3$$

and

$$A_3\Sigma A_3 = (\frac{k}{\lambda_2 tm})^2 AP_{31}P_{31}'A'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]AP_{31}P_{31}'A'$$

$$= (\frac{k}{\lambda_2 tm})^2 AP_{31}P_{31}'[(mrI - m^{-1}k^{-1}NN')(mr - m^{-1}k^{-1}NN')\sigma_2^2 + (mrI - m^{-1}k^{-1}NN')\sigma^2]P_{31}P_{31}'A'$$

$$= \frac{k}{\lambda_2 tm}(\sigma^2 + mk^{-1}\lambda_2 t\sigma_2^2)A_3$$

Let $B_3 = (\sigma^2 + mk^{-1}\lambda_2 t\sigma_2^2)^{-1}A_3$. Then $Y'B_3Y \sim \chi'^2(k_3, \lambda_3)$, where k_3 = rank of B_3 = rank of $A_3 = \mathrm{tr}\, A_3 = \frac{k}{\lambda_2 tm} \mathrm{tr}\, AP_{31}P_{31}'A' = g - 1$, and

$$\lambda_4 = \mu^2 J_{bkm}^1 AP_{31}P_{31}'A' C(\sigma) = 0$$

Hence

$$s_4 \sim (\sigma^2 + mk^{-1}\lambda_2 t\sigma_2^2)\ \chi_{g-1}^2$$

$$E(s_4) = (g-1)(\sigma^2 + mk^{-1}\lambda_2 t\sigma_2^2)$$

5. Distribution of $s_5 = (rm)^{-1}Y'AP_{32}P_{32}'A'Y$.

Let

$$A_4 = (rm)^{-1}AP_{32}P_{32}'A'.$$

Then

$$A_4A_4 = A_4$$

and

$$A_4 \not\Sigma A_4 = (rm)^{-2}AP_{32}P_{32}'A'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]AP_{32}P_{32}'A'$$

$$= (rm)^{-2}AP_{32}P_{32}'[(rmI - m^{-1}k^{-1}NN')\sigma_2^2 + (rmI - m^{-1}k^{-1}NN')\sigma^2]P_{32}P_{32}'A'$$

$$= (rm)^{-1}AP_{32}[mr\sigma_2^2 + \sigma^2]P_{32}'A'$$

$$= (mr\sigma_2^2 + \sigma^2)A_4$$

Let $B_4 = (\sigma^2 + mr\sigma_2^2)^{-1}A_4$. Then $Y'B_4Y \sim \chi'^2(k_4, \lambda_4)$, where k_4 = rank of B_4 = rank of $A_4 = \mathrm{tr}\, A_4 = \mathrm{tr}\,(mr)^{-1}AP_{32}P_{32}'A' = g(n-1)$, and

$$\lambda_4 = \mu^2 J^1_{bkm} A P_{32} P'_{32} A' J^{bkm}_1 C(0) = 0$$

Hence

$$s_5 \sim (\sigma^2 + mr\sigma_2^2) \; \chi^2_{g(n-1)}$$

$$E(s_5) = g(n-1)(\sigma^2 + mr\sigma_2^2)$$

6. Distribution of $s_6 = m^{-1} Y' F P_4 P_4' F' Y$ and its expected value are the same as in the BIB Design.

7. Distribution of $s_7 = Y' P_5 P_5' Y$ and its expected value are the same as in the BIB Designs.

8. Distribution of $s_8 = [k^{-2}(rk - \lambda_2 t)]^{1/2} Y' X_1 P_{22} P_{31}' A' Y$.

We know

$$P_{22}' = [m^2(rk - \lambda_2 t]^{-1/2} P_{31}' N$$

and so

$$s_8 = m^{-1} k^{-1} Y' X_1 N' P_{31} P_{31}' A' Y$$

Let

$$A_7 = m^{-1} k^{-1} X_1 N' P_{31} P_{31}' A'$$

Since A_7 is not symmetric, we may write $Y' A_7 Y = 2^{-1} Y' [A_7 + A_7'] Y$, then since $4^{-1}(A_7 + A_7')\Sigma(A_7 + A_7')$ is not equal to $2^{-1}(A_7 + A_7')$, s_8 is not distributed as χ^2 variate but as a linear combination of χ^2 variates; that is, $s_4 \sim \Sigma a_i \chi^2_{(1)}$ where a_i are the non-zero characteristic roots of $2^{-1}(A_7 + A_7')$.

$$E(s_8) = E\, m^{-1}k^{-1}Y'X_1N'P_{31}P_{31}'A'Y$$

$$= E\, \mathrm{tr}\, YY'X_1N'P_{31}P_{31}'A'm^{-1}k^{-1}$$

$$= (mk)^{-1}\mathrm{tr}\,[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]X_1N'P_{31}P_{31}'A'$$

$$= (mk)^{-1}\mathrm{tr}\,[A'X_2X_2'X_1N'P_{31}P_{31}'\sigma_2^2 + A'X_3X_3'X_1N'P_{31}P_{31}'\sigma_3^2]$$

$$= (mk)^{-1}\mathrm{tr}\, P_{31}'[mrI - m^{-1}k^{-1}NN']NN'P_{31}\sigma_2^2$$

$$= (mk)^{-1}[mrm^2(rk-\lambda_2 t) - m^3k^{-1}(rk-\lambda_2 t)^2]\sigma_2^2\, \mathrm{trace}\, I_{g-1}$$

$$= m^2k^{-2}(rk-\lambda_2 t)[rk - rk + \lambda_2 t]\sigma_2^2(g-1)$$

$$= m^2k^{-2}(rk-\lambda_2 t)(\lambda_2 t)(g-1)\sigma_2^2$$

<u>Semi Regular, Group Divisible, PBIB Designs.</u>

In this section we shall find the distributions and expectations of the statistics in the minimal set of sufficient statistics that were found for the semi-regular, group divisible, partially balanced incomplete block design.

1. Distribution of $s_1 = y...$

   $s_1 \sim N[\mu,\ (bkm)^{-1}(\sigma^2 + mk\sigma_1^2 + mr\sigma_2^2 + m\sigma_3^2)]$ as follows from s_1 for singular GD-PBIB Designs. $E(s_1) = \mu$

2. Distribution of $s_2 = (mk)^{-1}Y'X_1P_{21}P_{21}'X_1'Y$.

   Let

$$A_1 = (mk)^{-1}X_1P_{21}P_{21}'X_1'.$$

Then

$$A_1A_1 = A_1$$

and

$$A_1 \Sigma A_1 = (mk)^{-2} X_1 P_{21} P_{21}' X_1' [X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + X_3 X_3' \sigma_3^2 + \sigma^2 I] X_1 P_{21} P_{21}' X_1'$$

$$= (mk)^{-1} X_1 P_{21} P_{21}' X_1' [mk\sigma_1^2 + m\sigma_3^2 + \sigma^2]$$

$$= [\sigma^2 + mk\sigma_1^2 + m\sigma_3^2] A_1$$

Let $B_1 = [\sigma^2 + mk\sigma_1^2 + m\sigma_3^2]^{-1} A_1$. Then $Y' B_1 Y \sim \chi'^2(k_1, \lambda_1)$, where $k_1 = \text{rank of } B_1 = \text{rank of } A_1 = \text{tr } A_1 = \text{tr }(mk)^{-1} X_1 P_{21} P_{21}' X_1' = b - t + g - 1$, and

$$\lambda_1 \mu^2 J_{bkm}^1 X_1 P_{21} P_{21}' X_1' J_1^{bkm} C(\sigma) = 0$$

Hence

$$s_2 \sim (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2) \chi^2_{b-t+g-1}$$

$$E(s_2) = (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)(b - t + g - 1)$$

3. Distribution of $s_3 = (mk)^{-1} Y' X_1 P_{23} P_{23}' X_1' Y$.

Let

$$A_2 = (mk)^{-1} X_1 P_{23} P_{23}' X_1' .$$

Then

$$A_2 A_2 = A_2$$

and

$$A_2 \Sigma A_2 = (mk)^{-2} X_1 P_{23} P_{23}' X_1' [X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + X_3 X_3' \sigma_3^2 + \sigma^2 I] X_1 P_{23} P_{23}' X_1'$$

$$= (mk)^{-1}[\sigma^2 + mk\sigma_1^2 + mk^{-1}(r-\lambda_1)\sigma_2^2 + m\sigma_3^2]X_1P_{23}P_{23}'X_1'$$

$$= [\sigma^2 + mk\sigma_1^2 + mk^{-1}(r-\lambda_1)\sigma_2^2 + m\sigma_3^2]A_2$$

Let $B_2 = [\sigma^2 + mk\sigma_1^2 + mk^{-1}(r-\lambda_1)\sigma_2^2 + m\sigma_3^2]^{-1}A_2$. Then $Y'B_2Y \sim \chi'^2(k_2, \lambda_2)$ where k_2 = rank of B_2 = rank of $A_2 = \mathrm{tr}\, A_2 = (mk)^{-1}\mathrm{tr}\, X_1'P_{23}P_{23}'X_1' = g(n-1)$ and

$$\lambda_3 = \mu^2 J_{bkm}^1 X_1P_{23}P_{23}'X_1'J_1^{bkm}\, C(\sigma) = 0$$

Hence

$$s_3 \sim [\sigma^2 + mk\sigma_1^2 + mk^{-1}(r-\lambda_1\sigma_2^2 + m\sigma_3^2]\ \chi^2_{g(n-1)}$$

$$E(s_3) = g(n-1)[\sigma^2 + mk\sigma_1^2 + mk^{-1}(r-\lambda_1)\sigma_2^2 + m\sigma_3^2]$$

4. Distribution of $s_4 = (mr)^{-1}Y'AP_{31}P_{31}'A'Y$

Let

$$A_3 = (mr)^{-1}AP_{31}P_{31}'A'.$$

Then

$$A_3A_3 = A_3$$

and

$$A_3\Sigma A_3 = (mr)^{-2}AP_{31}P_{31}'A'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]AP_{31}P_{31}'A'$$

$$= (mr)^{-1}[\sigma^2 + mr\sigma_2^2]AP_{31}P_{31}'A'$$

$$= [\sigma^2 + mr\sigma_2^2]A_3$$

Let $B_3 = [\sigma^2 + mr\sigma_2^2]^{-1}A_3$. Then $Y'B_3Y \sim \chi'^2(k_3, \lambda_3)$, where $k_3 =$

rank of B_3 = rank of A_3 = tr A_3 = $(mr)^{-1}$ tr $AP_{31}P'_{31}A'$ = g - 1, and

$$\lambda_4 = \mu^2 J_1^{bkm} AP_{31}P'_{31}A' J_1^{bkm} C(0) = 0$$

Hence

$$s_4 \sim (\sigma^2 + mr\sigma_2^2)\, \chi^2_{(m-1)}$$

$$E(s_4) = (g-1)(\sigma^2 + mr\sigma_2^2)$$

5. Distribution of $s_5 = (mv)^{-1} Y'AP_{32}P'_{32}A\,Y$

Let

$$A_4 = (mv)^{-1} AP_{32}P'_{32}A'$$

Then

$$A_4 A_4 = A_4$$

and

$$A_4 \Sigma A_4 = (mv)^{-2} AP_{32}P'_{32}A'[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I] AP_{32}P'_{32}A'$$

$$= (mv)^{-1}[\sigma^2 + mv\sigma_2^2] AP_{32}P'_{32}A'$$

$$= [\sigma^2 + mv\sigma_2^2] A_4$$

Let $B_4 = [\sigma^2 + mv\sigma_2^2]^{-1} A_4$. Then $Y'B_4Y \sim \chi'^2(k_4, \lambda_4)$, where k_4 = rank of B_4 = rank of A_4 = tr A_4 = $(mv)^{-1}$ tr $AP_{32}P'_{32}A'$ = g(n - 1), and

$$\lambda_5 = \mu^2 J^1_{bkm} AP_{32}P'_{32}A' J_1^{bkm} C(\sigma) = 0$$

Hence

$$s_5 \sim (\sigma^2 + mv\sigma_2^2)\, \chi^2_{[g(n-1)]}$$

$$E(s_5) = [\sigma^2 + mv\sigma_2^2][g(n-1)]$$

6. The distribution of $s_6 = m^{-1}Y'FP_4P_4'F'Y'$ and its expectation are the same as those for BIB Designs.

7. The distribution of $s_7 = Y'P_5P_5'Y'$ and its expectation are the same as those for BIB Designs.

8. Distribution of $s_8 = [m^2k^{-2}(r-\lambda_1)]^{1/2}Y'X_1P_{23}P_{32}'A'Y$. We know

$$P_{23}' = [m^2(r-\lambda_1)]^{-1/2}P_{32}'N$$

$$s_8 = k^{-1}Y'X_1N'P_{32}P_{32}'A'Y$$

Let

$$A_7 = k^{-1}X_1N'P_{32}P_{32}'A' .$$

Since A_7 is not symmetric, we may write $Y'A_7Y = 2^{-1}Y'(A_7 + A_7')Y$. Then since $4^{-1}(A_7 + A_7')\Sigma(A_7 + A_7') \neq 2^{-1}[A_7 + A_7']$, s_8 is not distributed as χ^2 variate, but as a linear combination of χ^2 variates. That is, $s_8 \sim \Sigma a_i\chi^2_{(1)}$ where a_i are the non-zero characteristic roots of $2^{-1}(A_7 + A_7')\Sigma$.

$$E(s_8) = k^{-1}E\,\mathrm{tr}\,(Y'X_1N'P_{32}P_{32}'A'Y)$$

$$= k^{-1}\mathrm{tr}\,E(YY'X_1N'P_{32}P_{32}'A')$$

$$= k^{-1}\mathrm{tr}\,[X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + X_3X_3'\sigma_3^2 + \sigma^2 I]X_1N'P_{32}P_{32}'A'$$

$$= k^{-1}\mathrm{tr}\,P_{32}'A_1'X_2X_2'X_1N'P_{32}\sigma_2^2$$

$$= k^{-1}\mathrm{tr}\,P_{32}'(rmI - m^{-1}k^{-1}NN')NN'P_{32}\sigma_2^2$$

$$= k^{-1} tr [rm^3(r-\lambda_1) - k^{-1}m^3(r-\lambda_1)^2] I_{g(n-1)}\sigma_2^2$$

$$= k^{-2}m^3(r-\lambda_1)[rk - r + \lambda_1] \, tr \, I_{g(n-1)}\sigma_2^2$$

$$= g(n-1)m^3(r-\lambda_1)[rk - r + \lambda_1]k^{-2}\sigma_2^2$$

Regular GD-PBIB Designs.

In this section we shall derive the distributions and expectations of the statistics in the minimal set of sufficient statistics that were found for the regular group divisible partially balanced incomplete block designs.

1. Distribution of $s_1 = y\ldots$ and its expectation will correspond to those of s_1 for S-GD-PBIB Designs.
2. Distribution of $s_2 = (mk)^{-1}Y'X_1P_{21}P'_{21}X'_1Y$

Let

$$A_1 = (mk)^{-1}X_1P_{21}P'_{21}X'_1.$$

Then

$$A_1A_1 = A_1$$

and

$$A_1\Sigma A_1 = (mk)^{-2}X_1P_{21}P'_{21}X'_1[X_1X'_1\sigma_1^2 + X_2X'_2\sigma_2^2 + X_3X'_3\sigma_3^2 + \sigma^2 I]X_1P_{21}P'_{21}X'_1$$

$$= (mk)^{-1}(\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)X_1P_{21}P'_{21}X'_1$$

$$= (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)A_1$$

Let $B_1 = (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)^{-1}A_1$. Then $Y'B_1Y \sim \chi'^2(k_1, \lambda_1)$, where

k_1 = rank of B_1 = rank of A_1 = tr A_1 = tr $(mk)^{-1}X_1P_{21}P_{21}'X_1'$ = b - t, and

$$\lambda_1 = \mu^2 J^1_{bkm} X_1 P_{21} P_{21}' X_1' J_1^{bkm}\, C(0) = 0$$

Hence

$$s_2 \sim (\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)\, \chi^2_{(b-t)}$$

$$E(s_2) = (b-t)(\sigma^2 + mk\sigma_1^2 + m\sigma_3^2)$$

3. The distribution of $s_3 = (mk)^{-1}Y'X_1P_{22}P_{22}'X_1'Y$ and its expectation will correspond to those of s_3 for S-GD-PBIB Designs.

4. The distribution of $s_4 = (mk)^{-1}Y'X_1P_{23}P_{23}'X_1'Y$ and its expectation will correspond to those of s_3 for SR-GD-PBIB Designs.

5. The distribution of $s_5 = \frac{k}{\lambda tm} Y'AP_{31}P_{31}'A'Y$ and its expectation will correspond to those of s_4 for S-GD-PBIB Designs.

6. The distribution of $s_6 = (mv)^{-1}Y'AP_{32}P_{32}'A'Y$ and its expectation will correspond to those of s_5 for SR-GD-PBIB Designs.

7. The distribution of $s_7 = m^{-1}Y'FP_4P_4'F$ and its expectation will correspond to those of s_6 for S-GD-PBIB Designs.

8. The distribution of $s_8 = Y'P_5P_5'Y$ and its expectation will correspond to those of s_7 for S-GD-PBIB Designs.

9. The distribution of $s_9 = [k^{-2}(rk-\lambda_2 t)]^{1/2}Y'X_1P_{22}P_{31}'A'Y$ and its expectation will correspond to those of s_8 for S-GD-PBIB Designs.

10. The distribution of $s_{10} = [k^{-2}(r-\lambda_1)]^{1/2}Y'X_1P_{23}P_{32}'A'Y$ and its expectation will correspond to those of s_8 for SR-GD-PBIB Designs.

APPENDIX IV

Now we shall determine the pairwise independence of statistics in the minimal set.

In order to determine pairwise independence, we shall make use of the well known theorem:

If the bkm x 1 vector Z is distributed as the multivariate normal with mean μ and covariance matrix Σ and if $Z_1, Z_2, \ldots, Z_q$ are subvectors of Z such that $Z = (Z_1, Z_2, \ldots, Z_q)$, then a necessary and sufficient condition that the subvectors are jointly independent is that all the sub-matrices Σ_{ij} $(i \neq j)$ be equal to the null matrix.

In the balanced incomplete block design, we defined the vector Y and transformed Y to Z by the relation $Z = P'Y$. Then

$$Z \sim MVN[P'\tilde{\mu}, P'\Sigma P].$$

We then formed a partition of Z into $(Z_1, Z_2, Z_3, Z_4, Z_5, Z_6)$. The form of $P'\Sigma P$ is as given in Table I and is the covariance matrix of Z.

By making use of the above theorem, we have Z_1, Z_2, Z_5, Z_6, as mutually independent and they are independent of Z_3 and Z_4 and that Z_3 and Z_4 are not independent. We can have the following relationship.

$s_1 = Z_1$

$s_2 = Z_2'Z_2$

$s_3 = Z_3'Z_3$

$s_5 = Z_4'Z_4$

$s_6 = Z_5'Z_5$

$s_7 = Z_6'Z_6$

$s_4 = Z_3'Z_4$

Hence we conclude that the statistics in the minimal set of sufficient statistics are pairwise independent except for the pairs (s_3, s_4), (s_3, s_5) and (s_4, s_5).

The Singular, Group Divisible PBIB Design.

Following the procedure given in previous section and examining Table XIII, we have the results as stated in Corollary 2.3.

The Semi-Regular, Group Divisible PBIB Design.

Following a procedure similar to that of the first section and examining Table X, we have the results as stated in Corollary 3.3.

The Regular, Group Divisible PBIB Design.

Again following the procedure of the first section and examining Table XII, we have the results as stated in Corollary 4.3.

APPENDIX V

In what follows we shall try to associate each of the statistics in the minimal set with block-treatment and interaction sum of squares.

(1) s_1. This statistic is the mean of all observations in the vector Y and is the unbiased estimate of μ.

(2) $s_3 = [km^3(r-\lambda)]^{-1} Y'X_1N'P_3P_3'NX_1'Y$. The quantity $NX_1'Y$ is a t x 1 vector of elements T_j (say) where T_j is the total of all blocks containing treatment j. P_3P_3' can be replaced by $(I - t^{-1}J)$. Making this substitution, we have

$$s_3 = [km^3(r-\lambda)]^{-1} Y'X_1N'(I - t^{-1}J)NX_1'Y$$

$$= [km^3(r-\lambda)]^{-1} [Y'X_1N'NX_1'Y - t^{-1}Y'X_1N'JNX_1'Y]$$

$$= [km^3(r-\lambda)]^{-1} [\Sigma T_j^2 - t^{-1}(kY...)^2]$$

$$= [km^3(r-\lambda)]^{-1} \Sigma (T_j - T.)^2$$

where $T. = t^{-1}\Sigma T_j$ and $Y... = J_{bkm}^1 Y$.

(3) $s_5 = \frac{k}{\lambda tm} Y'AP_3P_3'A'Y$. If we replace P_3P_3' by $I - t^{-1}J$, we have

$$s_5 = \frac{k}{\lambda tm} Y'A(I - t^{-1}J)A'Y = \frac{k}{\lambda tm} Y'AA'Y$$

Consider $A'Y = (X_2' - m^{-1}k^{-1}NX_1')Y$. This we shall denote by Q_j's and it has the same conventionally known interpretation as we have one observation per cell. Therefore,

$$s_5 = \frac{k}{\lambda tm} \Sigma Q_j^2$$

(4) $s_6 = m^{-1}Y'P_4FF'P_4Y$. The way in which we have picked P_4, assures us that $s_6 = m^{-1}Y'FF'Y$. This is true since P_4' is bk-b-t+1 x bk orthogonal vectors of the bk x bk orthogonal matrix which diagonalizes the idempotent matrix $m^{-1}F'F$ which has rank bk-b-t+1. Let us call this orthogonal matrix 0. Let

$$0' = \begin{bmatrix} P_4' \\ P_{41}' \end{bmatrix}$$

where P_4' is bk-b-t+1 x bk and P_{41}' is b+t-1 x bk orthogonal vectors. Since

$$0'm^{-1}F'F0 = \begin{bmatrix} I_{bk-b-t+1} & \phi \\ \phi & \phi \end{bmatrix}$$

we have

$$m^{-1}P_{41}'F'FP_{41} = \phi$$

Therefore,

$$m^{-1}Y'FP_4P_4'F'Y = m^{-1}Y'F[I - P_{41}P_{41}'F'Y = m^{-1}Y'FF'Y$$

If we substitute

$$F' = X_3' - m^{-1}k^{-1}M'X_1' - m^{-1}\lambda^{-1}t^{-1}k(L' - m^{-1}k^{-1}M'N')(X_2' - m^{-1}k^{-1}NX_1')$$

then

$$m^{-1}Y'FF'Y = Y'[m^{-1}X_3X_3' - m^{-1}k^{-1}X_1X_1' - \frac{k}{\lambda tm}AA']Y$$

But the right hand side is the interaction sum of squares as shown below.

$$R[\mu, \tau, \beta, (\beta\tau)] = \sum_{ij} \frac{Y_{ij.}^2}{n_{ij}} \ ; \text{ where } Y_{ij.} = \sum_k y_{ijk}$$

$$R[\mu, \tau, \beta] = m^{-1}k^{-1}\sum_{i=1}^{b} Y_{i..} + \frac{k}{\lambda tm}\sum_{j=1}^{t} Q_j^2 \ ; \text{ where } Y_{i..} = \sum_j \sum_k y_{ijk}$$

Therefore,

$$R(\beta\tau \mid \mu, \tau, \beta] = R[\mu, \tau, \beta, (\beta\tau)] - R(\mu, \tau, \beta)$$

$$= \sum \frac{Y_{ij.}^2}{n_{ij}} - m^{-1}k^{-1}\sum_{i=1}^{b} Y_{i..} - \frac{k}{\lambda tm}\sum_{j=1}^{t} Q_j^2$$

$$= m^{-1}Y'X_3X_3'Y - m^{-1}k^{-1}Y'X_1X_1'Y - \frac{k}{\lambda tm} Y'AA'Y$$

$$= m^{-1}Y'FF'Y$$

Therefore,

$$s_6 = m^{-1}Y'X_3X_3'Y - m^{-1}k^{-1}Y'X_1X_1'Y - \frac{k}{\lambda tm} Y'AA'Y$$

$$= m^{-1}[\sum_{n=1}^{bk} C_2^2 - k^{-1}\sum_{i=1} B_i^2 - \frac{k}{\lambda t}\sum Q_j^2]$$

where C_n is the n-th element of $X_3'Y$.

(5) $s_7 = YP_5'P_5'Y$. In view of the above arguments we can infer that s_7 is the intra-block error.

VITA

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