A STUDY OF THE LIMIT CONCEPT IN THE SMSG REVISED SAMPLE TEXTBOOKS

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CHAPTER I

INTRODUCTION

Within the past few years, several new approaches to mathematics instruction at the high school level have been proposed. Probably, the best single representative of the many proposals comes from the School Mathematics Study Group in the form of sample textbooks. Due to the magnitude of the project undertaken by the School Mathematics Study Group, that of developing, testing and revising sample textbooks for high school mathematics, and because of the size, quality and influence of the membership, the sample textbooks can be expected to be of considerable influence on the secondary mathematics curriculum of the future.

The topic of limits is vital in mathematics and a necessary consideration in the teaching of mathematics at the secondary school level. The treatment of the theory of limits remains a controversial topic and one which is generally omitted by textbooks even though teachers and mathematicians are aware of the significance of this topic to the high school mathematics curriculum. Because of the importance of this topic and this set of textbooks, the writer elected to study the limit concept relative to the School Mathematics Study Group's revised sample textbooks.

Introduction To The School Mathematics Study Group

The School Mathematics Study Group (henceforth called the

SMSG) was organized in the Spring of 1958 under the directorship of Professor E. G. Begle. The group includes outstanding mathematicians from colleges and universities, teachers of mathematics at all levels, experts in education, and representatives of science and technology. Among these members are found representatives from many organizations concerned with the improvement of mathematics in our schools such as the Secondary School Curriculum Committee of the National Council of Teachers of Mathematics, the University of Illinois Committee on School Mathematics, the Commission on Mathematics of the College Entrance Examination Board, and the University of Maryland Mathematics Project ([57] , pp. 454-59).

Organization of this group began in 1958 at Yale University under the directorship of Professor E. G. Begle. Members of this group organized into writing teams and met in the summer of 1959 at the University of Colorado and at the University of Michigan. Their purpose was

> to develop a curriculum and teaching materials based upon the best available knowledge of mathematics, pedagogy, and the needs of our society. ([1] , p. 616.)

As a result of this meeting, sample textbooks were published and distributed during the years of 1959 and 1960. After these were tested by about 100 teachers and 8, 500 students, the group met again in the summer of 1960 at Stanford University to revise the textbooks ([57] , pp. 454-59). These revised sample textbooks were considered in this study.

Need for the Study

At the present time some mathematics educators agree that topics on limits should be included in the secondary school mathematics curriculum. The Third Annual Symposium on Engineering Mathematics was attended by approximately 280 persons who were primarily teachers of high school mathematics and college instructors of mathematics. At this meeting a questionnaire that was given to each individual included the following items:

> 14. Check those topics in the following list that you feel should definitely be taught regularly in the college preparatory program in high school.
>
> | | |
> |---|---|
> | a. inequalities | g. determinants |
> | b. absolute value | h. group theory |
> | c. algebra of sets | i. field theory |
> | d. limit concept | j. differential calculus |
> | e. vectors | k. statistics |
> | f. probability | l. integral calculus |
>
> ([30] , pp. 114.)

Item d, limit concept, was selected by 57% of the college teachers and 80% of the high school teachers, giving a combined total of 73%. These percentages are below the ones for items a, b, c, and e, of which a, inequalities, and b, absolute value, are not new to the curriculum ([30] , pp. 113-18).

The Commission on Mathematics of the College Entrance Examination Board includes the study of limits in their proposed program of study for both the eleventh and twelfth grades. They state that the notion of limit is one of the most important ideas in all mathematics ([10] , p. 64).

An intuitive study of limits is included in the revised sample textbooks of the SMSG. In order that merits and correctness of this approach could be more fully considered, an analytical investigation and rigorous treatment of those discussions and arguments that involve a limit were needed. These books use such terminology as "approaches," "close," "sufficiently small," "large enough." This approach to the teaching of the limit concept should not be the only one considered.

> The (ϵ, δ)-definition of limit is the result of more than a hundred years of trial and error, and embodies in a few words the result of persistent effort to put this concept on a sound mathematical basis. Only by limiting processes can the fundamental notions of the calculus—derivative and integral—be defined. But a clear understanding and a precise definition of limits had long been blocked by an apparently insurmountable difficulty .... The problem was to attach a precise mathematical meaning to the idea that f(x) "tends to" or "approaches" a fixed value as x moves toward x_1. ([12] , p. 305.)

This thesis, therefore, provides an opportunity for the following groups of individuals to gain experience in conjunction with and in addition to that provided by the revised sample textbooks:

1. <u>Authors of Textbooks.</u> It has long been recognized that treatments of the limit concept have been inadequate in secondary school mathematics.

   > It is greatly to be desired that our textbook writers and the teachers who use them have a proper conception of this fundamental notion. ([59] , p.202.)

   This study provides authors with a brief rigorous interpretation of the discussions that utilize a limit in order that they can more quickly and more surely decide on the amount of rigor they consider to be pedagogically desirable as content for a textbook in high school mathematics.

2. <u>Teachers of Mathematics.</u> Kenneth E. Brown of the United States Office of Education provides evidence which implies that a great many teachers in secondary school mathematics may lack the training that would lead to a thorough understanding of the concept of a limit. This conclusion is drawn from the numbers of teachers

reported to have less than thirty credit hours, no recent courses, or no graduate credit in college mathematics ([6] , pp. 7-8). For any teacher in this category, the need is great to gain a good understanding of the limit processes that are found in the sample textbooks.

> ...for how can one teach what he does not know? By mastery I mean more that the ability to perform. In addition, I mean that the teacher must know under what conditions the concept, or process, or theorem or formula can be applied. Further, I believe that the teacher must know the genesis of what he teaches, especially when there is deductive basis for the "what." ([24] , p. 28.)

3. Students of Mathematics. The availability of material such as that found in this study can greatly assist those academically talented or keenly interested students who desire more than an intuitive demonstration.

   Also, there are those who say, "Calculus is a proper high school subject" ([16] , p. 451). If this hypothesis is accepted, it would indicate a need for the concept of a limit to be taught.

> The concepts of a limit and an infinitesimal and the application of these concepts are vital to an understanding of the calculus and are indeed the foundation stones upon which the calculus is built. ([8] , p. 587.)

Statement of the Problem

How can a rigorous treatment of the limit processes that are used in the revised sample textbooks of the School Mathematics Study Group be embedded into the mathematical structure of these books? In order to answer this question the following sub-problems had to be satisfactorily solved:

1. The topics that involve arguments depending upon a limit process had to be identified.

2. The remaining topics and their relationship to these arguments that involve limits had to be noted.

3. Finally, mathematically rigorous treatments had to be presented, using only those concepts and processes which the textbooks have previously presented.

If, however, the order of topics was found to be inadequate for the above stated task to be completed, a reordering of the topics was to be given. Then within this new order the aforementioned tasks were to be completed.

Purpose

The purpose of this study was to devise an alternate approach to the concept of a limit as it appears in the revised sample textbooks of the SMSG. The objective was to provide an analysis and a resolution of the difficulties involved in presenting rigorous arguments in place of the intuitive arguments. Thereby the aspects of the concept of a limit could be more completely considered and the implications of the sample textbooks could be more broadly extended.

Theoretical Framework

Because of the influence of the SMSG, the sample textbooks that have been produced can be considered as an important contribution to the secondary school mathematics curriculum. However, the point of view expressed in these textbooks should not be considered the only approach to good mathematics. It is stated in the Foreword to Part 1 of each textbook:

> It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as

> a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. ([39] , Foreword.)

The topics that use a limit process in these textbooks are generally treated in an intuitive manner without using a definition of a limit. In support of this position the Commentary for Teachers urges that the words "limit" and "approaches" not be used when discussing the tangent to a curve ([46] , p. 65).

However, a contradictory point of view is expressed by Lehi T. Smith based upon a study completed in 1959. Smith gave an objective test of limit problems to two groups of students in grades seven through twelve. One group had received three class hours of instruction on limits whereas the other group had not. On the basis of the results of his study, he is prompted to conclude:

> Since evidence indicates that junior-senior high school students can profit from experience with limits, reason dictates that the concept of limits be formally discussed in presentations of those topics which require a concept of limits for full understanding. ([48] , p. 59.)

In regard to those topics of the high school curriculum that involve limits, Smith says:

> They need to be identified as limit situations and the characteristics of these situations which make them problems of limits need to be discussed. Only through this suggested approach will full understanding of these topics be achieved. ([48] , p. 59.)

The question of how to treat the concept of a limit, however, remains an unsettled question. Howard F. Fehr suggests the following questions should be answered before reform can take place in mathematics education:

> To what extent can we be more than intuitive and short of rigorous in our presentation? The following questions suggest themselves for consideration.

> a.) What treatment should be given to limits and continuity? Are ϵ and δ methods appropriate? What degree of rigor is necessary? ([15] , p. 429.)

One point of view concerning rigor is expressed by Morris Kline. Regarding the development of the irrational numbers which utilizes the concept of a limit, he skeptically queries:

> If it took mathematicians so long to arrive at the logical concept of an irrational number, can we believe that young people will appreciate it at once? ([25] , p. 422.)

Continuing his discussion he indicates the opinion that rigor "does little good;" has accomplished nothing except "that the mathematicians salved their consciences;" requires a "capacity to appreciate ... [that] must be developed;" and possesses a meaning which is in itself controversial ([25] , p. 422-23).

In an article directed as a reply to Kline, Albert E. Meder, Jr. quotes Kline as having said:

> Of course the level of rigor must suit the age and maturity of the student. But this does not mean to dispense with it entirely. I think you have missed the point on logic. Actually it is not lost on young people. My experience has been that they lap it up. ([33] , p. 431.)

An amount of agreement with this endorsement of rigor is expressed by Edwin E. Moise, a member of the School Mathematics Study Group:

> In high school teaching, rigor ought to take the form of disclosure, the form of candor. Rigor is saying what you really mean. I think it is a mistake to suppose that this always makes things harder for the student. ([35] , p. 439.)

These seemingly controversial points of view have important significance to the secondary school mathematics curriculum and especially to this specific study.

> Such differences of opinion stem largely from differences both in mathematical interest and in educational outlook. It is neither possible nor desirable to terminate these controversies. Our educational system can benefit by vigorous debate and by diverse experimental studies tending to resolve some of the points at issue. A great deal of hard work, bold experimentation, and thoughtful study will indeed be needed to clarify the picture of what we should and can do by way of curricular reform. ([9] , p. 69.)

Review of Related Research

Research regarding the limit concept as a topic of secondary school mathematics has been very rare. Only in recent years have scientific studies been conducted and reported on this subject.

Arthur H. Steinbrenner completed a study in 1955 that was an analysis of historical development and existing practices in teaching continuity. The relation of his study to this one is inferred by his statement: "Two concepts fundamental to an understanding of mathematical continuity are limits and irrational numbers" ([49] , p. 12). He found that the limit concept was not adequately presented in secondary school mathematics textbooks and that discussion of a limit was generally omitted ([49] , pp. 78-118).

From 1955 through 1960, only one other doctoral study had been completed concerning the concept of a limit. This study by Smith was cited above.

Definitions of Terms

1. A "limit process" refers to the means by which a number or set of numbers is defined or shown to be the limit of a sequence, or function. This also can be called a "limiting process." (It seems to this writer that calling the process of finding a limit a

limiting process connotes that it is the process itself that reaches a limit. This is similar to calling the process of finding the sum of two numbers a numbering process.) Topics in the textbooks that include limits or a limit process are called "limit topics."

2. A "concept" of a limit is composed of the terms and logical structure of the definitions and theorems of a limit, the definitions and theorems themselves, and of problems relating to these.
3. A "mathematical structure" is a pattern or particular organization of definitions, assumptions and theorems, and their logical interrelationships.
4. There are many levels of rigor in mathematics and philosophy and hence, many definitions of the word "rigorous." The word rigorous when used in this study refers to that level which is accepted in elementary analysis. It requires that assumptions, definitions and theorems be explicitly stated and that theorems be justified by previously stated assumptions, definitions and theorems.
5. A definition, postulate, or proof is said to be "embedded" into a mathematical structure if its component parts are shown to be established topics in keeping with an established order of topics of the mathematical structure.

Hypotheses

The hypotheses of this study were:

1. The structure of the textbooks is adequate to allow a rigorous treatment of those discussions and arguments that involve limits to be embedded into the structure without a reordering of the topics.
2. The arguments, intuitive or otherwise, that use a limit process are

logically correct when interpreted into rigorous mathematical definitions, postulates, and proofs.

3. The valid treatments of the limit topics deviate to a considerable degree from those found in college calculus textbooks.

Assumptions

The assumptions for this study were:

1. Mathematically rigorous treatment of topics is a necessary consideration in the development of and a continuously significant factor in a modern curriculum in mathematics.
2. The revised sample textbooks of the SMSG are an adequate source for those topics that are necessary in a good secondary school mathematics program.
3. Even though the conventional "(ϵ, δ)-notation" is used, the implication of this study pertains to most other accepted treatments of the topics involving limits.

   > ... when it comes to checking the existence of a limit in actual scientific procedure it is the (ϵ, δ) -definition that must be applied. ([12] , p. 306.)

4. All limit processes actually used in the sample textbooks can be identified by phrases such as: sufficiently close, as close as we please, closer and closer, tends to, approaches, limiting value; if x or n is sufficiently small, small enough, close to zero, large enough, sufficiently large, etc.
5. A rigorous treatment of a concept should not be presented prior to an introduction to deductive reasoning and the nature of proof.

Scope and Limitations

Because of the writer's acquaintance with other works of the

SMSG, it was realized that the scope of this study could have been greatly varied. These texts include material that is commonly not mentioned in conventional textbooks, such as the basic concepts of the calculus. Therefore, it was to be expected that many topics would be found in these books that would require advanced college mathematics to treat completely in a mathematically rigorous manner.

Thus in order to focus this study on the secondary school mathematics curriculum some arguments and discussion of limits were omitted. The decision of what should and should not be treated at each grade level was determined by the SMSG. In their revised sample textbooks they selected those topics to be mentioned without proof or any type of substantiating argument, and they chose other topics or subtopics to be discussed or justified by a selected type of argument. This study was limited to the topics of the latter category.

While this thesis is concerned with the presentation of a rigorous treatment of limits, it is not proposed that the proofs given in this thesis are the only ones that can be made. Rather, proofs are given and sources of proofs are cited to exhibit a particular argument by which the stated theorems or desired conclusions can be justified. Although some advanced mathematics textbooks are given as references, the writer is not suggesting that the passages cited are of extreme difficulty.

While the concern of this thesis is with the secondary school mathematics curriculum, the topic itself confines considerations to the later grades. Although the limit concept is inherent in the development of the real numbers, it is seldom mentioned prior to the deductive study of geometry. Therefore, this study focuses its attention on the mathematics of high school grades nine through twelve. The revised sample textbooks for these grades consist of the following publications in the _Mathematics for High School_ series: _First Course in Algebra_ (3 parts), _Geometry_ (3 parts), _Intermediate Mathematics_ (3 parts), and _Elementary Functions_ (2 parts). Also, associated with each part of each textbook is a _Commentary for Teachers_ that consists of answers to problems and additional remarks and discussions for teachers (see [39] through [45]).

Henceforth, in the body of this thesis reference to these books will be made as exemplified by the following:

Mathematics for High School

First Course in Algebra (Part 1)

Chapter 3, Section 4

is referred to as

First Course in Algebra 1, 3-4

Likewise

Mathematics for High School

Geometry (Part 2)

Commentary for Teachers

Chapter 3, Section 4

is referred to as

Geometry 2, _Commentary for Teachers_, 3-4.

If however, reference is made to this teachers' commentary within the discussion entitled _Geometry 2_, 3-4, the reference is simply to the _Commentary for Teachers_.

Procedure

The revised sample textbooks of the SMSG were analyzed in order to determine the following:

1. The arguments which use a limit process. They were identified by such phrases as: the function approaches, the limiting value is, approximately equal for large enough x, when x is sufficiently small, the function gets closer and closer, the function can be made as close as we please by choosing x close enough to a, and by similar statements.
2. The arrangement of topics and the method of presentation in relationship to the limit arguments.

The concern of the study was then directed toward the problem of determining and presenting mathematically rigorous arguments that would be in keeping with the structure of the textbooks.

1. The arguments and discussions that use limit processes were interpreted into the terminology of the (ϵ, δ)-definition of a limit. For example, to say that "f(x) is as close as we please to the number L if x is sufficiently close to a," or "f(x) approaches L as x gets close to a," it was stated that "for every $\epsilon > 0$ there exists $\delta > 0$ such that

   $$|f(x) - L| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta."$$

   To say that "for x sufficiently large" or "for x large enough" that such and such happens it was stated that "there exists N such that if $x > N$" then such and such happens.
2. On the basis of these arguments and discussions, definitions, postulates, and theorems were determined and used in the rigorous treatment of the presentations.

3. The definitions, postulates, and proofs of theorems were constantly compared with the structure in order to assure that any concept or process to be used had been previously presented.

Organization of the Thesis

The body of the thesis consists of two chapters devoted to the analysis and treatment of the limit processes found in the SMSG textbooks, followed by Chapter IV which contains the definitions and theorems required in the treatment but not found in the analysis, and, finally, by a summary of findings. Because limits were found more frequently and in greater detail in _Elementary Functions_ than in the other textbooks, Chapter III is devoted to _Elementary Functions_ and Chapter II to the preceding textbooks.

In Chapters II and III, theorems and definitions are numbered according to chapter, section and order of occurrence. This was found to be possible because, in the textbooks treated, no two topics involving a limit were found to have the same chapter and section number. Thus, Theorem A5-17.2 is the second theorem presented in the treatment of the Appendix, Chapter 5 Section 17. The following is a list of the topics in each textbook that are treated in this thesis:

Geometry: 15-3, 15-4, 15-5, 16-5.

Intermediate Mathematics: 1-8, 1-9, 1-10, 5-4, 6-5, 6-6, A9-1, A9-2, A9-3, A9-4, A9-8, A10-9, A13-4, A13-5, A13-6.

Elementary Functions: 2-3, 2-5, 3-1, 3-3, 3-4, 3-5, 3-11, A2-12, A3-12, A3-13, 4-5, 4-6, 4-7, 4-12, 5-10, A4-16, A4-18, A5-17.

The treatments given in Chapters II and III usually refer to the textbooks for supporting statements, definitions, or theorems. However, if a treatment requires a supporting definition or theorem that is not mentioned or inferred in the textbooks, it will be given in Chapter IV. Because other definitions and theorems of the report are numbered by at least three digits, no confusion results by numbering the entries in Chapter IV consecutively and by referring to them by their assigned number.

Appendix A contains pertinent letters received by the writer in answer to specific questions posed to members of the SMSG writing group and to other qualified mathematicians. Reference is given in the body of the report to the appropriate letter at that time when it is pertinent to the material being presented.

Summary

The SMSG revised sample textbooks for high school mathematics were selected as a source of those topics that involve the concept of a limit and also as an example of a widely accepted presentation of the topics. Each argument or discussion that involves a limit has been studied to determine assumptions, definitions, and theorems required by these presentations. Furthermore, a search has been made for a treatment of these presentations in which the required conditions are explicitly stated, and justified when necessary. Herein are reported the results of this study.

CHAPTER II

THE LIMIT TOPICS FOUND IN *FIRST COURSE IN ALGEBRA*, *GEOMETRY*, AND *INTERMEDIATE MATHEMATICS*

Treated in this chapter are those courses which are conventionally a part of the high school mathematics curriculum. They include *First Course in Algebra*, *Geometry*, and *Intermediate Mathematics*.

First Course in Algebra-General Introduction

The importance of this textbook to the present study does not lie in the discussions that involve a limit concept. Rather, these books offer a background of mathematical concepts to be used in subsequent topics.

No discussion was found that involved the limit concept. Topics that indirectly pertain to this study were found to be the following:

1. An introduction to the real numbers. This includes the fundamental properties of real numbers except for the Completeness Property.
2. An introduction to $|x|$ and inequalities.
3. An introduction to the concept of functions and the graph of a function.

Geometry-General Introduction

The use of limits in the study of elementary geometry is minimized by the SMSG material. By using the Birkhoff approach and by carefully selecting postulates and definitions, many proofs that in conventional texts require a limit concept are avoided here ([35] , p. 439).

The prime example is found in what is commonly referred to as the "incommensurable case." In common high school textbooks used today and in the past, consideration of incommensurable line segments is either with limits or it is omitted because limits are needed [49] .

However, in the SMSG Geometry course the Distance Postulate (page 34), the Ruler Postulate (page 36), the Ruler Placement Postulate (page 40), the Angle Measurement Postulate (page 80), the Angle Addition Postulate (page 81), and the Supplement Postulate (page 82) associate the real numbers with geometry. Thereby, proofs can be made using properties of real numbers and the consideration of incommensurable segments and angles can be avoided ([47] , p. 86).

Another topic of geometry that is frequently treated by a limit process is the area of a rectangle (for a typical argument see [7] , p. 182). In the SMSG Geometry the area of a rectangle, A = bh, is postulated (page 322). Then the area of a square is found as a special case of the area of a rectangle, and again the need for a limit process is avoided.

In presenting the basic concepts of solid geometry, Cavalieri's Principle is encountered (page 558). In conventional texts a limit process is used to prove this theorem while in the SMSG text this principle is postulated.

Limits are not, however, avoided completely. It will be pointed out below that the limit concept as well as the word itself is used in considering the measures associated with circles.

Geometry 3, 15-3

In this section the textbook introduces the notion of a limit. It is stated on page 526: "It seems reasonable to suppose that if you want to measure C [the circumference of a circle] approximately, you can do it by inscribing a regular polygon with a large number of sides and then measuring the perimeter [p] of the polygon." The paragraph continues with the definitional statement: "If we decide how close we want p to be to C, we ought to be able to get p this close to C merely by making n large enough ... [we write] $p \longrightarrow C$ and we say p approaches C as a limit."

No other discussion is given to define a limit; hence from the above statement the following definition is drawn.

Definition 15-3.1. Let $\{p_n\}$ be a sequence of numbers and c be a real number. The limit of $\{p_n\}$ is C, $\lim_{n\to\infty} p_n = C$, if for every $\epsilon > 0$ there exist a natural number N such that if $n > N$, then

$$|p_n - C| < \epsilon .$$

This, the conventional (ϵ, N)-definition of the limit of a sequence, would precede a discussion of sequences if presented here. Not until Intermediate Mathematics page 754 is the limit of a sequence discussed as a topic. Therefore, if the treatment of limits is to be done with rigor, a reordering of topics is needed at this point.

The word limit is used to define the circumference of a circle as "the limit of perimeters of the inscribed regular polygons." This statement is symbolized in the textbook by

$$p \longrightarrow C,$$

and in this thesis by

$$\lim_{n\to\infty} p_n = C. \tag{1}$$

(This symbol is used to avoid confusion later with the symbol for a function.) However, it should be noted that the textbook is tacitly assuming that this limit exists. No discussion whatsoever is given to suggest that this statement is even to be considered. Comments on this limit were obtained from Edwin E. Moise, and Walter Prenowitz, members of the SMSG writing group ([47], p. 142) and from F. A. Sherk, Professor of Mathematics, University of Toronto, and Merrill Shanks, Professor of Mathematics, Purdue University. The comments are reproduced in Appendix A.

Utilization of the limit definition of circles and also the _Commentary for Teachers_ infer that a rigorous treatment of the succeeding topics will require a knowledge of sequences. Therefore, throughout the treatment of geometry given in this study it is assumed that Definition 15-3.1 and Theorems A13-4.1 through A13-4.9 are available (see _Intermediate Mathematics_, Chapter 13).

Using the above quoted definition of circumference, the textbook suggests on page 527 that π be defined as the ratio $\frac{C}{2r}$.
To show that the ratio of $\frac{C}{2r}$ is constant for every circle, a limit is required to prove $\frac{C}{2r} = \frac{C'}{2r'}$ where C, r and C', r' are the circumferences and radii of any two circles. The textbook adequately proves that

$$\frac{p}{r} = \frac{p'}{r'}$$

where p and p' are the perimeters of regular n-gons inscribed in the respective circles. A limit argument is needed, as will be pointed out in the following theorem, to justify the concluding remark "$p \longrightarrow C$, by definition and $p' \longrightarrow C'$, by definition. Therefore, $\frac{C}{r} = \frac{C'}{r'}$;"

<u>Theorem 15-3.1.</u> If r, r', p_n, p_n' are radii of circles and perimeters of inscribed regular n-gons respectively, and if

$$\frac{p_n}{r} = \frac{p_n'}{r'}\,, \qquad (2)$$

then $\frac{C}{r} = \frac{C'}{r'}$ where C and C' are the circumferences of the respective circles.

Proof: By the uniqueness of limits, Theorem A13-4.1, and (2),

$$\lim_{n\to\infty} \frac{p_n}{r} = \lim_{n\to\infty} \frac{p_n'}{r'}\,.$$

By Theorem A13-4.3,

$$\frac{1}{r} \lim_{n\to\infty} p_n = \frac{1}{r'} \lim_{n\to\infty} p_n'\,,$$

but $\lim_{n\to\infty} p_n = C$ and $\lim_{n\to\infty} p_n' = C'$ by definition of circumference of a circle. Therefore, $\frac{C}{r} = \frac{C'}{r'}$ which was to be proved.

The symbol π is then defined to be the common ratio $\frac{C}{2r}$. Hence, $C = 2\pi r$.

<u>Geometry 3</u>, 15-4

In a manner similar to that of the preceding section, area of a circle is defined on page 531 to be the "limit of the areas of the inscribed regular polygons. Thus

$$A_n \longrightarrow A\text{'' or } \lim_{n\to\infty} A_n = A.$$

Again the textbook tacitly assumes that this limit exists. The justification of the existence of this limit is a problem similar and related to the existence of the limit of a sequence of perimeters. It is not treated in this thesis except for comments from capable mathematicians which are found in Appendix A.

The textbook provides an argument to derive the formula

$A = \pi r^2$. The first consideration is to show "$a \longrightarrow r$" which in the form of a theorem is:

Theorem 15-4.1. $\lim_{n\to\infty} a_n = r$ where a_n is the apothem of a regular n-gon inscribed in a circle of radius r.

Proof: As hinted by the textbook the Pythagorean theorem can be used to express a_n in terms of r and s_n, the length of a side of the inscribed n-gon. Hence,

$$a_n = \sqrt{r^2 - (\frac{s_n}{2})^2}$$

But, since $p_n = ns_n$ and $\lim_{n\to\infty} p_n = c$, then $\lim_{n\to\infty} ns_n = c$. This implies that $\lim_{n\to\infty} s_n$ can be no number except 0 by Theorem 1.

Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{r^2 - (\frac{s_n}{2})^2} = r \qquad (1)$$

by Theorems A13-4.3, and A13-4.5 and Theorem 2.

The second consideration of the argument is justification of a statement presented here as a theorem.

Theorem 15-4.2. With a, p, r, and c defined as above,

$$\lim_{n\to\infty} \frac{1}{2} a_n p_n = \frac{1}{2} rc.$$

Proof: By (1) and Definition 15-3.1

$$\lim_{n\to\infty} a_n = r \text{ and } \lim_{n\to\infty} p_n = c.$$

Therefore, by Theorem A13-4.5,

$$\lim_{n\to\infty} a_n p_n = rc,$$

and by Theorem A13-4.3,

$$\lim_{n\to\infty} \frac{1}{2} a_n p_n = \frac{1}{2} rc.$$

This conclusion could be stated: $\lim_{n\to\infty} A_n = A = \pi r^2$.

Geometry 3, 15-5

Continuing in the manner of the previous two sections, the length of an arc is defined here as a limit. It is suggested that an arc AB of a circle with center Q be partitioned by points A, P_1, P_2, $\cdots$ P_{n-1}, B such that the angle between successive points be $\frac{1}{n} \cdot m\widehat{AB}$. The length of arc $\widehat{AB}$ is then defined to be "the limit of $AP_1 + P_1P_2 + \cdots + P_{n-1}B$ as we take n larger and larger."

Here again, as in the previous two sections, the textbook tacitly assumes that such a limit exists. In addition to those references previously cited a discussion of the existence of this limit can be found in Johnson and Kiokemeister's Calculus ([21] , pp. 234-35).

Geometry 3, 16-5

This section is concerned with the volume and surface area of a sphere. It is proved by Cavalieri's Principle that the volume of a sphere is $\frac{4}{3}\pi r^3$; then this formula and the notion of a limit are used to derive a formula for the surface area.

On page 571 the following argument is given: "Given a sphere of radius r, form a slightly larger sphere of radius r + h." This forms a shell of thickness h whose volume can be computed and called V. It is suggested that surface area is "approximately, $S = \frac{V}{h}$," and "as h gets smaller and smaller, we have

$$\frac{V}{h} \longrightarrow s."$$

This discussion infers that h and V are both continuous variables. Because limits of sequences have been assumed, a restriction is placed here on the values of h. Henceforth, let it be understood that $h = \frac{1}{n}$ where n is a positive integer.

Although a definition of S is suggested by the Commentary for Teachers on page 448, none is made in the textbook. Therefore, for the purposes of this study, the following definition is given:

Definition 16-5.1. If V and $\frac{1}{n}$ are the volume and thickness of a spherical shell of inner radius r, then

$$S = \lim_{n \to \infty} nV,$$

where S is called the surface area of the sphere of radius r.

The textbook finds that the ratio of $\frac{V}{h}$, which is nV if $\frac{1}{n} = h$, is

$$nV = \frac{4}{3}\pi\left(3r^2 + 3r\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right)^2\right).$$

Now, by Theorem A13-4.4 and because $\lim_{n \to \infty} \frac{1}{n} = 0$, it follows that

$$\lim_{n \to \infty} nV = 4\pi r^2 = S.$$

The definition given above can be generalized or similar definitions can be given to find the surface area of a right circular cylinder and cone. Problem 11 page 573 is to find the surface area of a right circular cylinder, and the Commentary for Teachers on page 450 outlines an approach to finding the surface area of a right circular cone.

Intermediate Mathematics-General Introduction

This textbook includes topics from conventional high school and college algebra and plane trigonometry.

Limits are to be found in the discussion of properties of the real numbers and related topics such as the development of the logarithmic and exponential functions. Also, as was mentioned previously, limits of sequences are discussed in the latter part of this textbook.

Intermediate Mathematics 1, 1-8

Although there are no statements that resemble a discussion of the limit concept in this section, it does involve infinite decimals. Because infinite decimals and the development of real numbers are dependent upon the limit concept, this section was analyzed.

To this point, the textbook has discussed special properties of rational numbers in preparation for an introduction to the development of the real number system. On page 69, however, discussion is given which is premature. It is suggested that "each repeating decimal expression represents a rational number." The proof of the statement is exemplified by an accompanying example which requires that

$$(10^3)(0.\overline{123}) = 123.\overline{123} \tag{1}$$

($.\overline{ab}$ means $.abababab \cdots ab \cdots$).

If the discussion cited above is to be independent of the properties of the real number system, then it does not justify the statement to be proved. Consider the example; if it is not known that $1.\overline{123}$ is a rational number then the product of this number by a rational number is yet to be defined. Hence, the argument has assumed the repeating decimal to be a rational number. This discussion would be valid if it were known that $0.\overline{123}$ is a real number and that the product of two real numbers is as stated in (1).

Intermediate Mathematics 1, 1-9

The system of real numbers is introduced in this section by a discussion of their development and a statement of basic properties. Again as in the previous section, the relevance of this section to the present study is through the real number system. The concept of a limit and infinite sequences can be used to develop the real numbers from the rationals as is done in Goffmann's Real Functions ([18] , pp. 28-41), and Thurston's The Number-System [54] .

The last of the basic properties "0_7 (R)" (hereafter called the Completeness Property) is necessary for the considerations of this study and is therefore stated below in slightly different form.

Completeness Property. If $\{a_n\}$ and $\{b_n\}$ are two sequences of real numbers with the properties

(i) $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$,

(ii) $b_0 \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq \cdots$,

(iii) $a_n \leq b_n$, for every natural number n,

(iv) $\lim_{n \to \infty} (b_n - a_n) = 0$

then there is one and only one real number c such that $a_n \leq c \leq b_n$ for every natural number n ([50] , p. 95).

The textbook states for property (iv):

$$"b_n - a_n < \frac{1}{10^n}, \text{ for every natural number n."}$$

This statement is inadequate as exemplified by the following example: Consider

$$a_n = -\frac{1}{n},\ b_n = \frac{1}{n} \text{ and } c = 0.$$

Now properties (i), (ii), and (iii) are satisfied but

$$b_n - a_n = \frac{2}{n} > \frac{1}{10^n},$$

and property (iv) as stated by the textbook fails. However, for every natural number N there exists a natural number $n = (2)10^N$ such that

$$b_n - a_n < \frac{1}{10^N}$$

which is equivalent to the statement

$$\lim_{n\to\infty} (b_n - a_n) = 0.$$

In order that the discussions involving limits in Geometry could be treated, the properties of sequences and limits of sequences had to be assumed. Likewise, from this point on through Intermediate Mathematics the definitions and theorems concerning sequences and limits of sequences are required. Therefore, the definitions and theorems to be found in the treatment of limits of sequences (Intermediate Mathematics 3, Chapter 13) will be cited throughout the treatment of Intermediate Mathematics.

Intermediate Mathematics 1, 1-10

As an example of the properties of the real numbers that are not shared by the rational numbers, the solution of $x^n = a$ is discussed. In regard to these properties the Commentary for Teachers on page 51 states: "The real number system is a system 'closed' under limiting processes." Because no discussion is given of limits in the textbook, this topic is not treated in this study. However, more information about the Completeness Property can be found in many college textbooks ([12] , pp. 68-72, [50] , pp. 89-95, [18] , p.41, [54] , pp. 29-33).

Intermediate Mathematics 1, 6-4

Reference is given in this section to "limiting forms" of conic sections. Because the meaning of this term is vague, and because specific examples of the conic sections are considered in the next section, it is the next section that is treated in this thesis.

Intermediate Mathematics 1, 6-5

The reference to "limiting form" is given on page 336 where it is stated that a circle is a limiting form of an ellipse because "If e is very close to 0, $b = a\sqrt{1 - e^2}$ is very close to a. In fact ··· the ellipse becomes more and more like a circle; so that the circle is a limiting form of an ellipse." At this point, if the discussion is to be verified as follows, Definition 1 and Theorems 3 through 12 are required. The ellipse under discussion is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

By Theorems 8, 4, 10, and 12, it follows that

$$\lim_{e \to 0} a\sqrt{1 - e^2} = a.$$

Therefore, $\lim_{e \to 0} b = a$ and equation (1), if $b = a$, would become

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1.$$

which is the equation of a circle.

Intermediate Mathematics 2, 6-6

In regard to the asymptotes of a hyperbola it is stated on page 345 that "the curve gets closer and closer to these lines [asymptotes] as x increases." This statement could not be adequate as a definition of asymptote. Consider as an example $y = \frac{1}{x}$ for $x > 0$. The graph gets "closer and closer" to the lines $y = -1$, $x = -2$, but these lines are not asymptotes (see Figure 1).

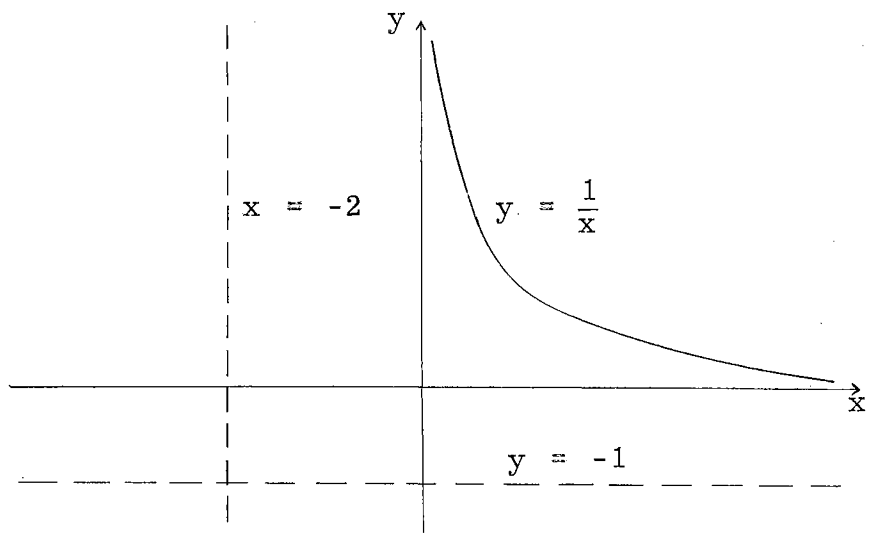


Figure 1.

However, the statement quoted above suggests the notion of a limit in defining an asymptote of a hyperbola. Therefore, the following definitions are given:

Definition 6-6.1. The linear equation $y = g(x)$ is that of a non-vertical asymptote of the function defined by $y = f(x)$ if

$$\lim_{x \to \infty} (f(x) - g(x)) = 0 \text{ or } \lim_{x \to -\infty} (f(x) - g(x)) = 0.$$

Definition 6-6.2. The linear equation $x = k$ is that of a vertical asymptote of the function defined by $y = f(x)$ if

$$\lim_{x \to k^+} f(x) = \pm\infty \text{ or } \lim_{x \to k^-} f(x) = \pm\infty.$$

These definitions involve the limit of functions. Such limits are defined and theorems are given in Chapter IV.

Under these definitions the statement, "If we take large values for x, then y in the first quadrant is nearly equal to $\frac{b}{a}x$," suggests the following theorem:

Theorem 6-6.1. If the equation of a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

then $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are the equations of the asymptotes of the hyperbola.

Proof: To use Definition 6-6.1 consider

$$f(x) = \frac{b}{a}\sqrt{x^2 - a^2} \text{ and } F(x) = -\frac{b}{a}\sqrt{x^2 - a^2}.$$

Now

$$\frac{b}{a}(\sqrt{x^2 - a^2} - x) = \frac{b}{a}\,\frac{-a^2}{\sqrt{x^2 - a^2} + x}$$

and

$$\lim_{x\to\infty}\left(\frac{b}{a}\,\frac{-a^2}{\sqrt{x^2 - a^2} + x}\right) = 0$$

by Theorems13 through 18. Therefore,

$$\lim_{x\to\infty}\left(\frac{b}{a}\sqrt{x^2 - a^2} - \frac{b}{a}x\right) = 0.$$

So if $y = f(x)$ is the equation of the part of the hyperbola which lies in the first and second quadrants, then by Definition 6-6.1 $y = \frac{b}{a}x$ is the equation of the asymptote to the hyperbola in the first quadrant. A similar argument implies that $y = \frac{b}{a}x$ is the equation of the asymptote of the hyperbola in the third quadrant and $y = -\frac{b}{a}x$ is the equation of the asymptote of hyperbola in the second and fourth quadrants.

Definition 6-6.2 can be used to determine the vertical asymptote of the hyperbola $yx = k$. The equation, written $y = \frac{k}{x}$, shows the asymptote to be $x = 0$ by Theorems 19 and 20.

Intermediate Mathematics 2, A9-1

Chapter 9 of the Appendix to Intermediate Mathematics 2 is devoted to a discussion of logarithms and exponents. This first section, on page 455, suggests that the area of "the shaded region be used to define ··· the new [logarithm] function" (see Figure 2).

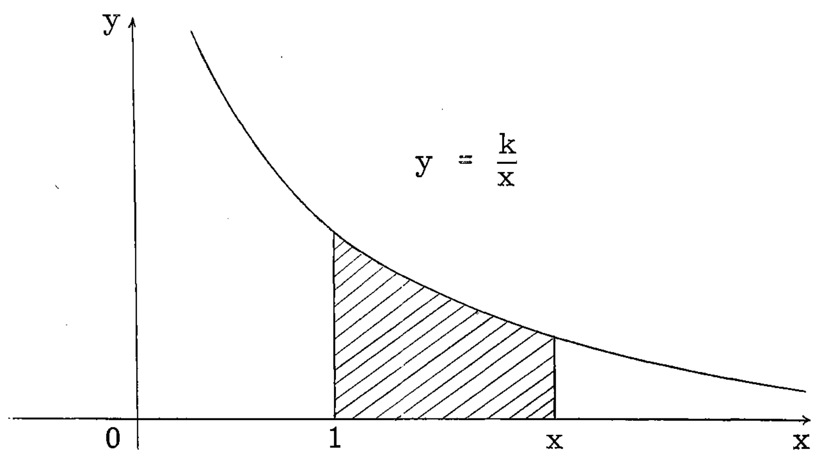


Figure 2.

It must be noted that area has not been defined for figures not previously considered in geometry. Therefore, the discussion to follow in the textbook and hence in this thesis is based upon the students intuitive notion of area and visual analysis of the pictures. A rigorous approach to this same topic in which the integral is used to define area can be found in calculus texts such as those by Thomas ([53] , pp. 287-296) and Taylor ([51] , pp. 301-314).

Intermediate Mathematics 2, A9-2

On page 471 the concept of area under a curve is hinted to be a limit process. Rectangles are used by the textbook to estimate the area under the curve. In regard to the rectangles it is stated: "If a large number of rectangles is used, the sum of their areas is very close to the area under the curve."

Because area has been defined in terms of the students visual interpretation of the pictures, it must be assumed that this statement is to be accepted on the same basis. Otherwise, a development of integral calculus would need to be presented where there is no discussion in the textbook that could be interpreted to give such a development.

<u>Intermediate Mathematics 2</u>, A9-3

In the discussion of the properties of the logarithm function, on page 477 it is said to have a graph that is a "continuous curve." The only explanation given for this term is that it "follows from the fact that the graph has no breaks or jumps in it." The limit definition of continuity is given in Chapter IV. Using the intuitive concept of area and Definition 2 of continuity the continuity of the lograithm function can be proved as a theorem.

<u>Theorem A9-3.1.</u> The logarithm function is continuous.

Proof: Considering the assumed definition that ln x ($\log_e x = \ln x$) is the area above the x-axis under the graph of $y = \frac{1}{x}$ between $x = 1$ and a fixed value $x > 1$, it follows by visual analysis of Figure 3 that if $x > 1$ and $|h| < x - 1$, then

$$\left|\ln (x + h) - \ln x\right| < |h| \frac{1}{x},$$

or

$$\left|\ln (x + h) - \ln x\right| < |h| \frac{1}{x + h}.$$

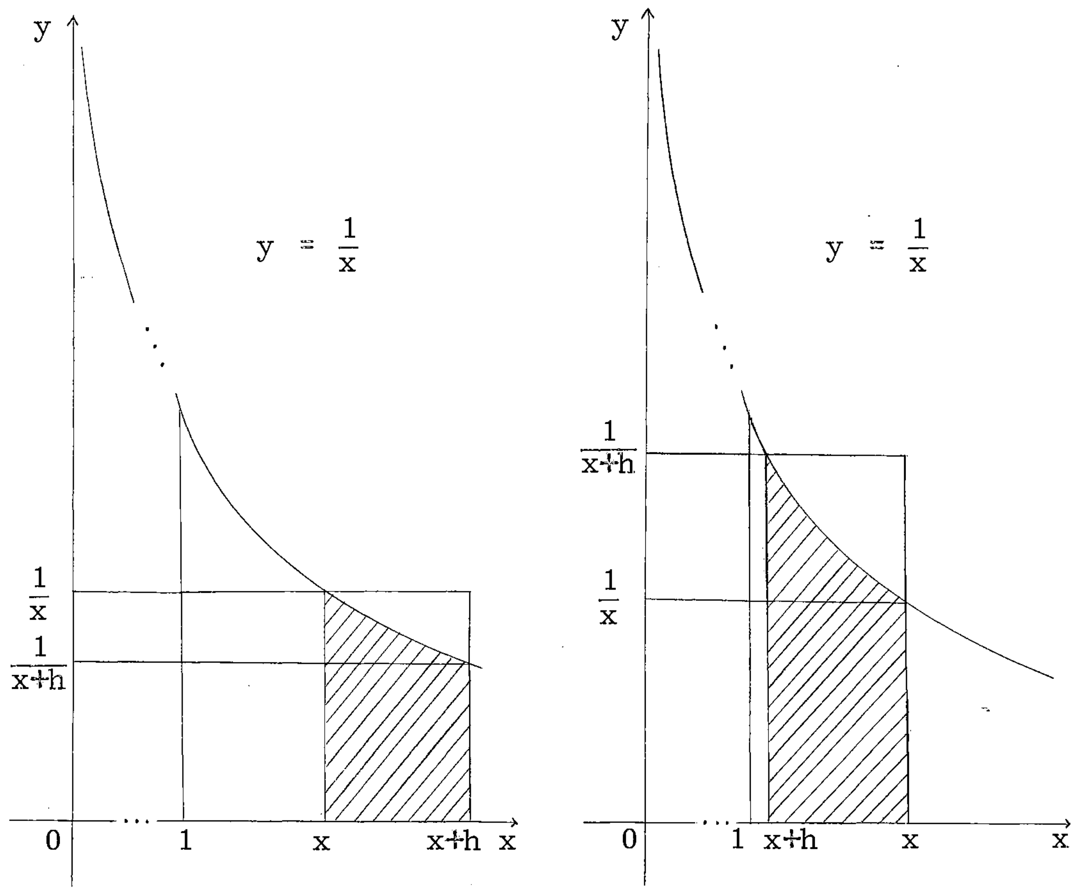


Figure 3

In either case, for every $\epsilon > 0$ there exists

$\delta = \min\left(\frac{x - 1}{2}, \frac{\epsilon x}{1 + \epsilon}\right) > 0$ such that if $0 < |h| < \delta$, then

$$\frac{|h|}{x} < \frac{\delta}{x - \delta} \text{ and } \frac{|h|}{x + h} \leq \frac{\delta}{x - \delta},$$

or

$$|\ln (x + h) - \ln x| < \epsilon .$$

Hence, $\lim_{h \to 0} \ln (x + h) = \ln x$ by Definition 1. Therefore, by Definition 2 and Theorem 23, the function defined by $y = \ln x$ is continuous for $x > 1$.

Now consider $x = 1$ and $h > 0$ in the left hand graph of Figure 3. Using again the assumed concept of area, it follows that

$$|\ln (1+h) - \ln 1| < |h| \text{ for every } h \neq 0.$$

Hence, by Definition 1,

$$\lim_{h \to o^+} \ln (1 + h) = \ln 1 = 0. \qquad (1)$$

For $x = 1$ and $-1 < h < 0$ write

$$\lim_{h \to o^-} \ln (1+h) = \lim_{h \to o^-} (-\ln \frac{1}{1+h}).$$

Note that if $-1 < h < 0$ then $\frac{1}{1+h} > 1$ and the function defined by $y = \ln \frac{1}{1+h}$ for $\frac{1}{1+h} > 1$ is continuous. Therefore, by Theorems 24 and 7 it follows that

$$\lim_{h \to o^-} \ln (1+h) = -\ln \lim_{h \to o^-} \frac{1}{1+h},$$

and by Theorems 8 and 11 that

$$\lim_{h \to o^-} \ln (1+h) = -\ln 1 = 0. \qquad (2)$$

Therefore, by Theorem 25, (1) and (2) combine to give

$$\lim_{h \to o} \ln (1+h) = \ln 1,$$

so the function defined by $y = \ln x$ is continuous at $x = 1$.

Finally, to show the function defined by $y = \ln x$ is continuous for $0 < x < 1$ use again $\ln x = -\ln \frac{1}{x}$. Note that if $0 < x < 1$, then $\frac{1}{x} > 1$ and the function defined by $y = \ln \frac{1}{x}$ is continuous for every $\frac{1}{x} > 1$. Therefore, as in the preceding paragraph, if $0 < a < 1$

$$\lim_{x \to a} \ln x = \lim_{x \to a} (-\ln \frac{1}{x}) = -\ln \frac{1}{a} = \ln a.$$

Hence, by Definitions 2 and 3 it follows now that the function defined by $y = \ln x$ is continuous for every $0 < x < 1$.

This completes the proof because every $x > 0$ has been considered and the function has been proved continuous in each case.

In general, $y = \log_a x$ defines a continuous function. Because

$$\log_a x = \frac{\ln x}{\ln a},$$

it follows by Definition 2 and Theorem 11 that the function defined by $y = \log_a x$ is continuous for every $x > 0$ when a is any positive number.

Also, on page 477 an "important consequence of this property" of continuity is given: If $x_1 < x_2$ and c is any number such that $\log x_1 < c < \log x_2$, then there is a number x_0 such that $x_1 < x_0 < x_2$ and $\log x_0 = c$." Because this statement is discussed later under the special name "The Location Theorem," no treatment is given at this point in this thesis (see <u>Elementary Functions 1</u>, page 59).

<u>Intermediate Mathematics 2</u>, A9-4

In this section the properties of the logarithmic function are extended and reviewed. In addition to the references to continuity and asymptotes the notion of a limit is used in a new way.

On page 485 it is stated that as "x increases without limit, y also increases without limit on the graph of $y = \log x$." From the accompanying discussion the following definition is drawn for this statement:

<u>Definition A9-4.1.</u> A function defined by $y = f(x)$ is said to have a graph that increases without limit as x increases without limit, $\lim_{x \to \infty} f(x) = \infty$ if, for every number $N > 0$ there exists a number $k > 0$ such that if $x > k$, then $f(x) > N$. (This definition is essentially the same as Definition 10 which was required in 6-5.)

From the discussion in the textbook and hence this definition, it might be inferred that a function whose graph is increasing without limit is monotonic. This is not necessarily true and is not actually implied. For example consider the graphs of $y = x^2$ and $y = x + 2 \sin x$ which are increasing without limit as x increases without limit but are not monotonic.

To show that the graph of $y = \log x$ increases without limit as x increases without limit the textbook on page 484 considers $\log 2^n$ as a specific set of values. The requirements of the above definition can be met by considering an increasing sequence of values. Thus, because $\log 2^n > 0$, for any number $N > 0$, there exist a smallest integer that is greater than $\frac{N}{\log 2}$; it can be called k. Therefore, if $n > k$, then

$$\log 2^n = n \log 2 > k \log 2 > N.$$

Because $\log x = y$ has a graph that is strictly increasing, this argument implies that

$$\lim_{x \to \infty} f(x) = \infty.$$

Discussion is also given concerning the vertical line $x = 0$, as the asymptote of the graph of $y = \log x$. The textbook explains that as "x decreases toward zero, y decreases without limit." In the form of Definition 6-6.2, this means that

$$\lim_{x \to 0+} \log x = -\infty.$$

Specifically $y = \log 2^{-n}$ is considered. For any number $M < 0$ there exists a smallest positive integer greater than $\frac{-M}{\log 2}$ which can be called k. Therefore, if $n > k$ and $\log 2 > 0$ then

$$\log 2^{-n} = -n \log 2 < -k \log 2 < M.$$

Thus,

$$\lim_{n \to \infty} \log 2^{-n} = -\infty,$$

by Definition 10. But because the graph of $y = \log x$ is strictly increasing and because $\log x$ is defined for every $x > 0$, then

$$\lim_{x \to 0^+} \log x = -\infty.$$

(For consideration of $\lim_{n \to \infty} 2^{-n} = 0$, see Theorem 26.)

<u>Intermediate Mathematics 2</u>, A9-8

This section is devoted to the definition of the exponential function as the inverse of the logarithm function. No limit discussion is given but continuity is mentioned on page 525. It is suggested there that the exponential function has a continuous graph because the logarithm function has a continuous graph.

A proof of the continuity of an inverse function of a continuous function is given in Johnson and Kiokemeister's calculus textbook ($\boxed{21}$, p. 254). The proof utilizes Definition 2 of continuity in an argument that involves the limit concept.

Because the textbook refers to a continuous graph as one with no holes or jumps and the graph of an inverse function as one which is symmetric to the line $y = x$, continuity of the graph of the inverse function is inferred in the textbook by visual analysis of the pictures and not by a limit process.

<u>Intermediate Mathematics 2</u>, A10-9

On page 593 reference is given to the vertical lines $x = \pm \frac{\pi}{2}$ being asymptotes to the graph of $y = \tan x$. Using Definition 6-6.2 this can be proved in the following theorem:

<u>Theorem A10-9.1</u>. The lines given by $y = \pm (2n - 1)\frac{\pi}{2}$, where n is a natural number, are asymptotes of the graph of $y = \tan x$.

Proof: Write $\tan x = \sin x \left(\frac{1}{\cos x}\right)$. Then by the previous discussion and graphs of $y = \sin x$ and $y = \cos x$ it is assumed that

$$\lim_{x \to \frac{\pi}{2}} \sin x = 1 \text{ and } \lim_{x \to \frac{\pi}{2}} \cos x = 0,$$

and also

$$\cos x > 0 \text{ if } -\frac{\pi}{2} < x < \frac{\pi}{2},$$

and

$$\cos x < 0 \text{ if } \frac{\pi}{2} < x < \frac{3\pi}{2}.$$

By Theorems 21, 27 and 28 it follows that

$$\lim_{x \to \frac{\pi}{2}^+} (\sin x)\left(\frac{1}{\cos x}\right) = -\infty,$$

and

$$\lim_{x \to \frac{\pi}{2}^-} (\sin x)\left(\frac{1}{\cos x}\right) = \infty,$$

so $x = \frac{\pi}{2}$ is the equation of a vertical asymptote of the graph of $y = \tan x$. In a similar manner $x = (2n - 1)\frac{\pi}{2}$ for any natural number n will be an asymptote.

Intermediate Mathematics 3, A13-4

In chapter 13, of the Appendix to Intermediate Mathematics 3, series and sequences are defined and considered. This section is devoted to an intuitive definition of limit and some assumed properties of limits.

In keeping with the established procedure of this thesis, the phrases found in the textbook's definition of a limit of a sequence are stated here in terms of ϵ and N. Thus the phrase, "a_n becomes and remains arbitrarily close to A as n gets larger and larger," gives rise to the following (ϵ, N)-definition that is given in the treatment of Geometry and restated here.

Definition 15-3.1. A sequence $\{a_n\}$ is said to have a limit A if for every $\epsilon > 0$ there exists a natural number $N > 0$ such that if $n > N$, then $|a_n - A| < \epsilon$.

Because no proofs are given in the textbook for the important limit theorems found on pages 758 and 759, they are repeated here and numbered in keeping with other theorems of this thesis. Because the proofs of many of these theorems are given by John F. Randolph in the 23rd Yearbook of the National Council of Teachers of Mathematics ([37] , pp. 208-16), they are not repeated here. However, the theorems listed include not only those cited by the textbook but also those proved by Randolph in proving the ones stated by the textbook. (In addition to the conventional (ϵ, N) or (e, N) definition he also uses an equivalent definition. His proofs are readily rewritten in (ϵ, N) form.)

On page 757 the following theorem is suggested:

Theorem A13-4.1. If a sequence has a limit A, then no other number $B \neq A$ is the limit of the sequence ([37] , p. 206).

On page 758 of the textbook a theorem is stated which is restated below, numbered in keeping with other theorems of this thesis, and proved.

Theorem A13-4.2. If $\{c_n\}$ is a constant sequence,

$$c_1 = c_2 = \ldots = c_n = \ldots,$$

then

$$\lim_{n \to \infty} c_n = c.$$

Proof: Definition 15-3.1 is immediately satisfied because for every $\epsilon > 0$ there exists $N = 1$ such that if $n > N$, then

$$|c - c| = 0 < \epsilon.$$

The following theorems are either found on page 759 or are necessary for their proofs:

Theorem A13-4.3. If $\lim_{n\to\infty} a_n = A$ and c is a real number, then

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \ ([37], \text{p. } 213).$$

Theorem A13-4.4. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n \ ([37], \text{p. } 213).$$

Theorem A13-4.5. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

$$\lim_{n\to\infty} (a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) \ ([37], \text{p. } 213).$$

Theorem A13-4.6. If $\lim_{n\to\infty} b_n = B \neq 0$ then there exists a natural number N such that if $n > N$, then $b_n \neq 0$ ([37], p. 214).

Theorem A13-4.7. If $\lim_{n\to\infty} b_n = B \neq 0$, then

$$\lim_{n\to\infty} \frac{B}{b_n} = 1 \ ([37], \text{p. } 214).$$

Theorem A13-4.8. If $\lim_{n\to\infty} b_n = B \neq 0$, then

$$\lim_{n\to\infty} \left(\frac{1}{b_n}\right) = \frac{1}{B} \ ([37], \text{p. } 214).$$

Theorem A13-4.9. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B \neq 0$, then

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B} \ ([37], \text{p. } 215).$$

These theorems are used throughout the remainder of Chapter 13 in the textbook. Also, the remaining sections are based directly on the concept of a limit and specifically on the theorems cited above.

Intermediate Mathematics 3, A13-5

The sum of an infinite series is defined in terms of the limit of a sequence of partial sums. On page 771 infinite series are given for e^x, sin x, and cos x. Because no discussion is given concerning the justification of these series, and because these series are treated in greater detail in Elementary Functions, they are not considered here.

Intermediate Mathematics 3, A13-6

In order to draw conclusions about the sum of an infinite geometric series the textbook considers some examples from which the following theorem and proof are taken:

Theorem A13-6.1. If $a_1 + \sum_{k=1}^{\infty} a_1 r^k$ is an infinite geometric series, then the series converges if (i) $|r| < 1$ or (ii) $a_1 = 0$ and diverges if (iii) $|r| \geq 1$ when $a_1 \neq 0$.

Proof: Previously it was shown that if $\{s_n\}$ is the sequence of partial sums, then

$$s_n = a_1 \left(\frac{1 - r^n}{1 - r}\right). \tag{1}$$

The sequence of partial sums would be

$$0, 0, 0, \cdots, 0, \cdots \text{ if } a_1 = 0 \text{ as in (ii)}$$

or

$$a_1, a_1, \cdots, a_1, \cdots \text{ if } r = 0.$$

In either case, by Theorem A13-4.2 these sequences converge.

If $|r| < 1$ and $r \neq 0$, then $\frac{1}{r} > 1$, and $r^n = \left(\frac{1}{1/r}\right)^n = \frac{1}{(1/r)^n}$. By Theorems 29 and 30

$$\lim_{n\to\infty} r^n = \lim_{n\to\infty} \frac{1}{(1/r)^n} = 0.$$

Therefore, in (1), $\lim_{n\to\infty} s_n = a_1 \left(\frac{1}{1 - r}\right)$ by Theorems A13-4.3 and A13-4.4. (An alternate proof of (i), in which mathematical induction is required, can be found in What is Mathematics? ([12] , pp. 64-5).)

To show (iii), that there is no sum of an infinite series if $|r| \geq 1$, special cases are considered. For example, if $r = 1$ and $a_1 \neq 0$ the geometric series becomes

$$a_1 + a_1 + a_1 + \cdots + a_1 + \cdots .$$

Assuming that this series had a sum would be assuming that the sequence of partial sums na_1 had a limit. But for every number N, $|na_1| > N$ whenever $n > \frac{N}{|a_1|}$ so na_1 could not have a limit.

If $r = -1$ the series would be

$$a_1 - a_1 + a_1 - a_1 + \cdots + (-1)^{n+1} a_1 + \cdots ,$$

and the sequence of partial sums would be

$$a_1, 0, a_1, 0, a_1, 0, \cdots .$$

There is no number L such that

$$|a_n - L| < \frac{a_1}{2} \text{ for } n > N \text{ for any N.}$$

Hence, there is no limit.

If $|r| > 1$, the sequence of partial sums is given by (1). Consider $r > 1$; then by Theorems 29, 31, and 32, $\lim_{n \to \infty} r^n = \infty$ and $\lim_{n \to \infty} s_n = \pm \infty$ depending on the sign of a_1.

Consider $r < -1$. The sequence of alternate terms, $s_{2n} = a_1 \left(\frac{1 - r^{2n}}{1 - r}\right)$, can be treated as above, so that $\lim_{n \to \infty} s_{2n} = \pm\infty$ depending again on the sign of a_1. Therefore, no number L could be the sum in this case because for ever natural number N the alternate terms s_{2n} for $2n > N$ are increasing or decreasing without limit. That is for every natural number N there exists an even number $2k > N$ such that $|s_{2k} - L| > 1$ so that L cannot be a limit.

Summary of Major Points

It was found that First Course in Algebra was presented from an intuitive level with no discussions that involve the concept of a limit. A major part of Geometry was found to be void of any discussions that involve limits. Because Birkhoff's postulates were accepted, many topics that depend on limits were treated by other means.

Not until the measure of the circumference and area of a circle and related topics were presented in Geometry were limits discussed. The discussion, however, was found to precede a definition of sequence and limit, and the theorems about limits of sequences. Futhermore, it was found that the existence of limits were tacitly assumed. Not until Intermediate Mathematics were the properties of the real numbers including the Completeness Property presented and it was upon these properties that a proof of the existence of a limit would depend.

In Intermediate Mathematics asymptotes were discussed which suggested the need for definitions and theorems concerning

$$\lim_{x\to\infty} f(x) = 0, \lim_{x\to a+} f(x) = \pm\infty, \lim_{x\to a-} f(x) = \pm\infty.$$

The logarithm function was found to be introduced in the Appendix to Intermediate Mathematics 2 as the area under the graph of $y = \frac{1}{x}$. Although such area was not defined, a proof was given for this study to show that the logarithm function is continuous. The proof required a limit definition of continuity instead of the textbook's explanation which was: "the graph has no holes or gaps."

A definition of sequence, series and limit of a sequence was found in Intermediate Mathematics 3. Some theorems about limits mentioned without proof in the textbook were restated for this study and proved or sources for proofs were cited.

The final topic treated was concerned with convergence of a geometric series. A proof was given that required theorems which were proved in Chapter IV. However, if mathematical induction had been presented previously, a more common proof, which was cited, for the convergence of the series could have been utilized.

CHAPTER III

THE LIMIT TOPICS FOUND IN ELEMENTARY FUNCTIONS

The textbook Elementary Functions is designed to be used in the twelfth grade. It includes discussions of polynomial, exponential, logarithm, and circular functions which are designed to provide the student with a better background for the calculus without trespassing upon it as it is taught at the college level.

Elementary Functions 1, 2-3

On page 50, a polynomial function is said to have a graph that is "a continuous curve with no breaks or holes in it." This is the only description given in the student's book regarding continuity of a polynomial function. Also, the Commentary for Teachers on page 33 points out that, even though "continuity of polynomial functions is assumed," an explanation of the assumption would involve showing that the "graph of a polynomial function contains no holes or breaks." No other discussion is given. Definitions 2 and 3 of Chapter IV are the common definitions of a continuous function in which a limit is used. On the basis of these definitions the continuity of a graph can be defined (Definition 4). It should be noted, however, that so far in the textbooks, there is no distinction made between continuous functions and continuous curves.

Another topic is begun on page 52 where it is stated that a term

$2x^3$ dominates the polynomial function

$$f: x \longrightarrow 2x^3 - 3x^2 - 12x + 13$$

which in factored form becomes

$$f: x \longrightarrow 2x^3 \left(1 - \frac{3}{2x} - \frac{6}{x^2} - \frac{13}{2x^3}\right).$$

The reason given is that "$2x^3$ dominates all other terms for large $|x| \cdots$ [because] for sufficiently large values of $|x|$, the expression in parentheses has a value close to 1." Additional explanation of a similar nature is found on page 53 of the Commentary for Teachers. From this discussion the following definition and theorems are drawn:

Definition 2-3.1. A term t of a polynomial expression f(x) is said to dominate the polynomial function $f: x \longrightarrow f(x)$ as $|x|$ increases if

$$\lim_{x \to \infty} \frac{f(x)}{t} = \lim_{x \to -\infty} \frac{f(x)}{t} = 1.$$

Theorem 2-3.1. As indicated in the textbook, $t = a_n x^n$; the term of highest degree dominates the polynomial as $|x|$ increases.

Proof: Consider $f: x \longrightarrow a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

The same function could be written for $x \neq 0$ as

$$f: x \longrightarrow a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_0}{a_n x^n}\right)$$

so that

$$\frac{f(x)}{a_n x^n} = 1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_0}{a_n x^n}.$$

Hence, it follows from Definitions 5 and 6, and Theorems 13 through 16 that

$$\lim_{x \to \infty} \frac{f(x)}{a_n x^n} = \lim_{x \to -\infty} \frac{f(x)}{a_n x^n} = 1.$$

Therefore, $a_n x^n$ is the dominating term of f.

Regarding this definition the textbook states on page 52: "This means that the sign of f(x) will agree with the sign of the term of largest degree for large $|x|$." This satement is proved here as a theorem.

Theorem 2-3.2. If a term t dominates a polynomial $f: x \longrightarrow f(x)$ as $|x|$ increases, there exist numbers m and n such that if

$x > m$, then f(x) and t have the same sign,

and if $x < n$, then f(x) and t have the same sign.

Proof: If

$$\lim_{x \to \infty} \frac{f(x)}{t} = 1,$$

then by definition of this limit, there exists a number m such that if $x > m$, then

$$\left| \frac{f(x)}{t} - 1 \right| < 1 \text{ or } \frac{f(x)}{t} > 0.$$

Likewise if

$$\lim_{x \to -\infty} \frac{f(x)}{t} = 1,$$

then there exists a number n such that if $x < n$, then $\frac{f(x)}{t} > 0$. This means that f(x) and t have the same sign whenever $x > m$ or $x < n$.

It would not be correct that the dominating term t could be defined as follows: A term t of a polynomial expression f(x) is said to dominate the polynomial function $f: x \longrightarrow f(x)$ if there exist numbers m and n such that if

$x > m$, then f(x) and t have the same sign,

or if $x < n$, then f(x) and t have the same sign.

Although this is a necessary condition, as has been shown above, it is not a sufficient condition. Consider

$$f: x \longrightarrow x^4 + x^2 + 1.$$

Because $f(x) > 0$ and $x^2 > 0$ for all $x \neq 0$, m could be chosen any positive number, n chosen any negative number, and it would follow that if

$x > m$, then $f(x)$ and x^2 have the same sign,

and if $x < n$, then $f(x)$ and x^2 have the same sign.

However,
$$\lim_{x\to\infty} \frac{f(x)}{x^2} = \lim_{x\to -\infty} \frac{f(x)}{x^2} = \infty \neq 1.$$

Elementary Functions 1, 2-5

In order to locate zeros of polynomial functions, it must be known that the graph of the function crosses the x-axis. A statement is made on page 59, called "The Location Theorem," that provides the student with this information. Because the statement is given without proof or a justifying argument, it is not treated in this study. However, the proof of the statement, a special case of the Intermediate Value Theorem, is proved using a limit process in Taylor's Advanced Calculus ([50] , pp. 103-4).

Elementary Functions 1, 3-1

In the first topic of Chapter 3, "Introduction," brief remarks are given on a strictly intuitive level. Because every topic mentioned in the first section is discussed more fully in later sections, the rigorous treatment of limit processes is given in conjunction with the presentations found later in Chapter 3.

Elementary Functions 1, 3-3

The terms "tangent" and "best linear approximation" to a graph are used interchangably in this and later chapters. In Section 3-3, discussion is given from which three definitions are drawn.

On page 97 it is stated that

$$f(x) = 1 + x$$

is the best linear approximation to the graph of

$$f(x) = 1 + x - 4x^2 = 1 + (1 - 4x)x$$

at the point $P(0, f(0))$ because the expression $1 - 4x$ can be made to "lie as close to 1 as we please by making $|x|$ sufficiently small."

Hence, the first definition follows:

Definition 3-3.1. The equation $y = f(0) + mx$ is that of the best linear approximation to the graph of

$$f: x \longrightarrow f(x) \text{ at } P(0, f(0))$$

if $f(x)$ can be written

$$f(x) = f(0) + (q(x))x$$

where $$\lim_{x \to 0} q(x) = m.$$

A second definition, called the "wedge" interpretation (Commentary for Teachers, page 64), is extracted from page 99 where it is stated that "if we stay close enough to $x = 0$ the graph of $f: x \longrightarrow 1 + (1 - 4x)x$ lies between two lines$\cdots$which differ in direction as little as we please."

Definition 3-3.2. The equation $y = f(0) + mx$ is that of the best linear approximation to the graph of

$$f: x \longrightarrow f(x) \text{ at } P(0, f(0))$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, then the graph of $y = f(x)$ lies between the two lines

$$L_1 : y = f(0) + (m - \epsilon)x$$

and $$L_2 : y = f(0) + (m + \epsilon)x.$$

Algebraically this means that if $0 < |x| < \delta$, then

$$f(0) + (m - \epsilon)x < f(x) < f(0) + (m + \epsilon)x \text{ for } x > 0,$$

and $$f(0) + (m - \epsilon)x > f(x) > f(0) + (m + \epsilon)x \text{ for } x < 0.$$

The Commentary for Teachers on page 65 mentions that the definitional statements quoted above avoid difficulties connected with limits of quotients as are commonly found in college calculus textbooks [21], [51], [53]. The third definition of the best linear approximation, therefore, involves the limit of a quotient to which the Commentary for Teachers refers.

Definition 3-3.3. The equation $y = f(0) + mx$ is that of the best linear approximation to the graph of

$$f: x \longrightarrow f(x) \text{ at } P(0, f(0))$$

if $$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = m.$$

Theorem 3-3.1. Definitions 3-3.2 and 3-3.3 are equivalent.

Proof: By Definition 3-3.2, $y = f(0) + mx$ is the equation of the tangent to the graph of

$$f: x \longrightarrow f(x) \text{ at } P(0, f(0))$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$,

then $$f(0) + (m - \epsilon)x < f(x) < f(0) + (m + \epsilon)x \text{ for } x > 0,$$

and $$f(0) + (m - \epsilon)x > f(x) > f(0) + (m + \epsilon)x \text{ for } x < 0.$$

This means that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, the above two inequalities reduce to give

$$m - \epsilon < \frac{f(x) - f(0)}{x} < m + \epsilon,$$

or $$\left| \frac{f(x) - f(0)}{x} - m \right| < \epsilon.$$

Now, by Definition 1

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = m$$

and it follows that Definition 3-3.2 implies Definition 3-3.3.

Since each step of the above proof is reversable, a reversal of the steps would show that Definition 3-3.3 implies Definition 3-3.2. Hence these two definitions are equivalent.

Theorem 3-3.2. If f is a polynomial, Definitions 3-3.1 and 3-3.2 are equivalent.

Proof: Definition 3-3.1 requires that f be written as

$$f: x \longrightarrow f(x) = f(0) + (q(x))x.$$

In the case that f is a polynomial,

$$f(x) = a_0 + (a_1 + \cdots + a_n x^{n-1})x,$$

the definition is readily usable. Therefore, assume that definition 3-3.2 holds for $f(x)$. Hence, for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, then

$$f(0) + (m - \epsilon)x < f(0) + (q(x))x < f(0) + (m + \epsilon)x \text{ for } x > 0,$$

and

$$f(0) + (m - \epsilon)x > f(0) + (q(x))x > f(0) + (m + \epsilon)x \text{ for } x < 0.$$

This implies that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, the above two inequalities reduce to give

$$m - \epsilon < q(x) < m + \epsilon,$$

or $$|q(x) - m| < \epsilon.$$

Thus, $$\lim_{x \to 0} q(x) = m.$$

Therefore, Definition 3-3.1 is satisfied when Definition 3-3.2 is assumed.

By reversing the above steps the converse of this statement can be verified so it follows that Definition 3-3.1 and Definition 3-3.2 are equivalent if f is a polynomial.

Now, since it has been proved that Definitions 3-3.2 and 3-3.3 are equivalent, and Definitions 3-3.1 and 3-3.2 are equivalent if f is a polynomial, then it follows that Definitions 3-3.1 and 3-3.2 are equivalent if f is a polynomial. Therefore, the following theorem is proved:

Theorem 3-3.3. Definitions 3-3.1, 3-3.2, and 3-3.3 are logically equivalent when applied to polynomials.

In applying the above mentioned definitions to a polynomial function $f: x \longrightarrow a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, the following substitutions are required:

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \\ f(0) &= a_0, \\ m &= a_1, \\ q(x) &= (a_1 + a_2 x + \cdots + a_n x^{n-1}). \end{aligned}$$

On the basis of these formally stated definitions the following theorem, which is used by the textbooks, can be rigorously proved:

Theorem 3-3.4. If $f: x \longrightarrow a_0 + a_1 x + \cdots + a_n x^n$ is a polynomial function, the equation of the tangent or best linear approximation to the graph of f at $P(0, f(0))$ exists and is $y = a_0 + a_1 x$.

Proof: Consider

$$a_0 + a_1 x + \cdots + a_n x^n = a_0 + (a_1 + a_2 x + \cdots + a_n x^{n-1})\, x.$$

Now

$$f(0) = a_0,$$

$$q(x) = a_1 + a_2 x + \cdots + a_n x^{n-1},$$

and by Definition 1 and Theorems 4, 8, and 10.

$$\lim_{x \to 0} q(x) = a_1,$$

so that $y = a_0 + a_1 x$ is the best linear approximation by Definition 3-3.1.

Elementary Functions 1, 3-4

In this section two topics are discussed which involve limits. The first one concerns the dominating term of a polynomial for x in a neighborhood of 0 and the second refers to the best polynomial approximation to the graph of a function.

On page 101 an example is given to provide an answer to the question: "Which term dominates the situation and determines the shape [of the graph of the function] about $x = 0$?" Concerning the example

$$f: x \longrightarrow 1 + x + x^2 - 2x^3$$

it is said that "sufficiently near $x = 0$ the lower degree term x^2 dominates the higher degree term $-2x^3$ and that the graph of f has the same character as if the term $-2x^3$ were missing." These statements yield the following definitions and theorems:

Definition 3-4.1. A term t of a polynomial expression $f(x)$ is said to dominate the polynomial function $f: x \longrightarrow f(x)$ for x in a neighborhood of 0 (or about $x = 0$) if

$$\lim_{x \to 0} \frac{f(x)}{t} = 1.$$

It is to be noted that in this and some later sections of both the textbook and this thesis the definition is used to determine the dominating term of a polynomial that is a part of the polynomial under initial consideration. Consider the example cited above. The polynomial under initial consideration is

$$f: x \longrightarrow 1 + x + x^2 - 2x^3$$

The polynomial, a part of the one under initial consideration, that is dominated by x^2 for x in a neighborhood of 0 is

$$x \longrightarrow x^2 - 2x^3.$$

Theorem 3-4.1. The term of lowest degree is the only term that dominates a polynomial for x in a neighborhood of 0.

Proof: Consider

$$f: x \longrightarrow a_k x^k + a_{k+1} x^{k+1} + \cdots + a_n x^n,$$

where $k < n$ and $a_k \neq 0$. Now

$$a_k x^k + a_{k+1} x^{k+1} + \cdots + a_n x^n = a_k x^k \left(1 + \frac{a_{k+1}}{a_k} x + \cdots + \frac{a_n}{a_k} x^{n-k}\right),$$

so that

$$\lim_{x \to 0} \frac{f(x)}{a_k x^k} = \lim_{x \to 0} \left(1 + \frac{a_{k+1}}{a_k} x + \cdots + \frac{a_n}{a_k} x^{n-k}\right).$$

Now by Definition 1 and Theorems 4, 8, and 10.

$$\lim_{x \to 0} \frac{f(x)}{a_k x^k} = 1$$

so that $a_k x^k$ dominates the polynomial function. To show that the term of lowest degree is the only dominating term, consider any other term $a_m x^m$ where $a_m \neq 0$, $k < m \leq n$, of the polynomial expression. Then,

$$\frac{f(x)}{a_m x^m} = \frac{a_k}{a_m} x^{k-m} + \frac{a_{k+1}}{a_m} x^{k+1-m} + \cdots + \frac{a_n}{a_m} x^{n-m}$$

where at least $k - m < 0$ and for some number j, $k + j = m$.

$$\frac{f(x)}{a_m x^m} = \frac{(a_k/a_m)x^k + \cdots + (a_{k+j}/a_m)x^m + \cdots + (a_n/a_m)x^n}{x^m}$$

which by Definition 1 and Theorem 33 is increasing or decreasing without limit as x approaches 0 depending on the sign of a_k/a_m.

Theorem 3-4.2. If a term t dominates a polynomial $f: x \longrightarrow f(x)$

for x in a neighborhood of 0, there exists a number $\delta > 0$ such that if $0 < |x| < \delta$, then f(x) and t have the same sign.

Proof: If

$$\lim_{x \to 0} \frac{f(x)}{t} = 1,$$

then there exists a number $\delta > 0$ such that if $0 < |x| < \delta$, then

$$\left| \frac{f(x)}{t} - 1 \right| < 1$$

or

$$\frac{f(x)}{t} > 0.$$

This means that if $0 < |x| < \delta$ then f(x) and t have the same sign.

It would not be correct to define the dominating term t as follows: A term t of a polynomial expression f(x) is said to dominate the polynomial function $f: x \longrightarrow f(x)$ for x in a neighborhood of 0 if there exists a number $\delta > 0$ such that f(x) and t have the same sign whenever $0 < |x| < \delta$. Although this is a necessary condition, as has been shown above, it is not a sufficient condition. Consider

$$f: x \longrightarrow x^2 + 1.$$

Because $f(x) > 0$ and $x^2 > 0$ for every number x and any $\delta > 0$ where $0 < |x| < \delta$, then f(x) and x^2 have the same sign. However

$$\lim_{x \to 0} \frac{f(x)}{x^2} = \infty \neq 1.$$

Also, in this section of the textbook, examples are given of the best second degree approximation and the best third degree approximation to the graph of a polynomial at P(0, f(0)). A geometric as well as an algebraic interpretation are given.

On page 102 it is stated,

"$x \longrightarrow 1 + x + x^2$

is the best quadratic approximation to the graph of

$$f\colon x \longrightarrow 1 + x + x^2 - 2x^3\text{''} \text{ at } P(0, f(0))$$

because: "We can write

$$f(x) = 1 + x + x^2 - 2x^3$$

in the form $\qquad f(x) = 1 + x + (1-2x)x^2$

and note that 1-2x is arbitrarily close to 1 for $|x|$ small enough."
This is interpreted in the following definition:

Definition 3-4.2. The equation $g(x) = a_0 + a_1x + \cdots + a_rx^r$ is that of the best rth degree polynomial approximation to the graph of $f\colon x \longrightarrow f(x)$ at $P(0, f(0))$ if $f(x)$ can be written as

$$f(x) = a_0 + a_1x + \cdots + a_{r-1}x^{r-1} + (q(x))x^r$$

where $\qquad \lim_{x\to 0} q(x) = a_r.$

The second definition is drawn from pages 103 and 104. Two examples are used in which a best approximation is found, and in summary a generalized definitional statement is given for the best second degree approximation to the graph of a third degree polynomial:

"If $\qquad f\colon x \longrightarrow a_0 + a_1x + a_2x^2 + a_3x^3$

we can write $\qquad f(x) = a_0 + a_1x + (a_2 + a_3x)x^2$

and conclude that the graph lies between the graphs of

$$x \longrightarrow a_0 + a_1x + (a_2 + \epsilon)x^2$$

and $\qquad x \longrightarrow a_0 + a_1x + (a_2 - \epsilon)x^2$

for arbitrarily small ϵ, provided that $|a_3x| < \epsilon$." This is further generalized in the following definition:

Definition 3-4.3. The equation $g(x) = f(0) + a_1x + \cdots + a_rx^r$ is that of the best rth degree polynomial approximation to the graph of $f\colon x \longrightarrow f(x)$ at $P(0, f(0))$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, then the graph of f lies between the graphs of

$$g_1 : x \longrightarrow f(0) + a_1x + \cdots + (a_r + \epsilon)x^r$$

and $$g_2 : x \longrightarrow f(0) + a_1x + \cdots + (a_r - \epsilon)x^r.$$

Theorem 3-4.2. Definitions 3-4.2 and 3-4.3 are equivalent with regard to polynomial functions.

Proof: Consider for $r < n$,

$$f: x \longrightarrow a_0 + a_1x + \cdots + a_rx^r + \cdots + a_nx^n,$$

and $$g: x \longrightarrow f(0) + b_1x + \cdots + b_{r-1}x^{r-1} + b_rx^r.$$

Assume that g is the best rth degree polynomial approximation to the graph of f in the sense of Definition 3-4.3. Therefore, $f(0) = a_0$, and for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, the graph of f lies between

$$g_1: x \longrightarrow a_0 + b_1x + \cdots + b_{r-1}x^{r-1} + (b_r + \epsilon)x^r$$

and $$g_2: x \longrightarrow a_0 + b_1x + \cdots + b_{r-1}x^{r-1} + (b_r - \epsilon)x^r.$$

This means that either

$$g_1(x) < f(x) < g_2(x) \tag{1}$$

or $$g_1(x) > f(x) > g_2(x). \tag{2}$$

Hence, upon division by $x \neq 0$, either

$$b_1 + \cdots + (b_r - \epsilon)x^{r-1} < a_1 + \cdots + a_nx^{n-1} < b_1 + \cdots + (b_r + \epsilon)x^{r-1}$$

or

$$b_1 + \cdots + (b_r - \epsilon)x^{r-1} > a_1 + \cdots + a_nx^{n-1} > b_1 + \cdots + (b_r + \epsilon)x^{r-1}.$$

Now to show that $b_1 = a_1$, let $|b_1 - a_1| = k$.

By Theorems 3 through 10, it follows that

$$\lim_{x \to 0} (a_1 + \cdots + a_rx^{r-1} + \cdots + a_nx^{n-1}) = a_1$$

and $$\lim_{x \to 0} (b_1 + \cdots + (b_r \pm \epsilon)x^{r-1}) = b_1.$$

Hence, for every ϵ and δ in the sense of Definition 3-4.3, there is for every $\epsilon' > 0$ a number $\delta' > 0$, $\delta' < \delta$, such that if $0 < |x| < \delta'$, then

$$|a_1 + \cdots + a_r x^{r-1} + \cdots + a_n x^{n-1} - a_1| < \epsilon'$$

and $\quad |b_1 + \cdots + (b_r \pm \epsilon)x^{r-1} - b_1)| < \epsilon'$.

This means that if $\epsilon' = 1/3|b_1 - a_1| > 0$, that

$a_1 + \cdots + a_r x^{r-1} + \cdots + a_n x^{n-1}$ is within a distance of $1/3\ |b_1 - a_1|$ from a_1 for $0 < |x| < \delta'$. Simultaneously however, by the requirements imposed by Definition 3-4.3,

$a_1 + \cdots + a_r x^{r-1} + \cdots + a_n x^{n-1}$ is also within a distance of $1/3\ |b_1 - a_1|$ from b_1 for $0 < |x| < \delta'$. This is impossible unless $|b_1 - a_1| = 0$ or $b_1 = a_1$.

The same procedure can be repeated to show

$$b_2 = a_2,\ b_3 = a_3,\ \cdots,\ b_r = a_r.$$

Now g can be written

$$g\colon x \longrightarrow a_0 + a_1 x + \cdots + a_{r-1}x^{r-1} + a_r x^r$$

and (1) and (2) can be replaced by the restriction that either

$$a_0 + \cdots + (a_r - \epsilon)x^r < a_0 + \cdots + a_n x^n < a_0 + \cdots + (a_r + \epsilon)x^r$$

or

$$a_0 + \cdots + (a_r - \epsilon)x^r > a_0 + \cdots + a_n x^n > a_0 + \cdots + (a_r + \epsilon)x^r .$$

In either case, the inequalities reduce for $x \neq 0$ to

$$a_r - \epsilon < \frac{a_r x^r + \cdots + a_n x^n}{x^r} < a_r + \epsilon .$$

This means that

$$\lim_{x \to 0} (a_r + \cdots + a_n x^{n-r}) = a_r .$$

In the polynomial f, rewritten here in different form,

$$f: x \longrightarrow a_0 + a_1 x + \cdots + (a_r + \cdots + a_n x^{n-r})x^r,$$

it follows that

$$q(x) = a_r + \cdots + a_n x^{n-r}.$$

Hence, $$\lim_{x \to 0} q(x) = a_r,$$

which is the requirement of Definition 3-4.2. A reversal of the above steps would show that Definitions 3-4.2 would imply Definition 3-4.3. Hence the two definitions are equivalent.

Theorem 3-4.3. The best rth degree polynomial approximation to the graph of a polynomial function

$$f: x \longrightarrow a_0 + \cdots + a_r x^r + \cdots + a_n x^n$$

at P(0, f(0)) is

$$y = a_0 + a_1 x + \cdots + a_r x^r.$$

Proof: By factoring, f can be written

$$a_0 + a_1 x + \cdots + (a_r + \cdots + a_n x^{n-r})x^r.$$

By Definition 1 and Theorems 4, 8, and 10

$$\lim_{x \to 0} (a_r + \cdots + a_n x^{n-r}) = a_r,$$

so Definition 3-4.2 is satisfied and the theorem is proved.

Although the textbooks have not given (ϵ, δ) - definitions and proofs, Exercises 7 through 12 on page 105 ask the student to "show that for any ϵ however small it is possible to choose $|x|$ so that f(x) lies between"

$$a_0 + a_1 x + \cdots + (a_r + \epsilon)x^r$$

and $$a_0 + a_1 x + \cdots + (a_r - \epsilon)x^r.$$

(The addition of $\cdots +$ in the above expressions is necessary and was not included by the textbook. This problem is essentially that of finding the number δ that corresponds to an arbitrarily selected $\epsilon > 0$.)

Elementary Functions 1, 3-5

The discussion in this section is equivalent to that given in 3-3 and 3-4 with a substitution of variables that results from a linear transformation. For an arbitrary point P(h, f(h)) the statements in the textbook as well as the definitions and theorems in this study that involve the best approximations and the dominating term for x in a neighborhood of h would be altered by substituting x-h for x and h for 0. The discussion in the textbook suggests that

$$f(x) = a_0 + a_1x + \cdots + a_nx^n$$

be written in terms of x-h as

$$f(x) = a_0' + a_1'(x - h) + \cdots + a_n'(x - h)^n$$

which is a horizontal translation of

$$g(x) = a_0' + a_1'x + \cdots + a_n'x^n .$$

The aforementioned definitions and theorems then could have been given in terms of x-h so that h = 0 would have been a special case. Example: The best linear approximation to the graph of f at P(h, f(h)) would be

$$y = a_0' + a_1'(x - h)$$

because

$$f(x) = a_0' + (a_1' + \cdots + a_n'(x - h)^{n-h})(x - h)$$

and

$$\lim_{x \to h} (a_1' + a_2'(x - h) + \cdots + a_n'(x - h)^{n - h}) = a_1' .$$

The dominating term for x in a neighborhood of h for

$$g: x \longrightarrow a_r'(x - h)^r + \cdots + a_n'(x - h)^n$$

would be $a_r'(x - h)^r$ because

$$\lim_{x \to h} \frac{a_r'(x - h)^r + \cdots + a_n'(x - h)^n}{a_r'(x - h)^r} = 1.$$

(It follows from Definition 1 that $\lim_{x \to h} f(x) = \lim_{x - h \to 0} f(x)$.)

The textbook, however, more often refers to the approximations at $P(0, f(0))$ than at any other point. For this reason, and also because the definitions and theorems concerning $P(0, f(0))$ are more simple to state, because they are readily generalized, and because the textbook gives a more complete discussion concerning $P(0, f(0))$, they are used in this study instead of the general ones.

Elementary Functions 1, 3-11

This section is a summary of Chapter 3. Now that the textbook's discussions which involve the concept of a limit have been interpreted formally in this study, the summary could be written in the language of this new interpretation.

For example, on page 136 the textbook states: "If P is the point $(h, f(h))$ on the graph G of a polynomial function $f: x \longrightarrow f(x)$, there exists a straight line T through P which is called the tangent to G at P. T is the best linear approximation to G at P in the following sense $\cdots$."

The paragraph can be completed by: Let m be the slope of T and ϵ any arbitrarily small number. Then there exists $\delta > 0$ such that if $0 < |x - h| < \delta$,

$$f(h) + (m - \epsilon)(x - h) < f(x) < f(h) + (m + \epsilon)(x - h) \text{ for } x > h,$$

and

$$f(h) + (m - \epsilon)(x - h) > f(x) > f(h) + (m + \epsilon)(x - h) \text{ for } x < h.$$

Or by: Let m be the slope of T. Then

$$\lim_{x \to h} \frac{f(x) - f(h)}{x - h} = m.$$

Elementary Functions 1, A2-12

Curve fitting is the topic discussed in this section. In order to consider a curve through a finite set of points it is stated on page A-17 that "we would prefer to work with polynomials ··· for the purpose of fitting a continuous graph to a finite number of points." Here, as in previous sections, a polynomial function is assumed to have a "continuous graph."

On pages A-22 and A-23 a different type of continuity is suggested because here the term is used to refer to a function and not to its graph. However, the following definition is inferred by the discussion.

Definition A2-12.1. A function is continuous if it has a continuous graph.

An interpretation of the Weierstrass Approximation Theorem is also given which suggests that a limit would be needed if more discussion were to be given. A proof of this theorem which involves limits can be found in McShane's *Real Analysis* ([32] , pp. 88-89).

Elementary Functions 1, A3-12

This section is devoted to an introduction of the integral in an informal manner only, for it is stated that "the extended study of this key concept must ··· await further developments in your mathematical education." A more thorough treatment of the discussion in this section can be found in college calculus texts [21] , [51] , [53] .

On page A-29 an example is discussed in which the area under the graph of $y = x^2$, above the x-axis between $x = 0$ and a vertical line through $x > 0$, is assumed to be between an over-estimate

$$\left(\frac{x^3}{n^3}\right)\left(\frac{1}{6}\right)(2n^3 + 3n^2 + n)$$

and an under-estimate

$$\left(\frac{x^3}{n^3}\right)\left(\frac{1}{6}\right)(2n^3 - 3n^2 + n)$$

for every integer $n > 0$. (Area is not defined for figures not considered in Geometry. The student's intuition and visual analysis of the pictures are relied upon.) These estimates and the Completeness Property for real numbers are used to arrive at a value for the area desired. The argument given by the book, which involves statements such as "the difference between the estimates ... is small if n is large," is restated here as a theorem.

<u>Theorem A3-12.1.</u> There exists one and only one real number $A(x)$ such that

$$\left(\frac{x^3}{n^3}\right)\left(\frac{1}{6}\right)(2n^3 - 3n^2 + n) < A(x) < \left(\frac{x^3}{n^3}\right)\left(\frac{1}{6}\right)(2n^3 + 3n^2 + n)$$

for every integer $n > 0$.

Proof: (i) If $m > n > 0$ where m and n are integers, then $2m > m + n$, and $2mn \geq 2m$ because $n \geq 1$, and so $3mn > 2mn > m + n$. Hence,

$3 > \left(\frac{1}{m} + \frac{1}{n}\right)$, $3\left(\frac{1}{m} - \frac{1}{n}\right) < \left(\frac{1}{m^2} - \frac{1}{n^2}\right)$, and $\frac{3}{m} - \frac{1}{m^2} < \frac{3}{n} - \frac{1}{n^2}$,

so $$-\frac{3}{m} + \frac{1}{m^2} > -\frac{3}{n} + \frac{1}{n^2}.$$

Therefore, for $x > 0$

$$\left(\frac{x^3}{6}\right)\left(2 + \frac{-3}{m} + \frac{1}{m^2}\right) > \left(\frac{x^3}{6}\right)\left(2 + \frac{-3}{n} + \frac{1}{n^2}\right)$$

or

$$\left(\frac{x^3}{m^3}\right)\left(\frac{1}{6}\right)(2m^3 - 3m^2 + m) > \left(\frac{x^3}{n^3}\right)\left(\frac{1}{6}\right)(2n^3 - 3n^2 + n).$$

(ii) If $m > n > 0$ where m and n are integers, then

$$\frac{3}{m} < \frac{3}{n} \text{ and } \frac{1}{m^2} < \frac{1}{n^2}.$$

Therefore,

$$\left(\frac{x^3}{6}\right)\left(2 + \frac{3}{m} + \frac{1}{m^2}\right) < \left(\frac{x^3}{6}\right)\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

or

$$\left(\frac{x^3}{m^3}\right)\left(\frac{1}{6}\right)(2m^3 + 3m^2 + m) < \left(\frac{x^3}{n^3}\right)\left(\frac{1}{6}\right)(2n^3 + 3n^2 + n).$$

(iii) If $x > 0$ and n is a positive integer

$$\left(\frac{x^3}{n^3}\right)\left(\frac{1}{6}\right)(2n^3 - 3n^2 + n) < \left(\frac{x^3}{n^3}\right)\left(\frac{1}{6}\right)(2n^3 + 3n^2 + n) \text{ by } \frac{x^3}{n}.$$

(iv) This difference, $\frac{x^3}{n}$, is such that

$$\lim_{n\to\infty} \frac{x^3}{n} = 0.$$

Therefore, by the Completeness Property of the real numbers there exists one and only one number $A(x)$ such that for every n

$$\left(\frac{x^3}{n^3}\right)\left(\frac{1}{6}\right)(2n^3 - 3n^2 + n) < A(x) < \left(\frac{x^3}{n^3}\right)\left(\frac{1}{6}\right)(2n^3 + 3n^2 + n).$$

Now, by Theorem 34 and because

$$\lim_{n\to\infty} \left(\frac{x^3}{n^3}\right)\left(\frac{1}{6}\right)(2n^3 \pm 3n^2 + n) = \frac{x^3}{3},$$

it follows that $A(x) = \frac{x^3}{3}$.

Elementary Functions 1, A3-13

This section is devoted to the fundamental theorem of calculus and includes an argument for a special case. A more general proof can be found in college calculus texts such as the one by Johnson and Kiokemeister, ([21] , pp. 149 - 51). In the language of the textbook, A is the "area function" of the function f referred to in the previous section. The symbol A' is used for the "slope function" of the area function. Therefore, in these symbols the statement to be proved is $A' = f$.

The validity of this section, however, depends upon a statement found on page A-32 that is basic to this study: "If x-h is small enough, f(x) exceeds f(h) by any arbitrarily small amount, that is, for x-h small enough

$$f(x) < f(h) + \epsilon ."$$

The truth of this statement depends upon the function f being continuous as well as the definition of continuity itself. To this point in the textbooks, a function has been considered continuous if its graph has no "holes or breaks." Now Definition 2 is needed so that if f is assumed to be continuous, then $\lim_{x \to h} f(x) = f(h)$. Then it would follow that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - h| < \delta$, then $f(h) - \epsilon < f(x) < f(h) + \epsilon$. The argument in the textbook can be concluded by saying that for ever $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - h| < \delta$, then "the graph of A lies between the straight lines

$$y_1 = A(h) + f(h)(x - h)$$

and

$$y_2 = A(h) + (f(h) + \epsilon)(x - h))$$

for points near enough to P on the right side."

Because of the definition of continuity mentioned above, a similar argument for x on the left side of P would follow to give: For every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - h| < \delta$, then the graph of A lies between the straight lines

$$y_3 = A(h) + (f(h) - \epsilon)(x - h)$$

and $$y_1 = A(h) + f(h)(x - h).$$

for points near enough to P on the left.

These two statements summarize to give: For every $\epsilon > 0$ there exists $\delta > 0$ such if $0 < |x - h| < \delta$, then the graph A lies between

$$y_3 = A(h) + (f(h) - \epsilon)(x - h)$$

and $$y_2 = A(h) + (f(h) + \epsilon)(x - h).$$

This is Definition 3-3.2 of the statement: The equation $y = A(h) + f(h)(x - h)$ is that of the best linear approximation to the graph of A at $P(h, A(h))$. This proves the following theorem:

Theorem A3-13.1. If A is the area function associated with the function f, and if f is a continuous function, then $A' = f$.

Elementary Functions 2, 4-3

Integral and rational exponents have been used and discussed by the textbooks prior to this section. Now, arbitrary real exponents are to be introduced. The major part of this section is devoted to a discussion of $2^{\sqrt{2}}$ as an example of the way all other expressions 2^r can be defined. The major parts of the example are repeated in this report so that the example can be thoroughly treated.

It is given that $\{r_n\}$ is the sequence of the greatest (n + 1)-

digit numbers whose square is less than 2, thus

$$\{r_n\} = 1.4,\ 1.41,\ 1.414,\ 1.4142, \cdots . \tag{1}$$

Likewise, $\{s_n\}$ is the sequence of the smallest $(n + 1)$ - digit numbers whose square is greater than 2, thus

$$\{s_n\} = 1.5,\ 1.42,\ 1.415,\ 1.4143, \cdots . \tag{2}$$

On page 159 it is stated that "the difference $s_n - r_n$ can be made arbitrarily small." This means that $\lim_{n\to\infty} (s_n - r_n) = 0$, as surely it does because

$$s_n - r_n = \frac{1}{10^n} \tag{3}$$

which has 0 as a limit as n increases. The discussion concludes with the remark: "We $\cdots$ look at the intervals $2^{r_n} \leq y \leq 2^{s_n} \cdots$ which pinch down to a uniquely determined number, which we shall define as the number 2^x $[2^{\sqrt{2}}]$." This statement, which requires the conclusion of the Completeness Property, can be accepted if the hypothesis of the Completeness Property can be met. Necessary inequalities for meeting the hypothesis, however, are not given until page 161 where it is proved that

$$2^r < 2^s, \text{ if } r \text{ and } s \text{ are rational and } r < s. \tag{4}$$

Using these inequalities, the example is completed below and $2^{\sqrt{2}}$ is defined by the Completeness Property for real numbers.

Theorem 4-3.1. If $\{r_n\}$ and $\{s_n\}$ are as defined in (1) and (2), then $\{2^{r_n}\}$ and $\{2^{s_n}\}$ satisfy the hypothesis of the Completeness Property.

Proof: (i) $\{2^{r_n}\} = 2^{1.4},\ 2^{1.41},\ 2^{1.414},\ 2^{1.4142}, \ldots$

is such that

$$2^{1.4} < 2^{1.41} < 2^{1.414} < 2^{1.4142} < \cdots$$

by (4).

(ii) $\{2^{s_n}\} = 2^{1.5}, 2^{1.42}, 2^{1.415}, 2^{1.4143}, \ldots$

is such that

$$2^{1.5} > 2^{1.42} > 2^{1.415} > 2^{1.4143} > \ldots$$

by (4).

(iii) $2^{r_n} < 2^{s_n}$ because by (3) $r_n < s_n$, and (4).

(iv) $\lim_{n\to\infty} (2^{r_n} - 2^{s_n}) = 0$ for the following reason:

Because $s_n - r_n = \frac{1}{10^n}$, then

$$2^{s_n} - 2^{r_n} = 2^{r_n}(2^{\frac{1}{10^n}} - 1) < 2^2(2^{\frac{1}{10^n}} - 1),$$

and by (4)

$$2^{s_n} - 2^{r_n} > 0.$$

By Theorems A13-4.3 and A13-4.4 and Theorem 36

$$\lim_{n\to\infty} (2^2(2^{\frac{1}{10^n}} - 1)) = 0.$$

Therefore, by Theorem 34 since $0 < 2^{r_n} - 2^{s_n} < 2^2(2^{\frac{1}{10^n}} - 1)$ for every n,

$$\lim_{n\to\infty} (2^{r_n} - 2^{s_n}) = 0.$$

Now, by the Completeness Property, it can be concluded that there is a unique number y such that $2^{r_n} \leq y \leq 2^{s_n}$ for every n. This is the number defined to be $2^{\sqrt{2}}$.

In order to define 2^x for arbitrary real x, a short discussion is given on page 160. It is stated that "the number obtained [y, as in the previous example] is independent of choice of sequences." Even though the Commentary for Teachers page 159 says, "This development is presented at an intuitive level," the interpretation of the discussion regarding $2^{\sqrt{2}}$ yields the following definition:

Definition 4-3.1. If $\{r_n\}$ is a monotonic increasing sequence, and $\{s_n\}$ is monotonic decreasing sequence such that

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = x$$

then a^x is the uniquely determined number

$$a^{r_n} \leq a^x \leq a^{s_n} \text{ for every } n.$$

Such a definition cannot be used, however, until it is proved that $\{a^{r_n}\}$ and $\{a^{s_n}\}$ satisfy the requirements of the Completeness Property as was shown in the above example of $2^{\sqrt{2}}$. No discussion of any type is given in the textbooks regarding this question. Therefore, it is assumed that the definition and development of real numbers of the type a^x, where x is an arbitrary real number, is to be omitted in these textbooks and hence from this study. References are given in Intermediate Mathematics 1, Commentary for Teachers on page 49 regarding the development of real numbers in general; a^x is one type. Another approach that uses Cauchy sequences and limits can be found in Goffmann's Real Functions ([18] , pp. 28-45).

Therefore, in lieu of a justifying argument, the textbook assumes on page 164 that f: $x \longrightarrow a^x$ has a graph that is continuous, one-to-one, steadily increasing, and that the properties of exponents hold for irrational exponents the same as for rational exponents provided $a > 0$.

Elementary Functions 2, 4-6

The first part of this section is devoted to finding the equation of the tangent to the graph G of $y = 2^x$ at $P(0, 1)$. The method by which this is done is of considerable importance to this study. The section consists, mainly, of an argument to show that G lies in a "hatched region" (see Figure 4) so that Definition 3-3.2 can be used. It will be pointed out below, however, that the argument given by the textbook is dependent upon the assumption that the conditions of Definition 3-3.3 are given.

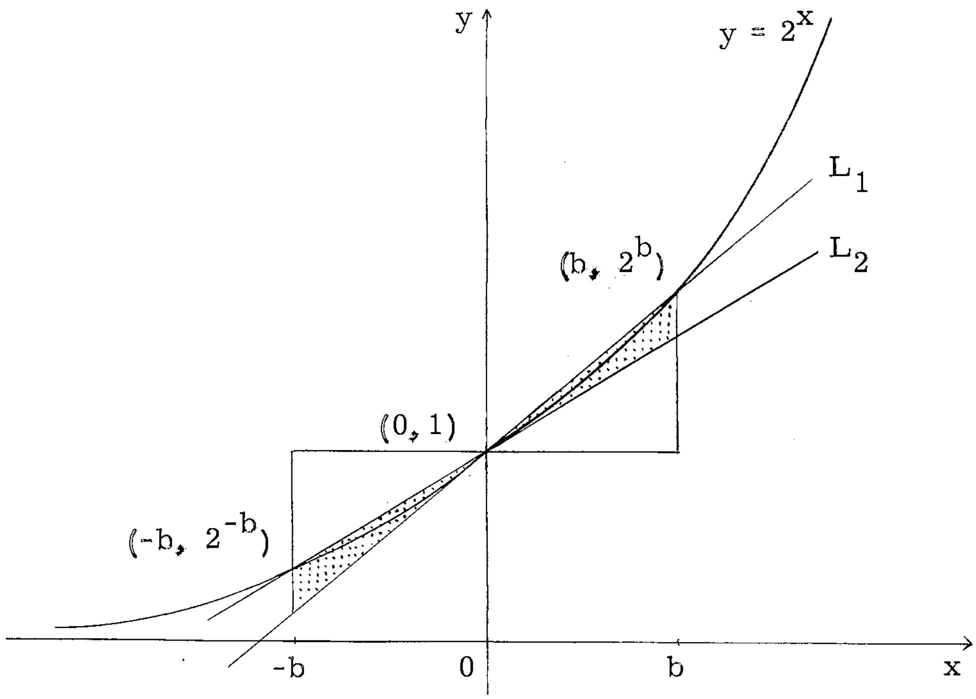


Figure 4

On pages 171 to 173 the discussion and exercises that are given are to prove what is stated here as a theorem. Although the theorem is not specifically stated in the textbook and of course δ is not used, a precise statement of the object of the discussion would provide the following theorem:

<u>Theorem 4-6.1.</u> For every number b such that $0 < b < 1$ there exists a number $\delta > 0$ such that if $0 < |x| < \delta$ then G, the graph of $y = 2^x$, lies in the "hatched region" (see Figure 4) between the lines

$$L_1: \ y = 1 + \frac{2^b - 1}{b} x,$$

$$L_2: \ y = 1 + \frac{2^{-b} - 1}{-b} x.$$

The argument found in the textbooks is correct except for failure to consider the sign of $1 - m^2x^2$ on page 172 and, similarly, $1 - \overline{m}^2x^2$ on page 169 of the Commentary for Teachers. It is stated that if $0 < x < b$,

$$\frac{1 - mx}{1 - m^2x^2} > 1 - mx$$

because "$1 - m^2x^2 < 1.$"

This statement is not necessarily true unless it is also stipulated that

$$0 < 1 - m^2x^2 \quad \text{or} \quad |x| < \left|\frac{1}{m}\right|.$$

Therefore, in this case, x must be chosen such that

$$0 < |x| < \min(b, \left|\frac{1}{m}\right|).$$

The completion of the textbook's argument, and hence the validity of the conclusion, is dependent upon the statement: "We expect that if b is small enough, the lines L_1 and L_2 will have slopes which differ by as little as we please." The slopes of the lines are

$$\frac{f(b) - f(0)}{b} \text{ and } \frac{f(-b) - f(0)}{-b}.$$

The sentence quoted is an assumption of the statement

$$\lim_{b\to o+} \frac{f(b) - f(0)}{b} = \lim_{-b\to o-} \frac{f(-b) - f(0)}{-b}.$$

By Theorem 25 if b is called x, then $\lim_{x\to o} \frac{f(x) - f(0)}{x}$ exists. Call this limit m and note that $f(0) = 1$. Thus, by Definition 3-3.3 it is assumed that $y = 1 + mx$ is the tangent to the graph of $y = 2^x$ at $P(0, 1)$.

Therefore, the argument given in the textbook is circular in that one definition which involves the limit of a quotient (Definition 3-3.3) is assumed in an argument given to show that the "wedge" definition (Definition 3-3.2) holds. However, it was proved in Theorem 3-3.1 that these two definitions are equivalent. A valid derivation of the equation of the tangent to the graph of $y = 2^x$, using Definition 3-3.3 can be found in the calculus textbook by Johnson and Kiokemeister ([21] , pp. 211-228).

Furthermore, it is the difference quotient found in Definition 3-3.3 that is used to approximate the slope of the tangent to the graph of $y = 2^x$ at $P(0, 1)$. This particular slope is henceforth, in the textbooks and this report, called k. On page 172, the example $b = 0.01$ is given and the slopes

$$\frac{f(b) - f(0)}{b} \quad \text{and} \quad \frac{f(-b) - f(0)}{-b}$$

are given as

$$\text{"}\frac{0.696}{0.01} \cdots \quad \text{and} \quad \frac{0.690}{0.01}\text{,"}$$

respectively. Thus, if $g: x \longrightarrow 2^x$, the equation of the tangent at $P(0, 1)$ is

$$y = kx + 1 \tag{1}$$

which the textbook writes as

$$\text{"}g(x) \approx kx + 1, \text{ for } |x| \text{ small."} \tag{2}$$

On page 173, general results are sought from the foregoing discussion. It is stated that for any $a > 0$, there exists a number α such that "$a = 2^\alpha$ so that

$$a^x = 2^{\alpha x} = g(\alpha x).\text{"}$$

Hence, "it follows that [as in (2)]

$$g(\alpha x) \approx k\alpha x + 1, \text{ for } |\alpha x| \text{ small."}$$

This statement is written in the form of (1) and proved here as a theorem.

Theorem 4-6.2. The equation $y = k\alpha x + 1$ is that of the tangent to the graph of $y = 2^{\alpha x} = a^x$ at P(0, 1).

Proof: By (1) with αx substituted for x and Definition 3-3.2 it follows that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, then

$$(k - \frac{\epsilon}{\alpha})(\alpha x) + 1 < 2^{(\alpha x)} < (k + \frac{\epsilon}{\alpha})(\alpha x) + 1 \text{ for } x > 0,$$

and $$(k - \frac{\epsilon}{\alpha})(\alpha x) + 1 > 2^{(\alpha x)} > (k + \frac{\epsilon}{\alpha})(\alpha x) + 1 \text{ for } x < 0.$$

Therefore,

$$(k\alpha - \epsilon)x + 1 < 2^{\alpha x} < (k\alpha + \epsilon)x + 1 \text{ for } x > 0,$$

and $$(k\alpha - \epsilon)x + 1 > 2^{\alpha x} < (k\alpha + \epsilon)x + 1 \text{ for } x < 0.$$

This means, by Definition 3-3.2, that the equation of the tangent to $y = 2^{\alpha x} = a^x$ at P(0, 1) is

$$y = k\alpha x + 1 \tag{3}$$

The final objective of this section is to define e as that number such that the graph of $y = e^x$ has as the equation of its tangent,

$$y = x + 1 \tag{4}$$

at P(0, 1). Hence, by (3) and (4) e is defined by $e = 2^{1/k}$.

"An important method for approximating the value of e, which would be expressed as follows

$$e \approx (1 + \frac{1}{n})^n \text{ for n large,"}$$

is given at the last of this section, on pages 174 and 175. In order that this statement "may be made plausable" the textbooks gives the following argument:

$$"e^x \approx 1 + x \text{ for } |x| \text{ near } 0,"$$

which become, for $x = \frac{1}{n}$

$$"e^{\frac{1}{n}} \approx 1 + \frac{1}{n}."$$

In the language of limits this means that

$$\lim_{x \to 0} e^x = \lim_{x \to 0} (1 + x) \text{ and } \lim_{n \to \infty} e^{\frac{1}{n}} = \lim_{n \to \infty} (1 + \frac{1}{n}).$$

These statements are indeed true but were found to be an inadequate basis to prove the required statement that

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n . \tag{5}$$

Consider for example,

$$\lim_{x \to 0} e^x = \lim_{x \to 0} (1 + 2x) \text{ and } \lim_{n \to \infty} e^{\frac{1}{n}} = \lim_{n \to \infty} (1 + \frac{2}{n}).$$

However, $e \neq \lim_{n \to \infty} (1 + \frac{2}{n})^n$ as can be shown by other means.

A rigorous development of the limit (5) can be found in the calculus textbook by Johnson and Kiokmeister ([21] , pp. 215-17) which is readily adaptable to use the Completeness Property as stated in this report.

Elementary Functions 2, 4-7

Having discussed in detail the tangent to the graph of $y = e^x$ at $P(0, 1)$, the textbook in this section generalizes the previous results to obtain the tangent to the graph of $y = e^x$ at $P(h, e^h)$. On page 178 it is stated that "we write $x = h + (x - h)$ and

$$e^x = e^{h + (x - h)} = e^h \cdot e^{x-h} .$$

For $|x - h|$ small enough, we use $[e^x \approx 1 + x]$ ... ." Using this suggestion the following statement is presented and proved as a theorem:

<u>Theorem 4-7.1.</u> The equation $y = e^h + e^h(x - h)$ is that of the tangent to the graph of $y = e^x$ at $P(h, e^h)$.

Proof: Let $y = 1 + x$ be the tangent to the graph of $y = e^x$ at $P(0, f(0))$. Hence, by Definition 3-3.2 it is given that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, then

$$1 + (1 - \frac{\epsilon}{e^h})x < e^x < 1 + (1 + \frac{\epsilon}{e^h})x \text{ for } x > 0,$$

and

$$1 + (1 - \frac{\epsilon}{e^h})x > e^x > 1 + (1 + \frac{\epsilon}{e^h})x \text{ for } x < 0.$$

Now making a linear transformation by substituting $x - h$ for x, the above becomes: For every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - h| < \delta$, then

$$1 + (1 - \frac{\epsilon}{e^h})(x - h) < e^{x - h} < 1 + (1 + \frac{\epsilon}{e^h})(x - h) \text{ for } x > h,$$

and

$$1 + (1 - \frac{\epsilon}{e^h})(x - h) > e^{x - h} > 1 + (1 + \frac{\epsilon}{e^h})(x - h) \text{ for } x < h.$$

Upon multiplication by $e^h > 0$, these inequalities can be written:

$$e^h + (e^h - \epsilon)(x - h) < e^x < e^h + (e^h + \epsilon)(x - h) \text{ for } x > h,$$

and

$$e^h + (e^h - \epsilon)(x - h) > e^x > e^h + (e^h + \epsilon)(x - h) \text{ for } x < h.$$

Therefore, by Definition 3-3.2, (with $x - h$ substituted for x) the equation of the tangent line to the graph of $y = e^x$ at $P(h, e^h)$ is $y = e^h + e^h(x - h)$.

<u>Elementary Functions 2</u>, 4-12

The discussion in this section, that refers to approximations for "$|x|$ small enough," is either repetition of what has been covered previously, or introduction to what will be covered later in A4-16 and A4-18. Therefore, this study follows the suggestion made by Commentary for Teachers (page 193): The answers to the questions which arise "are not answered in Section 4-12, but are discussed in the Appendix (Section 4-16) ··· and (Section 4-18)."

Elementary Function 2, 5-10

This section is devoted to finding the equations of the tangents to $y = \sin x$ and $y = \cos x$ at $P(0, f(0))$. Necessary inequalities are verified by the textbook; the three most important ones are:

if $\quad 0 < x < \frac{\pi}{2}$, then $x(1 - x^2) < \sin x < x$; (1)

if $\quad 0 > x > \frac{-\pi}{2}$, then $x(1 - x^2) > \sin x > x$; (2)

and if $\quad -\frac{\pi}{2} < x < \frac{\pi}{2}$, then $1 - x^2 < \cos x < 1$. (3)

The following theorems use an argument suggested by the textbooks and the definitions of tangent that are previously stated.

Theorem 5-10.1. The equation $y = x$ is that of the tangent to the graph of $y = \sin x$ at $P(0, 0)$.

Proof: $\lim_{x \to 0} (1 + x^2) = 1$ because for every $\epsilon > 0$ there exists $\delta = \sqrt{\epsilon} > 0$ such that if $0 < |x| < \delta$, then

$$|(1 - x^2) - 1| = |x^2| = x^2 < \epsilon,$$

and hence,
$$1 - x^2 > 1 - \epsilon.$$

This becomes, by (1) and (2): For every $\epsilon > 0$ there exists $\delta = \min(\frac{\pi}{2}, \sqrt{\epsilon}) > 0$ such that if $0 < |x| < \delta$, then

$$(1 - \epsilon)x < (1 - x^2)\, x < \sin x < x < (1 + \epsilon)x \quad \text{for } x > 0,$$

and
$$(1 - \epsilon)x > (1 - x^2)\, x > \sin x > x > (1 + \epsilon)x \quad \text{for } x < 0.$$

Therefore, by Definition 3-3.2, the equation of the tangent to the graph of $y = \sin x$ at $P(0, 0)$ is $y = x$.

Theorem 5-10.2. The equation $y = 1$ is that of the tangent to the graph of $y = \cos x$ at $P(0, 1)$.

Proof: $\lim\limits_{x \to 0} x = 0$ because for every $\epsilon > 0$ there exists $\delta = \epsilon > 0$ such that if $0 < |x| < \delta$, then

$$|x| < \epsilon \qquad \text{or} \qquad -\epsilon < x < \epsilon.$$

This becomes, by (3), for every $\epsilon > 0$ there exists $\delta = \min\left(\frac{\pi}{2}, \epsilon\right) > 0$ such that if $0 < |x| < \delta$, then

$$1 - \epsilon x < 1 - x^2 < \cos x < 1 < 1 + \epsilon x \quad \text{for } x > 0,$$

and $$1 - \epsilon x > 1 > \cos x > 1 - x^2 > 1 + \epsilon x \quad \text{for } x < 0.$$

Therefore, by Definition 3-3.2, the equation of the tangent to the graph of $y = \cos x$ is $y = 1$.

Elementary Functions 2, 5-11

The results of the previous section are to be generalized here to find the tangents to the graphs of $y = \sin x$ and $y = \cos x$ at a general point. This is done by writing x as $h + (x - h)$ and then using the trigonometric identities referred to as the "addition formulas." On page 278 it is given that

"$\sin[h + (x - h)] = \sin h \cos(x - h) + \cos h \sin(x - h)$, (1)

$\cos[h + (x - h)] = \cos h \cos(x - h) - \sin h \sin(x - h)$. (2)

We now replace $\cos(x - h)$ and $\sin(x - h)$ by their best linear approximations ⋯ and obtain the required tangent lines."

This procedure, however, requires the best linear approximation of the sum of two functions to be the sum of the best linear approximations, and the best linear approximation of the function g: $x \longrightarrow kf(x)$ to be k times the best linear approximation of f: $x \longrightarrow f(x)$. Because the textbook fails to consider these requirements, these two relationships are more exactly stated and proved below as theorems.

Theorem 5-11.1. If the equation of the tangent to the graph of f: $x \longrightarrow f(x)$ at $P(0, f(0))$ is $y = f(0) + mx$ and the equation of the tangent to the graph of g: $x \longrightarrow g(x)$ at $P(0, g(0))$ is $y = g(0) + nx$, then the equation of the tangent to the graph of $f + g$: $x \longrightarrow f(x) + g(x)$ at $P(0, f(0) + g(0))$ is

$$y = f(0) + g(0) + (m + n)x.$$

Proof: With regard to f, it can be stated by Definition 3-3.2 that for every $\epsilon > 0$ there exists $\delta_1 > 0$ such that if $0 < |x| < \delta_1$, then

$$f(0) + (m - \tfrac{\epsilon}{2})x < f(x) < f(0) + (m + \tfrac{\epsilon}{2})x \quad \text{for } x > 0,$$

and
$$f(0) + (m - \tfrac{\epsilon}{2})x > f(x) > f(0) + (m + \tfrac{\epsilon}{2})x \quad \text{for } x < 0.$$

With regard to g, it can be stated by Definition 3-3.2 that for every $\epsilon > 0$ there exists $\delta_2 > 0$ such that if $0 < |x| < \delta_2$, then

$$g(0) + (n - \tfrac{\epsilon}{2})x < g(x) < g(0) + (n + \tfrac{\epsilon}{2})x \quad \text{for } x > 0,$$

and
$$g(0) + (n - \tfrac{\epsilon}{2})x > g(x) > g(0) + (n + \tfrac{\epsilon}{2})x \quad \text{for } x < 0.$$

Now, for every $\epsilon > 0$ there exists $\delta = \min(\delta_1, \delta_2) > 0$ such that if $0 < |x| < \delta$, then both pairs of inequalities hold and can be added to give

$$f(0)+g(0) + (m+n-\epsilon)x < f(x)+g(x) < f(0)+g(0) + (m+n+\epsilon)x \text{ for } x > 0,$$

$$f(0)+g(0) + (m+n-\epsilon)x > f(x)+g(x) > f(0)+g(0) + (m+n+\epsilon)x \text{ for } x < 0.$$

Therefore, by Definition 3-3.2, $y = f(0) + g(0) + (m+n)x$ gives the best linear approximation of the sum, $f + g$: $x \longrightarrow f(x) + g(x)$, and is also the sum of the best linear approximations, $y = f(0) + mx$

and $y = g(0) + nx$.

Theorem 5-11.2. If the tangent to the graph of $f: x \longrightarrow f(x)$ at $P(0, f(0))$ is $y = f(0) + mx$, then the tangent to the graph of $g: x \longrightarrow kf(x)$ at $P(0, g(0))$ is $y = kf(0) + kmx$, for any real number k.

Proof: Let $k > 0$. By Definition 3-3.2, the hypothesis can be stated: For every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, then

$$f(0) + (m - \tfrac{\epsilon}{k})x < f(x) < f(0) + (m + \tfrac{\epsilon}{k})x \text{ for } x > 0,$$

and

$$f(0) + (m - \tfrac{\epsilon}{k})x > f(x) > f(0) + (m + \tfrac{\epsilon}{k})x \text{ for } x < 0.$$

The above inequalities can be multiplied by $k > 0$ to give

$$kf(0) + (km - \epsilon)x < kf(x) < kf(0) + (km + \epsilon)x \text{ for } x > 0,$$

and

$$kf(0) + (km - \epsilon)x > kf(x) > kf(0) + (km + \epsilon)x \text{ for } x < 0.$$

Therefore, it follows by Definition 3-3.2 that $y = kf(0) + kmx$ is the best linear approximation of $g: x \longrightarrow kf(x)$ at $P(0, g(0))$. For $k < 0$ the proof is essentially the same as that given above. If $k = 0$, the graph of g becomes the straight line $y = 0$ and the tangent coincides. Therefore, because every real number k has been considered, the theorem is proved.

The above theorems are readily generalized for $P(h, f(h))$ as was done in Elementary Functions 1, 3-5, so that the argument given in 5-11 is justified. Hence, it follows from (1) that the equation of the tangent to the graph of $y = \sin x$ at $P(h, \sin h)$ is

$$y = \sin h + (\cos h)(x - h),$$

and from (2) that the equation of the tangent to the graph of $y = \cos x$ at $P(h, \cos h)$ is

$$y = \cos h - (\sin h)(x - h).$$

Elementary Functions 2, A4-16

Previously, in section 4-12 the textbook conjectured that the best nth degree polynomial approximation to the graph of $f: x \longrightarrow e^x$ at $P(0, 1)$ is

$$g_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

The purpose of this section is to provide justification of this conjecture for $x > 0$.

The arguments given, however, were found to be incomplete and logically inadequate. After analyzing these arguments, this writer corresponded with Professor Donald E. Richmond of Williams College, Williamstown, Massachusetts, who is a member of the SMSG writing group ([47] , p. 143). Professor Richmond's letter that contains suggested proofs of the statements in question is reproduced in Appendix A of this report. In order that these arguments can be satisfactorily analyzed, they are reconsidered here.

In order to show that $y = g_n(x)$ is the equation of the best linear approximation of $f: x \longrightarrow e^x$ at $P(0, 1)$ when $x > 0$, the discussion suggests it is to be shown that the graph of f lies above

$$g_n: x \longrightarrow 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

for every $x > 0$, and below

$$h_n: x \longrightarrow 1 + x + \frac{x^2}{2!} + \cdots + \frac{cx^n}{n!} \qquad (1)$$

for some $x > 0$, depending on the number $c > 1$ (c is arbitrarily close to 1). This would satisfy Definition 3-3.2 if c-1 were called ϵ because then, for every $\epsilon > 0$ there would exists $\delta > 0$ such that if $0 < x < \delta$, then

$$g_n(x) < f(x) < h_n(x).$$

On page A-53, the discussion begins with the statements:

$$"g_n'(x) = g_n(x) - \frac{x^n}{n!},$$

$$g_n'(x) < g_n(x) \text{ when } x > 0.$$

The graph of the function $f: x \longrightarrow e^x$ climbs at such a rate that $f'(x)$ is always equal to $f(x)$ $[f'(x) = e^x]$. Since the graph of $g_n(x)$ climbs less rapidly it will fall below that of f." However, this last sentence is not justified on the basis of the given inequalities. If

$$f'(x) = f(x),$$

and

$$g_n'(x) < g_n(x),$$

it cannot be concluded that "the graph of $g_n(x)$ climbs less rapidly" than the graph of f, or

$$g_n'(x) < f'(x)$$

unless it is known that $g_n(x) < f(x)$. This is the relation to be proved and, therefore, it cannot be used in the argument. This proof can be completed, however, by other methods (see Appendix A).

Later on page A-53, $h_3(x)$ as special case of (1) is discussed. It is stated that: "$h_3(x) < g_3(x)$. It turns out that for sufficiently small positive values of x, $h_3(x)$ is also greater than e^x, as we now show. We wish to have $h_3(x)$ climb too fast to represent $x \longrightarrow e^x$ $[$or $h_3'(x) > f'(x)]$. This will be true if the slope is greater than the ordinate, that is,

$$h_3'(x) > h_3(x)."$$

However, this last sentence is not justified on the basis of the given inequalities. Even if

$$h_3'(x) > h_3(x) > g_3(x)$$

and

$$f'(x) = f(x),$$

it cannot be concluded that $h_3'(x) > f'(x)$ unless it is known that $h_3(x) > f(x)$ (which is to be proved), or $g_3(x) > f(x)$ (but it was proved that $g_n(x) < f(x)$). This relationship can be proved, as in the previous argument, by other methods (see Appendix A).

The remainder of this section is devoted to finding the number δ that corresponds to the number $c > 1$. If c is called $1 + \epsilon$ then for every $\epsilon > 0$ it would follow from the textbook's discussion that there exists a corresponding $\delta > 0$ which would be $n(\frac{\epsilon}{1+\epsilon})$. Hence, by Definition 3-3.2 the following theorem is justified for $x > 0$.

Theorem A4-16.1. The equation $g_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ is that of the best nth degree polynomial approximation to the graph of $y = e^x$ at $P(0, 1)$.

Elementary Functions 2, A4-18

This section is devoted to justifying approximations of the type

$$\text{"}\ln x \approx (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

for values of x near 1," or in general for $n = 1, 2, 3, \cdots$

$$\ln(1 + u) \approx u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots + (-1)^{n-1}\frac{u^n}{n}. \qquad (1)$$

On page A-60 it is stated: "It can be shown that in this case the error E made by replacing $\ln(1 + u)$ by . . .

$$\left[u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots + (-1)^{n-1}\frac{u^n}{n}\right]$$

is numerically less than $\frac{u^{n+1}}{n+1}$," that is

$$0 < E < \left|\frac{u^{n+1}}{n+1}\right| \text{ for } |u| < 1. \qquad (2)$$

No discussion is given in the textbook regarding the justification of (1), and it is stated on page A-61 that "a fuller justification of the methods discussed depends upon a knowledge of calculus." Specifically, a rigorous treatment in which ln $(1 + x)$ is used as an example can be found in Taylor's Advanced Calculus ([50] , pp. 115-116, 542-43).

It is the concern of this study to show that if (2) is accepted, the arguments included in the textbook are justified. First, consider the statements made on page A-61: "We note that if $u < 1$ then $\frac{u^{n+1}}{n+1}$ can be made arbitrarily small by choosing n large enough." This can be restated: if $|u| < 1$, then

$$\lim_{n\to\infty} \frac{u^{n+1}}{n+1} = 0. \tag{3}$$

Therefore, the desired conclusion, "a polynomial approximation of ln $(1 + u)$ can be found which is as accurate as you please," follows from (2) and (3) if $|u| < 1$.

Discussion is given concerning the approximations for fixed n and "for $|u|$ small" as in (1), but it is not argued that these approximations are the best nth degree polynomial approximations. However, if (2) is accepted, the textbooks could have continued with

$$\begin{aligned} &\ln(1+u) > u - \frac{u^2}{2} + \cdots + (-1)^{n-1}\frac{u^n}{n} - \left|\frac{u^{n+1}}{n+1}\right| \\ \text{and}\quad &\ln(1+u) < u - \frac{u^2}{2} + \cdots + (-1)^{n-1}\frac{u^n}{n} + \left|\frac{u^{n+1}}{n+1}\right|. \end{aligned} \tag{4}$$

Now if $1 > u > 0$, (4) becomes

$$\begin{aligned} &\ln(1+u) > u - \frac{u^2}{2} + \cdots + \left((-1)^{n-1} - \frac{n|u|}{n+1}\right)\frac{u^n}{n} \\ \text{and}\quad &\ln(1+u) < u - \frac{u^2}{2} + \cdots + \left((-1)^{n-1} + \frac{n|u|}{n+1}\right)\frac{u^n}{n}. \end{aligned} \tag{5}$$

If $-1 < u < 0$ and n is even, $|u^{n+1}| = -u^{n+1}$ and (4) becomes

$$\ln(1+u) > u - \frac{u^2}{2} + \cdots + ((-1)^{n-1} - \frac{n|u|}{n+1}) \frac{u^n}{n}$$

(6)

and $$\ln(1+u) < u - \frac{u^2}{2} + \cdots + ((-1)^{n-1} + \frac{n|u|}{n+1}) \frac{u^n}{n} .$$

If $-1 < u < 0$ and n is odd, $|u^{n+1}| = u^{n+1}$ and (4) becomes

$$\ln(1+u) > u - \frac{u^2}{2} + \cdots + ((-1)^{n-1} + \frac{n|u|}{n+1}) \frac{u^n}{n}$$

(7)

and $$\ln(1+u) < u - \frac{u^2}{2} + \cdots + ((-1)^{n-1} - \frac{n|u|}{n+1}) \frac{u^n}{n} .$$

But when n is fixed,

$$\lim_{u \to 0} \frac{n|u|}{n+1} = 0$$

and it follows that for every $\epsilon > 0$ there exists $\delta_1 > 0$ such that if $0 < |u| < \delta_1$, then

$$-\epsilon < \frac{n|u|}{n+1} < \epsilon .$$

Therefore, for every $\epsilon > 0$ there exists $\delta = \min(\delta_1, 1)$ such that if $0 < |u| < \delta$, then (5) and (6) can be written

$$\ln(1+u) > u - \frac{u^2}{2} + \cdots + ((-1)^{n-1} - \epsilon) \frac{u^n}{n}$$

and $$\ln(1+u) < u - \frac{u^2}{2} + \cdots + ((-1)^{n-1} + \epsilon) \frac{u^n}{n} ,$$

and (7) can be written

$$\ln(1+u) < u - \frac{u^2}{2} + \cdots + ((-1)^{n-1} - \epsilon) \frac{u^n}{n}$$

and $$\ln(1+u) > u - \frac{u^2}{2} + \cdots + ((-1)^{n-1} + \epsilon) \frac{u^n}{n} .$$

So, by Definition 3-4.3 the following theorem is proved:

Theorem A4-18.1. The equation $y = u - \frac{u^2}{2} + \cdots + (-1)^{n-1} \frac{u^n}{n}$, $n = 1, 2, 3, \cdots$, is that of the best nth degree polynomial approximation to the graph of $y = \ln(1 + u)$ at $P(0, 0)$.

Elementary Functions 2, A5-17

In this section polynomials are generated as the best approximation to the graphs of the functions sin and cos. On page A-81 it is stated that "we assert without proof that sin x is between any two successive polynomial approximations," that is,

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!} \tag{1}$$

and similarily,

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} < \cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \tag{2}$$

The proof of these statements can be found in Taylor's Advanced Calculus in which this particular example is discussed ($\boxed{50}$, p. 544).

It is also stated on page A-81: "We shall not prove that the polynomials written represent the best approximation possible for the degree chosen. (They do)." However, as a concern of this study, the statement is proved using (1) and (2).

Theorem A5-17.1. The equation $y = x - \frac{x^3}{3!} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$, $n = 1, 2, 3, \cdots$, is that of the best $(2n - 1)$st degree polynomial approximation to the graph of $y = \sin x$ at $P(0, 0)$.

Proof: Consider, as inferred in (1), if $x > 0$ and n is even, or $x < 0$ and n is odd, then

$$\sin x > x - \frac{x^3}{3!} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

and

$$\sin x < x - \frac{x^3}{3!} + \cdots + (-1)^{n} \frac{x^{2n+1}}{(2n+1)!}, \tag{3}$$

and if $x > 0$ and n is odd or $x < 0$ and n is even, then

$$\sin x > x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\text{and} \qquad \sin x < x - \frac{x^3}{3!} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \tag{4}$$

In (3) the second part of the inequality can be written

$$x - \frac{x^3}{3!} + \cdots + ((-1)^{n-1} + (-1)^n \frac{(2n-1)!\, x^2}{(2n+1)!}) \frac{x^{2n-1}}{(2n-1)!} .$$

Similarily the first part of (4) can be written

$$x - \frac{x^3}{3!} + \cdots + ((-1)^{n-1} + (-1)^n \frac{(2n-1)!\, x^2}{(2n+1)!}) \frac{x^{2n-1}}{(2n-1)!} .$$

Now, because for each fixed value of n,

$$\lim_{x \to 0} \frac{(2n-1)!\, x^2}{(2n+1)!} = 0$$

it follows that for every $\epsilon > 0$ there exists $\delta > 0$ such that if

$0 < |x| < \delta$ then

$$\frac{(2n-1)!\, x^2}{(2n+1)!} < \epsilon .$$

and therefore, in either (3) or (4),

$$\sin x > x - \frac{x^3}{3!} + \cdots + ((-1)^{n-1} - \epsilon) \frac{x^{2n-1}}{(2n-1)!}$$

$$\text{and} \qquad \sin x < x - \frac{x^3}{3!} + \cdots + ((-1)^{n-1} + \epsilon) \frac{x^{2n-1}}{(2n-1)!} ,$$

or

$$\sin x < x - \frac{x^3}{3!} + \cdots + ((-1)^{n-1} - \epsilon) \frac{x^{2n-1}}{(2n-1)!}$$

$$\text{and} \qquad \sin x > x - \frac{x^3}{3!} + \cdots + ((-1)^{n-1} + \epsilon) \frac{x^{2n-1}}{(2n-1)!} .$$

Hence, because the requirements of Definition 3-4.3 are satisfied, the theorem is proved.

Theorem A5-17.2. The equation $y = 1 - \frac{x^2}{2!} + \cdots + (-1)^{n-1} \frac{x^{2n}}{(2n)!}$, $n = 1, 2, 3, \cdots$, is that of the best (2n)th degree polynomial approximation to the graph of $y = \cos x$ at $P(0, 1)$.

Proof: Consider, as inferred by (2) if n is even, then

$$\cos x > 1 - \frac{x^2}{2!} + \cdots + (-1)^{n-1} \frac{x^{2n}}{(2n)!} \tag{5}$$

and $$\cos x < 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n+2}}{(2n+2)!},$$

and if n is odd, then

$$\cos x > 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n+2}}{(2n+2)!} \tag{6}$$

and $$\cos x < 1 - \frac{x^2}{2!} + \cdots + (-1)^{n-1} \frac{x^{2n}}{(2n)!}.$$

In (5) the second part of the inequality can be written

$$1 - \frac{x^2}{2!} + \cdots + \left((-1)^{n-1} + (-1)^n \frac{(2n)!x^2}{(2n+2)!}\right) \frac{x^{2n}}{(2n)!}$$

Similarily, the first part of (6) yields

$$1 - \frac{x^2}{2!} + \cdots + \left((-1)^{n-1} + (-1)^n \frac{(2n)!x^2}{(2n+2)!}\right) \frac{x^{2n}}{(2n)!}.$$

Using the fact that for every n,

$$\lim_{x \to 0} \frac{(2n)!x^2}{(2n+2)!} = 0$$

it follows that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, then

$$\frac{(2n)!x^2}{(2n+2)!} < \epsilon,$$

and therefore in either (5) or (6),

$$\cos x > 1 - \frac{x^2}{2!} + \cdots + \left((-1)^{n-1} - \epsilon\right) \frac{x^{2n}}{(2n)!}$$

$$\cos x < 1 - \frac{x^2}{2!} + \cdots + \left((-1)^{n-1} + \epsilon\right) \frac{x^{2n}}{(2n)!}.$$

Hence, because the requirements of Definition 3-4.3 are met, the theorem is proved.

Summary of Major Points

It was found that throughout Elementary Functions the concept of a "best approximation" of a graph of a function was used. In the case of a best linear approximation, it was shown in this study that this was equivalent to the conventional calculus definition of a tangent. The calculus definition of tangent (and derivative, called the slope function) was included in the study because it was tacitly assumed in an argument concerning the best linear approximation of the graph of $y = e^x$. The best rth degree polynomial approximation was defined as suggested by the textbook and used throughout the treatment of Elementary Functions.

Another topic that was treated in this study was the concept of a dominating term of a polynomial function. Definitions and theorems were presented for the case of x in a neighborhood of zero as well as for x increasing without bound.

Although the integral was not formally presented in Elementary Functions related discussions were found and treated. This included the computation of area under a curve and a proof of the fundamental theorem of calculus, both of which involved limits.

The explanation of continuity presented in Elementary Functions was that the "graph has no holes or jumps." It was found, however, that the limit definition is tacitly assumed and used in a proof of the fundamental theorem of calculus. In treating this theorem, the need for and use of the limit definition of continuity was explicitly stated.

Another major topic which was treated with limits was the justification of the textbook's discussion devoted to the definition of the real number $2^{\sqrt{2}}$. A complete discussion of the development of the exponential function is not given in the textbook but rather this example is given to suggest a development. Properties of the exponential function are assumed without a justifying argument by the textbook; hence, they were not treated in this report.

CHAPTER IV

AUXILIARY DEFINITIONS AND THEOREMS

The following definitions and theorems are presented for justification of some statements made in the previous chapters. When they were used in an argument they were referred to by number; hence, no discussion is needed in this chapter. The definitions and theorems of this chapter are located here rather than in the preceding chapters because they are not discussed by the textbooks and they would interrupt the continuity of the presentation if stated or proved at the source of need.

Theorem 1. If $\lim_{n\to\infty} ns_n = c > 0$, then $\lim_{n\to\infty} s_n = 0$.

Proof: If $\lim_{n\to\infty} ns_n = c$ then for every $\epsilon > 0$ there exists a natural number N such that if $n > N$, then

$$|ns_n - c| < \epsilon .$$

Select $\epsilon < c$, then

$$0 < c - \epsilon < ns_n < c + \epsilon ,$$

or

$$0 < \frac{c - \epsilon}{n} < s_n < \frac{c + \epsilon}{n} .$$

But for every $\epsilon > 0$ there exists a natural number N_1, the smallest positive integer greater than $\frac{c + \epsilon}{\epsilon}$, such that if $n > N_1$,

$$- \epsilon < 0 < s_n < \frac{c + \epsilon}{n} < \epsilon .$$

Thus, by definition

$$\lim_{n\to\infty} s_n = 0.$$

Theorem 2. If $\{a_n\}$ is a sequence of non-negative real numbers and $\lim_{n\to\infty} a_n = A > 0$, then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{A}$.

Proof: If $\lim_{n\to\infty} a_n = A$, then for every $\epsilon > 0$ there exist a natural number N such that if $n > N$, then

$$|a_n - A| < \epsilon^2.$$

Take $n > N$. If $a_n + A \geq \epsilon$, then

$$\left|\sqrt{a_n} - \sqrt{A}\right| = \frac{|a_n - A|}{\left|\sqrt{a_n} + \sqrt{A}\right|} < \frac{\epsilon^2}{\left|\sqrt{a_n} + \sqrt{A}\right|} \leq \frac{\epsilon^2}{\epsilon} = \epsilon .$$

If $a_n + A < \epsilon$, then

$$\left|\sqrt{a_n} - \sqrt{A}\right| < \left|\sqrt{a_n} + \sqrt{A}\right| < \epsilon .$$

Hence $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{A}$.

<u>Definition 1.</u> The limit of a function f at a is b, $\lim_{x\to a} f(x) = b$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - b| < \epsilon$.

<u>Definition 2.</u> A function f is said to be continuous at $x = a$ if f is defined in a interval containing $x = a$ and $\lim_{x\to a} f(x) = f(a)$.

<u>Definition 3.</u> A function f is said to be continuous on an interval $[a, b]$ if it is continuous at each point of the interval.

<u>Definition 4.</u> A graph G of a function f is called a continuous graph at a set of points $P(x, f(x))$, if the function f is continuous at the corresponding values of x.

The following Theorems 3 through 11 are proved in many calculus texts and are for this reason not proved here.

<u>Theorem 3.</u> $\lim_{x\to a} (mx + b) = ma + b$. ([21] , p. 35).

<u>Theorem 4.</u> If $\lim_{x\to a} f(x)$ exist, then $\lim_{x\to a} k\, f(x) = k \lim_{x\to a} f(x)$ for every number k ([21] , p. 35).

<u>Theorem 5.</u> $\lim_{x\to a} \frac{1}{x} = \frac{1}{a}$ for $a \neq 0$ ([21] , p. 39).

Theorem 6. $\lim_{x \to a} \sqrt{x} = \sqrt{a}$ for $a \geq 0$ ([21], p. 41).

Theorem 7. If the function f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(b)$ ([21], p. 48).

Theorem 8. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$ ([21], p. 48).

Theorem 9. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then $\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$ ([21], p. 48).

Theorem 10. If $\lim_{x \to a} f(x)$ exists and n is a positive number, then $\lim_{x \to a} f^n(x) = (\lim_{x \to a} f(x))^n$ ([21], p. 48).

Theorem 11. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and if $\lim_{x \to a} g(x) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$ ([21], p. 48).

Theorem 12. If $\lim_{x \to a} f(x)$ exists and n is a positive integer, then $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ ([21], p. 48).

Definition 5. The limit of the function f as x increases without bound is b, $\lim_{x \to \infty} f(x) = b$, if for every $\epsilon > 0$ there exists a number N such that if $x > N$, then $|f(x) - b| < \epsilon$.

Definition 6. The limit of the function f as x decreases without bound is b, $\lim_{x \to -\infty} f(x) = b$, if for every $\epsilon > 0$ there exists a number N such that if $x < N$, then $|f(x) - b| < \epsilon$.

The following Theorems 13 through 17 are stated without accompanying proofs because the proofs are but slight modifications of the ones given for Theorems 8 through 12 ([21], p. 170).

For Theorems 13 through 17 it is to be assumed that $\lim_{x \to \pm\infty} f(x)$ and $\lim_{x \to \pm\infty} g(x)$ exist.

Theorem 13. $\lim_{x \to \pm\infty} (f(x) \pm g(x)) = \lim_{x \to \pm\infty} f(x) \pm \lim_{x \to \pm\infty} g(x)$.

Theorem 14. $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = \lim_{x \to \pm\infty} f(x) \cdot \lim_{x \to \pm\infty} g(x)$.

Theorem 15. $\lim_{x \to \pm\infty} (f(x))^n = (\lim_{x \to \pm\infty} f(x))^n$.

Theorem 16. If $\lim_{x \to \pm\infty} g(x) \neq 0$, then $\lim_{x \to \pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to \pm\infty} f(x)}{\lim_{x \to \pm\infty} g(x)}$.

Theorem 17. $\lim_{x \to \pm\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to \pm\infty} f(x)}$.

In the following definition and in some definitions and theorems below, words or phrases will be found in parentheses. These allow two definitions or theorems that are closely related to be given simultaneously. One statement is obtained by omitting the phrases in parentheses; the other is obtained by considering the part in parentheses and omitting the phrases that immediately precede the parentheses.

Definition 7. The limit of a function f as x approaches a from the right (left) is b, $\lim_{x \to a+} f(x) = b$ ($\lim_{x \to a^-} f(x) = b$), if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < x - a < \delta$ $(0 < a - x < \delta)$, then $|f(x) - b| < \epsilon$.

Definition 8. The function f increases without limit as x approaches a from the right (left), $\lim_{x \to a^+} f(x) = \infty$ ($\lim_{x \to a^-} f(x) = \infty$), if for every number $M > 0$ there exists $\delta > 0$ such that if $0 < x - a < \delta$ $(0 < a - x < \delta)$, then $f(x) > M$.

Definition 9. The function f decreases without limit as x approaches a from the right (left), $\lim_{x \to a+} f(x) = -\infty$ ($\lim_{x \to a^-} f(x) = -\infty$), if for every number $M < 0$ there exists $\delta > 0$ such that if $0 < x - a < \delta$ $(0 < a - x < \delta)$, then $f(x) < M$.

Definition 10. The function f increases (decreases) without limit as x increases without limit, $\lim_{x \to \infty} f(x) = \infty$ ($\lim_{x \to \infty} f(x) = -\infty$), if for every $M > 0$ $(M < 0)$ there exists a number N such that if $x > N$ then $f(x) > M$ $(f(x) < M)$.

Definition 11. The function f increases (decreases) without limit as x decreases without limit, $\lim_{x\to-\infty} f(x) = \infty$ $(\lim_{x\to-\infty} f(x) = -\infty)$, if for every $M > 0$ $(M < 0)$ there exists a number N such that if $x < N$ then $f(x) > M$ $(f(x) < M)$.

Theorem 18. If $\lim_{x\to\infty} f(x) = \pm\infty$, then $\lim_{x\to\infty} \frac{k}{f(x)} = 0$ for every real number k.

Proof: If $k = 0$ and $\lim_{x\to\infty} f(x) = \infty$, then there exist some number N such that if $|x| > N$, then $|f(x)| > 0$ so $\left|\frac{k}{f(x)}\right| = 0$. Hence, $\lim_{x\to\infty} \frac{k}{f(x)} = 0$ if $k = 0$.

If $k \neq 0$ and $\lim_{x\to\infty} f(x) = \infty$, then for every $M > 0$ there exists a number N such that if $x > N$ then $f(x) > |k| M > 0$, or $\left|\frac{k}{f(x)}\right| < \frac{1}{M}$. Hence, for every $\epsilon > 0$, there exist $M > 0$ such that if $x > N$ then $\left|\frac{k}{f(x)}\right| < \epsilon$ for every $k \neq 0$. Now, $\lim_{x\to\infty} \frac{k}{f(x)} = 0$ for $k \neq 0$, and above $\lim_{x\to\infty} \frac{k}{f(x)} = 0$ for $k = 0$ so it follows that $\lim_{x\to\infty} \frac{k}{f(x)} = 0$ for every real number k when $\lim_{x\to\infty} f(x) = \infty$.

The proof of the theorem for $\lim_{x\to\infty} f(x) = -\infty$ is essentially the same as that given above. Hence, it is not given here.

Theorem 19. If $f(x) > 0$ for all $a < x < c$ and $\lim_{x\to a^+} f(x) = 0$, then $\lim_{x\to a^+} \frac{k}{f(x)} = \infty$ for every number $k > 0$.

Proof: If $\lim_{x\to a^+} f(x) = 0$, then for every $\epsilon > 0$ there exists $c > \delta > 0$ such that if $0 < x - a < \delta$, then $0 < f(x) < \epsilon$. Hence, if M is any positive number, there exists $c > \delta > 0$ such that if $0 < x - a < \delta$ then $0 < f(x) < \frac{k}{M}$ or $\frac{k}{f(x)} > M$ when $k > 0$. Therefore, $\lim_{x\to a^+} \frac{k}{f(x)} = \infty$.

Theorem 20. If $f(x) > 0$ for all $a > x > c$ and $\lim_{x\to a^-} f(x) = 0$, then

$\lim_{x \to a^-} \frac{k}{f(x)} = \infty$ for every number $k > 0$.

Proof: The proof is essentially the same as the one given above.

Theorem 21. If $f(x) < 0$ for all $a > x > c$ $(a < x < c)$ and $\lim_{x \to a^+} f(x) = 0$ $(\lim_{x \to a^-} f(x) = 0)$, then $\lim_{x \to a^+} \frac{k}{f(x)} = -\infty$ $(\lim_{x \to a^-} \frac{k}{f(x)} = -\infty)$ for every number $k > 0$.

Proof: The proof is essentially the same as the one given for Theorem 13.

Theorem 22. If $\lim_{x \to a^+} f(x) = \pm\infty$ $(\lim_{x \to a^-} f(x) = \pm\infty)$, then $\lim_{x \to a^+} k\,f(x) = \mp\infty$ $(\lim_{x \to a^-} k\,f(x) = \mp\infty)$, for every number $k < 0$.

Proof: Consider the case $\lim_{x \to a^+} f(x) = \infty$. This means that for every $M > 0$ there exists $\delta > 0$ such that if $0 < x - a < \delta$ then $f(x) > M$. Hence, if $-M$ is any negative number, there exists $\delta > 0$ such that if $0 < x - a < \delta$, then $f(x) > \frac{-M}{k}$ or $k\,f(x) < -M$ when $k < 0$. Therefore, $\lim_{x \to a^+} k\,f(x) = -\infty$.

The other parts of the theorem are proved in same manner.

Theorem 23. If $\lim_{x \to a} f(x) = f(a)$, then $\lim_{h \to 0} f(a + h) = f(a)$, and conversely.

Proof: Given that $\lim_{x \to a} f(x) = f(a)$, call $x = a + h$. By definition, it follows that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |(a + h) - a| < \delta$ then $|f(a + h) - f(a)| < \epsilon$. But $|(a + h) - a| = |h|$. Hence, $\lim_{h \to 0} f(a + h) = f(a)$. The converse follows in a like manner.

Theorem 24. If $\lim_{x \to a} f(x) = b$, then $\lim_{x \to a^+} f(x) = b$ and $\lim_{x \to a^-} f(x) = b$.

Proof: If $\lim_{x \to a} f(x) = b$, then by definition it follows that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - b| < \epsilon$. However, $0 < |x - a| < \delta$ if $0 < x - a < \delta$ and

and $0 < a - x < \delta$. Hence, the conditions of Definition 7 are satisfied under the hypothesis stated so $\lim_{x \to a^+} f(x) = b$ and $\lim_{x \to a^-} f(x) = b$.

Theorem 25. If $\lim_{x \to a^+} f(x) = b$ and $\lim_{x \to a^-} f(x) = b$, then $\lim_{x \to a} f(x) = b$.

Proof: Under the hypothesis it can be stated by Definition 7 that for every $\epsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $0 < x - a < \delta_1$ or $0 < a - x < \delta_2$, then $|f(x) - b| < \epsilon$. Therefore, for every $\epsilon > 0$ there exist $\delta = \min(\delta_1, \delta_2)$ such that if $0 < |x - a| < \delta$ then $|f(x) - b| < \epsilon$. Thus, by Definition 1, $\lim_{x \to a} f(x) = b$.

Theorem 26. $\operatorname{Lim}_{n \to \infty} 2^{-n} = 0$.

Proof: $\operatorname{Lim}_{n \to \infty} 2^{-n} = \lim_{n \to \infty} \frac{1}{2^n}$. Consider now 2^n. Assume there exists some number $M > 0$ such that $2^n \leq M$ for every n. But $2^n \leq M$ only if $\log 2^n \leq \log M$ for every number n and it was shown that for every number M' there exist a number n such that $\log 2^n > M'$. Therefore, for every number $M > 0$ there must exist some number N such that if $n > N$, $2^n > M > 0$ or $\frac{1}{2^n} < \frac{1}{M}$.

Furthermore, for every number $\epsilon > 0$ there exists a number $N > 0$ such that $\frac{1}{N} < \epsilon$. Therefore, combining the results above, it can be stated that for every $\epsilon > 0$ there exist a number $N > 0$ such that if $n > N$, then $\frac{1}{2^n} < \epsilon$. This by Definition 15-3.1 implies that $\lim_{n \to \infty} 2^{-n} = 0$.

Theorem 27. If $\lim_{x \to a^+} f(x) = b > 0$ $(b < 0)$ and $\lim_{x \to a^+} g(x) = \pm\infty$, then $\lim_{x \to a^+} f(x)g(x) = \pm\infty$ $(\lim_{x \to a^+} f(x)g(x) = \mp\infty)$.

Proof: Consider $\lim_{x \to a^+} f(x) = b > 0$ and $\lim_{x \to a^+} g(x) = -\infty$.

It must be shown that for every $M < 0$ there exists $\delta > 0$ such that if $0 < x - a < \delta$ then $f(x)g(x) < M$. Let M be negative number and

$M' = \frac{2}{b}M$. By the hypothesis it follows that there exists $\delta' > 0$ such that if $0 < x - a < \delta'$, then

$$|f(x) - b| < \frac{b}{2} \text{ or } \frac{b}{2} < f(x) < \frac{3b}{2},$$

and
$$g(x) < M' < 0.$$

Hence,
$$f(x)g(x) < f(x)\, M' < \frac{b}{2}\, M' = M.$$

Therefore, by Definition 9, $\lim_{x \to a^+} f(x)g(x) = -\infty$. The proofs of the other cases follow in the same manner.

Theorem 28. If $\lim_{x \to a^-} f(x) = b > 0$ $(b < 0)$ and $\lim_{x \to a^-} g(x) = \pm\infty$, then $\lim_{x \to a^-} f(x)g(x) = \pm\infty$ $(\lim_{x \to a^-} f(x)g(x) = \mp\infty)$.

Proof: The proofs are essentially the same as that given above.

Definition 12. The sequence $\{s_n\}$ increases (decreases) without limit, $\lim_{n \to \infty} s_n = \infty$ $(\lim_{n \to \infty} s_n = -\infty)$, if for every number $M > 0$ $(M < 0)$ there exists a number $N > 0$ such that if $n > N$, then $s_n > M$ $(s_n < M)$.

Theorem 29. If $x > 1$, then $\lim_{n \to \infty} x^n = \infty$.

Proof: In Intermediate Mathematics 2, A9-8 it is shown that $x_1 > x_2$ if and only if $\ln x_1 > \ln x_2$. Let M be any positive number, so $x^n > M$ if $\ln x^n > \ln M$. But $\ln x^n = n \ln x$ and since $\ln x > 0$, $n \ln x > \ln M$ whenever $n > \frac{\ln M}{\ln x}$. Therefore, for every number $M > 0$ there exist a number $N = \frac{\ln M}{\ln x} > 0$ such that if $n > N$ then $x^n > M$. This implies by Definition 12 that $\lim_{n \to \infty} x^n = \infty$.

Theorem 30. If $\lim_{n \to \infty} s_n = \infty$ and $s_n \neq 0$ for any n, then $\lim_{n \to \infty} \frac{k}{s_n} = 0$ for any real number k.

Proof: The proof is essentially the same as the proof of Theorem 18 and is not repeated here.

Theorem 31. If $\lim_{n\to\infty} s_n = \pm\infty$, then $\lim_{n\to\infty} (k + s_n) = \pm\infty$.

Proof: If $\lim_{n\to\infty} s_n = \infty$, it is as follows that for every $M' > 0$ there exist $N > 0$ such that if $n > N$ then $s_n > M'$. Let M be any positive number and $M' = M - k$. Therefore, under the condition above, $s_n > M - k$ or $k + s_n > M$. Hence, by Definition 12 it follows that $\lim_{n\to\infty} (k + s_n) = \infty$. The proof of the theorem for $\lim_{n\to\infty} (k + s_n) = -\infty$ is essentially the same.

Theorem 32. If $\lim_{n\to\infty} s_n = \pm\infty$ and k is a real number different from 0, then $\lim_{n\to\infty} ks_n = \pm\infty$ if $k > 0$ and $\lim_{n\to\infty} ks_n = \mp\infty$ if $k < 0$.

Proof: Consider $k > 0$ and $\lim_{n\to\infty} s_n = \infty$. Take $M > 0$. Under this assumption it can be stated that for every $M' > 0$ there exist $N > 0$ such that if $n > N$ then $s_n > M'$. Let $M' = \frac{M}{k}$. Therefore under the condition above $ks_n > M$. Hence, by Definition 12 it follows that $\lim_{n\to\infty} ks_n = \infty$ for $k \neq 0$.

The proofs of the other cases are essentially the same and are not repeated here.

Definition 13. The function f increases without limit as x approaches a, $\lim_{x\to a} f(x) = \infty$, if for every $M > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, then $f(x) > M$.

Theorem 33. If $\lim_{x\to 0} g(x) = k > 0$ and $\lim_{x\to 0} f(x) = \infty$, then

$$\lim_{x\to 0} \frac{g(x)}{f(x)} = 0.$$

Proof: Let ϵ be any positive number. Under the assumption that $\lim_{x\to 0} g(x) = k > 0$ it can be stated that there exists $\delta_1 > 0$ such that $|g(x) - k| < \frac{k}{2}$ or $\frac{k}{2} < g(x) < \frac{3k}{2}$. Also, since $\lim_{x\to 0} f(x) = \infty$, it follows that there exists $\delta_2 > 0$ such that $f(x) > \frac{3k}{2\epsilon} > 0$. When $\delta = \max(\delta_1, \delta_2)$ both inequalities hold to yield $0 < 1/f(x) < \frac{2\epsilon}{3k}$ so

$0 < \frac{k}{2}/f(x) < g(x)/f(x) < \frac{3k}{2}/f(x) < \epsilon$ or $-\epsilon < g(x)/f(x) < \epsilon$. Hence, the requirements of Definition 13 are met so $\lim_{x \to 0} \frac{g(x)}{f(x)} = 0$.

Theorem 34. If $\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = b$ and $r_n \leq c \leq s_n$ for every n, then $b = c$.

Proof: If $\lim_{n\to\infty} r_n = b$, and $\lim_{n\to\infty} s_n = b$ then for every $\epsilon > 0$ there exists $N_1 > 0$ and $N_2 > 0$ such that if

(a) $n > N_1$, then $|r_n - b| < \epsilon$, and

(b) $n > N_2$, then $|s_n - b| < \epsilon$.

Hence, for any $\epsilon > 0$ select $N = \max(N_1, N_2)$ so if $n > N$, (a) and (b) both hold. Assume $c \neq b$, then $|b - c| > 0$. However, there exists a number $N > 0$ such that if $n > N$, then

$$|r_n - b| < \frac{|b - c|}{3}, \; |s_n - b| < \frac{|b - c|}{3}.$$

This implies that if $r_n \leq c \leq s_n$, then c must be within $\frac{|b - c|}{3}$ of b for all $n > N$. This is impossible unless, of course, $|b - c| = 0$ or $b = c$.

Theorem 35. If $\{s_n\}$ is a positive monotone nonincreasing sequence that is bounded below, then there exist a real number b such that $\lim_{n\to\infty} s_n = b$.

Proof: Given $s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_n \geq \cdots$ and $M \geq 0$ such that $0 \leq M \leq s_n$ for every n, form new sequences $\{a_n\}$ and $\{b_n\}$ in the following manner: Consider the intervals $\left[M, \frac{s_1 + M}{2}\right]$, and $\left[\frac{s_1 + M}{2}, a_1\right]$. If there exists a term of $\{s_n\}$ in $\left[M, \frac{s_1 + M}{2}\right]$, let $b_1 = M$ and $a_1 = \frac{s_1 + M}{2}$; if not, let $b_1 = \frac{s_1 + M}{2}$ and $a_1 = s_1$. Next consider the intervals

$$\left[b_1, \frac{a_1 + b_1}{2}\right] \quad \text{and} \quad \left[\frac{a_1 + b_1}{2}, a_1\right] .$$

Again, if there exists a term of $\{s_n\}$ in $\left[b_1, \frac{a_1 + b_1}{2}\right]$, let $b_2 = b_1$ and $a_2 = \frac{a_1 + b_1}{2}$; if not, let $b_2 = \frac{a_1 + b_1}{2}$ and $a_2 = a_1$. Continue in this manner to generate the sequences $\{b_n\}$ and $\{a_n\}$ with the following properties:

(i) $b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_n \leq \cdots$

(ii) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$

(iii) $b_n \leq a_n$ by $\dfrac{s_1 - M}{2^n}$.

(iv) Because $a_n - b_n = \dfrac{s_1 - M}{2^n} < \dfrac{s_1 - M}{n}$, then

$\lim_{n\to\infty} (b_n - a_n) = 0$. Therefore, there exist a number c such that $b_n \leq c \leq a_n$ for every n.

Also, $\lim_{n\to\infty} a_n = c$ since for every $\epsilon > 0$ there exists a number $N > 0$ such that if $n > N$, then $|a_n - c| = a_n - c < a_n - b_n < \epsilon$.

Now to show that $\lim_{n\to\infty} s_n = c$, recall that by the construction of the sequence $\{a_n\}$ it follows that for every integer k there exists a number $N_k > 0$ such that if $n > N_k$ then $s_n \leq a_k$. Take any $\epsilon > 0$, then because $\lim_{n\to\infty} a_n = c$ there exist a number $N' > 0$ such that if k is any number $k > N'$ then $a_k - c < \epsilon$. But there exists a number N_k such that if $n > N_k$, then $s_n \leq a_k$. Therefore, by selecting $N = \max(N', N_k)$ it follows that $s_n - c < a_k - c < \epsilon$. Furthermore, $s_n > c$ otherwise there would be some term of $\{a_n\}$ less than c, and this is not so. Hence, it can be concluded that for every $\epsilon > 0$ there exists a number $N > 0$ such that if $n > N$ then

$|s_n - c| < \epsilon$ which defines $\lim_{n\to\infty} s_n = c$.

<u>Theorem 36.</u> $\lim_{n\to\infty} 2^{\frac{1}{10^n}} = 1$.

Proof: If $n > m > 0$ then $10^n > 10^m$ and $\frac{1}{10^n} < \frac{1}{10^m}$ so $2^{\frac{1}{10^n}} < 2^{\frac{1}{10^m}}$ as discussed in this report prior to the need for this theorem. Therefore, $2^{\frac{1}{10^n}}$ is a monotone decreasing sequence and since $2^{\frac{1}{10^n}} > 1$, it is bounded below. By Theorem 35, $\lim_{n\to\infty} 2^{\frac{1}{10^n}}$ exists, and can be called b.

From the definition of $\lim_{n\to\infty} s_n$, it follows readily that

$$b = \lim_{n\to\infty} 2^{\frac{1}{10^n}} = \lim_{n\to\infty} 2^{\frac{1}{10^{n-1}}}$$

so that $b = b^{10}$. Hence, $b = 0$ or $b = 1$. However, $2^{\frac{1}{10^n}} > 1$ so $b \neq 0$ and therefore $b = 1$.

Summary of Major Points

This chapter has presented the definitions and theorems cited in Chapter II and III but not presented in those chapters. The textbook gave no definitional statements or arguments from which these statements could be drawn. However, the rigorous treatments presented in this study required that these definitions be given and these theorems be proved.

It should be specifically noted that the proof of Theorem 36 required the establishment of the existence of a limit, which was proved in Theorem 35. These proofs were more difficult than similar theorems involving exponentials which were presented previously.

The reason for this was that the exponential function was defined in Intermediate Mathematics to be the inverse of the logarithm function while in Elementary Functions, where these theorems were required, the exponential function was defined prior to the defining of the logarithm function.

CHAPTER V

SUMMARY AND CONCLUSIONS

The problem of this study was to embed a rigorous treatment of the limit concept into the SMSG revised sample textbooks. The problem required an analysis of the textbooks to determine the discussions that involved limits. Such discussions were restated in more exact mathematical terms to yield definitions and theorems. The theorems for which a proof was suggested were proved in this study by arguments using material that was either presented by the textbooks or selected from other texts. Invalid arguments were identified and tacit assumptions were explicitly stated. Therefore, the objective of the study, which was to provide an analysis and a resolution of the difficulties involved in presenting rigorous limit arguments, has been realized.

Findings

On the basis of the presentations of the preceding chapters, the following findings seem to be warranted:

1. Without a reordering of the topics, the structure of *Geometry* and *Intermediate Mathematics* is not adequate for a rigorous treatment of those discussions and arguments that involve limits to be embedded. The theorems on limits of sequences required for *Geometry* 15-3, 15-4, 16-5, and for a rigorous statement of the Completeness Property in *Intermediate Mathematics* 1-9 are not given until *Intermediate Mathematics* A13-4. Also, if the continuity of the

logarithm function is to be proved as exemplified in this study, the inequalities in Intermediate Mathematics A9-3 require a slight but definite reordering.

A major omission of definitions was found in Intermediate Mathematics. The following types of limits are suggested: $\lim_{x \to a} f(x)$, $\lim_{x \to a+} f(x)$, $\lim_{x \to a-} f(x)$, $\lim_{x \to \infty} f(x)$, $\lim_{x \to -\infty} f(x)$, $\lim_{n \to \infty} s_n$; each of these limits could be called b, ∞, or $-\infty$. Therefore, eighteen definitions are required to define every combination of these symbols and the textbook offers discussion of only one definition.

If it is assumed that the definitions and theorems required for a rigorous treatment of the limits in Intermediate Mathematics are given, then Elementary Functions is adequate to complete the aforementioned task. Although the limit definition of a continuous function is required in proving a form of the fundamental theorem of calculus in A3-13 and it is not given in the textbook, the definition can be given in the language of the textbook and limits.

Similarly, in 5-11 the textbook assumes properties involving the tangent to the graph of the sum of two functions and of a function multiplied by a constant. Such an assumption is unnecessary because a proof can be given that requires only those concepts previously presented.

Another necessary addition that can be made within the structure is the derivation of the equation of the tangent to the graph of $y = a^x$ at $P(0, f(0))$ in Elementary Functions 4-6. Although the textbook tacitly assumes the quotient definition of derivative in an argument to show that the equivalent wedge definition holds, it is pointed out that the quotient definition may be proved from the concepts previously presented.

2. Arguments were found in both Intermediate Mathematics and Elementary Functions that were logically unsound. In Intermediate Mathematics 1-8 the argument given to show that a repeating decimal is a rational number assumes that the number is rational in the course of the argument. The Completeness Property of Intermediate Mathematics 1-9 is inadequate unless provisions are made for a generalization such as that given in this study. Also, the definitional statements given for asymptotes in Intermediate Mathematics 6-6 are inadequate.

In Elementary Functions the arguments associated with the following topics need corrections such as those suggested in this study: A3-13 which is devoted to a proof of a form of the fundamental theorem of calculus, 4-6 which includes a derivation of the equation of the tangent to the graph of $y = 2^x$ at $P(0, f(0))$, and A4-16 which consists of a derivation of the equation of the best rth degree polynomial approximation to the graph of $y = e^x$ at $P(0, f(0))$.

Care must be taken at some points to insure correct interpretation of the discussions given in the textbooks. Particularly, in Elementary Functions 2-3 following a definitional statement the textbook uses the phrase "this means" to introduce a condition that is necessary but not sufficient.

3. The valid treatments of the limit topics deviate little from those found in college calculus textbooks. The major points of deviation result from the somewhat uncommon statement of the Completeness Property, the definition of the equation of a tangent to the graph of a function at a point, and the definition of the equation of the best rth

degree polynomial approximation to the graph of a function at a point.

First Course in Algebra is not mentioned in any of the above findings. This is due to the fact that no limit discussions were found. Hence, there are no findings concerning limits except those which are vacuously satisfied.

Recommendations

The writer believes the following recommendations are supported by this study:

1. Teachers and students who are interested in the topic of limits in secondary school mathematics should examine or repeat this study in order to gain a broader understanding of the limit concepts and the problems and difficulties involved in treating this topic at the high school level.

2. Authors of textbooks who expect to utilize excerpts of the SMSG sample textbooks should investigate carefully the presentations and the structure to determine tacit assumptions, invalid arguments, and the relationships between the topics.

3. Authors of textbooks and teachers should make a rigorous analysis of their teaching materials , as exemplified by this study, in order to determine and correct possible imperfections that, otherwise, they might not have noticed.

4. Researchers in mathematics education should test the teachability of limits presented by (ϵ, δ) -arguments.

5. Researchers in mathematics education should test the desirability of the (ϵ, δ) -notion of limit to determine the time that could profitably be spent in such a study of the concept.

6. Researchers in mathematics education should determine and compare the advantages and disadvantages of teaching the SMSG approach to the concept of a derivative with the conventional calculus approach.

SELECTED BIBLIOGRAPHY

1. Begle, E. G. "The School Mathematics Study Group." The Mathe- Teacher, LI (December, 1958), 616-7.

2. Bell, E. T. The Development of Mathematics, First Edition. New York: McGraw-Hill Book Company, Incorporated, 1940.

3. Birkhoff, George David, and Ralph Beatley. Basic Geometry. Chicago: Scott Foresman and Company, 1940.

4. Boyer, Carl B. The History of the Calculus and Its Conceptual Development (The Concepts of the Calculus). New York: Dover Publications, Incorporated, 1949.

5. Bromwich, T. J. I'A. An Introduction to the Theory of Infinite Series. London: MacMillan and Company, Limited, 1908.

6. Brown, Kenneth E. "Teaching Load and Qualifications of Mathematics Teachers." The Mathematics Teacher, LIII (January 1960), 2-11.

7. Brumfiel, Charles F., Robert E. Eicholz, and Merrill E. Shanks. Geometry. Reading, Massachusetts: Addison-Wesley Publishing Company, Incorporated, 1960.

8. Butler, Charles H., and F. Lynwood Wren. The Teaching of Secondary Mathematics, Third Edition. New York: McGraw-Hill Book Company, Incorporated, 1960.

9. Cairns, Stewart Scott. "Mathematical Education and the Scientific Revolution." The Mathematics Teacher, LIII (February, 1960), 66-74.

10. Commission on Mathematics, Appendices. New York: College Entrance Examination Board, 1959.

11. ________. Program for College Preparatory Mathematics. New York: College Entrance Examination Board, 1959.

12. Courant, R., and H. Robbins. What is Mathematics? New York: Oxford University Press, 1941.

13. Dienes, P. The Taylor Series. New York: Dover Publications, Incorporated, 1957.

14. Fehr, Howard F. Secondary Mathematics, A Functional Approach for Teachers. Boston: D. C. Heath and Company, 1951.

15. ________. "New Thinking in Mathematical Education." The Mathematics Teacher, LIII (October, 1960), 424-29.

16. Fegurson, W. Eugene. "Calculus in the High School." The Mathematics Teacher, LIII (October, 1960), 451-53.

17. Gibson, George A. Advanced Calculus. New York: MacMillan Company, 1931.

18. Goffman, Casper. Real Functions. New York: Rinehart and Company, Incorporated, 1953.

19. Haag, Vincent H. Structure of Elementary Algebra. Vol. III of Studies in Mathematics (Preliminary Edition). Written for the School Mathematics Study Group. New Haven: Yale University, 1960.

20. Hardy, G. H. A Course of Pure Mathematics, Seventh Edition. London: Cambridge University Press, 1938.

21. Johnson, Richard E., and Fred L. Kiokemeister. Calculus. Boston: Allyn and Bacon, Incorporated, 1959.

22. Karst, Otto J. "The Limit." The Mathematics Teacher, LI (October 1958), 443-49.

23. Kemeny, John G. "Rigor Versus Intuition in Mathematics." The Mathematics Teacher, LIV (February 1961), 66-74.

24. Kinsella, John J. "Preparation in Mathematics of Mathematics Teachers." The Mathematics Teacher, LIII (January 1960), 27-32.

25. Kline, Morris. "The Ancient Versus the Moderns, A New Battle of the Books." The Mathematics Teacher, LI (October 1958), 418-27.

26. Kline, William E., Robert Oesterle and Leroy M. Wilson. Foundations of Advanced Mathematics. New York: American Book Company, 1959.

27. Knopp, Konrad. Theory and Application of Infinite Series. Second Edition. Translated from the second German edition and revised in accordance with the fourth by R. C. H. Young. New York: Hafner Publishing Company, 1951.

28. Landau, Edmund. Differential and Integral Calculus. New York: Chelsea Publishing Company, 1951.

29. Lennes, N. J. "The Treatment of Limits in Elementary Geometry." School Science and Mathematics, VI (January, 1906), 52-58.

30. Leissa, Arthur W. and Robert C. Fisher "A Survey of Teachers Opinions of a Revised Mathematics Curriculum." The Mathematics Teacher, LIII (February, 1960), 113-18.

31. Lytle, Ernest B. "Limits in Elementary Geometry." School Science and Mathematics, X (September, 1910), 530-32.

32. McShane, Edward James and Truman Arthur Botts. Real Analysis. Princeton: D. Van Nostrand Company, Incorporated, 1959.

33. Meder, Albert E., Jr. "The Ancient Versus the Moderns - A Reply." The Mathematics Teacher, LI (October, 1958), 428-33.

34. Merriman, Gaylord M. To Discover Mathematics. New York: John Wiley and Sons, Incorporated, 1942.

35. Moise, Edwin E. "The SMSG Geometry Program." The Mathematics Teacher, LIII (October, 1960), 437-42.

36. McKelvey, J. V. "The Teaching of Limits in High School." The American Mathematical Monthly, XXVIII (1921), 68-71.

37. Randolf, John F. "Limits." Insights Into Modern Mathematics, Twenty-third Yearbook. Washington, D. C. : National Council of Teachers of Mathematics, 1957, pp. 200-40.

38. Richmond, Donald E. Fundamentals of the Calculus. New York: McGraw-Hill Book Company, Incorporated, 1950.

39. School Mathematics Study Group. Mathematics for High School: First Course in Algebra. (Revised Edition) 3 Parts. New Haven: Yale University, 1960.

40. ________ ________. Commentary for Teachers. (Revised Edition) 3 Parts. New Haven: Yale University, 1960.

41. ________. Mathematics for High School: Geometry. (Revised Edition) 3 Parts. New Haven: Yale University, 1960.

42. ________ ________. Commentary for Teachers. (Revised Edition) 3 Parts. New Haven: Yale University, 1960.

43. ________. Mathematics for High School: Intermediate Mathematics. (Revised Edition) 3 Parts. New Haven: Yale University, 1960.

44. ________ ________. Commentary for Teachers. (Revised Edition) 3 Parts. New Haven: Yale University, 1960.

45. ________. Mathematics for High School: Elementary Functions. (Revised Edition) 2 Parts. New Haven: Yale University, 1960.

46. ________ ________. Commentary for Teachers. (Revised Edition) 2 Parts. New Haven: Yale University, 1960.

47. ________. Report of an Orientation Conference for SMSG Experimental Centers. Chicago, Illinois, September 19, 1959. New Haven: Yale University, 1959.

48. Smith, Louis Tingen. "The Role of Maturity in Acquiring a Concept of Limit in Mathematics." (Unpublished Ed. D. Dissertation, Stanford University, 1959).

49. Steinbrenner, Arthur Henry. "A Study of the Concept of Continuity for Teachers of Secondary School Mathematics." (Unpublished Ph. D. Dissertation, Columbia University, 1955).

50. Taylor, Angus E. Advanced Calculus. Boston: Ginn and Company, 1955.

51. ________. Calculus with Analytic Geometry. Englewood Cliffs, New Jersey: Prentice-Hall, Incorporated, 1959.

52. Theilman, Henry P. Theory of Functions of Real Variables. New York: Prentice-Hall, Incorporated, 1953.

53. Thomas, George B., Jr. Calculus and Analytic Geometry. Reading, Massachusetts: Addison-Wesley Publishing Company, Incorporated, 1951.

54. Thurston, H. A. The Number-System. London: Blackie and Son Limited, 1956.

55. Twenty-fourth Yearbook of the National Council of Teachers of Mathematics. The Growth of Mathematical Ideas. Grades K-12. Washington, D. C.: National Council of Teachers of Mathematics, 1959.

56. Twenty-third Yearbook of the National Council of Teachers of Mathematics. Insights Into Modern Mathematics. Washington D.C.: National Council of Teachers of Mathematics, 1957.

57. Wagner, John. "The Objectives and Activities of the School Mathematics Study Group." The Mathematics Teacher, LIII (October, 1960), 454-59.

58. Western, Donald W. and Vincent H. Haag. An Introduction to Mathematics. New York: Henry Holt and Company, Incorporated, 1959.

59. Young, John Wesley. Lectures on Fundamental Concepts of Algebra and Geometry. New York: The MacMillan Company, 1936.

APPENDIX A

REQUEST FOR INFORMATION CONCERNING THE EXISTANCE OF LIMITS ASSOCIATED WITH CIRCLES

May 4, 1961

(Inside Address)

Dear Sir:

In a doctoral study of the limit concept in secondary school mathematics and specifically in the SMSG revised sample textbooks, I have become stymied by a problem. A search of books in our department and library has been unfruitful. I hope that you will assist me.

The problem is to show by elementary geometry and algebra that the various limits associated with circles exist. More specifically, consider the following questions:

If a_n, p_n, A_n are the apothem, perimeter, and area of an inscribed regular n-gon, how can it be proved that $\lim a_n$, $\lim p_n$, and $\lim A_n$ exist? How can the sequences $\{a_n\}$, $\{p_n\}$, $\{A_n\}$ be proved to be monotone increasing?

How can these sequences be bounded above?

Your suggestions for proofs, articles or books to consult, or your opinion regarding the possibility of making such proofs will be greatly appreciated.

Sincerely yours,

Donald W. Hight

BROOKLYN COLLEGE

Brooklyn 10, New York

Department of Mathematics

May 19, 1961

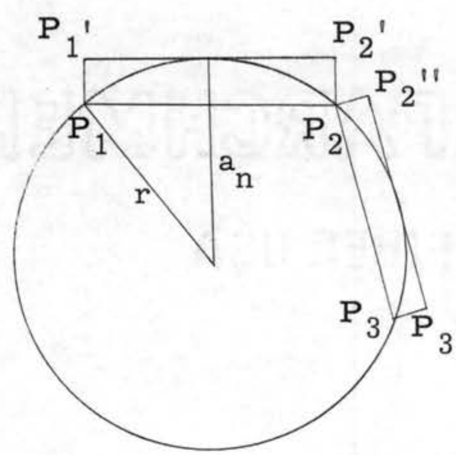
Dear Mr. Hight,

The question you pose is interesting - we justify the mensuration formulas in plane geometry by one procedure, but give the final rigorization later by methods of the calculus, instead of trying to carry through the original methods.

I think you would have gotten more help with the problem if you had consulted the preliminary edition of the SMSG Geometry text rather than the revised edition! The reason is that the original treatment was considered too difficult for the students and was softened in the revision. The prelim. edition pp. 516-517 justifies quite carefully that $a_n \longrightarrow r$. Let us go on from this. The justification on p. 515 that $P_n \longrightarrow C$ is perfectly satisfactory for the students, but has a subtle flaw mathematically. For the definition of the circumference merely says: If P_n converges its limit is called the circumference of the circle. Thus we must prove that P_n converges, which is not done in the SMSG text or commentary or anywhere else that I know.

To tackle this, study A_n first, it is easier. Let A_n be the area of the inscribed regular polygon $P_1 P_2 \cdots P_n$. Construct a kind of "circumscribed" polygon as follows:

At the midpoint of $\widehat{P_1P_2}$ draw a line tangent to the circle, and drop perpendiculars to this tangent from P_1, P_2 with respective feet P_1', P_2'.



Do this successively for $P_2P_3, \cdots$, and form the polygon

$P_1 P_2' P_2 P_2'' P_3' P_3 \cdots P_1$.

Let A_n' denote the area of this polygon.

then

$$A_n = \frac{1}{2} a_n p_n$$

$$A_n' = A_n + (r - a_n)p_n$$

and $$|A_n' - A_n| = (r - a_n)p_n .$$

Note that for every m and every n ($m, n \geq 3$)

$$A_n < A_m' .$$

Hence the set of A_n is bounded and has a L. U. B. which we may call A.

Then $$A_n \leq A \leq A_n'$$

so that $$|A_n - A| \leq |A_n' - A_n| = (r - a_n)p_n.$$

We know $r - a_n \longrightarrow 0$; we show $\{p_n\}$ bounded. We have

$$p_n = 2A_n / a_n.$$

Since $\{A_n\}$ is bounded and $a_n \longrightarrow r \neq 0$, we see that $\{p_n\}$ is bounded.

Then $$|A_n - A| \leq (r - a_n)p_n$$

implies $A_n \longrightarrow A$. Thus A_n truly converges.

Now $p_n = 2A_n / a_n$ implies that p_n also converges.

The usual procedures now can be applied to show $C = 2\pi r$ and $A = \pi r^2$.

This treatment doesn't require the proof that $\{p_n\}$, $\{A_n\}$ are monotonic, which seems very difficult by elementary methods. Using trigonometry and some calculus these sequences can now be proved monotonic. To validate the properties of the trigonometric functions (including their derivatives) in the familiar way we need the theory of arc length, but this can be justified by the method used above.

The Illinois Group doesn't use the definition Circumference = $\lim p_n$ but essentially takes it to be the least upper bound of the set of numbers p_n and in the commentary discusses arc length based on a related definition.

Sincerely,

Walter Prenowitz

HARVARD UNIVERSITY
GRADUATE SCHOOL OF EDUCATION

Lawrence Hall, Kirkland Street
Cambridge 38, Massachusetts
May 25, 1961

Mr. Donald W. Hight
Department of Mathematics
Oklahoma State University
Stillwater, Oklahoma

Dear Mr. Hight:

The easiest way for me to answer your inquiry is to send you a portion of the notes for my course at Harvard this year. I am sending you this under separate cover.

Sincerely,

Edwin E. Moise

P.S. I know of no easy way to show that a_n and p_n increase, the geometric proofs are hard, and the only real approach is to use the monotonicity of $\frac{\sin x}{x}$ near the origin.

E. E. M.

QUEEN'S UNIVERSITY
KINGSTON, ONTARIO

Dept. of Mathematics
May 26/61

Mr. D. W. Hight
Dept. of Mathematics
Oklahoma State University
Stillwater, Oklahoma

Dear Mr. Hight:

Please excuse the delay in answering your letter of May 4. This was partially due to the fact that I was in the process of moving for the summer upon its arrival, and partially to the fact that I can make no concrete suggestions on your problem.

The only reference I can think of at the moment is Courant and Robbins book "What is Mathematics?", and I'm not even too sure that that will be helpful.

As to the desirability or possibility of making the proofs you mentioned, I think that a certain balance will have to be made between intuition and rigor. A fully rigorous proof would, I fear, be much too subtle and high-powered. I personally am not opposed to an intuitive approach as long as the assumptions which are made are pointed out faithfully.

When Dr. Crowe arrives at O.S.U. for the Summer Institute, you might discuss the problem with him. I feel sure that he could advise you much better than I could.

Sincerely yours,

F. A. Sherk

THE UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL

May 21, 1961

DEPARTMENT OF MATHEMATICS

Dear Mr. Hight:

Your letter reached me here in Chapel Hill where I am for the summer. I'm sorry to be a bit slow answering but perhaps the comments below will be of some help - and that more can be achieved in conversation as I shall be in Stillwater in August for the mathematics meetings.

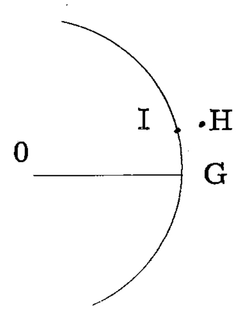
In the first place, it is doubtful if one can, in brief compass, prove all the necessary theorems connected with circles and arc length without recourse to fairly sophisticated limit considerations. The exposition in our text is aimed at clarity of concept rather than a selection of the easiest chain of theorems for full proof. For example we define circumference using regular polygons - because we think it conceptually best for the student - but it is not the best for proofs. I don't know of any place where a full treatment can be found. Below are some comments on the theorems you mentioned.

Theorems 11 - 12. As you have observed an elementary proof is hard and tedious. The hardest part is to prove that $p_{n+1} > p_n$. Using a bit of trigonometry it is relatively easy - but that is cheating. If we do not restrict to regular polygons then we are better off. Let p', p'' be the perimeter of two inscribed polygons. Let V be the vertices of the two polygons. Let p be the perimeter of the inscribed polygon make vertices V. Then $p' < p$ and $p'' < p$.

Since the perimeter of any inscribed polygon is $< 8r$ and since the inscribed polygons form a directed set (ordered by the inclusion relation on the vertices) lim p exists.

In the same way, Theorem 11 - 15 is for clarification of concepts not ease of proof. Observe that if we stick to polygons of 2^n sides then $s_4 < s_8 < \cdots < s_{2n} < \cdots < 4r^2$ and $\lim s_{2n}$ obviously exists.

Theorem 11 - 16 of course depends on $\overline{AC}$ being small in Fig. 11 - 12. If we allow any inscribed polygons we can get small sides as follows. Choose $\epsilon > 0$ choose H such that $\overline{GH} < \epsilon$ and H not on the line of G and O. On the ray OH choose I such that $\overline{OI} = r = \overline{OG}$. Then $\overline{IH} < \epsilon$ and $\overline{GI} < 2\epsilon$ by the triangle inequality. Thus there are inscribed polygons of arbitrarily small sides.



In Theorem 12 - 3 we can almost make a full proof. Certainly so if we assume that arclength = lim l_n exists as in Theorem 12 - 2. Then the "proof" in Thm 12-3 I think stands.

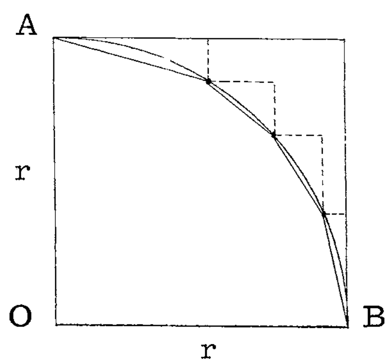
I hope to see you in August

Yours truly,

Merrill Shanks

P.S. This picture "almost proves that $p_n < 8r$ by the triangle inequality." There is the "nuisance" that A may not fall at the right point.

P.P.S. I shall be back at Purdue about June 4.



A note by the author of this thesis: The theorems to which Dr. Shanks refers are in the book Geometry written by Brumfiel, Eicholz, and Shanks.

LETTER OF INQUIRY CONCERNING THE DERIVATION OF THE BEST rth DEGREE POLYNOMIAL APPROXIMATION TO THE GRAPH OF $y = e^x$ at $P(0, 1)$.

March 18, 1961

Dr. D. E. Richmond
Department of Mathematics
Williams College
Williamstown, Massachusetts

Dear Sir:

As a part of a research project undertaken to complete a Doctor of Education degree, I am attempting to treat rigorously, those topics found in the SMSG revised sample textbooks that involve the concept of a limit. As an author of Elementary Functions (Part 2) (revised edition), your comment on the following discussion would be greatly appreciated. Or if another person should be contacted concerning this part of the book, please suggest to whom these questions should be directed.

There seems to be a flaw in the reasoning in regard to Functions 2, Topic 4-16, An Approximation for e^x, page A51. Specifically, on page A53 it is to be proven that $g_n(x) > e^x = f(x)$ for some x. Starting at the second paragraph "The first observation $\cdots$", it is shown that

$$g_n'(x) < g_n(x) \text{ when } x > 0$$

and it is known that if $f(x) = e^x$

$$f'(x) = f(x).$$

Can it be concluded that

$$g_n'(x) < f(x) \ ?$$

It would appear that it must be known that

$$g_n(x) < f(x)$$

A relationship that I could not show without using an induction argument.

Similarily at the bottom of this same page, if

$$h_3'(x) > h_3(x)$$

where $$h_3(x) < f(x) = f'(x),$$

how can it be concluded that

$$h_3'(x) > f'(x) \ ?$$

Is it possible to prove

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} < e^x < 1 + x + \frac{x^2}{2!} + \cdots + \frac{cx^n}{n!}, \ c > 1$$

using what has been given in the books? Also, what argument, if any, can be made in the case of $x < 0$?

Sincerely,

Donald W. Hight

WILLIAMS COLLEGE
Williamstown, Massachusetts

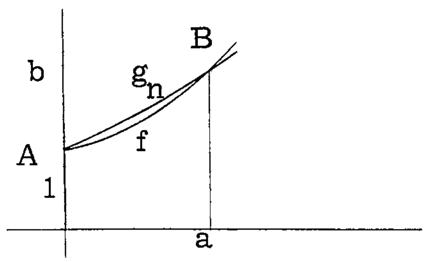
Department of Mathematics

April 6, 1961

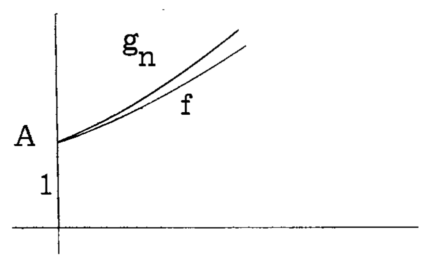
Mr. Donald W. Hight,

Thank you for your recent letter àpropos of the argument on page A-53, Elementary Functions, Part 2. We wrote this appendix during the last week and you are right in saying that the statement needs more justification. We had thought that an intuitive appeal was sufficient but I see that this is not the case. The following argument will perhaps do the job.

We know that $f(0) = g_n(0) (=1)$ and wish to prove that $f(x) < g_n(x)$ for all $x > 0$. Suppose first that for some $a > 0$, $g_n(a) = f(a)$. Then $g_n'(a) < f'(a)$ and the f graph must be below the g_n graph on the left of a and above it on the right of a. There is at most one intersection. The two possibilities which we wish to exclude are shown in the following figures



(one intersection)



(no intersection)

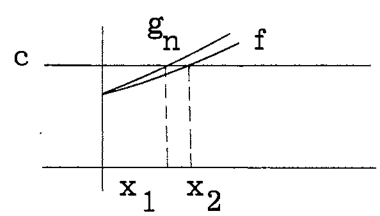
In either case there would be an interval, $1 < y < b$ for which the g_n graph would be to the left of the f graph. Let $y = c$ $(1 < c < b)$ intersect the graphs at x_1 and x_2 $(x_1 < x_2)$ as shown, then

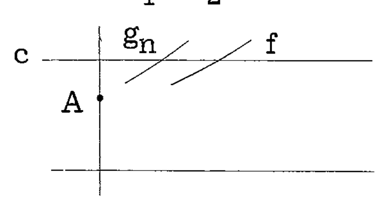
$$g_n(x_1) = f(x_2)$$

and

$$g_n'(x_1) < f'(x_2).$$

Hence the horizontal distance between the graphs increases as we decrease y. Since this holds at lower levels, it contradicts the fact that the graphs intersect at A.





It follows that the g_n graph is to the right of the f graph and hence below it.

It would probably be possible to change this to a direct proof. The whole thing would of course be simpler if we could assume a couple of theorems about derivatives.

Thank you for drawing this to my attention.

Sincerely yours,

Donald E. Richmond

P.S. Of course the same type of argument would apply further down the page.

VITA

Donald Wayne Hight

Candidate for the Degree of

Doctor of Education

Thesis: A STUDY OF THE LIMIT CONCEPT IN THE SMSG REVISED SAMPLE TEXTBOOKS

Major Field: Secondary Education, Mathematics

Biographical:

Personal Data: Born at Neodesha, Kansas, August 23, 1931, the son of Melvin B. and Opal N. Hight.

Education: Attended grade school in Neodesha, Kansas; graduated from Neodesha High School in 1949; received the Bachelor of Science in Education degree from Kansas State Teachers College, Pittsburg, Kansas with a major in Mathematics, in May, 1953; received the Master of Science degree in Mathematics from Oklahoma State University in 1958; completed requirements for the Doctor of Education degree in Secondary Education, Mathematics, in August, 1961.

Professional experience: Began teaching mathematics in January 1953 at Neodesha High School, Neodesha, Kansas; served in the United States Army from June 1953 to June 1955; returned to teaching mathematics in El Dorado High School and coaching basketball in El Dorado Junior College, El Dorado, Kansas until 1957; received a National Science Foundation Fellowship and became a fellow in the Supplementary Program for Teachers of High School Mathematics and Science; taught mathematics, including a special program for academically talented students, at Wichita High School Southeast, Wichita, Kansas from September 1958 until June 1960; was awarded a National Science Foundation Fellowship to attend three consecutive summer school sessions; served as a graduate assistant in the Mathematics Department of Oklahoma State University in 1961.

Professional organizations: Sigma Xi; The Mathematics Association of America; The National Council of Teachers of Mathematics; Kansas State Teachers Association; National Education Association.