SOME THEOREMS CONCERNING COMPACT CONTINUA IN THE PLANE

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CONTINUA IN THE PLANE

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PREFACE

The purpose of this paper is to examine certain properties of compact plane continua. In doing this the writer has obtained results in three somewhat different areas.

The first is that area which is concerned with a study of the properties of fixed point sets, and the properties of the space obtained after the omission of the fixed points. The writer generalizes some known theorems in this area.

The second area of this study is concerned with fixed points under periodic transformations. The writer shows a relationship between the study of the fixed point property under periodic transformations and the study of the fixed point property under isometric transformations. He shows also that a periodic transformation of the Euclidean plane into itself which leaves invariant a compact continuum that does not separate the plane must leave a point of the continuum fixed.

The last area is concerned with intersection properties of plane continua. The writer obtains a new set of sufficient conditions for the intersection of a collection of compact continua in the plane being non-empty. He shows that these conditions are equivalent to a known set of sufficient conditions. Using these new conditions he obtains a generalization of a well known theorem of Helly.

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TABLE OF CONTENTS

Chapter		Page
I.	ROTATION GROUPS UNDER MONOTONE TRANSFORMATIONS	1
II.	FIXED POINT THEOREMS	21
III.	INTERSECTION PROPERTIES OF PLANE CONTINUA	32

LIST OF ILLUSTRATIONS

Figure		Page
1.	Figure 1.	6
2.	Figure 2.	8
3.	Figure 3.	11
4.	Figure 4.	13
5.	Figure 5.	13
6.	Figure 6.	16
7.	Figure 7.	16
8.	Figure 8.	18
9.	Figure 9.	18
10.	Figure 10.	24
11.	Figure 11.	28

CHAPTER I

ROTATION GROUPS UNDER MONOTONE TRANSFORMATIONS

In 1937 Lucille Whyburn (1) published a paper in Fundamenta Mathematicae entitled ''Rotation Groups about a Set of Fixed Points.'' As a basis for her work she asked for a homeomorphism of a space S onto itself. The author generalizes her results by relaxing her hypothesis that the transformation be a homeomorphism to the condition that the transformation be a certain restricted monotone transformation. He proves that all of the theorems which she has shown true under homeomorphisms are still valid under this more general transformation. Many of the following theorems are direct generalizations of the theorems of Whyburn and will be so designated by an apostrophe following the number of the theorem. Proofs of theorems 9' through 15' follow along the same lines as those given by Whyburn, but in the interest of clarity the complete proofs are given.

The following are well known results which will be needed in the development of this chapter.

Definition. A monotone transformation $f: A \rightarrow B$ is a continuous transformation such that if $x \in f(A)$, then $f^{-1}(x)$ is connected.

Proposition A. Let A and B be compact T_1 spaces and f a monotone transformation such that $f(A) \subset B$. If $x \in f(A)$, then $f^{-1}(x)$ is a continuum.

Proposition B. If A is compact and $f(A) = B$ is continuous, then in order that f be monotone it is necessary and sufficient that the inverse of every connected set be connected.

In all the work of this chapter, unless otherwise specified, S is to mean a compact Hausdorff space; T will be a monotone transformation such that $T(S) = S$. If T allows fixed points in S such that K is this set of fixed points, then $T^{-1}(K) = K$.

Theorem 1. The transformation T is one-to-one on K.

Proof. Suppose this is not true, then there exist distinct points x and y in K such that $T(x) = T(y)$. Since x and y are in K they are fixed under T and therefore $T(x) = T(y) = x = y$. This contradicts the fact that x and y are distinct and completes the proof.

Theorem 2. Let C_1 be a component of S-K, then $T(C_1)$ is also a component of S-K.

Proof. By the hypothesis $T(C_1) \cap K = \emptyset$; hence $T(C_1) \subset S-K$. Since T is continuous, $T(C_1)$ is connected and hence is contained in some component of S-K. Let C_2 be the component of S-K containing $T(C_1)$. Now suppose that $T(C_1) \neq C_2$. Let $p \in C_2 - T(C_1)$. Since $T(S) = S$, there is a point y in S such that $y \notin C_1$, $y \notin K$, and $T(y) = p$. It follows then that y is in some component of S-K, say C_3. It follows that $T(C_3) \subset C_2$. By Proposition B, the set $T^{-1}(C_2)$ is connected and, by the hypothesis on T, is contained in S-K. Hence there must be a connected set in S-K containing $C_1 \cup C_3$. This contradicts the fact that C_1 and C_3 are components of S-K.

Theorem 3. If C_1 is a component of S-K, then $T^{-1}(C_1)$ is a component of S-K.

Proof. By hypothesis $T^{-1}(C_1) \subset$ S-K. Since $T^{-1}(C_1)$ is connected, it is contained in some component of S-K, say C_2. Now $T(C_2) \subset$ S-K, so suppose that $C_2 - T^{-1}(C_1) \neq \emptyset$. Let $x \in C_2 - T^{-1}(C_1)$. Then T(x) is in some component of S-K other than C_1. This contradicts the fact that continuous functions carry connected sets onto connected sets.

Theorem 4'. Let C_0 be a component of S-K, $C_n = T^n(C_o)$ for each integer n, and let G be the collection $\{C_n\}$. Then the collection $\{C_n\}$ forms a commutative group where the group operation is defined for any two elements C_i and C_j as $C_iC_j = T^j(C_i)$.

Proof. It follows directly from the definition that there is closure with respect to the operation. To show that the operation is associative consider the following: $C_i(C_jC_k) = C_i\left[T^k(C_j)\right] = C_i\left[T^k(T^j(C_0))\right] = C_i\left[T^{k+j}(C_0)\right] = C_iC_{k+j} = T^{k+j}(C_i) = T^{k+j+i}(C_0) = T^k(T^j(C_i)) = T^k(C_iC_j) = (C_iC_j)C_k$. To show that the operation is commutative consider the following: $C_iC_j = T^j(C_i) = T^j\left[T^i(C_0)\right] = T^{j+i}(C_0) = T^{i+j}(C_0) = T^i\left[T^j(C_0)\right] = T^i(C_j) = C_jC_i$. Now consider $C_jC_0 = T^0(C_j) = C_j$. It follows that C_0 serves as a right and left identity for every element of G. It is now shown that C_{-j} is an inverse for C_j. To do this, note that $C_jC_{-j} = T^{-j}(C_j) = T^{-j}\left[T^j(C_0)\right] = T^{j-j}(C_0) = C_0$. This completes the proof of the theorem.

Definition. A rotation group G_a will mean the group formed by the component C_a where C_a is the C_0 of Theorem 4'.

Definition. The order of a group will mean the number of components in the group if there is a finite number; otherwise the order will be infinite.

Theorem 5'. Every component of S-K lies in one and only one rotation group.

Proof. Suppose that $C_a \in G_b$, and $C_a \in G_d$ where G_d and G_b are distinct rotation groups. It follows then, for some two integers n and m, that $T^n(C_b) = C_a$ and $T^m(C_d) = C_a$. Consider $T^{n-m}(C_b) = T^{-m}(C_a) = T^{m-m}(C_d) = C_d$. Let C be any element of G_d. Then for some integer k, $T^k(C_d) = C$. It follows then that $T^{n-m+k}(C_b) = C$, and hence $C \in G_b$. Therefore, $G_b \supset G_d$, and in a similar way it can be shown that $G_d \supset G_b$. Hence it follows that $G_d = G_b$ which contradicts the supposition.

Lemma 1. If $T(x_0) = x_0$ and $\{x_i\}$ is a sequence of points converging to x_0, then $\{T^{-1}(x_i)\}$ is a sequence of sets converging to x_0.

Proof. Letting $L = \bigcup_{i=0}^{\infty} x_i$, it follows from the continuity of T that $T^{-1}(L)$ is closed and hence compact. Suppose first that $x_0 \notin \overline{\lim_i}\ T^{-1}(x_i)$. Then there is an open set U_0' about x_0 such that $U_0' \cap T^{-1}(L) = x_0$. Let U_i for $i > 0$ be an open set about x_i such that $U_i \cap (\bigcup_{\substack{j \neq i \\ j=0}}^{\infty} x_j) = \emptyset$. It follows that $U_i' = T^{-1}(U_i)$ is an open set containing $T^{-1}(x_i)$ such that $U_i' \cap (\bigcup_{\substack{j \neq i \\ j=0}}^{\infty} T^{-1}(x_j)) = \emptyset$. Hence the collection $\{U_i'\}$ including $i = 0$ is an open covering of $T^{-1}(L)$ which contains no finite subcovering. This contradicts the fact that $T^{-1}(L)$ is compact. Suppose now that there exists a point y in $\overline{\lim_i}\ T^{-1}(x_i)$ such that $y \neq x_0$. Then by the definition of the limit superior, given any open set U about y, U intersects infinitely many of the sets $T^{-1}(x_i)$ and hence T(U) contains an infinity of the points x_i. Therefore, T(y) is a limit point of the sequence $\{x_i\}$.

It is known that $T(y) \neq x_0$. This contradicts the fact that the sequence $\{x_i\}$ converges to x_0. Let $\{T^{-1}(x_{i_k})\}$ be any subsequence of $\{T^{-1}(x_i)\}$. It follows, from an argument similar to the one above, that $\overline{\lim_{i_k}} T^{-1}(x_{i_k}) = x_0$. Hence it follows that $\lim_i T^{-1}(x_i) = x_0$.

Theorem 6'. If C_i and C_j are two elements of a rotation group, then $\overline{C}_i - C_i = \overline{C}_j - C_j$.

Proof. It is now shown that $\overline{C}_i - C_i \subset \overline{C}_j - C_j$. Note that $C_i = T^i(C_0)$ and $C_j = T^j(C_0)$. Hence $C_j = T^{j-i}(C_i)$. First consider the case where $j-i = 0$. It then follows that $C_i = C_j$ and the theorem is proved. Next suppose that $j-i = k > 0$, then T^k is a continuous mapping of C_i onto C_j. If $x \in \overline{C}_i - C_i$, then there is a sequence of points $\{x_i\}$ in C_i which converge to x. By the continuity of T^k it follows that the sequence $\{T^i(x_i)\}$ converges to $T(x) = x$. Therefore, x is in $\overline{C}_j - C_j$. Finally suppose that $j-i = -k < 0$, in which case $T^{-k}(C_i) = C_j$, or $T^i(C_j) = C_i$. Note at this time that $(T^k)^{-1}$ is T^{-k}. Again let $\{x_i\}$ be a sequence of points from C_i converging to x in $\overline{C}_i - C_i$. Since T^k satisfies the conditions in Lemma 1, it follows that $\lim_i T^{-k}(x_i) = x$. Therefore x is in $\overline{C}_j - C_j$ since $T^{-k}(x_i) \subset C_j$ for each $i > 1$. In a similar manner it can be shown that $\overline{C}_j - C_j \subset \overline{C}_i - C_i$. Therefore it follows that $\overline{C}_i - C_i = \overline{C}_j - C_j$.

Corollary. If S is locally connected, then for any rotation group $F(\bigcup_i C_i) = \bigcup_i F(C_i) = F(C_k)$ for C_k a fixed element of the rotation group (where F(A) denotes the boundary of A relative to S).

Proof. Obviously $\bigcup_i F(C_i) = F(C_k)$ for each k since by Theorem 5' $F(C_i) = F(C_j)$ for any two elements C_i and C_j of the rotation group. Let x be in $\bigcup_i F(C_i)$. Then $x \notin C_i$ for any i, so $x \notin \bigcup_i C_i$. Therefore,

$x \in F(\bigcup_i C_i)$ and it follows that $\bigcup_i F(C_i) \subset F(\bigcup_i C_i)$. Now suppose $x \in F(\bigcup_i C_i)$ and $x \notin M = \bigcup_i F(C_i) = F(C_k)$. Then there exists an open set U_x containing x and an open set V_m containing M such that $U_x \cap V_m = \emptyset$; this follows from the compactness of M. Since $x \in F(\bigcup_i C_i)$, it is known that there is an open connected set $U \subset U_x$ about x such that $U \cap (\bigcup_i C_i) \neq \emptyset$, but $U \cap \left[\bigcup_i F(C_i)\right] = \emptyset$. It follows then that U intersects infinitely many of the components C_i. Then, since U is connected, either $U \subset C_i$ for some i which is a contradiction or $U \cap F(C_i) \neq \emptyset$ for some i which again is a contradiction. Therefore it follows that $F(\bigcup_i C_i) \subset \bigcup_i F(C_i)$, and finally that $F(\bigcup_i C_i) = \bigcup_i F(C_i) = F(C_k)$.

The following example shows that it is necessary to ask that S be locally connected.

Example. Let $S = \left[\bigcup_{n=1}^{\infty} C_n\right] \cup C_0$ where $C_n = \{(r, \theta): 0 \leq r \leq 1, \theta = 1/n\}$ and $C_0 = \{(r, 0): 0 \leq r \leq 1\}$. Define T in the following manner; $T(r, 1/n) = (r, 1/n+1)$ if $n \neq 0$, and $T(r, 0) = (r, 0)$. It is easy to see from Figure 1 that $F(\bigcup_{i=1}^{\infty} C_i) = C_0$; whereas, $\bigcup_{i=1}^{\infty} F(C_i) = F(C_i) = \{(0, 0)\}$.

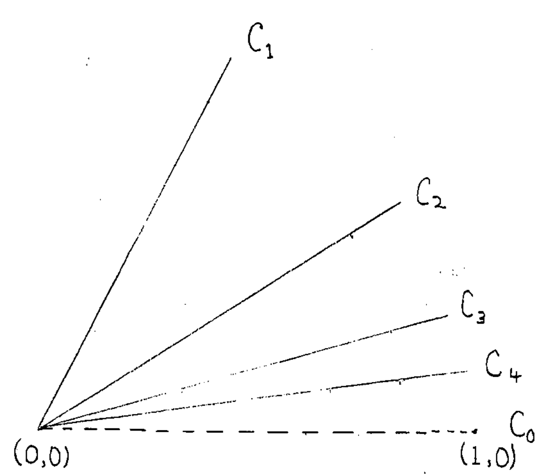

Fig. 1

Theorem 7. Let f be a monotone transformation of the unit interval I onto the set M. If M is non-degenerate then M is an arc.

Proof. M is connected and compact by the continuity of f. It is shown that $f(0) \neq f(1)$. Suppose $f(0) = f(1) = x$. Then $f^{-1}(x)$ is a compact continuum containing 0 and 1. This implies that $I \subset f^{-1}(x)$ and hence that M is degenerate. This contradicts the hypothesis of the theorem. Let y be a point of M such that $y \neq f(0)$ and $y \neq f(1)$. It will now be shown that y separates M. Suppose that y fails to separate M, then M-y is connected. It follows that $f^{-1}(M-y)$ is connected and contains the points 0 and 1. This contradicts the fact that $f^{-1}(y)$ separates I.

Theorem 8. Let f be a monotone transformation of a space M onto the unit interval I such that $f^{-1}(0)$ is a unique point of M. Then M is locally connected at $f^{-1}(0)$.

Proof. Let $\{U_i\}$ be a sequence of connected open sets converging to the point 0 in I. It follows by an argument similar to that of Lemma 1 and from the continuity of f that $\{f^{-1}(U_i)\}$ is a sequence of open sets converging to $f^{-1}(0)$. Since f is monotone, each set $f^{-1}(U_i)$ is connected. Now, given any open set V about $f^{-1}(0)$, there is a connected open set $f^{-1}(U_j)$ for some j such that $f^{-1}(U_j) \subset V$, and $f^{-1}(U_j)$ is connected. Hence M is locally connected at $f^{-1}(0)$ as required.

The following example will illustrate that the inverse image of an arc under a monotone transformation need not contain an arc joining every two points.

Example. Let the set M be the following set; $\{(x, y) : y = \sin 1/x, 0 < x \leq 1\} \cup \{(x, y) : x = 0, -1 \leq y \leq 1\}$. Let f be the perpendicular projection of M onto I. It is easily seen from Figure 2 that f is monotone, and there is no arc joining the points $(0, 0)$ and $(\pi/2, 0)$.

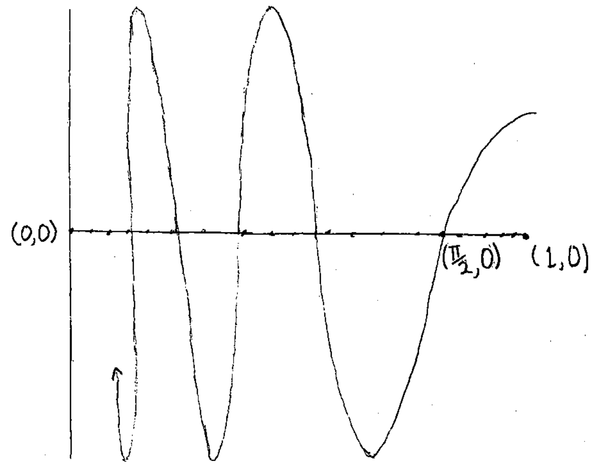

Fig. 2

Definition. A point p is said to be accessible from a set of points M provided there exists a simple arc $\widehat{xp}$ contained in $M \cup p$.

Theorem 9'. If C is an element of a finite rotation group G_a, any point p in F(C) which is accessible from C is accessible from any component of G_a.

Proof. Since G_a is finite, it follows that for any C_a in G_a $T^k(C) = C_a$ for some positive integer k. Let $\widehat{xp}$ be an arc from x to p such that $\widehat{xp} - p \subset C$. Then by Theorem 7, $T^k(\widehat{xp}) = \widehat{yp}$ is an arc such that $\widehat{yp} - p \subset C_a$. This completes the proof.

Theorem 10'. If C is an element of G_a (where G_a may not be finite), and p is in F(C) and accessible from C, then for any C_a, such that $T^k(C) = C_a$ for some positive integer k, p is accessible from C_a.

Definition. A point set M is said to have property S if for each $\epsilon > 0$, M is the union of a finite number of connected sets each of diameter less than ϵ. (2).

Definition. A plane Peano continuum is a subset of the plane which is the continuous image of the unit interval.

Lemma 2. Let M have property S, and let the closed set K separate M. Then, if C is a component of M-K such that C does not have property S, there exists an $\epsilon > 0$ and a sequence of points $\{p_n\}$ of C converging to a point p in $\overline{C}$-C such that no two of these points can be joined in C by a connected set of diameter less than ϵ.

Proof. Let it be assumed that C cannot be represented as a finite union of connected sets of C each of diameter less than δ. Let $\{g_n\}$ be a finite representation of M where each g_n is a connected set of M and of diameter less than $\delta/4$. It follows then that $(\bigcup_{n\in\sigma} g_n) \cap C$, where σ is finite, has infinitely many components. Let these components be denoted by $\{g_{n\alpha}\}_{n\in\sigma, \alpha\in\Lambda}$. Note that $\bar{g}_{n\alpha} \cap K \neq \emptyset$ for infinitely many $n\alpha$'s. Now choose some $g_{n\alpha}$ and call it g_1'. There exists some $g_{n\alpha}$, say g_2', distinct from g_1' such that g_1' cannot be joined to g_2' by a connected set in C of diameter less than $\delta/4$, and $g_2' \cap K \neq \emptyset$. Suppose this is not the case. It is obvious that there are sets $g_{n\alpha}$ which cannot be joined to g_1' by a connected set in C of diameter less than $\delta/4$. If this were not so, it would follow from the triangle inequality that C has property S contrary to the hypothesis. If each such $g_{n\alpha}$ has a closure which does not intersect K, then there can be at most a finite number of these. Therefore, represent C as the union of this finite collection plus the set obtained by connecting g_1' with the remaining $g_{n\alpha}$'s. This implies that C has property S and is a contradiction. Now given g_1' and g_2' choose g_3' such that neither g_1' or g_2' can be joined to g_3' by a connected set in C of diameter less than $\delta/4$. Again, if it were not possible to do this, C would have property S. Continue in this manner inductively to obtain an infinite sequence $\{g'_n\}$ of the components $g_{n\alpha}$ such that no two may be joined in C by a connected set of diameter less than $\delta/4$. Since $\bar{g}'_n \cap K \neq \emptyset$ for each n, choose p_1 in g'_1 such that $\rho(p_1, K) < 1$, and in general choose p_n in g'_n such that $\rho(p_n, K) < 1/n$. Now observe that $\overline{\bigcup_{n=1}^{\infty} p_n} \cap K \neq \emptyset$. Hence it is possible to choose a subsequence $\{p_{ni}\}$ of the sequence of points $\{p_n\}$ such that the subsequence $\{p_{ni}\}$ converges to a point p in K. Now let $\epsilon = \delta/4$ and the proof of the lemma is complete.

Theorem 11'. Let M be a plane Peano continuum and $T(M) = M$. If C is an element of a rotation group under T of order greater than 1, then C has property S.

Proof. Suppose C does not have property S. Then since M has property S it follows from Lemma 2 that there exists an $\epsilon > 0$ and a sequence of points $\{p_n\}$ of C converging to a point p of F(C) such that no two of these points can be joined in C by a connected set of diameter less than ϵ. Let N represent a circle with center p and diameter $\epsilon/4$. Without loss of generality it may be assumed that p_n lies inside N for each n. Let $\widehat{p_n p_{n+1}}$ denote an arc contained in C, joining p_n to p_{n+1}. There is such an arc since C is connected and locally connected. It follows that $\widehat{p_n p_{n+1}} \cap N \neq \emptyset$ since the diameter of $\widehat{p_n p_{n+1}}$ is greater than or equal to ϵ and the diameter of N is $\epsilon/4$. Let q_n be the first point of the intersection of $\widehat{p_n p_{n+1}}$ and N. Hence for each n the arc $\widehat{p_n q_n}$ is obtained such that $\widehat{p_n q_n} \cap N = q_n$. Note that $\widehat{p_n q_n} \cap \widehat{p_k q_k} = \emptyset$ if $n \neq k$ (illustrated in Figure 3) since otherwise there would be a connected set joining p_n and p_k having diameter less than ϵ. Let H denote $\overline{\lim_n}\, \widehat{p_n q_n}$. Observe first that H is a continuum since it is well known that the sequential limiting set of any sequence of connected sets is connected, Since p is in every limiting set of a subsequence of $\widehat{p_n q_n}$, it follows that H is a continuum. Obviously, H is contained in $\overline{C}$. Also note that H is non-degenerate since all but a finite number of the arcs $\widehat{p_n q_n}$ have diameters greater than $\epsilon/16$.

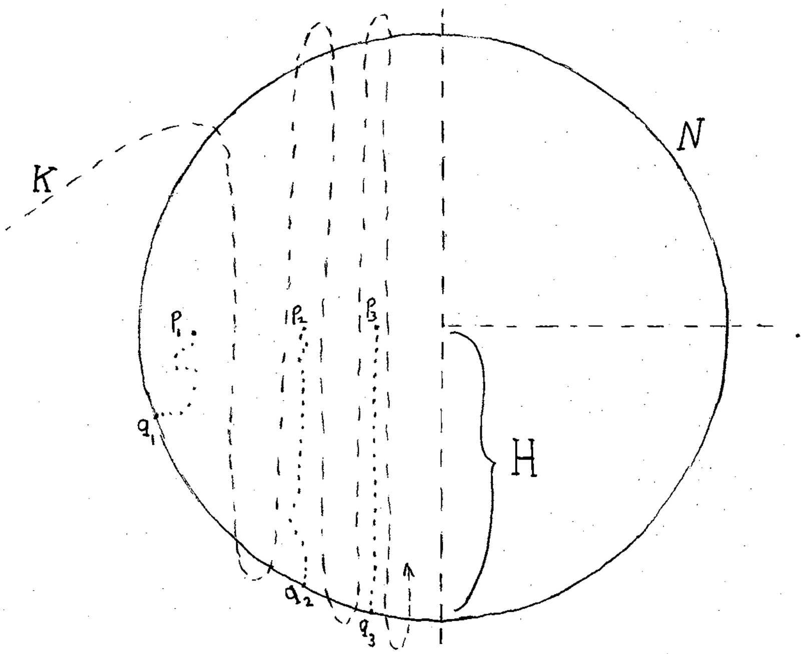

Fig. 3

Now to show that $H \subset \overline{C} - C \subset K$. Suppose that $H \cap C \neq \emptyset$, and the point b is in $H \cap C$. Since C is locally connected at b, there is a $\delta < \epsilon/4$ such that every point of C whose distance from b is less than δ can be joined to b by an arc whose diameter is less than $\epsilon/4$. Since b is in H, there exist arcs $\widehat{p_i q_i}$ and $\widehat{p_j q_j}$ such that $\widehat{p_i q_i}$ and $\widehat{p_j q_j}$ contain points p_i' and p_j', respectively, which can be joined to b by arcs of diameter less than $\epsilon/4$. It follows by the triangle inequality that the diameter of $\widehat{p_i' p_i} \cup \widehat{p_i' b} \cup \widehat{b p_j'} \cup \widehat{p_j' p_j}$ is smaller than ϵ. This contradicts the fact that p_j and p_i cannot be joined in C by a connected set of diameter less than ϵ. Therefore it follows that $H \subset F(C) \subset K$.

It is known that H contains p and a point of N, say q. Hence the diameter of H is greater than $\epsilon/16$. Let p' be a point of H such that $\rho(p', p) = \epsilon/16$. Let N' be a circle with center p' and diameter less than $\epsilon/16$. Let $\{p'_{nk}\}$ be a subsequence of $\{p_n\}$ such that no p'_{nk} is contained in N' plus its interior, and furthermore such that p' is in the sequential limiting set of the sequence of arcs $\{\widehat{p'_{nk}q_{nk}}\}$ where $\widehat{p'_{nk}q_{nk}} \subset \widehat{p_{nk}q_{nk}}$. Let $\{p''_{nk}\}$ be a sequence of points converging to p' such that p''_{nk} is in the arc $\widehat{p'_{nk}q_{nk}}$ and p''_{nk} is interior to N'. It follows then that there exists in the arc $\widehat{p'_{nk}q_{nk}}$ a sub-arc $\widehat{r_{nk}p''_{nk}s_{nk}}$ which intersects N' in only the points r_{nk} and s_{nk}. Since M is locally connected, there is a connected open subset of M about p' and contained in the interior of N'. It follows then that there exists an integer G such that if n and m are greater than G, then p''_n and p''_m may be joined by an arc of M contained in the interior of N' (refer to Figure 4).

Let n, m, and k be integers larger than G and consider the three arcs $\widehat{r_n p''_n s_n}$, $\widehat{r_m p''_m s_m}$, and $\widehat{r_k p''_k s_k}$. One of these arcs separate the other two in N' plus its interior. Without loss of generality suppose that $\widehat{r_m p''_m s_m}$ separates $\widehat{r_n p''_n s_n}$ from $\widehat{r_k p''_k s_k}$. Let $\widehat{p''_m p''_k}$ and $\widehat{p''_n p''_m}$ be arcs lying in N' plus its interior. The existence of these arcs has been demonstrated above. The arc $\widehat{p''_m p''_n}$ will contain a sub-arc $\widehat{u_m v_n}$ whose intersection with $\widehat{r_m p''_m s_m}$ is the point u_m and whose intersection with $\widehat{r_n p''_n s_n}$ is the point v_n. It is now shown that such an arc $\widehat{u_m v_n}$ does exist. Order the points on $\widehat{p''_m p''_n}$ from p''_m to p''_n. Let u_m be the last point of the intersection of $\widehat{p''_m p''_n}$ and $\widehat{r_m p''_m s_m}$. Let v_n be the first point of the intersection of the arc $\widehat{u_m p''_n}$ and $\widehat{r_n p''_n s_n}$ (refer to Figure 5).

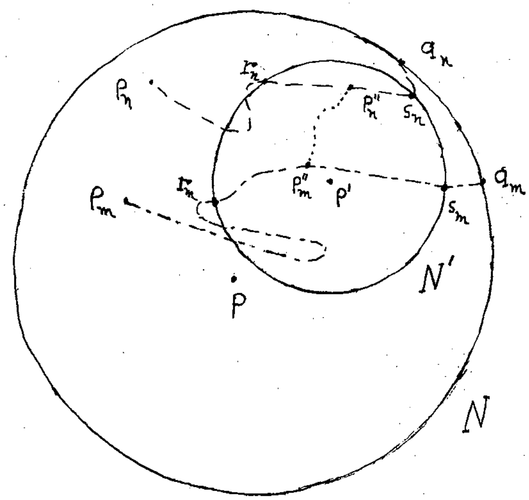

Fig. 4

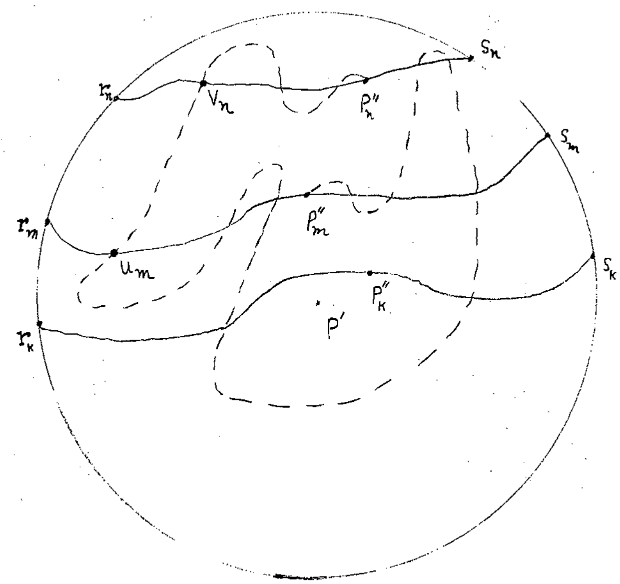

Fig. 5

In the manner described above, obtain the arc $\widehat{z_m w_k}$ such that $\widehat{z_m w_k} \cap \widehat{r_m p''_m s_m} = z_m$ and $\widehat{z_m w_k} \cap \widehat{r_k p''_k s_k} = w_k$. Since the diameter of the set $\widehat{p_m q_m} \cup \widehat{u_m v_n} \cup \widehat{p_n q_n}$ is less than ϵ, it follows that $\widehat{u_m v_n} \cap K \neq \emptyset$ and likewise $\widehat{z_m w_k} \cap K \neq \emptyset$. Let x_1 be the first point of $\widehat{u_m v_n} \cap K$ which is ordered from u_m to v_n. Let y_1 be the first point of $\widehat{z_m w_k} \cap K$ which is ordered from z_m to w_k. Now $\widehat{x_1 y_1}$ will denote the arc which is the union of the arcs $\widehat{u_m x_1}$, $\widehat{u_m z_m}$, and $\widehat{z_m y_1}$. It follows that $\widehat{x_1 y_1} - (x_1 \cup y_1)$ is contained in C and that x_1 is separated from y_1 in N' plus its interior by the arc $\widehat{r_m p''_m s_m}$. By Theorem 7 it follows that $T(\widehat{x_1 y_1})$ is an arc. Now $T(\widehat{x_1 y_1})$ must contain a point b_1 lying outside N'. To show this, observe that $T(x_1) = x_1$ and $T(y_1) = y_1$. Hence $T(x_1)$ is separated from $T(y_1)$ in N' plus its interior by the arc $\widehat{r_m p''_m s_m}$. It follows that either $T(\widehat{x_1 y_1})$ contains a point outside of N' or contains a point of $\widehat{r_m p''_m s_m}$. The latter case is impossible since $\widehat{r_m p''_m s_m} \subset C$ and by hypothesis the order of C is greater than 1. In this manner choose an infinite sequence of arcs $\{\widehat{x_n y_n}\}$ so that they converge sequentially to p', and such that $T(\widehat{x_n y_n})$ contains a point b_n lying outside N' plus its interior. By Lemma 1 the sequence of sets $\{T^{-1}(b_i)\}$ must converge to p'. However the sequence of points $\{T(T^{-1}(b_i))\} = b_i$ does not converge to $T(p') = p' \in N'$. This contradicts the continuity of T. Therefore, C must have property S.

Definition. If A is a subset of M, the M-boundary of A is the boundary of A intersected with M.

G. T. Whyburn (3) has shown that in order for every point of the M-boundary B of a bounded connected open subset R of a Peano continuum M to be accessible from R it is sufficient that R should have property S. It follows from this result of Whyburn's that under the hypothesis of Theorem 11', every point of F(C) is accessible from C.

G. T. Whyburn (4) has also shown that if every point of the closed and bounded point set K in the plane S is accessible from each of two mutually exclusive connected subsets R_1 and R_2 of S-K, then either K is a simple closed curve or there exists a simple continuous arc which contains K. He obtains as a corollary to this result the fact that, under the same conditions, there exists a simple closed curve which contains K.

Theorem 12'. If M is a plane Peano continuum and C is an element of a rotation group of order greater than 1, then F(C) is contained in some simple closed curve.

Proof. This result follows immediately from the conclusion of Theorem 10', and the results of Whyburn stated above.

Lemma 3. Let $\widehat{px}$, $\widehat{py}$, and $\widehat{pz}$ be arcs joining the point p to the distinct points x, y, and z, respectively. There exist arcs $\widehat{q_1x}$, $\widehat{q_1y}$, and $\widehat{q_1z}$ (possibly degenerate) joining the point q_1 to the points x, y, and z, respectively, such that $\widehat{q_1x} \cup \widehat{q_1y} \cup \widehat{q_1z} \subset \widehat{px} \cup \widehat{py} \cup \widehat{pz}$ and $\widehat{q_1x} \cap \widehat{q_1y} = q_1$, $\widehat{q_1x} \cap \widehat{q_1z} = q_1$, and $\widehat{q_1y} \cap \widehat{q_1z} = q_1$.

Proof. Let $\widehat{px}$ be ordered from p to x, and let q_x be the last point on $\widehat{px}$ and also in $\widehat{py} \cup \widehat{pz}$ (note that q_x may be the point p or the point x). It follows that q_x is a point of $\widehat{py}$ or a point of $\widehat{pz}$. Without loss of generality suppose that q_x is a point of $\widehat{py}$. Let the arc $\widehat{pz}$ be ordered from p to z, and let $\widehat{q_xy}$ be contained in $\widehat{py}$. First consider the case where $\widehat{q_xy} \cap \widehat{pz} \neq \emptyset$. If $\widehat{q_xy} \cap \widehat{pz} \neq \emptyset$, let q_y be the last point in this intersection relative to the ordering on $\widehat{pz}$. In this case let $q_1 = q_y$, and $\widehat{q_1q_x}$ be contained in $\widehat{q_xy}$. Now set $\widehat{q_1x} = \widehat{q_1q_x} \cup \widehat{q_xx}$, and let $\widehat{q_1z}$ and $\widehat{q_1y}$ be contained in $\widehat{pz}$ and $\widehat{q_xy}$, respectively. It is seen that if $\widehat{q_1z}$ and $\widehat{q_1x}$ have points other than q_1 in common, then either q_y is not the last point of the intersection of $\widehat{q_x y}$ and $\widehat{pz}$ or q_x is not the last point of intersection of $\widehat{px}$ and $\widehat{py} \cup \widehat{pz}$. This would contradict the construction of the arcs (note Figure 6). Also it is obvious that $\widehat{q_1 y} \cap \widehat{q_1 x} = q_1$ and $\widehat{q_1 y} \cap \widehat{q_1 z} = q_1$.

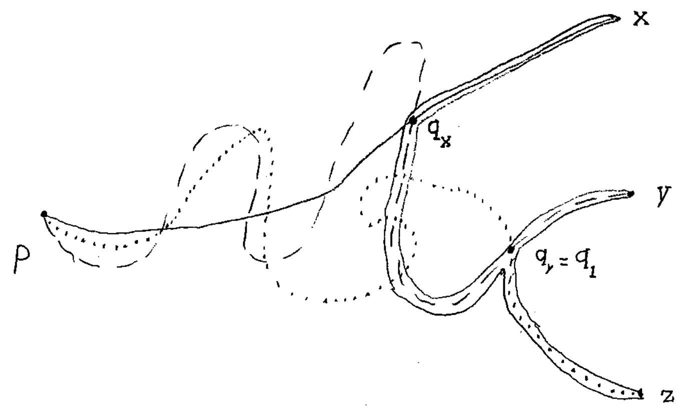

Fig. 6

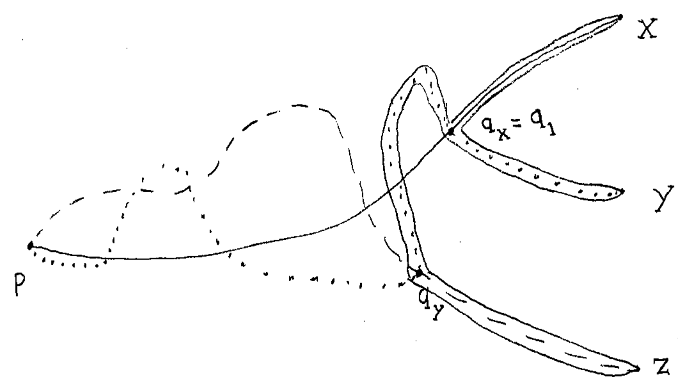

Fig. 7

Now consider the case where $\widehat{q_x y} \cap \widehat{pz} = \emptyset$. Let the arc $\widehat{pq_x}$ be contained in the arc $\widehat{py}$ and order $\widehat{pq_x}$ from p to q_x. Let q_y be the last point in the intersection of $\widehat{pq_x}$ and $\widehat{pz}$ (note Figure 7). Let $\widehat{q_y q_x}$ be contained in $\widehat{pq_x}$, and $\widehat{q_y z}$ be contained in $\widehat{pz}$. Set

$q_1 = q_x$, and $\widehat{q_1z} = \widehat{q_xq_y} \cup \widehat{q_yz}$. It is easy to see that $\widehat{q_1z} \cap \widehat{q_1x} = q_1$, $\widehat{q_1z} \cap \widehat{q_1y} = q_1$, and $\widehat{q_1y} \cap \widehat{q_1x} = q_1$. Therefore, the proof is complete.

Theorem 13'. Let M be a plane Peano continuum and $T(M) = M$ a monotone transformation which is one-to-one on the set of fixed points. If $F(C)$ contains more than two points, then the rotation group of C is of order less than or equal to 2.

Proof. Suppose the rotation group of C is greater than 2. Let D and E be in the rotation group of C where $T(C) = D \neq C$, $T^2(C) = T(D) = E \neq D$ and $E \neq C$. Let x, y, and z be distinct points of $F(C)$. Let p be a point of C. By the corollary above obtain arcs $\widehat{xp}$, $\widehat{yp}$ and $\widehat{zp}$ which are contained in C except for x, y, and z, respectively. Using Lemma 3, obtain three arcs $\widehat{q_1x}$, $\widehat{q_1y}$, and $\widehat{q_1z}$ (note that these arcs are non-degenerate since $q_1 \neq x$, y, or z) such that the intersection of any pair is q_1. By Theorem 7 it follows that $T(\widehat{q_1x})$, $T(\widehat{q_1y})$, and $T(\widehat{q_1z})$ are three arcs lying in D except for $T(x) = x$, $T(y) = y$, and $T(z) = z$, respectively. It is seen also that each of these arcs have in common with D the point $T(q_1)$. Again with the help of Lemma 3, obtain a point r_1 and three arcs $\widehat{r_1x}$, $\widehat{r_1y}$, and $\widehat{r_1z}$ lying in D except for the points x, y, and z, respectively, and such that the intersection of any pair of these arcs is the point r_1. In a similar manner obtain three arcs in E which satisfy the same conditions relative to E. Denote these by $\widehat{s_1x}$, $\widehat{s_1y}$, and $\widehat{s_1z}$ (note Figure 8).

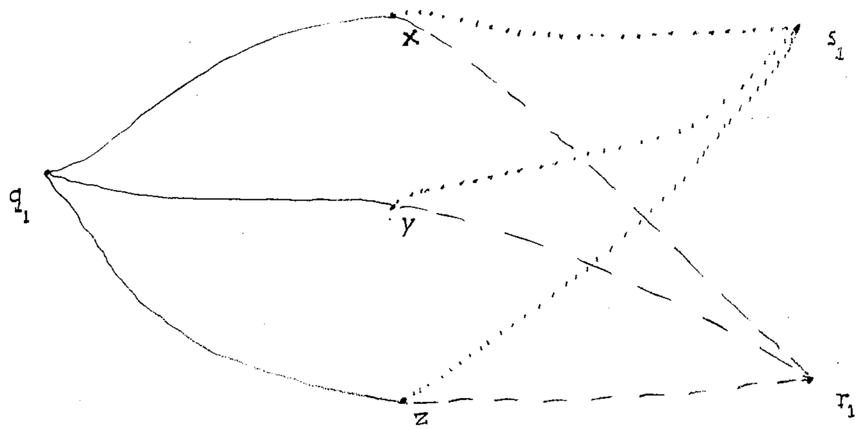

Fig. 8

Note now that $\widehat{r_1 x} \cup \widehat{q_1 x} \cup \widehat{r_1 z} \cup \widehat{q_1 z}$ forms a simple closed curve L. Without loss of generality assume that y is in the bounded component of the complement of L (Figure 9). It follows that the arc $\widehat{r_1 y} \cup \widehat{q_1 y}$ separates the interior of L into two components and hence the set $\widehat{r_1 y} \cup \widehat{q_1 y} \cup L$ separates the plane into three components. Denote these components by O_1, O_2, and O_3 in such a way that $x \notin \overline{O}_1$, $z \notin \overline{O}_2$, and $y \notin \overline{O}_3$. It is obvious that s_1 must lie in the interior of one of these components. Hence it follows that either x, y, or z is not accessible from s_1. This is a contradiction and hence the proof of the theorem is complete.

Note that C. Kuratowski (5) has shown that the graph described above and pictured in Figure 8 cannot lie in the plane.

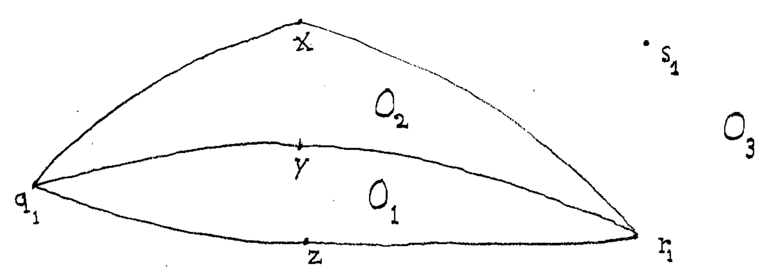

Fig. 9

Theorem 14'. Let M be a plane Peano continuum and $T(M) = M$. If G is an infinite rotation group generated by C, then F(C) reduces to one point and, for any preassigned positive number ϵ, $\delta(C_i) < \epsilon$ ($\delta(A)$ is the diameter of A and $C_i = T^i(C_o)$) for all but a finite number of subscripts i.

Proof. First show that $F(C_i)$ is connected. By the Corollary to Theorem 6 it follows that $F(\bigcup_i C_i) = \bigcup_i F(C_i) = F(C_k)$. Also it follows that $F(C_k) = \lim_i C_i$. If this were not the case $F(\bigcup_i C_i) \neq F(C_k)$ for each k. Hence it may be concluded that $F(C_i)$ is connected.

If $F(C_i)$ has more than one point, it must contain infinitely many points. Hence by Theorem 13 the order of G is less than 3. This is a contradiction and the proof that F(C) reduces to one point is complete.

The fact that $F(C_i) = \lim_i C_i$ and that $F(C_i)$ consists of one point leads to the conclusion that $\delta(C_i) < \epsilon$ for all but a finite number of subscripts i.

Theorem 15'. Let M be a two dimensional sphere and $T(M) = M$. If there is a rotation group under T of order greater than 1, then K is a simple closed curve.

Proof. Let G be a rotation group of order greater than 1. Let C_1 and C_2 be elements of G. It follows from Theorem 12' that $F(C_1)$ is contained in a simple closed curve J. Since C_1 is open in M, $F(C_1)$ separates M. Hence $F(C_1) = J$. Since $F(C_1) = F(C_2)$, it follows that $F(C_1) = F(C_2) = J$. Now it is shown that $C_1 \cup C_2 \cup J = M$. By the Jordan Curve Theorem $M - J = D_1 \cup D_2$ where D_1 and D_2 are mutually separated connected open sets. Suppose $C_1 \subset D_1$. It follows that $F(C_1) = F(C_2) = F(D_1)$ and hence $C_1 = D_1$. In the same way it can be shown that $C_2 = D_2$. Therefore, $C_1 \cup C_2 \cup J = M$ and $J = K$, so the proof is complete.

Corollary. Under the conditions of Theorem 15' there exists only one rotation group under T and this group is of order 2.

CHAPTER II

FIXED POINT THEOREMS

A well known unsolved problem in topology is the following: If M is a compact continuum in the plane and does not separate the plane, and T is a periodic transformation such that $T(M) = M$, does T necessarily allow a fixed point in M?

The question has been answered in the affirmative in certain cases. The main result along this line, obtained by P. A. Smith (6), is stated as follows: Let K be a point set in Euclidean m-space and T a topological transformation of K into itself of finite prime period p. If every continuous single-valued image in K of every sphere of dimension less than or equal $pm-m-1$ is deformable in K to a point, then T leaves fixed at least one point of K.

It has also been shown that if T is a one-to-one continuous and orientation preserving transformation of the Euclidean plane S onto itself which leaves a bounded continuum M invariant, and if M does not separate S, then some point of M is left fixed by T. This result was first obtained by M. L. Cartwright and J. E. Littlewood (7); later O. H. Hamilton (8) obtained the same result using a much shorter method.

The writer is able to obtain in this chapter the result that if T is a periodic transformation of the plane into itself which leaves a plane continuum M invariant, and if M does not separate the plane, then some point of M is left fixed by T. This result is not contained in either of the previously mentioned results, since a periodic transformation need neither be orientation preserving nor of prime period.

The writer shows that the answer to the general problem concerning the existance of fixed points in continua which do not separate the plane under periodic transformations is in the affirmative if every isometry I of Euclidean n-space into itself which leaves a unicoherent continuum M invariant necessarily leaves some point of M fixed.

The following theorem is basic to the development of this chapter.

Theorem 1. If T is a periodic transformation of a metric space S with metric ρ onto itself, then there exists a metric ρ' on S such that T is an isometry relative to the metric ρ'.

Proof. First define the function ρ' which is defined from S X S to the real numbers as follows: let $\rho'(x, y) = \max_n [\rho(T^n(x), T^n(y)):$ $n = 1, 2, 3, \ldots, m]$ where m is the period of T. Now show that ρ actually satisfies the conditions for a metric. Observe first that $\rho(T^n(x), T^n(y))$ is greater than or equal 0 for each n, and therefore $\rho'(x, y)$ is greater than or equal 0. Now show that $\rho'(x, y) = 0$ if and only if $x = y$. Suppose $x = y$, then $T^n(x) = T^n(y)$ for each n, and hence for each n it follows that $\rho(T^n(x), T^n(y)) = 0$. It may be concluded that $\rho'(x, y) = 0$. Suppose now that $\rho'(x, y) = 0$. It follows that $\rho(T^n(x), T^n(y)) = 0$ for each n and in particular when $n = m$. Since ρ is a metric, it follows that $T^m(x) = T^m(y)$. Now using the fact that T is periodic of period m, it is seen that $x = y$.

In order to show that $\rho'(x, y) = \rho'(y, x)$ observe that $\rho(T^n(x), T^n(y)) = \rho(T^n(y), T^n(x))$ for each n, and hence $\max_n \left[\rho(T^n(x), T^n(y))\right] = \max_n \left[\rho(T^n(y), T^n(x))\right]$. To prove that the triangle inequality is valid under ρ' note that $\rho(T^n(x), T^n(z)) \leq \rho(T^n(x), T^n(y)) + \rho(T^n(y), T^n(z))$ for each n. It follows that $\rho'(x, z) = \max_n \left[\rho(T^n(x), T^n(z))\right] \leq \max_n \left[\rho(T^n(x), T^n(y)) + \rho(T^n(y), T^n(z))\right] \leq \max_n \left[\rho(T^n(x), T^n(y))\right] + \max_n \left[\rho(T(y), T^n(z))\right] = \rho'(x, y) + \rho'(y, z)$. This completes the proof that ρ' is a metric.

In order to show that T is an isometry with respect to the ρ' metric observe that since T is periodic of period m, the collection of non-negative real numbers $\left\{ \rho(T^n(x), T^n(y)) : n = 1, 2, 3, \ldots, m \right\}$ is identical with the collection $\left\{\rho(T^{n+1}(x), T^{n+1}(y)) : n = 1, 2, 3, \ldots, m\right\}$.

The following example shows that if the distance between two points x and y is defined to be the $\min_n \left[\rho(T^n(x), T^n(y)) : n = 1, 2, 3, \ldots, m\right]$, then the resulting function need not be a metric (refer to Figure 10). It is easy to show that the triangle inequality is the only property which may fail to be satisfied.

Example. Let M be a subset of the plane such that $M = \left\{(0, 0), (4, 0), (5, 0), (5, 1)\right\}$. Define the transformation T of period 2 as follows; $T(0, 0) = (4, 0)$, $T(4, 0) = (0, 0)$, $T(5,1) = (5, 0)$ and $T(5, 0) = (5, 1)$. Note that under the induced function the distance between the points (0, 0) and (4, 0) is 4, whereas the distance between (0, 0) and (5, 1) plus the distance between (5, 1) and (4, 0) is $1 + \sqrt{2}$. Therefore, the triangle inequality does not hold.

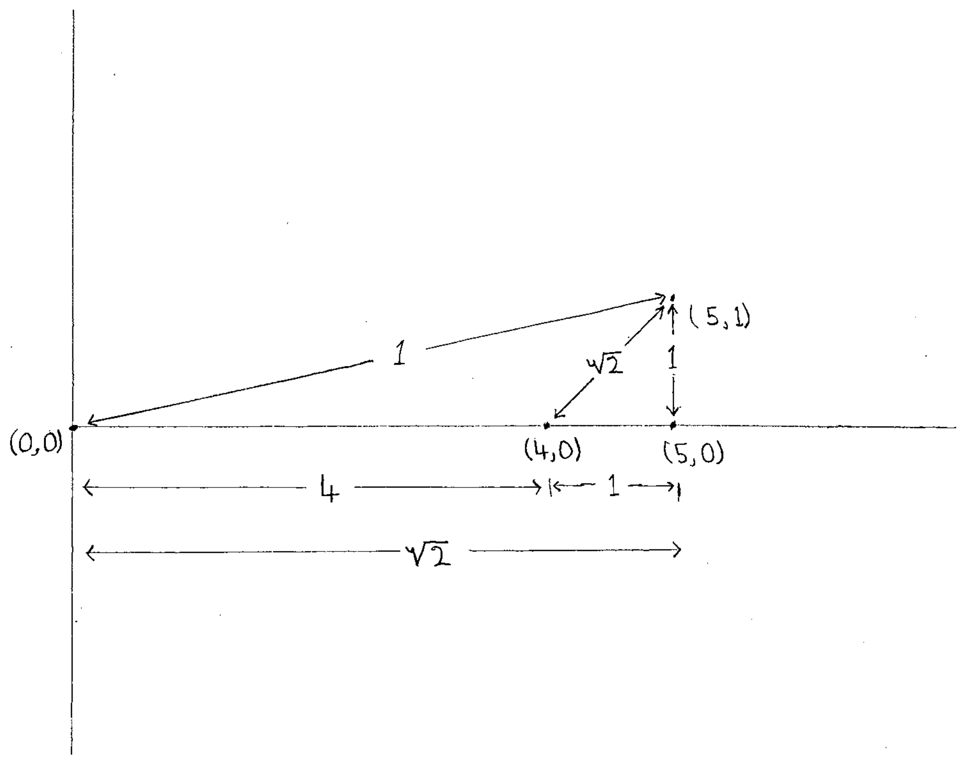

Fig. 10

Theorem 2. If S is a metric space with the metric ρ, and T and ρ' are defined as in Theorem 1, then the topology induced by ρ' is equivalent to the topology induced by ρ.

Proof. Let $S'_\epsilon(x)$ denote a spherical region about x of radius ϵ in the ρ' metric, and let $S_\epsilon(x)$ denote a spherical region about x of radius ϵ in the ρ metric. Take any spherical region $S_\epsilon(x)$ in the ρ metric. Observe that $\rho'(x, y) \geq \rho(x, y)$ for all x and y. Therefore it follows that $S'_\epsilon(x)$ is contained in $S_\epsilon(x)$.

Now it will be shown that given any spherical region $S'_\epsilon(x)$ in the ρ' metric, there exists a spherical region $S_\delta(x)$ in the ρ metric such that $S_\delta(x)$ is contained in $S'_\epsilon(x)$. Since T is continuous at each point x, it follows by definition that, given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that if $\rho(x, y) < \delta_\epsilon$ then $\rho(T(x), T(y)) < \epsilon$. In general it can be shown inductively that, given $\epsilon > 0$ and a positive integer m, there exists a number $\delta_{\epsilon, m} > 0$ such that if $\rho(x, y) < \delta_{\epsilon, m}$ then $\rho(T^n(x), T^n(y)) < \epsilon$ for $n = 0, 1, 2, \ldots, m$. Therefore it may be concluded that the region $S_{\delta_{\epsilon, m}}(x)$ is contained in $S'_\epsilon(x)$. This completes the proof of the theorem.

Theorem 3. If T is a periodic transformation of period k of a subset M of Euclidean n-space onto itself, then M can be imbedded in Euclidean nk-space in such a way that the mapping T', induced by T, of M'(M' is the result of imbedding M in nk-space) onto itself is an isometry under the usual metric of Euclidean space.

Proof. Let X represent the point $(x_1, x_2, \ldots, x_n)$ in n-space Let $X^1 = (x^1_1, x^1_2, \ldots, x^1_n)$ denote the image of X by T if X is in M. In general X^n will denote $T^n(X)$. Let $(X^1, X^2, \ldots, X^m)$ denote the point $(x^1_1, x^1_2, \ldots, x^1_n, x^2_1, x^2_2, \ldots, x^2_n, \ldots, x^m_1, x^m_2, \ldots, x^m_n)$ in Euclidean mn-space. Now define the transformation F such that F carries M into nk-space in the following manner, if X is in M, then $F(X) = (X, X_1, X_2, \ldots, X_{k-1})$. It will be shown that F is one-to-one and continuous on M and hence is a homeomorphism. Obviously, F is one-to-one. To show that F is continuous, note that T is continuous and hence given $\epsilon/\sqrt{k} > 0$ there exists $\delta > 0$ such if $\rho(X, Y) < \delta$ then $\rho(X_i, Y_i) < \epsilon/\sqrt{k}$ for $i = 1, 2, 3, \ldots, k$. Also observe that $\rho(F(X), F(Y)) = \left([\rho(X, Y)]^2 + [\rho(X_1, Y_1)]^2 + \ldots + [\rho(X_{k-1}, Y_{k-1})]^2\right)^{1/2}$. Therefore it follows that if $\rho(X, Y) < \delta$ then $\rho(F(Y), F(X)) < \left[k(\epsilon/\sqrt{k})^2\right]^{1/2} = (k\epsilon^2/k)^{1/2} = \epsilon$ and hence F is continuous. To clarify what is meant by the induced mapping T' it is defined in the following manner; if X is in M then $T'(X, X_1, X_2, \ldots, X_{k-1}) = (X_1, X_2, \ldots, X_{k-1}, X)$. It is easy to see that T' is a distance preserving transformation.

Note, in relation to the question of extending the transformation T', that the following result is well known. If I is an isometry which is defined on a subset of Euclidean space, then I may be extended as an isometry to all of the space.

Theorem 4. If T, T', M, and M' are defined as in Theorem 3, then T leaves a point of M invariant if and only if T' leaves a point of M' invariant.

Proof. Suppose that T leaves the point X of M invariant. Then $T'(F(X)) = (X_1, X_2, \ldots, X_{k-1}, X)$ where $X = X_i$ for $i = 1, 2, 3, \ldots, k$. It follows immediately that $T'(F(X)) = F(X)$. Suppose now that $T'(F(X)) = F(X)$. It then follows that $(X, X_1, X_2, \ldots, X_{k-1}) = (X_1 X_2, \ldots, X_{k-1}, X)$. It may be concluded that $X_i = X$ for $i = 1, 2, \ldots, k-1$ and, therefore, $T(X) = X$.

The following theorem is well known but in the interest of clarity a proof is given.

Theorem 5. Let C be a circle and T be a transformation of C onto itself such that T is a rotation which is not periodic on any point of C (that is, a rotation through some angle A such that $kA \neq 0 \pmod{2\pi}$ for any integer $k \neq 0$). Then the closure of the union of the points $T^i(x)$ for all integers i is C for any x in C.

Proof. Suppose there is an arc $\widehat{ab}$ on C such that $\widehat{ab} \cap \left[\overline{\bigcup_i T^i(x)}\right] = \emptyset$. Since $B = \left[\overline{\bigcup_i T^i(x)}\right]$ is a compact set, the arc $\widehat{ab}$ may be extended to an open arc $\widehat{cd}$ such that the points c and d belong to B.

The points c and d are either of the form $T^n(x)$ for some n or are limit points of the set $\bigcup_n T^n(x)$. Let $\rho(c, d) = \epsilon$. There exist integers k and k' such that $\rho(T^k(x), T^{k'}(x)) < \epsilon/2$ since B must possess a limit point. Choose $\delta > 0$ such that $\rho(T^k(x), T^{k'}(x)) > \delta$. Choose integers n and j such that $\rho(T^n(x), c) < \delta$, and $\rho(T^j(x), d) < \delta$. Obviously, T is both an orientation preserving and isometric transformation on C. Let C be oriented by the arc $\widehat{T^n(x)T^j(x)}$ from $T^n(x)$ to $T^j(x)$. Consider the case where the arc $\widehat{T^k(x)T^{k'}(x)}$ agrees with C in orientation from $T^k(x)$ to $T^{k'}(x)$. Let m = n-k, and consider the points $T^{k+m}(x)$ and $T^{k'+m}(x)$. Note that each of these points is in B. Furthermore since T is an isometry, $T^{k'+m}(x)$ must lie between c and d in the arc which is a contradiction. Now suppose that $\widehat{T^k(x)T^{k'}(x)}$ is negatively oriented. Then by letting m = j-k' the same contradiction is obtained. Therefore, the proof is complete.

The writer will make use of the following well known result. If I is an isometry of the plane, then I must be one of the following transformations: (1) Identity, (2) Rotation, (3) Translation, (4) Reflection, (5) Glide Reflection. The following theorem is one of the main results of this chapter. So far as is known, it is new.

Theorem 6. Let T be an isometry of a compact continuum M of the plane onto itself. If T is not periodic on any point of M, then $\bar{C}_x = \overline{\bigcup_i T^i(x)}$ is a continuum for any point x in M.

Proof. Consider the different types of isometries in the plane. The hypothesis that T is not periodic on any point of M excludes the types (1) and (4). The hypothesis that M is compact excludes types (3) and (5). Hence it follows that T must be a rotation about some point p not in M so that $T^i(x)$ lies on some circle whose center is p. Since T is not periodic on x, it follows that the angle of rotation A is such that $kA \neq 0$ (mod 2π) for any integer $k \neq 0$. Therefore, by Theorem 5 it follows that $\overline{C}_x = C$ where C is the circle with center p and radius $\rho(x, p)$. Therefore, $\overline{C}_x$ is a continuum.

Theorem 7. Let T be an isometry of a compact continuum M of the plane onto itself. If K denotes the set of fixed points of M under T and T is not periodic on any point x of M-K, then $\overline{C}_x = \overline{\bigcup_i T^i(x)}$ is a continuum for any point x of M.

Proof. The proof is essentially the same as that of Theorem 6.

The following example shows that it is possible to have a homeomorphism T of a compact continuum M of the plane onto itself such that T is not periodic on any point of M and such that there is a point x in M where $\overline{C}_x = \overline{\bigcup_i T^i(x)}$ is not connected.

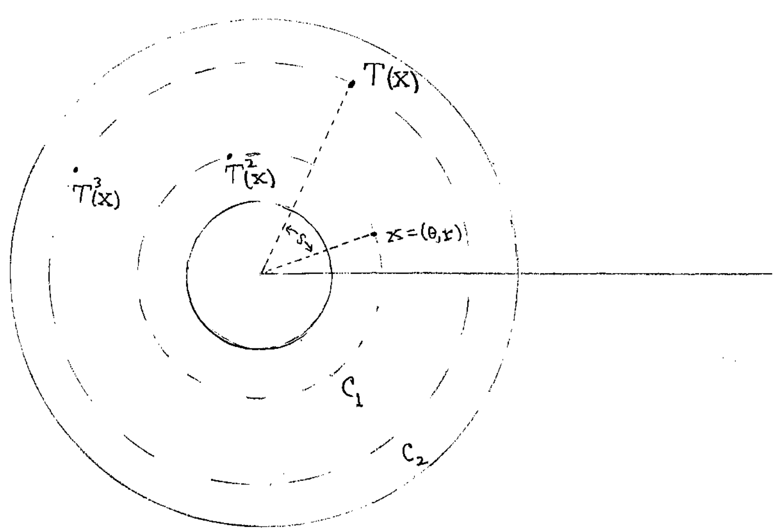

Fig. 11

Example. Let $M = \{(\theta, r): 1 \leq r \leq 3\}$. Define $T(\theta, r) = F[H(\theta, r)]$, where $H(\theta, r) = (\theta, 4-r)$ and $F(\theta, r) = (\theta + s, r)$ for a fixed rational number s. It can be shown using the results of Theorem 5 that $\overline{\bigcup_i T^i(\theta, r)}$ with $r \neq 2$ is the union of two circles C_1 and C_2 where C_1 has its center at the origin and has radius r, and C_2 has its center at the origin and has radius 4-r (refer to Figure 11).

It is known that if M is a continuum which does not separate the plane, then M is the intersection of a monotonic descending sequence of topological 2-cells. (9). Using this information it is possible to prove the following theorem.

Theorem 8. Let T be a periodic transformation of the plane into itself which leaves the compact continuum M invariant. If M does not separate the plane, then T leaves a point of M fixed.

Proof. Let p be the period of T, and M be the intersection of the monotonic descending sequence $\{C_i\}$ of topological 2-cells. Let $S_i = \bigcup_{j=0}^{p-1} T^j(C_i)$; it is now shown that $\bigcap_{i=1}^{\infty} S_i = M$. It is obvious that $\bigcap_{i=1}^{\infty} S_i \supset M$. To show that $\bigcap_{i=1}^{\infty} S_i \subset M$ assume that there exists a point x in $\bigcap_{i=1}^{\infty} S_i$ such that x is not in M. Then there exists an open set U about M such that x is not in U. Since T is continuous, there exists an open set V about M such that $T^k(V) \subset U$ for $k = 0, 1, 2, \ldots, p-1$. Since $\bigcap_{i=1}^{\infty} C_i = M$ it follows that there exists a C_j for some j such that $C_j \subset V$. Therefore, x is not in S_j. This is a contradiction and hence $\bigcap_{i=1}^{\infty} S_i = M$.

Let Q_i be defined as the topological 2-cell which contains $\bigcup_{j=0}^{p-1} T^j(C_i)$ and whose boundary $F(Q_i)$ is contained in the set $\bigcup_{j=0}^{p-1} T^j(F(C_i))$. There exists such a 2-cell by a known theorem. (9)

It is now shown that $\bigcap_{i=1}^{\infty} Q_i = M$. It is obvious that $\bigcap_{i=1}^{\infty} Q_i \supset M$. To show that $\bigcap_{i=1}^{\infty} Q_i \subset M$, suppose that there is a point x in $\bigcap_{i=1}^{\infty} Q_i$ such that x is not in M. It follows that x must be in a bounded component of $E^2 - S_i$ for each i (where E^2 denotes the plane). Let y be in the unbounded component of $E^2 - S_i$. It follows that S_i separates x from y for each i, and from a known theorem x is separated from y by $\bigcap_{i=1}^{\infty} S_i = M$. (9). This is a contradiction and hence it follows that $\bigcap_{i=1}^{\infty} Q_i = M$.

It is now shown that $T(Q_i) \subset Q_i$. From the definition of Q_i it is known that $T(F(Q_i)) \subset Q_i$. Let x be an interior point of Q_i such that T(x) is in Q_i. Assume that $T(Q_i)$ is not contained in Q_i. Then there is a point y in the interior of Q_i such that T(y) is not in the bounded component of $T(F(Q_i))$. Since Q_i is connected and x and y are interior points of Q_i there is an arc $\widehat{xy}$ from x to y contained in the interior of Q_i. It follows that $T(\widehat{xy})$ does not intersect $T(F(Q_i))$. Therefore, T(x) and T(y) are not separated by $T(F(Q_i))$. This contradicts the assumption that T(x) was in the bounded component of the complement of $T(F(Q_i))$ and that T(y) was in the unbounded component. It follows now by the Brouwer fixed point theorem that each Q_i contains a point which is fixed under T, and hence M must contain a point which is fixed under T.

Theorem 9. If T is an isometry of a compact continuum M of the plane into itself and M does not separate the plane, then T leaves a point of M fixed.

Proof. Again consider the different types of isometries in the plane. As noted above, any isometry of a subset of the plane into itself may be extended to an isometry of the whole plane into itself. Observe that neither of the types (3) or (5) transforms a compact set into itself and that (1) leaves every point fixed. Therefore, it is necessary to consider only types (2) and (4). First consider type (4), and let L represent the line about which the reflection occurs. If M intersects the line L then the theorem is true. If M does not intersect L then T(M) is separated from M by L. This is a contradiction of either the hypothesis that T transforms M into M or the hypothesis that M is a continuum. Now consider type (2). If T is a rotation about a point in M then the theorem is true. Suppose that T is a rotation about a point p not in M. It is easy to see that if T is periodic at some point of the plane other than p, then T is periodic at every point of the plane and the period would be the same for each point other than p. It would follow from Theorem 8 that M contains a point which is fixed under T. If T is not periodic on any point of the plane other than p, then by Theorem 5 it follows that for any point x of M the closure of the set of iterates of x is a circle containing p as its center (where the set of iterates of x is $\bigcup_i T^i(x)$). This contradicts the fact that p is not in M and that M does not separate the plane.

CHAPTER III

INTERSECTION PROPERTIES OF PLANE CONTINUA

In 1930 Eduard Helly (10) proved the following theorem: Let there be given in R^n (Euclidean n-space) any collection of cells. If the intersection of each k of these for $k = 1, 2, 3, \ldots, n$ is again a cell, and the intersection of each $n+1$ is not empty, then the intersection of all the cells of the collection is not empty and again a cell.

In 1957 Josef Molnar (11) proved the following generalization for the Euclidean plane: If in the plane an arbitrary number of simply connected, bounded and closed domains are given so that the intersection of every two is connected and the intersection of every three is non-empty, then the intersection of all the domains is not empty.

The writer shows that in the plane to require the intersection of every three be non-empty is equivalent to requiring the union of any three fail to separate the plane. In the remainder of this chapter the space will be the Euclidean plane.

The writer makes use of the following properties of the Euclidean plane:

Proposition A. Let A and B be subcontinua of the plane, neither of which separates the plane. Then $A \cup B$ does not separate the plane if and only if $A \cap B$ is connected. (12).

Proposition B. If C_{xy} is an oriented simple closed curve containing the points x and y ($x \neq y$), and furthermore H and K are compact disjoint sets such that $H \cap \widehat{xy} = \emptyset$ and $K \cap \widehat{yx} = \emptyset$ ($\widehat{ab}$ is to denote the simple arc on C where the orientation from a to b is positive if $a \neq b$), then there is a simple arc from x to y such that it is contained in the interior of C_{xy} except for x and y, and does not intersect $H \cup K$. (9).

The following theorem is the main result of this chapter.

Theorem 1. Let $\{C_\alpha\}$ be any collection of compact continua which do not separate the plane such that the intersection of any two is non-empty and connected. Then the union of any three fails to separate the plane if and only if the intersection of any three is non-empty.

Proof. Let the union of any three fail to separate the plane. Assume the intersection of some three, say C, D, and E, to be empty. Let $A_{cd} = C \cap D$ and in the same manner define the sets A_{ce} and A_{de}. Now consider $(C \cup D) \cap E$. This set must be connected by Proposition A since $C \cup D \cup E$ does not separate the plane. But $(C \cup D) \cap E = A_{ce} \cup A_{de}$ and hence is not connected by the assumption that $C \cap D \cap E = \emptyset$. This is a contradiction. Therefore, $C \cap D \cap E \neq \emptyset$.

Let the intersection of any three be non-empty. Assume that there exist some three sets, say C, D, and E, such that their union separates the plane. It is known that A_{cd}, A_{ce}, and A_{de} are each connected. Since $C \cap D \cap E \neq \emptyset$ it may be concluded that $A_{ce} \cup A_{cd} \cup A_{de}$ is connected. Let x be a point in the unbounded component of $S-(C \cup D \cup E)$ (where S denotes the Euclidean plane) and p be a point in one of the bounded components of $S-(C \cup D \cup E)$. Since $C \cup D$ does not separate the plane (this follows from proposition A), $S-(C \cup D)$ is connected and open. Therefore there exists a simple arc from x to p which intersects only E. Denote this arc by L_e and in a similar manner obtain L_d. Let x_{ed} be the last point on L_e in order from x to p such that x_{ed} is in the intersection of L_e and L_d, and also in the unbounded component of $S-(C \cup D \cup E)$. Let y_{ed} be the first point on L_e in order from x_{ed} to p in the intersection of L_e and L_d, and in a bounded component of $S-(C \cup D \cup E)$. Denote by K_{xy} the simple closed curve consisting of the union of the arc $\widehat{x_{ed}y_{ed}}$ contained in L_e and the arc $\widehat{x_{ed}y_{ed}}$ contained in L_d. The simple closed curve K_{xy} separates the plane into two components one of which contains the continuum C. Without loss of generality, suppose C is in the unbounded component. Let $B(K_{xy})$ denote the bounded component of $S-K_{xy}$. By Proposition B it follows that $E \cap D \cap B(K_{xy}) \neq \emptyset$. Otherwise x would not be separated from p by $C \cup D \cup E$, and hence $A_{ed} \subset B(K_{xy})$ since A_{ed} is connected. Therefore, conclude that A_{ed} and $A_{cd} \cup A_{ce}$ are separated sets and thereby reach a contradiction.

Corollary 1. If $\{C_\alpha\}$ is any collection of compact simply connected sets in the plane such that the intersection of any two is non-empty and connected, and the union of any three fails to separate the plane, then the intersection of the collection is non-empty.

Proof. This follows immediately from Theorem 1 and the result of Molnar.

As a special case of Corollary 1 the following result is obtained.

Corollary 2. If $\{C_\alpha\}$ is any collection of compact convex sets in the plane such that the intersection of any two is non-empty and the union of any three fail to separate the plane, then the intersection of the collection is non-empty.

It is now possible to prove a theorem which is somewhat more general than the theorem of Molnar.

Theorem 2. Let $\{C_\alpha\}$ be any collection of compact simply connected sets in the plane such that the intersection of any two is non-empty and the union of any three fails to separate the plane. Then the intersection of the collection is non-empty.

Proof. Assume the union of some two sets C_1 and C_2 separates the plane. Let C denote the union of the bounded components of $S-(C_1 \cup C_2)$. If C_β be any other set of the collection, then $C \subset C_\beta$. Otherwise $C_1 \cup C_2 \cup C_\beta$ would separate the plane. Note that the boundary of C, denoted by F(C), is contained in $C_1 \cup C_2$. Let D be a component of C. It follows then from the Brouwer Property that F(D) is connected. (12). Since $F(D) = (F(D) \cap C_1) \cup (F(D) \cap C_2)$, it follows that $F(D) \cap (C_1 \cap C_2) \neq \emptyset$. Choose p an element of the set $F(D) \cap (C_1 \cap C_2)$. Now since $F(D) \subset C_\beta$ for every β, it follows that p is an element of their intersection.

In the case that the union of no two sets separates the plane it follows that the intersection of any two sets is connected. The proof of the theorem is then completed by the use of Theorem 1.

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