MEMBRANE ANALYSIS OF CANTILEVERED PARABOLIC CONOIDAL SHELLS

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PREFACE

The membrane analysis of cantilevered parabolic conoidal shells is presented in this thesis. The partial differential equations of equilibrium are first solved as homogeneous equations in terms of both the internal forces and the stress function. Particular and complete solutions are obtained for the force variations due to uniform load, parabolic variation of dead load, and corrections corresponding to appreciable shell rise. Stress curves are plotted, and numerical examples are included showing the application of the theory presented.

I wish to express my indebtedness to Dr. K. S. Havner, not only for introducing to me the possibilities of the application of membrane theory to the analysis of cantilevered parabolic conoidal shells, but also for his valuable assistance and guidance throughout the preparation of this thesis.

To Mrs. Barbara Adams I wish to express my gratitude for her careful typing of the manuscript.

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NOMENCLATURE

$x, y, z,$	Coordinates of a point on the middle surface of the shell.
β, α	The angles between the middle surface of the shell and the projected plane when measured along the X and Y axes, respectively.
c	Rise of shell in Z-direction.
b	Span of shell in Y-direction.
a	Length of shell in X-direction.
$\overline{N}_x, \overline{N}_y, \overline{N}_z$	Normal and shearing forces on an element of the shell.
N_x, N_y, N_{xy}	Projected normal and shearing forces in the X-Y plane.
F	Airy's stress function.
G	$\frac{\partial F}{\partial x}$.
h	Thickness of the shell.
P_z	Intensity of vertical load on projected area.
P_0	Intensity of uniformly distributed load.
P	Weight of shell per unit area.
C	Compressive force in edge member.
A_i, B_i, C_i', K_i . .	Constants
A_s	Area of reinforcing steel.

CHAPTER I

INTRODUCTION

1-1. Historical Sketch.

The formulation of the equilibrium equations for the general shell of double curvature in terms of a stress function and projected forces was first accomplished by Pücher. (1). Extension of the general method to the problem of circular conoidal shells used as retaining walls was made by Torroja. (2). Another solution for circular conoidal shells was developed by Flügge. (3).

Soare developed solutions for conoidal shells of several different types, including circular, catenary, and parabolic conoids. (4). However, his solutions were limited to shells supported by edge members having lateral stiffness in the Y-direction. Candela indicated a possible means of estimating forces in the cantilevered conoidal shell by replacing the surface with a hyperbolic paraboloid. (5). Reissner suggested a solution of the differential equation of the conoidal shell in terms of the stress function and for deep shells developed a power series representation of the dead load in terms of nondimensional shell rise. (6).

1-2. Equilibrium Equations of Thin Shells.

An element from a general shell of double curvature is considered as shown in Figure 1-1. The displacements are assumed to be small and not to affect the equilibrium. Bending resistance is taken to be negligible, the loads being resisted by linear membrane stresses.

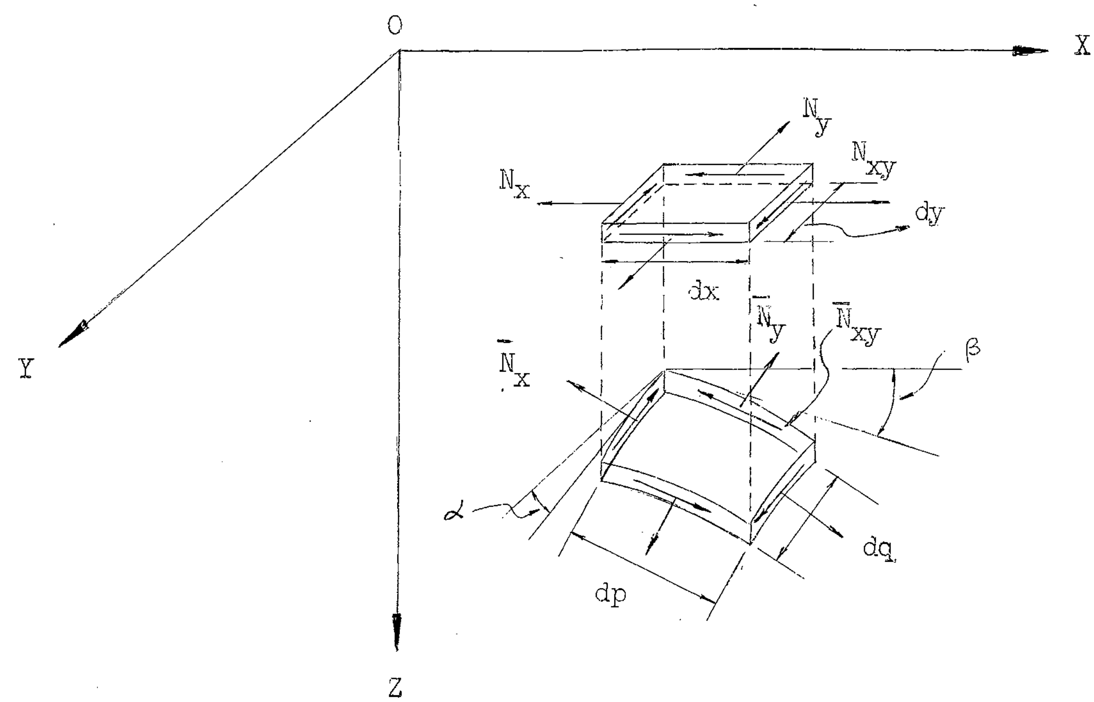

Figure 1-1

Real and Projected Elements of a Shell

The relations between the stresses on the projected element and the stresses on the real element are

$$N_x = \bar{N}_x \frac{\cos\beta}{\cos\alpha} \tag{1-1}$$

$$N_y = \bar{N}_y \frac{\cos\alpha}{\cos\beta} \tag{1-2}$$

$$N_{xy} = \bar{N}_{xy} \tag{1-3}$$

The condition of equilibrium in the X-direction requires that

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} + P_x = 0. \tag{1-4}$$

Similarly, in the Y-direction

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} + P_y = 0. \tag{1-5}$$

The summation of forces in the Z-direction yields

$$N_x \frac{\partial^2 z}{\partial x^2} + 2N_{xy} \frac{\partial^2 z}{\partial x \partial y} + N_y \frac{\partial^2 z}{\partial y^2} = -P_z + P_x \frac{\partial z}{\partial x} + P_y \frac{\partial z}{\partial y} \tag{1-6}$$

where P_x, P_y, and P_z are, respectively, the X, Y, and Z components of the lateral load per unit of projected area.

The solution of the equilibrium equations (1-4, 5, 6) is usually effected by reducing them to one single differential equation. This is accomplished by introducing Airy's stress function F, which is related to the membrane forces through the equations

$$N_x = \frac{\partial^2 F}{\partial y^2} - \int P_x \, dx \tag{1-7}$$

$$N_y = \frac{\partial^2 F}{\partial x^2} - \int P_y \, dy \tag{1-8}$$

$$N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}. \tag{1-9}$$

Introduction of Equation (1-7), Equation (1-8), and Equation (1-9) into Equation (1-6) gives a differential equation for F of the following form

$$\frac{\partial^2 F}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 z}{\partial x^2} = -P_z + P_x \frac{\partial z}{\partial x}$$

$$+ P_y \frac{\partial z}{\partial y} + \left(\int P_x \, dx \right) \frac{\partial^2 z}{\partial x^2} + \left(\int P_y \, dy \right) \frac{\partial^2 z}{\partial y^2} . \tag{1-10}$$

This is the well known equilibrium equation for shells of double curvature. Its solution for conoidal shells under various types of loading and with different edge conditions is shown in the following chapters.

1-3. Parabolic Conoidal Shell.

A conoidal surface is generated by the translation of a rectilinear generatrix, one end of which travels along a curve and the other end along a straight line. In the special case where the curve is a parabola, the Parabolic Conoid is developed.

There are two basic types of parabolic conoidal shells, as shown in Figure 1-2. The equations of the middle surfaces are

Type I: $$z = -\frac{c}{a}\,x\left(1 - \frac{y^2}{b^2}\right) \qquad (1\text{-}11)$$

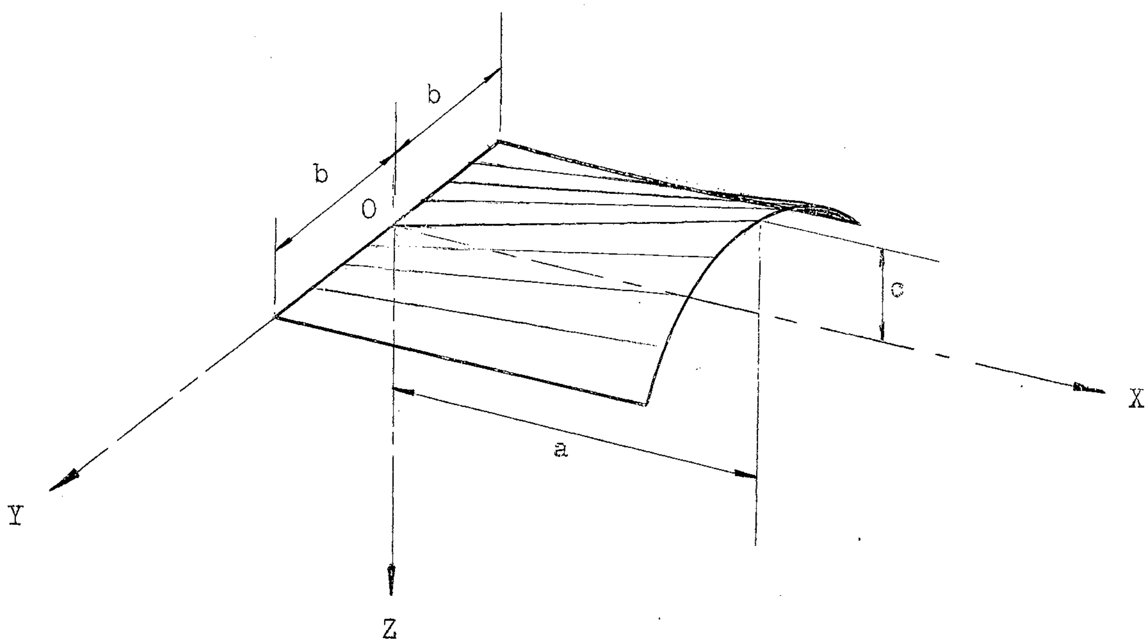

Figure 1-2a

Parabolic Conoid Shell Type I

Type II: $$z = \frac{c}{a}\,x\,\frac{y^2}{b^2} \qquad (1\text{-}12)$$

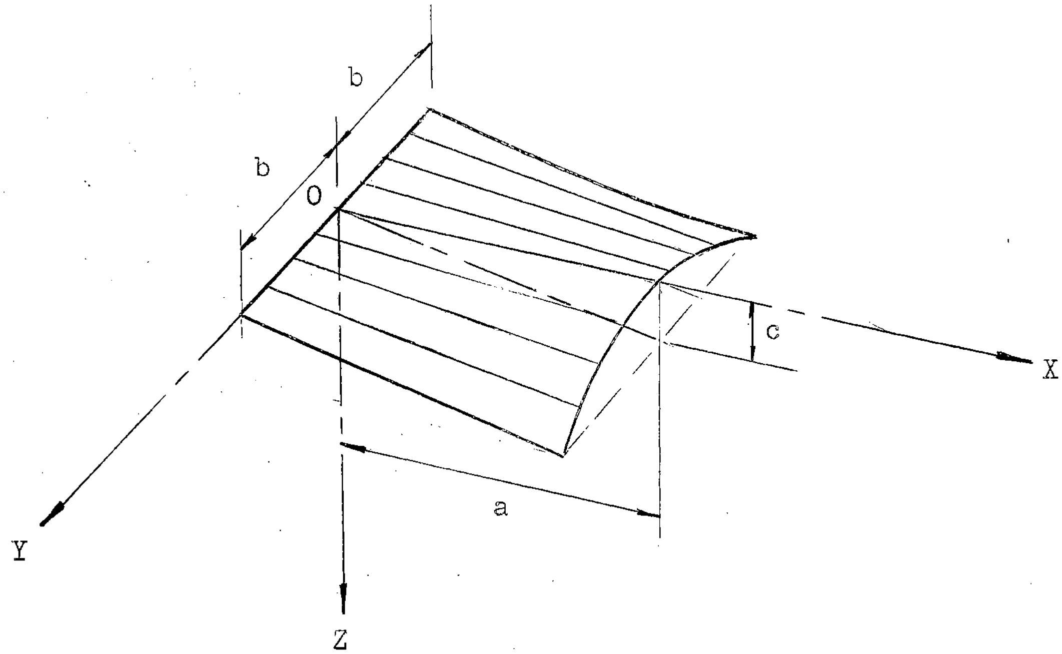

Figure 1-2b

Parabolic Conoid Shell Type II

Considering first Equation (1-11), the various derivatives are

$$\frac{\partial z}{\partial x} = -\frac{c}{a}\left(1 - \frac{y^2}{b^2}\right) \qquad \frac{\partial z}{\partial y} = \frac{2c}{ab^2}\,xy$$

$$\frac{\partial^2 z}{\partial x^2} = 0 \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{2c}{ab^2}\,y \qquad \frac{\partial^2 z}{\partial y^2} = \frac{2c}{ab^2}\,x \,. \tag{1-13}$$

From Equation (1-12) it is easily seen that the expressions for $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, and $\frac{\partial^2 z}{\partial x \partial y}$ are the same as shown above. Therefore, no matter which form of the parabolic conoidal shell is chosen, the same form of the partial differential equation of equilibrium is obtained.

Substituting Equations (1-13) into Equation (1-10), and considering vertical loads only, the final equation becomes

$$- 2y \frac{\partial^2 F}{\partial x \partial y} + x \frac{\partial^2 F}{\partial x^2} + \frac{ab^2}{2c} P_z = 0. \qquad (1\text{-}14)$$

Equation (1-14) is the governing differential equation for the parabolic conoidal shell.

CHAPTER II

SOLUTION OF THE HOMOGENEOUS EQUATIONS

2-1. Alternate Forms of the Equilibrium Equations.

From Equation's (1-4) through (1-6) and Equation (1-14), the equilibrium equations of the vertically loaded parabolic conoid in terms of the projected membrane forces are

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0 \tag{2-1}$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \tag{2-2}$$

$$2y\ N_{xy} + x\ N_y + \frac{ab^2}{2c}\ P_z = 0. \tag{2-3}$$

Differentiating Equation (2-3) with respect to X:

$$2y \frac{\partial N_{xy}}{\partial x} + x \frac{\partial N_y}{\partial x} + N_y = -\frac{ab^2}{2c} \frac{\partial P_z}{\partial x}.$$

Alternately, differentiating with respect to Y:

$$2N_{xy} + 2y \frac{\partial N_{xy}}{\partial y} + x \frac{\partial N_y}{\partial y} = -\frac{ab^2}{2c} \frac{\partial P_z}{\partial y} .$$

From the equilibrium Equation's (2-1) and (2-2)

$$\frac{\partial N_x}{\partial x} = -\frac{\partial N_{yx}}{\partial y} \qquad \Bigg| \qquad \frac{\partial N_y}{\partial y} = -\frac{\partial N_{xy}}{\partial x} .$$

Therefore, alternate forms of the final partial differential equations are:

$$x \frac{\partial N_y}{\partial x} - 2y \frac{\partial N_y}{\partial y} + N_y = -\frac{ab^2}{2c} \frac{\partial P_z}{\partial x} \qquad (2\text{-}4)$$

$$-x \frac{\partial N_{xy}}{\partial x} + 2y \frac{\partial N_{xy}}{\partial y} + 2N_{xy} = -\frac{ab^2}{2c} \frac{\partial P_z}{\partial y} \qquad (2\text{-}5)$$

$$x \frac{\partial G}{\partial x} - 2y \frac{\partial G}{\partial y} = -\frac{ab^2}{2c} P_z \qquad (2\text{-}6)$$

where

$$G = \frac{\partial F}{\partial x} \quad .$$

Before attempting to solve the partial differential equations of the conoidal shell for specific loads and specific boundary conditions, solutions of the homogeneous equations will be developed. In the following articles, the homogeneous form of each of Equation's (2-4, 5, 6) is investigated and the possible stress surfaces are determined. For the sake of completeness, each equation is solved, although any one is adequate to describe the theoretical distribution of stresses in the shell.

2-2. Solution of Homogeneous Equation in N_y.

The homogeneous equation in terms of N_y is:

$$x \frac{\partial N_y}{\partial x} - 2y \frac{\partial N_y}{\partial y} + N_y = 0. \qquad (2\text{-}7)$$

From the theory of linear, first order partial differential equations, the integral curves which generate the general solution can be determined from the ordinary differential equations

$$\frac{dx}{x} = \frac{dy}{-2y} = \frac{dN_y}{-N_y} \; .$$

Solving these equations:

$$\frac{dx}{x} = \frac{dN_y}{-N_y} \qquad \Bigg| \qquad \frac{dy}{-2_y} = \frac{dN_y}{-N_y}$$

$$xN_y = C_1 \qquad \Bigg| \qquad \frac{N_y}{y^{\frac{1}{2}}} = C_2 \quad .$$

Thus

$$f_1 \left(xN_y , \frac{N_y}{y^{\frac{1}{2}}}\right) = 0$$

is a solution of Equation (2-7).

The admissable forms of the function f_1 can be found by a trial method. One of the admissable solutions is:

$$N_y = C_1\, x^{-1} + C_2\, y^{\frac{1}{2}} \quad .$$

The stress surfaces corresponding to the terms in this expression are shown in Figure 2-1. The variation in the X-direction is hyperbolic, the variation in the Y-direction is parabolic.

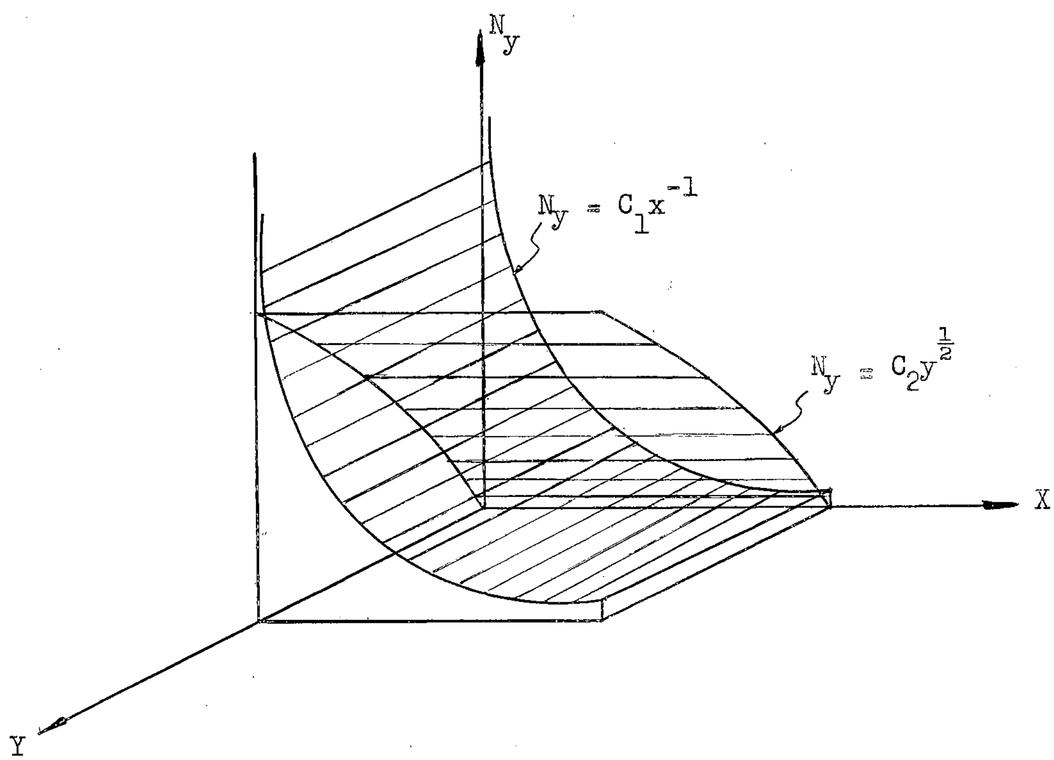

Figure 2-1

Stress Surfaces for N_y

Another admissable solution is

$$N_y = Cxy \quad .$$

This is also a hyperbolic surface, as shown in Figure 2-2.

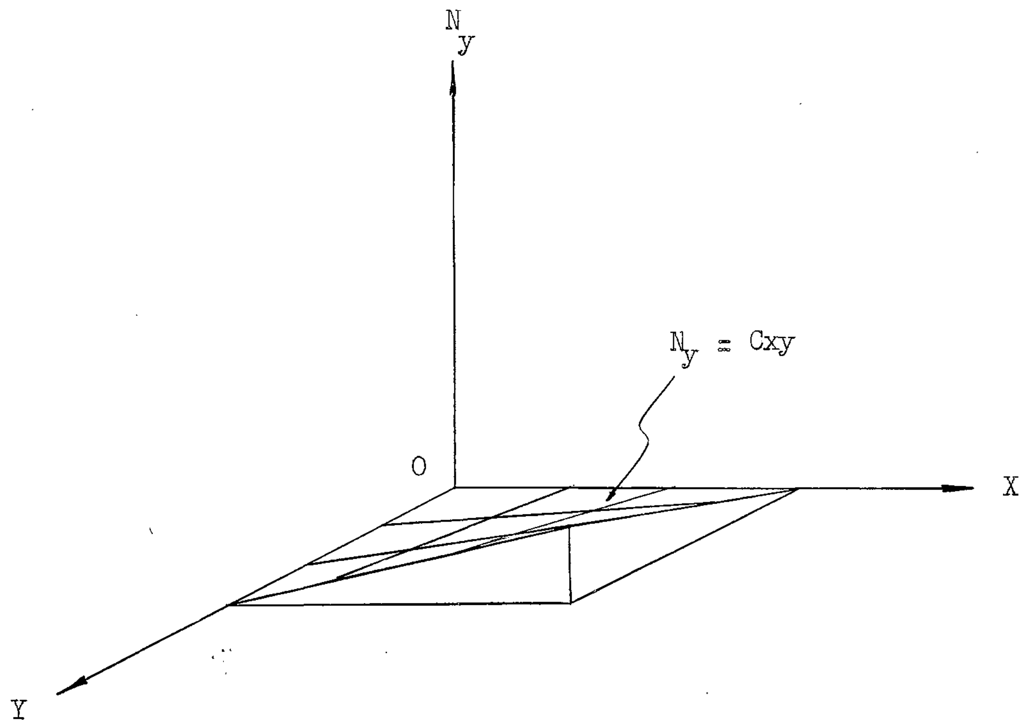

Figure 2-2

Stress Surface for N_y

Observing these admissable solutions, it is evident that a general solution may be achieved by assuming a stress surface of a higher order hyperbolic paraboloidal type with curved generatrices.

Thus let

$$N_y = C x^{K_1} y^{K_2} \quad .$$

Substituting into Equation (2-7), we have

$$x\ (C K_1\ x^{K_1 - 1}\ y^{K_2})\ - 2y\ (C K_2\ x^{K_1}\ y^{K_2 - 1})\ + C x^{K_1}\ y^{K_2} = 0\ .$$

Therefore

$$K_1 - 2K_2 + 1 = 0$$

$$K_1 = 2K_2 - 1$$

and the general equation for N_y becomes

$$N_y = Cx^{2K-1}\, y^K \quad . \tag{2-8}$$

It is seen that the first three stress surfaces are special forms of the above equation. The value of K will depend upon the type of loading and boundary conditions.

2-3. Solution of Homogeneous Equation in N_{xy}.

The homogeneous equation in terms of N_{xy} is:

$$-x \frac{\partial N_{xy}}{\partial x} + 2y \frac{\partial N_{xy}}{\partial y} + 2N_{xy} = 0 \quad . \tag{2-9}$$

The corresponding differential equations of the integral curves are:

$$\frac{dx}{-x} = \frac{dy}{2y} = \frac{dN_{xy}}{-2N_{xy}} \quad .$$

The solutions of these equations are obtained as before:

$$\frac{dx}{-x} = \frac{dN_{xy}}{-2N_{xy}} \quad \Bigg| \quad \frac{dy}{y} = \frac{dN_{xy}}{-2N_{xy}} \quad .$$

$$N_{xy}\, x^{-2} = C_1 \quad \Bigg| \quad N_{xy}\, y = C_2 \quad .$$

Thus

$$f_2\,(x^{-2} N_{xy}, y\, N_{xy}) = 0$$

is a solution of Equation (2-9).

One admissable solution of Equation (2-9) is

$$N_{xy} = C_3\, x^2 + C_4\, y^{-1}$$

with the stress surfaces shown in Figure 2-3.

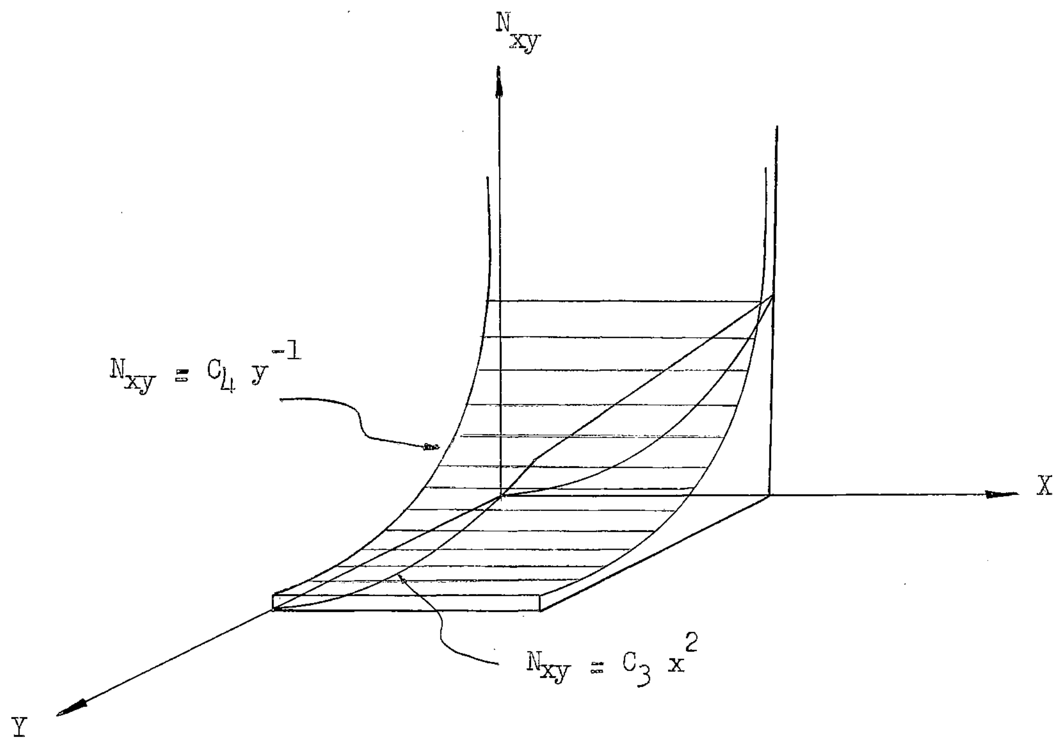

Figure 2-3

Stress Surfaces for N_{xy}

As before, the general solution can be assumed in the form

$$N_{xy} = C_1\, x^{K_1}\, y^{K_2} \quad .$$

Substituting into the Equation (2-9), we get

$$-x\left[C_1\, K_1\, x^{K_1-1}\, y^{K_2}\right] + 2y\left[C_1\, K_2\, x^{K_1}\, y^{K_2-1}\right] + 2C_1\, x^{K_1}\, y^{K_2} = 0 \ .$$

Therefore,

$$-K_1 + 2K_2 + 2 = 0$$

$$K_1 = 2(K_2 + 1)$$

and the general solution is

$$N_{xy} = C_1\, x^{2(K + 1)}\, y^K \quad . \tag{2-10}$$

2-4. Solution of Stress Function F.

The solution may also be determined from the stress function F. Since the equilibrium equation is of the first order in terms of the quantity

$$G = \frac{\partial F}{\partial x}$$

the same methods of solving the partial differential equation apply.

The homogeneous equation in terms of G is

$$x \frac{\partial G}{\partial x} - 2y \frac{\partial G}{\partial y} = 0 \tag{2-11}$$

The integral curves are obtained from the ordinary differential equations

$$\frac{dx}{x} = \frac{dy}{-2y} = \frac{dG}{0} \quad .$$

Proceeding as before the solutions for the integral curves are:

$$G = C_1 \qquad \Big| \qquad xy^{\frac{1}{2}} = C_2 \quad .$$

Therefore,

$$f_3\,(G,\ xy^{\frac{1}{2}}) = 0$$

is a solution of Equation (2-11), or

$$G = f_4\,(xy^{\frac{1}{2}}) \quad .$$

By the trial method, we can easily find the general forms of admissable solutions. Thus

$$G = C'\, x^{2K}\, y^K$$

or

$$G = C'' \ln (x^{2K} y^{K}) \quad .$$

The stress function is given by the equation

$$F = \int G \, dx + f(y) \quad .$$

The alternate forms of this equation become, upon integration:

$$F = C_1' x^{2K+1} y^{K} + f(y) \tag{2-12}$$

and

$$F = C_2' x \quad \ln (x^{2K} y^{K}) - 2K \quad + g(y) \tag{2-13}$$

Therefore, the general solution of stresses in the shell will be:

$$F = C_1' x^{2K+1} y^{K} + f(y)$$

$$\left.\begin{aligned} N_x &= C_1' K (K-1) x^{2K+1} y^{K-2} + f''(y) \\ N_y &= C_1' (2K+1)(2K) x^{2K-1} y^{K} \\ N_{xy} &= -C_1' (2K+1) K x^{2K} y^{K-1} \quad . \end{aligned}\right\} \tag{2-14}$$

The special solution of stress will be:

$$F = C_2' x \left[\ln (x^{2K} y^{K}) - 2K \right] + g(y)$$

$$\left.\begin{aligned} N_x &= -C_2' K \frac{x}{y^2} + g''(y) \\ N_y &= 2K C_2' \frac{1}{x} \\ N_{xy} &= -C_2' K \frac{1}{y} \quad . \end{aligned}\right\} \tag{2-15}$$

CHAPTER III

SPECIAL LOAD CONDITIONS

3-1. Uniform Load.

Considering a uniformly distributed live load of intensity $P_z = P_0$ to be acting on the parabolic conoidal shell, the governing differential equation in terms of the stress function becomes

$$x \frac{\partial G}{\partial x} - 2y \frac{\partial G}{\partial y} = - \frac{ab^2}{2c} P_0 \tag{3-1}$$

where as before

$$G = \frac{\partial F}{\partial x} \quad .$$

The solution of the homogeneous equation was obtained in the previous chapter. The particular solution may be determined by taking G to be either a logarithmic function of X or a logarithmic function of Y.

Thus, in the first case

$$G = B_1 \ln x \quad .$$

Substituting into Equation (3-1), we have

$$B_1 = - \frac{ab^2}{2c} P_0 \quad .$$

Therefore,

$$G_1 = - \frac{ab^2}{2c} P_0 \ln x$$

is a particular solution.

Similarly,

$$G = B_2 \ln y$$

and the constant B_2 is determined as

$$B_2 = \frac{ab^2}{4c} P_0 \ .$$

Therefore,

$$G_2 = \frac{ab^2}{4c} P_0 \ln y$$

is also a particular solution.

The alternate forms of the stress function can be obtained by integration. Thus

$$G_1 = -\frac{ab^2}{2c} P_0 \ln x \qquad\qquad G_2 = \frac{ab^2}{4c} P_0 \ln y$$

$$F = -\frac{ab^2}{2c} P_0 \, x \, (\ln x - 1) \qquad\qquad F_2 = \frac{ab^2}{4c} P_0 \, x \ln y \ .$$

The corresponding equations for the normal and shearing forces are:

$$F_1 = -\frac{ab^2}{2c} P_0 \, x \, (\ln x - 1)$$

$$\left.\begin{aligned} N_x &= 0 \\ N_y &= -\frac{ab^2}{2c} P_0 \frac{1}{x} \\ N_{xy} &= 0 \end{aligned}\right\} \qquad (3\text{-}2)$$

and

$$F_2 = \frac{ab^2}{4c} P_0 \, x \ln y$$

$$\left.\begin{aligned} N_x &= -\frac{ab^2}{4c} P_0 \frac{x}{y^2} \\ N_y &= 0 \\ N_{xy} &= -\frac{ab^2}{4c} P_0 \frac{1}{y} \ . \end{aligned}\right\} \qquad (3\text{-}3)$$

These solutions for the stresses in the shell satisfy different boundary conditions which will now be discussed.

In the case of a conoidal shell supported as shown in Figure 3-1, the approximate stress distribution is given by Equation (3-2). (4).

The lateral rigidity of the edge members is assumed to be developed by sufficient buttresses, lateral ties, or prestressing, and the entire load is carried by arching action in the Y-direction.

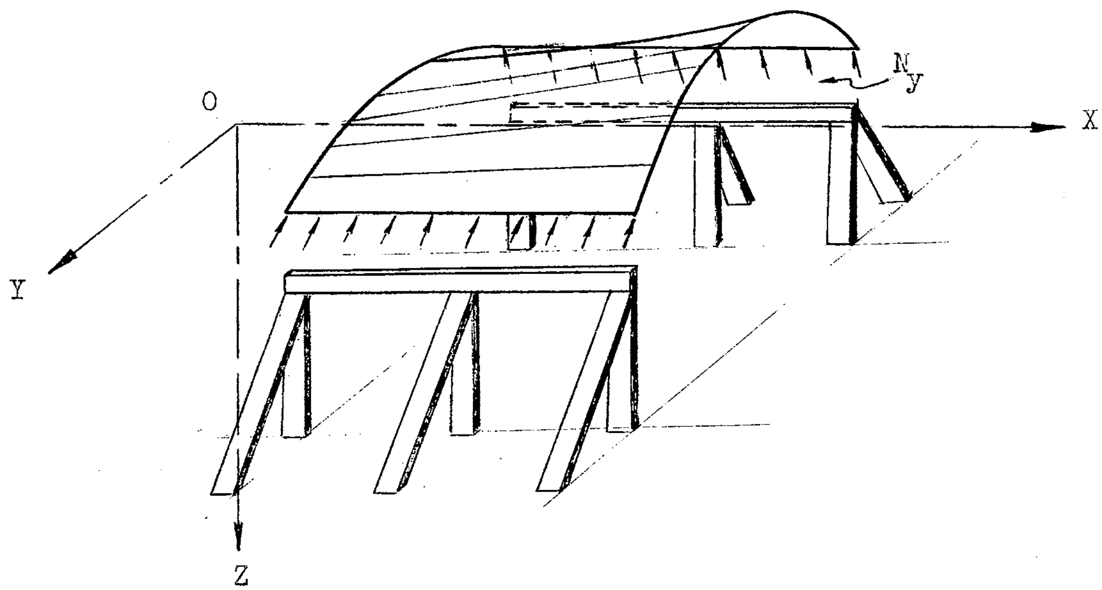

Figure 3-1

Conoidal Arch Shell Simply Supported

The force boundary conditions can be prescribed as

$$x = 0 \ , \ a \ ; \quad N_x = 0 \ , \ N_{xy} = 0$$

which are evidently consistent with expressions for stresses in Equation (3-2).

In the case of a uniformly loaded cantilever conoidal shell (Figure 3-2), however, the load cannot be carried by arch action as the edge members are assumed to have negligible lateral stiffness.

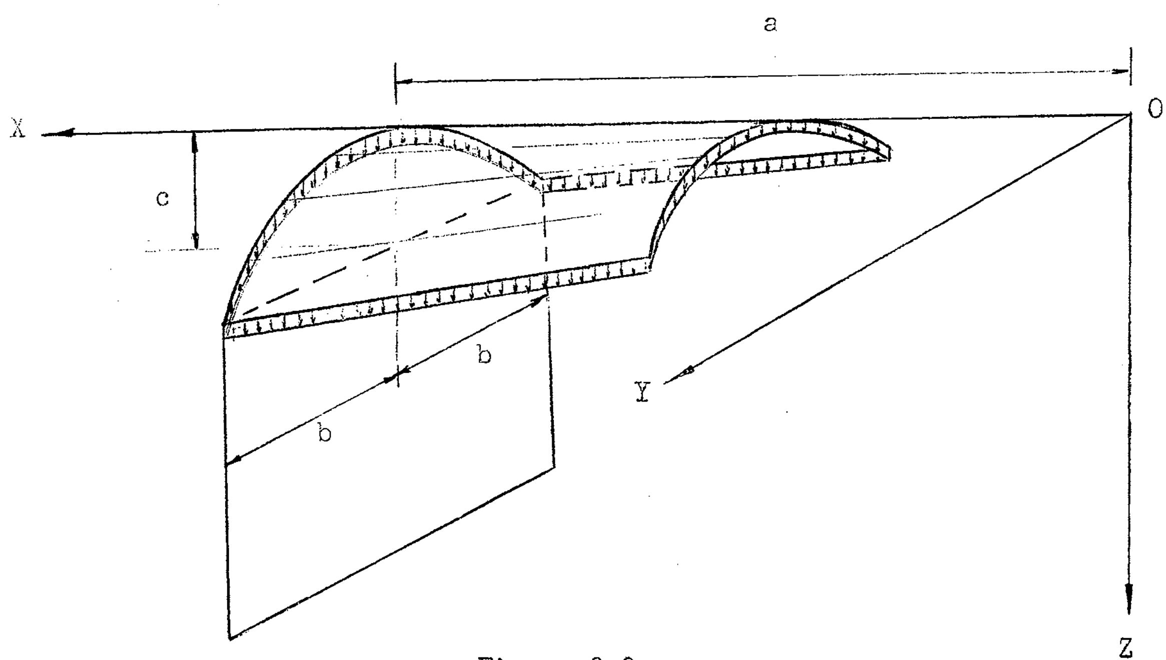

Figure 3-2

Uniformly Loaded Cantilevered Conoidal Shell

The force boundary conditions are

$$x = 0 \ , \quad N_x = 0$$

$$y = \pm b \ , \quad N_y = 0$$

and the stress distribution on the boundary is as shown in Figure 3-3.

The ribs supporting the arches must be designed to carry the tangential shear load imparted to them by the shell, and the vertical load must be carried by the components of the shearing and normal stresses.

Equation's (3-3) satisfy the boundary conditions for the cantilever shell, the load being carried by shearing and longitudinal forces. An investigation of the equations, however, indicates that they are not

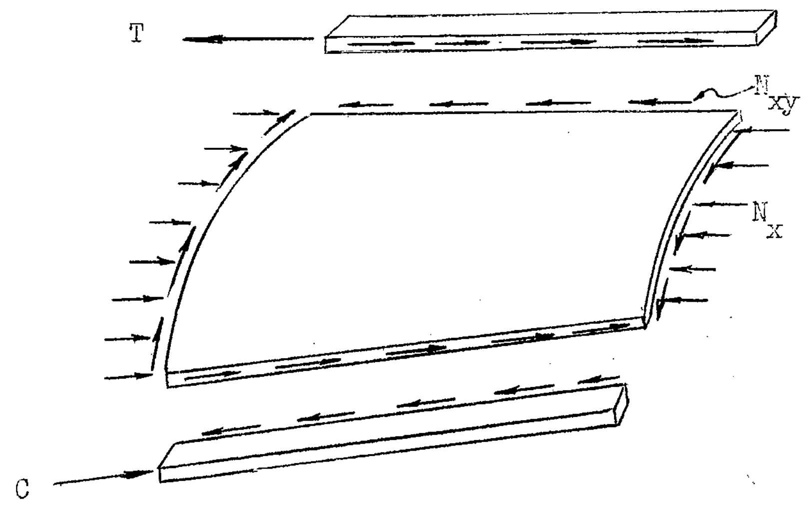

Figure 3-3

Edge Forces on the Cantilever Shell

valid near the crown of the conoidal shell where N_x and N_{xy} become increasingly large, approaching infinity as y approaches zero. That the equations yield infinite values along this line is dependent both upon the mathematical nature of the assumed membrane stresses under uniform load and upon the physical action of the membrane state.

Considering first the mathematical viewpoint, a study of the homogeneous and particular solutions for N_y shows that, for N_y to be zero along the longitudinal edges of the shell, it must be zero throughout the shell. Thus, no advantage can be taken of the curvature of the surface in the Y-direction, and the load must be carried by vertical components of shearing and normal stresses in the X-direction. Along the top of the cantilevered shell, however, the undeformed surface is flat in the X-direction and the N_x and N_{xy} forces lie in a horizontal plane. (Figure 3-4).

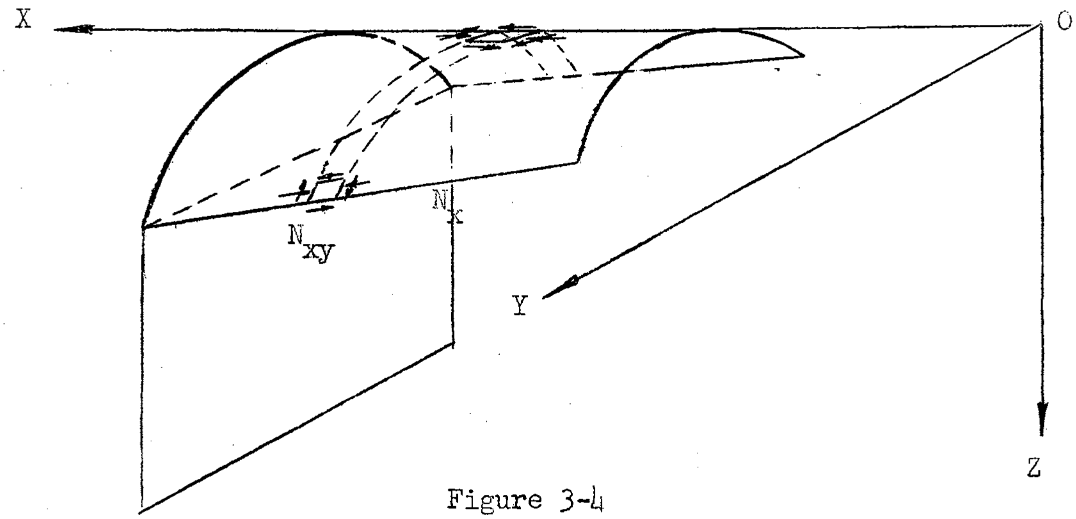

Figure 3-4

Stress Diagrams

Thus, if there are no compressive arch stresses in the Y-direction (as required mathematically) it is physically impossible for the uniform vertical load to be carried by linear membrane forces. Bending will take place in the central region of the shell (this portion acting similarly to a cantilever beam) and the load will be carried by transverse shearing forces. The membrane theory breaks down for the conoidal shell, therefore, even before deformation incompatibilities at the boundaries are considered. Nevertheless, Equation (3-3) may well yield an acceptable approximation for the true state of stress over a considerable portion of the surface of the shell. To determine the extent of this possible region of membrane action, and to determine the upper limit on the magnitude of the membrane forces near the crown of the shell, a linear bending theory would be necessary, which is beyond the scope of this thesis.

Some idea of the variation of the theoretical stresses can be obtained from Figure 3-5, where the dimensionless ratios $\frac{N_{xy}}{(-\frac{ab}{2c}P_0)}$ and $\frac{y}{b}$ are plotted against one another.

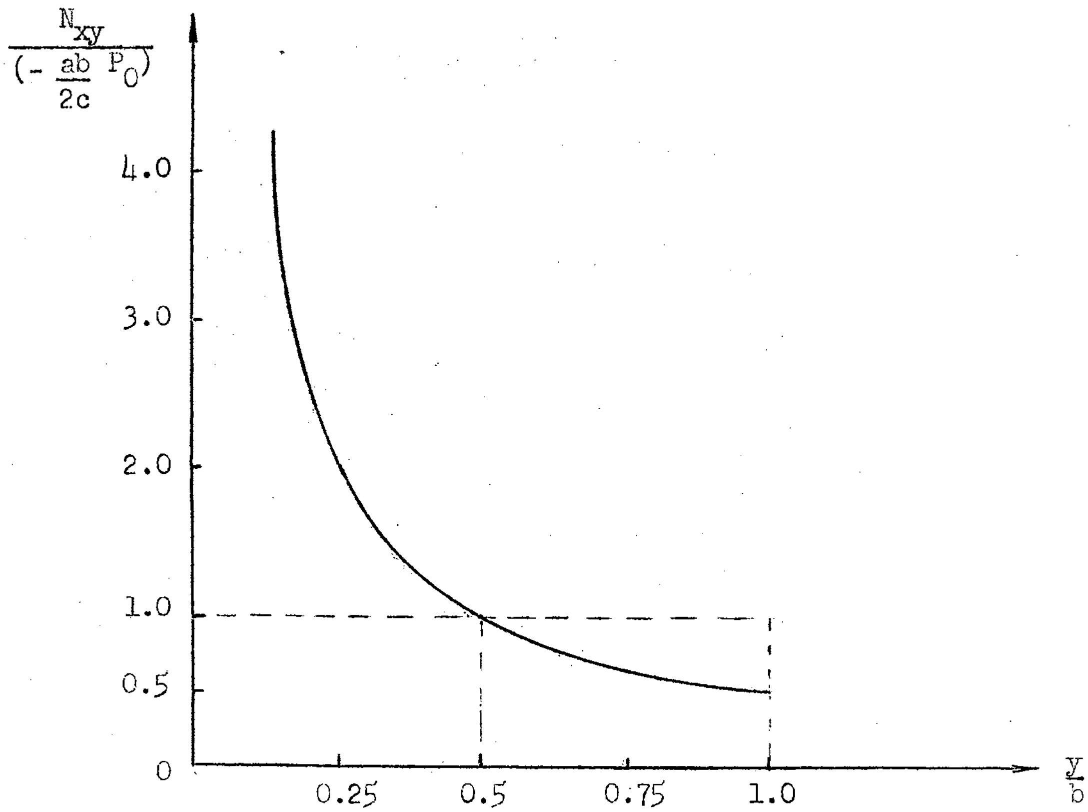

Figure 3-5

Shearing Force Variation

The quantity $-\frac{ab}{2c}P_0$ is the shearing force in a hyperbolic paraboloid having the same dimensions as the parabolic conoidal shell. It is evident from the figure that the membrane force in the conoidal shell is less in the outer one quarter of the transverse span 2b, greater in the middle one-half. Use will be made of Figure 3-5 in Chapter IV to estimate the forces in the edge members and in the shell.

3-2. Parabolic Loading.

A conoidal shell will in general be constructed with increasing thickness near the fixed end (Figure 3-6), thus requiring stresses due to an arbitrary variation to be superimposed upon the uniform load stresses. To determine these stresses, a parabolic variation of thickness is assumed because of its mathematical simplicity, and a shallow shell is considered (i.e. the force per unit of projected area and the weight per unit area of the shell can be taken as equal).

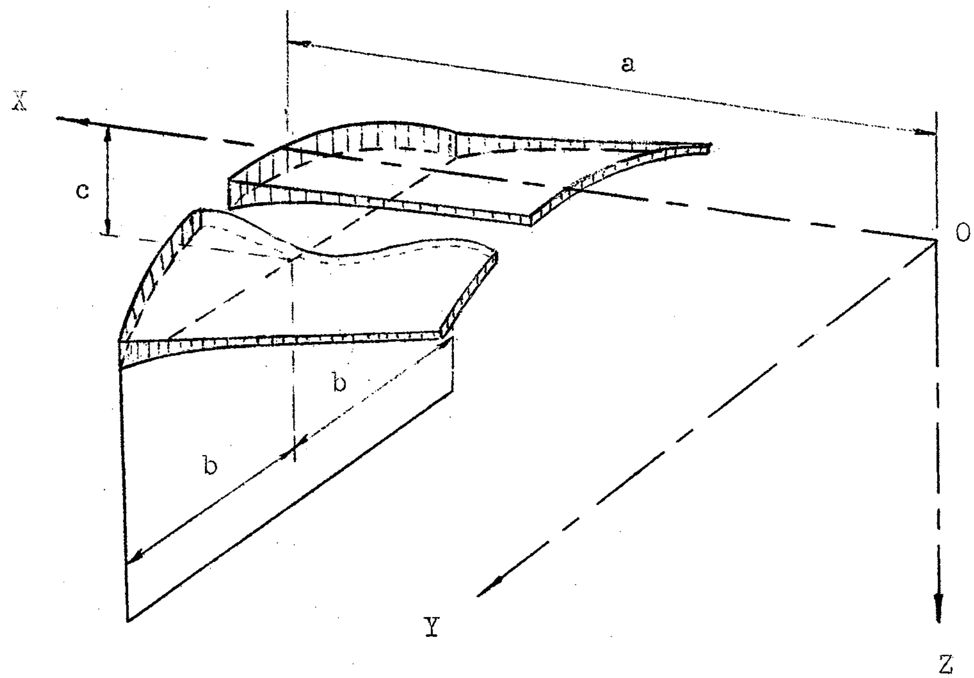

Figure 3-6

Conoidal Shell with Variable Thickness

The intensity of the additional load thus becomes

$$P_z = \frac{Ax^2}{2} \quad .$$

Where A is an arbitrary constant to be determined from the given variation of thickness.

Substituting this load condition into Equation (2-4)

$$x \frac{\partial N_y}{\partial x} - 2y \frac{\partial N_y}{\partial y} + N_y = - \frac{ab^2}{2c} Ax \ .$$

A particular solution can be obtained by taking $N_y = Bx$. Substituting, the constant B is

$$B = - \frac{ab^2}{4c} A$$

and

$$N_y = - \frac{ab^2}{4c} Ax \quad .$$

From Equation (2-2) the complete solution will be

$$N_y = C_1 \, x^{2K - 1} \, y^K - \frac{ab^2}{4c} Ax \tag{3-4}$$

Introducing the boundary condition $y = b$, $N_y = 0$, the constants C_1 and K can be determined. Thus

$$N_y = C_1 \, x^{2K - 1} \, b^K - \frac{ab^2}{4c} Ax = 0 \quad .$$

In order that this expression vanish for all values of x, let $K = 1$, from which

$$C_1 = \frac{ab}{4c} A \quad .$$

The final equation for the normal force N_y is

$$N_y = - \frac{ab}{4c} Ax \, (b - y) \quad . \tag{3-5}$$

The shearing force is obtained by integrating Equation (3-5):

$$N_{xy} = - \int \frac{\partial N_y}{\partial y} dx + f(y) \quad .$$

The function f (y) can be taken as zero and the final equation becomes

$$N_{xy} = - \frac{ab}{4c} \frac{Ax^2}{2} \quad . \tag{3-6}$$

Similarly,

$$N_x = - \int \frac{\partial N_{xy}}{\partial y} dx + g(y) \quad .$$

The function g (y) also can be taken as zero, from which

$$N_x = 0 \quad . \tag{3-7}$$

The solution of stresses given in Equation's (3-5), (3-6), and (3-7) gives very satisfactory results which are finite at all points of the shell.

3-3. Series Expansion.

In previous articles it has been assumed that the weight per unit area P is the same as the vertical intensity of load on the projected area. This is only approximately true. The actual intensity P_z is given by the equation

$$P_z = P \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \quad .$$

Substituting the values for the derivatives:

$$P_z = P \sqrt{1 + \left(\frac{c}{a} \frac{y^2}{b^2}\right)^2 + \left(\frac{2c}{ab^2} xy\right)^2} \quad .$$

This equation can be expanded in a power series as shown by Reissner. (6).

Thus

$$P_z = P\left\{1 + \frac{1}{2}\left[\left(\frac{c}{a}\frac{y^2}{b^2}\right)^2 + \left(\frac{2c}{ab^2}\,xy\right)^2\right] - \frac{1}{8}\left[\left(\frac{c}{a}\frac{y^2}{b^2}\right)^2 + \left(\frac{2c}{ab^2}\,xy\right)^2\right]^2 + \ldots\right\} . \tag{3-8}$$

In the case of a shell that is relatively shallow ($\frac{c}{a}, \frac{c}{b} \leq \frac{1}{8}$), the quantities $(\frac{c}{a})^2$ and $(\frac{c}{b})^2$ are sufficiently small to permit all but the first term on the right side of Equation (3-8) to be disregarded. With c so restricted, the first neglected term is less than 4 per cent of what is retained. All other terms are less significant still, and P_z may be taken equal to P in all calculations.

In the case of a deeper shell ($\frac{1}{8} < \frac{c}{a}, \frac{c}{b} < 1$), the third and all succeeding terms can still be neglected but the second term may prove significant. Considering a shell of constant thickness, the equilibrium equation (Equation 2-12) can be written

$$x\frac{\partial G}{\partial x} - 2y\frac{\partial G}{\partial y} = -\frac{ab^2}{2c}P\left\{1 + \frac{1}{2}\left[\left(\frac{c}{a}\frac{y^2}{b^2}\right)^2 + \left(\frac{2c}{ab^2}\,xy\right)^2\right]\right\} . \tag{3-9}$$

Denoting

$$A_1 = -\frac{ab^2}{2c}P \qquad A_2 = -\frac{c}{4ab^2}P \qquad A_3 = -\frac{c}{ab^2}P$$

Equation (3-9) becomes

$$x\frac{\partial G}{\partial x} - 2y\frac{\partial G}{\partial y} = A_1 + A_2\,y^4 + A_3\,x^2\,y^2 \quad . \tag{3-10}$$

The particular solution corresponding to the constant term A_1 was obtained in Article 3-1. The general solutions corresponding to the second and third terms are determined as follows.

The particular solutions of Equation (3-10) corresponding to the A_2 and A_3 load terms are:

$$G_2 = -\frac{A_2}{8}y^4 \qquad\qquad G_3 = -\frac{A_3}{2}x^2\,y^2 \quad .$$

Hence the stress functions are

$$F_2 = -\frac{A_2}{8} xy^4 \qquad\qquad F_3 = -\frac{A_3}{6} x^3 y^2$$

and the particular solutions for stresses become

$$N_x = -\frac{3A_2}{2} xy^2 - \frac{A_3}{3} x^3$$

$$N_y = -A_3 xy^2$$

$$N_{xy} = +\frac{A_2}{2} y^3 + A_3 x^2 y \quad .$$

From Equations (2-14), the complete solutions are

$$N_x = C_1' K (K-1) x^{2K+1} y^{K-2} + f'' (y) + \frac{3c}{8ab^2} Pxy^2 + \frac{c}{3ab^2} Px^3 \quad .$$

$$N_y = C_1' (2K+1)(2K) x^{2K-1} y^K + \frac{c}{ab^2} Pxy^2 \quad . \tag{3-11}$$

$$N_{xy} = -C_1' (2K+1) K x^{2K} y^{K-1} - \frac{c}{8ab^2} Py^3 - \frac{c}{ab^2} Px^2 y \quad .$$

For the cantilever shell with boundary conditions

$$x = 0 \; : \; N_x = 0 \quad , \quad y = b \; : \; N_y = 0$$

The values of C_1', K, and the function f" (y) can be determined. Thus

$$K = 1 \qquad\qquad C_1' = -\frac{c}{6ab} P \qquad\qquad f''(y) = 0$$

and the final equations become

$$N_x = \frac{3c}{8ab^2} Pxy^2 + \frac{c}{3ab^2} Px^3$$

$$N_y = -\frac{c}{ab} P\, xy \left[1 - \frac{y}{b} \right] \tag{3-12}$$

$$N_{xy} = \frac{c}{2ab} Px^2 - \frac{c}{8ab^2} Py^3 - \frac{c}{ab^2} Px^2\, y \quad .$$

Equation's (3-12) are the forces in the deep shell due to load correction.

CHAPTER IV

NUMERICAL EXAMPLES

4-1. General Notes.

Two examples are shown in the following articles.

In the first example, the variation of stresses in a cantilevered shell under uniform load is described and the theoretical forces in the edge regions are determined.

In the second example, stresses due to dead load are determined, using a parabolic variation of thickness. This results in a more satisfactory distribution of stresses which are finite at every point.

Units for various values are in terms of pounds, feet, or pounds per foot.

4-2. Example 1.

A cantilevered conoidal shell acted upon by a uniformly distributed load of intensity 41 psf. is considered. (Figure 4-1).

The dimensions of the shell are 20 x 20 feet in plan and the thickness is $3\frac{1}{4}$ inches.

The projected forces in the shell can be obtained from Equations (3-3).

$$N_x = - \frac{a^2 p}{4c} \frac{(\frac{x}{a})}{(\frac{y}{b})^2} = - \frac{20^2 \times 41}{4 \times 1.96} \frac{(\frac{x}{a})}{(\frac{y}{b})^2}$$

$$= - 2100 \frac{(\frac{x}{a})}{(\frac{y}{b})^2} \quad \text{pounds per foot.}$$

$$N_y = 0$$

$$N_{xy} = -\frac{ab}{4c} P \frac{1}{(\frac{y}{b})} = -\frac{20 \times 10}{4 \times 1.96} \times 41 \frac{1}{(\frac{y}{b})}$$

$$= -1020 \frac{1}{(\frac{y}{b})} \quad \text{pounds per foot.}$$

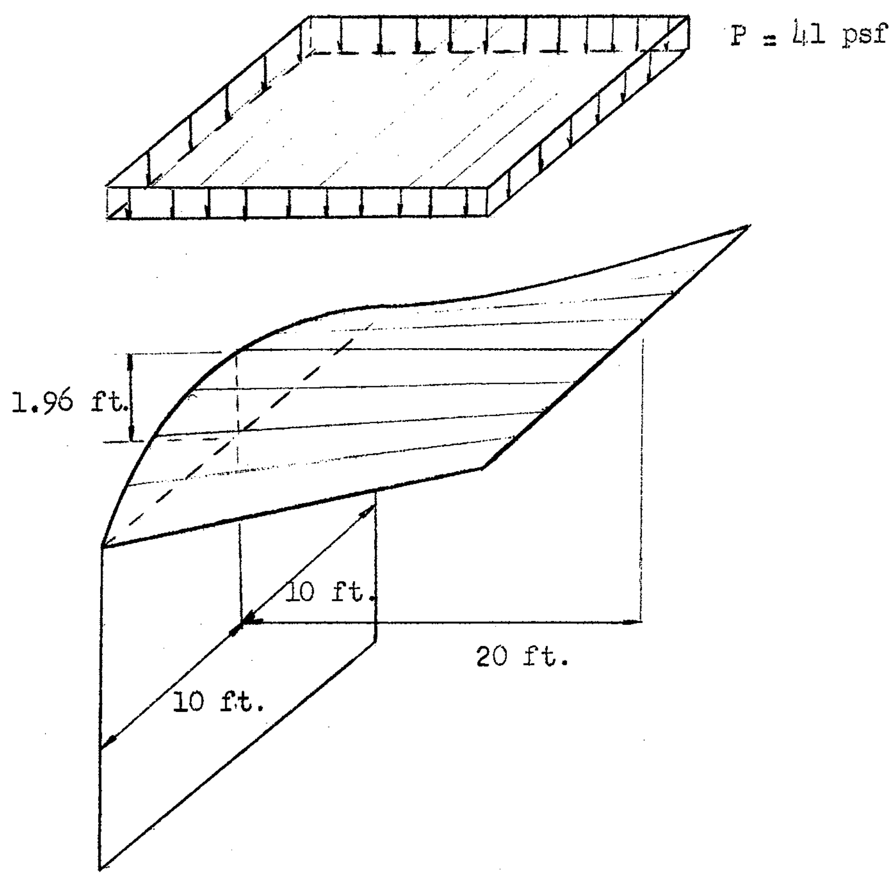

Figure 4-1

Uniformly Loaded Cantilever Shell

The forces in the shell are evaluated in Table I.

TABLE I

INTERNAL FORCES IN PARABOLIC

CONOIDAL SHELL

Forces	$\frac{x}{a}$	$\frac{y}{b}$						
		0.00	0.10	0.20	0.40	0.60	0.80	1.00
N_x	0.1	∞	-21000	-5250	-1380	-583	-318	-210
	0.2		-42000	-10500	-2760	-1166	-636	-420
	0.5		-105000	-27250	-6900	-2915	-1590	-1050
	0.8		-168000	-42000	-11040	-4664	-2524	-1680
	1.0		-210000	-52500	-13800	-5830	-3180	-2100
N_{xy}		∞	-10200	-5100	-2550	-1690	-1275	-1020

The stress distribution in the shell is shown in Figure 4-2. From the sharpness of rise of the theoretical curves, the validity of the membrane stresses may not extend inside the interval $\frac{y}{b} = 0 \longrightarrow 0.2$ for the shearing force and $\frac{y}{b} = 0 \longrightarrow 0.4$ for the normal force. In any case it will be assumed acceptable to take the upper limits for the shearing and normal forces to be at $\frac{y}{b} = 0.1$ and $\frac{y}{b} = 0.2$, respectively.

The unit compressive stress at the point $\frac{x}{a} = 1.0$, $\frac{y}{b} = 0.2$ is

$$N_x' = \frac{-52500}{12 \times 3.25} = -1350 \text{ psi.}$$

Which shows that, using ordinary concrete, with $f_c' = 1700$ psi., it would not be theoretically necessary to reinforce the interior of the shell for compression.

The unit shearing stress at the point $\frac{x}{a} = 1.0$, $\frac{y}{b} = 0.1$ is

$$N'_{xy} = \frac{-\ 10200}{12 \times 3.25} = -\ 262 \text{ psi.}$$

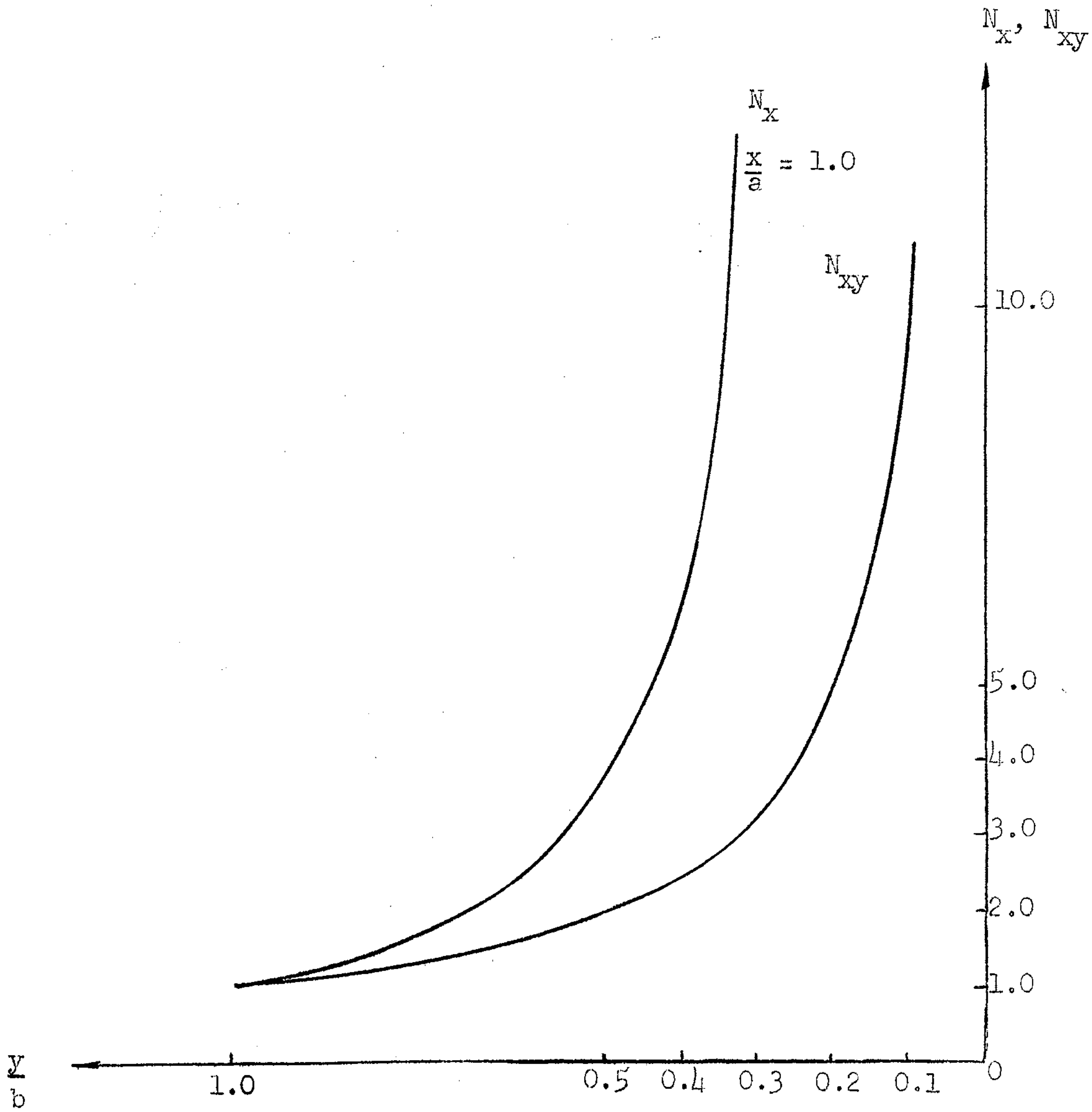

Figure 4-2

Stress Distribution for N_x, N_{xy}

The tensile steel reinforcement in the shell can be computed using Mohr's circle.

The theoretical tensile and compressive forces in the edge regions necessary to resist the shearing stresses can be computed from free body diagrams as shown in Figure 4-3. Thus

$$T = 10200 \times 20 \times 2 = 408000 \text{ pounds}$$

and

$$C = 1020 \times 20.1 = 20400 \text{ pounds.}$$

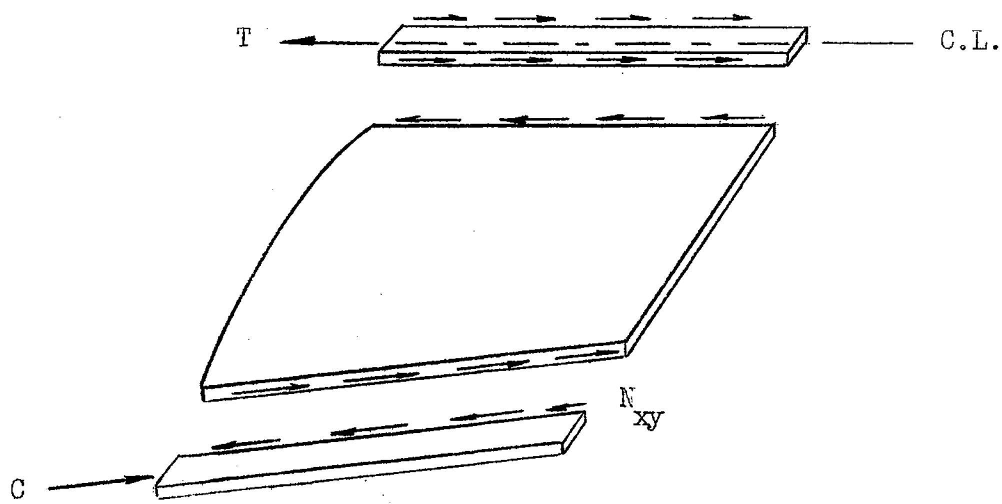

Figure 4-3

Stresses on Edge Member

4-3. Example 2.

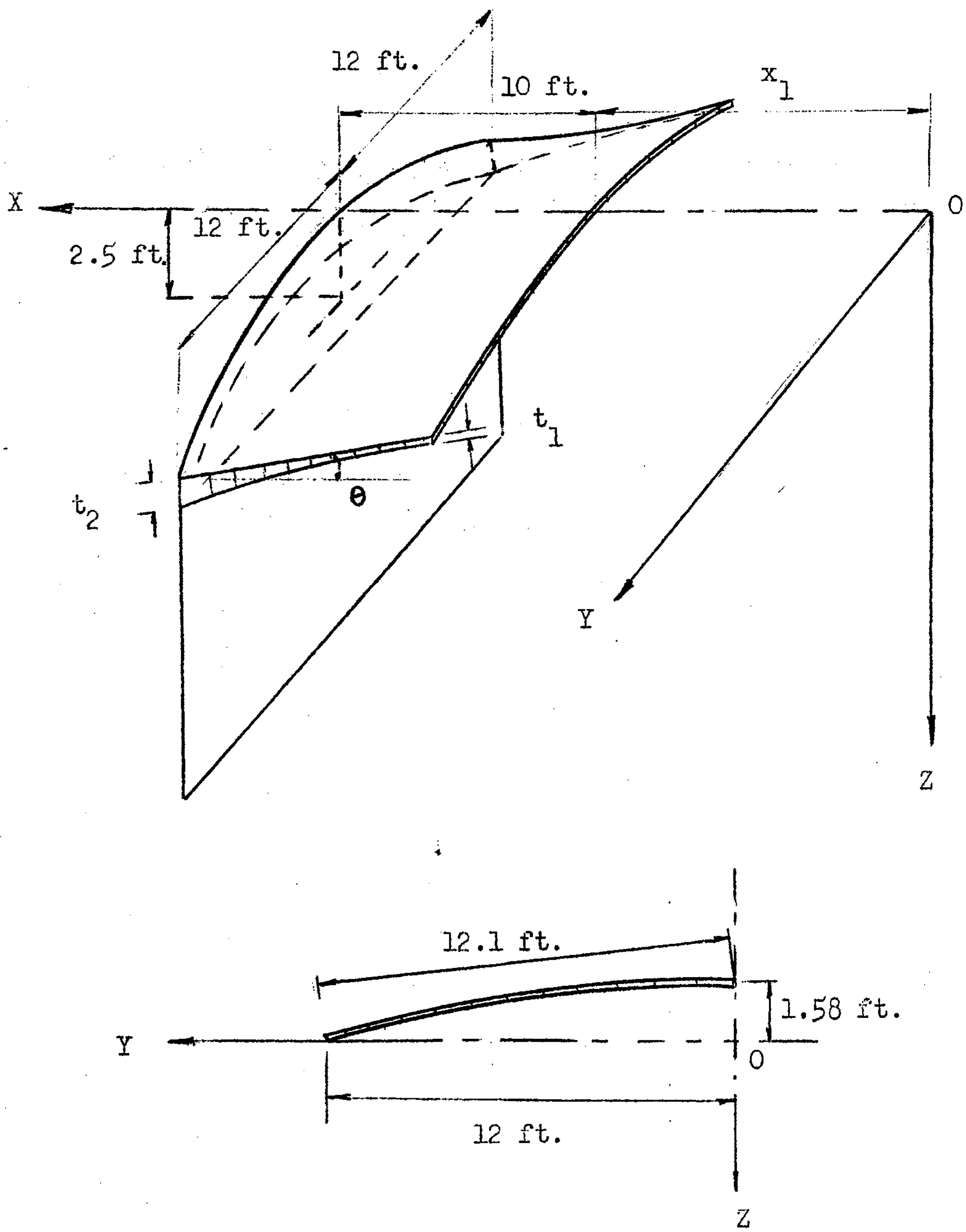

Figure 4-4

Cantilevered Shell with Parabolic Thickness Variation

A cantilevered conoidal shell of variable thickness and specified span, rise, and overhang (Figure 4-4) is to be designed such that the entire weight of the shell can be represented by a parabolic load function

$$P = A\frac{x^2}{2} \quad .$$

Taking the thickness t_2 at the wall to be two and one half times the thickness t_1 at the end of the shell:

$$\frac{t_1}{12} \times 150 = A\frac{{x_1}^2}{2}$$

$$\frac{2.5t_1}{12} \times 150 = A\frac{(10 + x_1)^2}{2} \quad .$$

Eliminating A and t_1 between these two equations and solving for x_1 yields

$$x_1 = 17.2 \text{ feet.}$$

Selecting the thickness $t_1 = 1.5$ inches, the constant A is found to be

$$A = 0.126 \quad \text{lb/ft}^4 \ .$$

If the force per unit of projected area and the weight per unit area of the shell are taken as equal, the maximum error in the representation of the load will occur at the corner x = a, y = b, and can be determined by substituting into Equation (3-8). Thus

$$P_z = P\left\{1 + \tfrac{1}{2}\left[(\tfrac{c}{a})^2 + (\tfrac{2c}{b})^2\right] . \ . \ . \ .\right\}$$

$$= P + 0.091\ P$$

from which

$$\text{Max. Error} = \frac{.091P}{1.091P} \times 100 = 8.36\% \quad .$$

This difference will be neglected and stresses computed by taking $P_z = P$.

From Equations (3-5) the internal forces in the shell are

$$N_y = -\frac{a^2b^2}{4c} A \left(\frac{x}{a}\right)\left(1 - \frac{y}{b}\right) = -\frac{(27.2)^2\,(12)^2}{4\,(2.5)}(0.126)\left(\frac{x}{a}\right)\left(1 - \frac{y}{b}\right)$$

$$= -1340\left(\frac{x}{a}\right)\left(1 - \frac{y}{b}\right)$$

$$N_{xy} = -\frac{a^3b}{8c} A \left(\frac{x}{a}\right)^2 = -\frac{(27.2)^3\,(12)}{8(2.5)}(0.126)\left(\frac{x}{a}\right)^2$$

$$= -1525\left(\frac{x}{a}\right)^2 .$$

$$N_x = 0 .$$

Curves representing the variation of forces are shown in Figure 4-5. The constants B_1 and B_2 are given as

$$B_1 = -1340 \qquad B_2 = -1525 .$$

The unit shearing stress is constant throughout the shell:

$$N_{xy}' = \frac{-1525\left(\frac{x}{a}\right)^2}{12 \times 3.75\left(\frac{x}{a}\right)^2} = -33.9 \text{ lb. per sq. in.}$$

The maximum unit normal stress will occur in the end of the shell at the crown $\left(\frac{x}{a} = 0.632,\ \frac{y}{b} = 0\right)$:

$$(N_y')_{max} = -\frac{1340(.632)}{12 \times 1.5} = -47.1 \text{ lb. per sq. in.}$$

The maximum unit normal stress at the wall is:

$$(N_y') = -\frac{1340}{12 \times 3.75} = -29.8 \text{ lb. per sq. in.}$$

The tensile force in the edge member at the crown can be computed from the free body diagram of Figure 4-6 and is equal to:

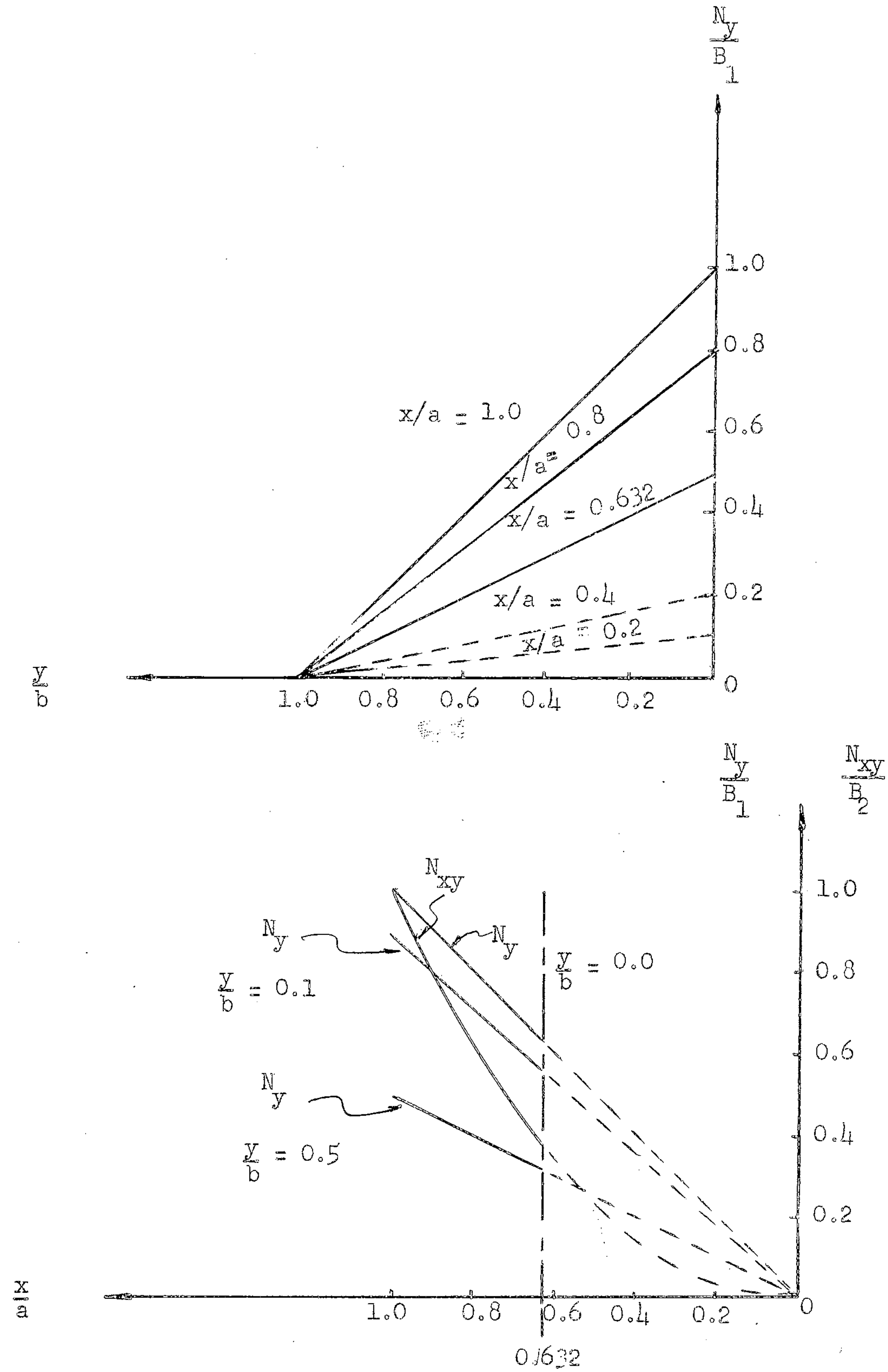

Figure 4-5

Stress Distribution for N_{xy}, N_y

$$T = 2 \int_{x_1}^{a} N_{xy}\, dx = + 2 \frac{1525}{(27.2)^2} \cdot \frac{1}{3} x^3 \Big|_{x_1}^{a}$$

$$= + 1.37 \left[(27.2)^3 - (17.2)^3 \right] = + 20700 \text{ lb.}$$

The steel area required in the edge member is:

$$A_s = \frac{20700}{20000} = 1.04 \text{ sq. in.}$$ (Allowable stress of steel = 20000 psi)

Six No. 4 bars are selected ($A_s = 1.18$ sq. in.). The compressive force in each of the inclined edge members is (Figure 4-6):

$$C = \int_{x_1}^{a} \frac{N_{xy}}{\cos \theta}\, dx = - 10400 \text{ lb.}$$

where $\cos \theta = 0.995$.

The area required in the edge member is

$$A_s = \frac{10400}{20000} = 0.520 \text{ sq. in.}$$

Three No. 4 bars are selected ($A_s = 0.50$). The compressive force C_1 in the edge member at the end of the shell is approximately:

$$C_1 = \int_{0}^{y_1} N_{xy}\, dy = \int_{0}^{12.1} (- 1370) \left(\frac{x}{a}\right) \left(1 - \frac{y}{b}\right) dy$$

$$= (- 1370)(0.462)\left(y - \frac{y^2}{2b}\right) \Big|_{0}^{12.1}$$

$$= 3800 \text{ lb.}$$

Steel area required

$$A_s = \frac{3800}{20000} = 0.19 \text{ sq. in.}$$

From static equilibrium of the entire shell, the tensile and compressive forces in the edge members can readily be checked.

The force diagram in the shell is shown in Figure 4-6.

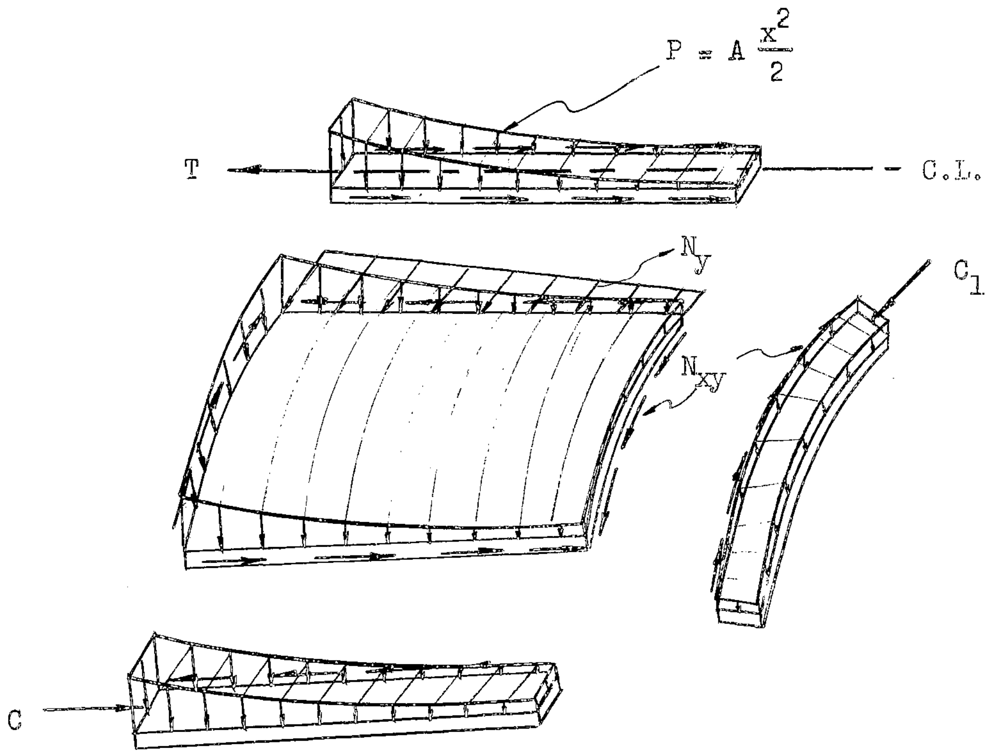

Figure 4-6

Force Diagram

CHAPTER V

SUMMARY AND CONCLUSIONS

The analysis of cantilevered parabolic conoidal shells by linear membrane theory is presented in this thesis. Points of major significance may be summarized as follows:

1. The general solution of the homogeneous equations may be achieved by assuming a stress surface of a higher order hyperbolic paraboloidal type with curved generatrices.
2. The particular solution depends upon both the type of loading and the boundary conditions, yielding infinite values for stresses along the crown of a cantilevered shell under uniform load.
3. The infinite values of stress in the uniformly loaded shell can be explained both physically and mathematically, indicating the breakdown of the membrane theory and the necessity for bending stresses even before deformation incompatibilities are considered.
4. A cantilevered shell designed with a parabolic variation of thickness gives satisfactory results, finite at all points, for dead load stresses.
5. The significance of the shell rise in determining corrections to the stresses can be evaluated by expanding the load intensity in a power series.

The membrane theory of the cantilevered parabolic conoidal shell is not adequate for all load conditions. In the case of uniform load it yields values for stresses which increase without limit near the crown, causing considerable incompatibilities between stresses in edge members and stresses in the shell. Only an adequate bending theory, beyond the scope of this thesis, could predict the region of validity of the membrane stresses.

In the case of dead load, however, varying the thickness of the shell according to a parabolic equation so completely alters the theoretical membrane stress distribution in the shell that it would seem worthwhile as a topic for further investigation. The shearing stress is constant throughout the shell and less than the stress in a comparable hyperbolic paraboloid.

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