

THE GENERAL STRING POLYGON
METHOD

By
Chien Min Wu
Bachelor of Science
National Southwest Associated
University
China
1945

Submitted to the Faculty of the
Graduate School of the
Oklahoma State University
in partial fulfillment of the
requirements for the degree of
MASTER OF SCIENCE
May, 1961

OCT 11 1961

THE GENERAL STRING POLYGON
METHOD

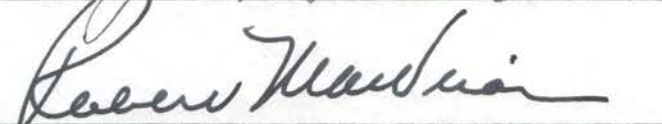
Thesis Approved:



Thesis Adviser



Roger L. Haudus



Dean of the Graduate School

472903

PREFACE

The material presented in this thesis is the outgrowth of the seminar lectures presented by Professor Jan J. Tuma in the Spring of 1960. The literature survey and the general theory recorded in the introduction were prepared by Professor Tuma.

The application of string polygon method to the analysis of single span rigid frames, with members of variable cross-section, was reported by John T. Oden.

The general theory of the string polygon, in terms of the energy due to bending moments, shearing forces, and normal forces, is presented in this thesis.

The writer wishes to express his indebtedness and gratitude to Professor Jan J. Tuma for his invaluable aid and guidance in preparing this thesis. The writer also expresses his appreciation to Professor Roger L. Flanders for his acting as the author's major adviser, and for his advice and thorough reading of the manuscript.

An acknowledgment of thankfulness is also due Miss Velda D. Davis for her exceptional skill in typing of the manuscript.

Paul C. M. Wu

TABLE OF CONTENTS

Chapter	Page
I. INTRODUCTION	1
II. THEORY OF GENERAL STRING POLYGON	3
1. Basic Derivation	3
2a. Angular Load Function due to Bending Moment.	15
2b. Angular Load Function due to Shear	18
2c. Linear Load Function due to Normal Force	21
2d. Change in Slope of the String Polygon	23
2e. Linear Displacements of String Polygon Influenced by Normal Forces	24
III. ELASTIC WEIGHTS	26
IV. NUMERICAL EXAMPLES	30
1. General Notes	30
2. Example No. 1	31
3. Example No. 2	36
V. CONCLUSION	42
A SELECTED BIBLIOGRAPHY	44

LIST OF TABLES

Table	Page
2-1. Interpretation of Angular Constants	11
3-1. Relation of the end Condition Between Real Structure and the Corresponding Conjugate Structures	29
4-1. a. End Moments, end Shears and Normal Forces	32
b. Elastic Constants	32
4-2. a. Elastic Constants	37
b. Moments, Shearing Forces and Normal Forces	38
c. Linear Deformations due to Normal Force . .	39

LIST OF FIGURES

Figure		Page
2-1.	Bent Member	4
2-2.	Moment, Shear, Normal Force Diagram due to Actual Loading	5
2-3.	Virtual Loads	6
2-4.	Shear and Bending Diagrams due to Virtual Loads	8
2-5.	Interpretation of $\tau_{jix}^{(M)}$	15
2-6.	Interpretation of $\tau_{jiy}^{(M)}$	17
2-7.	Interpretation of $\tau_{jix}^{(V)}$	18
2-8.	Interpretation of $\tau_{jiy}^{(V)}$	20
2-9.	Interpretation of $\Omega_{ijx}^{(N)}$	21
2-10.	Interpretation of $\Omega_{ijy}^{(N)}$	22
2-11.	Linear Displacement Under General Loading . . .	24
3-1.	The Relation Between a Real Structure and the Corresponding Conjugate Structure	27
4-1.	Three Member Cantilever Beam	31
4-2.	Conjugate Structure	34
4-3.	Symmetrical Gabled Frame	36

NOMENCLATURE

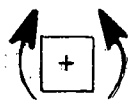
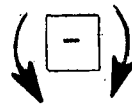
b	Width of Beam
d_j	Length of Bent Member ij
d_{jx}	Horizontal Projection of d_j
d_{jy}	Vertical Projection of d_j
u, u', v, v', x, y	Coordinates of Cross-Section
BM	Bending Moment due to Load
BV	Shearing forces due to Loads
F_{ij}, F_{ji}	Angular Flexibilities
G_{ij}, G_{ji}	Carry-Over Value
M_i, M_j, M_k	End Moments
$N_{ij}, N_{ji}, N_{jk}, N_{kj}$	End Thrusts
$V_{ij}, V_{ji}, V_{jk}, V_{kj}$	End Shears
\bar{M}	Moment of the Elastic Weights
\bar{P}	Elastic Weight
BN	Normal Force due to Loads
τ	Angular Load Function
π	Angle That a Bent Member Makes With the Horizontal
ρ_i	Angle Between Extensions of Bent Members ij and jk
ϕ_j	Change in Change of Slope of Bent Line ijk

NOMENCLATURE (Continued)

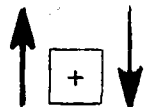

Δ_{ij} Linear Deformation of Member ij

γ_b Shape Factor

SIGN CONVENTION

Moment:  

Normal Force: + tension - compression

Shearing Force:  

CHAPTER I

INTRODUCTION

The idea of elastic weight and the application of the elastic weights was introduced in the middle of the last century by Otto Mohr (1). The extension of the application of elastic weights and a methodical classification of elastic weights was performed by Müller Breslau (2), (3). The study of deformation of beams by means of elastic weights was extensively presented by Wanke (4) and Chmelka (5). The development of the joint elastic weights, in terms of end moments for strips of small length, may be found in work of Kaufmann (6).

In this country, the application of finite elastic weights was shown by Hardy Cross as his Column Analogy (7) and by Michalos as the Column, Shear and Torsion Analogy (8).

The generalization of the joint elastic weight expression and the application of these joint elastic weights, in connection with the string polygon, was developed by Tuma (9) and extended by his students, Chu (10), Oden (11), and Boecker (12), to the solution of many special problems.

The application of the string polygon method requires calculation of angular constants, which are now available

in various publications. (13) (14).

In this thesis, the effort has been made to derive the general expressions for the elastic weights in terms of the bending moments, shearing forces, and normal forces. This leads to the representation of the elastic weight as a vector force and vector moment. This elastic weight is then applied to the conjugate structure. The shear of the conjugate structure is equal to the slope of the real structure and the bending moment of the conjugate structure is the deflection of the real structure along the line of the vector bending moment. The application is illustrated by two examples.

The nomenclature is assembled in the front part of this thesis.

The sign convention of statics is used in formation of equilibrium conditions and elasto-static equations.

The sign convention of deformation is used for the calculation of cross-section elements. The signs of vectors are governed by the right hand rule.

CHAPTER II

THEORY OF GENERAL STRING POLYGON

The general string polygon theory for bent members is developed in this chapter. All the influences of the bending moments, shearing forces and normal forces are considered.

2-1 Basic Derivation

A bent member, ijk , loaded by a general system of loads is considered (Figure 2-1). The cross-section of the member $ij(jk)$ is given by ordinates $u, u'(v, v')$ measured from the respective ends. The cross-sectional elements at a given section are:

bending moments $M_u(M_v)$,
shearing forces $V_u(V_v)$, and
normal forces $N_u(N_v)$.

The geometry of each member is given by the slope π and the length d . The horizontal projection of each d is d_x and the vertical projection of the same length is d_y . Due to the action of loads, the bent member ijk displaces to the position $i'j'k'$ and the change in change of the slope at j is denoted by ϕ_j as shown in Figure 2-1. The

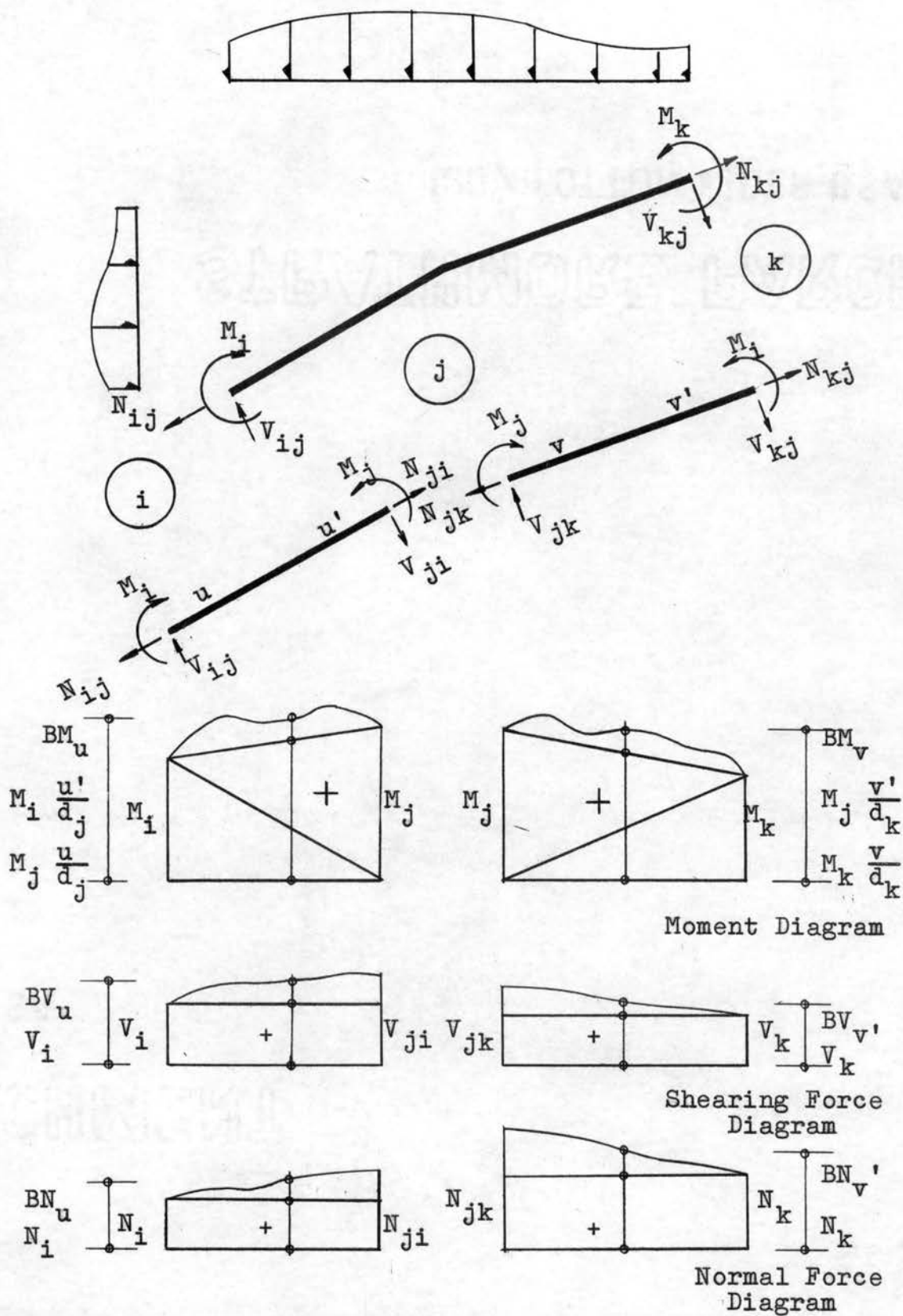


Figure 2-2. Moment, Shear, Normal Diagram due to Actual Loading

where BM_u and BM_v are the bending moments due to loads at the section u and v respectively.

The shearing force at the section u is:

$$V_u = V_i + BV_u \quad (2-2a)$$

and, at section v is:

$$V_v = V_k + BV_v, \quad (2-2b)$$

where BV_u, BV_v , are the shearing forces due to loads on the segment u and v' respectively.

The normal force at the section u is:

$$N_u = N_i + BN_u \quad (2-3a)$$

and at section v is:

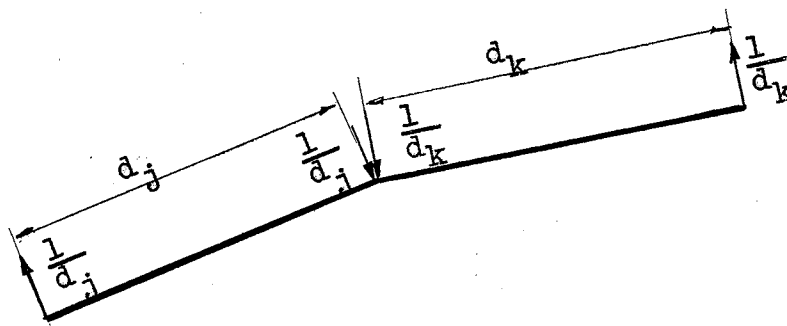
$$N_v = N_k + BN_v, \quad (2-3b)$$

where BN_u, BN_v , are the normal forces due to loads on the segment u and v' respectively.

For the purpose of determining δ_j , the virtual loads

$$\frac{1}{d_j} \text{ and } \frac{1}{d_k}$$

are applied on the member ijk as shown at the Figure 2-3.



2-3. Virtual Loads

The normal force at u and v due to these virtual loads is equal to zero, which indicates that the normal force has no direct influence on the formation of ϕ_j . On the other hand, the shearing force and bending moment do influence the formation of ϕ_j and their diagrams are shown in Figure 2-4.

The bending moment at section u due to the virtual loads is:

$$(M_u) = \frac{u}{d_j} \quad (2-4a)$$

and at section v is:

$$(M_v) = \frac{v'}{d_k} \quad (2-4b)$$

And, the shearing force at section u due to the virtual loads is:

$$(V_u) = \frac{1}{d_j} \quad (2-5a)$$

and, at section v is:

$$(V_v) = -\frac{1}{d_k} \quad (2-5b)$$

From the theory of virtual work, the change in slope due to bending moments and shearing forces is:

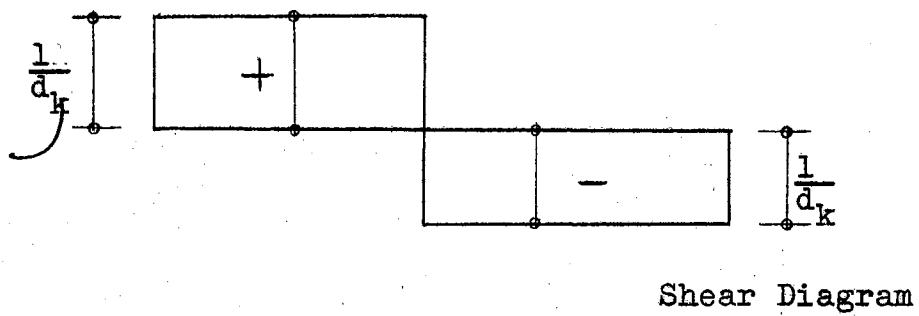
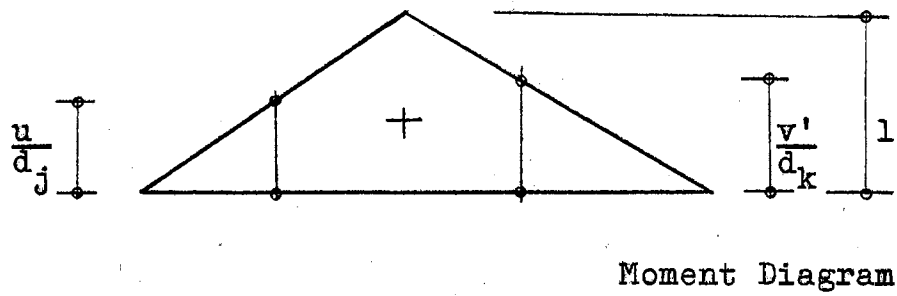
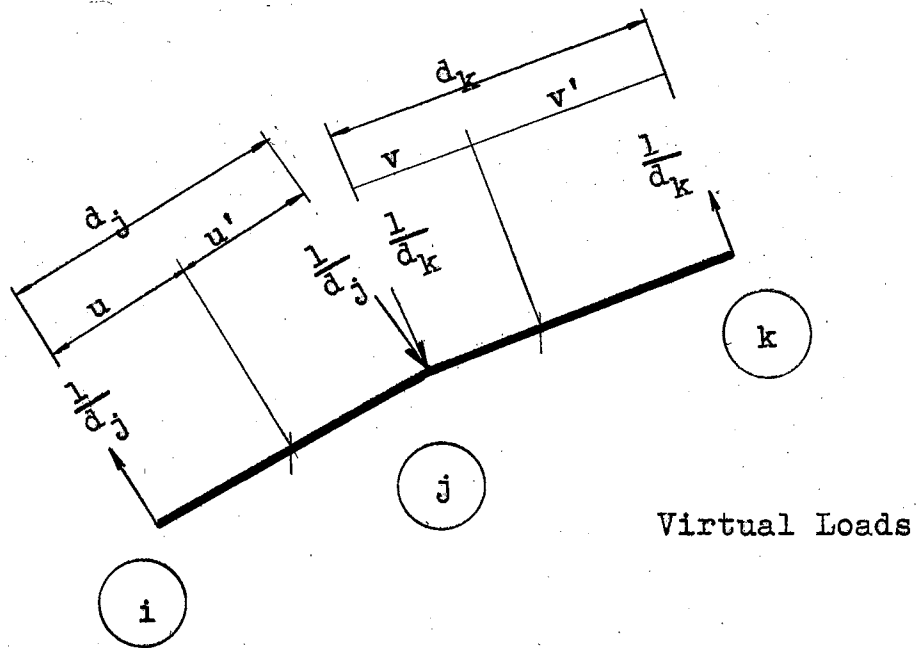


Figure 2-4. Shearing Bending Diagrams due to Virtual Loads

$$\begin{aligned} \emptyset_j = & \int_i^j \frac{M_u(M_u)du}{EI_u} + \mathcal{H} \int_i^j \frac{V_u(V_u)du}{A_u G} + \int_j^k \frac{M_v(M_v)dv}{EI_v} \\ & + \mathcal{H} \int_j^k \frac{V_v(V_v)dv}{A_v G} . \end{aligned} \quad (2-6)$$

In terms of equations 2-4 and 2-5, the equation 2-6 will become:

$$\begin{aligned} \emptyset_j = & M_i \int_i^j \frac{u'udu}{d_j^2 EI_u} + M_j \left(\int_i^j \frac{u^2 du}{d_j^2 EI_u} + \int_j^k \frac{v'^2 du}{d_k^2 EI_v} \right) \\ & + M_k \int_j^k \frac{v'v dv}{d_k^2 EI_v} + \int_i^j \frac{BM_u u du}{d_j EI_u} + \int_j^k \frac{BM_v v' dv}{d_k EI_v} \\ & + V_i \mathcal{H} \int_i^j \frac{du}{d_j A_u G} - V_k \mathcal{H} \int_j^k \frac{dv}{d_k A_v G} + \mathcal{H} \int_i^j \frac{BV_u du}{d_j A_u G} \\ & - \mathcal{H} \int_j^k \frac{Bv_v' dv}{d_k A_v G} \end{aligned} \quad (2-7a)$$

$$\text{or, } \emptyset_j = M_i G_{ij}^{(M)} + M_j \Sigma F_j^{(M)} + M_k G_{kj}^{(M)} + \Sigma \tau_j^{(M)} + V_i G_{ij}^{(v)}$$

$$+ V_k G_{kj}^{(v)} + \Sigma \tau_j^{(v)} . \quad (2-7b)$$

The angular constants in equation (2-7) can be interpreted in terms of a simple beam as shown in Table 2-1.

The normal force will cause a linear elongation or contraction of each member. These deformations are:

$$\begin{aligned}\Delta_{ij}^{(N)} &= \int_i^j \frac{N_u du}{EA_u} \\ &= \int_i^j \frac{N_{ij} du}{EA_u} + \int_i^j \frac{BN_u du}{EA_u}\end{aligned}\quad (2-8a)$$

and:

$$\begin{aligned}\Delta_{jk}^{(N)} &= \int_j^k \frac{N_v dv}{EA_v} \\ &= \int_j^k \frac{N_{jk} dv}{EA_v} + \int_j^k \frac{BN_v dv}{EA_v}\end{aligned}\quad (2-8b)$$

where N_{ij} , N_{jk} are end thrust at i and j respectively,

BN_u is the normal components of loads on the segment u ,

BN_v is the normal components of loads on the segment v .

TABLE 2-1
INTERPRETATION OF ANGULAR CONSTANTS

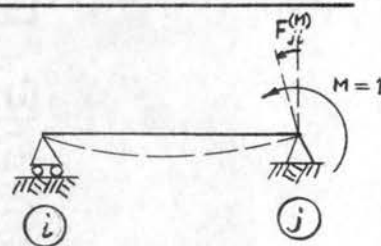
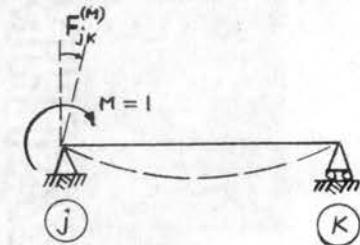
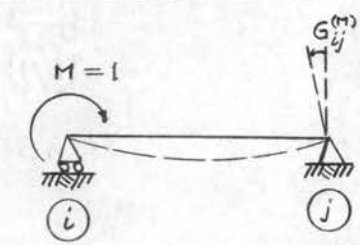
Term	Name	Value	Physical Meaning	Illustration
$F_{ji}^{(M)}$	The angular flexibility due to moment.	$\int_i^j \frac{u^2 du}{d_j^2 EI_u}$	The end slope of a simple beam ij at j due to a unit moment applied at that end.	
$F_{jk}^{(M)}$	The angular flexibility due to moment.	$\int_j^k \frac{v^2 dv}{d_k^2 EI_v}$	The end slope of a simple beam jk at j due to a unit moment applied at that end.	
$G_{ij}^{(M)}$	The angular carry over value due to moment.	$\int_i^j \frac{uu' du}{d_j^2 EI_u}$	The end slope of a simple beam ij at j due to a unit moment applied at the far end i.	

TABLE 2-1 (Continued)

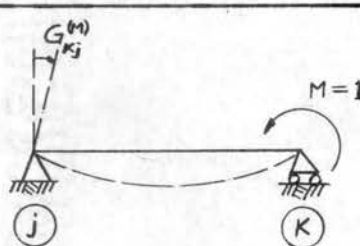
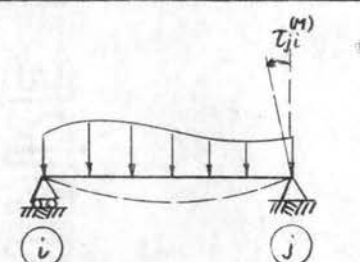
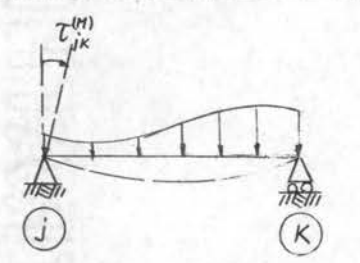
Term	Name	Value	Physical Meaning	Illustration
$G_{kj}^{(M)}$	The angular carry over value due to moment.	$\int_j^k \frac{vv'dv}{d_k^2 EI_v}$	The end slope of a simple beam jk at j due to a unit moment applied at the far end k .	
$\tau_{jk}^{(M)}$ ji	The angular load function due to moment.	$\int_i^j \frac{BM_u u du}{d_j EI_u}$	The end slope of a simple beam ij at j due to a moment influence of the loads.	
$\tau_{jk}^{(M)}$	The angular load function due to moment.	$\int_j^k \frac{BM_v v' dv}{d_k EI_v}$	The end slope of a simple beam jk at j due to a moment influence of the loads.	

TABLE 2-1 (Continued)

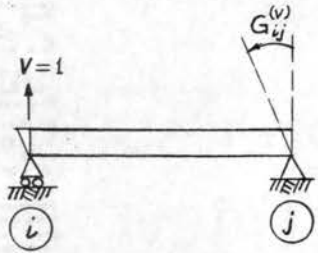
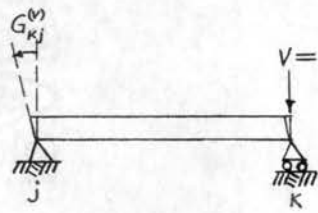
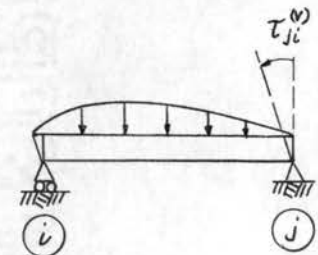
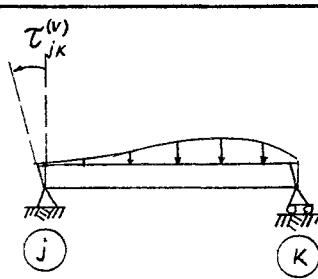
Term	Name	Value	Physical Meaning	Illustration
$G_{ij}^{(V)}$	The angular carry over value due to shear.	$\mathcal{H} \int_i^j \frac{du}{d_j A_u G}$	The angular slope of a simple beam ij at j due to a unit shearing force applied at far end i .	
$G_{kj}^{(V)}$	The angular carry over value due to shear.	$-\mathcal{H} \int_j^k \frac{dv}{d_k A_v G}$	The angular slope a simple beam jk at j due to a unit shearing force applied at far end k .	
$\tau_{ji}^{(V)}$	The angular load function due to shear.	$\mathcal{H} \int_i^j \frac{BV_u du}{d_j A_u G}$	The end slope of a simple beam ij at j due to a shearing influence of the loads.	

TABLE 2-1 (Continued)

Term	Name	Value	Physical Meaning	Illustration
$\tau_{jk}^{(V)}$	The angular load function due to shear.	$-\mathcal{L}_0 \int_j^k \frac{BV_v dv}{d_k A G}$	The end slope of a simple beam jk at j due to a shearing influence of the loads.	

The physical interpretation of equations (2-8a) and (2-8b) is self-evident and does not need to be explained.

2-2a Angular Load Function due to Bending Moment $\tau^{(M)}$

Consider the segment ij of the member ijk loaded only by a system of vertical loads (Figure 2-5); it is desirable to evaluate $\tau_{ji}^{(M)}$ in terms of horizontal or vertical coordinates, since loads are usually applied in these directions. It is necessary to imagine the horizontal projection of member ij as $i'j'$ as shown in the Figure 2-5.

$\tau_{jik}^{(M)}$ may be defined as the slope of the simple beam $i'j'$ at j' due to loads, $i'j'$ being the horizontal projection of ij .

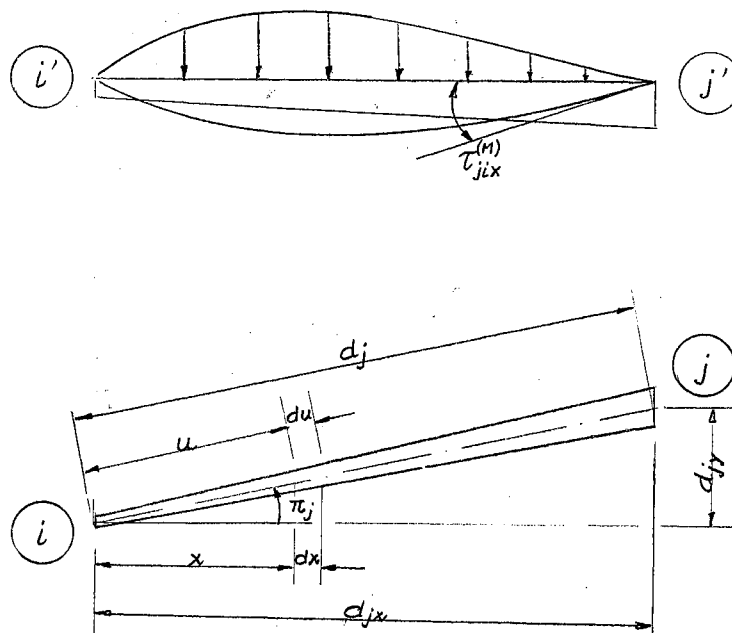


Figure 2-5. Interpretation of $\tau_{jik}^{(M)}$

If the unloaded member ij is naturally inclined at an angle π_j as shown in Figure 2-5, it follows that:

$$du = \frac{dx}{\cos \pi_j} \quad (2-9)$$

....., and defining $\tau_{jix}^{(M)}$ by

$$\tau_{jix}^{(M)} = \int_i^j \frac{BM_x x dx}{d_{jx} EI_u} \quad (2-10)$$

it is seen that

$$\tau_{ji}^{(M)} = \int_i^j \frac{BM_u u du}{d_j EI_u} = \frac{1}{\cos \pi_j} \int_i^j \frac{BM_x x dx}{d_{jx} EI_u} \quad (2-11a)$$

$$\text{or} \quad \tau_{ji}^{(M)} = \frac{1}{\cos \pi_j} \tau_{jix} \quad (2-11b)$$

In a similar manner, the angular load-functions for ij due to the action of horizontal loads only may be evaluated. $\tau_{ijy}^{(M)}$ is defined as the slope of the simple beam $i''j''$ at j'' due to loads where $i''j''$ is the vertical projection of ij as shown in Figure 2-6.

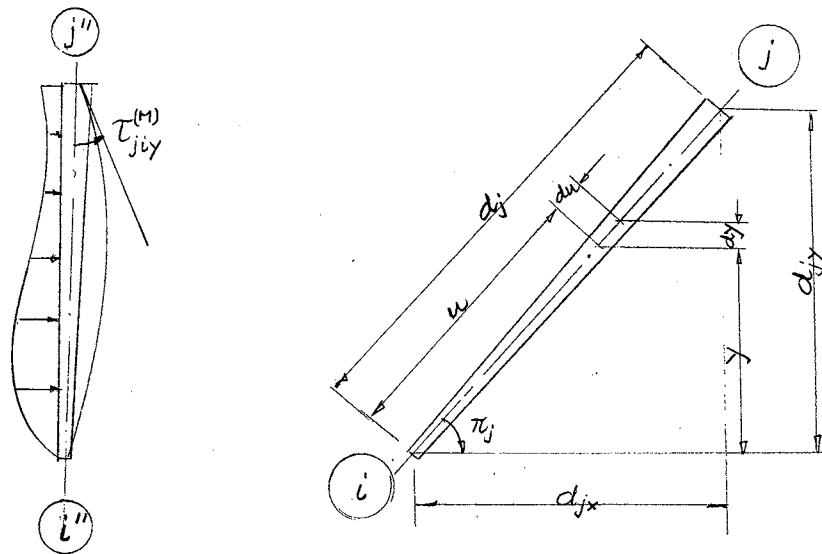


Figure 2-6. Interpretation of $\tau_{j iy}^{(M)}$

It is seen that

$$\tau_{j iy}^{(M)} = \int_i^j \frac{BM_y dy}{d_{jy} EI_u} \quad (2-12)$$

Since $du = \frac{dy}{\sin \pi_j}$ it follows that

$$\tau_{ji}^{(M)} = \frac{1}{\sin \pi_j} \int_i^j \frac{BM_y dy}{d_{jy} EI_u} \quad (2-13a)$$

or

$$\tau_{ji}^{(M)} = \frac{1}{\sin \pi_j} \tau_{j iy} \quad (2-13b)$$

and $BV_u = BW_x \cos \pi_j$

$$du = \frac{dx}{\cos \pi_j}$$

defining $\tau_{jix}^{(v)}$ by

$$\tau_{jix}^{(v)} = \mathcal{H} \int_i^j \frac{BW_x dx}{d_{jx} AuG} .$$

It is seen that

$$\tau_{ji}^{(v)} = \mathcal{H} \int_i^j \frac{BV_u du}{d_j AuG} = \mathcal{H} \cos \pi_j \int_i^j \frac{BW_x dx}{d_{jx} AuG} \quad (2-14a)$$

or $\tau_{ji}^{(v)} = \tau_{jix}^{(v)} \cos \pi_j \quad (2-14b)$

also $\tau_{jk}^{(v)} = \tau_{jkx}^{(v)} \cos \pi_k . \quad (2-15)$

$\tau_{ijy}^{(v)}$ is defined as the slope of simple beam i"j" at j" due to horizontal loads where i"j" is the vertical projection of ij.

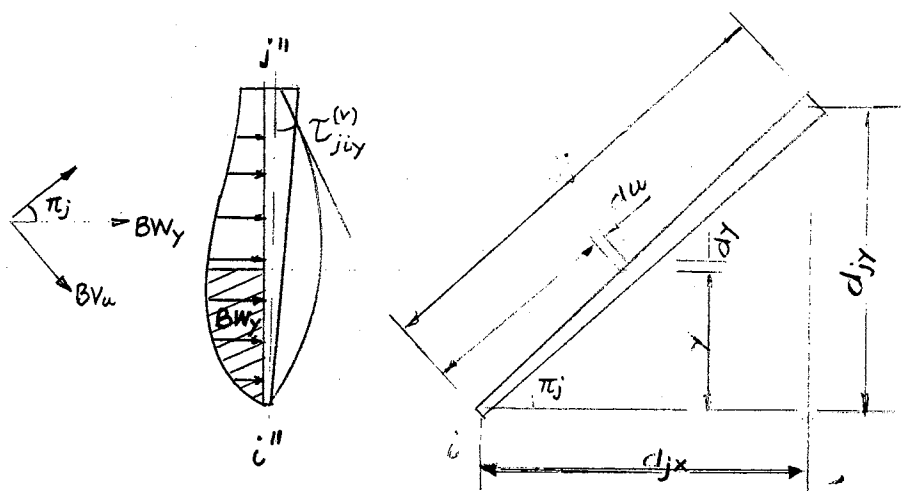


Figure 2-8. Interpretation of $\tau_{jiy}^{(v)}$

where BW_y = Horizontal loads on the segment y
 BV_u = Perpendicular loads on the segment u of
the member ij

and $BV_u = BW_y \sin \pi_j$

$$du = \frac{dy}{\sin \pi_j}$$

$$\text{Therefore, } \tau_{ji}^{(v)} = \mathcal{H} \int_i^j \frac{BV_u du}{d_j AuG} = \mathcal{H} \sin \pi_j \int_i^j \frac{BW_y dy}{d_{jy} AuG} \quad (2-16a)$$

$$\text{or } \tau_{ji}^{(v)} = \tau_{jiy}^{(v)} \sin \pi_j \quad (2-16b)$$

$$\tau_{jk}^{(v)} = \tau_{jky}^{(v)} \sin \pi_k \quad (2-17)$$

2-2c Linear Load Function due to Normal Force Ω_{ij}

Consider the bent member ij loaded only by a system of vertical load as shown in Figure 2-9.

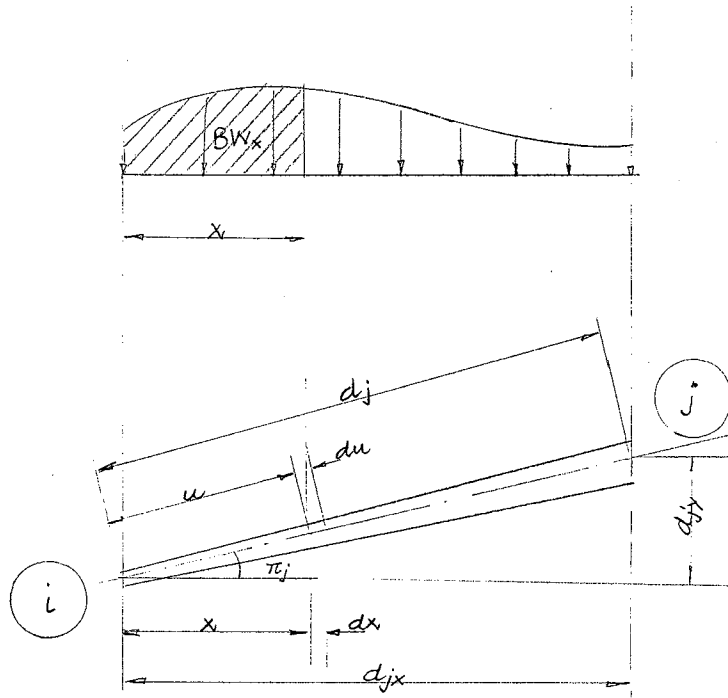


Figure 2-9. Interpretation of $\Omega_{ijx}^{(N)}$

The linear load function

$$\begin{aligned}
 \Omega_{ij}^{(N)} &= \int_i^j \frac{BN_u du}{EA_u} = \int_i^j \frac{BW_x \sin \pi_j}{EA_u} \frac{dx}{\cos \pi_j} \\
 &= \tan \pi_j \int_i^j \frac{BW_x dx}{EA_u}
 \end{aligned} \tag{2-18}$$

where BN_x = Vertical loads on the segment x

BN_u = Normal loads on the segment u of the member

ij

and $BN_u = BW_x \sin \pi_j$

$$du = \frac{dx}{\cos \pi_j} .$$

In a similar manner, consider the bent member ij loaded only by a system of horizontal loads as shown in Figure 2-10.

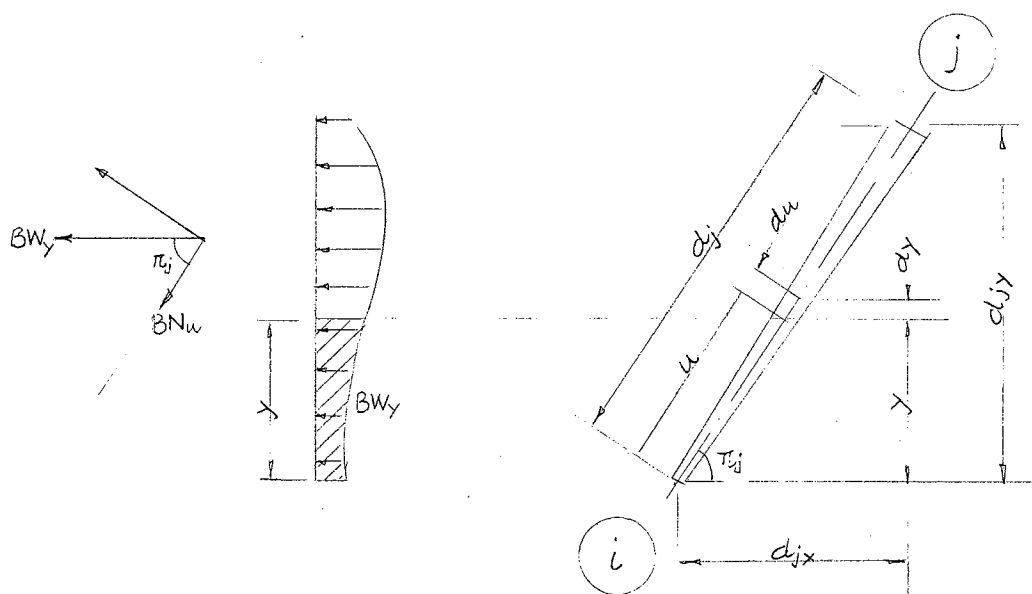


Figure 2-10. Interpretation of $\Omega_{ijy}^{(N)}$

The linear load function

$$\begin{aligned}\Omega_{ij}^{(N)} &= \int_i^j \frac{BN_u du}{EA_u} = \int_i^j \frac{BW_y \cos \pi_j \frac{dy}{\sin \pi_j}}{EA_u} \\ &= \cot \pi_j \int_i^j \frac{BW_y dy}{EA_u}\end{aligned}\quad (2-19)$$

where BW_y = Horizontal loads on the segment y

BN_u = Normal loads on the segment u of the member ij

$$BN_u = BW_y \cos \pi_j$$

$$du = \frac{dy}{\sin \pi_j}$$

2-2d Change in Slope of the String Polygon ϕ_j for Vertical and Horizontal Loads

The change in slope of the string polygon due to the vertical loads becomes:

$$\begin{aligned}\phi_j &= M_i G_{ij}^{(M)} + M_j \Sigma F_j^{(M)} + M_k G_{kj}^{(M)} + \frac{\tau_{jix}^{(M)}}{\cos \pi_j} \\ &+ \frac{\tau_{jkx}^{(M)}}{\cos \pi_k} + V_i G_{ij}^{(V)} + V_k G_{kj}^{(V)} + \tau_{jix} \cos \pi_j \\ &+ \tau_{jkx} \cos \pi_k\end{aligned}\quad (2-20)$$

And the change in slope of the string polygon due to the horizontal loads becomes:

$$\begin{aligned} \varnothing_j = & M_i G_{ij}^{(M)} + M_j \Sigma F_j^{(M)} + M_k G_{kj}^{(M)} + \frac{\tau_{jly}}{\sin \pi_j} + \frac{\tau_{jky}}{\sin \pi_k} \\ & + V_i G_{ij}^{(V)} + V_k G_{kj}^{(V)} + \tau_{jly} \sin \pi_j + \tau_{jky} \sin \pi_k . \end{aligned} \quad (2-21)$$

2-2e Linear Displacements of String Polygon Influenced by Normal Forces

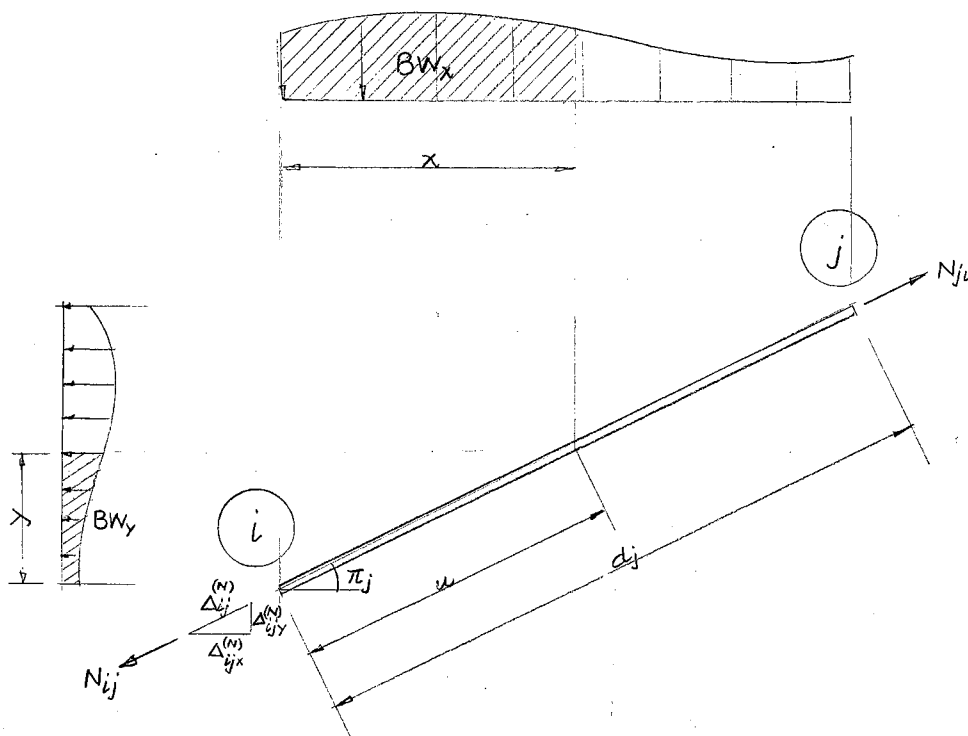


Figure 2-11. Linear Displacement Under General Loading

The defined linear displacement due to the influence of the normal forces, which the general loading condition of vertical and horizontal is considered in Figure 2-11.

$$\Delta_{ijx}^{(N)} = \Delta_{ij}^{(N)} \cos \pi_j \quad \left| \quad \Delta_{ijy}^{(N)} = \Delta_{ij}^{(N)} \sin \pi_j \right. \quad (2-22)$$

or

$$\Delta_{ijx} = N_{ij} \int_i^j \frac{dx}{A_u E} + \sin \pi_j \int_i^j \frac{BW_x dx}{A_u E} + \cos \pi_j \cot \pi_j \cdot \int_i^j \frac{BW_y dy}{A_u E} \quad (2-23)$$

and

$$\Delta_{ijy} = N_{ij} \int_i^j \frac{dy}{A_u E} + \sin \pi_j \tan \pi_j \int_i^j \frac{BW_x dx}{A_u E} + \cos \pi_j \cdot \int_i^j \frac{BW_y dy}{A_u E} \quad (2-24)$$

CHAPTER III

ELASTIC WEIGHTS

The change in change of slope ϕ_j of bent line ijk of Figure 1-1 at j is an angular deformation at j and it can be treated as a vector force normal to the plane of the bent member:

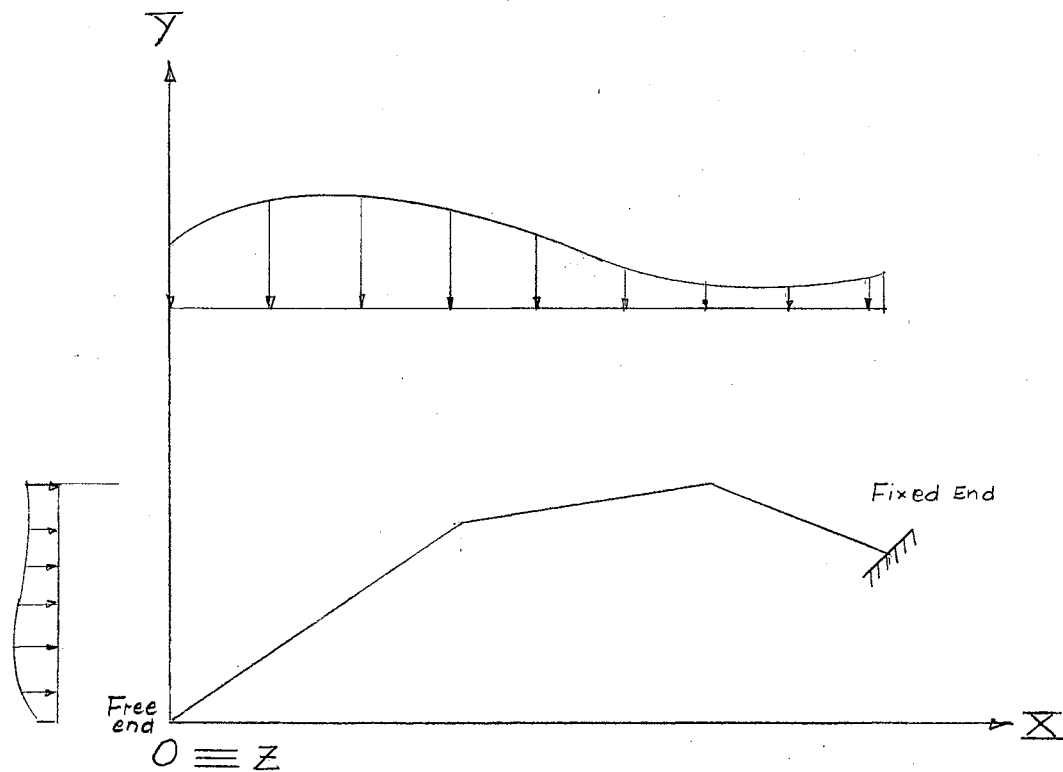
$$\phi_j = \bar{P}_j \quad (3-1)$$

The linear deformations Δ_{ij} and Δ_{jk} can be represented as vector moments acting at i and j , respectively.

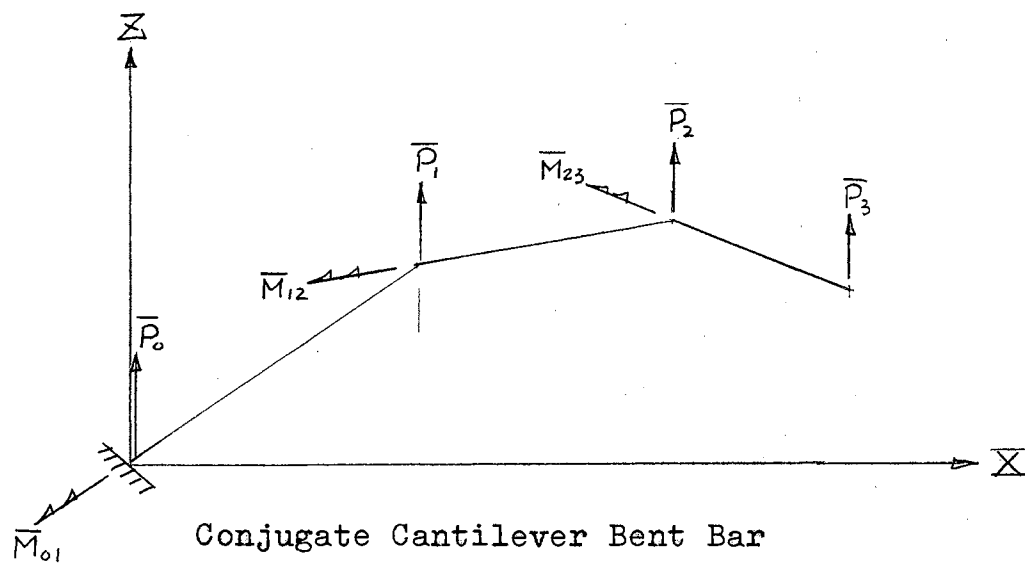
$$\Delta_{ij} = \bar{M}_{ij} \quad \left| \quad \Delta_{jk} = \bar{M}_{jk} \quad (3-2) \right.$$

The angular and linear deformations can be treated as elastic weights applied on the conjugate structure and used in elasto-static equations for the calculation of deformations.

The relationship between a real structure and a conjugate structure is shown in Figure 3-1.



Real Cantilever Bent Bar



Conjugate Cantilever Bent Bar

Figure 3-1. The Relation Between a Real Structure and the Corresponding Conjugate Structure

From the relationship of the real structure to the conjugate structure, the following analogies may be stated:


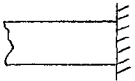
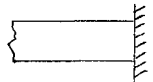


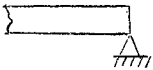
1. The shear of the conjugate structure at any point represents the slope of the real structure at the same point.
2. The bending moment of the conjugate structure at the certain point about a given axis is the displacement of the real structure along that axis.

This two laws hold for any polygon of any set of members without regard to its end conditions and type of loading. The end conditions of the real structure and the end conditions of the corresponding conjugate structure are related to each other as shown in Table 3-1.

The application of the string polygon equations (2-7) and (2-8) as elastic weights defined by equations (3-1) and (3-2) to a solution of a specific problem is shown in the last part of this thesis.

TABLE 3-1

RELATION OF THE END CONDITION
BETWEEN REAL STRUCTURE AND
THE CORRESPONDING CONJUGATE
STRUCTURE

End Condition	Real Structure	Conjugate Structure
1	 Free end	 Fixed end
2	 Fixed end	 Free end
3	 Simple Supported end	 Simple Supported end

CHAPTER IV

NUMERICAL EXAMPLES

4-1. General Notes

The following illustrative examples comprise this chapter of the thesis and describe the numerical procedure of analysis by the general string polygon method. Dimensions are chosen to simplify calculation and units for various values are in terms of Kips, feet and Kip-feet.

4-2. Example No. 1

A three-member cantilever beam of constant cross section is considered. (See Figure 4-1.) The angular and linear deflections at the free end 1 are required in terms of modulus of elasticities, E , G , moment of inertia I , and area of the constant cross section A . The shape factor α is assumed to be 1.

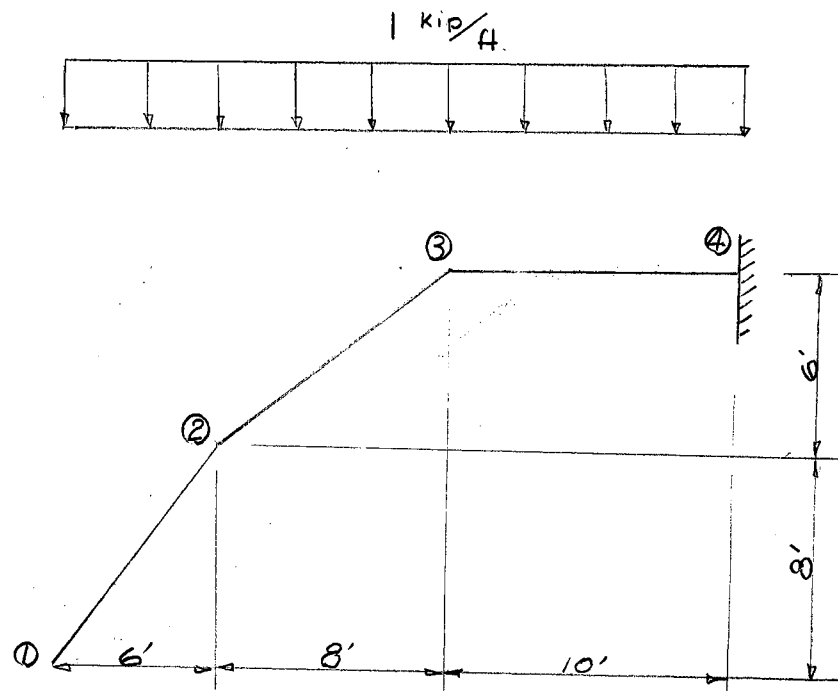


Figure 4-1. Three-Member Cantilever Beam

(a) End moments, end shears and end normal forces:

Evaluating joint (j)	End moment (k-ft)			End shearing force (kip)		End normal force (kip)	
	M_i	M_j	M_k	V_{ij}	V_{kj}	N_{ij}	N_{kj}
1	0	0	-18	0	$-\frac{18}{5}$	0	$+\frac{24}{5}$
2	0	-18	-98	0	$-\frac{56}{5}$	0	$+\frac{42}{5}$
3	-18	-98	-288	$-\frac{24}{5}$	-24	$+\frac{18}{5}$	0
4	-98	-288	0	-14	0	0	0

(b) Elastic constants

Evaluating joint (j)	$G^M(EI)$		$F^M(EI)$		$\tau^M(EI)$		$G^V(AG)$		$\tau^V(AG)$	
	ij	kj	ji	jk	ji	jk	ij	kj	ji	jk
1	0	$\frac{5}{3}$	0	$\frac{10}{3}$	0	$\frac{45}{3}$	0	-1	0	$-\frac{9}{5}$
2	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{10}{3}$	$\frac{10}{3}$	$\frac{45}{3}$	$\frac{80}{3}$	1	-1	$-\frac{9}{5}$	$-\frac{16}{5}$
3	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{10}{3}$	$\frac{10}{3}$	$\frac{80}{3}$	$\frac{125}{3}$	1	-1	$-\frac{16}{5}$	$-\frac{25}{5}$
4	$\frac{5}{3}$	0	$\frac{10}{3}$	0	$\frac{125}{3}$	0	1	0	$-\frac{25}{5}$	0

(c) Elastic weights

$$\bar{P}_1 = \phi_1 = (-18)\left(\frac{5}{3EI}\right) + \frac{45}{3EI} + \left(-\frac{18}{5}\right)\left(-\frac{1}{AG}\right) - \frac{9}{5AG} = -\frac{45}{3EI} + \frac{9}{5AG}$$

$$\begin{aligned}\bar{P}_2 = \phi_2 &= (-18)\left(\frac{20}{3EI}\right) + (-98)\left(\frac{5}{3EI}\right) + \frac{125}{3EI} + \left(-\frac{56}{5}\right)\left(-\frac{1}{AG}\right) - \frac{25}{5AG} \\ &= -\frac{725}{3EI} + \frac{31}{5AG}\end{aligned}$$

$$\begin{aligned}\bar{P}_3 = \phi_3 &= (-18)\left(\frac{5}{3EI}\right) + (-98)\left(\frac{20}{3EI}\right) + (-288)\left(\frac{5}{3EI}\right) + \frac{205}{3EI} \\ &\quad + \left(-\frac{24}{5}\right)\left(\frac{1}{AG}\right) + (-24)\left(-\frac{1}{AG}\right) - \frac{41}{5AG} = -\frac{3285}{3EI} + \frac{55}{5AG}\end{aligned}$$

$$\begin{aligned}\bar{P}_4 = \phi_4 &= (-98)\left(\frac{5}{3EI}\right) + (-288)\left(\frac{10}{3EI}\right) + \frac{125}{3EI} + (-14)\left(\frac{1}{AG}\right) - \frac{25}{5AG} \\ &= -\frac{3245}{3EI} - \frac{95}{5AG}\end{aligned}$$

$$\bar{M}_{12}^N = \Delta_{12} = \tan \pi \int_0^6 \frac{BW_x dx}{EA} = \frac{8}{6} \int_0^6 \frac{x dx}{EA} = \frac{8}{6EA} \left| \frac{x^2}{2} \right|_0^6 = \frac{8 \times 18}{6EA} = \frac{24}{EA}$$

$$\Delta_{12x} = \frac{24}{EA} \times \frac{3}{5} = \frac{72}{5EA}$$

$$\Delta_{12y} = \frac{24}{EA} \times \frac{4}{5} = \frac{96}{5EA}$$

$$\bar{M}_{23}^N = \Delta_{23} = \left(\frac{18}{5}\right) \int_0^{10} \frac{du}{EA} + \tan \pi_2 \int_0^8 \frac{x dx}{EA}$$

$$= \frac{180}{5EA} + \frac{6 \times 32}{8EA} = \frac{36}{EA} + \frac{24}{EA} = \frac{60}{EA}$$

$$\Delta_{23x} = \frac{60}{EA} \times \frac{4}{5} = \frac{240}{5EA}$$

$$\Delta_{23y} = \frac{60}{EA} \times \frac{3}{5} = \frac{180}{5EA}$$

(d) Conjugate system

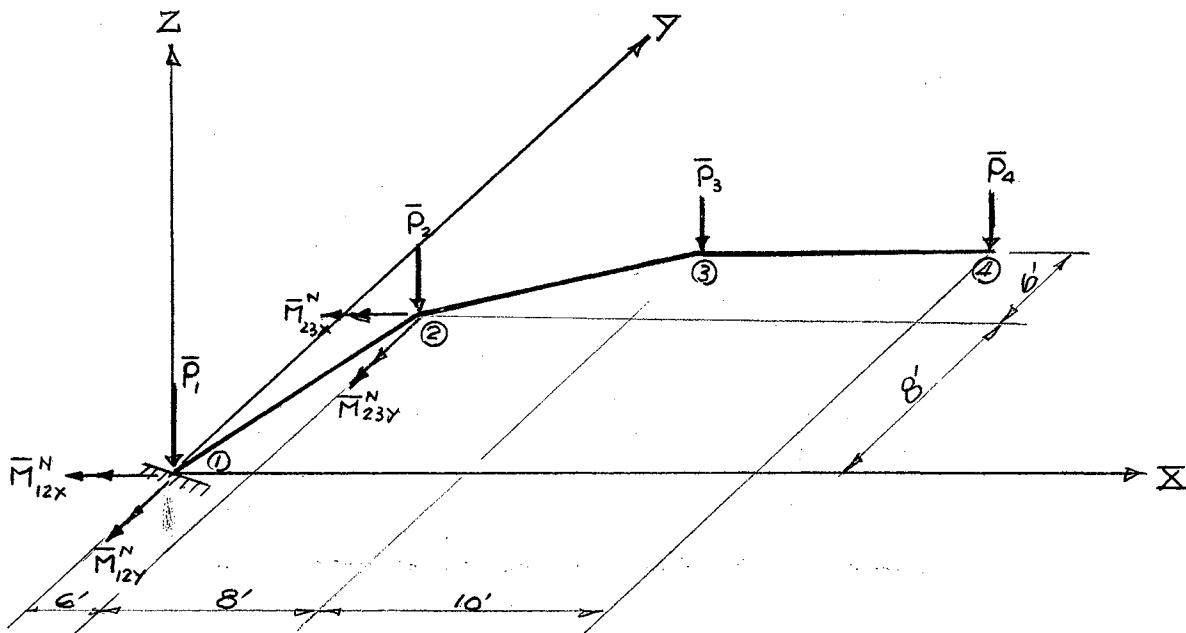


Figure 4-2. Conjugate Structure

(e) Deformations at end 1

$$\phi_{14} = \Sigma \bar{P} = - \frac{7300}{3EI} - \frac{9}{5AG} \quad \left(\begin{array}{l} \text{Angular deformation at} \\ \text{end 1 with respect to} \\ \text{end 4} \end{array} \right)$$

$$\Delta_{14x} = \Sigma \bar{P}_y + \Sigma \bar{M}_x$$

$$= \left(\frac{-725}{3EI} + \frac{31}{5AG} \right) (8) + \left(\frac{-6530}{3EI} - \frac{40}{5AG} \right) (14) + \frac{72}{5EA} + \frac{240}{5EA}$$

$$= - \frac{97220}{3EI} - \frac{312}{5AG} + \frac{312}{5EA} \quad \left(\begin{array}{l} \text{Horizontal linear de-} \\ \text{formation at end 1} \end{array} \right)$$

$$\Delta_{14y} = \Sigma \bar{P}_x + \Sigma \bar{M}_y$$

$$= \left(- \frac{725}{3EI} + \frac{31}{5AG} \right) (6) + \left(- \frac{3285}{3EI} + \frac{55}{5AG} \right) (14)$$

$$+ \left(- \frac{3245}{3EI} - \frac{95}{5AG} \right) (24) - \frac{96}{5EA} - \frac{180}{5EA} = - \frac{138145}{3EI}$$

$$- \frac{1324}{5AG} - \frac{276}{5EA} \quad \left(\begin{array}{l} \text{Vertical linear de-} \\ \text{formation at end 1} \end{array} \right)$$

4-3. Example 2.

Find the reactions at A and E of the symmetrical gabled frame shown in Figure 4-3.

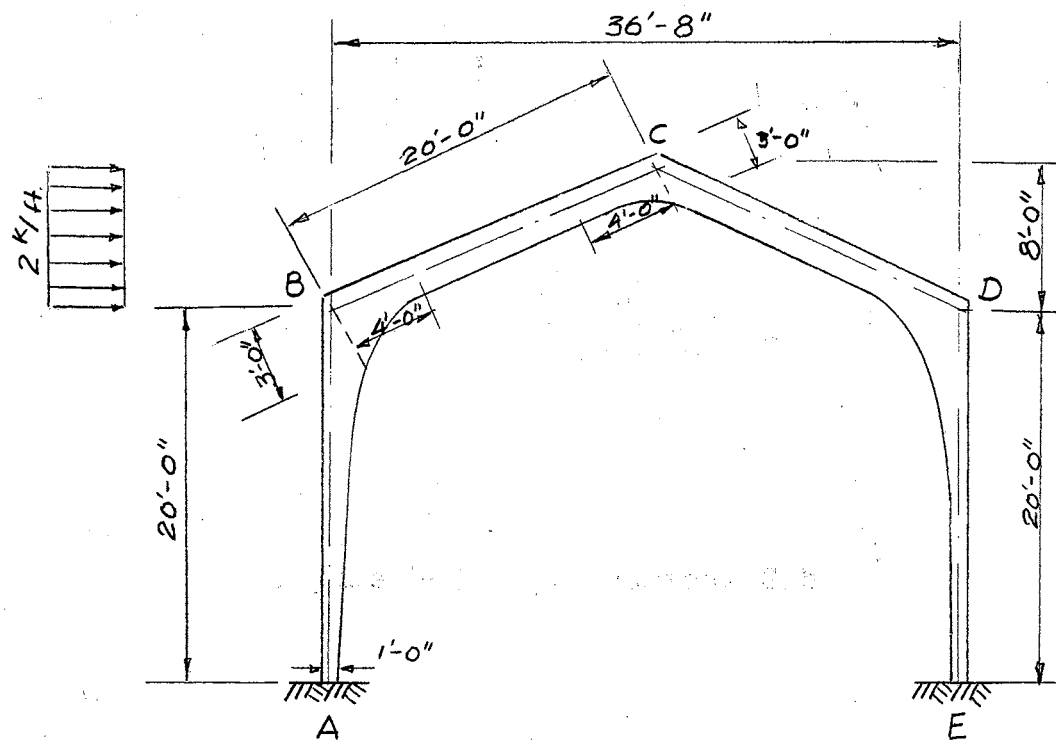


Figure 4-3. Symmetrical Gabled Frame

A. Elastic Constants

Joint	$G^{(M)}_{ijE}$	$\Sigma F^{(M)}_{jE}$	$G^{(M)}_{kjE}$	$G^{(V)}_{ijE}$	$G^{(V)}_{kjE}$	$\Sigma \tau^{(M)}_{jE}$	$\tau^{(V)}_{jE}$
A	-	55.94	14.86	-	-1.79	-	-
B	14.86	20.14	4.86	1.79	-1.22	163.50	-4.00
C	4.86	16.66	4.86	1.22	-1.22	150.20	+4.00
D	4.86	20.14	14.86	1.22	-1.79	-	-
E	14.86	55.94	-	1.79	-	-	-

B. Moments, Shearing Forces, and Normal Forces

Joint	Moment	Shearing force		Normal force	
		V_{ij}	V_{kj}	N_{ij}	N_{kj}
A	M_A	-	$-R_{AX}$	-	$-R_{AY}$
B	$M_A - 20 R_{AX}$	$-R_{AX}$	$-.4 R_{AX}$ $+ .916 R_{AY} - 6.4$	$-R_{AY}$	$-.916 R_{AX}$ $-.4 R_{AY} - 14.6$
C	$M_A - 28 R_{AX}$ $+ 18.33 R_{AY} - 64$	$-.4 R_{AX} + .916 R_{AY}$	$+.4 R_{AX} + .916 R_{AY}$ $+ 6.4$	$-.916 R_{AX} - .4 R_{AY}$	$-.916 R_{AX}$ $+ .4 R_{AY} - 14.6$
D	$M_A - 20 R_{AX}$ $+ 36.67 R_{AY} + 64$	$+.4 R_{AX} + .916 R_{AY}$ $+ 6.4$	$R_{AX} + 16$	$-.916 R_A + .4 R_{AY}$ $- 16.4$	$+R_{AY}$
E	$M_A + 36.67 R_{AY}$ $+ 384$	$R_{AX} + 16$	-	$+R_{AY}$	-

C. Linear Deformations due to Normal Force

Member	AB	BC	CD	DE
$\int_0^1 \frac{ds}{A}$	13.54	9.23	9.23	13.54
$\int_0^1 \frac{wds}{A}$	-	75.9	-	-

$$\bar{M}_{AB} = \Delta_{AB}^{(N)} = -13.54 R_{AY}$$

$$\bar{M}_{BC} = \Delta_{BL}^{(N)} = (-0.916 R_{AX} - 0.4 R_{AY} - 14.6)(9.23) - 75.9$$

$$\bar{M}_{CD} = \Delta_{CD}^{(N)} = (-0.916 R_{AX} + 0.4 R_{AY} - 14.6)(9.23)$$

$$\bar{M}_{DE} = \Delta_{DE}^{(N)} = 13.54 R_{AY}$$

$\bar{M}_{ABX} = 0$	$\bar{M}_{ABY} = -13.54 R_{AY}$
$\bar{M}_{BCX} = -7.8 R_{AX} - 3.4 R_{AY} - 193.5$	$\bar{M}_{BCY} = -3.4 R_{AX} - 1.5 R_{AY} - 84.3$
$\bar{M}_{CDX} = -7.8 R_{AX} + 3.4 R_{AY} - 124$	$\bar{M}_{CDY} = -3.4 R_{AX} + 1.5 R_{AY} - 53.9$
$\bar{M}_{DEX} = 0$	$\bar{M}_{DEY} = 13.54 R_{AY}$
$\Sigma \bar{M}_X = -15.6 R_{AX} - 317.5$	$\Sigma \bar{M}_Y = -6.8 R_{AX} - 138.2$

D. Elastic Weights

$$\bar{P}_A = \phi_A = 55.94 M_A + 14.84 (M_A - 20 R_{AX}) + 1.79 R_{AX} \quad .$$

$$\begin{aligned} \bar{P}_B = \phi_B = & 14.86 M_A + 20.14 (M_A - 20 R_{AX}) + 4.86 (M_A - 28 R_{AX} \\ & + 18.33 R_{AY} - 64) - 1.79 R_A - 1.22 (-.4 R_{AX} + .916 \\ & \times R_{AY} - 6.4) + 163.5 - 4 \quad . \end{aligned}$$

$$\begin{aligned} \bar{P}_C = \phi_C = & (M_A - 20 R_{AX}) 4.86 + (M_A - 28 R_{AX} + 18.33 R_{AY} \\ & - 64) (16.66) + (M_A - 20 R_{AX} + 36.37 R_{AY} + 64) 4.86 \\ & + (-.4 R_{AX} + .916 R_{AY}) (1.22) + (.4 R_{AX} + .916 R_{AY} \\ & + 6.4) (-1.22) + 154 \quad . \end{aligned}$$

$$\begin{aligned} \bar{P}_D = \phi_D = & (M_A - 28 R_{AX} + 18.33 R_{AY} - 64) (4.86) + (M_A \\ & - 20 R_{AX} + 36.67 R_{AY} + 64) (20.14) + (M_A + 36.67 R_{AY} \\ & + 384) (14.86) + (.4 R_{AX} + .916 R_{AY} + 6.4) (1.22) \\ & + (R_{AX} + 16) (-1.79) \quad . \end{aligned}$$

E. Elastic Equations

$$\Sigma \bar{P} = 0$$

$$247.7 M_A - 2332 R_{AX} + 4546 R_{AY} + 28375 = 0 \quad (1)$$

$$\Sigma \bar{P}_Y + \Sigma \bar{M}_X = 0$$

$$2332 M_A - 40116 R_{AX} + 42862 R_{AY} + 113578 = 0 \quad (2)$$

$$\Sigma \bar{P}_X + \Sigma \bar{M}_Y = 0$$

$$4542 M_A - 42739 R_{AX} + 154480 R_{AY} + 1055327 = 0 \quad (3)$$

F. Resultants

Solving equations 1, 2, and 3, gives the following:

$$R_{AX} = 8.4684 \quad \text{Kip} \quad \leftarrow$$

$$R_{AY} = 7.5225 \quad \text{Kip} \quad \downarrow$$

$$M_A = 56.1181 \quad \text{K. ft.} \quad \curvearrowright$$

Applying the equation of static, gives the following:

$$R_{EX} = 7.5316 \quad \text{Kip} \quad \leftarrow$$

$$R_{EY} = 7.5225 \quad \text{Kip} \quad \downarrow$$

$$M_E = 51.3749 \quad \text{K. ft.} \quad \curvearrowright$$

G. Comparison of Resultants

This problem was solved by Oden (11), using the angular elastic weight due to bending moment only. The comparison of this writer's resultants with Oden's resultants is as follows:

	Writer's	Oden's
R_{AX}	8.4684 Kips	8.48 Kips
R_{AY}	7.5225 Kips	7.55 Kips
M_A	56.1181 Kip ft.	56.02 Kip ft.
R_{EX}	7.5316 Kips	7.52 Kips
R_{EY}	7.5225 Kips	7.55 Kips
M_E	51.3749 Kip ft.	51.20 Kip ft.

CHAPTER V

CONCLUSION

The application of the general string polygon theory to the analysis of co-planar bent members is developed in this thesis.

The significant points of this study are summarized as follows:

1. The general expression for the joint elastic weight due to deformation of bending, shear and normal force is represented by a vector force and a vector moment.
2. These joint elastic weights are applied on the conjugate structure and are in equilibrium with the end conditioning elements (reactions of the conjugate structure).
3. The shear of conjugate structure is the slope of real structure; also, the bending moment of the conjugate structure about a given line is the deflection of the real structure along the same line.
4. Because the end conditioning elements are forming the static equilibrium with the elastic weights, three independent static equations are available for each conjugate structure. These equations are

called the elasto-static equations.

5. The inclusion of the influence of the shearing forces and normal forces in the analysis of deformation of bent members loaded in their own plane does not cause any special difficulty by this method.

A SELECTED BIBLIOGRAPHY

- (1). Mohr, O. "Behandlung der Elastischen Als Seillinie." Zeitschr D. Architekt. u. Ing. Vereins Zu, Hannover, Germany (1868).
- (2). Müller-Breslau, H. F. B. "Bietrag Zur Theorie Des Fachwerks." Zeitschr D. Architekt. u. Ing., Hannover, Germany (1885).
- (3). Müller-Breslau, H. F. B. Die Graphische Statik Der Baukonstruktionen, Vol. II, Part 2, 2nd Ed., Leipzig (1925), pp. 337-365.
- (4). Wanke, J. Zür Berechnung Der Formänderungen Vollwandiger Tragwerke. Der Stahlbau (1939), No. 23, 24.
- (5). Chmelka, F. "Näherung Formein." Der Stahlbau (1940), No. 23, 24.
- (6). Kaufmann, W. Statik Der Tragwerke. 4th Ed. (1957), Berlin, pp. 144-153.
- (7). Michalos, J. Theory of Structural Analysis and Design (1958), pp. 20-37.
- (8). Cross, Hardy and N. D. Morgan. Continuous Frames of Reinforced Concrete (1945), pp. 26-76.
- (9). Tuma, J. J. "Carry-Over Procedures Applied to Civil Engineering Problems." Lecture Notes, C. E. 620 - Seminar, Oklahoma State University, Stillwater, Oklahoma (Spring, 1959).
- (10). Chu, S. L. "Beam Constants by the String Polygon Method." M. S. Thesis, Oklahoma State University, Stillwater, Oklahoma (Summer, 1959).
- (11). Oden, J. T. "The Analysis of Fixed-end Frames With Bent Members by the String Polygon Method." M. S. Thesis, Oklahoma State University, Stillwater, Oklahoma (Summer, 1960).

- (12). Boecker, H. C. "Analysis of Pinned-end Frames With Bent Member by the String Polygon Method."
M. S. Thesis, Oklahoma State University,
Stillwater, Oklahoma (Summer, 1960).
- (13). Guldan, R. Die Cross-Methode, Springer Verlag Wien
(1955), pp. 131-138.
- (14). Tuma, J. J., T. Lassley and S. French. "Analysis
of Continuous Beam Bridges, Vol. I, Carry-Over
Procedure." School of Civil Engineering
Research Publication, Oklahoma State University,
Stillwater, Oklahoma (No. 3, 1959).

VITA

Chien Min Wu

Candidate for the Degree of
Master of Science

Thesis: THE GENERAL STRING POLYGON METHOD

Major Field: Civil Engineering

Biographical:

Personal Data: Born in Malang, Indonesia, May 11, 1924, the son of Pek Tsing Wu and Yung Song Wang of China.

Education: Graduated from Tien Nan High School, Kuming Yunnan, China in July, 1941; received the degree of Bachelor of Science in Civil Engineering from National Southwest Associated University (associate of National Peking University, National Tising Hua University and Nankan University during the World War II, China) Kuming Yunnan, China, July, 1945; did graduate study in Vienna Technical University, Vienna, Austria, 1958; completed the requirements for the Master of Science Degree in May, 1961.

Professional experience: Assistant Engineer, Tien Tsin-Pukow Railway Administration, China, 1946-1948; high school teacher, Indonesia, 1948-1955; lecturer, Gamalial University, Indonesia, 1955-1957; engineer, Mayred and Company, Linz, Austria, 1958.