TRANSLATIONAL SHELLS BY

FINITE DIFFERENCES

By

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1953

Submitted to the faculty of the Graduate School of
the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
MASTER OF SCIENCE
May, 1961

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PREFACE

The analysis of translational shells in the extensional state, using finite difference equations to approximate the derivatives, is presented in this thesis. The partial differential equation in finite difference form is developed in terms of both the stress function and internal forces. The system of simultaneous difference equations is solved by the combined use of the Algebraic Carry-Over method and Relaxation. In the Algebraic Carry-Over solution, the unknowns have been eliminated by successively deleting points in the network and evaluating modified carry-over factors between retained points. Numerical examples are included showing the application of the theory presented.

I wish to express my appreciation to Dr. K. S. Havner for his assistance, guidance, and interest in the preparation of this thesis.

To my wife, Cecile, my sons, Jeff and Ronnie, and to my parents, Mr. and Mrs. Iver Tilden, I wish to express my gratitude and indebtedness for many things.

Thanks also go to Mrs. June Daniel for her careful typing of the manuscript.

D. I. Tilden

D. I. Tilden

Stillwater, Okla.
August 1, 1960

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NOMENCLATURE

x, y, z, ...	Coordinates of a point on the middle surface of the shell.
ψ, ϕ	The angles between the middle surface of the shell and the projected plane when measured along the X and Y axes, respectively.
h_x, h_y	Total rise of the shell on the X and Y axes, respectively.
L_x, L_y	One-half the total length of the shell in the X and Y directions, respectively.
N_x, N_y, N_{xy}	Normal and shearing forces on an element of the shell.
$\overline{N}_x$, $\overline{N}_y$, $\overline{N}_{xy}$	Projected normal and shearing forces in the X-Y plane.
F	Airy's stress function.
a, b, a_1, b_1	Original carry-over factors.
$A', B' \cdots a', b' \cdots$ . .	Carry-over factors modified once.
$A'', B'', C'', a'', b'', c''$, .	Carry-over factors modified twice.
A''', a'''.	Final carry-over factors.
$\overline{N}y^*$.	Starting value representing internal force in the Y direction at a point.
$\lambda, \lambda^*, \lambda^{**}, \lambda^{***}$. .	Carried-over starting value at a point.

p . . . . Load intensity.

Δ_x, Δ_y. . . . . . Interval of the finite difference network in the X and Y directions, respectively.

r_x, r_y . . . . Radius of curvature of the shell in the X and Y directions, respectively.

t . . . . Thickness of shell.

σ_1, σ_2. . . . . . Major and minor principal stresses.

f_c . . . . Compressive stress in concrete.

A_s . . . . Area of reinforcing steel.

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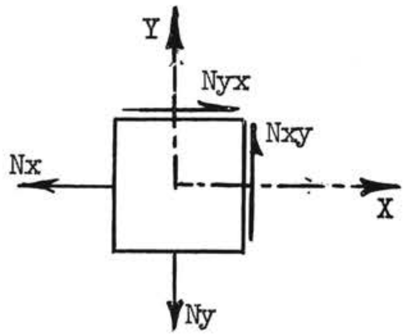


Normal forces, Nx and Ny, are considered positive when they are tensile.

Shearing forces, Nxy and Nyx, are considered positive when they create tension in the diagonal direction of increasing values of x and y.

CHAPTER I

INTRODUCTION

1-1. Historical Study.

In 1931, Pucher, in his dissertation, presented the extensional solution for shells of double curvature. In this solution he made use of projected forces and Airy's stress function in solving the partial differential equations (1). He also presented a series solution for stresses in an elliptical paraboloid.

Flügge (2), in 1950, discussed the analysis of translational shells by finite differences, using relaxation to solve the simultaneous equations. Arup and Jenkins (3) made use of the stress function, finite differences, and relaxation in the analysis of a circular translational shell constructed in 1950. They evaluated the tension in the edge beams by numerical integration of normal forces across the shell. Flügge and Geyling (4) confirmed the validity of the extensional solution for elliptical paraboloids.

In 1956, Salvadori (5), in analyzing a circular translational shell, used coarse finite difference networks in evaluating the stress function with final values determined by extrapolation. In 1957, Parme (6) gave a detailed account of the analysis of the elliptical paraboloid. He presented numerical tables, prepared using a trigonometric series solution of the partial differential equation, for determination of shell stresses.

The solution of the finite difference equations by infinite geometric series for the two dimensional, second order problem was presented by Tuma, Havner, and French (7) in 1958. The idea of extending the Algebraic Carry-Over method to the solution of translational shells was proposed by Havner (8) in shell lectures delivered in 1959. The method of successive elimination of points for the solution of differential equations in finite difference form was applied to circular plate problems by Havner (9) in 1960.

1-2. Membrane Equations of Equilibrium.

An element of a shell, projected into the X-Y plane is shown in Figure 1-1. To simplify the equations of equilibrium, internal forces on the shell element have been transferred to the projected element. Internal forces are shown in the positive sense.

The shell is considered to be in the extensional (membrane) state, thus, the forces on the element are membrane forces and flexural action is considered negligible. Also, stresses arising from deflection of the shell are ignored.

The relationships between the projected internal forces and actual internal forces may be determined by geometry. The final equations are

$$\left.\begin{aligned} \overline{N}y &= Ny\,\frac{\cos\phi}{\cos\psi} \\ \overline{N}x &= Nx\,\frac{\cos\psi}{\cos\phi} \\ \overline{N}xy &= Nxy. \end{aligned}\right\} \qquad (1\text{-}1)$$

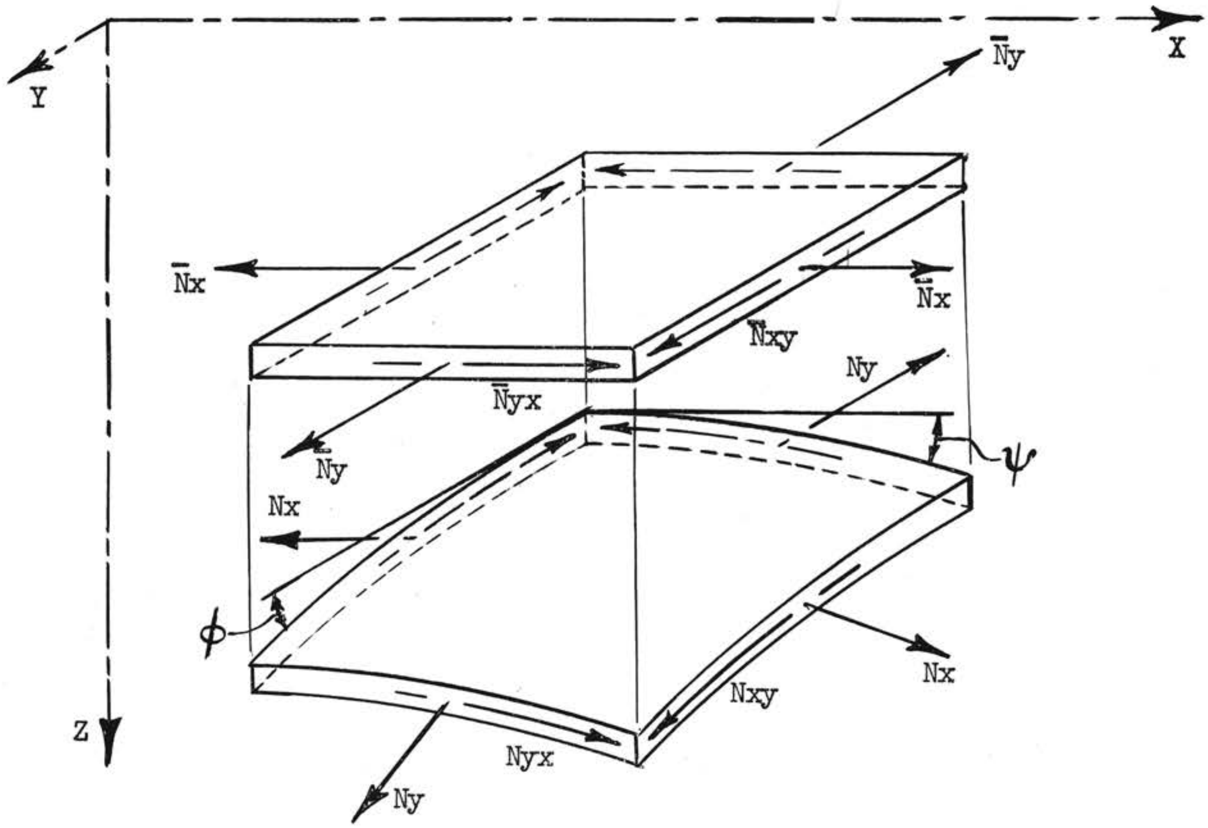


Figure 1-1

Element of Shell Projected in X-Y Plane

The development of the equations of equilibrium appears in many references and is not repeated here. The final three equations considering vertical load only acting on the shell are

$$\frac{\partial \bar{N}x}{\partial x} + \frac{\partial \bar{N}yx}{\partial y} = 0. \tag{1-2}$$

$$\frac{\partial \bar{N}y}{\partial y} + \frac{\partial \bar{N}xy}{\partial x} = 0. \tag{1-3}$$

$$\bar{N}x \frac{\partial^2 z}{\partial x^2} + \bar{N}y \frac{\partial^2 z}{\partial y^2} + 2\,\bar{N}xy \frac{\partial^2 z}{\partial x\,\partial y} = -\,p. \tag{1-4}$$

where p is a function of x and y only.

The equilibrium equations contain three dependent variables, $\bar{N}x$, $\bar{N}y$, $\bar{N}xy$, each depending on two independent variables, x and y. To simplify the solution of the internal forces, Pucher (1) made use of the stress function in which the internal forces are described as follows:

$$\left.\begin{aligned} \bar{N}y &= \frac{\partial^2 F}{\partial x^2} \\ \bar{N}x &= \frac{\partial^2 F}{\partial y^2} \\ -\bar{N}xy &= \frac{\partial^2 F}{\partial x\,\partial y}\,. \end{aligned}\right\} \tag{1-5}$$

Substituting Equation (1-5) into Equation (1-4) yields

$$\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 F}{\partial x\,\partial y}\frac{\partial^2 z}{\partial x\,\partial y} = -p. \tag{1-6}$$

Equation (1-6) is the governing differential equation in terms of the Stress Function.

1-3. Shells of Translation.

The surface of a translational shell is generated by moving one curve along and at right angles to another curve as shown in Figure 1-2.

The equation of the surface is given by

$$z = f_1(x) + f_2(y). \tag{1-7}$$

Differentiating Equation (1-7) twice with respect to x yields

$$\frac{\partial^2 z}{\partial x^2} = f_1''(x). \tag{1-8}$$

Similarly, with respect to y

$$\frac{\partial^2 z}{\partial y^2} = f_2''(y). \tag{1-9}$$

The mixed derivitive becomes

$$\frac{\partial^2 z}{\partial x \; \partial y} = 0. \qquad (1\text{-}10)$$

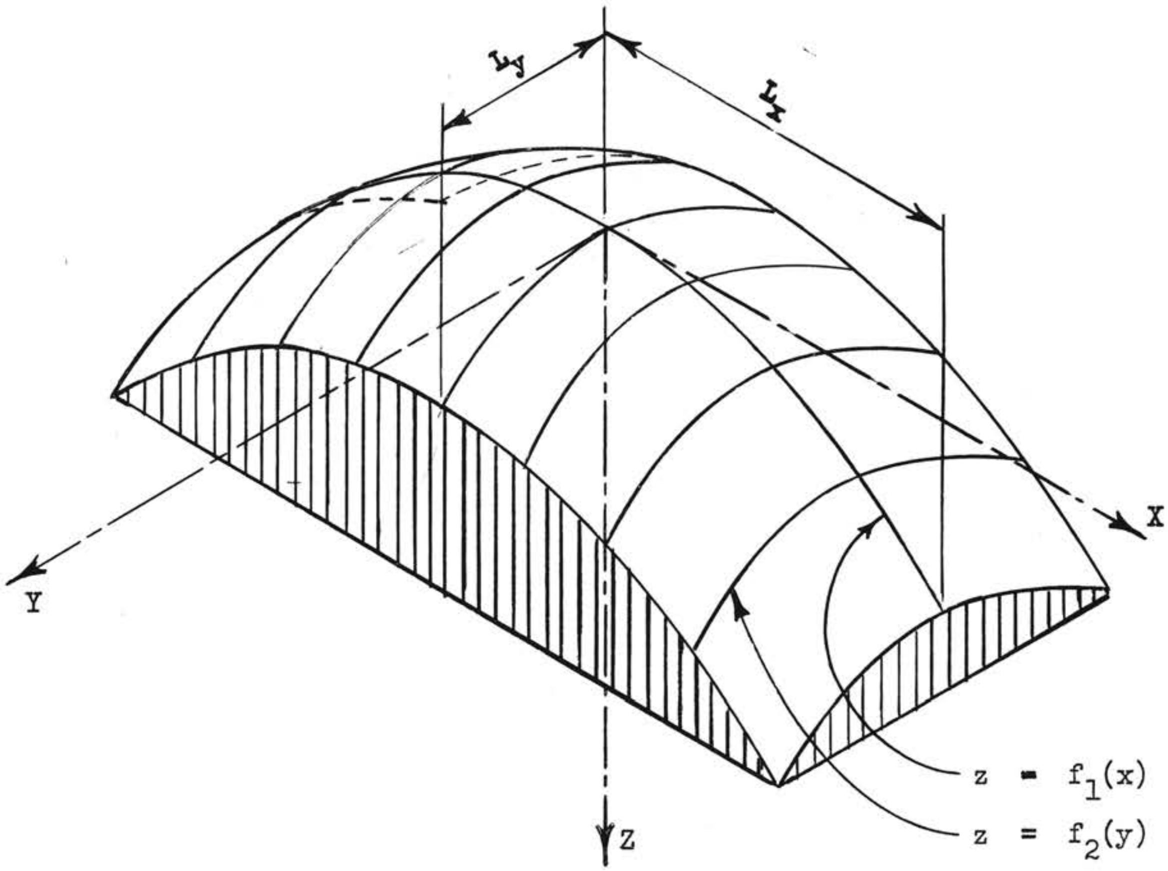


Figure 1-2

Shell of Translation

Equation (1-10) shows that the twist of the undeformed surface is zero, which is typical of translational shells.

<u>1-4. Boundary Conditions.</u>

The shell is supported by edge beams that are considered incapable of vertical deflection, and incapable of resisting lateral forces. Therefore, at the boundary

$$x = \pm L_x, \quad \bar{N}x = 0$$

and at the boundary

$$y = \pm L_y, \quad \bar{N}y = 0.$$

The translational shell is incapable of resisting load by shearing forces only, being that the twist of the undeformed surface is zero. At the corners of the shell an ambiguity arises as the direct forces are zero by the boundary conditions, and the shearing force, although incapable, must provide resistance to the load. From a theoretical standpoint, the shear becomes infinite in magnitude at the corners. From a practical standpoint, normal forces and bending moments will occur in the region near the corner, and all values will be finite.

CHAPTER II

FINITE DIFFERENCE EQUATIONS

2-1. Introduction.

For most shells of double curvature, an algebraic solution of the differential equation is quite intractable, and one must resort to approximate methods, such as finite differences. Even in the case of the elliptical paraboloid, which is amenable to an algebraic series solution, there are certain advantages in using finite differences by virtue of the simplicity of computation. The finite difference equations necessary for the solution of the differential equation of any translational shell are developed in this chapter.

2-2. The Finite Difference Equations in Terms of the Stress Function.

The finite difference approximation of $\frac{\partial^2 F}{\partial x^2}$ at the point, i,j (Figure 2-1) is

$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{\Delta x^2} (F_{i+1,j} - 2F_{i,j} + F_{i-1,j}). \tag{2-1}$$

Similarly in the Y direction

$$\frac{\partial^2 F}{\partial y^2} = \frac{1}{\Delta y^2} (F_{i,j+1} - 2F_{i,j} + F_{i,j-1}). \tag{2-2}$$

The mixed derivative of the Stress Function becomes

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{1}{4 \Delta x \Delta y} (F_{i+1,j+1} - F_{i-1,j+1} - F_{i+1,j-1} - F_{i-1,j-1}). \tag{2-3}$$

Substituting Equations (2-1, 2, 3) into Equation (1-6) and employing the relationships expressed in Equations (1-8, 9, 10) results in

$$\frac{f_1''(x)}{\Delta y^2}(F_{i,j+1} - 2F_{i,j} + F_{i,j-1}) + \frac{f_2''(y)}{\Delta x^2}(F_{i+1,j} - 2F_{i,j} + F_{i-1,j}) = -p \,. \qquad (2\text{-}4)$$

In a form suitable for iteration, Equation (2-4) becomes

$$F_{i,j} = \left[\begin{matrix} a_1(F_{i+1,j} + F_{i-1,j}) \\ b_1(F_{i,j+1} + F_{i,j-1}) \end{matrix}\right\} + F_{i,j}^{*} \qquad (2\text{-}5)$$

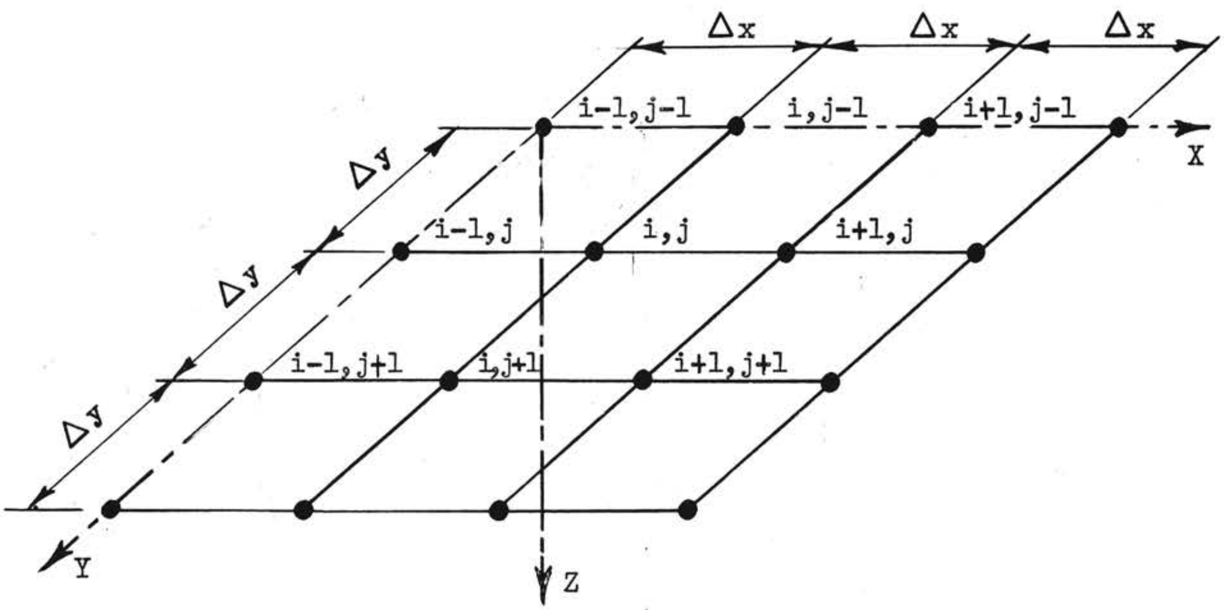


Figure 2-1

Finite Difference Network

where

$$a_1 = \frac{1}{2\left(1 + \frac{\Delta x^2 f_1''(x)}{\Delta y^2 f_2''(y)}\right)}$$

$$b_1 = \frac{\frac{\Delta x^2 f_1''(x)}{\Delta y^2 f_2''(y)}}{2\left(1 + \frac{\Delta x^2 f_1''(x)}{\Delta y^2 f_2''(y)}\right)}$$

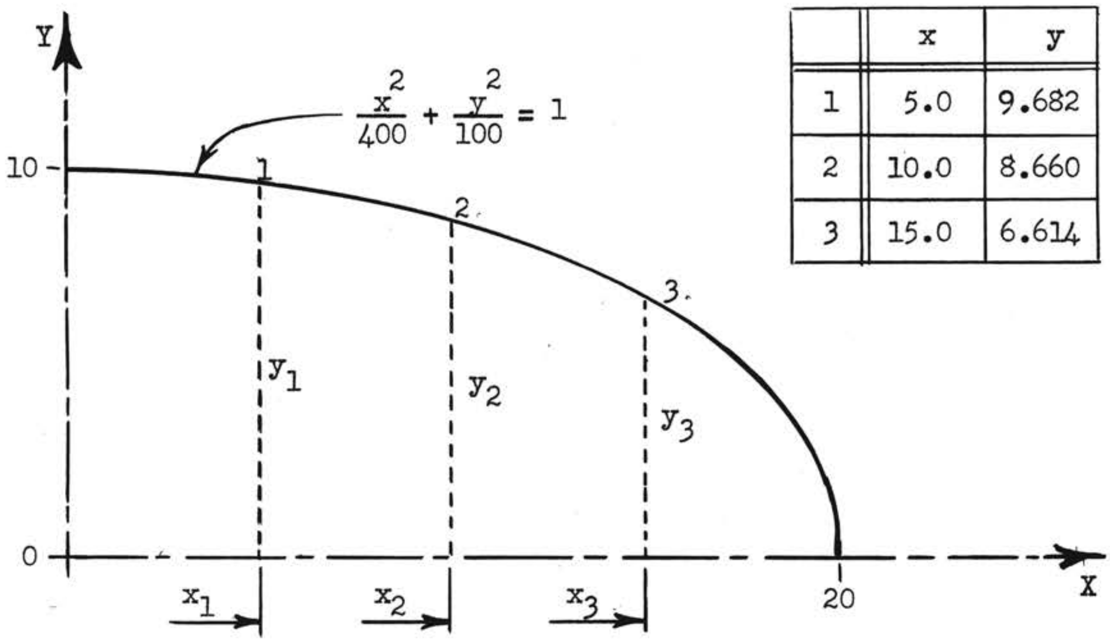
$$F^*_{i,j} = \frac{p\,\Delta x^2}{2f_2''(y)\left(1 + \frac{\Delta x^2 f_1''(x)}{\Delta y^2 f_2''(y)}\right)} \quad .$$

After obtaining the solution for the Stress Function at each pivotal point in the network through a system of similtaneous equations of the type (2-5), the stresses may be computed as follows:

Substitututing Equation (1-5) into Equations (2-1, 2, 3) results in

$$\left.\begin{aligned} \bar{N}y_{i,j} &= \frac{1}{\Delta x^2}\left(F_{i+1,j} - 2F_{i,j} + F_{i-1,j}\right) \\ \bar{N}x_{i,j} &= \frac{1}{\Delta y^2}\left(F_{i,j+1} - 2F_{i,j} + F_{i,j-1}\right) \\ \bar{N}xy_{i,j} &= \frac{1}{4\Delta x\Delta y}\left(F_{i+1,j+1} - F_{i-1,j+1} - F_{i+1,j-1} + F_{i-1,j-1}\right. \end{aligned}\right\} \quad (2\text{-}6)$$

Solution of the governing differential equation by the use of the Stress Function is subject to criticism. This is because the internal forces are equal to the second difference in "F", and a small error in "F" can result in a large error in the internal force. This is clearly shown in the following coplanar problem.



	x	y
1	5.0	9.682
2	10.0	8.660
3	15.0	6.614

Figure 2-2

Ellipse in X-Y Plane

Figure 2-2 shows a curve following the equation of an ellipse. Values of x and y satisfying the equation are shown in the inset on the figure.

The finite difference approximation of $\frac{\partial^2 y}{\partial x^2}$ at the point 2 is

$$-\frac{\partial^2 y_2}{\partial {x_2}^2} = \frac{1}{\Delta x^2}(y_1 - 2y_2 + y_3).$$

Noting that $\Delta x = 5$ and substituting the appropriate values of y,

$$-\frac{\partial^2 y_2}{\partial {x_2}^2} = \frac{1}{25}(9.682 - 17.32 + 6.614) = 0.041.$$

Considering y_2 to be in error by 5 per cent the new value is taken as 9.093, and

$$-\frac{\partial^2 y_2}{\partial {x_2}^2} = \frac{1}{25}(9.682 - 18.186 + 6.614) = 0.076.$$

When the error in y_2 is 5 per cent, the second difference of y_2 is in error 85 per cent. Consequently, extreme accuracy is necessary in evaluation of "F". To avoid this, it is usually advantageous to work directly with the internal forces in finite difference form.

2-3. The Finite Difference Equations in Terms of Internal Forces.

The third equilibrium equation for shells of translation may be written

$$\overline{N}x\, f_1''(x) + \overline{N}y\, f_2''(y) = -p. \tag{2-7}$$

Dividing Equation (2-7) by $f_1''(x)$

$$\overline{N}x + \overline{N}y \frac{f_2''(y)}{f_1''(x)} = -\frac{p}{f_1''(x)}. \tag{2-8}$$

Differentiating Equation (2-8) twice with respect to x yields

$$\frac{\partial^2 \overline{N}x}{\partial x^2} + \frac{\partial^2 \overline{N}y}{\partial x^2}\left(\frac{f_2''(y)}{f_1''(x)}\right) + 2\frac{\partial \overline{N}y}{\partial x}\cdot\left[\frac{\partial}{\partial x}\left(\frac{f_2''(y)}{f_1''(x)}\right)\right] + \overline{N}y\frac{\partial}{\partial x^2}\left(\frac{f_2''(y)}{f_1''(x)}\right)$$

$$= -\frac{\partial^2}{\partial x^2}\left(\frac{p}{f_1''(x)}\right). \tag{2-9}$$

Differentiating Equation (1-2) with respect to x results in

$$\frac{\partial^2 \overline{N}x}{\partial x^2} + \frac{\partial^2 \overline{N}xy}{\partial x\, \partial y} = 0. \tag{2-10}$$

Differentiating Equation (1-3) with respect to y yields

$$\frac{\partial^2 \overline{N}y}{\partial y^2} + \frac{\partial^2 \overline{N}xy}{\partial x\, \partial y} = 0. \tag{2-11}$$

From Equations (2-10) and (2-11)

$$\frac{\partial^2 \bar{N}x}{\partial x^2} = \frac{\partial^2 \bar{N}y}{\partial y^2} \quad . \tag{2-12}$$

Substituting Equation (2-12) into Equation (2-9) gives

$$\frac{\partial^2 \bar{N}y}{\partial y^2} + \frac{\partial^2 \bar{N}y}{\partial x^2}\left(\frac{f_2''(y)}{f_1''(x)}\right) + 2\,\frac{\partial \bar{N}y}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{f_2''(y)}{f_1''(x)}\right)\right] + \bar{N}y\,\frac{\partial^2}{\partial x^2}\left(\frac{f_2''(y)}{f_1''(x)}\right)$$

$$= -\frac{\partial^2}{\partial x^2}\left(\frac{p}{f_1''(x)}\right) \quad . \tag{2-13}$$

Equation (2-13) is the governing differential equation in terms of the internal force, $\bar{N}y$.

The finite difference approximation of $\frac{\partial^2 \bar{N}y}{\partial y^2}$ at the point i,j (Figure 2-1), is

$$\frac{\partial^2 \bar{N}y}{\partial y^2} = \frac{N_{i,j+1} - 2N_{i,j} + N_{i,j-1}}{\Delta y^2} \tag{2-14}$$

where for convenience the symbol $\bar{N}y$ has been replaced by the symbol, N.

Similarly in the X direction,

$$\frac{\partial^2 \bar{N}y}{\partial x^2} = \frac{N_{i+1,j} - 2N_{i,j} + N_{i-1,j}}{\Delta x^2} \quad . \tag{2-15}$$

The finite difference approximation of $\frac{\partial \bar{N}y}{\partial x}$ at the point, i,j, is

$$\frac{\partial \bar{N}y}{\partial x} = \frac{N_{i+1,j} - N_{i-1,j}}{2\,\Delta x} \quad . \tag{2-16}$$

Substituting Equations (2-14,15,16) into Equation (2-13) results in,

$$\frac{N_{i,j-1} - 2N_{i,j} + N_{i,j-1}}{\Delta y^2} \quad + \quad \frac{f_2''(y)}{f_1''(x)}\left(\frac{N_{i+1,j} - 2N_{i,j} + N_{i-1,j}}{\Delta x^2}\right)$$

$$+ \frac{\partial}{\partial x}\left(\frac{f_2''(y)}{f_1''(x)}\right)\left(\frac{N_{i+1,j} - N_{i-1,j}}{\Delta x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{f_2''(y)}{f_1''(x)}\right)(N_{i,j})$$

$$= - \frac{\partial^2}{\partial x^2}\left(\frac{p}{f_1''(x)}\right)_{ij} . \quad (2\text{-}17)$$

In a form suitable for iteration Equation (2-17) becomes

$$N_{i,j} = \left\{\begin{matrix} a_{i+1,ij}\, N_{i+1,j} - a_{i-1,ij} N_{i-1,j} \\ b_{ij}\, (N_{i,j+1} + N_{i,j-1}) \end{matrix}\right\} + N_{ij}^{*} . \quad (2\text{-}18)$$

where

$$a_{i+1,ij} = \frac{1}{D_{i,j}}\left[\left(\frac{f_2''(y)}{f_1''(x)}\right)_{i,j} + \frac{\partial}{\partial x}\left(\frac{f_2''(y)}{f_1''(x)}\right)_{i,j} \Delta x\right]$$

$$a_{i-1,ij} = \frac{1}{D_{i,j}}\left[\left(\frac{f_2''(y)}{f_1''(x)}\right)_{i,j} - \frac{\partial}{\partial x}\left(\frac{f_2''(y)}{f_1''(x)}\right)_{i,j} \Delta x\right]$$

$$b_{i,j} = \frac{\Delta x^2}{\Delta y^2 D_{i,j}}$$

$$N_{i,j}^{*} = \frac{\Delta x^2}{D_{i,j}}\left[\frac{\partial^2}{\partial x^2}\left(\frac{p}{f_1''(x)}\right)_{i,j}\right]$$

$$D_{i,j} = 2\,\frac{\Delta x^2}{\Delta y^2} + 2\left(\frac{f_2''(y)}{f_1''(x)}\right)_{i,j} - \frac{\partial^2}{\partial x^2}\left(\frac{f_2''(y)}{f_1''(x)}\right) \Delta x^2 .$$

The value of the internal force in the Y direction may now be computed at each pivotal point through a system of simultaneous equations of the type (2-18). Knowing $\overline{N}y$, the internal force in the X direction may be obtained by direct substitution into Equation (2-8). The internal shearing force, $\bar{N}xy$, however, cannot be obtained directly but may be determined as follows:

Considering the case of a symmetrically loaded, symmetrical translational shell, it is evident that the shearing force, $\bar{N}xy$, equals zero on the axis of symmetry.

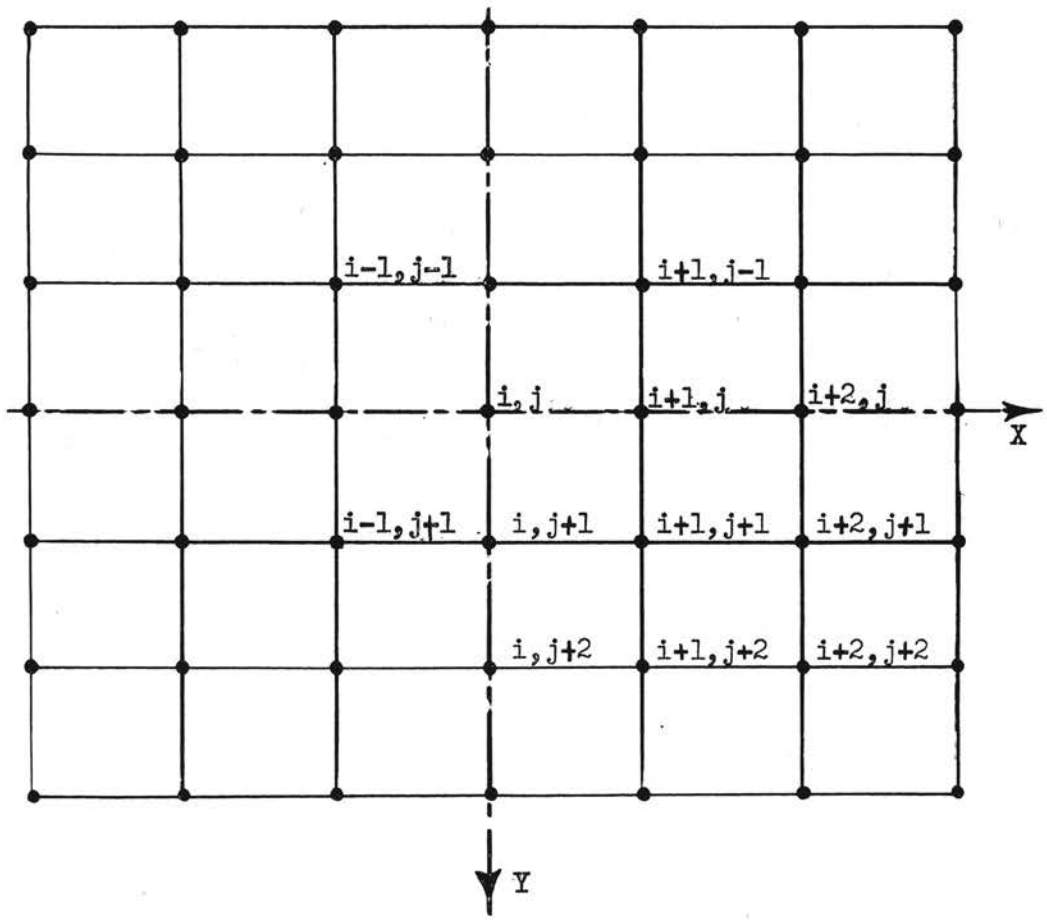


Figure 2-3

Symmetrical Finite

Difference Network

Equation (1-3) is now written in finite difference form for the point, i+1, j+1.

$$\frac{(\bar{N}y)_{i+1,j} - (\bar{N}y)_{i+1,j+2}}{2\,\Delta y} = \frac{(\bar{N}xy)_{i+2,j+1} - (\bar{N}xy)_{i,j+1}}{2\,\Delta x} \,. \tag{2-19}$$

Considering the values of $\bar{N}y$ known, the value of $\bar{N}xy$ at i+2, j+1 can be computed from Equation (2-19) as $\bar{N}xy$ at i, j+1 is equal to zero.

Equation (1-3) is now written in finite difference form for the point i, j+1 as follows

$$-\frac{(\bar{N}y)_{i,j+2} - (\bar{N}y)_{ij}}{2\,\Delta y} = \frac{(\bar{N}xy)_{i+1,j+1} - (\bar{N}xy)_{i-1,j+1}}{2\,\Delta x} \,. \tag{2-20}$$

The left-hand side of the above equation is known and the right-hand side may be reduced to one unknown by symmetry, that is

$$(\bar{N}xy)_{i+1,j+1} = -(\bar{N}xy)_{i-1,j+1} \,. \tag{2-21}$$

In a similar manner values of $\bar{N}xy$ at all points in the shell may be determined.

When the shell is subjected to an unsymmetrical load, the shearing force, $\bar{N}xy$, is no longer zero on the axis of symmetry. There is no starting point on the shell that can be used for the elimination of unknowns. Sufficient equations of the type (2-19) are available to obtain a solution if forward or backward differences are used. However, from an accuracy standpoint, it is deemed advisable to evaluate $\bar{N}xy$ for the unsymmetrical case by use of the Stress Function.

CHAPTER III

THE SOLUTION OF THE FINITE DIFFERENCE EQUATIONS

3-1. Methods of Solution.

Many methods are available for the solution of the finite difference network. However, all methods fall into just two categories:

(a) Approximate Solutions

(b) Exact Solutions.

All solutions of finite difference equations are approximate in that the accuracy of the solution is dependent on how well the derivative of the function is represented by a finite difference equation. However, in referring to a method as an "Approximate Solution," it is intended that this be considered a solution gained through a process of iteration or successive approximation. The accuracy of "Approximate Solutions" is dependent upon the number of cycles of iteration in the procedure. In contrast, "Exact Solutions" yield results that are dependent only on the interval of the finite difference network chosen.

Of the "Approximate Solutions" available, the Relaxation method has the greatest application to shell analysis. This method has the advantage of simplicity; but to use the method effectively, a technique must be acquired that can be gained only through considerable experience working relaxation networks. A significant reduction in labor can be realized if a reasonably accurate estimate of initial values is possible.

"Exact Solutions" are only applicable to systems having a limited number of unknowns. Therefore, it is necessary to use a coarse grid. Of the "Exact Solutions" available, the Algebraic Carry-Over method has excellent application to finite difference networks. This method will be fully described in Part 3 of this chapter.

All methods of solution require considerable labor when a fine lattice is used. As the interval (Δx, Δy) is halved, the number of simultaneous equations is increased by four. Therefore, it is important to use the largest network interval that is practical.

It will be shown in the numerical example in Chapter IV that, for translational shells, a coarse network analyzed by the Algebraic Carry-Over method yields excellent results for the direct internal forces. However, being that the direct forces are discontinuous on the boundary at the corner, the shearing forces are subject to appreciable error. To reduce this error, a finer network is necessary and solution by relaxation becomes applicable. The relaxation solution, however, is very short if the initial values are based on the final values gained through the Algebraic Carry-Over procedure.

3-2. Reduction of the Finite Difference Network by Symmetry and Boundary Conditions.

A. The Complete Network. A forty-nine point network (Fig. 3-1) has been chosen for analysis. The boundaries of the network are coincident with the boundaries of the translational shell. Carry-over factors conforming to Equation (2-18) are shown in Figure 3-1.

B. Reduction of the Network by the Condition of Symmetry. Considering the case of the symmetrically loaded, symmetrical translational shell, the carry-over factors and internal forces are symmetrical to both the x and y axis. By taking advantage of symmetry, the network can be

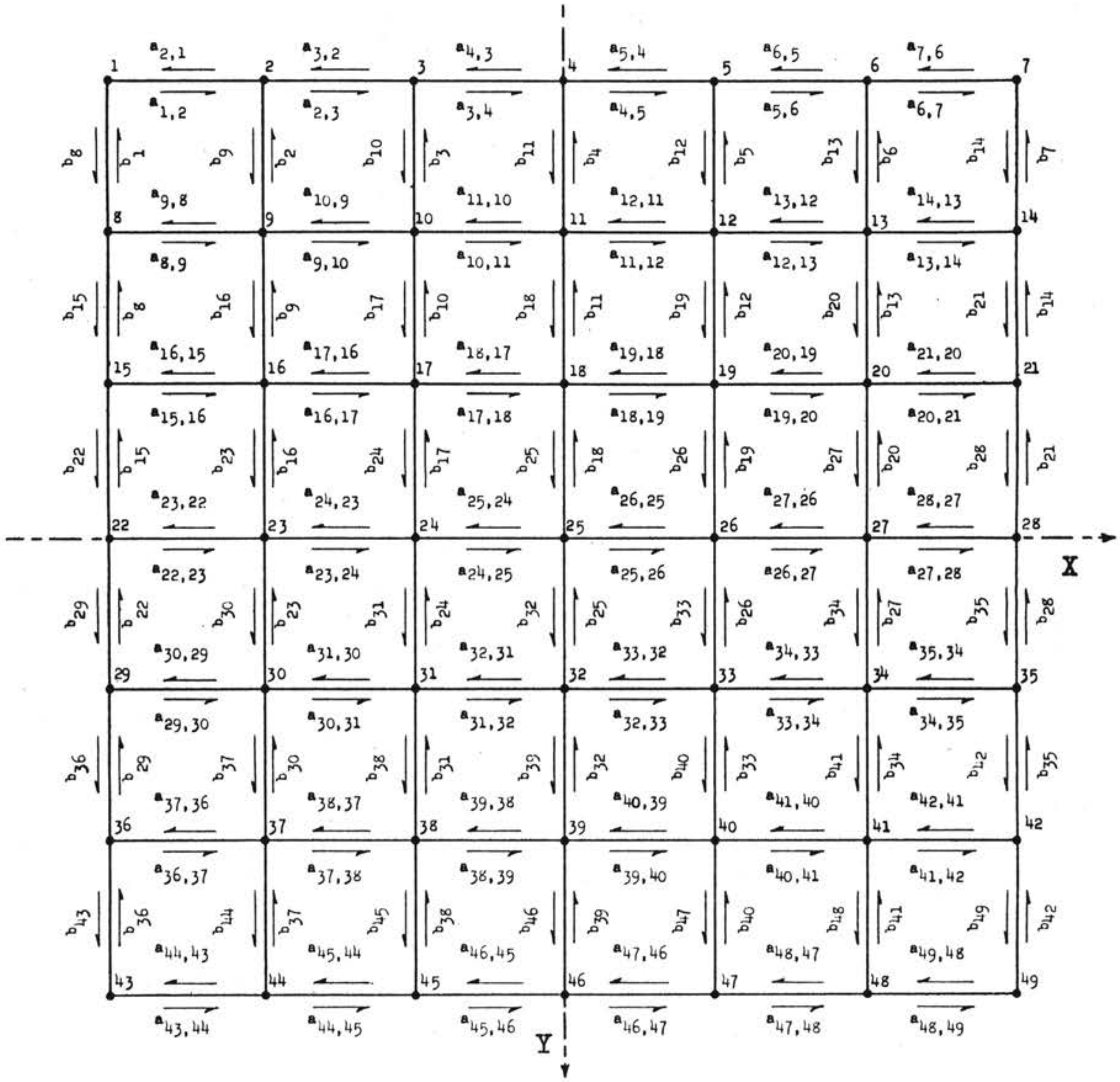


Figure 3-1

Forty-Nine Point Network

reduced from 49 unknowns to 16 unknowns or pivotal points as shown in Figure 3-2. Carry-over factors that contribute to final values on the axis of symmetry must be modified when the network reduction would otherwise effect the final values at such points. As an example, the point 32 (Figure 3-1) receives contributions of $a_{31,\ 32}\overline{N}y_{31}$ from point 31, and $a_{33,\ 32}\overline{N}y_{33}$ from point 33. From symmetry

$$\begin{aligned} a_{33,\ 32} &= a_{31,\ 32} \\ \overline{N}y_{33} &= \overline{N}y_{31} \end{aligned} \qquad (3\text{-}1)$$

Therefore, it is permissible to let the contribution to point 32 equal the modified carry-over value $2(a_{33,\ 32}\overline{N}y_{33})$. Figure 3-2 shows the reduced network with the required modified carry-over factors.

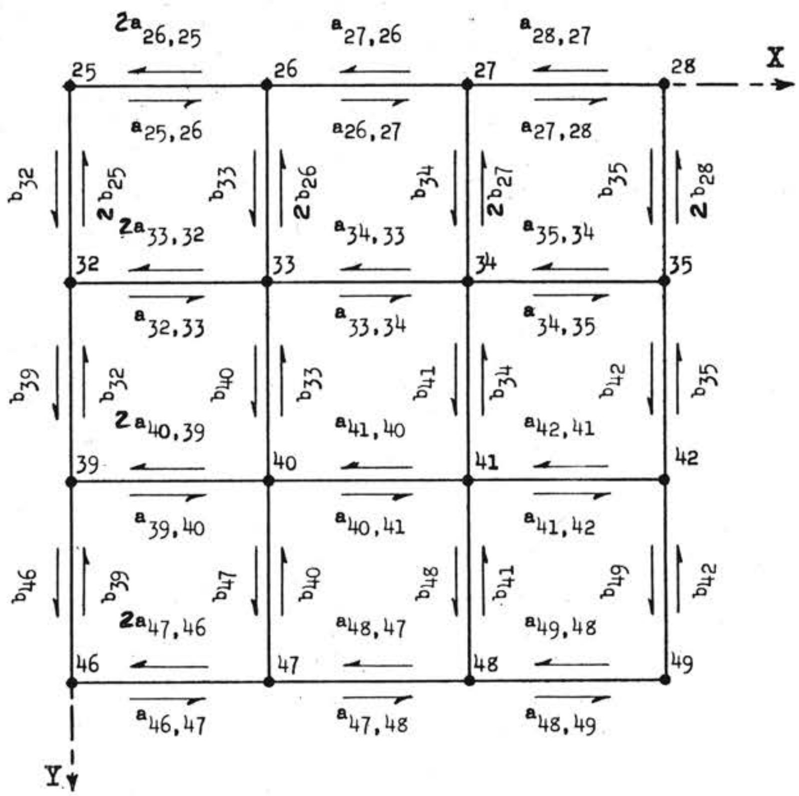


Figure 3-2

Sixteen-Point Network

C. Reduction of the Network from Boundary Conditions. Boundary conditions dictate that $\bar{N}y$ equals zero at points 46, 47, 48, and 49. Therefore, carry-over factors to, from, and between these points must equal zero. From Equation (2-8), it is noted that $\bar{N}y$ is a prescribed value at points 28, 35, and 42:

$$\bar{N}y = -\frac{p}{f_2''(y)} \quad . \tag{2-8a}$$

Since the value is prescribed, iteration can in no way effect $\bar{N}y$ at points 28, 35, and 42. Therefore, it follows that all carry-over factors to these points must equal zero. That is

$$a_{27,\,28} = a_{34,\,35} = a_{41,\,42} = 2b_{28} = b_{35} = b_{42} = 0 \, . \tag{3-2}$$

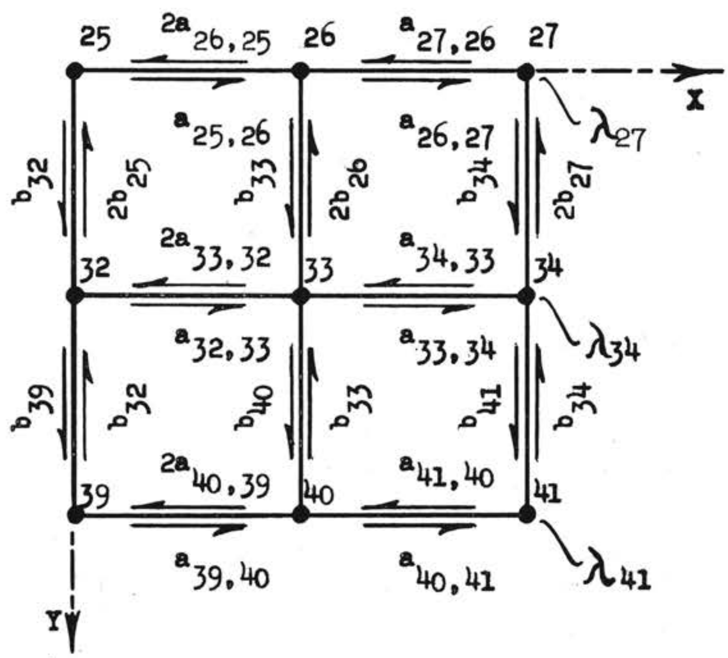


Figure 3-3

Nine-Point Network

From Equation (2-18), the expression for $\bar{N}y_{27}$ may be written

$$\bar{N}y_{27} = a_{28,\,27}\bar{N}y_{28} + a_{26,\,27}\bar{N}y_{26} + 2b_{27}\bar{N}y_{34} + \bar{N}y^{*}_{27} \qquad (2\text{-}18a)$$

where $a_{28,\,27}\bar{N}y_{28}$ is a known value similar in form to the starting value, $\bar{N}y^{*}_{27}$. Denoting the carried-over starting value as λ, the values of at points 27, 34, and 41 become

$$\left.\begin{aligned}\lambda_{27} &= a_{28,\,27}\bar{N}y_{28}\\ \lambda_{34} &= a_{35,\,34}\bar{N}y_{35}\\ \lambda_{41} &= a_{42,\,41}\bar{N}y_{42}\ .\end{aligned}\right. \qquad (3\text{-}3)$$

The sixteen-point network has been reduced to a nine-point network as shown in Figure 3-3.

3-3. Reduction of the Finite Difference Network by Removal of Points.

A. Reduction by Removal of Alternate Points. In Figure 3-3 all odd-numbered points may be removed from the network and the difference equations at these points may be incorporated into the difference equations at the retained points. This may be accomplished by either direct substitution or by a much shorter method. As a means of explanation and proof of the shorter method, the expression for the internal force in the Y direction at point 32 will be developed by substitution as follows.

The finite difference approximation of the force, $\bar{N}y_{32}$, is

$$\bar{N}y_{32} = 2(a_{33,\,32}\,\bar{N}y_{33}) + b_{32}(\bar{N}y_{25} + \bar{N}y_{39}) + \bar{N}y^{*}_{32}. \qquad (3\text{-}4)$$

The finite difference approximations of $\bar{N}y$ at the deleted points, 25, 33, and 39, are now written:

$$\left.\begin{aligned}\bar{N}y_{25} &= 2(a_{26,\,25}\bar{N}y_{26}) + 2(b_{25}\bar{N}y_{32}) + \bar{N}y^{*}_{25}\\ \bar{N}y_{33} &= a_{32,\,33}\bar{N}y_{32} + a_{34,\,33}\bar{N}y_{34} + b_{33}(\bar{N}y_{26} + \bar{N}y_{40}) + \bar{N}y^{*}_{33}\\ \bar{N}y_{39} &= 2(a_{40,\,39}\bar{N}y_{40}) + b_{39}\bar{N}y_{32} + \bar{N}y^{*}_{39}\ .\end{aligned}\right\} \qquad (3\text{-}5)$$

Substituting Equation (3-5) into (3-4), transposing, and arranging terms results in

$$\overline{N}y_{32} = A'\overline{N}y_{34} + C'\overline{N}y_{26} + D'\overline{N}y_{40} + \lambda^*_{32} \qquad (3\text{-}6)$$

where the modified carry-over factors are

$$A' = \frac{1}{X_{\overline{32}}} (2a_{33,\ 32}\, a_{34,\ 33})$$

$$C' = \frac{1}{X_{\overline{32}}} (2a_{33,\ 32}\, b_{33} + 2a_{26,\ 25}\, b_{32})$$

$$D' = \frac{1}{X_{\overline{32}}} (2a_{33,\ 32}\, b_{33} + 2a_{40,\ 39}\, b_{32})$$

$$\lambda^*_{32} = \frac{1}{X_{\overline{32}}} \left[2a_{33,\ 32}\overline{N}y^*_{33} + b_{32}(\overline{N}y^*_{25} + \overline{N}y^*_{39}) \right] + \overline{N}y^*_{32}$$

$$X_{\overline{32}} = 1 - 2a_{33,\ 32}\, a_{32,\ 33} - 2b_{25}\, b_{32} - b_{32}\, b_{39}.$$

The force, $\overline{N}y_{32}$, is now in terms of forces in the y direction at retained points in the network. Expressions for $\overline{N}y$ at the other retained points may be developed in a similar manner.

The shorter method, previously mentioned, takes advantage of carry-over, and algebraic series principles. The Equation (3-6) can be written by observation if the following rules are adopted.

1. The numerator of the modified carry-over factor is the product of the original carry-over factors along the path through the point removed or the sum of the products along the paths if two points are removed.

2. The denominator of the modified carry-over factor is the sum of the algebraic power series forming between the retained point and the adjacent deleted points in the network when the starting value (λ) at the retained point equals unity.

Rule No. 1 can be illustrated by investigating the modified carry-over factor between points 26 and 32.

The product of the original carry-over factors through point 25 is $2a_{26,\,25}b_{32}$. Similarily, through point 33 the product is $b_{33}2a_{33,\,32}$. The sum of the products is identically equal to the numerator of C' in Equation (3-6).

Rule No. 2 can be illustrated by assigning a starting value (λ) at point 32 and carrying this value over to the deleted points and back to point 32. This procedure results in

$$\lambda' = \lambda + \lambda\,(2a_{33,\,32}\,a_{32,\,33} + b_{32}\,2b_{25} + b_{32}\,b_{39}).$$

Letting the terms in parenthesis equal β and repeating the carry-over procedure n times results in a value at point 32 of

$$\lambda^{n} = \lambda + \lambda\left[\beta + \beta^{2} + \beta^{3} + \ldots\ldots\beta^{n}\right] .$$

It is obvious that a power series has developed, and when n becomes infinite and when (λ) equals unity, the denominator, denoted $X_{\overline{32}}$, becomes

$$X_{\overline{32}} = 1 - 2a_{33,\,32}\,a_{32,\,33} - b_{32}\,2b_{25} - b_{32}\,b_{39}$$

which is identically equal to the value determined by substitution in Equation (3-6).

Expressions for $\bar{N}y$ at points 26, 34, and 40 obtained by this method are given as follows

$$\bar{N}y_{26} = c'\,\bar{N}y_{32} + B'\,\bar{N}y_{40} + E'\,\bar{N}y_{34} + \lambda^{*}_{26} \tag{3-7}$$

$$\bar{N}y_{34} = e'\,\bar{N}y_{26} + a'\,\bar{N}y_{32} + g'\,\bar{N}y_{40} + \lambda^{*}_{34} \tag{3-8}$$

$$\bar{N}y_{40} = b'\,\bar{N}y_{26} + d'\,\bar{N}y_{32} + G'\,\bar{N}y_{34} + \lambda^{*}_{40} \ . \tag{3-9}$$

where

$$a' = \frac{a_{32,\,33}\, a_{33,34}}{X_{\overline{34}}}$$

$$b' = \frac{b_{33}\, b_{40}}{X_{\overline{40}}}$$

$$c' = \frac{2b_{25}\, a_{25,\,26} + a_{32,\,33}\, 2b_{26}}{X_{\overline{26}}}$$

$$d' = \frac{b_{39}\, a_{39,\,40} + a_{32,\,33}\, b_{40}}{X_{\overline{40}}}$$

$$e' = \frac{a_{26,\,27}\, b_{34} + b_{33}\, a_{33,34}}{X_{\overline{34}}}$$

$$g' = \frac{a_{40,\,41}\, b_{34} + b_{33}\, a_{33,\,34}}{X_{\overline{34}}}$$

$$B' = \frac{b_{33}\, 2b_{26}}{X_{\overline{26}}}$$

$$E' = \frac{2b_{27}\, a_{27,\,26} + a_{34,\,33}\, 2b_{26}}{X_{\overline{26}}}$$

$$G' = \frac{a_{34,\,33}\, b_{40} + b_{41}\, a_{41,\,40}}{X_{\overline{40}}}$$

$$X_{\overline{26}} = 1 - 2a_{26,\,25}\, a_{25,\,26} - b_{33}\, 2b_{26} - a_{27,\,26}\, a_{26,\,27}$$

$$X_{\overline{34}} = 1 - b_{41}\, b_{34} - a_{34,\,33}\, a_{33,\,34} - b_{34}\, 2b_{27}$$

$$X_{\overline{40}} = 1 - b_{40}\, b_{33} - 2a_{40,\,39}\, a_{39,\,40} - a_{41,\,40}\, a_{40,\,41}$$

$$\lambda_{26}^{*} = \frac{1}{X_{\overline{26}}}\left[a_{25,\,26}\overline{N}y_{25}^{*} + 2b_{26}\overline{N}y_{33}^{*} + a_{27,\,26}\left(\overline{N}y_{27}^{*} + \lambda_{27}\right) + \overline{N}y_{26}^{*}\right]$$

$$\lambda_{34}^{*} = \frac{1}{X_{\overline{34}}}\left[b_{34}\left(\lambda_{27} + \overline{N}y_{27}^{*} + \lambda_{41} + \overline{N}y_{41}^{*}\right) + a_{33,\,34}\overline{N}y_{33}^{*} + \lambda_{34} + \overline{N}y_{34}^{*}\right]$$

$$\lambda_{40}^{*} = \frac{1}{X_{\overline{40}}}\left[a_{39,\,40}\overline{N}y_{39}^{*} + b_{40}\overline{N}y_{33}^{*} + a_{41,\,40}\left(\overline{N}y_{41}^{*} + \lambda_{41}\right) + \overline{N}y_{40}^{*}\right].$$

The network reduced by removal of alternate points is shown in Figure 3-4.

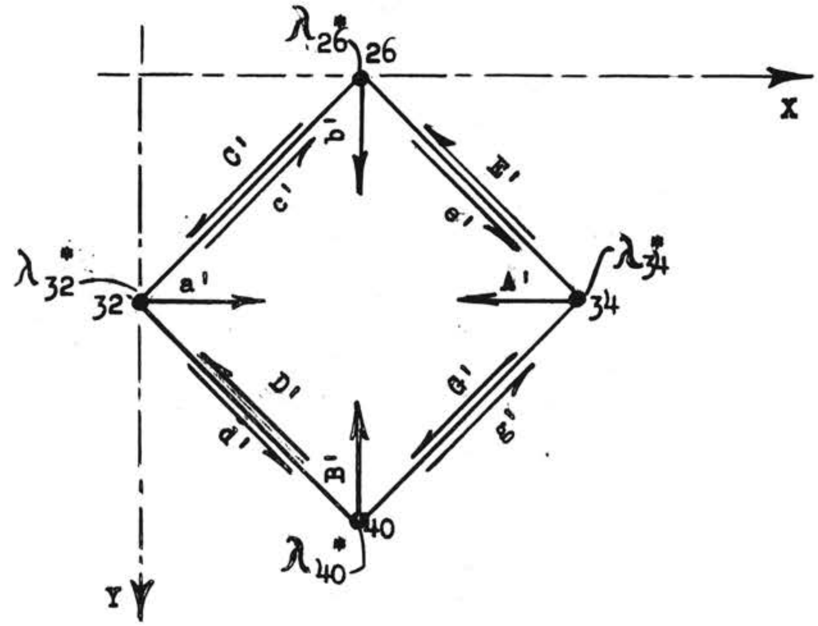


Figure 3-4

Four-Point Network

B. <u>Reduction by Removal of Sucessive Points.</u> A single point may be deleted in the network and equations may be written in terms of the remaining points in precisely the same manner as described for the removal of alternate points. In Figure 3-4, point 32 is deleted from the network. The expressions for $\bar{N}y$ at the remaining points follow

$$\bar{N}y_{26} = a'' \, \bar{N}y_{40} + C'' \, \bar{N}y_{34} + \lambda_{26}^{**} \tag{3-10}$$

$$\bar{N}y_{34} = c'' \, \bar{N}y_{26} + B'' \, \bar{N}y_{40} + \lambda_{34}^{**} \tag{3-11}$$

$$\bar{N}y_{40} = A'' \, \bar{N}y_{26} + b'' \, \bar{N}y_{34} + \lambda_{40}^{**} \tag{3-12}$$

where

$$a'' = \frac{B' + c'D'}{1 - c'C'} \qquad A'' = \frac{b' + d'C'}{1 - d'D'}$$

$$b'' = \frac{G' + d'A'}{1 - d'D'} \qquad B'' = \frac{g' + a'D'}{1 - a'A'}$$

$$c'' = \frac{e' + a'C'}{1 - a'A'} \qquad C'' = \frac{E' + c'A'}{1 - c'C'}$$

$$\lambda_{26}^{**} = \frac{c'\lambda_{32}^{*} + \lambda_{26}^{*}}{1 - c'C'}$$

$$\lambda_{34}^{**} = \frac{a'\lambda_{32}^{*} + \lambda_{34}^{*}}{1 - a'A'}$$

$$\lambda_{40}^{**} = \frac{d'\lambda_{32}^{*} + \lambda_{40}^{*}}{1 - d'D'} \quad .$$

The reduced network is shown schematically in Figure 3-5.

In a similar manner, point 34 may be deleted. Expressions for $\overline{N}y$ at the remaining two points follow

$$\overline{N}y_{26} = a''' \overline{N}y_{40} + \lambda_{26}^{***} \tag{3-13}$$

$$\overline{N}y_{40} = A''' \overline{N}y_{26} + \lambda_{40}^{***} \tag{3-14}$$

where

$$a''' = \frac{a'' + B''C''}{1 - c''C''}$$

$$A''' = \frac{A'' + b''c''}{1 - b''B''}$$

$$\lambda_{26}^{***} = \frac{C''\lambda_{34}^{**} + \lambda_{26}^{**}}{1 - c''C''}$$

$$\lambda_{40}^{***} = \frac{b''\lambda_{34}^{**} + \lambda_{40}^{**}}{1 - b''B''} \quad .$$

The reduced and final network is shown in Figure 3-6.

Solving Equations (3-13) and (3-14) simultaneously yields

$$\overline{N}y_{26} = \frac{a'''\lambda_{40}^{***} + \lambda_{26}^{***}}{1 - a'''A'''} \tag{3-15}$$

$$\overline{N}y_{40} = \frac{A'''\lambda_{26}^{***} - \lambda_{40}^{***}}{1 - a'''A'''} \quad . \tag{3-16}$$

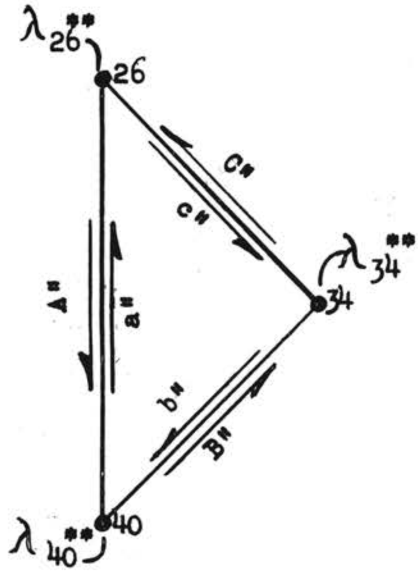


Figure 3-5

Three-Point Network

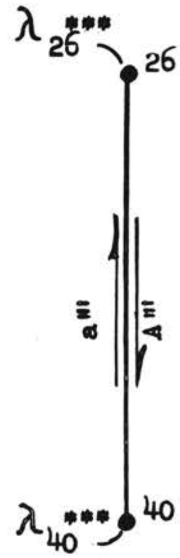


Figure 3-6

Two-Point Network

Reviewing, Equations (3-15) and (3-16) represent the projected internal force in the Y direction at pivotal points 26 and 40, respectively, that are compatible with a known projected internal force in the Y direction at pivotal points, 28, 35, and 42.

Now that $\bar{N}y$ at points 26 and 40 can be computed, $\bar{N}y$ at all points in the network can be evaluated. This can be accomplished by either direct substitution into the developed simultaneous equations or by observing Figures 3-6, 5, 4, and 3 and making the required carry-over computations. This latter method will be clearly demonstrated in the numerical example in Chapter IV.

CHAPTER IV

NUMERICAL EXAMPLES

4-1. Example Problem No. 1.

It is required to compute the internal forces, Ny, Nx, and Nxy, and the design stresses acting in the elliptical paraboloid shell shown in Figure 4-1. The shell is subjected to a uniform dead load and live load of 60 pounds per square foot. The edge beams on the shell provide negligible lateral rigidity and are considered incapable of vertical displacement. The thickness of the shell is three inches.

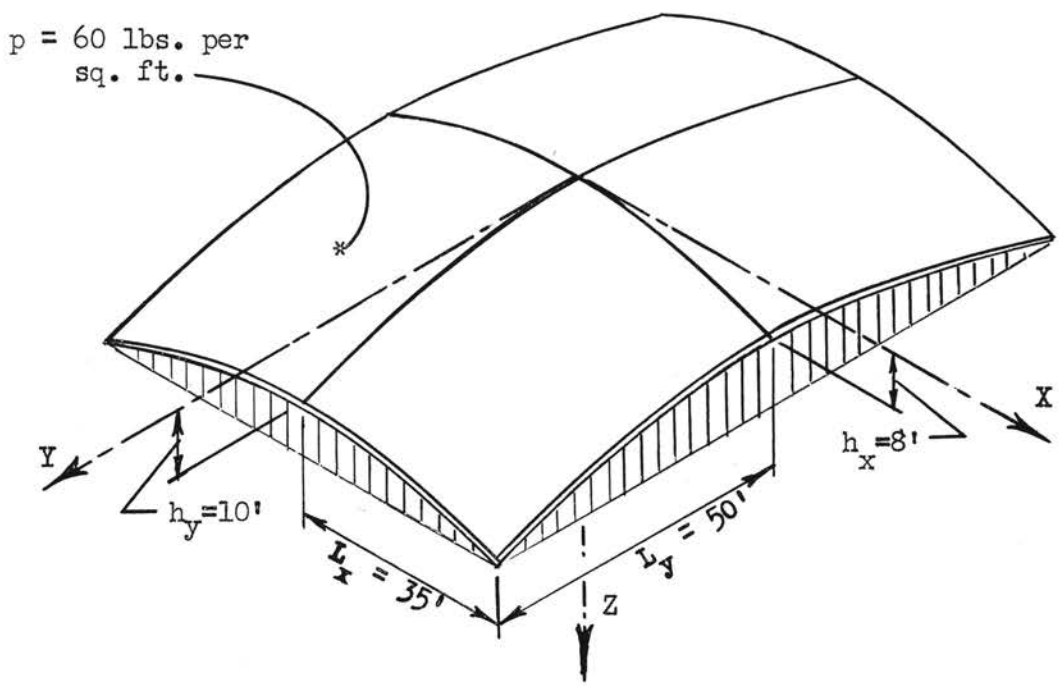


Figure 4-1

Elliptical Paraboloid

The equation for the elliptical paraboloid shown in Figure 4-1 is

$$z = \frac{h_y\, y^2}{L_y^{\,2}} + \frac{h_x\, x^2}{L_x^{\,2}} .$$

Substituting the known quantities:

$$z = \left(\frac{10}{50^2}\right) y^2 + \left(\frac{8}{35^2}\right) x^2$$

and from Equations (1-8) and (1-9):

$$f_1''(x) = \frac{16}{35^2} \qquad f_2''(y) = \frac{20}{50^2} .$$

From Equation (2-18)

$$a = \left(\frac{20}{50^2}\right)\left(\frac{35^2}{16}\right) \div 2\left[\frac{\Delta x^2}{\Delta y^2} + \left(\frac{20}{50^2}\right)\left(\frac{35^2}{16}\right)\right]$$

$$b = \frac{\Delta x^2}{\Delta y^2} \div 2\left[\frac{\Delta x^2}{\Delta y^2} + \left(\frac{20}{50^2}\right)\left(\frac{35^2}{16}\right)\right]$$

Choosing a 49 pivotal point network

$$\Delta x = \frac{35}{3} \qquad \Delta y = \frac{50}{3}$$

then

$$a = \left(\frac{20}{16}\right)\left(\frac{35}{50}\right)^2 \div 2\left(\frac{35}{50}\right)^2 \left(1 + \frac{20}{16}\right) = 0.2777$$

$$b = \left(\frac{35}{50}\right)^2 \div 2\left(\frac{35}{50}\right)^2 \left(1 + \frac{20}{16}\right) = 0.2222.$$

It is noted that for the elliptical paraboloid, the carry-over factors, a and b, are constant over the domain of the shell. Evaluation of the modified carry-over factors follows.

From Equations (3-6, 7, 8, 9)

$$x_{\overline{32}} = 1 - 2(.2777)^2 - 3(.2222)^2 = 0.6977$$

$$x_{\overline{26}} = 1 - 3(.2777)^2 - 2(.2222)^2 = 0.6699$$

$$x_{\overline{34}} = 1 - (.2777)^2 - 3(.2222)^2 = 0.7748$$

$$x_{\overline{40}} = 1 - 3(.2777)^2 - (.2222)^2 = 0.7193$$

$$A' = \frac{2(.2777)^2}{.6977} = 0.221 \qquad a' = \frac{(.2777)^2}{.7748} = 0.0995$$

$$B' = \frac{2(.2222)^2}{.6699} = 0.1474 \qquad b' = \frac{(.2222)^2}{.7193} = 0.0686$$

$$C' = \frac{4(.2777)(.2222)}{.6977} = 0.3538 \qquad c' = \frac{4(.2777)(.2222)}{.6699} = 0.3684$$

$$D' = C' = 0.3538 \qquad d' = \frac{2(.2777)(.2222)}{.7193} = 0.1716$$

$$E' = c' = 0.3684 \qquad e' = \frac{2(.2777)(.2222)}{.7748} = 0.1593$$

$$G' = d' = 0.1716 \qquad g' = e' = 0.1593.$$

From Equations (3-10, 11, 12)

$$A'' = \frac{.0686 + (.3538)(.1716)}{1 - (.3538)(.1716)} = 0.1376$$

$$B'' = \frac{.1593 + (.0995)(.3538)}{1 - (.2210)(.0995)} = 0.1989$$

$$C'' = \frac{.3684 + (.2210)(.3684)}{1 - (.3538)(.3684)} = 0.5172$$

$$a'' = \frac{.1474 + (.3538)(.3684)}{1 - (.3538)(.3684)} = 0.3193$$

$$b'' = \frac{.1716 + (.2210)(.1716)}{1 - (.3538)(.1716)} = 0.2230$$

$$c'' = B'' = 0.1989.$$

From Equations (3-13) and (3-14)

$$A''' = \frac{.1376 + (.2230)(.1989)}{1 - (.2230)(.1989)} = 0.1903$$

$$a''' = \frac{.3193 + (.1989)(.5172)}{1 - (.1989)(.5172)} = 0.4706.$$

All the carry-over factors have now been evaluated. The starting values at all points are now computed.

From Equation (2-13), the starting value, $\bar{N}y^*$, at all interior points is zero, because the second derivative of the load function is zero. At pivotal points on a line, $y = L_y$, boundary conditions dictate that $\bar{N}y = 0$. On a line, $x = L_x$, boundary conditions dictate that $\bar{N}x = 0$ and from Equation (2-8)

$$\bar{N}y \left(\frac{20}{50^2}\right) \left(\frac{35^2}{16}\right) = -60 \left(\frac{35^2}{16}\right)$$

therefore

$$\bar{N}y^*_{28} = \bar{N}y^*_{35} = \bar{N}y^*_{42} = -7500 \text{ lbs. per ft.}$$

From Equation (3-3)

$$\lambda_{27} = \lambda_{34} = \lambda_{41} = .2777(-7500) = -2082.7 \text{ lbs. per ft.}$$

From Equations (3-6, 7, 8, and 9)

$$\lambda^*_{26} = \frac{.2777}{.6699}(-2082.7) = -863.5 \text{ lbs. per ft.}$$

$$\lambda^*_{32} = 0$$

$$\lambda^*_{34} = \frac{.2222}{.7748}(2)(-2082.7) - 2082.7 = -3882.6 \text{ lbs. per ft.}$$

$$\lambda^*_{40} = \frac{.2777}{.7193}(-2082.7) = -804.1 \text{ lbs. per ft.}$$

From Equations (3-10, 11, 12)

$$\lambda^{**}_{26} = \frac{-863.5}{1 - (.3538)(.3684)} = -992.9 \text{ lbs. per ft.}$$

$$\lambda^{**}_{34} = \frac{-3882.6}{1 - (.2210)(.0995)} = -3969.9 \text{ lbs. per ft.}$$

$$\lambda^{**}_{40} = \frac{-804.1}{1 - (.3538)(.1716)} = -856.1 \text{ lbs. per ft.}$$

From Equations (3-13) and (3-14)

$$\lambda_{26}^{***} = \frac{-992.9 + .5172\ (-3969.9)}{1 - .1989\ (.5172)} = -3395.4 \text{ lbs. per ft.}$$

$$\lambda_{40}^{***} = \frac{-856.1 + .2230\ (-3969.9)}{1 - .2230\ (.1989)} = -1822.3 \text{ lbs. per ft.}$$

The values of λ^{***} are substituted into Equations (3-15) and (3-16)

$$\bar{N}y_{26} = \frac{(.4706)(-1822.3) - 3395.4}{1 - (.4706)(.1903)} = -4671.1 \text{ lbs. per ft.}$$

$$\bar{N}y_{40} = \frac{(.1903)(-3395.4) - 1822.3}{1 - (.4706)(.1903)} = -2711.0 \text{ lbs. per ft.}$$

Values of $\bar{N}y$ at the remaining points may be determined by carry-over principles. For example, in the determination of $\bar{N}y$ at point 34, reference is made to Figure 3-5. The values of $\bar{N}y$ at points 26 and 40 are multiplied by the carry-over constants, c'' and B'', respectively. These two quantities are summed and the result is added to the starting value at point 34 which is λ_{34}^{**}. In equation form this reads

$$\bar{N}y_{34} = c''\bar{N}y_{26} + B''\bar{N}y_{40} + \lambda_{34}^{**}.$$

Substituting known values

$$\bar{N}y_{34} = .1989\ (-4671.1) + .1989\ (-2711.0) - 3969.9$$
$$= -5438.2 \text{ lbs. per ft.}$$

$\bar{N}y$ can be evaluated at other pivotal points in the same manner.

$\bar{N}x$ may be evaluated by direct substitution into Equation 28. For example, the value of the projected internal force in the x direction at point 34 is

$$\bar{N}x_{34} = -\left[-5438.2\left(\frac{20}{50^2}\right)\ \left(\frac{35^2}{16}\right) \quad - 60\ \left(\frac{35^2}{16}\right)\right]$$
$$= -1262.8 \text{ lbs. per ft.}$$

$\bar{N}x$ can be evaluated at other points in a similar manner.

As discussed in Chapter II, $\bar{N}xy$ is zero at points 25, 26, 27, 28, 32, and 39. Also, $\bar{N}xy$ approaches infinity at pivotal point 49. $\bar{N}xy$ at point 34 may be evaluated by substituting into Equation (2-19) as follows

$$(\bar{N}y_{40} - \bar{N}y_{26}) \frac{1}{2\Delta y} = -(\bar{N}xy_{34} - \bar{N}xy_{32}) \frac{1}{2\Delta x} \ .$$

Substituting known quantities

$$(-2711.0) - (-4671.1) \frac{3}{100} = -(\bar{N}xy_{34} - 0) \frac{3}{70}$$

$$\bar{N}xy_{34} = -(1960.1)0.7 = -1372.1 \text{ lbs. per ft.}$$

By following a similar procedure, $\bar{N}xy$ may be evaluated at all points in the network.

Rounded values of the projected normal forces, $\bar{N}x$ and $\bar{N}y$, and the shearing forces, $\bar{N}xy$, for all points in the network are tabulated in Table 4-1.

Comparing the results in Table 4-1 with the classical series solution of the partial differential equations (6), it is found that the percentage error in $\bar{N}x$ and $\bar{N}y$ varies from about one per cent to four per cent. Also, the percentage error in $\bar{N}xy$ varies from about three per cent to thirty per cent, increasing in magnitude as the corner is approached. This accuracy is certainly adequate for the normal forces, but the higher percentage errors in $\bar{N}xy$ are not acceptable. Reasonably accurate values of $\bar{N}xy$ are essential so that the distribution of load to the edge beam may be ascertained.

	Force		
Pivotal Point	$\bar{N}x$	$\bar{N}y$	$\bar{N}xy$
25	-1965	-4290	0
26	-1735	-4670	0
27	-1045	-5795	0
28	0	-7500	0
32	-2260	-3815	0
33	-2015	-4210	- 690
34	-1265	-5440	-1370
35	0	-7500	-1915
39	-3170	-2325	0
40	-2935	-2710	-1335
41	-2115	-4045	-2945
42	0	-7500	-5145
46	-4595	0	0
47	-4595	0	-2200
48	-4595	0	-5135
49	0	0	∞

Table 4-1

Projected Normal and Shearing Forces

To reduce the error in $\bar{N}xy$ a finer network must be employed. In

$\frac{Y}{L_y}$ \ $\frac{X}{L_x}$	0.000	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000 X
0.000	+35, +9, +17, 0, +4 -4290, +17 **-4273**	-47, 0, +14, 0, +5, +9, 0 -4300, -47, +14, +9 **-4324**	+23, +10, +21, +11, +15, 0, -3, -1, -3 -4500, +15 **-4585**	+39, 0, -12, -8, -11, 0, -6 -4780, +39, -11 **-4752**	+26, +11, -9, -12, 0, -3, -5 -5130, -12 **-5142**	+10, +23, +5, +2, 0, -4 -5600 **-5600**	+46, 0, -8, 0, -2, -6 -6200, +46, -8 **-6162**	+18, -10, +3, -5, -7, 0 -6800, -7 **-6807**	0 -7500 **-7500**
0.125	-25, -7, -3 -4200 **-4200**	+42, +32, 0, -6, -3, -1, +1 -4290, +32 **-4258**	-16, -23, -14, -21, 0, +3, 0 -4400, -21 **-4421**	-22, -13, -26, 0, -6, -10, -12, 0 -4650, -26, -12 **-4688**	-47, 0, -11, -18, -13, 0, -3, -6, -8 -5000, -47, -13 **-5060**	-28, -41, 0, -5, -9, 0, -3 -5500, -41, -9 **-5550**	+6, -12, -2, -13, -19, 0, -5, -7, -9, 0 -6100, -19, -9 **-6128**	-65, 0, -14, -19, 0, -2, -4, -7 -6700, -65, -19 **-6784**	0 -7500 **-7500**
0.250	-25, -11, -8, -2 -4000 **-4000**	-5, +6, -3, +4, +10, 0 -4070, +10 **-4060**	-46, -32, 0, -5, +3, +6, 0 -4200, -32 **-4232**	-16, +1, -8, -14, -8, -2, -5, -3 -4500 **-4500**	-4, +11, +21, 0, +7, +4, +6 -4900, +21 **-4879**	+1, +14, +5, -2, +4, +2 -5400 **-5400**	-10, -27, 0, -4, -7, -9, -11 -5980, -27 **-6007**	-49, -63, 0, -7, -11, 0, -2 -6650, -63, -11 **-6724**	0 -7500 **-7500**
0.375	+36, +62, 0, +14, 0, +2 -3740, +62, +14 **-3664**	+31, +48, 0, +17, +27, 0, +4, +6, +9 -3800, +48, +27 **-3725**	+62, 0, +22, +35, +28, +35, 0, +7, +10, 0 -4000, +62, +35, +10 **-3893**	+62, +79, 0, +19, +27, 0, +10, 0, +3, +6 -4300, +79, +27, +10 **-4184**	+47, +69, 0, +16, +21, +30, 0, +7, +10, 0 -4700, +69, +30, +10 **-4591**	+40, +59, 0, +8, +5, +8, +6 -5200, +59 **-5141**	-5, -17, -1, -7, -10, 0, -3 -5800, -10 **-5810**	+29, +11, -3, -9, -11, 0, -3 -6600, -11 **-6611**	0 -7500 **-7500**
0.500	-5, +9, +5, +8, 0 -3170, +8 **-3162**	-27, -16, -9, -3, -1 -3220 **-3220**	-21, -7, +1, +4, +6 -3400 **-3400**	-29, -12, -1, +5, +7 -3700, +12 **-3688**	+15, +30, +41, 0, +7, +9, +11, +9 -4170, +41 **-4129**	-24, -39, -26, -16, -5, -8, 0 -4700, -8 **-4708**	-32, -55, 0, -8, -10, -12, 0, -2, -4 -5400, -55, -12 **-5467**	-82, 0, -15, -28, 0, -4, -6, -9 -6300, -82, -28 **-6410**	0 -7500 **-7500**
0.625	-26, -35, -19, 0, +2 -2500, -19 **-2519**	+38, +30, 0, -5, -1 -2600, +30 **-2570**	+6, +14, 0 -2750, +14 **-2736**	+7, -12, +1, -2, +2, +5 -3020 **-3020**	+48, 0, +13, +4, +13, +10, 0 -3500, +48, +10 **-3442**	+34, +47, 0, -3, 0, -2, -4 -4100, +47 **-4053**	-16, +5, -7, -24, -11, 0, -4, -7, -9, 0 -4900, -11, -9 **-4920**	-43, -61, 0, -7, -13, -16, 0, -2 -6000, -61, -16 **-6077**	0 -7500 **-7500**
0.750	-16, -36, -43, 0, -4, -6 -1700, -43 **-1743**	-36, -, -12, -5, -7 -1750, -36 **-1786**	+41, +17, +7, +3, 0, +3 -1925 **-1925**	-86, 0, -11, -16, 0, -4 -2050, -86, -16 **-2152**	-27, -51, -40, 0, -11, -15, 0, +2, 0 -2470, -40, -15 **-2525**	-24, +2, +12, +1, -3, -6, -7 -3100 **-3100**	+95, 0, -8, -10, 0, +4, +2 -4100, +95, -10 **-4015**	-43, -17, -30, 0, +3, -1, -4 -5400, -30 **-5430**	0 -7500 **-7500**
0.875	-33, 0, -9, 0, -4 -850, -33, -9 **-892**	+13, +5, -4, -7, -9, 0 -910, -9 **-919**	-6, -12, 0, -2 -980, -12 **-992**	+11, -8, -22, 0, -3, -6, -9 -1100, -22 **-1122**	-40, -49, 0, -6, -7, -10, 0 -1300, -49, -10 **-1359**	+6, -8, -5, 0, -3 -1750, -5 **-1755**	-8, +13, +17, +15, 0, -3 -2500, +15 **-2485**	+22, +15, 0, +3 -4000, +15 **-3985**	0 -7500 **-7500**
1.000 Y	0 0	0 0	0 0	0 0	0 0	0 0	0 0	0 0	0 0

Table 4-2

Relaxation Procedure

Table 4-2, the Relaxation method has been used to solve a network involving 64 unknowns. The relaxation operator has been modified to allow the use of the carry-over factors, a and b. Initial values at each pivotal point have been estimated using final values in Table 4-1 as a guide. Final rounded values of $\bar{N}y$ at the eighth points of the shell have been transferred to Table 4-3. Values of $\bar{N}x$ and $\bar{N}xy$ have been computed as before and also appear in Table 4-3.

		Value of $\frac{y}{L_y}$				
$\frac{x}{L_x}$	Force	0.00	0.25	0.50	0.75	1.00
0.00	$\bar{N}y$	-4275	-4000	-3160	-1745	0
	$\bar{N}x$	-1975	-2145	-2655	-3527	-4595
	$\bar{N}xy$	0	0	0	0	0
0.25	$\bar{N}y$	-4585	-4230	-3400	-1925	0
	$\bar{N}x$	-1785	-2000	-2510	-3415	-4595
	$\bar{N}xy$	0	- 395	- 810	-1155	-1350
0.50	$\bar{N}y$	-5140	-4880	-4130	-2525	0
	$\bar{N}x$	-1445	-1605	-2065	-3050	-4595
	$\bar{N}xy$	0	- 750	-1625	-2485	-3055
0.75	$\bar{N}y$	-6160	-6005	-5465	-4015	0
	$\bar{N}x$	- 820	- 915	-1245	-2135	-4595
	$\bar{N}xy$	0	-1035	-2385	-4090	-6085
1.00	$\bar{N}y$	-7500	-7500	-7500	-7500	0
	$\bar{N}x$	0	0	0	0	0
	$\bar{N}xy$	0	-1155	-2760	-5555	∞

Table 4-3

Projected Normal and Shearing Forces

The finer network has reduced the error in $\bar{N}xy$ considerably in spite of the premature curtailment of the iteration process. For example, at

$y = L_y$, $x = 0.75\ L_x$

$\bar{N}xy$ (Table 4-3) = -6085 lbs. per ft.

$\bar{N}xy$ (Classical Solution) = -5600 lbs. per ft.

Percentage error = 8.7%.

The percentage error in $\bar{N}xy$ increases along the boundary as the corner is approached. The reason for this error becomes obvious if the variation of the internal force is plotted for one quadrant of the shell. Figure 4-2 shows the variation of the internal force, $\bar{N}y$. It is noted that the first derivative of $\bar{N}y$ with respect to y (slope) increases at

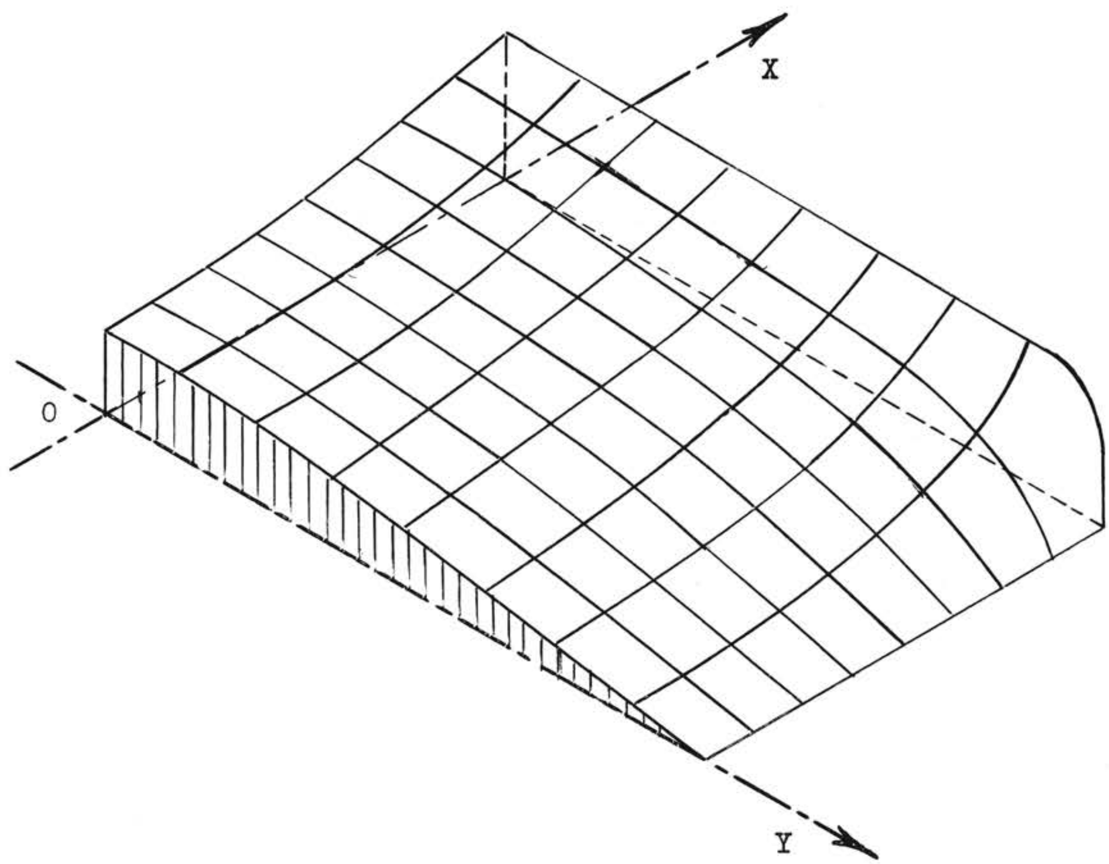


Figure 4-2

Variation-Internal Force $\bar{N}y$

an accelerated rate in the region near the corner as the boundary, $y = L_y$, is approached. Accordingly, the first derivative in finite difference form is not a good approximation of the true value.

To increase the accuracy in the corner region a finer network is required. However, it is not deemed necessary to evaluate $\bar{N}y$ at points on the finer network by finite differences. Having an understanding of the force variation, interpolation of $\bar{N}y$ should yield sufficiently accurate results. In this problem, interpolation has not been used, and the error in $\bar{N}xy$ has been considered tolerable.

The projected forces, $\bar{N}x$, $\bar{N}y$, and $\bar{N}xy$, can be transferred to the middle surface of the shell by substitution into Equation (1-1).

$$Ny = \frac{\cos\psi}{\cos\phi}\bar{N}y$$

$$Nx = \frac{\cos\phi}{\cos\psi}\bar{N}x$$

$$Nxy = \bar{N}xy$$

The coefficient can be expressed as a function of x and y as follows

$$\frac{\cos\phi}{\cos\psi} = \sqrt{\frac{1 + \left[2\left(\frac{h_x}{L_x}\right)\left(\frac{x}{L_x}\right)\right]^2}{1 + \left[2\left(\frac{h_y}{L_y}\right)\left(\frac{y}{L_y}\right)\right]^2}} \cdot$$

Recognizing that the coefficient is a maximum at $\frac{x}{L_x} = 1$, $\frac{y}{L_y} = 0$ and a minimum at $\frac{x}{L_x} = 0$, $\frac{y}{L_y} = 1$, the value of the coefficient for the extreme conditions follows

$$\left(\frac{\cos\phi}{\cos\psi}\right)_{min} = \sqrt{\frac{1}{1 + (2 \times \frac{10}{50})^2}} = 0.93 \cdot$$

$$\left(\frac{\text{Cos}\,\phi}{\text{Cos}\,\psi}\right)_{\text{max}} = \sqrt{\frac{1 - (2 \times \frac{8}{35})^2}{1}} = 1.10.$$

Therefore, the maximum difference in magnitude between the projected force and actual force on the shell is 10 per cent. Being that direct stresses in the shell are never critical, this difference can be ignored.

Design Stresses.

From Table 4-3, it is noted that the maximum design stresses will occur at the boundary and near the corner. It is also noted that the shearing force, $\bar{N}xy$, has a considerable influence on the principal forces.

It has been developed that $\bar{N}xy$ at the corner theoretically approaches infinity. That is

$$\bar{N}xy_{\substack{x \to L_x \\ y \to L_y}} = \to \infty .$$

As previously explained, values of all the internal forces in the corner region are not reliable. The point where these forces do become reliable is largely a matter of judgement. Parme (6) has suggested that the cut-off points for design forces along the boundary should be located at

$$\frac{x}{L_x} = \frac{L_x - 0.4\sqrt{r_x\, t}}{L_x}$$

$$\frac{y}{L_y} = \frac{L_y - 0.4\sqrt{r_y\, t}}{L_y}$$

where

t = shell thickness

r_x = radius of curvature of the shell in the x direction

$$= \frac{\left[1 + \left(\frac{\partial z}{\partial x}\right)^2\right]^{3/2}}{\frac{\partial^2 z}{\partial x^2}} = \frac{35^2}{16}\left[1 + \left(\frac{16}{35}\right)^2\right]^{3/2} = 102 \text{ ft.}$$

r_y = radius of curvature in the y direction

$$= \frac{\left[1 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{3/2}}{\frac{\partial^2 z}{\partial y^2}} = \frac{50^2}{20}\left[1 + \left(\frac{20}{50}\right)^2\right]^{3/2} = 156 \text{ ft.}$$

Substituting known quantities

$$\frac{x}{L_x} = \frac{35 - 0.4\sqrt{102 \times .25}}{35} = 0.94$$

$$\frac{y}{L_y} = \frac{50 - 0.4\sqrt{156 \times .25}}{50} = 0.95.$$

The force, $\bar{N}xy$, is known at the sixteenth points of the shell. Consequently, curves may be drawn showing the variation in shear force on a coordinate line and values at any point may be taken from the curve. This has been done in Figure 4-3 for the variation in shear along the boundaries. The cut-off lines are also shown and the design forces are noted.

Maximum and minimum principal forces at the cut-off points will be computed. Design forces at $\frac{x}{L_x} = 0.94$, $\frac{y}{L_y} = 1$ are

$\bar{N}x$ = -4595 lbs. per ft.

$\bar{N}y$ = 0

$\bar{N}xy$ = -11,900 lbs. per ft.

Letting σ_1 and σ_2 represent the maximum and minimum principal forces respectively

$$\sigma_1 = \frac{-4595}{2} - \sqrt{(-11{,}900)^2 + (-2297.5)^2} = -14{,}417 \text{ lbs. per ft.}$$

$$\sigma_2 = \frac{-4595}{2} + \sqrt{(-11{,}900)^2 + (-2297.5)^2} = +\ 9{,}822 \text{ lbs. per ft.}$$

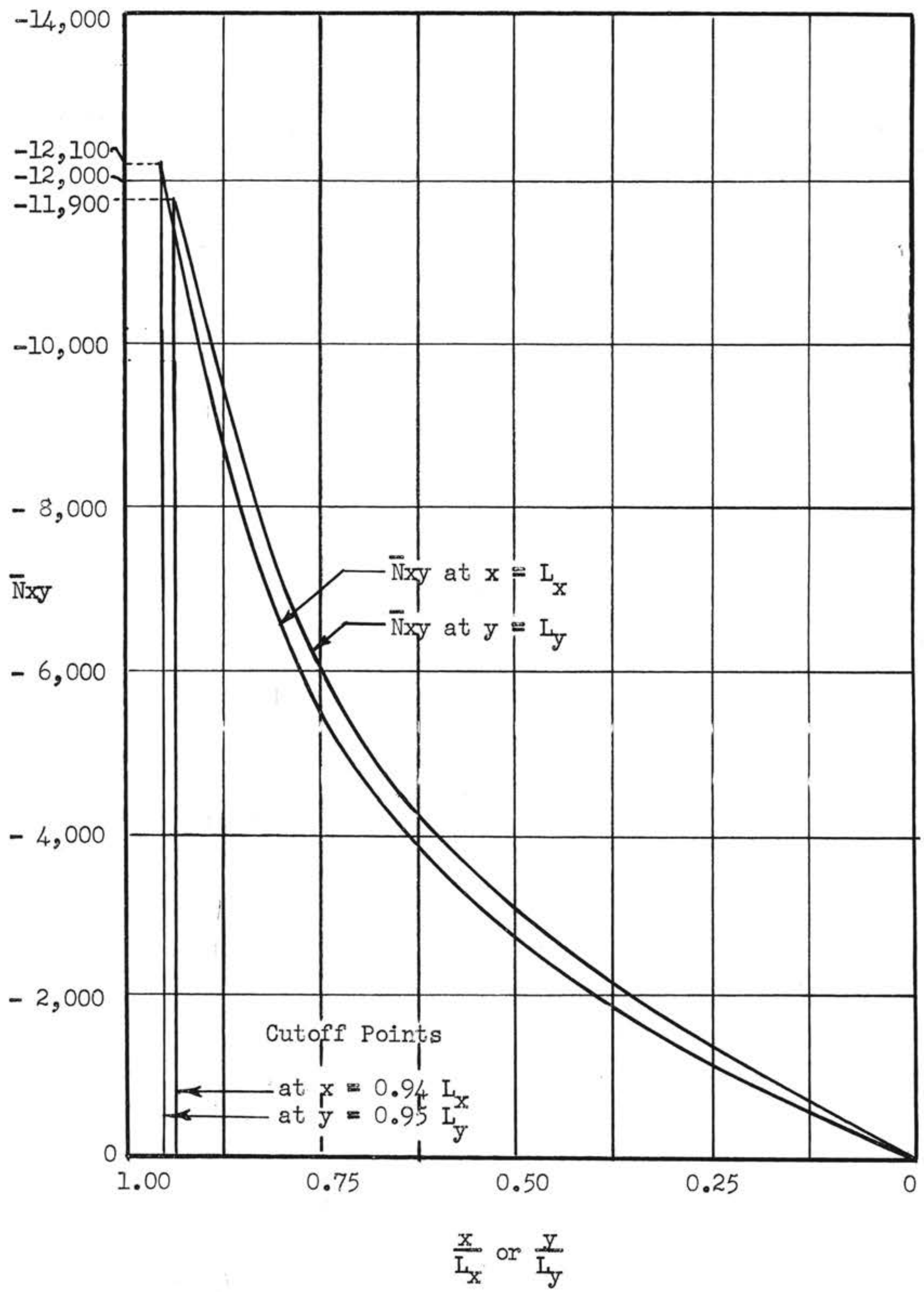


Figure 4-3

Tangential Shearing Force

At Shell Boundary

At the point $\frac{x}{L_x} = 1$, $\frac{y}{L_y} = 0.95$

$\bar{N}x = 0$

$\bar{N}y = -7500$ lbs. per ft.

$\bar{N}xy = -12{,}100$ lbs. per ft.

The principal forces become

$$\sigma_1 = \frac{-7500}{2} - \sqrt{(-12{,}100)^2 + (-3750)^2} = -16{,}418 \text{ lbs. per ft.}$$

$$\sigma_2 = \frac{-7500}{2} + \sqrt{(-12{,}100)^2 + (-3750)^2} = +8{,}918 \text{ lbs. per ft.}$$

The maximum compressive stress in the concrete is therefore

$$f_c = \frac{-16{,}418}{3 \times 12} = -456 \text{ lbs. per sq. in.}$$

which is considerably less than the allowable concrete stress.

Assuming an allowable reinforcing steel stress of 20,000 lbs. per sq. in., the tension reinforcement required at the most critical point is

$$A_s = \frac{+9{,}822}{20{,}000} = 0.49 \text{ sq. in. per ft.}$$

From Table 4-3, it is noted that both the compressive and tensile stress reduce in magnitude regressing from the corner in any direction. Other points on the shell should be similarily checked to establish cut-off points for the tensile reinforcement.

4-2. Example Problem No. 2.

It is required to outline the procedure for analysis of a translational shell subjected to a load varying uniformly in intensity.. The shell is to be an elliptical paraboloid with physical dimensions as given in Example Problem No. 1. Consider the shell to be subjected to a uniform base load of 40 pounds per square foot and a varying load increasing in intensity in the x and y direction at a rate of 0.5 pounds per square foot per linear foot. The loading diagram is shown in Figure 4-4.

The equation of the load function in terms of x and y is

$$p = 40 + 0.5\ (x + y)$$

From Equation (2-13), the starting value, $\bar{N}y^*$, at all interior points is zero, because the second derivative of the load function is zero. At pivotal points on a line, $y = L_y$, boundary conditions dictate that $\bar{N}y^*$ equals zero. On a line, $x = L_x$, boundary conditions dictate that $\bar{N}x = 0$ and by substituting into Equation(2-8)

$$\bar{N}y = \frac{-P}{f_2''y} = -\frac{50^2}{20} \left[40 + 0.5(x + y) \right] \quad .$$

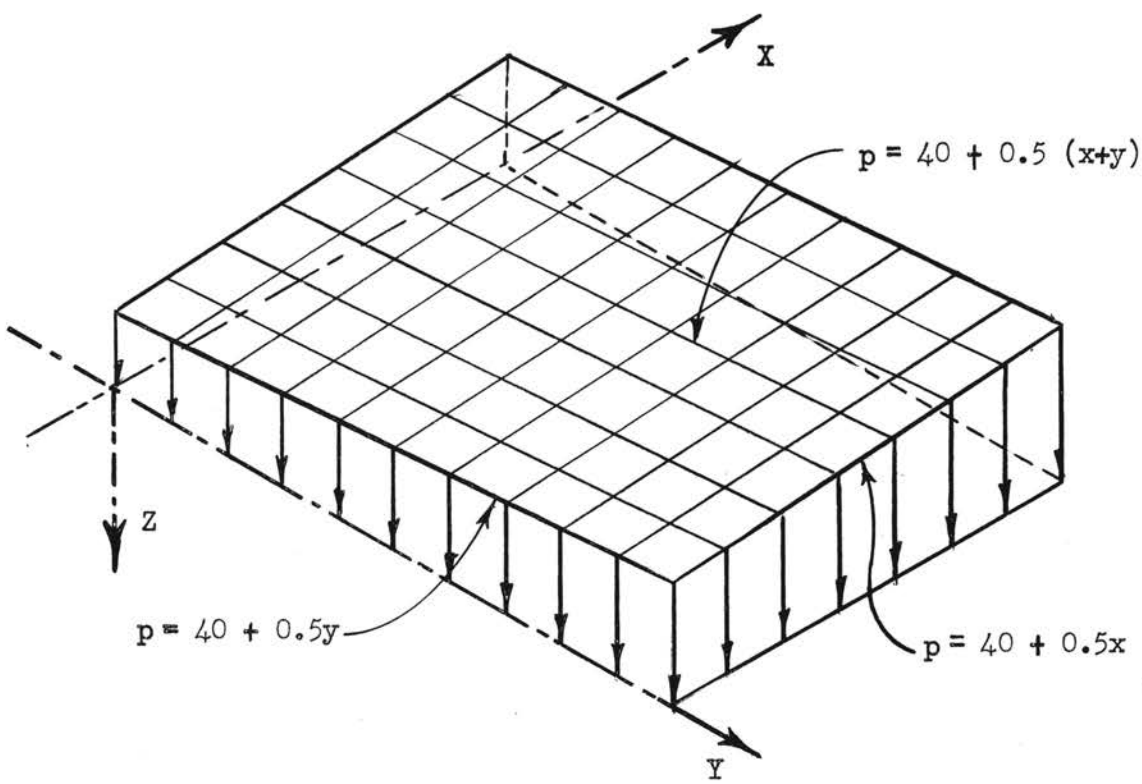


Figure 4-4

Uniformly Varying Load

For the same forty-nine point network used in Example Problem No. 1, the starting values at the points, 28, 35, and 42, become

$$\bar{N}y_{28}^{*} = \frac{50^2}{20} \left[40 + 0.5\ (35) \right] = -7187 \text{ lbs. per ft.}$$

$$\bar{N}y_{35}^{*} = \frac{50^2}{20} \left[40 + 0.5\ (35 + \frac{50}{3}) \right] = -8229 \text{ lbs. per ft.}$$

$$\bar{N}y_{42}^{*} = \frac{50^2}{20} \left[40 - 0.5\ (35 + \frac{100}{3}) \right] = -9271 \text{ lbs. per ft.}$$

With the exception of the preceding changes the procedure for analysis is identical to the procedure presented in Example Problem No. 1.

CHAPTER V

SUMMARY AND CONCLUSIONS

A new procedure for analysis of translational shells by finite differences is presented. The steps in analysis are summarized as follows.

1. The finite difference equation governing the variation of internal force $\bar{N}y$ in the general translational shell is formulated.

2. A basic difference network of forty-nine pivotal points is chosen and reduced to nine points from boundary conditions and symmetry.

3. The nine-point network is solved by Algebraic Carry-Over, employing the method of successive elimination of points and corresponding modification of carry-over factors between retained points.

4. The algebraic expressions for $\bar{N}y$ are evaluated for a specific shell and used for initial starting values on a finer network solved by relaxation.

The solution of the coarse network by Algebraic Carry-Over has excellent application to shells. The unique method of pivotal point elimination for modifying carry-over factors is marked by its simplicity and the small possibility of mechanical error.

In applying the method to a specific translational shell, the algebraic solution on the forty-nine point network was found to yield values for normal forces which differed only one to four per cent from those of the classical series solution. Accurate values of the shearing forces were obtained by advancing to a finer net and using the results from the algebraic solution as starting values at the pivotal points, an approach which served to considerably reduce the effort usually expended in the relaxation process. Should greater accuracy by desired in the area of discontinuity, a finer network could be effected by interpolation.

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