ANALYSIS OF PRESTRESSED CONCRETE CONTINUOUS BEAMS BY THE CARRY-OVER MOMENT METHOD

By

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ANALYSIS OF PRESTRESSED CONCRETE

CONTINUOUS BEAMS BY THE

CARRY-OVER MOMENT

METHOD

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PREFACE

The work in this thesis is an extension of the Carry-Over Methods developed at the Oklahoma State University. The topic was selected by the author from the Structural Engineering Seminar conducted by Professor Tuma in Spring, 1969.

The author wishes to express his gratitude to the following persons:

To Prof. Jan J. Tuma, his major adviser, who not only provided him with a graduate assistantship, but also gave invaluable guidance and encouragement for advanced study.

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R. K. M.

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NOMENCLATURE

dx . . . . Small distance along the beam.

e . . . . Eccentricity of the prestressing cable.

i, j, k . . . . Beam supports.

m_j . . . . Starting moment at j.

$m - 1, m, m + 1$ . . . . Intermediate points on a beam.

r_{ij} . . . . Carry-over factor from i to j.

w . . . . Intensity of distributed load.

x, x' . . . . Distances measured to a section from the left and right supports respectively.

A, B, C . . . . Beam supports.

BM . . . . Bending moment due to loads on a simple span.

E . . . . Modulus of elasticity.

F_{ij}, F_{ji} . . . . Angular flexibilities.

G_{ij}, G_{ji} . . . . Angular carry-over values.

H_0 . . . . Initial prestressing force.

H_n . . . . Final prestressing force.

I . . . . Moment of inertia.

L_j, L_k . . . . Lengths of spans ij, jk.

$M_x^{(i)}$ . . . . Bending moment at a section, distance x from origin at i.

M_i, M_j . . . . Final bending moments at supports i, j.

$M_i^{(j)}$ . . . . Final bending moment at support i due to a unit starting moment at j.

$P^{(H)}$ . . . . Equivalent concentrated real load due to prestress.

$\overline{P}^{(H)}$ . . . . Equivalent concentrated elastic load due to prestress.

$\overline{R}$ . . . . Reaction of a conjugate beam.

U_{ijk} . . . . Strain energy of member ijk.

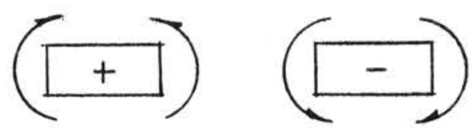
$V_{m,\, m+1}$ . . . . Shear in the segment m, m+1.

$\tau_{ij}^{(H)}$ . . . . Angular prestress function.

$\tau_{ij}^{(L)}$ . . . . Angular load function.

SIGN CONVENTION

Bending Moment



Eccentricity e + if above the centroidal axis.

- if below the centroidal axis.

CHAPTER I

INTRODUCTION

The analysis of prestressed concrete beams has been done before this, by the classical methods - the Area Moment, Virtual Work, Slope Deflection, etc. The main factor of study in this subject has been the inclusion of the effect of the prestressing force.

R. B. B. Moorman's (1) concept of "Equivalent Load" due to prestress has proved itself valuable, especially in the analysis of continuous prestressed concrete beams by the method of moment distribution.

This study shows the extension to prestressed beams of the Carry-Over Moment Method for analysis of continuous beams originated by J. J. Tuma (2). This method fundamentally studies the effect of the bending moment at one support, on those at the adjacent supports. In this method the analysis is based on the flexibilities of the spans.

In Chapter II is presented the main derivation of the Three-Moment Equation in carry-over form. Considering the support moments as redundants, this equation is derived by minimizing the total strain energy of the structure. The angular flexibilities, load functions and prestress functions are also defined in this chapter.

The Carry-Over Procedure and the modifications required in the carry-over factors for special cases are explained in Chapter III.

The calculation of the prestress functions is specially studied in Chapter IV and exact and approximate methods presented.

The procedure of analyzing prestressed concrete continuous beams by this method is illustrated by two numerical examples presented in Chapter V.

The final chapter summarizes the study and shows the conclusions drawn.

CHAPTER II

DERIVATION OF THE THREE MOMENT EQUATION IN CARRY-OVER FORM

A general case of a prestressed concrete continuous beam is considered (Fig. 1).

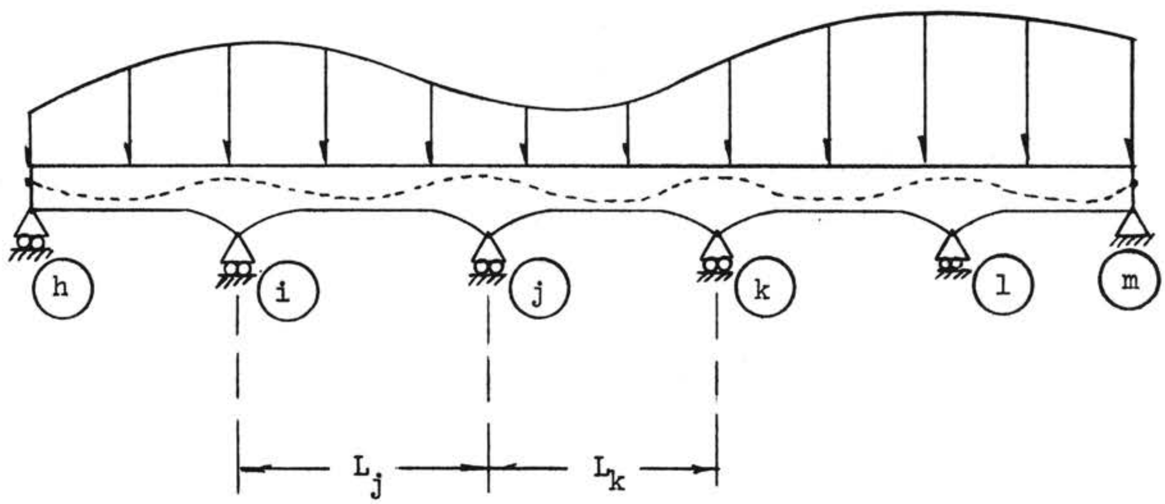


Fig. 1

A Prestressed Concrete Continuous Beam with General Loading

The supports are assumed to be rigid. The cross-section of the beam can be variable and the system of external load considered is perfectly general. The slope of the prestressing cable is small, and, hence, the horizontal component of the prestressing force H is taken to be constant and equal to H.

(a) Statics and Free Body Diagrams

The free body diagrams of spans ij and jk, isolated from the beam, are shown in Fig. 2. M_i, M_j, and M_k are the internal bending moments developed at supports i, j, and k, respectively.

The bending moment on a cross-section at any distance x, on either span, can be obtained by superimposing the effect of external load, bending moments M_i, M_j, M_k and the prestressing force H acting at an eccentricity e_x at the section.

(b) Bending Moments

Thus,

$$M^{(i)}_{x = 0 \rightarrow L_j} = BM^{(i)}_x + M_i \frac{x'}{L_j}^{(i)} + M_j \frac{x}{L_j}^{(i)} + He_x \tag{1a}$$

$$M^{(j)}_{x = 0 \rightarrow L_k} = BM^{(j)}_x + M_j \frac{x'}{L_k}^{(j)} + M_k \frac{x}{L_k}^{(j)} + He_x \tag{1b}$$

(c) Strain Energy Expression and Application of Castigliano's Theorem

The total strain energy U_{ijk} of these two spans ij and jk can be expressed as the sum of the strain energies U_{ij} and U_{jk}. The strain energy of volume change due to temperature or moisture content change is

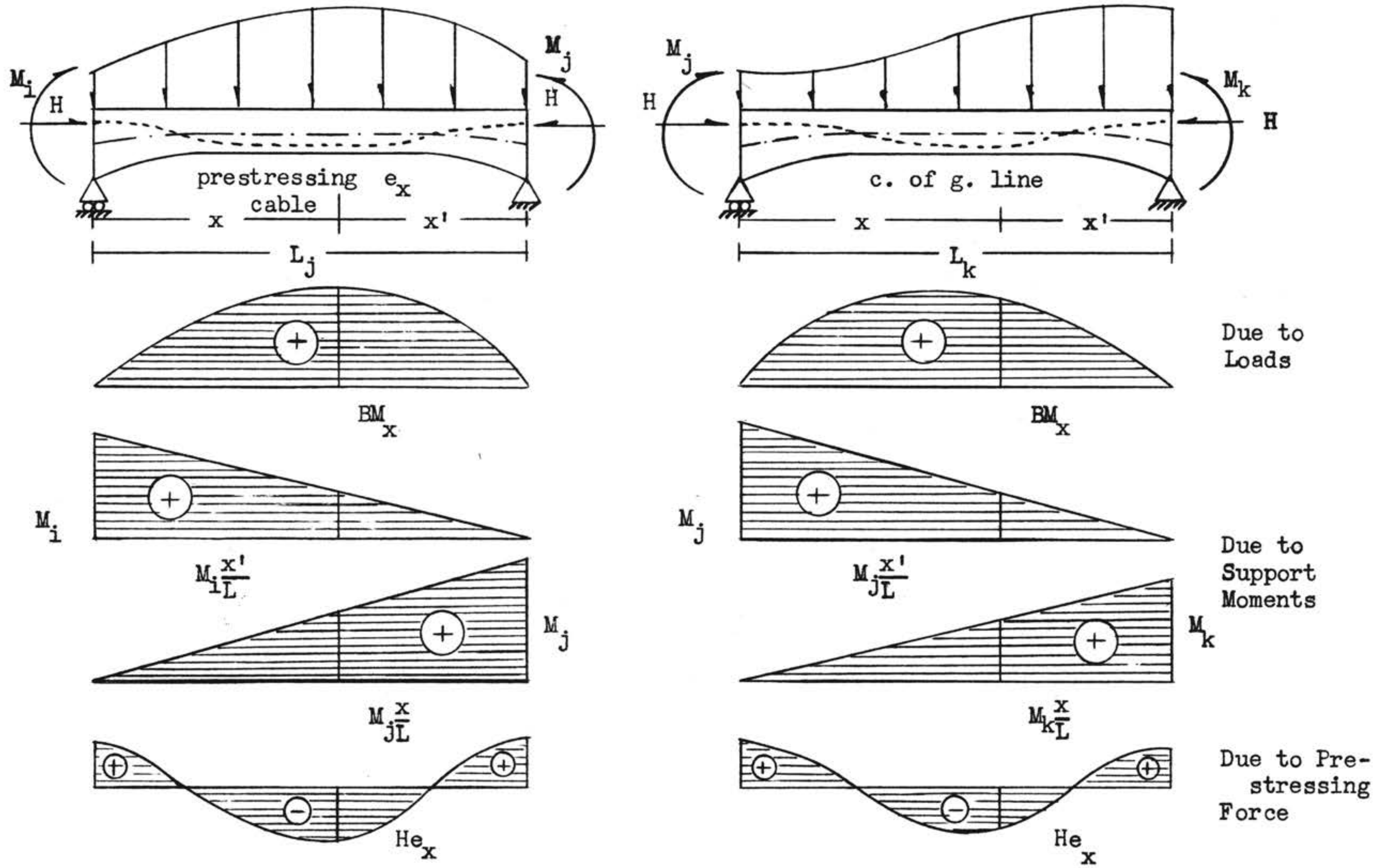


Fig. 2

Free Body Diagrams for Spans ij and jk -

Bending Moments

not considered. The strain energy due to normal forces and shearing forces is small and therefore neglected. Considering only the strain energy due to bending, the expression for strain energy becomes

$$U_{ijk} = U_{ij} + U_{jk}$$

where

$$U_{ij} = \int_i^j \frac{\left[M_x^{(i)}\right]^2 dx}{2EI_x} \tag{2a}$$

and

$$U_{jk} = \int_j^k \frac{\left[M_x^{(j)}\right]^2 dx}{2EI_x} \tag{2b}$$

Considering M_j as a redundant moment, by Castigliano's Theorem,

$$\frac{\partial U_{ijk}}{\partial M_j} = \frac{\partial U_{ij}}{\partial M_j} + \frac{\partial U_{jk}}{\partial M_j} = 0 \tag{3}$$

Now

$$\frac{\partial U_{ij}}{\partial M_j} = \int_i^j \frac{M_x^{(i)} \frac{\partial M_x^{(i)}}{\partial M_j} dx}{EI_x} \tag{4a}$$

and

$$\frac{\partial U_{jk}}{\partial M_j} = \int_j^k \frac{M_x^{(j)} \frac{\partial M_x^{(j)}}{\partial M_j} dx}{EI_x} \tag{4b}$$

Also from (1)

$$\frac{\partial M_x^{(i)}}{\partial M_j} = \frac{x^{(i)}}{L_j} \tag{5a}$$

and

$$\frac{\partial M_x^{(j)}}{\partial M_j} = \frac{x'^{(j)}}{L_k} \tag{5b}$$

Substituting the values from (5) and (1) in (4) and adding, equation (3) becomes

$$\int_i^j \frac{\left[BM_x^{(i)} + M_i \frac{x'^{(i)}}{L_j} + M_j \frac{x^{(i)}}{L_j} + He_x\right] \frac{x^{(i)}}{L_j}}{EI_x} dx$$

$$+ \int_j^k \frac{\left[BM_x^{(j)} + M_j \frac{x'^{(j)}}{L_k} + M_k \frac{x^{(j)}}{L_k} + He_x\right] \frac{x'^{(j)}}{L_k}}{EI_x} dx = 0$$

Expanding and rearranging, the equation becomes

$$\int_i^j \frac{BM_x^{(i)}\, x^{(i)}\, dx}{L_j\, EI_x} + M_i \int_i^j \frac{x^{(i)}\, x'^{(i)}\, dx}{L_j^2\, EI_x} + M_j \int_i^j \frac{x^{(i)2}\, dx}{L_j^2\, EI_x}$$

$$+ \int_i^j \frac{He_x\, x^{(i)}\, dx}{L_j EI_x} + \int_j^k \frac{BM_x\, x'^{(j)}\, dx}{L_k\, EI_x} + M_j \int_j^k \frac{x'^{(j)2}\, dx}{L_k^2\, EI_x}$$

$$+ M_k \int_j^k \frac{x^{(j)}\, x'^{(j)}\, dx}{L_k^2\, EI_x} + \int_j^k \frac{He_x\, x'^{(j)}\, dx}{L_k\, EI_x} = 0$$

The integrals in the above equation have definite and important physical interpretations, as explained later. Using the notations defined in (d), the equation becomes

$$\tau_{ji}^{(L)} + M_i G_{ij} + M_j F_{ji} + \tau_{ji}^{(H)}$$

$$+ \tau_{jk}^{(L)} + M_j F_{jk} + M_k G_{kj} + \tau_{jk}^{(H)} = 0$$

which can be rewritten as

$$M_i G_{ij} + M_j \sum F_j + M_k G_{kj} + \sum \tau_j^{(L)} + \sum \tau_j^{(H)} = 0 \qquad (6)$$

where

$$\sum F_j = F_{ji} + F_{jk}$$

$$\sum \tau_j^{(L)} = \tau_{ji}^{(L)} + \tau_{jk}^{(L)}$$

$$\sum \tau_j^{(H)} = \tau_{ji}^{(H)} + \tau_{jk}^{(H)}$$

Equation (6) is the general form of the three moment equations for a prestressed concrete continuous beam. The constants F, G and τ, introduced in Eq. (6), are defined in the following article.

(d) Angular Flexibilities, Load Functions and Prestress Functions

The integrals seen above have the following meaning:

$$G_{ij} = \int_i^j \frac{x^{(i)} \, x'^{(i)} \, dx}{L_j^2 \, EI_x} = \text{End slope of simple beam ij at j due to a unit couple applied at i}$$

$$G_{kj} = \int_j^k \frac{x^{(j)} \, x'^{(j)} \, dx}{L_k^2 \, EI_x} = \text{End slope of simple beam jk at j due to a unit couple applied at k}$$

$$F_{ji} = \int_i^j \frac{x^{(i)2} \, dx}{L_j^2 \, EI_x} = \text{End slope of simple beam ij at j due to a unit couple applied at j}$$

$$F_{jk} = \int_j^k \frac{x'^{(j)2} \, dx}{L_k^2 \, EI_x} = \text{End slope of simple beam jk at j due to a unit couple applied at j}$$

$$\tau_{ji}^{(L)} = \int_i^j \frac{BM_x^{(i)} \, x^{(i)} \, dx}{L_j \, EI_x} = \text{End slope of simple beam ij at j due to gravity loads}$$

$$\tau_{jk}^{(L)} = \int_j^k \frac{BM_x^{(j)} \, x'^{(j)} \, dx}{L_k \, EI_x} = \text{End slope of simple beam jk at j due to gravity loads}$$

$$\tau_{ji}^{(H)} = \int_i^j \frac{He_x \; x^{(i)} \, dx}{L_j \; EI_x} = \text{End slope of simple beam ij at j due to prestressing force}$$

$$\tau_{jk}^{(H)} = \int_j^k \frac{He_x \; x'^{(j)} \, dx}{L_k \; EI_x} = \text{End slope of simple beam jk at j due to prestressed force}$$

The above interpretations are further clarified by the sketches in Fig. 3.

The beam functions G's and F's depend only upon the geometry of the beam. They are available ready calculated for many common cases. The load functions $\tau^{(L)}$'s are also available calculated for many common loading conditions. The prestress functions $\tau^{(H)}$'s may be different for every case. The calculation of $\tau^{(H)}$ values is discussed in Chapter IV.

(e) <u>The Three Moment Equation in Carry-Over Form</u>

Dividing throughout by $\sum F_j$ in Equation (6) and rearranging

$$M_j = - \frac{\sum \tau_j^{(L)}}{\sum F_j} - \frac{\sum \tau_j^{(H)}}{\sum F_j} - M_i \frac{G_{ij}}{\sum F_j} - M_k \frac{G_{kj}}{\sum F_j}$$

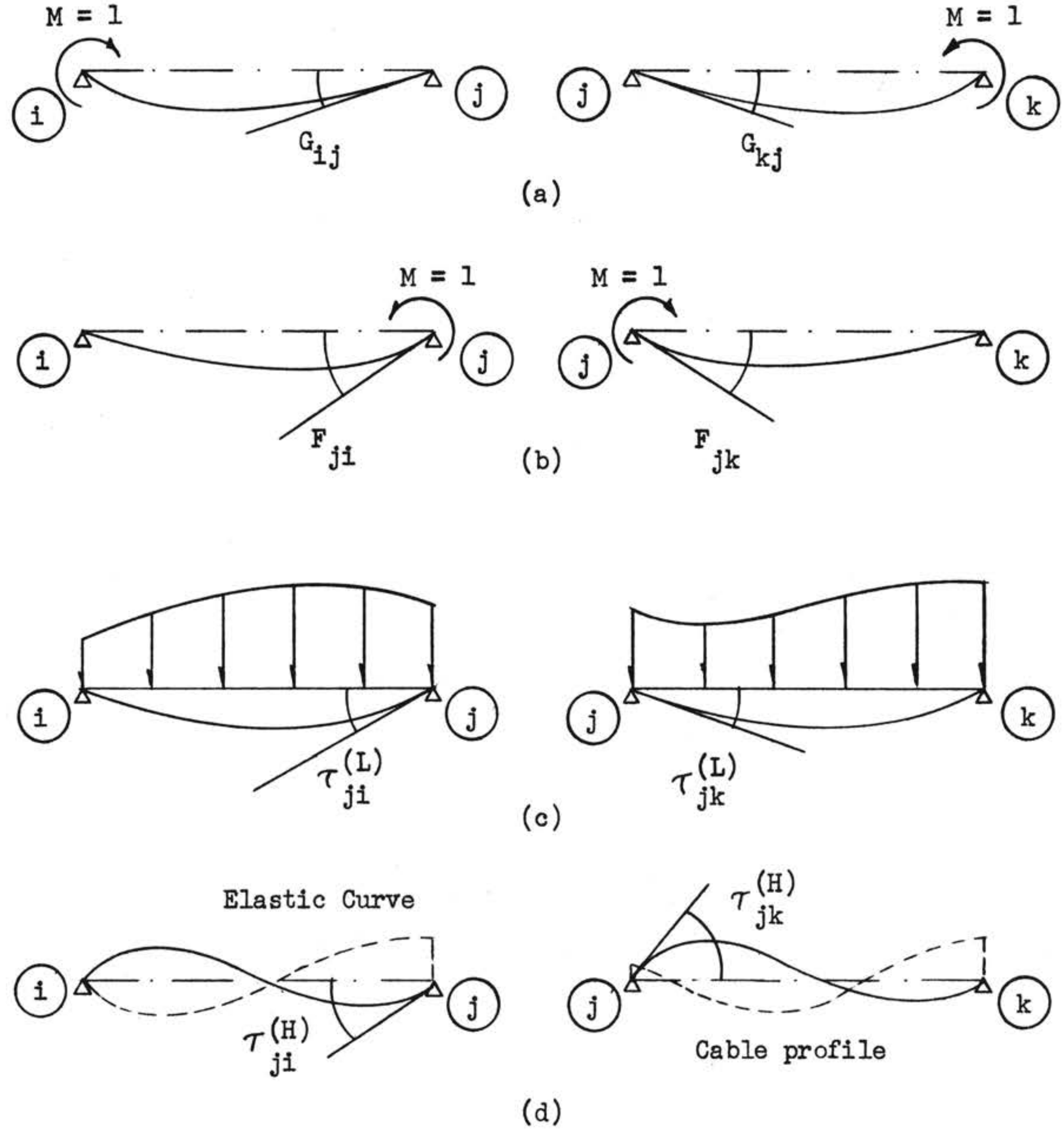


Fig. 3

Angular Functions

(a) Carry-Over Values
(b) Flexibilities
(c) Load Functions
(d) Prestress Functions

Defining the carry-over functions

$$-\frac{\sum \tau_j^{(L)}}{\sum F_j} = m_j^{(L)} \quad (7a), \qquad -\frac{\sum \tau_j^{(H)}}{\sum F_j} = m_j^{(H)} \quad (7b)$$

$$m_j^{(L)} + m_j^{(H)} = m_j = \text{Starting moment at } j$$

$$-\frac{G_{ij}}{\sum F_j} = r_{ij} = \text{Carry-over moment factor, } i \text{ to } j \quad (8a)$$

$$-\frac{G_{kj}}{\sum F_j} = r_{kj} = \text{Carry-over moment factor, } k \text{ to } j \quad (8b)$$

the equation for M_j becomes

$$M_j = m_j + M_i r_{ij} + M_k r_{kj} \quad (9)$$

The final bending moment at any support is thus expressed as a function of external load and the effects of bending moments at its adjacent supports. When all the supports are considered together, an iterative numerical procedure called the Carry-Over Moment Method can be set up, as discussed in the next chapter, to obtain the final bending moments at all supports.

CHAPTER III

CARRY-OVER PRECEDURE

AND

MODIFICATIONS FOR SPECIAL END CONDITIONS

The numerical carry-over precedure can be conveniently presented in a tabular form (Fig. 4).

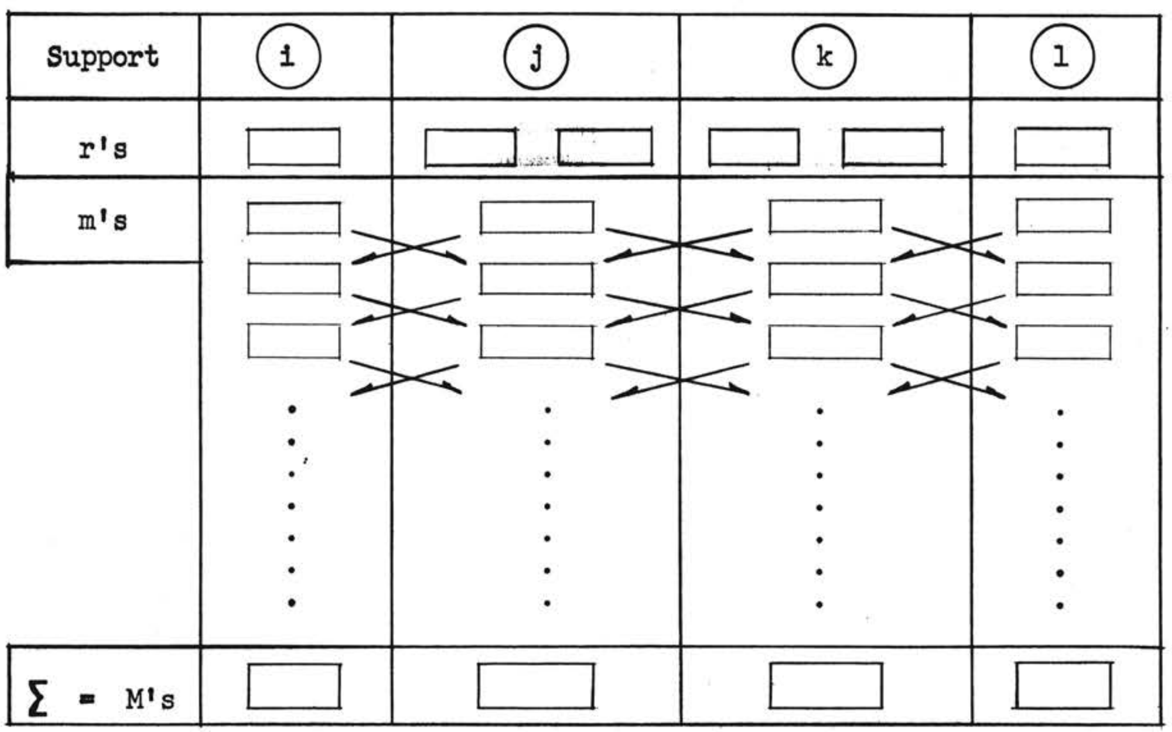


Fig. 4

A Typical Carry-Over Pattern

The starting moments m's are the starting values. The moment at every support is carried over to its adjacent supports, on multiplying by the carry-over factors. The carry-over factors are always less than unity, so the moment for each support results in a converging series, the total sum of which is the final moment M at the support.

Unit starting values can be used to calculate the influence on other supports, as shown in the illustrative example. The effect of prestressing load or any other load only, can be found by using the corresponding starting moments.

The numerical procedure can be carried out to a desired degree of accuracy. The procedure also has a numerical control in as much as all final moments computed thus must satisty Equation 9.

The effect of various conditions of fixity at the end supports of the beam is as follows:

(1) When an outer end is simply supported (Fig. 5(a)) the carry-over factors between the first inner support to the outer support are zero. An externally applied couple at support 0 may be treated an as external load, to calculate m_1.

(2) When an outer end in fixed (Fig. 5(b)) regular carry-over exists between the fixed end and the first inner support. In this case, however,

$$M_0 = m_0 + M_1 r_{10} \tag{10}$$

where $$m_0 = - \frac{\tau_{01}}{F_{01}} \tag{11}$$

and $$r_{10} = - \frac{G_{10}}{F_{01}} \tag{12}$$

(3) For an overhanging support (Fig. 5(c)) M_1 is calculated from statics and

$$r_{21} = 0 \quad , \quad r_{12} = -\frac{G_{12}}{\sum F_2}$$

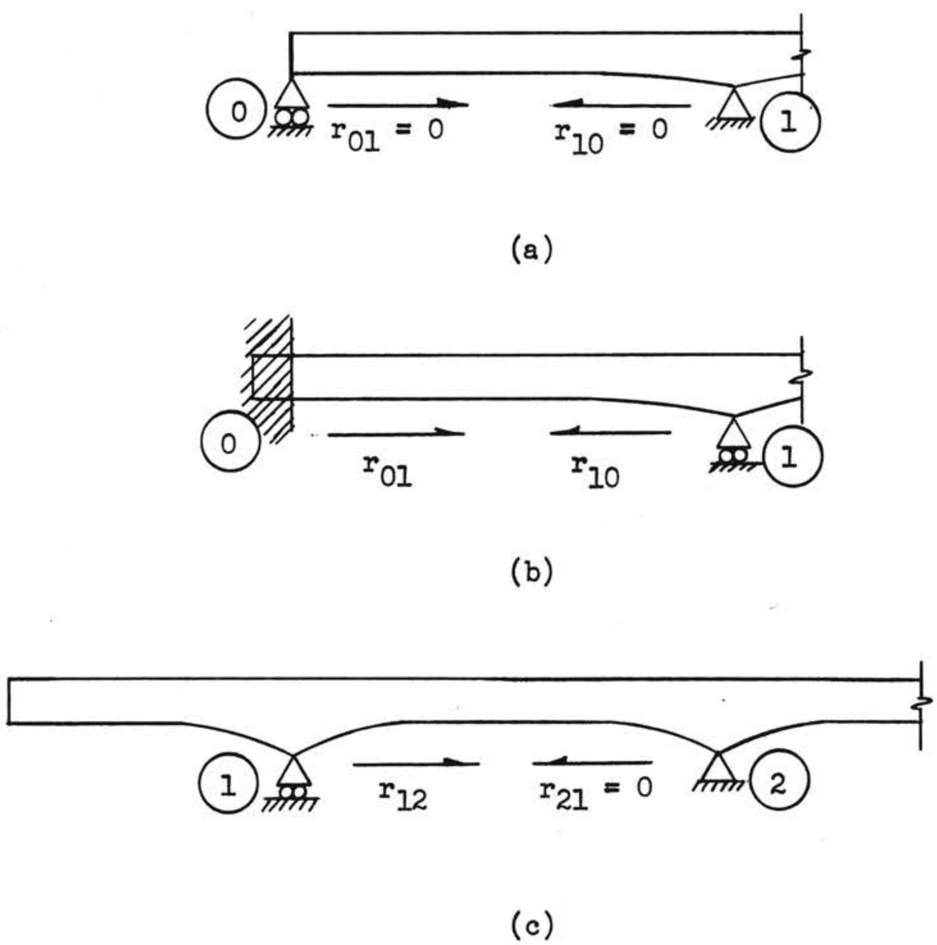


Fig. 5

Effect of End Conditions on

Carry-Over Factors

CHAPTER IV

EVALUATION OF $\tau^{(H)}$'S - END SLOPES OF SIMPLE BEAMS DUE TO PRESTRESS

The prestress functions $\tau^{(H)}$ have been defined in Chapter II as follows

$$\tau^{(H)}_{ji} = \int_i^j \frac{He_x x^{(i)}\, dx}{L_j\, EI_x} = \text{End slope of simple beam ij at j due to prestressing force H.}$$

$$\tau^{(H)}_{jk} = \int_j^k \frac{He_x x'^{(j)}\, dx}{L_k\, EI_x} = \text{End slope of simple beam jk at j due to prestressing force H.}$$

These formulas are convenient to use when e_x is defined mathematically as a function of x. When this is not the case or when the desired accuracy permits close approximation, the following methods can be used.

(a) Equivalent Elastic Load

The end slopes $\tau^{(H)}$'s are equal to the reactions of a conjugate beam loaded by $\frac{He}{EI}$ diagram. Considering the beam to be divided in several small lengths, the equivalent concentrated elastic load at any point m can be approximated as shown below (Fig. 6).

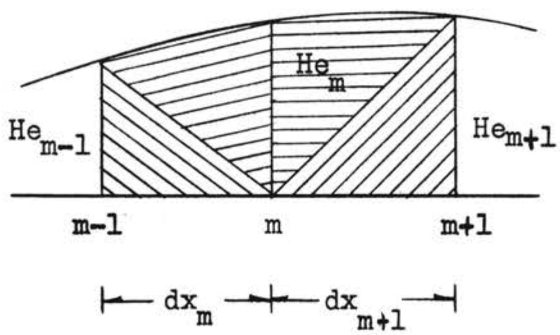


Fig. 6

Equivalent Concentrated Elastic Load

at a Point

Assuming a straight line variation of the He diagram within a segment,

$$\bar{P}_m^{(H)} = \frac{1}{EI_m}\left[\frac{1}{3}\frac{He_{m-1}\,dx_m}{2} + \frac{2}{3}\frac{He_m\,dx_m}{2} + \frac{2}{3}\frac{He_m\,dx_{m+1}}{2} + \frac{1}{3}\frac{He_{m+1}\,dx_{m+1}}{2}\right]$$

$$= \frac{H}{6EI_m}\left[e_{m-1}\,dx_m + 2\,e_m\,(dx_m + dx_{m+1}) + e_{m+1}\,dx_{m+1}\right] \qquad (13)$$

If $\qquad dx_m = dx_{m+1} = dx$,

$$\bar{P}_m^{(H)} = \frac{H\,dx}{6EI_m}\,(e_{m-1} + 4\,e_m + e_{m+1}) \qquad (14)$$

Using a set of such elastic loads for any span ij,

$$\tau_{ij}^{(H)} = \bar{R}_{ij} = \sum \frac{\bar{P}_m^{(H)} x'_m}{L_j} \tag{15}$$

$$\tau_{ji}^{(H)} = \bar{R}_{ji} = \sum \frac{\bar{P}_m^{(H)} x_m}{L_j} \quad . \tag{16}$$

(b) Equivalent Real Load

R. B. B. Moorman (1) applied the equation

$$\frac{d^2M}{dx^2} = w \tag{17}$$

to the moment due to prestress and presented the concept of equivalent load due to prestress, defined by

$$H \frac{d^2e_x}{dx^2} = w^{(H)} \quad . \tag{18}$$

If e_x is a mathematically defined function of x, $w^{(H)}$ can be calculated easily. A second degree parabolic variation of e_x has a uniformly distributed load as its equivalent. The effect of the prestress moment can, as well, be approximated by a set of equivalent concentrated loads acting along the beam. These are evaluated, in terms of the eccentricities, under various assumptions.

Consider the beam to be divided into several small lengths, each equal to dx. Assuming a linear variation of $w^{(H)}$ over each length, (Fig. 7) the equivalent concentrated load to any point m is equal to

$$P_m^{(H)} = \frac{1}{3}\,\frac{w_{m-1}^{(H)}\,dx}{2} + \frac{2}{3}\,\frac{w_m^{(H)}\,dx}{2} + \frac{2}{3}\,\frac{w_m^{(H)}\,dx}{2} + \frac{1}{3}\,\frac{w_{m+1}^{(H)}\,dx}{2}$$

$$= \frac{dx}{6}\left(w_{m-1}^{(H)} + 4\,w_m^{(H)} + w_{m+1}^{(H)} \right)$$

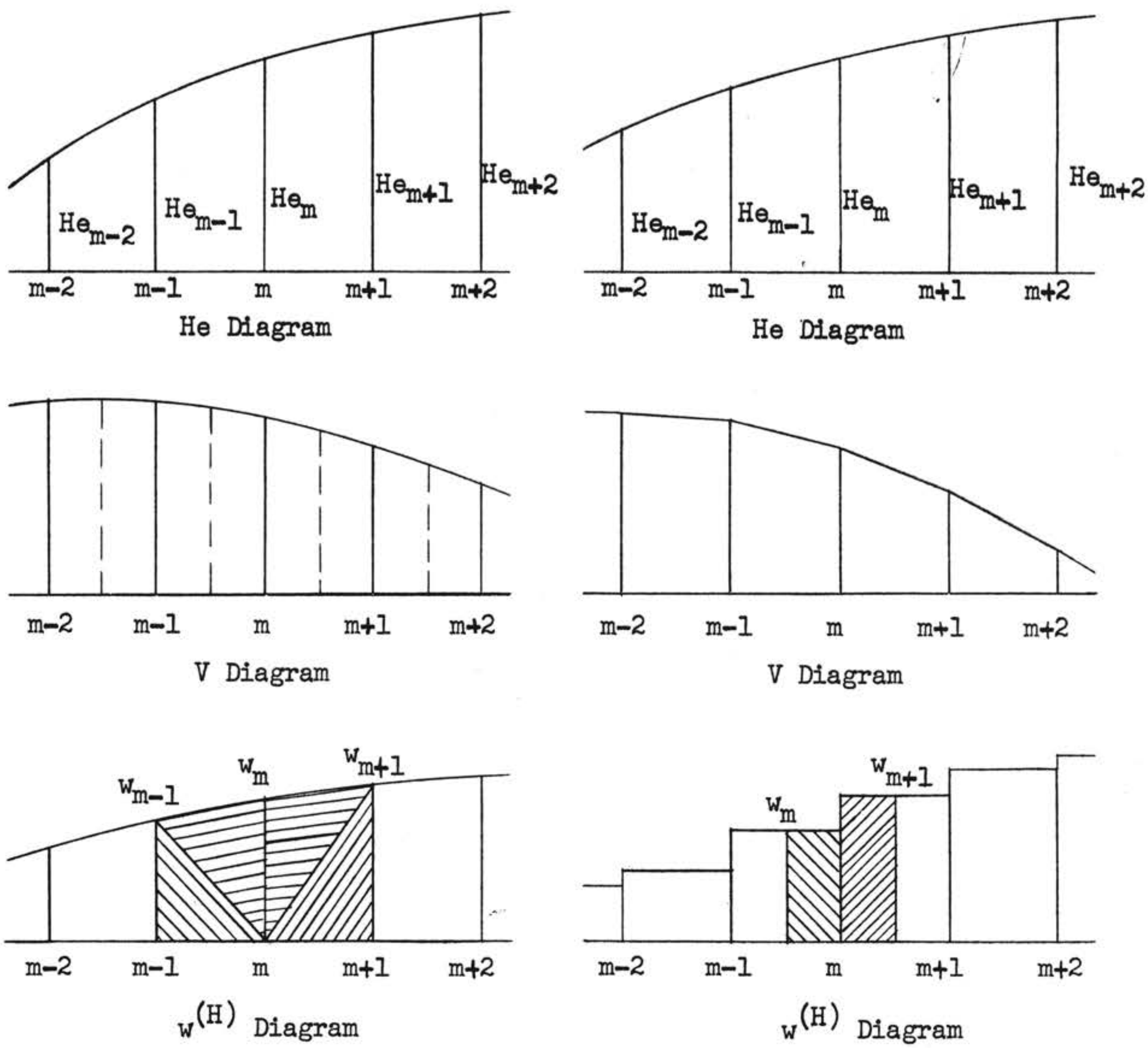


Fig. 7

Equivalent Concentrated Real Load

Replacing the w's by the second derivative with respect to x of the moment due to prestress, and using finite defference approximations,

$$P_m^{(H)} = \frac{Hdx}{6}\left[\frac{e_{m-2} - 2\, e_{m-1} + e_m}{(dx)^2} + 4\,\frac{e_{m-1} - 2\; e_m + e_{m+1}}{(dx)^2} + \frac{e_m - 2e_{m+1} + e_{m+2}}{(dx)^2}\right]$$

$$\therefore \quad P_m^{(H)} = \frac{H}{6dx}\left(e_{m-2} + 2\, e_{m-1} - 6\, e_m + 2\, e_{m+1} + e_{m-2}\right) \qquad (19)$$

The assumption of a linear variation of $w^{(H)}$ within each segment corresponds to assuming a third degree parabolic variation of the He diagram within a segment.

Alternatively, a uniform intensity $w^{(H)}$ can as well be assumed within a segment. For this the He diagram assumes a parabolic variation of second degree within a segment. For this assumption (Fig. 7)

$$P_m^{(H)} = \frac{dx}{2}\left(w_m^{(H)} + w_{m+1}^{(H)}\right)$$

$$= \frac{dx}{2}\left[\frac{V_m^{(H)} - V_{m-1}^{(H)}}{dx} + \frac{V_{m+1}^{(H)} - V_m^{(H)}}{dx}\right]$$

$$= \frac{1}{2}\left[V_{m+1}^{(H)} - V_{m-1}^{(H)}\right]$$

Using the central finite difference approximations:

$$P_m^{(H)} = \frac{H}{2}\left[\frac{e_{m+2} - e_m}{2\ dx} - \frac{e_m - e_{m-2}}{2\ dx}\right]$$

$$P_m^{(H)} = \frac{H}{4dx}\left(e_{m-2} - 2\, e_m + e_{m+2}\right) \qquad (20)$$

Finally for a simple assumption of linear variation of the He diagram within a segment the equivalent concentrated load at any point m can be calculated by evaluating the change of shear produced at m as shown in Fig. 8.

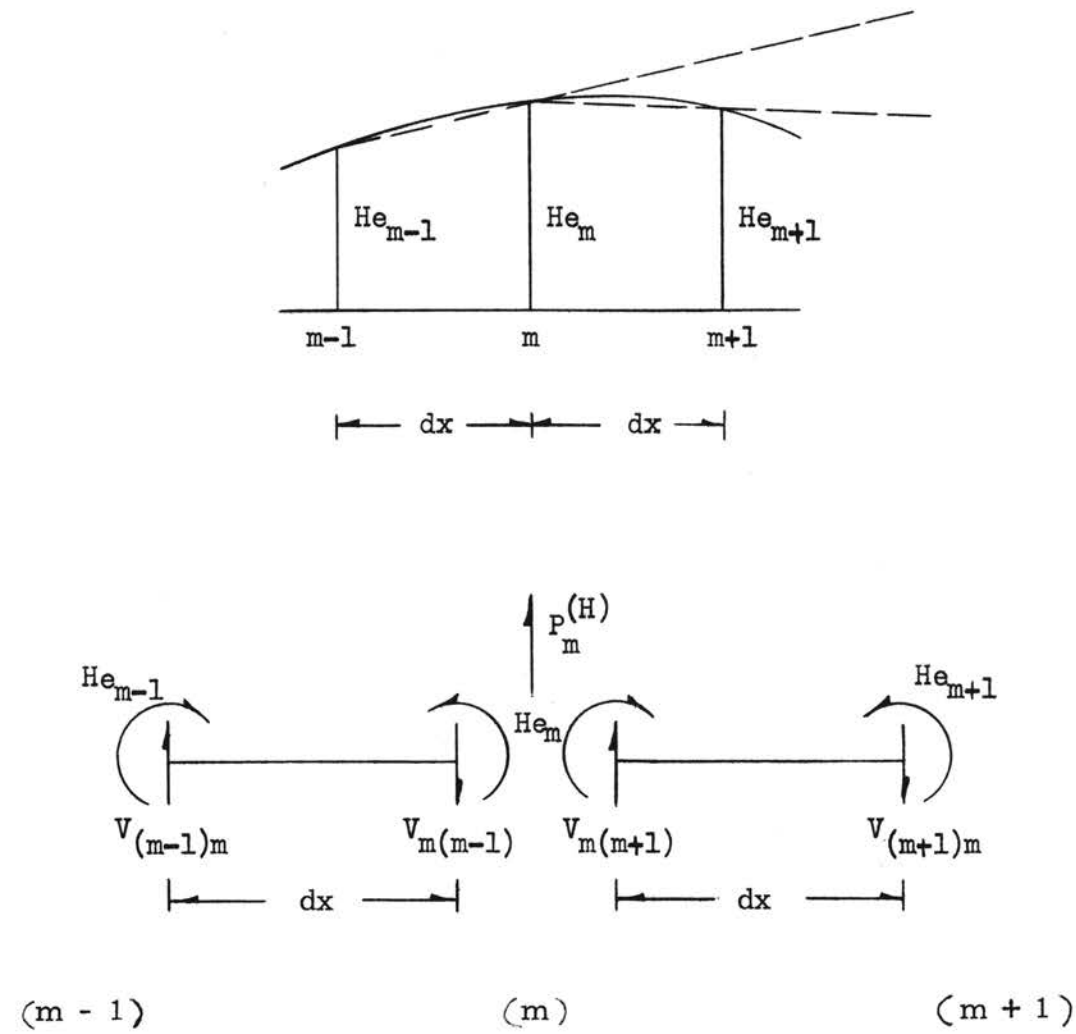


Fig. 8

Equivalent Concentrated Real Load

For this case, considering the equilibrium of the isolated segments,

$$P_m^{(H)} = V_{m(m+1)} - V_{m(m-1)}$$

$$= \frac{He_{m+1} - He_m}{dx} - \frac{He_m - He_{m-1}}{dx}$$

$$= \frac{H}{dx}\,(e_{m-1} - 2\,e_m + e_{m+1}) \tag{21}$$

The equations 14, 19, 20, and 21 are developed for any point within the span. They, however, need modification for points near the end of a span. For any span divided into n segments (Fig. 9), the values of $\overline{P}_o^{(H)}$ through $\overline{P}_n^{(H)}$ or the values of $P_1^{(H)}$ through $P_{n-1}^{(H)}$ need to be calculated.

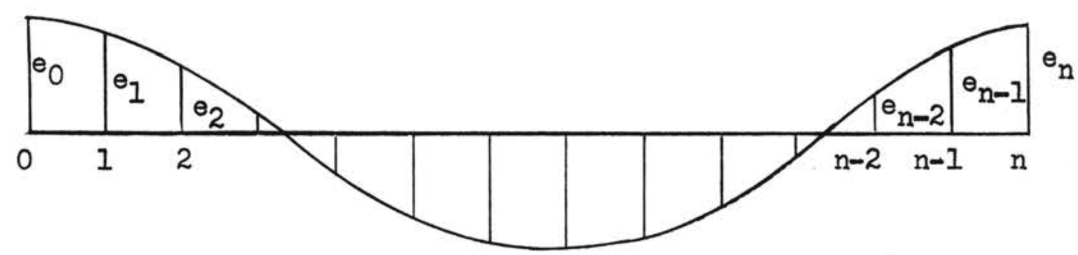


Fig. 9

A Span Divided in Equal Small Lengths

The equations for $P_m^{(H)}$ are modified by calculating the eccentricities beyond the span on the assumption that the He diagram continues with the same rate of change of slope. The results in terms of known eccentricities are presented in Table I.

It may be noted that the positive sign is assigned to $P_m^{(H)}$ when acting in the upward direction.

TABLE I

FORMULAS MODIFIED FOR

SPECIAL CASES

Modified Formulas	From Equation
$\bar{P}_0^{(H)} = \frac{H\,dx}{6EI_0}(2e_0 + e_1)$ $\bar{P}_n^{(H)} = \frac{H\,dx}{6EI_n}(e_{n-1} + 2e_n)$	(14)
$P_1^{(H)} = \frac{H}{6dx}(5e_0 - 9e_1 + 3e_2 + e_3)$ $P_{n-1}^{(H)} = \frac{H}{6dx}(e_{n-3} + 3e_{n-2} - 9e_{n-1} + 5e_n)$	(19)
$P_1^{(H)} = \frac{H}{4dx}(3e_0 - 5e_1 + e_2 + e_3)$ $P_{n-1}^{(H)} = \frac{H}{4dx}(e_{n-3} + e_{n-2} - 5e_{n-1} + 3e_n)$	(20)

The $\tau^{(H)}$ values for any span can be calculated as end slopes caused by these equivalent loads and end moments due to eccentricities at the ends. While using a set of concentrated loads, advantage can be taken of the reciprocal relationship between the deflection at a point due to a unit end moment and the end slope due to a unit load at that point. This procedure is convenient when presented in a tabular form, as can be seen from the illustrative example.

CHAPTER V

NUMERICAL EXAMPLES

General Note:

Two numerical examples are presented to illustrate the Carry-Over Moment Method.

A four-span continuous prestressed concrete beam of variable cross section is considered first. The angular beam functions are calculated using the method of finite strips. In calculating the load functions and the prestress functions, the reciprocal relationship is utilized. The use of approximate methods to evaluate the prestress functions is illustrated in this example. The carry-over procedure is shown using the actual starting moments.

In Example II, a three-span continuous prestressed concrete beam of constant cross section is analyzed. The beam constants and the angular functions due to load and prestress are evaluated by the exact formulas. In considering various load conditions, the use is illustrated of the carry-over procedure for unit starting moments.

Units of kips, feet and kip-feet are used in both problems.

Example I

A four-span continuous symmetrical beam of variable cross-section is considered. (Fig. 10). The relative EI values and the prestress eccentricities for points every four feet apart on spans AB and BC are given in Tables II (a) and II (b), respectively. The prestressing force is 250 ks.

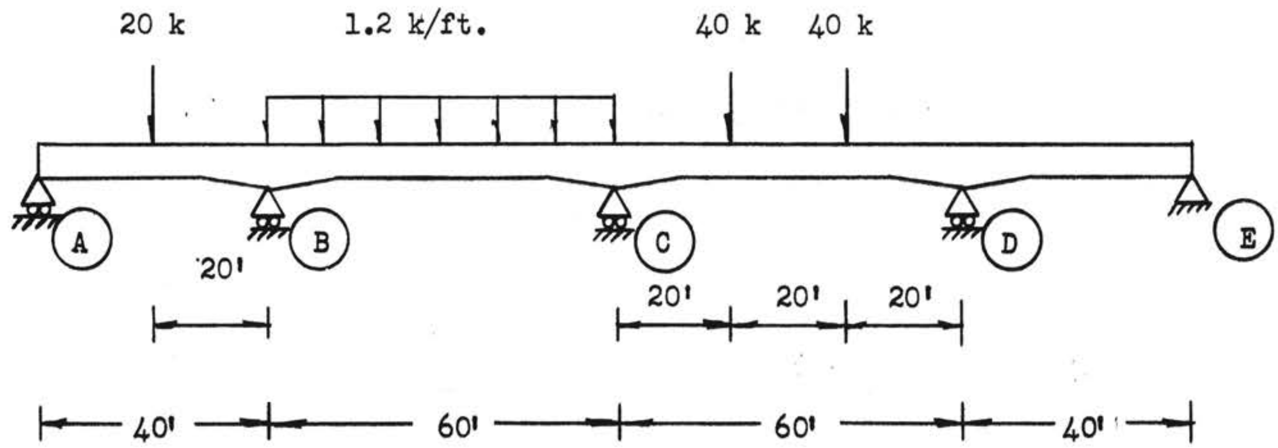


Fig. 10

A Four-Span Continuous Prestressed Concrete Beam of

Variable Cross Section.

TABLE II (a)

DATA FOR SPAN AB

m	EI	e ins	m	EI	e ins	m	EI	e ins
0	1.00	0.00	4	1.00	- 9.60	8	1.93	+ 5.00
1	1.00	-3.60	5	1.00	-10.00	9	3.36	+14.00
2	1.00	-6.40	6	1.00	- 8.67	10	5.42	+19.00
3	1.00	-8.40	7	1.00	-4.67	-	-	-

TABLE II (b)

DATA FOR SPAN BC

m	EI	e ins	m	EI	e ins
0, 15	5.42	+19.00	4, 11	2.82	- 5.20
1, 14	4.42	+16.10	5, 10	2.82	-10.00
2, 13	3.56	+ 9.90	6, 9	2.82	-13.20
3, 12	2.82	+ 1.20	7, 8	2.82	-14.80

1. Angular Beam Constants

The calculations for the angular flexibilities and carry-over values are shown in tabular form in Tables III (a) and III (b) for spans AB and BC, respectively. The method of conjugate beam is used to find the end slopes. The deflection of the elastic curve due to the applied unit moment are entered in the last columns as $\bar{M}_x$ values. By the reciprocal relationship these are the end slopes due to a unit load applied at the corresponding points. From the tables

$$F_{BA} = F_{DE} = + \frac{8.63}{EI}$$

$$F_{BC} = F_{CB} = F_{CD} = F_{DC} = + \frac{6.455}{EI}$$

$$\sum F_B = \sum F_D = + \frac{15.085}{EI}$$

$$\sum F_C = + \frac{12.910}{EI}$$

$$G_{BC} = G_{CB} = G_{CD} = G_{DC} = + \frac{3.396}{EI}$$

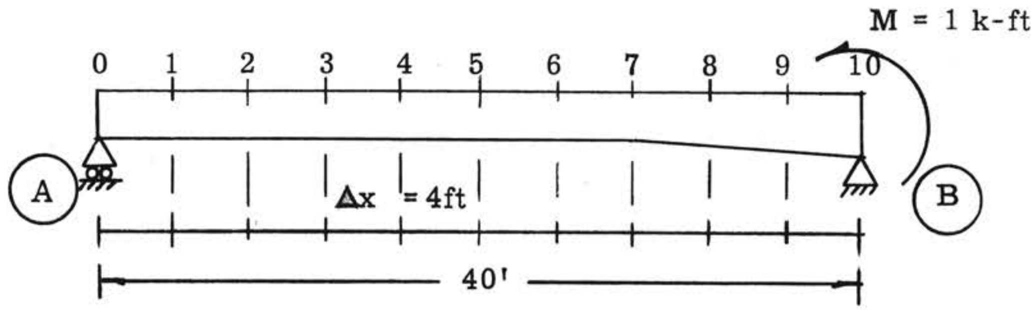
2. Carry-Over Factors

$$r_{BC} = r_{DC} = - \frac{3.396}{12.910} = -0.263$$

$$r_{CB} = r_{CD} = - \frac{3.396}{15.085} = -0.225$$

TABLE III (a)

ANGULAR FUNCTIONS - SPAN AB

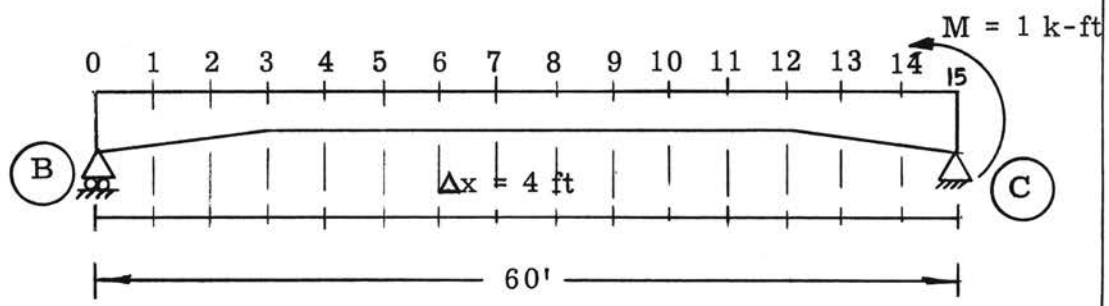


$$M_x = \frac{x}{L}, \quad \bar{P}_x = \frac{x}{L}\frac{dx}{EI_x}, \quad F_{BA} = \sum \bar{P}_x \frac{x}{L}$$

m	$\frac{x}{L}$	$\frac{dx\ EI}{EI_x}$	$\bar{P}_x$ EI	$\bar{P}_x \frac{x}{L}$ EI	$\bar{M}_x$ EI
0	0.0	4.00	0.00	0.00	00.00
1	0.1	4.00	0.40	0.04	24.12
2	0.2	4.00	0.80	0.16	46.68
3	0.3	4.00	1.20	0.36	66.04
4	0.4	4.00	1.60	0.64	80.60
5	0.5	4.00	2.00	1.00	88.76
6	0.6	4.00	2.40	1.44	88.92
7	0.7	4.00	2.80	1.96	79.48
8	0.8	2.07	1.66	1.33	58.84
9	0.9	1.19	1.07	0.96	31.56
10	1.0	0.74	0.74	0.74	0.00
			$\sum$	8.63	

TABLE III (b)

ANGULAR FUNCTIONS - SPAN BC



$$M_x = \frac{x}{L}, \quad \bar{P}_x = \frac{x}{L}\frac{dx}{EI_x}, \quad F_{CB} = \sum \bar{P}_x \frac{x}{L}, \quad G_{BC} = \sum \bar{P}_x \frac{x'}{L}$$

m	$\frac{x}{L}$	$\frac{x'}{L}$	$\frac{dx\ EI}{EI_x}$	$\bar{P}_x\ EI$	$\bar{P}_x \frac{x}{L}\ EI$	$\bar{P}_x \frac{x'}{L} EI$	$\bar{M}_x\ EI$
0	.000	1.000	0.740	0.000	0.000	0.000	0.000
1	.067	.933	0.900	0.060	0.004	0.056	13.584
2	.133	.867	1.120	0.149	0.020	0.129	26.928
3	.200	.800	1.418	0.284	0.057	0.227	39.676
4	.267	.733	1.418	0.379	0.101	0.278	51.288
5	.333	.667	1.418	0.472	0.157	0.315	61.384
6	.400	.600	1.418	0.567	0.227	0.340	69.592
7	.467	.533	1.418	0.662	0.309	0.353	75.532
8	.533	.467	1.418	0.756	0.403	0.353	78.824
9	.600	.400	1.418	0.851	0.511	0.340	79.120
10	.667	.333	1.418	0.946	0.631	0.315	75.980
11	.733	.267	1.418	1.039	0.762	0.278	69.056
12	.800	.200	1.418	1.134	0.907	0.227	57.976
13	.867	.133	1.120	0.971	0.842	0.129	42.360
14	.933	.067	0.900	0.840	0.784	0.056	22.860
15	1.000	.000	0.740	0.740	0.740	0.000	0.000
				$\sum$	6.455	3.396	

3. Load Functions

The end slopes due to loads are calculated using the influence values given in Tables III (a) and III (b). The distributed load on span BC is replaced by equivalent concentrated loads placed at every four-foot distance. Thus

$$\tau_{BA}^{(L)} = + \frac{1775.2}{El}$$

$$\tau_{BC}^{(L)} = \tau_{CB}^{(L)} = + \frac{3668.0}{El}$$

$$\tau_{CD}^{(L)} = \tau_{DC}^{(L)} = + \frac{5494.6}{El}$$

$$\tau_{DE}^{(L)} = 0 \ .$$

4. Prestress Functions

The end slopes $\tau^{(H)}$'s are evaluated using the approximate methods discussed in Chapter IV. Tables IV (a) and IV (b) show $P_m^{(H)}$ values, calculated using Equations (19), (20), and (21) for spans AB and BC, respectively. $\Delta x = 4$ ft.

Using these values of $P_m^{(H)}$ and the moments due to the eccentricities at ends of simple spans AB and BC, $\tau^{(H)}$ values are calculated. These $\tau^{(H)}$ values and an additional value for each span, obtained by using Equation (14), are entered in Table V.

Tables IV (a), IV (b), and V are worked out to show the comparative results under various approximations.

For the purpose of this example, values obtained by using Equation (20) are taken and the calculations completed.

TABLE IV (a)

EQUIVALENT CONCENTRATED LOADS DUE TO

PRESTRESS - SPAN AB

m	$P_m^{(H)}$ using		
	Eq. (19)	Eq. (20)	Eq. (21)
1	+ 4.167	+ 4.167	+ 4.167
2	+ 4.167	+ 3.646	+ 4.167
3	+ 4.167	+ 4.167	+ 4.167
4	+ 4.974	+ 5.378	+ 4.167
5	+ 9.019	+ 9.023	+ 9.010
6	+15.694	+16.589	+13.906
7	+21.424	+17.370	+29.531
8	- 0.877	+ 0.430	- 3.490
9	-17.943	-16.497	-20.833

TABLE IV (b)

EQUIVALENT CONCENTRATED LOADS DUE TO

PRESTRESS - SPAN BC

m	$P_m^{(H)}$ using Eq. (19)	$P_m^{(H)}$ using Eq. (20)	$P_m^{(H)}$ using Eq. (21)
1, 14	-16.493	-16.146	-17.187
2, 13	- 9.549	- 7.812	-13.021
3, 12	+ 7.205	+ 4.818	+11.979
4, 11	+ 8.941	+ 9.245	+ 8.333
5, 10	+ 8.333	+ 8.333	+ 8.333
6, 9	+ 8.333	+ 8.333	+ 8.333
7, 8	+ 8.333	+ 8.333	+ 8.333

TABLE V

$\tau^{(H)}$ VALUES BY VARIOUS METHODS

Using	$\tau_{BA}^{(H)}$ - SPAN AB			$\tau_{BC}^{(H)} = \tau_{CB}^{(H)}$ - SPAN BC		
	Due To $P_m^{(H)}$	Due To End Moments	Total	Due To $P_m^{(H)}$	Due To End Moments	Total
Eq. (19)	-4252	+3416	- 836	-4187	+3899	-288
Eq. (20)	-4140	+3416	- 724	-4123	+3899	-224
Eq. (21)	-4427	+3416	-1011	-4314	+3899	-415
Eq. (14)	-	-	-1089	-	-	-544

All Values to be Divided by EI

$$\tau_{BA}^{(H)} = \tau_{DE}^{(H)} = -\frac{724}{El}$$

$$\tau_{BC}^{(H)} = \tau_{CB}^{(H)} = \tau_{CD}^{(H)} = \tau_{DC}^{(H)} = -\frac{224}{El}$$

5. Total Starting Moments

$$m_B = -\frac{\sum \tau_B^{(L)+(H)}}{\sum F_B} = -\frac{4495.2}{15.085} = -298.0 \text{ kip. ft.}$$

$$m_C = -\frac{\sum \tau_C^{(L)+(H)}}{\sum F_C} = -\frac{8714.6}{12.910} = -675.0 \text{ kip. ft.}$$

$$m_D = -\frac{\sum \tau_D^{(L)+(H)}}{\sum F_D} = -\frac{4546.6}{15.085} = -301.4 \text{ kip. ft.}$$

6. Carry-Over Procedure (Please see the carry-over table on the next page)

Numerical Control

M_B = -298.0 -0.225 (-586.5) = -166.0 o.k.

M_C = -675.0 -0.263 (-166.0 - 169.4) = -586.8 o.k.

M_D = -301.4 -0.225 (-586.5) = -169.4 o.k.

Joint	B	C	D
r	-0.263	-0.225 -0.225	-0.263
m	-298.0	-675.0	-301.4
		+ 78.4	
		+ 79.3	
		-517.3	
	+116.4		+116.4
		- 30.6	
		- 30.6	
		- 61.2	
	+ 13.8		+ 13.8
		- 3.6	
		- 3.6	
		- 7.2	
	+ 1.6		+ 1.6
		- 0.4	
		- 0.4	
	+ 0.2	- 0.8	+ 0.2
∑ = M	-166.0	-586.5	-169.4

7. Final Moments

From the Carry-Over Table the final moments at the supports are

M_B = -166.0 kip. ft.

M_C = -586.5 kip. ft.

M_D = -169.4 kip. ft.

Example II

A three-span continuous prestressed concrete symmetrical beam (Fig. 11) of constant cross-section is analyzed for all possible combinations of given loadings that produce maximum and minimum bending moments in the spans. The prestressing cable profile is shown in Fig. 12 The intensities of the loads are

$w^{(g)}$ = 0.6 k/ft. (self weight)

$w^{(D)}$ = 0.8 k/ft. (slab load)

$w^{(L)}$ = 0.6 k/ft. (live load)

The prestressing force is

H_0 = 660 kips

H_n = 528 kips

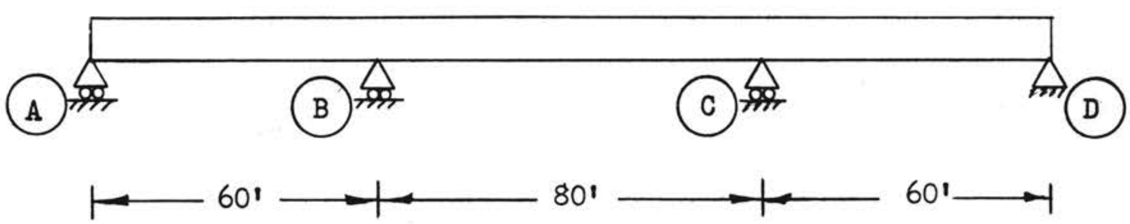


Fig. 11

A Continuous Three-Span Beam

of Constant Cross Section

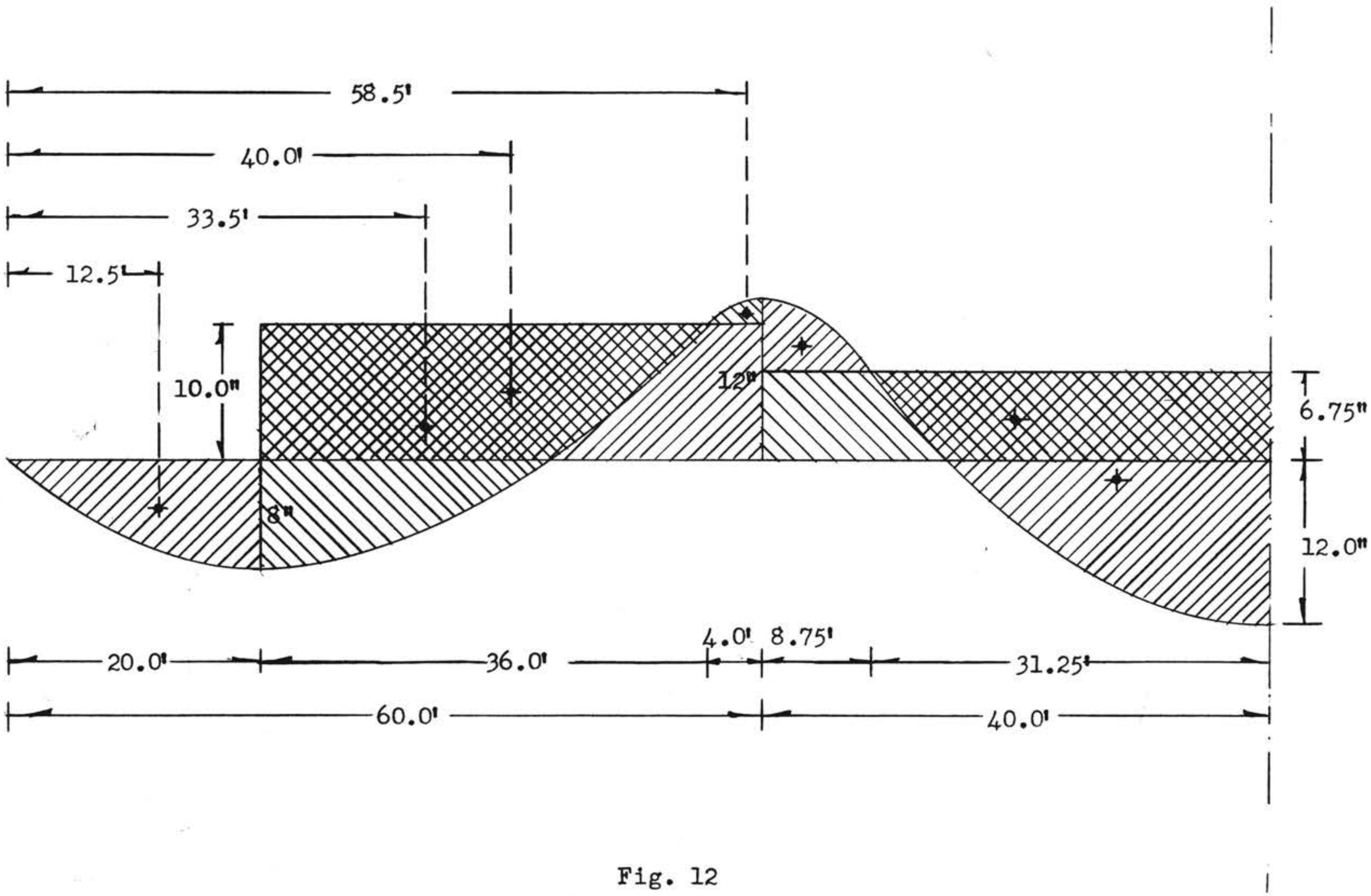


Fig. 12

Prestressing Cable Profile

1. Angular Flexibilities

$$F_{ij} = F_{ji} = \int_i^j \frac{x^2 dx}{L_j^2 EI} = \frac{L_j}{3EI}$$

$$\therefore \; F_{BA} = F_{CD} = \frac{60}{3EI} \qquad \frac{20}{EI}$$

$$F_{BC} = F_{CB} = \frac{80}{3EI} \qquad \frac{26.67}{EI}$$

$$\sum F_B = \sum F_C = \frac{46.67}{EI}$$

2. Angular Carry-Over Values

$$G_{ij} = G_{ji} = \int_i^j \frac{xx' dx}{L_j^2 \; EI} = \frac{L_j}{6EI}$$

$$\therefore \; G_{BC} = G_{CB} = \frac{80}{6EI} = \frac{13.33}{EI}$$

3. Carry-Over Factors

$$r_{BC} = -\frac{G_{BC}}{\sum F_C} = -\frac{13.33}{46.67} = -0.286$$

$$r_{CB} = -\frac{G_{CB}}{\sum F_B} = -\frac{13.33}{46.67} = -0.286$$

4. Carry-Over Procedure

For $m_B = +1.000$ — For $m_C = +1.000$

Joint	B	C
r	-0.286	-0.286
m	+1.000	
		-0.286
	+0.082	
		-0.023
	+0.007	
		-0.002
M	+1.089	-0.311

B	C
-0.286	-0.286
	+1.000
-0.286	
	+0.082
-0.023	
	+0.007
-0.002	
-0.311	+1.089

Numerical control

$$M_B^B = +1.000 - 0.286\ (-0.311) = +1.089 = M_C^C$$

$$M_C^B = 0.0 \quad - 0.286\ (+1.086) = -0.311 = M_B^C$$

5. Final Moments in Terms of the Starting Moments

$$M_B = +1.089\ m_B - 0.311\ m_C$$

$$M_C = +1.089\ m_C - 0.311\ m_B$$

6. Actual Starting Moments

$$\tau_{BA}^{(H)} = \tau_{CD}^{(H)} = \int_A^B \frac{He_x x dx}{LEI_x}$$

$$\tau_{BC}^{(Hn)} = \tau_{CB}^{(Hn)} = -\frac{3960}{EI}$$

The τ's due to uniformly distributed load on a span is calculated using $\tau_{ij} = \tau_{ji} = \frac{wL_i^3}{24EI}$, for $w^{(g)}$, $w^{(D)}$, $w^{(L)}$ and all results entered in Table VI.

TABLE VI

τ'S DUE TO LOADS AND PRESTRESS

τ Due To	$\tau_{BA} = \tau_{CD}$	$\tau_{BC} = \tau_{CB}$
$w^{(g)}$	$+\frac{5400}{EI}$	$+\frac{12800}{EI}$
$w^{(D)}$	$+\frac{7200}{EI}$	$+\frac{17067}{EI}$
$w^{(L)}$	$+\frac{5400}{EI}$	$+\frac{12800}{EI}$
H_0	$+\frac{464.44}{EI}$	$-\frac{4950}{EI}$
H_n	$+\frac{371.55}{EI}$	$-\frac{3960}{EI}$

The actual starting moments are now computed for the following conditions of loading:

Condition (a): $w^{(g)}$ and H_0 for all spans

Condition (b): $w^{(g)}$, $w^{(D)}$, H on all spans and $w^{(L)}$ on spans AB and BC

Condition (c): $w^{(g)}$, $w^{(D)}$, H on all spans and $w^{(L)}$ on span BC only

Condition (d): $w^{(g)}$, $w^{(D)}$, H on all spans and $w^{(L)}$ on span CD only

Condition (e): $w^{(g)}$, $W^{(D)}$, H on all spans and $w^{(L)}$ on spans AB and CD.

Using the appropriate values of $\sum \mathcal{T}$'s and $\sum F$'s, the starting moments are computed and recorded in Table VII.

TABLE VII

STARTING MOMENTS FOR VARIOUS CONDITIONS OF LOADING

Condition of Loading	m_B using		m_C using	
	H_0	H_n	H_0	H_n
(a)	- 293.9	- 313.1	- 293.9	- 313.1
(b)	-1203.9	-1223.1	-1088.1	-1107.3
(c)	-1088.1	-1107.3	-1088.1	-1107.3
(d)	- 813.9	- 833.1	- 929.7	- 948.9
(e)	- 929.7	- 948.9	- 929.7	- 948.9

7. Final Moments

The final moments at B and C are computed using these starting moments and the results are shown in Table VIII. The purpose, for which these conditions of loading are useful, is also indicated in the table.

$$= \frac{1}{LEI}\begin{bmatrix}\text{Static moment of the He diagram} \\ \text{on AB about A}\end{bmatrix}$$

$$= \frac{H}{60EI}\Big[-(\tfrac{2}{3})(20)(\tfrac{8}{12})(12.5) - (\tfrac{2}{3})(36)(\tfrac{18}{12})(33.5)$$
$$+(40)(\tfrac{10}{12})(40) + (\tfrac{2}{3})(4)(\tfrac{1}{6})(58.6)\Big]$$

$$= +\frac{0.7037\ H}{EI}$$

$$\therefore\ \tau_{BA}^{(H_0)} = \tau_{CD}^{(H_0)} = \frac{464.44}{EI}$$

$$\tau_{BA}^{(H_n)} = \tau_{CD}^{(H_n)} = \frac{371.55}{EI}$$

$$\tau_{BC}^{(H)} = \tau_{CB}^{(H)} = \int_B^C \frac{He_x x'dx}{LEI}$$

$$= \frac{1}{LEI}\begin{bmatrix}\text{Static moment of the He} \\ \text{diagram on BC about C}\end{bmatrix}$$

$$= \frac{1}{EI}\left[\frac{1}{2}\ (\text{Area of the He diagram on BC})\right]$$

$$= \frac{H}{EI}\Big[(\tfrac{2}{3})(8.75)(\tfrac{5.25}{12}) + (40)(\tfrac{6.75}{12})$$
$$- (\tfrac{2}{3})(31.25)(\tfrac{18.75}{12})\Big]$$

$$= -\frac{7.50\ H}{EI}$$

$$\therefore\ \tau_{BC}^{(H_0)} = \tau_{CB}^{(H_0)} = -\frac{4950}{EI}$$

TABLE VIII

FINAL MOMENTS M_B AND M_C

Condition of Loading	M_B using H_0	M_B using H_n	M_C using H_0	M_C using H_n	To Be Used for Computing
(a)	-228.7	-243.6	-228.7	-243.6	Stresses before loading
(b)	-972.6	-987.6	-810.5	-825.5	Maximum negative M_B
(c)	-846.5	-861.5	-846.5	-861.5	Maximum positive moment in span BC Minimum positive moment in spans AB and CD
(d)	-597.2	-612.1	-759.3	-774.3	Minimum negative M_B
(e)	-723.3	-738.2	-723.3	-738.2	Minimum positive moment in span BC Maximum positive moment in spans AB and CD

CHAPTER VI

SUMMARY AND CONCLUSIONS

A method of analysis of prestressed concrete continuous beams, using the Carry-Over Moment Method, is presented in this thesis.

The relation between the final bending moments at any three consecutive supports is established in terms of the functions of given loads, prestress data and angular beam functions. The angular flexibilities, carry-over values, load functions and prestress functions are defined, and their physical meaning is explained.

The actual carry-over procedure is explained as a numerical procedure of successive approximation that can be carried out to a desired degree of accuracy. Basically, in this case, it solves a set of three moment equations by an iterative process.

The Carry-Over Method, originated by Prof. Jan J. Tuma, obviously finds a great advantage over the conventional methods for the analysis of continuous beams. The superiority is further amplified in cases involving large numbers of spans where a direct solution of a set of simultaneous equations would be highly cumbersome. The carry-over factors are usually small, and since there is only one column for the values of the moment at each support, the carry-over tables are compact. The procedure is simple and has good physical meaning.

By virtue of the angular flexibilities and load functions being readily available for many common cases, and in light of all methods discussed in this thesis to evaluate the prestress functions, the application of the Carry-Over Moment Method to the analysis of prestressed concrete continuous beams should be found easily adaptable.

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