THE DOOLITTLE TECHNIQUE:
HISTORY AND APPLICATION

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INTRODUCTION

This paper is an attempt to integrate into one source the application of the "Abbreviated Doolittle Technique" for solving a system of normal equations.

Although the "Abbreviated Doolittle" is primarily a method for obtaining the solution to a set of equations, we shall point out many statistical computations that come about in the process of applying this method to a set of normal equations. Hereafter, when we refer to a set of normal equations, we will mean that the coefficient matrix is symmetric.

The approach in this presentation will be confined mostly to examples of this method applied to the more common statistical designs. We will confine the examples to normal equations containing a small number of unknowns to expedite the reading. This will be without loss of generality since the "Technique" is identical for larger systems of equations.

CHAPTER I

A BRIEF HISTORY OF THE "DOOLITTLE TECHNIQUE"

In most computational work involving systems of linear equations one is confronted with the problem of solving n equations in n unknowns. This problem, prior to the advent of the numerous present day mechanical computing aids was imposing as n became large.

Concerning this problem M. H. Doolittle, an employee in the U. S. Coast and Geoditic Survey Office, presented a paper dated November 9, 1878 [1].

We will present this original method in a simple example, as we intend to do for all applications. Keep in mind these applications may be easily extended to n equations in n unknowns. This example will be set up exactly as the method was presented by Doolittle in the above paper.

Suppose we have the following system of three equations in three unknowns:

$$\begin{aligned} 0 &= 2x + 4y + 2z - 6 \\ 0 &= 4x + 10y + 2z - 18 \\ 0 &= 2x + 2y + 12z - 16 \end{aligned} \qquad (1.1)$$

This solution for x, y, and z is found as follows.

Table I

Original Doolittle for solving Equations (1. 1)

A							B				
1	2	3	4	5	6	7	1	2	3	4	5
			x	y	z				y	z	
1	1		2	4	2	-6	1	3	10	2	-18
2	2	-.5	x=	-2	-1	3	2	4	-8	-4	12
3	5			2	-2	-6	3	7		12	16
4	6	-.5		y=	1	3	4	8		-2	6
5	10				8	16	5	9		-2	-6
6	11	-.125			z=	-2					

line

2 $x = -2y - z + 3$ $x = 3$

4 $y = z + 3$ $y = 1$ (1.1.1)

6 $z = -2$ $z = -2$

The first column in sub-tables A and B of Table I given the number of the line and the second column, the order of procedure.

The coefficients and absolute term of equation 1 are entered in line 1, columns 4 to 7 of Table A. The negative reciprocal of the coefficient of x, is entered in line 2, column 3. All the remaining numbers in line 1 are multiplied by this reciprocal and the products entered in line 2. This gives the value of x as an explicit function of y and z.

The coefficients and absolute term of equation 2, (omitting the coefficient of x, already in the first equation in Table A), are now written in line 1, Table B. The coefficient of y and all the following numbers of line 1, Table A, are now multiplied by the coefficient [illegible] y in line 2 of Table A, and the products are written in line 2, Table B. The algebraic sum of line 1 and 2, Table B, is now entered in line 3, Table A, and line 4 is formed from line 3, exactly as line 2 was formed from line 1.

Omitting the coefficients x and y, the remaining coefficients, and the absolute value term of equation 3, is entered in line' 3 of Table B. The coefficients of z and the following numbers in line 1 and 3, Table A, are respectively multiplied by the coefficients of z in lines 2 and 4, these products entered in line 4 and 5, Table B. The algebraic sum of lines 3, 4, and 5 of Table B, is now entered in line 5, Table A.

Now, if there were other equations and unknowns, this process would be repeated. In our example, the last number in column 7, Table A, is an approximation of the unknown z. We can now take the explicit equations in lines 2, 4, and 6, Table A, for the approximate solution to the set of equations (see equations 1. 1. 1). We have used approximate solution here in the sense that if we had chosen to use decimal equivalent numbers instead of fractions, then we would obtain only an approximation rather than an exact solution. Decimal notation is used when computations are done on a desk calculator or when the "technique" is programmed for an electronic computer.

We can summarize by stating that this "technique" is merely a systematic, mechanical procedure to obtain the solution for n equations in n unknowns. The procedure being restricted to row operations on the system.

CHAPTER II

PROCEDURE FOR APPLYING THE "ORDINARY" AND "ABBREVIATED DOOLITTLE"

2.1. The Ordinary Doolittle

As stated before, the Doolittle technique is a method of obtaining the solution to a set of n equations in n unknowns. Before proceeding further, let us define the terminology concerning this method, when applied to a system of normal equations.

Suppose we have the system of normal equations $X'X\hat{\beta} = X'Y$. This system of equations is in matrix notation where the matrices are defined as follows:

(1) $X'X$ is an $n \times n$ matrix of constant coefficients and will be referred to as the coefficient matrix. Often we are interested in finding the inverse of this matrix, and the procedure for finding the inverse will be discussed later. This matrix is always symmetric about the main diagonal.

(2) $\hat{\beta}$ is an $n \times 1$ vector of unknown parameters, whose elements we wish to estimate by application of the Doolittle method.

(3) $X'Y$ is an $n \times 1$ vector of constants.

For example, consider the following set of three equations in three unknowns.

$$\begin{aligned} 2\hat{\beta}_1 + 4\hat{\beta}_2 + 2\hat{\beta}_3 &= 6 \\ 4\hat{\beta}_1 + 10\hat{\beta}_2 + 2\hat{\beta}_3 &= 18 \\ 2\hat{\beta}_1 + 2\hat{\beta}_2 + 12\hat{\beta}_3 &= -16 \end{aligned} \qquad (2.1)$$

Using the notation described above, this system becomes

$$\begin{bmatrix} 2 & 4 & 2 \\ 4 & 10 & 2 \\ 2 & 2 & 12 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ -16 \end{bmatrix} \tag{2. 1a}$$

or

$$[X'X]\,[\,\hat{\beta}\,] = [\,X'Y\,] \tag{2. 1b}$$

or

$$X'X\hat{\beta} = X'Y \quad . \tag{2. 1c}$$

Equations 2. 1, 2. 1a, 2. 1b, and 2. 1c, are equivalent expressions.

We now wish to investigate both the ordinary Doolittle and the Abbreviated Doolittle when applied to such a system of normal equations. We want to solve for approximations to the elements of the vector of unknowns $\hat{\beta}$. We define the "forward solution" as follows. This is the procedure that triangularizes $X'X$. We can say then that the forward solution transforms $X'X$ so that $\hat{\beta}$ may be obtained. The actual obtaining of $\hat{\beta}$ is a consequence of, rather than a part of, the forward solution.

Many times we are interested in obtaining the inverse of $X'X$. This can be accomplished at the same time the forward solution is carried out. The only difference is a tableau change. The procedure for obtaining the inverse of $X'X$ is defined to be the "backward solution." We will investigate this procedure later in this chapter.

Using the system of equations (2.1) for an example, let us apply the ordinary Doolittle to obtain a solution for the β_i. The layout for the forward solution is as follows. Note that only the elements of the coefficient matrix and the constant vector are used in the layout.

Table II

Ordinary Doolittle Technique

		X'X Column				
		1	2	3		
Instruction	Row	β_1	β_2	β_3	X'Y	Check Column
	R_1	2	4	2	6	14
	R_2	4	10	2	18	34
	R_3	2	2	12	-16	0
$1/2\ R_1$	R_4	1	2	1	3	7
R_2-2R_1	R_5	0	2	-2	6	6
R_3-R_1	R_6	0	-2	10	-22	-14
R_4	R_7	1	2	1	3	7
$1/2\ R_5$	R_8	0	1	-1	3	3
$R_6 + R_5$	R_9	0	0	8	-16	-8
R_7	R_{10}	1	2	1	3	7
R_8	R_{11}	0	1	-1	3	3
$1/8R_9$	R_{12}	0	0	1	-2	-1

The prupose of the technique is to triangularize the X'X matrix, using row operations on the system. This is known as the forward solution. It is a well known fact, that row operations on such a system do not affect the resulting solution. The Instruction column in Table II merely explains what operations on the rows of the system we are to perform.

If we now consider R_{10}, R_{11}, and R_{12}, in Table II, we have the following system with X'X now upper triangular. From this stem $\hat{\beta}$ may be obtained as follows:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$$

or

$$\hat{\beta}_1 + 2\hat{\beta}_2 + \hat{\beta}_3 = 3 \tag{2. 2}$$

$$\hat{\beta}_2 - \hat{\beta}_3 = 3 \tag{2. 3}$$

$$\hat{\beta}_3 = -2 \tag{2. 4}$$

Equation (2. 4) gives us $\hat{\beta}_3$. Substituting this in Equation (2. 3), we can obtain $\hat{\beta}_2$, then substituting in Equation (2. 2) for $\hat{\beta}_2$ and $\hat{\beta}_3$, we obtain $\hat{\beta}_1$, thus, we have solved for the $\hat{\beta}$ vector with the following result.

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \tag{2. 5}$$

The check column is the sum of the numbers in each row. The same operations are performed on this column as on other elements. That is, the elements of this column are treated as members of their respective rows. After each operation, the check column must be the sum of the elements of that particular row. This serves as a check against arithmetic mistakes. We will discuss the validity of this later.

2.2 The Abbreviated Doolittle

We now apply the Abbreviated Doolittle to the same system. The primary difference being that since the coefficient matrix $X'X$ is symmetric with respect to the main diagonal, it is not necessary to use the coefficients below the main diagonal.

The layout for the Abbreviated Doolittle is shown in Table III.

The computational outline for Table III is as follows:

(1) Write the $X'X$ matrix as shown, omitting all elements below the main diagonal. Designate these rows as R_1', R_2', R_3'. Note the check column sum includes the omitted elements of $X'X$.

(2) Write down the first row of the $X'X$ matrix to obtain R_1.

(3) Divide all elements of R_1 by the leading element of R_1 to obtain r_1. R_1 and r_1 will be called the computational rows associated with R_1'.

(4) Determine pivotal multiplier for obtaining R_2. This will be the second element in r_1. In particular, the element marked with a square.

(5) To obtain the elements of R_2, subtract the product of the pivotal multiplier and each element of R_1 from the respective elements

Table III

Abbreviated Doolittle Method for Solving Equations (2. 1)

Instructions	Row	X' X Column			X'Y	Check Row Sum
		1	2	3		
	R_1'	2	4	2	6	14
	R_2'		10	2	18	34
	R_3'			-12	-16	0
R_1'	R_1	2	4	2	6	14
$1/2\ R_1$	r_1	1	$\boxed{2}$	$\textcircled{1}$	3	7
$R_2' - \boxed{2} R_1$	R_2		2	-2	6	6
$1/2\ R_2$	r_2		1	$\textcircled{-1}$	3	3
$R_3' - [\textcircled{-1} R_2 + \textcircled{1} R_1]$	R_3			8	-16	-8
$1/8\ R_3$	r_3			1	-2	-1

of R_2', omitting all elements of R_1 to the left of the pivotal multiplier. These are omitted because only zeros are obtained if they are used.

(6) Again obtain r_2 by dividing R_2 by the leading element.

(7) Determine pivotal multipliers for obtaining R_3. These will be the third elements in r_1 and r_2. In particular, the elements marked with a circle.

(8) To obtain the elements of R_3 find the product of the pivotal multiplier in r_2 with R_2 again omitting all elements of R_2 to the left of the

multiplier. Subtract this sum from the respective elements of R_3'. This gives R_3. r_3 is obtained by dividing R_3 by its leading element.

It is important to note that, in this procedure, it requires two computational rows for each row of X'X. The second of which is always obtained from the first by dividing each element of the first by its leading element. We will refer to R_1 and r_1, R_2 and r_2. . . R_n and r_n as sets of computational rows.

To facilitate the extension to n equations in n unknowns we will do a three by three system in general notation.

Consider Table IV which will be analagous to Table III.

Notw: (1) For the general computational row term R_{ij}, we have $i = (1, 2, \ldots, n)$ and $j = (1, 2, \ldots, n+1)$ where n is the number of unknowns in the system. Now for $i > 1$ the following formula yields any R_{ij}.

$$R_{ij} = R_{ij}' - \sum_{k=1}^{i-1} r_{k_i} R_{k_j}$$

For example, suppose we want R_{33} in Table (2. 2). Using the formula we have $R_{33} = R_{33}' - [r_{13}R_{13} + r_{23}R_{23}] = 12 - [\textcircled{-1} \cdot (-2) + \textcircled{1} \cdot (2)] = 8.$

It may seem that determining the pivotal multipliers is somewhat difficult, but we shall now formulate a rule to determine these multipliers.

Rule for Determining Pivotal Multipliers.

The number of multipliers for R_n' will be $(n-1)$. These will be the elements in the n-th column of rows $r_{n-1}, r_{n-2}, \ldots, r_1$.

Table IV

Abbreviated Doolittle in General

Instructions	Row	X' X Column			X'Y	Check (Row Sum)
		1	2	3		
	R_{1j}'	R_{11}'	R_{12}'	R_{13}'	R_{14}'	$\Sigma_j R_{1j}'$
	R_{2j}'		R_{22}'	R_{23}'	R_{24}'	$\Sigma_j R_{2j}'$
	R_{3j}'			R_{33}'	R_{34}'	$\Sigma_j R_{3j}'$
R_1'	R_{1j}	R_{11}	R_{12}	R_{13}	R_{14}	$\Sigma_j R_{1j}'$
$\frac{R_{1j}}{R_{11}}$	r_{1j}	r_{11}	$\boxed{r_{12}}$	$\enclose{circle}{r_{13}}$	r_{14}	$\Sigma_j r_{1j}$
$R_{2j}' - r_{12}R_{1j}$	R_{2j}		$R_{22}' - r_{12}R_{12}$	$R_{23}' - r_{12}R_{13}$	$R_{24}' - r_{12}R_{14}$	$\Sigma_j R_{2j}$
$\frac{R_{2j}}{R_{22}}$	r_{2j}		1	$\enclose{circle}{\frac{R_{23}}{R_{22}}} = r_{23}$	$\frac{R_{24}}{R_{22}} = r_{24}$	$\Sigma_j r_{2j}$
$R_{33}' - [r_{13}R_{1j} + r_{23}R_{2j}]$.	R_{3j}			$R_{33}' - [r_{23}R_{23} + r_{13}R_{13}]$	$R_{34}' - [r_{23}R_{24} - r_{13}R_{14}]$	$\Sigma_j R_{3j}$
$\frac{R_{3j}}{R_{33}}$	r_{3j}			1	$\frac{R_{34}}{R_{33}} = r_{34}$	$\Sigma_j r_{3j}$

Example:

Suppose we want the pivotal multipliers for R_5' in a seven by seven system. We know then that there will be four in all and they will be the elements in the fifth column of rows r_4, r_3, r_2, r_1.

Returning to Table III we note that r_1, r_2, and r_3, give

$$\begin{bmatrix} 1 & 2 & 1 \\ & 1 & -1 \\ & & 1 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$$

exactly as in the ordinary Doolittle and again

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

Also, note that it has taken three less computational rows in the Abbreviated Doolittle as opposed to the Ordinary Doolittle. We again point out that although this has been presented in the special case of three equations in three unknowns, it is most readily extended to n equations in n unknowns. The procedure is identical.

Using the Doolittle Method to Obtain the Inverse of X'X

As we shall see later, many times the inverse of X'X is desired. This can be obtained, by what we shall refer to as the backward solution of the Abbreviated Doolittle.

To obtain the backward solution, we again apply the forward solution with this tableau change; in addition we write the n x n identity matrix to the right of X'Y in Table III and treat each element as members of R'_1, R_2', R_3'. The check column is obviously different, since each R_n' now is increased by one. We will refer to this new section as the $[X'X]^{-1}$ section. The forward solution is now carried out on this new layout just as in Table III.

Table V

The Abbreviated Doolittle Method for Solving Equations (2. 1) and Determining the Inverse of the Coefficient Matrix

Instructions	Row	X' X Column			X'Y	$[X'X]^{-1}$ Column			Check (Row Sum)	Row Identi-fication
		1	2	3		1	2	3		
	R_1'	2	4	2	6	1	0	0	15	
	R_2'		10	2	18	0	1	0	35	
	R_3'			12	-16	0	0	1	1	
R_1'	R_1	2	4	2	6	1	0	0	15	T_1
$1/2\ R_1$	r_1	1	$\boxed{2}$	①	3	$\boxed{1/2}$	0	0	15/2	t_1
$R_2' - \boxed{2}R_1$	R_2		2	-2	6	-2	1	0	5	T_2
$1/2\ R_2$	r_2		1	$\textcircled{-1}$	3	$\boxed{-1}$	1/2	0	5/2	t_2
$R_3' - \textcircled{-1}R_2 - \textcircled{1}R_1$	R_3			8	-16	-3	1	1	-9	T_3
$1/8\ R_3$	r_3			1	-2	$\boxed{-3/8}$	1/8	1/8	-9/8	t_3

$$[X'X]^{-1} = \begin{array}{|c|c|c|c|} \hline \frac{29}{8} & -\frac{11}{8} & -\frac{3}{8} & s_1 \\ \hline -\frac{11}{8} & \frac{5}{8} & \frac{1}{8} & s_2 \\ \hline -\frac{3}{8} & \frac{1}{8} & \frac{1}{8} & s_3 \\ \hline \end{array}$$

From Table V we obtain the backward solution which gives $[X'X]^{-1}$, the inverse of $X'X$.

1. Using the part of the $[X'X]^{-1}$ section, starting with Row T_1 and ending with row t_3, we proceed as follows; recalling that if $X'X$ is symmetric, then $[X'X]^{-1}$ is also symmetric.

2. To obtain the first row of $[X'X]^{-1}$, we choose as pivotal multipliers, the elements in column one and rows t_3, t_2, and t_1. The first row of $[X'X]^{-1}$ is designated s_1 and the pivotal multipliers are distinguished by a square. The elements of row s_1 are the respective sum of products of the pivotal multipliers and the corresponding elements of the rows T_3, T_2, and T_1, specifically, the first element of s_1 is $\boxed{-3/8} \cdot (-3) + \boxed{-1} \cdot (-2) + \boxed{1/2} \cdot 1 = \frac{29}{8}$; the second element is $\boxed{-3/8} \cdot 1 + \boxed{-1} \cdot 1 + \boxed{1/2} \cdot 0 = -11/8$, etc.

3. For the second row of $[X'X]^{-1}$, designated s_2, we choose as pivotal multipliers the elements in column two and rows t_3, t_2, and t_1. These elements will be designated with diamonds. Again, as in the forward solution, we will not multiply any element of any row to the left of the column in which the pivotal multiplier appears. The elements of s_2 are obtained by obtaining the sum of the products of the pivotal multipliers and the respective elements of rows T_3, T_2, and T_1. That is the second element of s_2, (the first already obtained due to symmetry)

is equal to $\langle 1/8 \rangle \cdot 1 + \langle 1/2 \rangle \cdot 1 + \langle 0 \rangle \cdot 0 = 5/8$; the third element of s_2 is $\langle 1/2 \rangle \cdot 1 + \langle 1/2 \rangle \cdot 0 + \langle 0 \rangle \cdot 0 = 2/8$.

4. The first and second elements of s_3 are now available due to symmetry and to find the third element of s_3, we choose the pivotal multipliers in column 3, rows t_3, t_2, and t_1, designated with triangles. The third element in s_3 then is $\overline{1/8} \cdot 1 + \overline{0} \cdot 0 + \overline{0} \cdot 0 = 1/8$.

We can now formulate the manner of finding the pivotal multipliers in general.

(1) The number of pivotal multipliers for each row of $[X'X]^{-1}$ is n where n is the number of unknowns in the system.

(2) The location of these multipliers will be in the same number column as the member of the row of $[X'X]^{-1}$ we are seeking and they are located in the t_n, t_{n-1}, ... , t_2, t_1 rows of the $[X'X]^{-1}$ section of the layout.

Consider the $[X'X]^{-1}$ section of Table V. We can write this in general notation as in Table VI.

Note: (1) For the general s_{ij} or T_{ij} we have $i = (1, 2, \ldots, n)$ and $j = (1, 2, \ldots, n)$ where n is the number of unknowns in the system.

The following formula yields any s_{ij} in $[X'X]^{-1}$

$$s_{ij} = \sum_{k=1}^{n} t_{ki} T_{kj}$$

For example, suppose we want s_{13} in Table V. Using the formula we have $s_{13} = t_{11}T_{13} + t_{21}T_{23} + t_{31}T_{33} = \boxed{1/2} \cdot 0 + \boxed{-1} \cdot 0 + \boxed{-3/8} \cdot 1 = -3/8$.

To fully understand why this method works, we will appeal to some fundamental concepts of the theory of matrices.

Table VI

$[X'X]^{-1}$ Section in General

$[X'X]^{-1}$			Row Identification
1	2	3	
1	0	0	
0	1	0	
0	0	1	
T_{11}	T_{12}	T_{13}	T_{1j}
t_{11}	t_{12}	t_{13}	t_{1j}
T_{21}	T_{22}	T_{23}	T_{2j}
t_{21}	t_{22}	t_{23}	t_{2j}
T_{31}	T_{32}	T_{33}	T_{3j}
t_{31}	t_{32}	t_{33}	t_{3j}

s_{11}	s_{12}	s_{13}	
s_{21}	s_{22}	s_{23}	$= [X'X]^{-1}$
s_{31}	s_{32}	s_{33}	

Consider the matrix multiplication, BA. The matrix B is called a pre-multiplier of the matrix A. Likewise in the multiplication of AB, B is called the post-multiplier of A. In either case, we can say that the matrix B transforms the matrix A and the product can be considered a transformation of the matrix A.

It is important to note that a pre-multiplication of a matrix A by a matrix B will result in a series of row operations, being performed on the rows of matrix A. That is, in the process of matrix multiplication, the rows of the product matrix will be linear combinations of the rows of matrix A, in fact the particular linear combinations dictated by the elements of the rows in the pre-multiplier matrix B.

Illustration:

$$\begin{array}{ccccc} B & & A & & C \\ \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} & & \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} & = & \begin{bmatrix} 1 & -3 \\ 5 & 6 \end{bmatrix} \end{array}$$

Note: (1) The first row of the product matrix C is the first row of matrix A minus the second row of matrix A. This following from the first row of the pre-multiplier B. $(1)(2) + (-1)(1) = 1$ and $(1)(1) + (-1)(4) = (-3)$.

(2) The second row of C is two times the first row of A plus the second row of A. $(2)(2) + (1)(1) = 5$ and $(2)(1) + (1)(4) = 6$.

It follows in the same manner, that if A is post-multiplied by a matrix B, then in effect the columns of the product matrix will be linear combinations of the columns of the matrix A. These combinations being dictated by the columns of the post-multiplier B.

Also, if we are given a square matrix A, it is possible to transform this matrix to the identity matrix by row and column operations.

We will do this by row and column operations. Suppose we set the given matrix A between two identity matrices I_L and I_R thusly: $I_L A I_R$. Now we proceed to do column and row operations on the matrix A to transform it into the identity matrix, simultaneously performing the identical row operations on I_L and column operations on I_R, that are performed on A. That is, if row one in A is subtracted from row two in A, then row one in I_L is subtracted from row two in I_L and if column one in A is added to column three in A, then column one in I_R is added to column three in I_R. We are using the two identity matrices as recorders of the row and column operations necessary to transform A to the identity matrix. After completing all necessary row and column operations on A, if we now designate the resultant I_L matrix as T' and resultant I_R matrix as t, we assert with reference the following:

$$T'At \doteq I .$$

This is given as a theorem in most texts on elementary theory of matrices [2].

With this theory in mind let us now investigate the Abbreviated Doolittle as given in Table V. Consider the rows T_1, T_2, and T_3 of the $[X'X]^{-1}$ section of the table. Use these rows to construct the n x n matrix which we shall call T', then:

$$T' = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix}$$

Now consider rows t_1, t_2, and t_3 of the same [illegible] of Table V and use these rows to form the $n \times n$ matrix, we shall call t', then:

$$t' = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{1}{2} & 0 \\ -\frac{3}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} \frac{1}{2} & -1 & -\frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{8} \\ 0 & 0 & \frac{1}{8} \end{bmatrix}$$

Referring again to the discussion previously on pre-multipliers and post-multipliers we find that

$$T'X'Xt = I$$

then

$$X'X = [T']^{-1}\, t^{-1}$$

$$[X'X]^{-1} = tT'$$

where T' is the row recording matrix and t is the column recording matrix found by reducing X'X to the identity matrix by row and column operations. This proves the validity of the Abbreviated Doolittle for finding the inverse of X'X.

We now point out a few facts that will be referred to later.

It is interesting to note that T'X'X = R where R is the matrix composed of Rows R_1, R_2, and R_3 of the X'X section of Table V, and

$$R = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 8 \end{bmatrix}$$

hence T traingularizes X'X. Also $t'X'X = r$ where r is [illegible] matrix composed of rows r_1, r_2, and r_3 of the X'X section and

$$r = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

hence t' also triangularizes X'X.

A diagonal matrix is a matrix with zeros for every element except the main diagonal.

Now $T'X'XT = D_1$, where D_1 is a diagonal matrix with the diagonal elements being the leading elements of rows R_1, R_2, and R_3. Then

$$D_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Also $t'X'Xt = D_2$, where D_2 is diagonal and whose diagonal elements are the inverse elements of D_1. Then

$$D_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix}$$

hence, $D_1D_2 = I$.

It is interesting to note that the determinant of X'X is equal to the determinant of D_1 which is simply the product of the diagonal elements of D_1.

We will refer to these facts later as we discuss the application of the Doolittle method to statistical designs.

To verify the validity of the check column, we can look at each line of Table V as being an expression equated to the sum of the coefficients of the particular line. Now in the process of obtaining the forward solution we are merely performing the same arithmetic operations to both sides of the above equations. Hence, the equality prevails. If it does not, then we will not have performed the same operation to both sides of the equation and this will indicate an arithmetic error in our computations.

CHAPTER III

THE APPLICATION OF THE "ABBREVIATED DOOLITTLE" TO REGRESSION

3.1. Introduction and Definitions

In this chapter we shall investigate what the "Abbreviated Doolittle" will do for us in the analysis of what is often called "multiple" regression. It is not the intent here to delve into the complexities of the statistical theory concerning regression [3]. However, we will discuss the basic assumptions and properties of regression, along with the necessary nomenclature that is essential to understanding what the "Abbreviated Doolittle" yields computationally in a regression analysis.

Consider the matrix model

$$Y = X\beta + e \tag{3.1.1}$$

where Y is an n x 1 matrix, X is n x p matrix, β is an p x 1 matrix and e is an n x 1 matrix. We are concerned here with what is referred to as the full rank case. By this we shall mean that the rank of X is $p \leqslant n$.

We now distinguish between simple and multiple regression. If $p = 2$, then an observation from the Y vector in (3.1.1) can be written

$$y_i = \beta_0 + \beta_1 X_i + e_i \tag{3.1.2}$$

If $p > 2$, then an observation from the Y vector in (3.1.1) can be written

$$y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \ldots + \beta_p X_{pi} + e_i \qquad (3.1.3)$$

Equation (3.1.2) is the model for a "simple" regression while Equation (3.1.3) is the model for "multiple" regression. It is easily seen that "simple" regression is just a special case of "multiple" regression. We will present the material as applied to "multiple" regression only, since this is the general regression case.

3.2. Multiple Regression

Suppose we have the following system of normal equations.

$$X'X\hat{\beta} = X'Y \qquad (3.2.1)$$

where,

$$X'X = \begin{array}{ccc} X_0 & X_1 & X_2 \\ \left[\begin{matrix} 4 & 8 & 12 \\ 8 & 18 & 23 \\ 12 & 23 & 46 \end{matrix}\right. & & \left.\right] \end{array} \qquad \hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} \qquad X'Y = \begin{bmatrix} 20 \\ 42 \\ 56 \end{bmatrix}$$

The format for the forward solution is given in Table VII. We will omit the instruction column since it will be exactly the same as presented in Chapter II. (See Table VII).

The forward solution yields the following values for $\hat{\beta}$.

$$\hat{\beta}_3 = -6/19 \qquad ?$$

$$\hat{\beta}_2 = 1 + 1/2\,\hat{\beta}_3 = 16/19$$

$$\hat{\beta}_1 = \overset{5}{2} \overset{-}{+} 2\hat{\beta}_2 + 3\hat{\beta}_3 = 81/19$$

$$5 - 2\hat{\beta}_2 + 3\hat{\beta}_3$$

We then obtain the prediction equation.

Table VII

Forward and Backward Solutions to Equations (3. 2. 1)

Row	X'X			X'Y		$[X'X]^{-1}$			Check	Row
R_1'	4	8	12	20		1	0	0	45	
R_2'		18	23	42		0	1	0	92	
R_3'			46	56		0	0	1	138	
R_1	4	8	12	20	CPID$_1$	1	0	0	45	T_1
r_1	1	$\boxed{2}$	③	5		$\frac{1}{4}$	0	0	$\frac{35}{4}$	t_1
R_2		2	-1	2	CPID$_2$	-2	1	0	2	T_2
r_2		1	$\enclose{circle}{-\frac{1}{2}}$	1		-1	$\frac{1}{2}$	0	1	t_2
R_3			$\frac{19}{2}$	-3	CPID$_3$	-4	$\frac{1}{2}$	1	4	T_3
r_3			1	$-\frac{6}{19}$		$-\frac{8}{19}$	$\frac{1}{19}$	$\frac{2}{19}$	$\frac{8}{19}$	t_3

$$[X'X]^{-1} = \begin{bmatrix} \frac{299}{76} & -\frac{23}{19} & -\frac{8}{19} \\ -\frac{23}{19} & \frac{10}{19} & \frac{1}{19} \\ -\frac{8}{19} & \frac{1}{19} & \frac{2}{19} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$$

$$\hat{Y} = 81/19 + 16/19\,X_1 - 6/19\,X_2 \qquad (3.2.2)$$

from which, for any given values of X_1 and X_2 we may find $\hat{Y}$. This is the multiple regression of Y on X_1 and X_2.

Now let us look at equation $X'X\hat{\beta} = X'Y$ in the light of its relationship to sample quantities. We have

$$X'X = \begin{bmatrix} n & \Sigma X_1 & \Sigma X_2 \\ \Sigma X_1 & \Sigma X_1^2 & \Sigma X_1 X_2 \\ \Sigma X_2 & \Sigma X_1 X_2 & \Sigma X_2^2 \end{bmatrix} \qquad \hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} \qquad X'Y = \begin{bmatrix} \Sigma y_i \\ \Sigma X_1 y_i \\ \Sigma X_2 y_i \end{bmatrix}$$

$Y'Y = \sum_j y_i^2$ is called the total sum of squares. $\hat{\beta}'X'Y$ is called the reduction due to β and $\dfrac{[Y'Y - \hat{\beta}'X'Y]}{n - p}$ is an estimate of σ^2 or the experimental error used in the analysis of variance for testing hypotheses. The reduction due to β is denoted by $R(\beta)$.

Returning to Table VII we define the term "Cross Product in the Doolittle." This is the product of the two elements in the X'Y column associated with each pair of computational rows. The number of these products will be the same as the number of parameters associated with the X'X matrix. We designate the first cross product $CPID_1$, the second cross product $CPID_2$, etc. For example, $CPID_1 = 5 \cdot 20 = 100$ (See Table VII, $CPID_1$). In general, $CPID_i = R_{i,\,n+1} \cdot r_{i,\,n+1}$ where n is the number of parameters in β.

We shall show that the $\sum_i CPID_i = \hat{\beta}'X'Y = R(\beta)$. Consider Equation (3.2.1),

$$X'X\hat{\beta} = X'Y$$

then

$$\hat{\beta} = [X'X]^{-1}X'Y$$

from the theory in Chapter II

$$tT' = [X'X]^{-1}$$

This gives

$$\hat{\beta} = tT'X'Y$$

and

$$\hat{\beta}' = Y'XTt'$$

Hence

$$\hat{\beta}'X'Y = Y'XtT'X'Y$$

since

$$Tt' = tT'$$

Now examine the right side of the two equations,

$$T'X'X\hat{\beta} = T'X'Y$$

and

$$t'X'X\hat{\beta} = t'X'Y$$

$T'X'Y$ is the $p \times 1$ vector of the elements in the $X'Y$ column and rows $R_1, R_2, \ldots, R_p$. $t'X'Y$ is the $p \times 1$ vector of the elements [illegible] $X'Y$ column and rows $r_1, r_2, \ldots, r_p$.

Hence:

$$[t'X'Y]'T'X'Y = \underline{\sum_i CPID_i} = Y'XtT'X'Y = \hat{\beta}'X'Y = \underline{R(\beta)}$$

For example, we obtain $R(\beta)$ from Table VII,

$$R(\beta) = \sum_{i=1}^{3} CPID_i = CPID_1 + CPID_2 + CPID_3$$
$$= 5 \cdot 20 + 2 \cdot 1 + (-3)(-6/19)$$
$$= 102\ 18/19$$

We obtain the Error Sum of Squares for the analysis of variance since,

$$\text{Error Sum of Squares} = \text{Total Sum of Squares} - R(\beta)$$
$$E_{ss} = T_{ss} - R(\beta)$$
$$E_{ss} = \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} CPID_i$$

where $\sum_{i=1}^{n} y_i^2 = Y'Y$.

The standard computing formula for the corrected total sum of squares is

$$\sum_{i=1}^{n} y_i^2 - \frac{(\sum_{i=1}^{n} y_i)^2}{n} .$$

Now in Table VII we see that $CPID_1$ is equal to

$$(\sum_i y_i)\ \frac{(\sum_i^n y_i)}{n} = \frac{(\sum_i y_i)^2}{n}$$

Hence $CPID_1$ is the correction factor. This is sometimes referred to as the reduction in sum of squares due to the mean and is designated $R(\mu)$ or $R(\beta_1)$. $CPID_2$ is the reduction in sum of squares due to β_2 adjusted for β_1 designated $R(\beta_2 | \beta_1)$ and $CPID_3$ is the reduction due to β_3 adjusted for β_1 and β_2 which is designated $R(\beta_3 | \beta_1 \beta_2)$.

This information is most useful for testing hypothesis concerning the β vector.

For example, suppose we wish to test the hypothesis that $\beta_3 = 0$. For this we need $R(\beta_3 | \beta_1 \beta_2)$. To test the hypothesis that any particular elements of $\beta = 0$, we need to obtain the reduction in sum of squares due to these elements adjusted for the remaining elements. In this case $R(\beta_3 | \beta_1 \beta_2) = \mathrm{CPID}_3$.

Now suppose we wanted to test $\beta_2 = \beta_3 = 0$. For this we need to obtain $R(\beta_2 \beta_3 | \beta_1)$. This is equal to $\mathrm{CPID}_3 + \mathrm{CPID}_2$ from Table VII.

From this we see that we can obtain the adjusted reductions in any ✓ order we wish by merely rearranging X'X such that the cross products in the doolittle for the parameters of interest occur last in the forward solution.

If we are interested in setting confidence intervals on the point estimates of the β_i, or linear combinations of the β_i, [illegible] e covariance matrix of the $\hat{\beta}$ vector since $\mathrm{Cov}(\beta) = \sigma^2 [X'X]^{-1}$. The backward solution given in Table VII yields $[X'X]^{-1}$.

For example, suppose we want an estimate of:

1. Variance of $\hat{\beta}_1$. This is $\hat{\sigma}^2 s_{11} = \hat{\sigma}^2\ 299/16$.
2. The Covariance of $(\hat{\beta}_1\ \hat{\beta}_2)$. This is $\hat{\sigma}^2 s_{12} = \hat{\sigma}^2 s_{21} = \hat{\sigma}^2\ (-23/19)$

In general:

1. $\widehat{\mathrm{Var}}\ (\hat{\beta}_i) = \hat{\sigma}^2 s_{ii}$
2. $\widehat{\mathrm{Cov}}\ (\hat{\beta}_i, \hat{\beta}_j) = \hat{\sigma}^2 s_{ij} \quad i \neq j$
3. Standard $\widehat{\mathrm{Error}}$ of $(\hat{\beta}_i) = \sqrt{\sigma^2 s_{ii}}$

We can obtain the multiple correlation coefficient R^2 from Table VII by the following formula.

$$R^2 = \frac{\sum_i \mathrm{CPID}_i}{\sum_i y_i^2 - \mathrm{CPID}_1}$$

Note that since $CPID_1$ is $R(\mu)$ the denominator of the formula for R^2 is total sum of squares adjusted.

As was stated before it is possible to get the adjusted reduction for any parameter or combination of parameters by rearranging the model such that these parameters appear last. This will make the CPID associated with these parameters occur last in the forward solution. To clarify this, consider Table VII. The arrangement of the model is $Y = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + e$. Hence, the last CPID = $CPID_3$ is $R(\beta_3 | \beta_1, \beta_2)$ and $R(\beta_2, \beta_3 | \beta_1)$ is the sum of the last two CPID = $CPID_3 + CPID_2$.

Suppose, however, that we have completed the forward and backward solution for a particular arrangement of the model and then find that we want the adjusted reduction of one of the parameters or combination of parameters whose cross products are not last in the forward solution. It is not necessary to rearrange the X'X matrix (the model) and again go through the forward solution. To illustrate this we shall rearrange the model to $Y = \beta_2 X_2 + \beta_3 X_3 + \beta_1 X_1 + e$ and carry out the forward solution. For this arrangement we are interested in $R(\beta_1 | \beta_2, \beta_3)$ and $R(\beta_1, \beta_3 | \beta_2)$ (See Table VIII).

From Table VIII we obtain

$$\begin{aligned} \hat{\beta}_1 &= 81/19 \\ \hat{\beta}_3 &= 1 + 32/299\, \hat{\beta}_1 = -6/19 \\ \hat{\beta}_2 &= 1 + 23/18\, \hat{\beta}_3 + 4/9\, \hat{\beta}_1 = 16/19 \end{aligned} \tag{1}$$

The same as from the forward solution of Table VII.

$$R(\beta) = \sum_i CPID_i = 102\ 18/19 \tag{2}$$

Table VIII

Forward Solution To Equation (3. 2. 1)

Row	X'X			X'Y	$[X'X]^{-1}$			Check	Row
R_1'	18	23	8	42	1	0	0	92	
R_2'		46	12	56	0	1	0	138	
R_3'			4	20	0	0	1	45	
R_1	18	23	8	42	1	0	0	92	T_1
r_1	1	$\frac{23}{18}$	$\frac{4}{9}$	$\frac{7}{3}$	$\frac{1}{18}$	0	0	$\frac{46}{9}$	t_1
R_2		$\frac{299}{18}$	$\frac{16}{9}$	$\frac{7}{3}$	$-\frac{23}{18}$	1	0	$\frac{184}{9}$	T_2
r_2		1	$\frac{32}{299}$	$\frac{42}{299}$	$-\frac{23}{299}$	$\frac{18}{299}$	0	$\frac{368}{299}$	t_2
R_3			$\frac{76}{299}$	$\frac{324}{299}$	$-\frac{92}{299}$	$-\frac{32}{299}$	1	$\frac{575}{299}$	T_3
r_3			1	$\frac{81}{19}$	$-\frac{23}{19}$	$-\frac{8}{19}$	$\frac{299}{76}$	$\frac{575}{76}$	t_3

The same as from Table VII

$$R(\beta_1 | \beta_2 \beta_3) = CPID_3 = (81/19)\ (324/299) = 4.62$$

and

$$R(\beta_1, \beta_2 | \beta_2) = CPID_3 + 'CPID_2 = (81/19)(324/299) + (7/3)(24/299)$$
$$= 4.94$$

We shall show how these two values can be obtained from Table VII.

Consider Table VII. Suppose we wanted $R(\beta_1 | \beta_2\beta_3)$. This is not available from the forward solution as given in Table VII. Using the value for $\hat{\beta}_1$ and $[X'X]^{-1}$ we have

$$\begin{aligned} R(\beta_1 | \beta_2\beta_3) &= \hat{\beta}_1' [s_{11}]^{-1}\hat{\beta}_1 \\ &= [81/19]\ [76/299][81/19] \\ &= [81/19][324/299] \\ &= 4.62 \end{aligned}$$

Exactly what we obtained for $R(\beta_1 | \beta_2\beta_3)$ from Table VIII.

In general the adjusted reduction sum of squares for any parameter or combination of parameters is obtained by the following relation:

$$\text{Adjusted Reduction Sum of Squares} = B^{*\prime}Z^{-1}B^* \qquad (3.2.2)$$

where B^* is a column vector of estimates of the parameters of interest and Z^{-1} is the inverse of the matrix obtained by partitioning $[X'X]^{-1}$ according to the rows and columns associated with the elements of B^*.

To illustrate, this suppose we want the $R(\beta_1, \beta_3 | \beta_2)$ from Table VII. We partition $[X'X]^{-1}$ to find Z and

$$Z = \begin{bmatrix} \frac{299}{76} & -\frac{8}{19} \\ -\frac{8}{19} & \frac{2}{19} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{13} \\ s_{31} & s_{33} \end{bmatrix}$$

and

$$Z^{-1} = \begin{bmatrix} \frac{76}{171} & \frac{304}{171} \\ \frac{304}{171} & \frac{5681}{342} \end{bmatrix}$$

Now $\hat{\beta}_1 = 81/19$ and $\hat{\beta}_3 = -6/19$, hence,

$$B^* = \begin{bmatrix} 81/19 \\ -6/19 \end{bmatrix}$$

Then using Equation (3. 2. 2) we have

$$R(\beta_1\beta_3 \mid \beta_2) = B^{*\prime}Z^{-1}B^*$$

$$= [\frac{81}{19}, -\frac{6}{19}] \begin{bmatrix} \frac{76}{171} & \frac{304}{171} \\ \frac{304}{171} & \frac{5681}{342} \end{bmatrix} \begin{bmatrix} \frac{81}{19} \\ -\frac{6}{19} \end{bmatrix}$$

$$= \frac{610812}{123462} = 4.94$$

Exactly what Table VIII yields for $R(\beta_1, \beta_3 \mid \beta_2)$.

In summary we will list the information obtained from the forward and backward solutions of the Abbreviated Doolittle when applied to multiple linear regression.

We obtain the following:

1. A solution for the vector of unknowns β.
2. Reduction in sum of squares due to β.
3. Reduction in sum of squares for any parameters or combination of parameters adjusted for the remaining parameters.
4. The multiple correlation coefficient R^2.
5. Covariance matrix for the vector β.

CHAPTER IV

THE APPLICATION OF THE "ABBREVIATED DOOLITTLE" TO THE TWO-WAY CLASSIFICATION

4.1. Introduction and Definitions

In this chapter we shall investigate the computational information that is obtained by applying the "Doolittle Technique" to the following two-way classification models:

(1) Without interaction;

(2) Without interaction and with a co-variable;

(3) With interaction.

These models are classified as experimental design models of less than full rank. By this we mean that in the model denoted by the matrix equation

$$Y = X\beta + e \qquad (4.1.1)$$

the rank of the X matrix is $k < p$ where p is the number of parameters in the model. This is referred to as Model 4 [3]. We are interested in the model in the light of unequal sub-class numbers and missing data both of which complicate ordinary methods of computing necessary statistical information.

In the model of less than full rank, X is of dimension $(n \times p)$ $p \leq n$ of rank $k < p$ hence $X'X$ is of dimension $(p \times p)$ of rank $k < p$. This means that $X'X$ is singular and has no inverse. A unique solution for

β in equation (4.1.1) does not exist and we cannot obtain unbiased estimates of the β_i as we did in Chapter II where we were considering the model of full rank. However, we will show that we can obtain estimates of linear combinations of the β_i.

4.2. Two-way Classification Model without Interaction

Consider the model

$$y_{ijk} = \mu + \tau_i + \beta_j + e_{ijk} \qquad \left\{ \begin{array}{l} i = 1, 2, 3 \\ j = 1, 2 \\ k = 1, 2, \ldots, n_{ij} \end{array} \right. \tag{4.2.1}$$

where y_{ijk} is the k-th observation in the ij-th cell; μ, τ_i, β_j, are unknown parameters; and e_{ijk} are random variables with the conventional distributional properties. The ij-th cell contains n_{ij} observations. Suppose for our example we are interested in the τ_i. We then rewrite model (4.2.1) as $y_{ijk} = \mu + \beta_j + \tau_i + e_{ijk}$. The X matrix with the columns labelled with their respective parameters is

$$X = \begin{array}{c} \begin{array}{cccccc} \mu & \beta_1 & \beta_2 & \tau_1 & \tau_2 & \tau_3 \end{array} \\ \left[\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \end{array}$$

The observed observations in the statistical layout for model (4. 2. 1) are given in Table IX and the number of observations per cell layout is given in Table X.

Table IX

Statistical Layout for Model (4. 2. 1)

$\tau \backslash \beta$	1	2	$Y_{i\cdot\cdot}$
1	4 4	3	11
2	5	8 8 9	30
3	6 5	8	19
$Y_{\cdot j\cdot}$	24	36	$60 = Y_{\cdot\cdot\cdot}$

Table X

Observations per cell Layout for Model (4. 2. 1)

$\tau \backslash \beta$	1	2	$N_{i\cdot}$
1	2	1	3
2	1	3	4
3	2	1	3
$N_{\cdot j}$	5	5	$10 = N_{\cdot\cdot}$

The normal equations $X'X\hat{\gamma} = X'Y$ are

$$\begin{array}{llcll}
\mu: & 10\hat{\mu} + 5\hat{\beta}_1 + 5\hat{\beta}_2 + 3\hat{\tau}_1 + 3\hat{\tau}_2 + 3\hat{\tau}_3 & = & 60 & = Y_{\cdot\cdot\cdot} \\
\beta_1: & 5\hat{\mu} + 5\hat{\beta}_1 \qquad\quad + 2\hat{\tau}_1 + \hat{\tau}_2 + 2\hat{\tau}_3 & = & 24 & = Y_{\cdot 1\cdot} \\
\beta_2: & 5\hat{\mu} \qquad\quad + 5\hat{\beta}_2 + \hat{\tau}_1 + 3\hat{\tau}_2 + \hat{\tau}_3 & = & 36 & = Y_{\cdot 2\cdot} \\
\tau_1: & 3\hat{\mu} + 2\hat{\beta}_1 + \hat{\beta}_2 + 3\hat{\tau}_1 & = & 11 & = Y_{1\cdot\cdot} \\
\tau_2: & 4\hat{\mu} + \hat{\beta}_1 + 3\hat{\beta}_2 \qquad + 4\hat{\tau}_2 & = & 30 & = Y_{2\cdot\cdot} \\
\tau_3: & 3\hat{\mu} + 2\hat{\beta}_1 + \hat{\beta}_2 \qquad\qquad + 3\hat{\tau}_3 & = & 19 & = Y_{3\cdot\cdot}
\end{array} \qquad (4.\ 2.\ 2)$$

and

$$X'X = \begin{bmatrix} 10 & 5 & 5 & 3 & 4 & 3 \\ 5 & 5 & 0 & 2 & 1 & 2 \\ 5 & 0 & 5 & 1 & 3 & 1 \\ 3 & 2 & 1 & 3 & 0 & 0 \\ 4 & 1 & 3 & 0 & 4 & 0 \\ 3 & 2 & 1 & 0 & 0 & 3 \end{bmatrix} \quad X'Y = \begin{bmatrix} 60 \\ 24 \\ 36 \\ 11 \\ 30 \\ 19 \end{bmatrix} \quad \gamma = \begin{bmatrix} \mu \\ \beta_1 \\ \beta_2 \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

The forward solution for this system is given in Table XI. (See Table XI)

The backward solution is of no importance since $[X'X]^{-1}$ does not exist. As was stated before the X'X matrix is singular. This means that there is a linear dependence between the rows of X'X and that there is at least one row that is a linear combination of the other rows. We note that the forward solution will detect these dependencies by yielding zero computational rows when the dependency is encountered. For example, if we inspect the normal equations (4. 2. 2) we see that the sum of the rows for the β_i is equal to the equation for μ. Hence, when the last β_i equation (third row) of X'X is processed in the forward solution we get computational rows equal to zero. (See R_3 and r_3 of Table XI). Also in Equations (4. 2. 2) we see that the sum of the rows for τ_i is equal to the μ equation. Again when the last row for the τ_i (sixth row) of X'X is reached in the forward solution we obtain zero computational rows. (See R_6 and r_6 of Table XI.) Whenever a zero computational row occurs we continue to the next row.

From Table XI we can obtain the following information:

(1) From r_6 we get $\tilde{\tau}_3 = 0$

(2) From r_5 we get $\tilde{\tau}_2 = 7/15$

(3) From r_4 we get $\tilde{\tau}_1 = -8/3$

Table XI

Forward Solution to Equations (4. 2. 2)

Row	X'X						X'Y	Check
R'_1	10	5	5	3	4	3	60	90
R'_2		5	0	2	1	2	24	39
R'_3			5	1	3	1	36	51
R'_4				3	0	0	11	20
R'_5					4	0	30	42
R'_6						3	19	28
R_1	10	5	5	3	4	3	60	90
r_1	1	$\boxed{\frac{1}{2}}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{3}{10}$	6	$\frac{90}{10}$
R_2		$\frac{5}{2}$	$-\frac{5}{2}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	-6	-6
r_2		1	-1	$\frac{1}{5}$	$-\frac{2}{5}$	$-\frac{1}{5}$	$-\frac{12}{5}$	$-\frac{12}{5}$
R_3			0	0	0	0	0	0
r_3			0	0	0	0	0	0
R_4				2	-1	-1	$-\frac{29}{5}$	$-\frac{29}{5}$
r_4				1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{29}{10}$	$-\frac{29}{10}$
R_5					$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{7}{10}$	$\frac{7}{10}$
r_5					1	-1	$\frac{7}{15}$	$\frac{7}{15}$
R_6						0	0	0
r_6						0	0	0

Note: [This is one of many solutions to the system, hence we use the notation $\tilde{\tau}_i$ instead of $\hat{\tau}_i$ which implies that we have an unbiased estimate of the τ_i.]

In fact:

$$\tilde{\tau}_3 \text{ estimates } \tau_3 - \tau_3$$

$$\tilde{\tau}_2 \text{ estimates } \tau_2 - \tau_3$$

$$\tilde{\tau}_1 \text{ estimates } \tau_1 - \tau_3$$

2. We find $R(\gamma) = R(\mu, \beta_j, \tau_i) = \sum_{i=1}^{6} \text{CPID}_i$ where

$$\sum_{i=1}^{6} \text{CPID}_i = (60)(6) + (-6)(-12/5) + (0)(0) + (-29/5)(-29/10) + (7/10)(7/15) + (0)(0)$$

$$= 380.32$$

3. We find $R(\tau \mid \mu, \beta) = \sum_{i=4}^{6} \text{CPID}_i$ where

$$\sum_{i=4}^{6} \text{CPID}_i = (-29/5)(-29/10) + (7/10)(7/15) + (0)(0) = 17.15$$

Under the assumption of no interaction

$$\frac{1}{n - p}\left[\sum_{ijk} y_{ijk}^2 - R(\gamma)\right]$$

is equal to the error mean square and we can use this information in our analysis of variance to test the hypothesis $\tau_1 = \tau_2 = \tau_3$. The AOV is given in Table XII. (See Table XII)

We shall now present an alternate method for applying the "Doolittle" to Equations (4. 2. 2). The objective is the same, that of testing H_o: $\tau_1 = \tau_2 = \tau_3$. Since there are two dependencies in the system given by Equations (4. 2. 2), we may impose the non-estimable conditions

Table XII

AOV for Testing H_o: $\tau_1 = \tau_2 = \tau_3$

Source	d.f.	Sum of Squares	Mean Square
Total	10	400.00	
$R(\beta)$	4	300.32	
$R(\tau \mid \mu\beta)$	2	17.15	8.57
Error	6	19.68	3.28

$\hat{\mu} = 0$ and $\hat{\tau}_3 = 0$. The first equation for μ and last equation for τ_3 will be omitted. The resulting system is of full rank and is given by the following equations:

$$\begin{array}{llcl} \hat{\beta}_1: & 5\hat{\beta}_1 + 2\hat{\tau}_1 + \hat{\tau}_2 & = 24 & = Y_{\cdot 1 \cdot} \\ \hat{\beta}_2: & 5\hat{\beta}_2 + \hat{\tau}_1 + 3\hat{\tau}_2 & = 36 & = Y_{\cdot 2 \cdot} \\ \hat{\tau}_1: & 2\hat{\beta}_1 + \hat{\beta}_2 + 3\hat{\tau}_1 & = 11 & = Y_{1 \cdot \cdot} \\ \hat{\tau}_2: & \hat{\beta}_1 + 3\hat{\beta}_2 + 4\hat{\tau}_2 & = 30 & = Y_{2 \cdot \cdot} \end{array} \qquad (4.2.3)$$

and

$$X'X = \begin{bmatrix} 5 & 0 & 2 & 1 \\ 0 & 5 & 1 & 3 \\ 2 & 1 & 3 & 0 \\ 1 & 3 & 0 & 4 \end{bmatrix} \qquad X'Y = \begin{bmatrix} 24 \\ 36 \\ 11 \\ 30 \end{bmatrix}$$

The forward solution for this system is given in Table XIII.

Table XIII

Forward Solution to Equations (4. 2. 3)

Row	X'X				X'Y	Check
R_1'	5	0	2	1	24	32
R_2'		5	1	3	36	45
R_3'			3	0	11	17
R_4'				4	30	38
R_1	5	0	2	1	24	32
r_1	1	$\boxed{0}$	$\enclose{circle}{\frac{2}{5}}$	$\triangle\frac{1}{5}$	$\frac{24}{5}$	$\frac{32}{5}$
R_2		5	1	3	36	45
r_2		1	$\enclose{circle}{\frac{1}{5}}$	$\triangle\frac{3}{5}$	$\frac{36}{5}$	$\frac{45}{5}$
R_3			2	-1	$-\frac{29}{5}$	$-\frac{24}{5}$
r_3			1	$\triangle -\frac{1}{2}$	$-\frac{29}{10}$	$-\frac{24}{10}$
R_4				$\frac{3}{2}$	$\frac{7}{10}$	$\frac{22}{10}$
r_4				1	$\frac{7}{15}$	$\frac{22}{15}$

From Table XIII we obtain:

$$\text{(1)}\qquad R(\gamma) = \sum_{i=1}^{4} \mathrm{CPID}_i = (24)(24/5) + (36)(36/5) + (-29/5)(-29/10) + (7/10)(7/15)$$

$$= 380.32$$

$$(2) \quad R(\tau \mid \mu\beta) = \sum_{j=3}^{4} CPID_i = (-29/5)(-29/10) + (7/10)(7/15) = 17.15$$

These of course are the same as obtained from Table XI.

Previously we have been only interested in testing H_0: $\tau_1 = \tau_2 = \tau_3$. This by itself is not all the information we would like from the model. The τ_i are not estimable but we may obtain estimates of contrasts of the τ_i which may be of interest and also set confidence intervals on these contrasts.

We shall present the application of the Abbreviated Doolittle as a computational aid in finding these contrasts and also finding the estimates of variance and covariance necessary for setting confidence intervals on these estimates.

Consider model (4. 2. 1) in matrix notation and we have

$$Y = X\gamma + e$$

where the dimensions of the matrices are as follows:

Matrix	General Dimension	Dimension for Model (4. 2. 1)
Y	N x 1	10 x 1
X	N x b+t+1	10 x 6
γ	(b+t+1) x 1	6 x 1
X_0	N x 1	10 x 1
X_1	N x b	10 x 2
X_2	N x t	10 x 3
β	b x 1	2 x 1
τ	t x 1	3 x 1
e	N x 1	10 x 1

where

N = No. observations in the experiment;

$X = (X_0, X_1, X_2)$;

$\gamma' = (\mu, \beta', \tau')$;

b = Number of Blocks;

t = Number of Treatments.

We define J^n_m to be an n x m matrix whose elements are all ones. We can write the model (4. 2. 1) as

$$Y = J^N_1 \mu + X_1 \alpha + X_2 \tau + e$$

or

$$Y = X_1(J^b_1 \mu + \alpha) + X_2 \tau + e$$

since $X_1 J^b_1 = J^N_1$. Now define $(J^b_1 \mu + \alpha) = \beta$ and we have

$$Y = X_1 \beta + X_2 \tau + e \qquad (4. 2. 3)$$

Now consider the results of this on the system given by model (4. 2. 1).

(1) The μ column of the X matrix will be absorbed by the β_i columns, hence the dimensions of X will be 10 x (3+2). The μ column is omitted.

(2) We then partition $X = (X_1 | X_2)$ where:

  (a) X_1 is 10 x 2 partition of the X matrix. It contains the columns labelled β_1 and β_2.

  (b) X_2 is 10 x 3 partition of the X matrix. It contains the columns labelled τ_1, τ_2, and τ_3.

The normal equations (4. 2. 3) become,

(1) $$X_1'X_1\hat{\beta} + X_1'X_2\hat{\tau} = X_1'Y$$

(2) $$X_2'X_1\hat{\beta} + X_2'X_2\hat{\tau} = X_2'Y$$

Solving (1) for $\hat{\beta}$ we get

$$\hat{\beta} = [X_1'X_1]^{-1}[X_1'Y - X_1'X_2\hat{\tau}]$$

Substituting this in (2) we have

$$[X_2'X_1][X_1'X_1]^{-1}[X_1'Y - X_1'X_2\hat{\tau}] + X_2'X_2\hat{\tau} = X_2'Y$$

(2) $$[X_2'X_2 - X_2'X_1[X_1'X_1]^{-1}X_1'X_2]\hat{\tau} = X_2'Y - X_2'X_1[X_1'X_1]^{-1}X_1'Y$$

In (2)

$$\left.\begin{aligned} &\text{(a)}\quad \text{Let } X_2'X_2 - X_2'X_1[X_1'X_1]^{-1}X_1'X_2 = A \\ &\text{(b)}\quad \text{and } X_2'Y - X_2'X_1[X_1'X]^{-1}X_1'Y = q \end{aligned}\right\} \qquad (4.\ 2.\ 4)$$

Then (2) can be written

$$A\tilde{\tau} = q$$

where the dimensions of the matrices are as follows:

Matrix	Dimension	Dimension for Model (4. 2. 1)
A	t x t	3 x 3
$\tilde{\tau}$	t x 1	3 x 1
q	t x 1	3 x 1

Consider now the X'X matrix for the normal equations in model (4. 2. 3) as applied to model (4. 2. 1).

$$X'X = \left[\begin{array}{c|c} X_1'X_1 & X_1'X_2 \\ \hline X_2'X_1 & X_2'X_2 \end{array}\right] = \left[\begin{array}{cc|ccc} 5 & 0 & 2 & 1 & 2 \\ 0 & 5 & 1 & 3 & 1 \\ \hline 2 & 1 & 3 & 0 & 0 \\ 1 & 3 & 0 & 4 & 0 \\ 2 & 1 & 0 & 0 & 3 \end{array}\right]$$

We note the following:

(1) $X_1'X_1$ is diagonal with the elements on the diagonal being the number of observations in each block.

(2) $X_1'X_2$ is the transpose of the matrix composed of the number of observations in each cell. (See Table X). This is called the "Incidence Matrix."

We will now build the A matrix in the $A\tilde{\tau} = q$ system by a special application of the "Doolittle Technique." (See Table XIV)

The procedure for Table XIV is as follows:

(1) R_1, r_1, R_2, r_2 are obtained as usual for the forward solution.

(2) The elements of row A_1 in the X'X section are the elements of the first row of the "A" matrix. The element in row A_1 of the X'Y section is the first element of "q".

(3) The A_1 row is obtained in the same manner as though we were carrying out the forward solution on two sets of computational rows R_1, r_1, and R_2, r_2.

For example, we will determine A_1. The pivotal multipliers are the elements designated by circles.

Table XIV

"Doolittle Technique" for Finding the "A" and "q" Matrices for Model (4. 2. 1)

Row	X'X					X'Y
	β_1	β_2	τ_1	τ_2	τ_3	
R_1'	5	0	2	1	2	24
R_2'		5	1	3	1	36
A_1'			3	0	0	11
A_2'				4	0	30
A_3'					3	19
R_1	5	0	2	1	2	24
r_1	1	0	$\frac{2}{5}$	$\frac{1}{5}$	$\boxed{\frac{2}{5}}$	$\frac{24}{5}$
R_2		5	1	3	1	36
r_2		1	$\frac{1}{5}$	$\frac{3}{5}$	$\boxed{\frac{1}{5}}$	$\frac{36}{5}$
A_1			2	-1	-1	$-\frac{29}{5}$
A_2			-1	2	-1	$\frac{18}{5}$
A_3			-1	-1	2	$\frac{11}{5}$

(a) 1st element of A_1 = 1st element of the 1st row of the "A" matrix = a_{11}

$$= 3 - [\left(\frac{1}{5}\right) \cdot 1 + \left(\frac{2}{5}\right) \cdot 2]$$

$$= 2$$

(b) 2nd element of A_1 = 2nd element of the 1st row of the "A" matrix = a_{12}

$$= 0 - [\left(\frac{1}{5}\right) \cdot 3 + \left(\frac{2}{5}\right) \cdot 1]$$

$$= -1$$

(c) 3rd element of A_1 = 3rd element of the 1st row of the "A" matrix = a_{13}

$$= 0 - [\left(\frac{1}{5}\right) \cdot 1 + \left(\frac{2}{5}\right) \cdot 2]$$

$$= -1$$

(d) 4th element of A_1 = 1st element of "q" = q_1

$$= 11 - [\left(\frac{1}{5}\right) \cdot 36 + \left(\frac{2}{5}\right) \cdot 24]$$

$$= -29/5$$

(4) The A_2 row is obtained the same except we subtract the sum of the products of the pivotal multipliers and rows R_1 and R_2 from the respective elements of A_2'. These multipliers are designated by diamond

(a) 1st element of A_2 = 2nd element of A_1 = $a_{21} = a_{12}$, since "A" is symmetric.

(b) 2nd element of A_2 = 2nd element of the 2nd row of "A" = a_{22}

$$= 4 - \left[\left\langle \frac{3}{5} \right\rangle \cdot 3 + \left\langle \frac{1}{5} \right\rangle \cdot 1 \right]$$

$$= 2$$

(c) 3rd element of A_2 = 3rd element of 3rd row of "A" = a_{23}

$$= 0 - \left[\left\langle \frac{3}{5} \right\rangle . 1 + \left\langle \frac{1}{5} \right\rangle \cdot 2 \right]$$

$$= -1$$

(d) 4th element of A_2 = 2nd element of "q" = q_2

$$= 30 - \left[\left\langle \frac{3}{5} \right\rangle \cdot 36 + \left\langle \frac{1}{5} \right\rangle \cdot 24 \right]$$

$$= 18/5$$

(5) We obtain A_3 by subtracting the products of the pivotal multipliers, and rows R_1 and R_2 from the respective elements of $A_3^{!}$. These multipliers are designated by squares.

(a) 1st element of A_3 = 3rd element of A_1; $a_{13} = a_{31}$

(b) 2nd element of A_3 = 3rd element of A_2; $a_{32} = a_{23}$

(c) 3rd element of A_3 = 3rd element of the 3rd row of "A" = a_{33}

$$= 3 - \left[\boxed{\frac{1}{5}} \cdot 1 + \boxed{\frac{2}{5}} \cdot 2 \right]$$

$$= 2$$

(d) 4th element of A_3 = 3rd element of "q" = q_3

$$= 19 - \left[\boxed{\frac{1}{5}} \cdot 36 + \boxed{\frac{2}{5}} \cdot 24 \right] = 11/15$$

From the special application of the "Doolittle Technique" given in Table XIV we can obtain the following system of t equations (in this case t = 3) in t unknowns given by:

$$A \tilde{\tau} = q$$

or

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \tilde{\tau}_1 \\ \tilde{\tau}_2 \\ \tilde{\tau}_3 \end{bmatrix} = \begin{bmatrix} -29/5 \\ 18/5 \\ 11/5 \end{bmatrix}$$

or

$$\begin{aligned} 2\tilde{\tau}_1 - \tilde{\tau}_2 - \tilde{\tau}_3 &= -\frac{29}{5} \\ -\tilde{\tau}_1 + 2\tilde{\tau}_2 - \tilde{\tau}_3 &= \frac{18}{5} \\ -\tilde{\tau}_1 - \tilde{\tau}_2 + 2\tilde{\tau}_3 &= \frac{11}{5} \end{aligned}$$

In this special case it is seen that the diagonal elements of A are all equal and the off diagonal elements are ± 1. This does not necessarily occur in all designs. The diagonal elements may be unequal. Likewise, the off diagonals may be values other than ± 1. However, it will be found that the sum of the elements in any row of the A matrix will always be equal to zero.

To understand why this scheme works we will verify Equations (4. 2. 4) by applying the "Doolittle" to a general system. Consider Table XIV in general notation as given in Table XV.

Table XV

Computation of "A" and "q" in General

Row	X'X					X'Y
	β_1	β_2	τ_1	τ_2	τ_3	
R_1'	$X_1'X_1$		$X_1'X_2$			$X_1'Y$
R_2'						
A_1'			$X_2'X_2$			$X_2'Y$
A_2'						
A_3'						
R_1	R_{11}	R_{12}	R_{13}	R_{14}	R_{15}	R_{16}
r_1		r_{12}	r_{13}	r_{14}	$\boxed{r_{15}}$	r_{16}
R_2		R_{22}	R_{23}	R_{24}	R_{25}	R_{26}
r_2		r_{22}	r_{23}	r_{24}	$\boxed{r_{25}}$	r_{26}
A_1			a_{11}	a_{12}	a_{13}	q_1
A_2			a_{21}	a_{22}	a_{23}	q_2
A_3			a_{31}	a_{32}	a_{33}	q_3

Now comparing Table XV with Table XIV we see that $[X_1'X_1]^{-1}$ is diagonal and equal to

$$\begin{bmatrix} 1/5 & 0 \\ 0 & 1/5 \end{bmatrix}$$

Look at

$$[X_1'X_1]^{-1}X_1'X_2 = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} r_{13} & r_{14} & r_{15} \\ r_{23} & r_{24} & r_{25} \end{bmatrix}$$

and

$$X_2'X_1[X_1'X_1]^{-1}X_1'X_2 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Now

$$X_1'X_2 = \begin{bmatrix} R_{13} & R_{14} & R_{15} \\ R_{23} & R_{24} & R_{25} \end{bmatrix}$$

and

$$X_2'X_1 = \begin{bmatrix} R_{13} & R_{23} \\ R_{14} & R_{24} \\ R_{15} & R_{25} \end{bmatrix}$$

Then the Doolittle application in Table XV gives

$$X_2'X_2 - \begin{bmatrix} R_{13} & R_{23} \\ R_{14} & R_{24} \\ R_{15} & R_{25} \end{bmatrix}\begin{bmatrix} r_{13} & r_{14} & r_{15} \\ r_{23} & r_{24} & r_{25} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

which from Table XIV becomes

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

or

$$X_2'X_2 - X_2'X_1[X_1'X_1]^{-1}X_1'X_2 = A$$

which is the same as Equation (a) in Equations (4. 2. 4).

Returning to Table XV we see that

$$X_1'Y = \begin{bmatrix} R_{16} \\ R_{26} \end{bmatrix}$$

and that the "Doolittle Technique" gives

$$X_2'Y - \begin{bmatrix} r_{13} & r_{23} \\ r_{14} & r_{24} \\ r_{15} & r_{25} \end{bmatrix} \begin{bmatrix} R_{16} \\ R_{26} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

which from Table XIV becomes

$$\begin{bmatrix} 11 \\ 30 \\ 19 \end{bmatrix} - \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & 3/5 \\ 2/5 & 1/5 \end{bmatrix} \begin{bmatrix} 24 \\ 36 \end{bmatrix} = \begin{bmatrix} -29/5 \\ 18/5 \\ 11/5 \end{bmatrix}$$

or

$$X_2'Y - X_2'X_1[X_1'X_1]^{-1}X_1'Y = q$$

which is the same as Equation (b) in Equations (4. 2. 4).

We have verified that the special application of the "Abbreviated Doolittle" will build the $A\tilde{\tau} = q$ system of equations.

4. 3. Application of the "Abbreviated Doolittle" to $A\tilde{\tau} = q$.

The system $A\tilde{\tau} = q$ has t equations in t unknowns. The rank of the matrix A is (t - 1) hence there is no unique solution for $\tilde{\tau}_i$. We shall discuss with reference [~~4~~3] the methods of solving this system for estimable contrasts of the τ_i and finding point estimates, along with the estimates of the variance and covariance, of these contrasts.

Since the rank of A in $A\tilde{\tau} = q$ is (t - 1) and the dimension of A is (t x t) suppose we impose the restriction that $\sum_{i=1}^{t} \tilde{\tau}_i = 0$. This may be written $J_t'\tilde{\tau} = 0$.

Then $A\tilde{\tau} = q$ can be written

$$\begin{bmatrix} A & J_t' \\ J_1^t & 0 \end{bmatrix} \begin{bmatrix} \tilde{\tau} \\ \tilde{\tau}_0 \end{bmatrix} = \begin{bmatrix} q \\ 0 \end{bmatrix}$$

or

$$A^*\tau^* = q^*$$

where

$$A^* = \begin{bmatrix} A & J_t' \\ J_1^t & 0 \end{bmatrix} \quad \text{and} \quad [A^*]^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Now the dimension of A* is [t+1 x t+1] and it can be shown that the rank of A* is t+1. Hence A* is non-singular and $[A^*]^{-1}$ exists. We can apply the Abbreviated Doolittle to this system and obtain estimates of the τ_i from the forward solution and the variance-covariance constant matrix from the backward solution.

It can be shown that the estimates of the $\tilde{\tau}_i$ will be unbiased estimates of $\tau_i - \bar{\tau}.$, since $E(\tilde{\tau}_i) = \tau_i - \bar{\tau}.$; also $\hat{\sigma}^2 B_{11}$ will give estimates of the variance and covariance of the $\widehat{\tau_i - \bar{\tau}}$. [4].

Using the A matrix obtained in Table XIV we shall apply the Doolittle Technique to find the variance-covariance constant matrix and the estimates of $(\tilde{\tau}_i)$.

From Table XIV we get

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad q^* = \begin{bmatrix} -29/5 \\ 18/5 \\ 11/5 \\ 0 \end{bmatrix} \quad A^* = \begin{bmatrix} 2 & -1 & -1 & 1 \\ -1 & 2 & -1 & 1 \\ -1 & -1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Before applying the forward solution to the system $A^*\tau^* = q^*$ we note that the A matrix appears as a leading principal minor of the A^* matrix, [the 3 x 3 matrix in the upper left hand corner of A^*]. Now from our previous knowledge of the Doolittle procedure we know that zero computational rows occur when the third row of A^* is processed in the forward solution (since A is singular). This can be alleviated by interchanging the third and fourth rows and columns of A^*. A^* then becomes A^{**} where

$$A^{**} = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -1 & 2 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 2 \end{bmatrix} \quad q^{**} = \begin{bmatrix} q_1 \\ q_2 \\ 0 \\ q_3 \end{bmatrix} \quad \text{and} \quad \tilde{\tau}^{**} = \begin{bmatrix} \tilde{\tau}_1 \\ \tilde{\tau}_2 \\ \tilde{\tau}_0 \\ \tilde{\tau}_3 \end{bmatrix}$$

We shall now apply the forward and backward solution to $A^{**}\tilde{\tau}^{**} = q^{**}$ from which we can obtain the following information:

1. Estimates of the $\tilde{\tau}_i$ where $E(\tilde{\tau}_i) = \tau_i - \bar{\tau}$.
2. $[A^*]^{-1}$ from which we get B_{11} and hence covariance of $(\tilde{\tau}_i, \tilde{\tau}_j)$ which is equal to $B_{11}\sigma^2$.
3. $R(\tau \mid \mu\ \beta)$ for testing H_0: $\tau_1 = \tau_2 = \ldots = \tau_t$.

The tableaux is given in Table XVI. (See Table XVI)

From Table XVI we get $[A^*]^{-1}$ from $[A^{**}]^{-1}$ by interchanging the third and fourth rows and columns of $[A^{**}]^{-1}$. Exactly the same row and column interchange that was performed on A* to get A** is used.

$$[A^*]^{-1} = \begin{bmatrix} 4/18 & -2/18 & -2/18 & 6/18 \\ -2/18 & 4/18 & -2/18 & 6/18 \\ -2/18 & -2/18 & 4/18 & 6/18 \\ 6/18 & 6/18 & 6/18 & 0 \end{bmatrix}$$

From $[A^*]^{-1}$ we obtain B_{11} which will be the t x t matrix in the upper left hand corner of $[A^*]^{-1}$. In our example,

$$B_{11} = \begin{bmatrix} 4/18 & -2/18 & -2/18 \\ -2/18 & 4/18 & -2/18 \\ -2/18 & -2/18 & 4/18 \end{bmatrix}$$

From Table XVI we get the following:

$$(1)\quad R(\tau \mid \mu\ \beta) = \sum_{i=1}^{t+1} CPID_i \qquad (t+1 = 4)$$

$$= (33/10)(11/15) + (-22/20)(22/10) + (7/10)(7/15) + (-29/5)(-29/10)$$

$$= 17.5$$

which is the same as was obtained from Table XI and Table XIII.

Table XVI

Forward and Backward Solution to $A^{**}\tilde{\tau}^{**} = q^{**}$

Row	A^{**}				q^{**}	$[A^{**}]^{-1}$				Check
R'_1	2	-1	1	-1	$-\frac{29}{5}$	1	0	0	0	$-\frac{19}{5}$
R'_2		2	1	-1	$\frac{18}{5}$	0	1	0	0	$\frac{28}{5}$
R'_3			0	1	0	0	0	1	0	4
R'_4				2	$\frac{11}{5}$	0	0	0	1	$\frac{21}{5}$
R_1	2	-1	1	-1	$-\frac{29}{5}$	1	0	0	0	$-\frac{19}{5}$
r_1	1	$\enclose{circle}{-\frac{1}{2}}$	$\frac{1}{2}$	$\boxed{-\frac{1}{2}}$	$-\frac{29}{10}$	$\frac{1}{2}$	0	0	0	$-\frac{19}{10}$
R_2		$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{7}{10}$	$\frac{1}{2}$	1	0	0	$\frac{37}{10}$
r_2		1	1	$\boxed{-1}$	$\frac{7}{15}$	$\frac{1}{3}$	$\frac{2}{3}$	0	0	$\frac{37}{10}$
R_3			-2	-3	$\frac{22}{10}$	-1	-1	1	0	$\frac{22}{10}$
r_3			1	$\boxed{-\frac{3}{2}}$	$-\frac{22}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{22}{10}$
R_4				$\frac{9}{2}$	$\frac{33}{10}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	1	$\frac{93}{10}$
r_4				1	$\frac{11}{10}$	$-\frac{1}{9}$	$-\frac{1}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{31}{15}$

$$[A^{**}]^{-1} = \begin{array}{cccc} \frac{4}{18} & -\frac{2}{18} & \frac{6}{18} & -\frac{2}{18} \\ -\frac{2}{18} & \frac{4}{18} & \frac{6}{18} & -\frac{2}{18} \\ \frac{6}{18} & \frac{6}{18} & 0 & \frac{6}{18} \\ -\frac{2}{18} & -\frac{2}{18} & \frac{6}{18} & \frac{4}{18} \end{array}$$

(2) Each CPID in the Doolittle is associated with a parameter of the normal equations. These parameters can be identified with the columns of X'X. In A** we have added the J_1^t column and will identify the "dummy" parameter $\tilde{\tau}_0$ with this column.

(a) r_4 yields $\tilde{\tau}_3 = 11/15$, hence $(\widehat{\tau_3 - \bar{\tau}.}) = 11/15$.

(b) r_3 yields $\tilde{\tau}_0 = 0$ [the "dummy" parameter].

(c) r_2 yields $\tilde{\tau}_2 = 18/15$, hence $(\widehat{\tau_2 - \bar{\tau}.}) = 18/15$.

(d) r_1 yields $\tilde{\tau}_1 = -29/15$, hence $(\widehat{\tau_1 - \bar{\tau}.}) = -29/15$.

(3) Consider B_{11} in general notation, then

$$B_{11} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Since $B_{11}\sigma^2$ is the variance-covariance matrix for the $(\tau_i - \bar{\tau}.)$ which are estimated by the $\tilde{\tau}_i$, we get estimates of variances and covariances of the $(\tau_i - \bar{\tau}.)$ thusly:

(a) $\text{Var}\,(\tilde{\tau}_1) = \text{Var}\,(\widehat{\tau_1 - \bar{\tau}.}) = b_{11}\hat{\sigma}^2 = (4/18)\hat{\sigma}^2$

$\text{Var}\,(\tilde{\tau}_2) = \text{Var}\,(\widehat{\tau_2 - \bar{\tau}.}) = b_{22}\hat{\sigma}^2 = (4/18)\hat{\sigma}^2$

$\text{Var}\,(\tilde{\tau}_3) = \text{Var}\,(\widehat{\tau_3 - \bar{\tau}.}) = b_{33}\hat{\sigma}^2 = (4/18)\hat{\sigma}^2$

In general $\text{Var}\,(\tilde{\tau}_i) = \text{Var}\,(\widehat{\tau_i - \bar{\tau}.}) = b_{ii}\hat{\sigma}^2$, where the b_{ii} are the diagonal elements of B_{11}. Also

(b) $\text{Cov}\,(\tilde{\tau}_1, \tilde{\tau}_2) = \text{Cov}[(\tau_1 - \bar{\tau}.), (\tau_2 - \bar{\tau}.)] = -2/18\ \hat{\sigma}^2$

$\text{Cov}\,(\tilde{\tau}_1, \tilde{\tau}_3) = \text{Cov}[(\tau_1 - \bar{\tau}.), (\tau_2 - \bar{\tau}.)] = -2/18\ \hat{\sigma}^2$

$$\text{Cov}\ (\tilde{\tau}_2, \tilde{\tau}_3) = \text{Cov}[(\tau_2 - \bar{\tau}.), (\tau_3 - \bar{\tau}.)] = -2/18\ \hat{\sigma}^2$$

In general, Cov $(\tilde{\tau}_i, \tilde{\tau}_j) = \text{Cov}[(\tau_i - \bar{\tau}.), (\tau_j - \bar{\tau}.)] = b_{ii}\hat{\sigma}^2\ (i \neq j)$, where b_{ij} are the off-diagonal elements of B_{11}.

With this information we can set confidence intervals on $(\tau_i - \bar{\tau}.)$ or linear combinations thereof.

We will now present an alternate method of applying the "Doolittle Technique" to the system $A\tilde{\tau} = q$. The results will be the same as just discussed.

We shall show that if we form the matrix $\bar{A} = A + 1/t\ J$ and apply the forward and backward solution to the system $\bar{A}\tilde{\tau} = q$, we obtain the $\tilde{\tau}_i$ where $E(\tilde{\tau}_i) = (\tau_i - \bar{\tau}.)$ from the forward solution and $[\bar{A}]^{-1} = B_{11} + \frac{1}{t}J$ from the backward solution. B_{11} is defined as before.

Applying this to $A\tilde{\tau} = q$ as given on page 49 we have,

$$\bar{A} = A + \frac{1}{t}J = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 7/3 & -2/3 & -2/3 \\ -2/3 & 7/3 & -2/3 \\ -2/3 & -2/3 & 7/3 \end{bmatrix}$$

and $\bar{A}\tilde{\tau} = q$ is

$$\begin{bmatrix} 7/3 & -2/3 & -2/3 \\ -2/3 & 7/3 & -2/3 \\ -2/3 & -2/3 & 7/3 \end{bmatrix} \begin{bmatrix} \tilde{\tau}_1 \\ \tilde{\tau}_2 \\ \tilde{\tau}_3 \end{bmatrix} = \begin{bmatrix} -29/5 \\ 18/5 \\ 11/5 \end{bmatrix}$$

Note that $\bar{A}$ is full rank hence $[\bar{A}]^{-1}$ exists.

The tableaux for the forward and backward solution of the system $\bar{A}\tilde{\tau} = q$ is given in Table XVII.

Table XVII

Forward and Backward Solution of $\overline{A}\,\tilde{\tau} = q$

Row	$\overline{A}$			q	$[\overline{A}]^{-1}$			Check
R_1'	$\frac{7}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{29}{5}$	1	0	0	$-\frac{19}{5}$
R_2'		$\frac{7}{3}$	$-\frac{2}{3}$	$\frac{18}{5}$	0	1	0	$\frac{28}{5}$
R_3'			$\frac{7}{3}$	$\frac{11}{5}$	0	0	1	$\frac{21}{5}$
R_1	$\frac{7}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{29}{5}$	1	0	0	$-\frac{19}{5}$
r_1	1	$-\frac{2}{7}$	$\boxed{-\frac{2}{7}}$	$-\frac{87}{35}$	$\frac{3}{7}$	0	0	$-\frac{57}{5}$
R_2		$\frac{45}{21}$	$-\frac{18}{21}$	$\frac{68}{35}$	$\frac{2}{7}$	1	0	$\frac{158}{35}$
r_2		1	$\boxed{-\frac{18}{45}}$	$\frac{68}{75}$	$\frac{2}{15}$	$\frac{21}{45}$	0	$\frac{158}{75}$
R_3			$\frac{9}{5}$	$\frac{33}{25}$	$\frac{2}{5}$	$\frac{2}{5}$	1	$\frac{123}{25}$
r_3			1	$\frac{33}{45}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{5}{9}$	$\frac{41}{15}$

$$[\overline{A}]^{-1} = \begin{array}{|ccc|} \frac{5}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{5}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{5}{9} \\ \hline \end{array}$$

From Table XVII we obtain the following:

(1) r_3 yields $\tilde{\tau}_3 = 11/15 = (\widehat{\tau_3 - \bar{\tau}.})$

r_2 yields $\tilde{\tau}_2 = 18/15 = (\widehat{\tau_2 - \bar{\tau}.})$

r_1 yields $\tilde{\tau}_1 = -29/15 = (\widehat{\tau_1 - \bar{\tau}.})$

The same as from Table XVII

(2) $[\bar{A}]^{-1} = B_{11} + \frac{1}{t} J$. Hence $B_{11} = [\bar{A}]^{-1} - \frac{1}{t} J$.

Then

$$B_{11} = \begin{bmatrix} 5/9 & 2/9 & 2/9 \\ 2/9 & 5/9 & 2/9 \\ 2/9 & 2/9 & 5/9 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4/18 & -2/18 & -2/18 \\ -2/18 & 4/18 & -2/18 \\ -2/18 & -2/18 & 4/18 \end{bmatrix}$$

The same as before and $\mathrm{Cov}(\tau_i - \bar{\tau}.) = B_{11}\sigma^2$. Hence we obtain the estimates of the variance and covariance of $(\tau_i - \bar{\tau}.)$, just as we did before.

(3) From Table XVII we get

$$R(\tau \mid \mu\ \beta) = \sum_{i=1}^{t} \mathrm{CPID}_i \qquad (t = 3)$$

$$= (33/25)(33/45) + (68/75)(68/35) + (-29/5)(-87/35)$$

$$= 17.5$$

This is the same from Tables XI, XIII, and XVI.

We shall now present the application of the "Abbreviated Doolittle" to $A\tilde{\tau} = q$ such that we obtain unbiased estimates of $(\tau_i - \bar{\tau}_t)$, where $i \neq t$ and τ_t is arbitrarily chosen from the τ_i.

For example, consider the system $A\tilde{\tau} = q$ obtained in Table XIV. Suppose we are interested in obtaining estimates of the $(\tau_i - \tau_t)$, and the covariance of these estimates.

Suppose also we choose $\tau_t = \tau_3$ then we will be interested in estimates of $(\tau_i - \tau_3)$ $(i \neq 3)$.

As stated before A is less than full rank; hence, it is singular and A^{-1} does not exist. We then impose the non-estimable condition $\tau_3 = 0$ and omit the row and column identified with $\tilde{\tau}_3$ in the system $A\tilde{\tau} = q$ from Table XIV. We will designate this reduced system $\bar{\bar{A}}\tilde{\tau} = q$ where

$$\bar{\bar{A}} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}; \qquad \tilde{\tau} = \begin{bmatrix} \tilde{\tau}_1 \\ \tilde{\tau}_2 \end{bmatrix}; \qquad q = \begin{bmatrix} -29/5 \\ 18/5 \end{bmatrix}$$

This system will be of full rank and the $E(\tilde{\tau}_i) = (\tau_i - \tau_3)$ and $\text{Cov}\,(\tau_i - \tau_3) = [\bar{\bar{A}}]^{-1}\sigma^2$.

The tableaux for the forward and backward solution of the system $\bar{\bar{A}}\tilde{\tau} = q$ is given in Table XVIII. (See Table XVIII)

From Table XVIII we obtain the following:

(1) $$R(\tau \mid \mu\,\beta) = \sum_{i=1}^{t-1} \text{CPID}_i \qquad (t - 1 = 2)$$

$$= (-29/5)(-29/10) + (7/10)(7/15)$$

$$= 17.15$$

which is the same as from Tables XI, XIII, XVI, and XVII.

(2) r_2 yields $\tilde{\tau}_2 = 7/15$, hence $(\widehat{\tau_2 - \tau_3}) = 7/15$;

r_1 yields $\tilde{\tau}_1 = -8/3$, hence $(\widehat{\tau_1 - \tau_3}) = -8/3$;

which is the same as from Table XI.

(3) Since $\text{Cov}\,(\tau_i - \tau_t) = [\bar{\bar{A}}]^{-1}\sigma^2$, we obtain estimates:

(a) $\text{Var}\,(\tilde{\tau}_1) = \text{Var}\,(\widehat{\tau_1 - \tau_3}) = 2/3\,\hat{\sigma}^2$

$\text{Var}\,(\tilde{\tau}_2) = \text{Var}\,(\widehat{\tau_2 - \tau_3}) = 2/3\,\hat{\sigma}^2$

In general, $\text{Var}\,(\tilde{\tau}_i) = \text{Var}\,(\widehat{\tau_i - \tau_t}) = \bar{a}_{ii}\sigma^2$ $(i \neq t)$, where the a_{ii} are the diagonal elements of $[\bar{A}]^{-1}$.

Table XVIII

Forward and Backward Solution of $\bar{\bar{A}}\tilde{\tau} = q$

Row	$\bar{\bar{A}}$		$\underline{q}$	$[\bar{\bar{A}}]^{-1}$		Check
R_1'	2	-1	$-\frac{29}{5}$	1	0	$-\frac{19}{5}$
R_2'		2	$\frac{18}{5}$	0	1	$\frac{28}{5}$
R_1	2	-1	$-\frac{29}{5}$	1	0	$-\frac{19}{5}$
r_1	1	$\boxed{-\frac{1}{2}}$	$-\frac{29}{10}$	$\frac{1}{2}$	0	$-\frac{19}{10}$
R_2		$\frac{3}{2}$	$\frac{7}{10}$	$\frac{1}{2}$	1	$\frac{37}{10}$
r_2		1	$\frac{7}{15}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{37}{15}$

$$[\bar{\bar{A}}]^{-1} = \begin{array}{|cc|} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \hline \end{array}$$

(b) $\text{Cov}\,(\tilde{\tau}_1, \tilde{\tau}_2) = \text{Cov}\,[\widehat{\tau_1 - \tau_3}, \widehat{\tau_2 - \tau_3}] = 1/3\,\hat{\sigma}^2$

In general, $\text{Cov}\,(\tilde{\tau}_i, \tilde{\tau}_j) = \text{Cov}\,[\widehat{\tau_i - \tau_t}, \widehat{\tau_j - \tau_t}]$ $(i \neq j \neq t)$

$$= \bar{a}_{ij}\hat{\sigma}^2$$

where the $\bar{a}_{ij}$ are the off-diagonal elements of $[\bar{\bar{A}}]^{-1}$,

For the two-way classification with no interaction given by model (4.2.1) we can apply the "Abbreviated Doolittle" to obtain the statistical information in the following summary.

(1) Table XI yields $R(\gamma)$, $R(\tau \mid \mu, \beta)$, and estimates of $(\tau_i - \tau_t)$. Thus we can test H_o: $\tau_1 = \tau_2 = . . . = \tau_t$.

(2) Table XIII yields $R(\gamma)$ and $R(\tau \mid \mu, \beta)$ and again we can test H_o: $\tau_1 = \tau_2 = . . . = \tau_t$.

(3) Table XIV is the application of the "Doolittle" that builds the (t x t) system $A\tilde{\tau} = q$.

(4) Table XVI yields $R(\tau \mid \mu, \beta)$, estimates of $(\tau_i - \bar{\tau}.)$, and the variance-covariance constant matrix for these estimates.

(5) Table XVII is an alternate application to obtain the same information as in (4).

(6) Table XVIII yields $R(\tau \mid \mu, \beta)$, estimates of $(\tau_i - \tau_t)$ $(i \neq t)$, and the variance-covariance constant matrix for these estimates.

4. 3. Co-variable in the Two-Way Classification Without Interaction

Uncontrolled environmental conditions may affect both experimental error and estimates of the treatment effects. If the proper assumptions can be met and the environmental conditions can be measured even approximately, some adjustments can be made, often increasing the information in the experiment. An appropriate statistical method is known as covariance [4].

In this section we shall present an application of the "Abbreviated Doolittle" to covariance analysis.

Consider the model

$$Y_{ijk} = \mu + \alpha Z_{ijk} + \tau_i + \beta_j + e_{ijk} \qquad \begin{cases} i = 1, 2, 3 \\ j = 1, 2 \\ k = 1, 2, \ldots, n_{ij} \end{cases} \qquad (4. 3. 1)$$

where

(1) y_{ijk} is the k-th observation in the ij-th cell.

(2) Z_{ijk} is the k-th observation of the co-variable Z which appears in the ij-th cell.

(3) μ, τ_i, β_i, α, are unknown parameters.

(4) e_{ijk} are random variables with the conventional distributional properties.

(5) the ij-th cell contains n_{ij} observations.

The statistical layout for model (4. 3. 1) is given in Table XIX and the observation per cell layout is given in Table XX.

Table XIX

Statistical Layout for model (4. 3. 1)

β_j / τ_i	1		2			
	Z	Y	Z	Y	$Z_{i..}$	$Y_{i..}$
1	2 3	4 4	1	2	6	10
	Z	Y	Z	Y		
2	3	5	5 6 7	7 8 9	21	29
	Z	Y	Z	Y		
3	4 4	6 5	missing		8	11
$Z_{.j.}$	16		19		$Z_{..} = 35$	
$Y_{.j.}$		24		26		$Y_{...} = 50$

Table XX

Observations per cell Layout for Model (4. 3. 1)

β_j / τ_i	1		2		
	Z	Y	Z	Y	$N_{i\cdot}$
1	2	2	1	1	3
	Z	Y	Z	Y	
2	1	1	3	3	4
	Z	Y	Z	Y	
3	2	2	0	0	2
$N_{\cdot j}$	5		4		$N_{\cdot\cdot} = 9$

Note: There is an observation of the co-variable Z associated with every observation of Y.

Suppose we are interested in testing H_o: $\tau_1 = \tau_2 = \tau_3$ in the model (4. 3. 1). For this we need $R(\tau \mid \mu\ \beta\ \alpha)$. The $CPID_i$ associated with the t_i should appear last in the forward solution. We will, for computational expediency, absorb the μ equation and arrange columns of the X matrix such that the column (α) associated with the co-variable will appear just before the columns associated with the τ_i. The τ_i columns appear last. We write model (4. 3. 1) as

$$\overset{Y}{Y}_{ijk} = \gamma_j + \alpha X_{ijk} + \tau_i + e_{ijk}$$

where $\gamma_j = (\mu + \beta_j)$.

We point out the difference in the X matrix for this model as compared to previously discussed two-way classification models. The observed value of the covariable Z is placed in the α column rather than zeros or ones as with other parameters.

The X'X and X'Y matrices in general notation are:

$$X'X = \begin{bmatrix} N._1 & 0 & Z._1. & N_{11} & N_{21} & N_{31} \\ 0 & N._2 & Z._2. & N_{12} & N_{22} & N_{32} \\ Z._1. & Y._2. & Z^2... & Z_1... & Z_2.. & Z_3.. \\ N_{11} & N_{12} & Z_1.. & N_1. & 0 & 0 \\ N_{21} & N_{22} & Z_2.. & 0 & N_2. & 0 \\ N_{31} & N_{32} & Z_3.. & 0 & 0 & N_3. \end{bmatrix} \quad X'Y = \begin{vmatrix} Y._1. \\ Y._2. \\ Z'Y \\ Y_1.. \\ Y_2.. \\ Y_3.. \end{vmatrix}$$

where the dot in the subscript means that we have the sum over that subscript.

For example,

$N._1$ = Number of observations in $\beta_1 = \sum_{i=1}^{t} n_{i1}$

N_{11} = Number of observations in β_1 and τ_1

$Z._2.$ = Sum of the co-variable Z in $\beta_2 = \sum_{i=1}^{t} \sum_{k=0}^{n_{ij}} Z._{i2k}$

Using the data in Table XIX we have

$$X'X = \begin{bmatrix} 5 & 0 & 16 & 2 & 1 & 2 \\ 0 & 4 & 19 & 1 & 3 & 0 \\ 16 & 19 & 165 & 6 & 21 & 8 \\ 2 & 1 & 6 & 3 & 0 & 0 \\ 1 & 3 & 21 & 0 & 4 & 0 \\ 2 & 0 & 8 & 0 & 0 & 2 \end{bmatrix} \quad X'Y = \begin{bmatrix} 24 \\ 26 \\ 127 \\ 10 \\ 29 \\ 11 \end{bmatrix} \qquad (4.3.2)$$

We now apply the Abbreviated Doolittle to this system. The tableaux is given in Table XXI.

Table XXI

Forward Solution Equations (4. 3. 2)

Row	X'X						X'Y	Check
R'_1	5	0	16	2	1	2	24	50
R'_2		4	19	1	3	0	26	53
R'_3			165	6	21	8	127	362
R'_4				3	0	0	10	22
R'_5					4	0	29	58
R'_6						2	11	23
R_1	5	0	16	2	1	2	24	50
r_1	1	0	$\frac{16}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\boxed{\frac{2}{5}}$	$\frac{24}{5}$	10
R_2		4	19	1	3	0	26	53
r_2		1	$\frac{19}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\boxed{0}$	$\frac{26}{4}$	$\frac{53}{4}$
R_3			$\frac{471}{20}$	$-\frac{103}{20}$	$\frac{71}{20}$	$\frac{8}{5}$	$-\frac{1466}{20}$	$-\frac{199}{4}$
r_3			1	$-\frac{103}{20}$	$\frac{71}{471}$	$\boxed{\frac{32}{471}}$	$-\frac{1466}{471}$	$-\frac{995}{471}$
R_4				$\frac{388}{471}$	$-\frac{176}{471}$	$-\frac{212}{471}$	$-\frac{10423}{471}$	$-\frac{10423}{471}$
r_4				1	$-\frac{176}{388}$	$\boxed{-\frac{212}{388}}$	$-\frac{10423}{388}$	$-\frac{10423}{388}$
R_5					$\frac{328}{388}$	$-\frac{328}{388}$	$\frac{2216}{388}$	$\frac{2216}{388}$
r_5					1	-1	$\frac{2216}{328}$	$\frac{2216}{328}$
						0	0	0
						0	0	0

From Table XXI we obtain

(1) $R(\mu, \beta, \alpha, \tau) = \sum_{i=1}^{6} CPID_i$

(2) $EMS = \frac{1}{n - p}[\Sigma y_{ijk}^2 - R(\mu, \beta, \alpha, \tau)]$

(3) $R(\tau \mid \mu \beta \alpha) = \sum_{i=4}^{6} CPID_i$

From this information we can construct the AOV to test $H_0: \tau_1 = \tau_2 = \tau_3$. (The procedure is the same as discussed previously.)

We will now construct the $A\tilde{\tau} = q$ system by the previously discussed special application of the "Abbreviated Doolittle."

The tableaux is given in Table XXII. (See Table XXII)

From Table XXII we get $A\tilde{\tau} = q$ where

$$A = \begin{bmatrix} \frac{388}{471} & -\frac{176}{471} & -\frac{212}{471} \\ -\frac{176}{471} & \frac{478}{471} & -\frac{302}{471} \\ -\frac{212}{471} & -\frac{302}{471} & \frac{514}{471} \end{bmatrix} \qquad q = \begin{bmatrix} -\frac{10423}{471} \\ \frac{7418}{471} \\ \frac{3005}{471} \end{bmatrix}$$

All theory previously discussed pertaining to the system $A\tilde{\tau} = q$ is applicable. After the proper restrictions are imposed, we can apply the Abbreviated Doolittle to $A\tilde{\tau} = q$ and find the following:

(1) Estimates of $(\tau_i - \tau_t)$. $(i \neq t)$

(2) Variance and Covariance of (1).

(3) Estimates of $(\tau_i - \bar{\tau}.)$.

(4) Variance and Covariance of (3).

(5) $R(\tau \mid \mu \beta \alpha)$.

Table XXII

"Doolittle Technique" for Finding the "A" and "q" Matrices for Model (4. 3. 1)

Row	X'X						X'Y	Check
R_1'	5	0	16	2	1	2	24	50
R_2'		4	19	1	3	0	26	53
R_3'			165	6	21	8	127	362
A_1'				3	0	0	10	22
A_2'					4	0	29	58
A_3'						2	11	23
R_1	5	0	16	2	1	2	24	50
r_1	1	0	$\frac{16}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\boxed{\frac{2}{5}}$	$\frac{24}{5}$	10
R_2		4	19	1	3	0	26	53
r_2		1	$\frac{19}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\boxed{0}$	$\frac{26}{4}$	$\frac{53}{4}$
R_3			$\frac{471}{20}$	$-\frac{103}{20}$	$\frac{71}{20}$	$\frac{8}{5}$	$-\frac{1466}{20}$	$-\frac{199}{4}$
r_3			1	$-\frac{103}{471}$	$\frac{71}{471}$	$\boxed{\frac{32}{471}}$	$-\frac{1466}{471}$	$-\frac{995}{471}$
A_1				$\frac{388}{471}$	$-\frac{176}{471}$	$-\frac{212}{471}$	$-\frac{10423}{471}$	
A_2				$-\frac{176}{471}$	$\frac{478}{471}$	$-\frac{302}{471}$	$-\frac{7418}{471}$	
A_3				$-\frac{212}{471}$	$-\frac{302}{471}$	$\frac{514}{471}$	$\frac{3005}{471}$	

4.4. Two-Way Classification With Interaction

In this section we will investigate the following model,

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + e_{ijk} \qquad \begin{array}{l} i = 1,\ 2,\ 3 \\ j = 1,\ 2 \\ k = 1,\ 2,\ \ldots, n_{ij} \end{array} \qquad (4.4.1)$$

where y_{ijk} is the k-th observation in the ij-th cell; μ, τ_i, β_j, $(\tau\beta)_{ij}$, are unknown parameters; and e_{ijk} are random variables with the conventional distributional properties. The ij-th cell contains n_{ij} observations.

The normal equations for model (4.4.1) are:

$$\begin{array}{llcl}
\mu: & N_{\cdot\cdot}\hat{\mu} + \sum_j N_{\cdot j}\hat{\beta}_j + \Sigma N_{i\cdot}\hat{\tau}_i + \sum_{ij} N_{ij}(\widehat{\tau\beta})_{ij} & = & Y_{\cdot\cdot\cdot} \\
\tau_i: & N_{i\cdot}\hat{\mu} + \sum_j N_{ij}\hat{\beta}_j + N_{i\cdot}\hat{\tau}_i + \sum_j N_{ij}(\widehat{\tau\beta})_{ij} & = & Y_{i\cdot\cdot} \\
\beta_j: & N_{\cdot j}\hat{\mu} + N_{\cdot j}\hat{\beta}_j + \sum_i N_{ij}\hat{\tau}_i + \sum_i N_{ij}(\widehat{\tau\beta})_{ij} & = & Y_{\cdot j\cdot} \\
(\tau\beta)_{ij}: & N_{ij}\hat{\mu} + N_{ij}\hat{\beta}_j + N_{ij}\hat{\tau}_i + N_{ij}(\widehat{\tau\beta})_{ij} & = & Y_{ij\cdot}
\end{array}$$

We will use the same statistical layout and data as given in Table IX.

The normal equations then become as shown on the following page. We will absorb μ as before. The Forward solution is given in Table XXIII. (See Table XXIII)

$$\begin{array}{ll}
\mu: & 10\hat{\mu}+5\hat{\beta}_1+5\hat{\beta}_2+3\hat{\tau}_1+4\hat{\tau}_2+3\hat{\tau}_3+2\widehat{(\tau\beta)}_{11}+\widehat{(\tau\beta)}_{12}+\widehat{(\tau\beta)}_{21}+3\widehat{(\tau\beta)}_{22}+2\widehat{(\tau\beta)}_{31}+\widehat{(\tau\beta)}_{32}=60=Y_{\cdot\cdot\cdot} \\
\beta_1: & 5\hat{\mu}+5\hat{\beta}_1+2\hat{\tau}_1+\hat{\tau}_2+3\hat{\tau}_3+2\widehat{(\tau\beta)}_{11}+\widehat{(\tau\beta)}_{21}+2\widehat{(\tau\beta)}_{31}=24=Y_{\cdot 1\cdot} \\
\beta_2: & 5\hat{\mu}+5\hat{\beta}_2+\hat{\tau}_1+3\hat{\tau}_2+\hat{\tau}_3+\widehat{(\tau\beta)}_{12}+3\widehat{(\tau\beta)}_{22}+\widehat{(\tau\beta)}_{32}=36=Y_{\cdot 2\cdot} \\
\tau_1: & 3\hat{\mu}+2\hat{\beta}_1+\hat{\beta}_2+3\hat{\tau}_1+2\widehat{(\tau\beta)}_{11}+\widehat{(\tau\beta)}_{12}=11=Y_{1\cdot\cdot} \\
\tau_2: & 4\hat{\mu}+\hat{\beta}_1+3\hat{\beta}_2+4\hat{\tau}_2+\widehat{(\tau\beta)}_{21}+3\widehat{(\tau\beta)}_{22}=30=Y_{2\cdot\cdot} \\
\tau_3: & 3\hat{\mu}+2\hat{\beta}_1+\hat{\beta}_2+3\hat{\tau}_3+2\widehat{(\tau\beta)}_{31}+\widehat{(\tau\beta)}_{32}=19=Y_{3\cdot\cdot} \\
(\tau\beta)_{11}: & 2\hat{\mu}+2\hat{\beta}_1+2\hat{\tau}_1+2\widehat{(\tau\beta)}_{11}=8=Y_{11\cdot} \\
(\tau\beta)_{12}: & \hat{\mu}+\hat{\beta}_2+\hat{\tau}_1+\widehat{(\tau\beta)}_{12}=3=Y_{12\cdot} \\
(\tau\beta)_{21}: & \hat{\mu}+\hat{\beta}_1+\hat{\tau}_2+\widehat{(\tau\beta)}_{21}=5=Y_{21\cdot} \\
(\tau\beta)_{22}: & 3\hat{\mu}+3\hat{\beta}_2+3\hat{\tau}_2+3\widehat{(\tau\beta)}_{22}=26=Y_{22\cdot} \\
(\tau\beta)_{31}: & 2\hat{\mu}+2\hat{\beta}_1+2\hat{\tau}_3+2\widehat{(\tau\beta)}_{31}=11=Y_{31\cdot} \\
(\tau\beta)_{32}: & \hat{\mu}+\hat{\beta}_2+\hat{\tau}_3+\widehat{(\tau\beta)}_{32}=8=Y_{32\cdot}
\end{array}$$

Table XXIII

Forward Solution for Model (4. 4. 1)

Row	X'X											X'Y	Check
R'_1	5	0	2	1	2	2	0	1	0	2	0	24	39
R'_2		5	1	3	1	0	1	0	3	0	1	36	51
R'_3			3	0	0	2	1	0	0	0	0	11	20
R'_4				4	0	0	0	1	3	0	0	30	42
R'_5					3	0	0	0	0	2	1	19	28
R'_6						2	0	0	0	0	0	8	14
R'_7							1	0	0	0	0	3	6
R'_8								1	0	0	0	5	8
R'_9									3	0	0	26	35
R'_{10}										2	0	11	17
R'_{11}											1	8	11
R_1	5	0	2	1	2	2	0	1	0	2	0	24	39
r_1	1	0	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	0	$\frac{1}{5}$	0	$\frac{2}{5}$	0	$\frac{24}{5}$	$\frac{39}{5}$
R_2		5	1	3	1	0	1	0	3	0	1	36	51
r_2		1	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{1}{5}$	0	$\frac{3}{5}$	0	$\frac{1}{5}$	$\frac{36}{5}$	$\frac{51}{5}$
R_3			2	-1	-1	$\frac{6}{5}$	$\frac{4}{5}$	$-\frac{2}{5}$	$-\frac{3}{5}$	$-\frac{4}{5}$	$-\frac{1}{5}$	$-\frac{29}{5}$	$-\frac{29}{5}$
r_3			1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{6}{10}$	$\frac{4}{10}$	$-\frac{2}{10}$	$-\frac{3}{10}$	$-\frac{4}{10}$	$-\frac{1}{10}$	$-\frac{29}{10}$	$-\frac{29}{10}$

Table XXIII (Continued)

Row	X'X								X'Y	Check
R_4	$\frac{15}{10}$	$-\frac{15}{10}$	$\frac{2}{10}$	$-\frac{2}{10}$	$\frac{6}{10}$	$\frac{9}{10}$	$-\frac{8}{10}$	$-\frac{7}{10}$	$\frac{7}{10}$	$\frac{7}{10}$
r_4	1	-1	$\frac{2}{15}$	$-\frac{2}{15}$	$\frac{6}{15}$	$\frac{9}{15}$	$-\frac{8}{15}$	$-\frac{7}{15}$	$\frac{7}{15}$	$\frac{7}{15}$
R_5		0	0	0	0	0	0	0	0	0
r_5		0	0	0	0	0	0	0	0	0
R_6			$\frac{68}{150}$	$-\frac{68}{150}$	$-\frac{36}{150}$	$\frac{36}{150}$	$-\frac{32}{150}$	$\frac{32}{150}$	$\frac{268}{150}$	$\frac{268}{150}$
r_6			1	-1	$-\frac{36}{68}$	$\frac{36}{68}$	$-\frac{32}{68}$	$\frac{32}{68}$	$\frac{268}{68}$	$\frac{268}{68}$
R_7				0	0	0	0	0	0	0
r_7				0	0	0	0	0	0	0
R_8					$\frac{24}{68}$	$-\frac{24}{68}$	$-\frac{24}{68}$	$\frac{24}{68}$	$-\frac{20}{68}$	$-\frac{20}{68}$
r_8					1	-1	-1	1	$-\frac{20}{24}$	$-\frac{20}{24}$
R_9						0	0	0	0	0
r_9						0	0	0	0	0
R_{10}							0	0	0	0
r_{10}							0	0	0	0
R_{11}								0	0	0
r_{11}								0	0	0

From Table XXIII we obtain,

(1) $R[\mu, \beta, \tau, (\tau\beta)] = \sum_{i=1}^{11} CPID_i = 387.62$

(2) $R[(\tau\beta) | \mu \beta \tau] = \sum_{i=1}^{11} CPID_i = 7.29$

(3) $EMS = \frac{1}{n-p} [\sum_{ijk} y_{ijk}^2 - R[\mu, \beta, \tau, (\tau\beta)]$

With this information we can test H_0: the $(\tau\beta)$ interaction is zero.

Returning to the X'X matrix obtained from the normal equations for model (4.4.1) we see that the last six rows contain a diagonal matrix. These are the six rows associated with the $(\tau\beta)_{ij}$. (See lower right hand corner of X'X.) This means that the rank of X'X is at least six.

Since we have a 12 x 12 system where the rank of the coefficient matrix is six, we have the liberty of imposing six restrictions on the system. The six restrictions we choose will be non-estimable functions. These conditions when imposed will reduce the system dimension to 6 x 6.

We know that if we sum the rows of X'X associated with the β_j, the sum is equal to the row associated with μ. We then, as previously shown, absorb μ and delete the row and column of X'X associated with μ. (See Table XXIII). This is the first condition we impose.

Likewise the sum of the rows associated with the τ_i is equal to the μ row, so we let a particular $\tau_i = 0$, say $\tau_3 = 0$. We then delete the row and column of X'X associated with τ_3. This is the second condition we impose.

The degrees of freedom associated with interaction $(\tau\beta)_{ij}$ is (b-1)(t-1) where b is the number of β_j and t is the number of τ_i. In our case b = 2,

$t = 3$. The number of degrees of freedom associated with interaction is the number of rows of X'X for which the "Abbreviated Doolittle" will not give zero computational rows. For example, in model (4. 4. 1) $(b-1)(t-1) = (1)(2) = 2$, hence in Table XXIII we see that we have only two non-zero computational rows for the six $(\tau\beta)_{ij}$ rows of X'X. (See rows R_6, r_6, and R_8, r_8 of Table XXIII). We then set four of the $(\tau\beta)_{ij} = 0$ and delete the rows and columns of X'X that are associated with these $(\tau\beta)_{ij}$.

From this we can say that the number of rows and columns associated with the $(\tau\beta)_{ij}$ which will not be deleted is equal to $(b-1)(t-1)$. This product, $(b-1)(t-1)$, is also the degrees of freedom associated with the interaction. The number of rows and columns in X'X that may be deleted is $bt - (b-1)(t-1)$.

We will formulate a rule for choosing the rows and columns associated with the $(\tau\beta)_{ij}$ that will remain in the X'X matrix.

(1) Construct the incidence matrix table with general notation in each cell. The layout for our particular model is given in Table X.

$\tau \backslash \beta$	1	2	
1	N_{11}	N_{12}	$N_{1\cdot}$
2	N_{21}	N_{22}	$N_{2\cdot}$
3	N_{31}	N_{32}	$N_{3\cdot}$
	$N_{\cdot 1}$	$N_{\cdot 2}$	

(2) For each N_{ij} we can associate a $(\tau\beta)_{ij}$. For example, we associate $(\tau\beta)_{11}$ and N_{11}, $(\tau\beta)_{12}$ with N_{12}, etc.

We may use the following rule for determining the rows and columns of the $(\tau\beta)_{ij}$ section of X'X that are not to be deleted.

Cross out the last row and last column of the incidence matrix. Take the $(\tau\beta)_{ij}$ associated with the remaining N_{ij}. The rows and columns of X'X associated with these elements will not be deleted. This means that the $(\tau\beta)_{ij}$ rows and columns of X'X corresponding to the N_{ij} in the last row and column of the incidence matrix will be deleted.

We will apply the rule to our example. Crossing out the last row and last column of the incidence matrix we see that we have deleted N_{12}, N_{22}, N_{32}, and N_{31}. We have left N_{11} and N_{21}. This means that in X'X we will delete the rows and columns corresponding to $(\tau\beta)_{12}$, $(\tau\beta)_{22}$, $(\tau\beta)_{22}$, and $(\tau\beta)_{31}$.

Since N_{11} and N_{21} remain in the incidence matrix, the rows and columns of X'X associated with $(\tau\beta)_{11}$ and $(\tau\beta)_{21}$ remain in the reduced matrix. (Note that (t-1)(b-1) rows and (t-1)(b-1) columns remain in the X'X matrix.)

Applying these deletion rules to the normal equations for model (4.4.1) we delete the following rows and columns:

(1) $\hat{\mu}$

(2) $\hat{\tau}_3$

(3) $\widehat{\tau\beta}_{12}$, $\widehat{\tau\beta}_{22}$, $\widehat{\tau\beta}_{32}$, and $\widehat{\tau\beta}_{31}$.

The resulting matrices are:

$$X'X = \begin{array}{cccccc} \beta_1 & \beta_2 & \tau_1 & \tau_2 & (\tau\beta)_{11} & (\tau\beta)_{21} \\ \hline 5 & 0 & 2 & 1 & 2 & 1 \\ 0 & 5 & 1 & 3 & 0 & 0 \\ 2 & 1 & 3 & 0 & 2 & 0 \\ 1 & 3 & 0 & 4 & 0 & 1 \\ 2 & 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{array} \qquad X'Y = \begin{bmatrix} 24 \\ 36 \\ 11 \\ 30 \\ 8 \\ 5 \end{bmatrix}$$

Note that the corresponding element of X'Y is deleted when a row of X'X is deleted.

We will now apply the forward solution to this reduced system. This is given in Table XXIV. (See Table XXIV)

From Table XXIV we obtain,

$$(1) \quad R(\mu, \beta, \tau, \tau\beta) = \sum_{i=1}^{6} CPID_i = 387.62$$

$$(2) \quad R[\tau\beta \,|\, \mu\, \beta\, \tau) = \sum_{i=5}^{6} CPID_i = 7.29$$

$$(3) \quad EMS = \frac{1}{n-p} \left[\sum_{ijk} y_{ijk}^2 - R(\mu, \beta, \tau, \tau\beta) \right)$$

This is the same information as from Table XXIII.

From this we see that by imposing the proper restrictions we can considerably reduce the size of the original system without sacrificing any information.

The particular model (4.4.1) that we have used for an example has no missing plots. The procedure for applying the Abbreviated Doolittle to such a model with missing data is the same with the exception of the deletion rule for the incidence matrix.

Table XXIV

Forward Solution to the Reduced System of Model (4. 4. 1)

Row	X'X						X'Y	Check
R'_1	5	0	2	1	2	1	24	35
R'_2		5	1	3	0	0	36	45
R'_3			3	0	2	0	11	19
R'_4				4	0	1	30	39
R'_5					2	0	8	14
R'_6						1	5	8
R_1	5	0	2	1	2	1	24	35
r_1	1	0	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\boxed{\frac{1}{5}}$	$\frac{24}{5}$	$\frac{35}{5}$
R_2		5	1	3	0	0	36	45
r_2		1	$\frac{1}{5}$	$\frac{3}{5}$	0	$\boxed{0}$	$\frac{36}{5}$	$\frac{45}{5}$
R_3			2	-1	$\frac{6}{5}$	$-\frac{2}{5}$	$-\frac{29}{5}$	$-\frac{20}{5}$
r_3			1	$-\frac{1}{2}$	$\frac{6}{10}$	$\boxed{-\frac{2}{10}}$	$-\frac{29}{10}$	$-\frac{20}{10}$
R_4				$\frac{3}{2}$	$\frac{2}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{30}{10}$
r_4				1	$\frac{4}{30}$	$\boxed{\frac{12}{30}}$	$\frac{14}{30}$	$\frac{60}{30}$
R_5					$\frac{34}{75}$	$-\frac{18}{75}$	$\frac{134}{75}$	$\frac{150}{75}$
r_5					1	$\boxed{-\frac{18}{34}}$	$\frac{134}{34}$	$\frac{150}{34}$
R_6						$\frac{90}{255}$	$-\frac{75}{255}$	$\frac{15}{255}$
r_6						1	$-\frac{75}{90}$	$\frac{15}{90}$

We shall state and give an example of a rule for determining the rows and columns of the $(\tau\beta)_{ij}$ section of X'X that are not to be deleted in a model with missing plots. For example, consider the following incidence matrix:

τ \ β	1	2	3
1	N_{11}	Missing	N_{13}
2	Missing	N_{21}	Missing
3	N_{31}	N_{32}	Missing
4	N_{41}	N_{42}	N_{43}

where

(1) Number of β's = b = 3;

(2) Number of τ's = t = 4;

(3) Degrees of freedom for interaction = (b-1)(t-1) - Number of missing plots = 6 - 4 = 2.

Rule Concerning Incidence Matrix:

(1) Strike out all rows that contain only one N_{ij}.

(2) In the remaining matrix strike out all columns that contain only one N_{ij}.

(3) Repeat (1) and (2) respectively until the remaining matrix contains no row or column with only one N_{ij}.

(4) In the first row of the remaining matrix circle all elements except the last element in this row.

(5) Now strike out this top row.

(6) Repeat (1), (2), (3), and (4) with the remaining matrix until all rows and columns are crossed out.

(7) The $(\tau\beta)_{ij}$ which correspond to the circled N_{ij} will remain in the reduced X'X matrix.

Let us apply the rule to our example.

Apply (1): Strike out row two leaving the following matrix:

N_{11}		N_{13}
	$\xcancel{N_{22}}$	
N_{31}	N_{32}	
N_{41}	N_{42}	N_{43}

N_{11}		N_{13}
N_{31}	N_{32}	
N_{41}	N_{42}	N_{43}

Apply (2): There are no columns containing only one N_{ij}.

Apply (3): This step is not necessary for this example.

Apply (4): We circle N_{11} and strike out the top row leaving:

$\textcircled{N_{11}}$		N_{13}
N_{31}	N_{32}	
N_{41}	N_{42}	N_{43}

N_{31}	N_{32}	
N_{41}	N_{42}	N_{43}

Apply (1): This is not applicable.

Apply (2): This is not applicable.

Apply (3): This is not applicable.

Apply (4): We circle N_{31} and strike out the top row leaving:

N_{31}	N_{32}	
N_{41}	N_{42}	N_{43}

N_{41}	N_{42}	N_{43}

Apply (1): This is not applicable.

Apply (2): We see that applying (2) three times will delete the remaining matrix.

Since we circled N_{11} and N_{31} in applying the rule, we do not delete the rows and columns in X'X associated with $\tau\beta_{11}$ and $\tau\beta_{31}$. (Note the number of rows and columns not deleted equals the number of degrees of freedom for interaction.)

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2 The last equation of the system (1.1) should be

$$0 = 2x + 2y + 12z + 16$$

11 Step eight should read: To obtain the elements of R_3 find the sum of the products of the pivotal multipliers in r_i with R_i where $i = 1, 2$. Again omitting all elements of R_i to the left of the multipliers.

25 The solutions for β at the bottom of the page are:

$$\hat{\beta}_3 = -6/19$$
$$\hat{\beta}_2 = 1 + 1/2\, \hat{\beta}_3 = 16/19$$
$$\hat{\beta}_1 = 5 - 2\hat{\beta}_2 + 3\hat{\beta}_3 = 81/19$$

29 The formula for R^2:

$$R^2 = \frac{\sum_{i=2}^{n} CPID_i}{\Sigma y_i^2 - CPID_1}$$

53 The Reference [4] should be [3]

65 The model at the bottom of the page is

$$Y_{ijk} = \gamma_j + \alpha Z_{ijk} + \tau_i + e_{ijk}$$