PLASTIC DEFORMATION ANALYSIS OF FRAMES AT ULTIMATE LOAD BY THE STRING POLYGON METHOD

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BY THE STRING POLYGON METHOD

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PREFACE

In June, 1960, the author attended a National Science Foundation Seminar for Civil Engineering Teachers at Oklahoma State University. The String Polygon Method was introduced as a method of elastic analysis in this seminar.

During the fall semester, the author attended a seminar on plasticity, in which problems of deformation of plastic-elastic beams and frames were discussed. It was at this time that it occurred to the author that the String Polygon Method could be applied, to the deformation analysis of elastic-plastic beams and frames, in a manner which would yield very direct solution.

After consultation with his advisors, the author undertook the writing of this thesis.

The author wishes to express his appreciation to the following individuals who generously gave of their time and talent to aid him during the last two years of graduate study:

To Professor Roger L. Flanders for introducing the theory of plasticity and serving as major advisor.

To Professor Jan J. Tuma, for many hours of instruction and consultation, and for invaluable aid in the literature survey for this thesis.

To Professor R. E. Means for his advice and encouragement.

To Professor James D. Gillespie, and Messrs. J. T. Oden and R. K. Munshi for their friendship and consultation during the period of this study.

To his wife who conscientiously typed the manuscript and assumed many family responsibilities to allow time for study and for her encouragement throughout this period of study.

And to the entire faculty and staff of the Department of Civil Engineering for a very enjoyable period of association.

F. N. G.

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NOMENCLATURE

$a, b, c,$
$1, 2, 3,$ = polygon vertex point designation

BM_u, BM_v = bending moment of simple beam due to loads, at point u, v, respectively

D = reciprocal of flexural rigidity

E = modulus of elasticity

F = ultimate load

F_{ij} = angular flexibility function

G_{ij} = angular carry over function

h = vertical dimension

I = moment of inertia

i, j, k = general index points

L_i, L_j
d_i, d_j = length of beam segments ij and jk respectively

M_i, M_j, M_k = bending moment at points i, j, k, respectively

n = any integer

$\overline{P}_{Ej}$ = elastic weight applied at point j

$\overline{P}_{Pj}$ = plastic weight applied at point j

$\overline{R_j}$ = conjugate reaction at point j

S_{bg} = length between points b and g

$\overline{x}, \overline{y}$ = moment arm measured parallel to x, y, axis respectively

α, β = coefficient of length

Δ_{jx}, Δ_{jy} = deflection components of point j, in x and y directions respectively

ϕ_j = the change in the deflection angle of the polygon due to plastic hinge, real hinge and/or elastic rotation at any point j

ϕ_{ij} = end slope of simple beam segment ij at end i

θ_j = the deflection angle of the undeformed closed polygon at any point j

ω = angle designation

Σ = summation

τ_{ij} = angular load function

$\Sigma M @_{\overline{ij}}$ = summation of moments of forces or conjugate weights about line ij

CHAPTER I

INTRODUCTION

The String Polygon probably was conceived by Archimedes, however, it is usually attributed, in its basic form to Varignon, who studied the loaded string, and introduced the concept of the polygon of equilibrium. Culmann discarded the material string, and used the polygon of equilibrium as a tool of analysis, thus laying the foundation for the development of graphic statics as an effective means of analysis.

Mohr (15) represented the elastic curve of a straight beam as a differential string polygon in connection with his concept of the conjugate beam loaded with differential angle changes known as elastic weights.

The Joint Loads Concept was introduced by Muller-Breslau (16, 17). In his definition of joint loads, the influence of loads on the elements was neglected and only the influence of moment, shear, and axial force was considered.

By adding the angular load function to the joint load, Tuma (1) generalized the String Polygon Method, and related it to the Three Moment Equation. This generalization greatly increases the effectiveness of the method, since elements of any length or curvature may be used, with exact results.

Deformation analysis of frames at ultimate load is important. It is the basis of approximate working load deformation analysis. When materials having a limited rotation capacity are used, the magnitude of plastic rotation is often critical and, therefore, must be determined.

Difficulty arises in the deformation analysis by the commonly used Slope Deflection Method since it is necessary to solve many simultaneous equations and, by trial and error, establish the last hinge to form. Lee (27) generalized the Conjugate Beam Method, and called it the Conjugate Frame Method. This method provides three independent equations which are identical to the String Polygon equilibrium equations. Lee further recognized that a fourth rational condition is obtained from the direction of plastic hinge rotations of the collapse mechanism. Thus, adequate equations are available and, usually the last hinge to form is obtained by inspection of these equations.

The Conjugate Frame Method is somewhat tedious because of the differential elastic weights, which operate in two coordinate directions and necessitate the computation of moment arms from the centroid of each segment of the moment diagram to the axis of moments for each conjugate moment equation. The sign convention is also two-phased and involved.

The String Polygon Method is an efficient tool of analysis for many structural problems. Recent investigators (2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14) have extended the present concept of the String Polygon, to many phases of elastic analysis.

The String Polygon approach simplifies the expression for differential elastic weights by concentrating their effect in the form of joint elastic weights at convenient points, thus eliminating the necessity for computing moment arms for conjugate moments.

The computation of elastic weights is made by substitution into the three moment equation, and is further simplified by means of beam constants which are available, for members of constant or variable cross section. (2, 5, 6, 26.)

The method is applied to plastic structures on the verge of collapse, and is perfectly general as to variation of cross section since this variation is taken into account by proper evaluation of the elastic weights.

In Chapter Two of this thesis, the general theory of the String Polygon method is restated to include the deformation effects of plastic hinges. Chapter Three is devoted to examples, in which the elastic deformation is neglected, thereby providing an alternate method to Instantaneous Centers Method which is commonly used to determine the mechanism angle relationships. Chapter Four is devoted to examples of single bay frames, and Chapter Five is devoted to examples of complex frames. The sixth chapter summarizes and concludes the study.

CHAPTER II

THEORY OF THE STRING POLYGON

2-1 GENERAL

Important, well-known relationships exist between the basic rules of closed polygon geometry and the basic rules of statics.

Under certain conditions, these relationships allow the problems of geometry to be solved by the familiar processes used in the solution of the problems of statics.

Planar structural analysis problems, among others, fall within these conditions if the small deflection theory is permissable.

2-2 ASSUMPTIONS

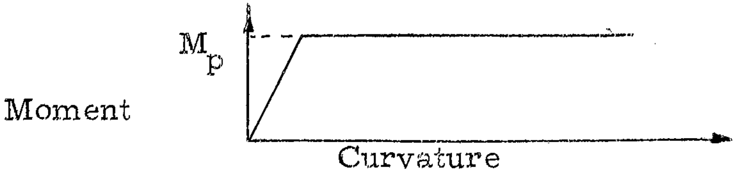
1) The usual assumptions of structural analysis apply to the determination of elastic constants.

2) The change in length of structural members is small and may be neglected.

3) The length of plastic hinges is small in comparison to the length of members, and may be considered to occur at a point.

4) The plastic and elastic angle changes are small, and the Sine and Tangent of the angle are taken as the angle itself.

5) The structural material is perfectly plastic, and the moment-curvature relationship is as shown in Fig. 2-2.



Idealized moment-curvature relationship

Fig. 2-2

2-3 GEOMETRIC RELATIONSHIPS

Consider the frame or structural panel of Fig. 2-3a, which describes the closed polygon (1, 2, 3, 4, 5, 1), having deflection angles $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$.

Under the influence of applied loads, the structure deforms to a new position, on which the points (1, 2', 3', 4', 5, 1) lie. The deformed structure is thus represented by straight lines (string lines) connecting the prime points of the deformed polygon.

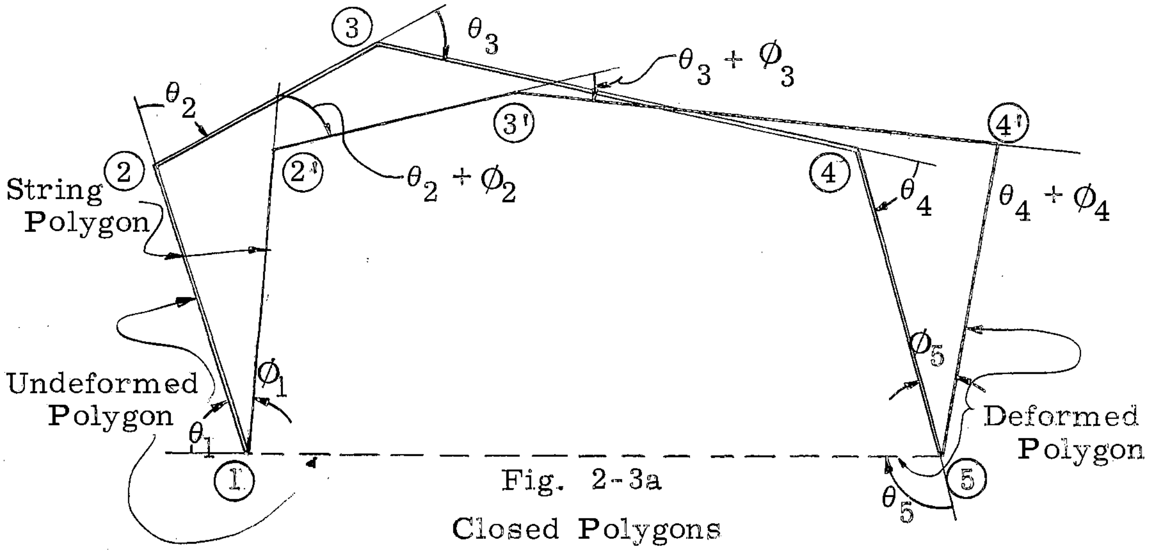


Fig. 2-3a
Closed Polygons

From plane geometry, the sum of the deflection angles of the deformed and undeformed polygon are given by equations 2-3a and 2-3b respectively.

$$(\theta_1+\phi_1) + (\theta_2+\phi_2) + (\theta_3+\phi_3) + (\theta_4+\phi_4) + (\theta_5+\phi_5) = 2\pi \qquad \text{Eq. 2-3a}$$

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 = 2\pi \qquad \text{Eq. 2-3b}$$

subtracting Eq. 2-3b from Eq. 2-3a yields:

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 = 0 \qquad \text{Eq. 2-3c}$$

This expression is for a five sided polygon; however, the concepts are perfectly general. It is therefore evident that:

> In the general case of an n-sided closed polygon which undergoes deformation, the algebraic sum of all angle changes must be equal to zero for geometric compatability,

or mathematically:

$$\sum_{1}^{n} \phi_n = 0 \qquad \text{Eq. 2-3d}$$

and is analogous to:

> in the general case of a system of parallel forces, the algebraic sum of all forces must be equal to zero for static equilibrium,

or mathematically:

$$\sum_{1}^{n} P_n = o \qquad \text{Eq. 2-3e}$$

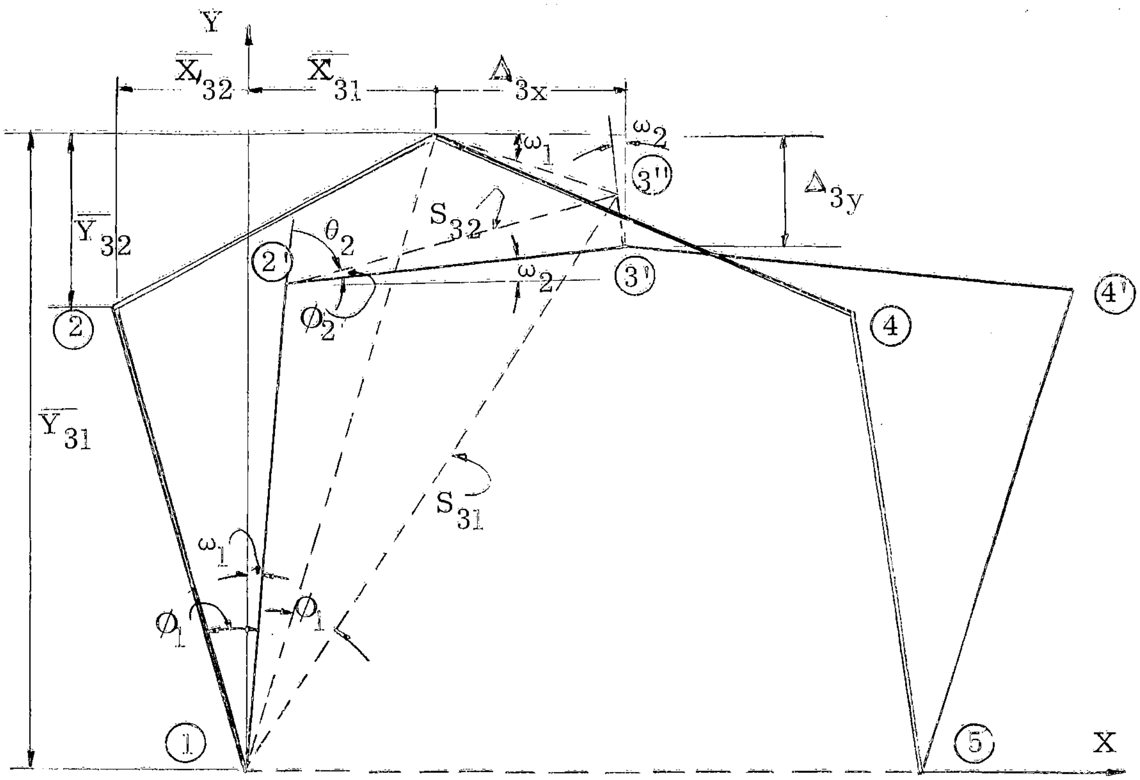


Fig. 2-3b
String Polygon

Next, consider the string polygon of Fig. 2-3b. The angle between lines (1, 2) and (1, 2') is ϕ_1. Similarly the angle between lines (2, 3) and (2', 3') is $\phi_1 + \phi_2$, etc. For the general case, it is seen that:

> The angle between respective original and deformed lines of the polygon, is equal to the sum of the angle changes to the left or right of that line.

which is analogous to:

> The internal shear force in a beam or frame is equal to the sum of the forces to the left or right of that point.

The distance between the original and prime points of the string polygon represent the absolute displacement of the respective points of the structure.

The displacement of point 3 due to ϕ_1, is (3-3''), and from small angle geometry:

$$(3-3'') = \phi_1 \cdot S_{31}$$

the X component of (3-3'') is:

$$(3-3'')_x = (3-3'') \sin \omega_1 = \phi_1 S_{31} \sin \omega_1$$

and the Y component of (3-3'') is

$$(3-3'')_y = (3-3'') \cos \omega_1 = \phi_1 S_{31} \cos \omega_1$$

but since $\overline{X}_{31} = S_{31} \cos \omega_1$ and $\overline{Y}_{31} = S_{31} \sin \omega_1$

the direction components of (3-3'') are:

$$(3-3'')x = \phi_1 \overline{Y}_{31}$$

$$(3-3'')y = \phi_1 \overline{X}_{31}$$

Similarly the displacement due to ϕ_2 is (3''-3') and:

$$(3''-3') = \phi_2 \quad S_{32}$$

the X component of (3''-3') is

$$(3''-3')_x = \phi_2 \overline{Y}_{32}$$

the Y component of (3''-3') is

$$(3''-3')_y = \phi_2 \overline{X}_{32}$$

Superimposing the deformation components due to ϕ_1, and ϕ_2 :

$$\Delta_{3x} = \phi_1 \overline{Y}_{31} + \phi_2 \overline{Y}_{32} \qquad \Delta_{3y} = \phi_1 \overline{X}_{31} + \phi_2 \overline{X}_{32}$$

In the general case, the deformation components, of any point n, are:

$$\Delta_{nx} = \sum_{1}^{n} \phi_n \overline{Y}_n \qquad \Delta_{ny} = \sum_{1}^{n} \phi_n \overline{X}_n$$

It is then evident that:

> The displacement component of any point on the polygon, is equal to the sum of the moments of the changes in deflection angles of the polygon, about a line passing thru that point of the original polygon, parallel to the direction of the desired displacement component, of all such angle changes which lie on one side of the displacement line.

which is analogous to:

> The bending moment at any point of a beam or frame, is equal to the sum of the moments of forces about that point of the beam or frame, of all forces which lie on one side of the point.

A further analogy may be made since the displacement at any point n is common to both sides of the polygon, therefore, for an n sided polygon the displacement in any direction z is:

$$\Delta_{mz} = \sum_{1}^{m} \phi_m \overline{Z}_m = \sum_{n}^{m} \phi_m \overline{Z}_m$$

or

$$\sum_{1}^{n} \phi_m \overline{Z}_m = 0$$

therefore:

> The algebraic sum of moments of changes in deflection angles of a closed polygon, about any line in the plane of the polygon, must be equal to zero for geometric compatability.

which is analogous to:

> The algebraic sum of moments of a system of parallel forces acting on a plane, about any line in that plane, must be equal to zero for static equilibrium.

2-4 ELASTIC WEIGHTS

One of the major advantages of the String Polygon Method is due to the fact that points on the polygon may be selected arbitrarily, from the geometry of the frame, usually at corners, abrupt changes in cross-section, and real or plastic hinges. It is therefore necessary to transfer the effect of elastic deformation which occurs between these selected points to the points on the polygon. This transformation is accomplished by means of joint elastic weights.

The basic stress analysis of a frame may be accomplished by the elementary theory of plasticity, which reduces most frames to statically determinate ones. Moment diagrams are thus available, from which elastic deformation is determined.

An expression for the joint elastic weight at any point j, may be derived by considering the beam segments adjacent to point j, Fig. 2-4a.

The segments ij and jk are straight beam segments, but may have any variation in cross-section and are subjected to general loads and end moments.

From the Fig. 2-4a, it is seen that ϕ_j is the change in the deflection angle at point j, and is thus the elastic weight, and

$$\phi_j = \phi_{ji} + \phi_{jk} \qquad \text{Eq. 2-4a}$$

It is also evident that ϕ_{ji} is the end slope of beam segment ij at end j due to moments, and ϕ_{jk} is the end slope of beam segment jk at end j due to moments.

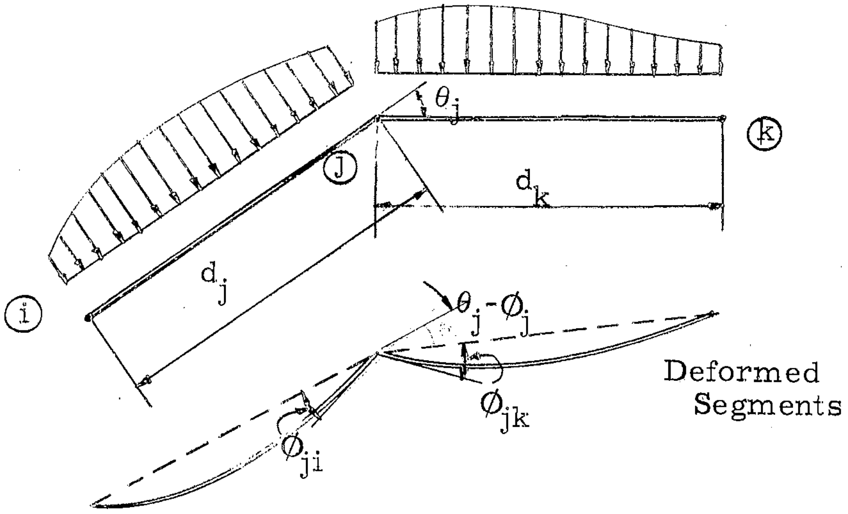


Fig. 2-4a
Beam segments adjacent to point j.

Taking free bodies of the beam segments, and dividing the moment diagrams into three parts as shown in Fig. 2-4b, the end slopes of each segment may be written using the area-moment relationships, thus:

$$\phi_{ji} = M_i \int_i^j \frac{uu' \, du}{d_j^2 \, EI_u} + M_j \int_i^j \frac{u^2 \, du}{d_j^2 \, EI_u} + \int_i^j \frac{BM_u \, u \, du}{d_j \, EI_u}$$

$$\phi_{jk} = M_j \int_j^k \frac{v'^2 \, dv}{d_k^2 \, EI_v} + M_k \int_j^k \frac{vv' \, dv}{d_k^2 \, EI_v} + \int_j^k \frac{BM_v \, v' \, dv}{d_k \, EI_v}$$

denoting the integrals by:

$$\tau_{ji} = \int_i^j \frac{BM_u \, u \, du}{d_j \, EI_u} \qquad \tau_{jk} = \int_j^k \frac{BM_v \, v' \, dv}{d_k \, EI_v}$$

$$F_{ji} = \int_i^j \frac{u^2 \, du}{d_j^2 \, EI_u} \qquad F_{jk} = \int_j^k \frac{v'^2 \, dv}{d_k^2 \, EI_v}$$

$$G_{ji} = \int_i^j \frac{uu' \, du}{d_j^2 \, EI_u} \qquad G_{jk} = \int_j^k \frac{vv' \, dv}{d_k^2 \, EI_v}$$

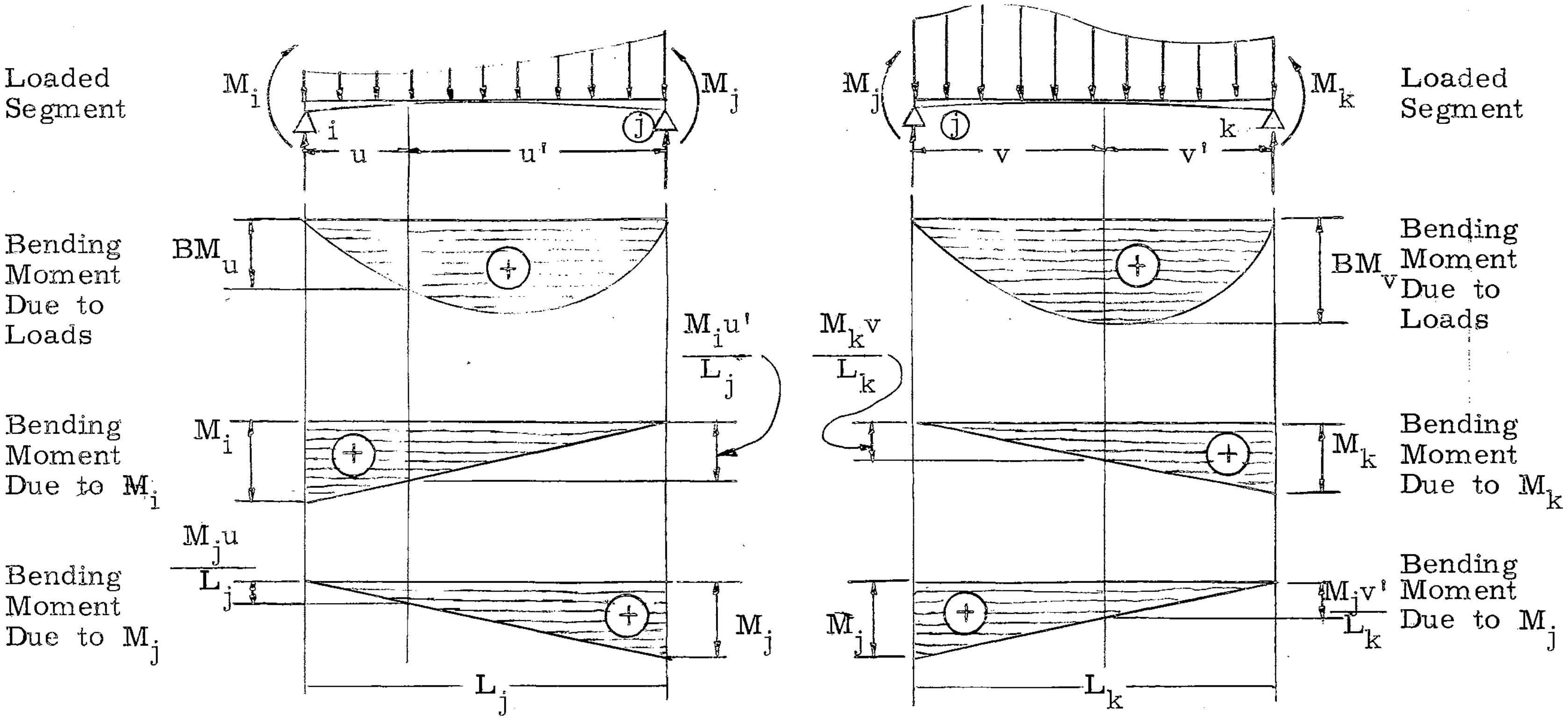


Fig. 2-4b

Free Bodies $\overline{ij}$ and $\overline{jk}$

Eq. 2-4a becomes:

$$\overline{P}_{Ej} = \phi_j = M_i\, G_{ij} + M_j\, \Sigma\, F_j + M_k\, G_{kj} + \Sigma\, \tau_j \qquad \text{Eq. 2-4b}$$

Where $\Sigma\, F_j = F_{ji} + F_{jk}$ and $\Sigma\, \tau_j = \tau_{ji} + \tau_{jk}$

Thus the equation of the joint elastic weight, Eq. 2-4b, is seen to be identical to the familiar Three Moment Equation. The quantities τ, F, and G have the following physical interpretation:

<u>Angular Load Function τ_{ji} (τ_{jk})</u> is
the end slope of the simple beams ij (jk) at j due to loads

<u>Angular Flexibility F_{ji} (F_{jk})</u> is
The end slope at j of the simple beam ij (jk) due to unit moment applied at j.

<u>Angular Carry-Over Value G_{ij} (G_{kj})</u> is
the end slope of the simple beam ij (jk) at j due to unit moment applied at i, (k).

If the cross-section of each member is different but constant between two joints, the following simplifications are possible:

$$F_{ji} = \frac{L_j}{3\ EI_j} \qquad\qquad F_{jk} = \frac{L_k}{3\ EI_k}$$

$$G_{ij} = \frac{L_j}{6\ EI_j} \qquad\qquad G_{kj} = \frac{L_k}{6\ EI_j}$$

The load functions τ_{ji} and τ_{jk} for the most common load conditions reduce to the expressions shown in Table 2-1.

2-5 PLASTIC WEIGHTS

Plastic weights are defined as the changes in the deflection angle of the polygon due to plastic rotation. Since the points of plastic rotation are known from the collapse mechanism, and are selected to be points on the polygon, no transformation is necessary.

These plastic rotations are taken as redundants, and their variation is such that geometric compatability is provided.

2-6 CONJUGATE REACTIONS

Conjugate reactions are defined as the changes in deflection angles of the polygon due to real hinge rotations. Except for the distinguishing symbol, they are treated identically to plastic weights.

2-7 VECTOR NOTATION

Since the changes in the deflection angles ϕ are analogous to forces, as has been shown in article 2-3, it is convenient to represent the angle changes by vectors. This is easily accomplished since all angle changes lie in the plane of the frame, and are, therefore, directly additive.

Vectors which represent rotational quanities have a direction perpendicular to the plane of the rotation; thus elastic and plastic weights will be represented by vectors perpendicular to the plane of the frame or panel.

2-8 SIGN CONVENTION

Bending moments are plotted on the tension side of the member; thus, a moment diagram lying on the inside of the polygon is positive, and those moments outside are negative for that particular panel.

Elastic weights will carry the sign of the bending moment.

Plastic weights for each particular panel are positive if the interior angle of the polygon at that point is increased and negative if the interior angle is decreased.

Weights which are applied to the conjugate frame are positive upward and negative downward when the frame is drawn in the horizontal plane.

2-9 GENERAL APPLICATION

Deformation analysis of single or multiple panel frames, at impending collapse, may be effectively carried out by the String Polygon Method.

The usual methods of plastic design are used to determine member sizes and provide the basic geometry of the collapse mechanism.

Plastic hinges, real hinges, and the corners of the frame or panel are selected as points on the polygon. If it is known in advance that the deformation of additional points are required, those points may also be selected as points on the polygon.

By means of Eq. 2-4b, and the moment diagram, the elastic weights for all selected points on the polygon are computed.

Plastic weights and conjugate reactions are redundant; however, their sign is known from the collapse mechanism. It is convenient to apply these redundants to the conjugate frame in their proper sense, thereby requiring the solution of the equilibrium equations to yield a positive sign for the plastic and real hinge rotations. Only the plastic weight representing the last hinge to form, when equated to zero, will yield a positive sign for all of the remaining values.

The conjugate frame is then drawn in the horizontal plane and all redundants applied to the conjugate frame in their proper sense, in a vertical plane.

Three independent equilibrium equations may then be written for each panel. It is usually most convenient to take moments about three sides of the conjugate frame. A brief inspection will usually determine which sides to select for the most simplified equations. Most of the equilibrium equations will contain only two unknowns each. In many cases, equating the last hinge to form to zero, will reduce the simultaneous equation set to explicit form. The values of plastic and real hinge rotations are obtained directly by solving the equilibrium equations.

Which is real deflection

The conjugate bending moment is the distance between the original polygon and the deformed string line polygon. If the deflection of the originally selected points are required, it is only necessary to determine the conjugate moment at that point about a line parallel to the direction of the desired deflection. If the deflection of some intermediate point is required, it is necessary to add the deflection of the simple beam segment due to loads at that point, to the conjugate bending moment at that same point. The direction of the deflection is determined rationally.

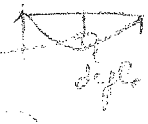


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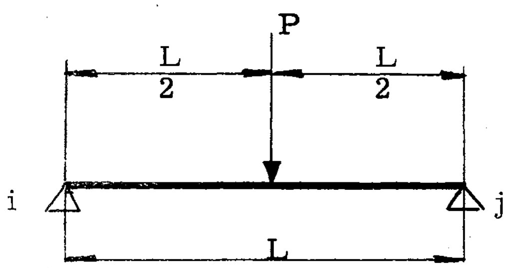
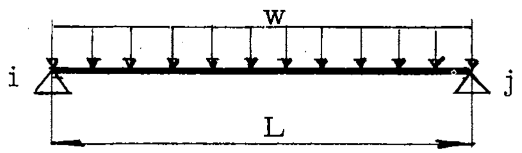
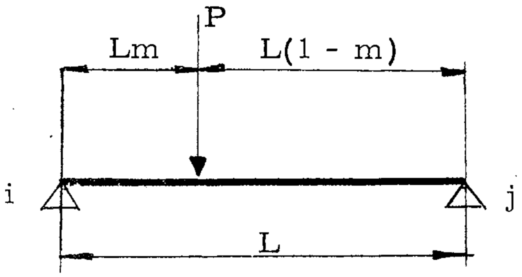
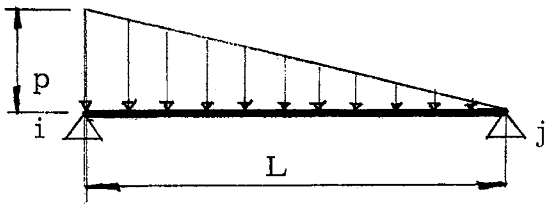
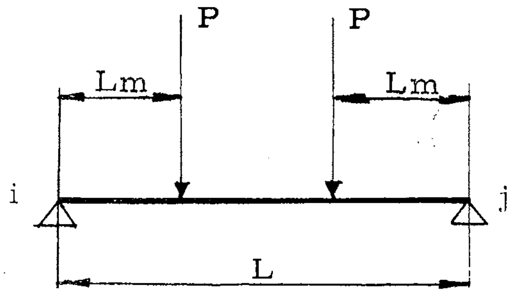
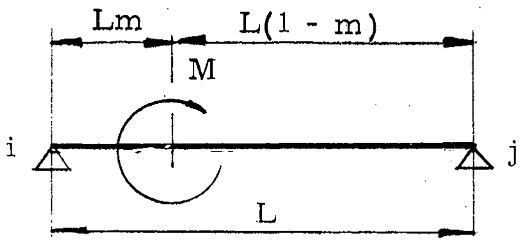
Table 2-1 Load Functions

 $\tau_{ij} = \frac{PL^2}{16\ EI}$ $\tau_{ji} = \frac{PL^2}{16\ EI}$	 $\tau_{ij} = \frac{wL^3}{24\ EI}$ $\tau_{ji} = \frac{wL^3}{24\ EI}$
 $\tau_{ij} = \frac{PL^2 m(1 - m)(2 - m)}{6\ EI}$ $\tau_{ji} = \frac{PL^2 m(1 - m^2)}{6\ EI}$	 $\tau_{ij} = \frac{pL^3}{45\ EI}$ $\tau_{ji} = \frac{7pL^3}{360\ EI}$
 $\tau_{ij} = \frac{PL^2 m(1 - m)}{2\ EI}$ $\tau_{ji} = \frac{PL^2 m(1 - m)}{2\ EI}$	 $\tau_{ij} = \frac{ML(3m^2 - 6m + 2)}{6\ EI}$ $\tau_{ji} = \frac{ML(1 - 3m^2)}{6\ EI}$

CHAPTER III

MECHANISM ANGLE RELATIONSHIPS

3-1 GENERAL

The elementary theory of plastic design by the mechanism method requires the relative magnitude of the plastic and real hinge rotations. These relationships are quickly and effectively determined by the String Polygon Method.

In common practice, the effects of elastic deformation are assumed to have negligable influence on the relative rotations. For the String Polygon Method, this assumption is equivalent to assuming that the elastic weights are equal to zero. The redundant plastic weights and conjugate reactions are placed on the conjugate frame with the same direction. The direction of rotation is thus indicated by the sign of the values. Three independent equilibrium equations are obtained for each panel by setting the sum of moments of conjugate weights about three sides of that panel equal to zero. Three of the four redundants may then be found in terms of the fourth by solving the equations simultaneously.

Examples 3-2 and 3-3 illustrate this procedure for single and multiple panel frames.

3-2 EXAMPLE OF SINGLE PANEL GABLE FRAME

The mechanism angle relationships are found for the frame with the assumed collapse mechanism shown in Fig. 3-2a.*

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* This example is worked by the Instantaneous Center Method on pages 6, 7, of Ref. (29).

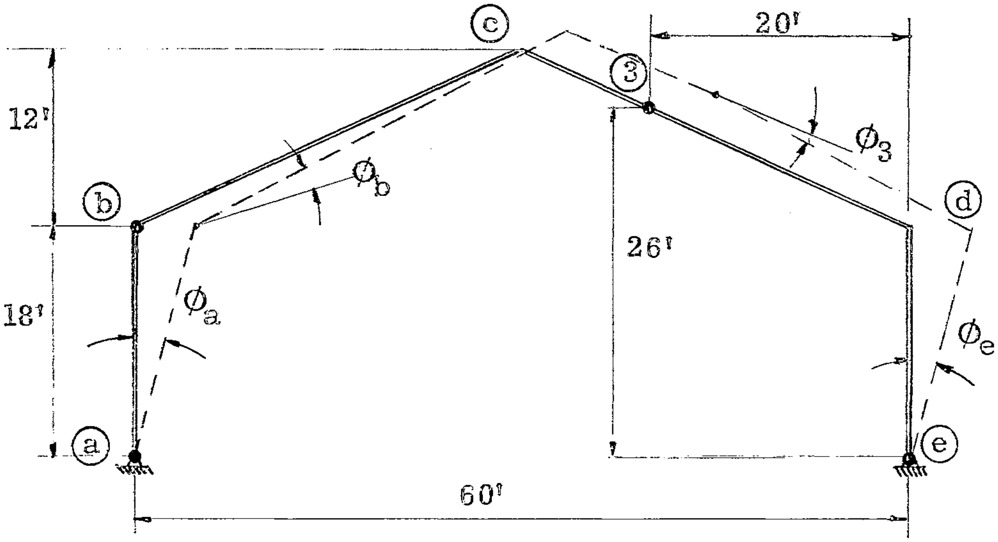


Fig. 3-2a

Gable Frame and Mechanism

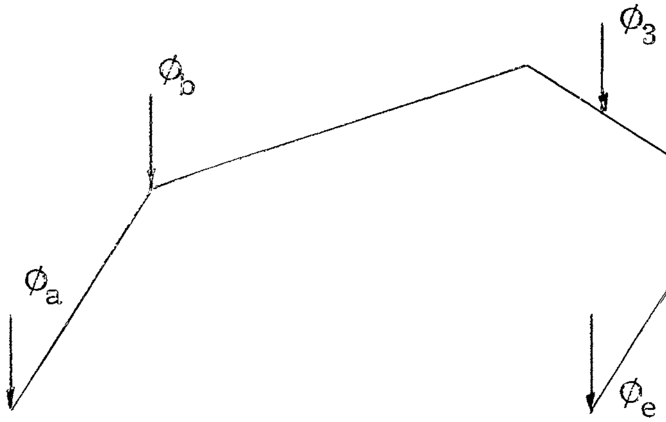


Fig. 3-2b

Conjugate Frame

$$\sum M @ \overline{ae} = 18\phi_b + 26\phi_3 = 0$$

$$\sum M @ \overline{ab} = 40\phi_3 + 60\phi_e = 0$$

$$\sum M @ \overline{ed} = 20\phi_3 + 60\phi_a + 60\phi_b = 0$$

Solution of these equations in terms of ϕ_a yield:

$\phi_b = -1.3\ \phi_a$ $\qquad$ $\phi_3 = .9\ \phi_a$ $\qquad$ $\phi_e = -.6\ \phi_a$

3-3 EXAMPLE OF THREE PANEL GABLE FRAME

The relations between plastic rotations are found for the frame and assumed mechanism shown in Fig. 3-3a in terms of the rotation at point a.

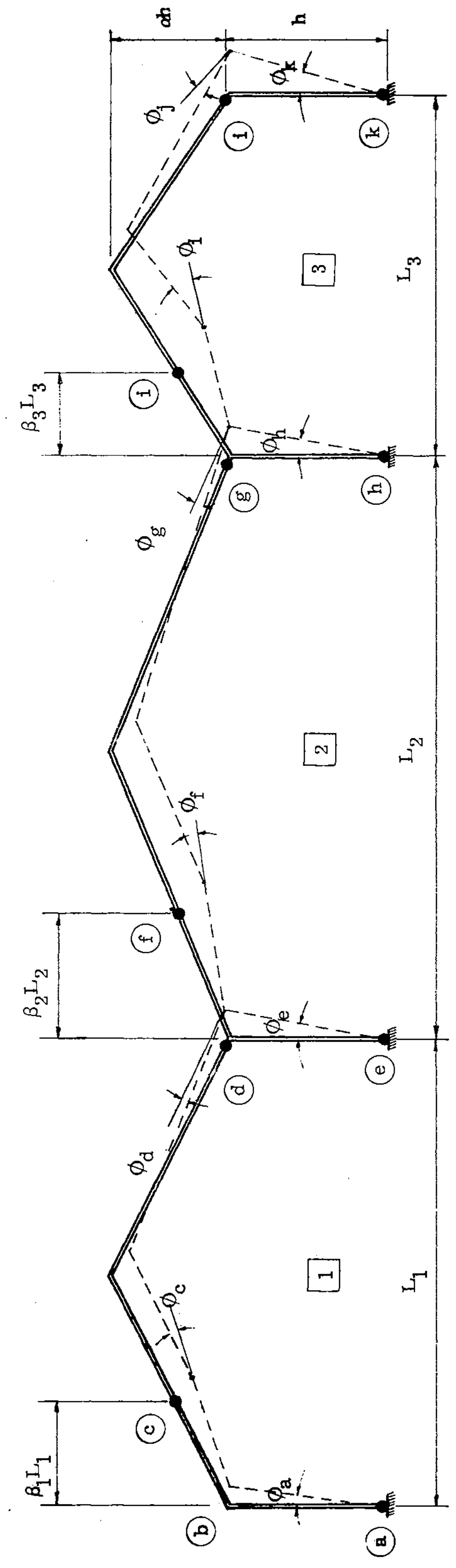


Fig. 3-3a

Three Panel Gable Frame and Mechanism

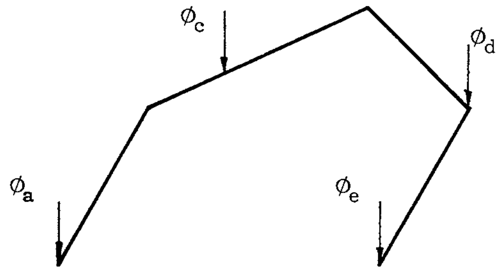


Fig. 3-3b
Conjugate Panel 1

The equilibrium equations are:

$$M@_{\overline{ab}} = \beta_1 L_1 \phi_c + L_1 \phi_d + L_1 \phi_e = 0$$

$$M@_{\overline{ae}} = h\phi_d + (h + 2\alpha\beta_1 h)\,\phi_c = 0$$

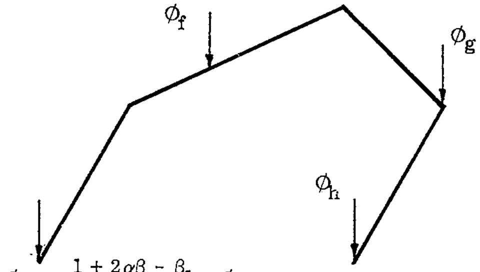
$$M@_{\overline{ed}} = L_1 \phi_a + (1 - \beta_1)\, L_1 \phi_c = 0$$

Solving these equations in terms of ϕ_a:

$$\phi_c = - \left(\frac{1}{1-\beta_1}\right) \phi_a$$

$$\phi_d = - (1 + 2\alpha\beta_1)\,\phi_c = \frac{(1 + 2\alpha\beta_1)}{1 - \beta_1} \phi_a$$

$$\phi_e = - \beta_1 \phi_c - \phi_d = - \frac{(1 + 2\alpha\beta_1)}{1 - \beta_1} \phi_a$$



$$\phi_e = \frac{1 + 2\alpha\beta - \beta_1}{1 - \beta_1} \phi_a$$

Fig. 3-3c
Conjugate Panel 2

The equilibrium equations are:

$$M@_{\overline{ed}} = \beta_2 L_2 \phi_f + L_2 \phi_g + L_2 \phi_h = 0$$

$$M@_{\overline{eh}} = (h + 2\alpha\beta_2 h)\,\phi_f + h\phi_g = 0$$

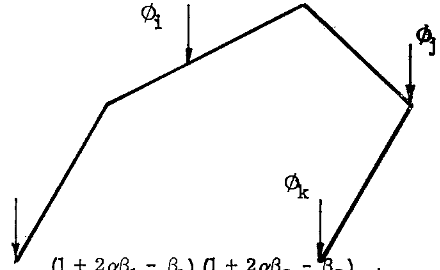
$$M@_{\overline{gh}} = (1 - \beta_2)\, L_2 \phi_f + L_2 \phi_e = 0$$

Solving:

$$\phi_f = - \frac{1 + 2\alpha\beta_1 - \beta_1}{(1 - \beta_1)(1 - \beta_2)} \phi_a$$

$$\phi_g = \frac{(1 + 2\alpha\beta_1 - \beta_1)(1 + 2\alpha\beta_2)}{(1 - \beta_1)(1 - \beta_2)} \phi_a$$

$$\phi_h = - \frac{(1 + 2\alpha\beta_1 - \beta_1)(1 + 2\alpha\beta_2 - \beta_2)}{(1 - \beta_1)(1 - \beta_2)} \phi_a$$



$$\phi_h = \frac{(1 + 2\alpha\beta_1 - \beta_1)(1 + 2\alpha\beta_2 - \beta_2)}{(1 - \beta_1)(1 - \beta_2)} \phi_a$$

Fig. 3-3d
Conjugate Panel 3

The equilibrium equations are:

$$M@_{\overline{hg}} = \beta_3 L_3 \phi_i + L_3 \phi_j + L_3 \phi_k = 0$$

$$M@_{\overline{hk}} = (h + 2\alpha\beta_3 h)\,\phi_i + h\phi_j = 0$$

$$M@_{\overline{kj}} = (1 - \beta_3)\, L_3 \phi_i + L_3 \phi_h = 0$$

Solving:

$$\phi_i = - \frac{(1 + 2\alpha\beta_1 - \beta_1)(1 + 2\alpha\beta_2 - \beta_2)}{(1 - \beta_1)(1 - \beta_2)(1 - \beta_3)} \phi_a$$

$$\phi_j = \frac{(1 + 2\alpha\beta_1 - \beta_1)(1 + 2\alpha\beta_2 - \beta_2)(1 + 2\alpha\beta_3)}{(1 - \beta_1)(1 - \beta_2)(1 - \beta_3)} \phi_a$$

$$\phi_k = - \frac{(1 + 2\alpha\beta_1 - \beta_1)(1 + 2\alpha\beta_2 - \beta_2)(1 + 2\alpha\beta_3 - \beta_3)}{(1 - \beta_1)(1 - \beta_2)(1 - \beta_3)} \phi_a$$

CHAPTER IV

DEFORMATION ANALYSIS OF SINGLE PANEL FRAMES

4-1 GENERAL

The deformation analysis of single panel frames may be accomplished by means of the String Polygon Method. The deformation analysis begins with known loads, beam sections, the moment diagram, and the collapse mechanism. The angular functions are computed and the elastic weights evaluated. The conjugate frame is then loaded and the equilibrium equations are written by setting the summation of moments of the conjugate weights about three sides of the frame equal to zero. The last hinge to form is found by inspection of the equilibrium equations. By setting the plastic weight corresponding to the last hinge to form equal to zero, the three simultaneous equations are reduced to explicit form and solution is made by direct substitution. The plastic weights are equal to the plastic rotation of the hinges measured in radians.

The deflection of any point originally selected as a point on the polygon is determined by evaluating the bending moment of the conjugate frame at that point.

4-2 EXAMPLE OF SINGLE PANEL PORTAL FRAME

The frame of Fig. 4-2a is analyzed by the String Polygon Method for plastic rotations at points e and g, and the deflection of points d, g, e, i.**

**This example is worked by the Slope Deflection Method on pages 100, 103, of Ref. (28). The frame was tested to failure and reported by Schilling, Schutz, and Beedle, Ref. (31).

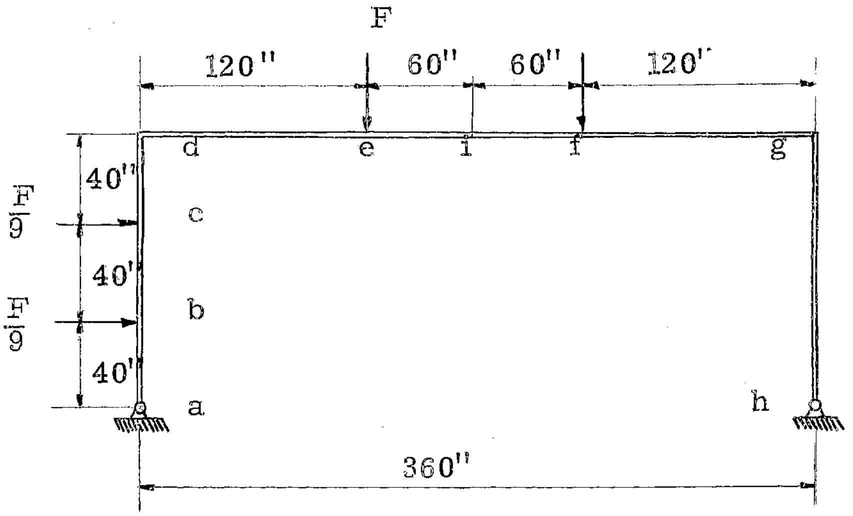


Fig. 4-2a

One Panel Portal Frame

An analysis by the elementary theory of plasticity indicates a collapse load of F = 29.9 kips for a uniform beam section whose yield moment is M_p = 1925 inch kips, and flexural rigidity is EI = 80.39×10^5.

The collapse mechanism is formed by the real hinges at a and h, and plastic hinges at e and g.

The moment diagram is as shown in Fig. 4-2b.

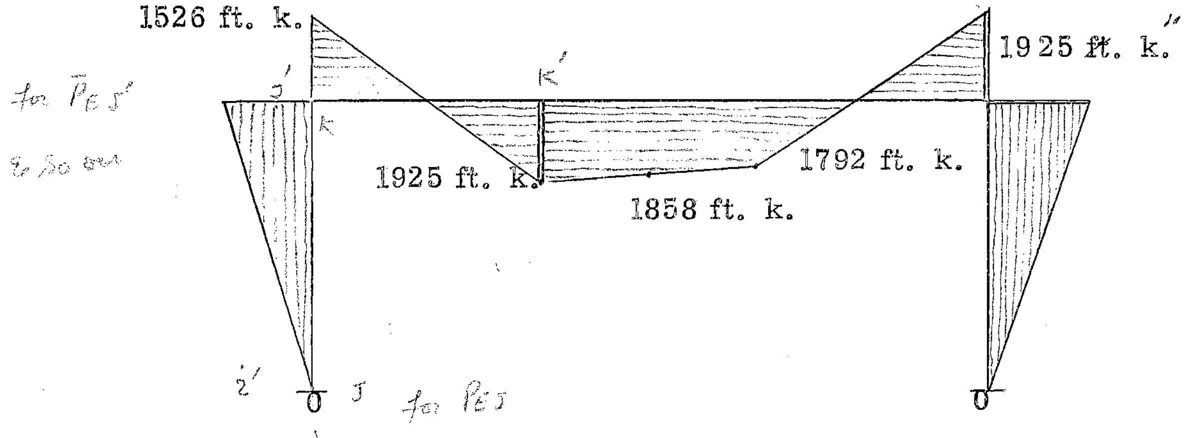


Fig. 4-2b
Moment Diagram

The points a, d, e, i, g, h, are selected as points on the polygon.

The angular functions F, G, and τ are tabulated as coefficients of $\frac{1}{EI}$ in Table 4-1.

TABLE 4-1

P- 2-13
L = 120" *

Beam Segment	$F_{ij(ji)}$	$G_{ij(ji)}$	τ_{ij}	τ_{ji}
ad	40 $= \frac{L}{3}$	20 $= \frac{L}{6}$	+5, 315	+5, 315
de	40	20	0	0
ei	20	10	0 L = 180"	0
ig	60	30	+59, 600	+47, 700
gh	40	20	0	0

The elastic weights for each point are determined by means of Eq. 2-4b in Table 4-2 as coefficients of $\frac{1}{EI}$.

See Moment also Page 6-2

TABLE 4-2

Point	M_iG_{ij}	$M_j\Sigma F_j$	M_kG_{kj}	$\Sigma\tau_j$	$\overline{P}_{Ej}$
a	0	0	-1526(20)	+5, 315	-25, 205
d	0	-1526(80)	+1925(20)	+5, 315	-78, 265
e	-1526(20)	+1925(60)	+1858(10)	0	+103, 560
i	+1925(10)	+1858(80)	-1925(30)	+59, 600	+169, 740
g	+1858(30)	-1925(100)	0	+47, 700	- 89, 060
h	-1925(20)	0	0	0	- 38, 500

The conjugate structure is then as shown Fig. 4-2c.

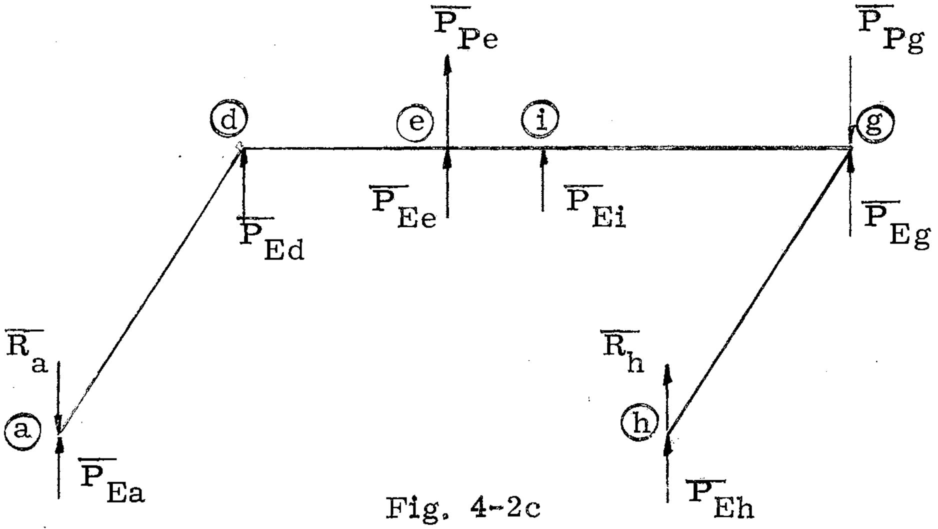


Fig. 4-2c
Conjugate Frame

The equilibrium equations may then be written by taking moments of all conjugate forces about three sides of the conjugate frame, thus:

$$\sum M@_{\overline{ah}} = 120(\overline{P}_{Ed} + \overline{P}_{Ee} + \overline{P}_{Ei} + \overline{P}_{Eg} + \overline{P}_{Pe} - \overline{P}_{Pg}) = 0$$

Substituting values from Table 4-2:

Plastic — all at plastic hinges

$$\overline{P}_{Pe} - \overline{P}_{Pg} = -\frac{105,975}{EI} \qquad \text{Eq. 4-2a}$$

Similarly:

$$\sum M@_{\overline{hg}} = 180\,\overline{P}_{Ei} + 240(\overline{P}_{Ee} + \overline{P}_{Pe}) + 360(\overline{P}_{Ed} + \overline{P}_{Ea} - \overline{R}_a) = 0$$

Substituting values:

$$2\,\overline{P}_{Pe} - 3\,\overline{R}_a = -\frac{151,320}{EI} \qquad \text{Eq. 4-2b}$$

and similarly:

$$\sum M@_{\overline{dg}} = 120(\overline{P}_{Ea} - \overline{R}_a + \overline{P}_{Eh} + \overline{R}_h) = 0$$

Substituting values:

$$\overline{R}_a - \overline{R}_h = -\frac{63,705}{EI} \qquad \text{Eq. 4-2c}$$

Inspection of Eq. 4-2a, b, c, shows that only $\overline{P}_{Pe}$ may be equated to zero, leaving all other redundants equal to positive values. Therefore, $\overline{P}_{Pe}$ must be the last hinge to form.

Equating $\overline{P}_{Pe}$ to zero reduces Eq. 4-2a, b, c, to explicit form, and their solution yields:

$$\overline{P}_{Pg} = \frac{105,975}{EI} = .0132 \text{ Radians}$$

$$\overline{R}_a = \frac{50,440}{EI} = .0063 \text{ Radians}$$

$$\overline{R}_h = \frac{114,145}{EI} = .0142 \text{ Radians}$$

The deflections may be determined since they are equal to the conjugate bending moments. Those deflections which are required are computed as follows:

$$\Delta_{dx} = \Delta_{gx} = \overline{M}_d = \overline{M}_g = 120(\overline{R}_a - \overline{P}_{Ea})$$

$$= \frac{9,070,000}{EI} = 1.13 \text{ inches}$$

$$\Delta_{ey} = \overline{M}_e = 120(\overline{R}_a - \overline{P}_{Ea} - \overline{P}_{Ed})$$

$$= \frac{18,460,000}{EI} = 2.29 \text{ inches}$$

$$\Delta_{iy} = \overline{M}_i = 180(\overline{R}_a - \overline{P}_{Ea} - \overline{P}_{Ed}) - 60(\overline{P}_{Ee})$$ Since $P_{Pe} = 0$

$$= \frac{21,500,000}{EI} = 2.67 \text{ inches}$$

4-3 EXAMPLE OF SINGLE PANEL GABLE FRAME

The horizontal displacements of points b and e, and the plastic rotation of hinges are determined at the instant of collapse for the frame shown in Fig. 4-3a.

An analysis by the elementary theory of plasticity indicates that for the ultimate load shown, a uniform section whose yield moment M_o = 182 ft. kips, is required.

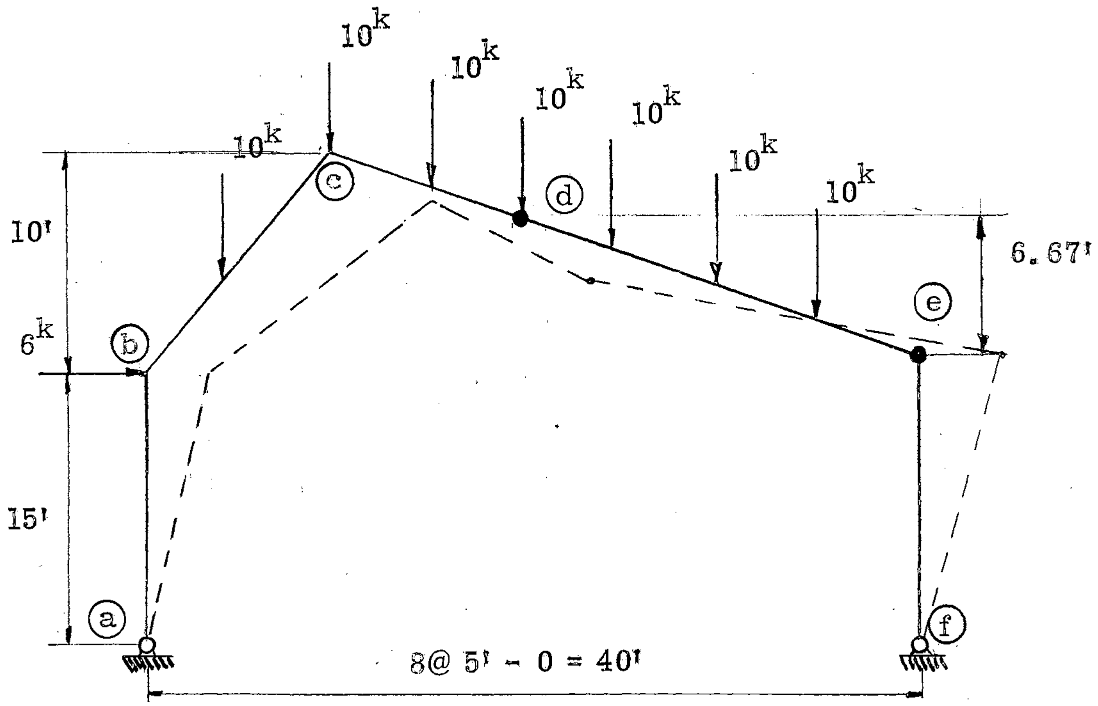


Fig. 4-3a

Single Panel Gable Frame

The collapse mechanism is formed by real hinges at a and f, and plastic hinges at d and e.

The moment diagram is as shown in Fig. 4-3b.

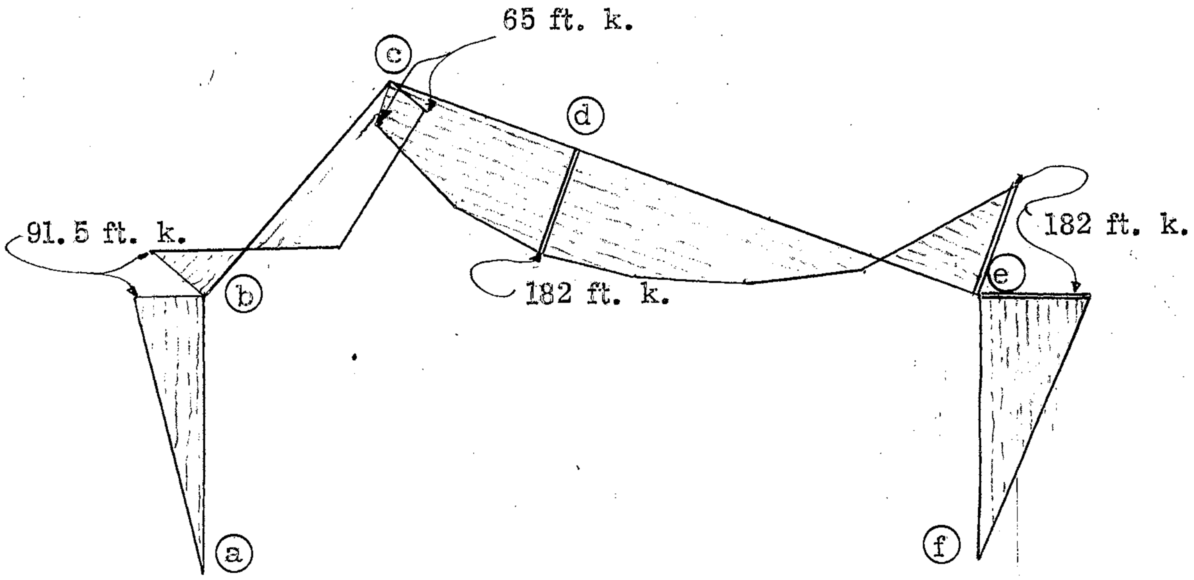


Fig. 4-3b

Moment Diagram

The points a, b, c, d, e, f, are selected as points on the polygon.

The angular functions are tabulated as coefficients of $\frac{1}{EI}$ in Table 4-3.

The elastic weights for each point are determined by means of Eq. 2-4b, and are tabulated as coefficients of $\frac{1}{EI}$ in Table 4-4.

TABLE 4-3

Beam Segment	$F_{ij(ji)}$	$G_{ij(ji)}$	τ_{ij}	τ_{ji}
ab	5	2.5	0	0
bc	4.71	2.36	+ 88.4	+ 88.4
cd	3.51	1.76	+ 65.9	+ 65.9
de	7.03	3.51	+659.0	+659.0
ef	5	2.5	0	0

TABLE 4-4

Point	$G_{ij}M_i$	$\Sigma F_j M_j$	$G_{kj}M_j$	$\Sigma \tau_j$	$\overline{P}_{Ej}$
a	0	0	-2.5 x 91.5	0	-229
b	0	-9.71 x 91.5	+2.36 x 65	+ 88.4	-646.6
c	-2.36 x 91.5	+ 8.22 x 65	+1.76 x 182	+154.3	+793.0
d	+1.76 x 65	+10.54 x 182	-3.51 x 182	+724.9	+2118.8
e	+3.51 x 182	-12.03 x 182	0	+659.0	-891.6
f	-2.5 x 182	0	0	0	-455.

The conjugate structure is then as shown in Fig. 4-3c.

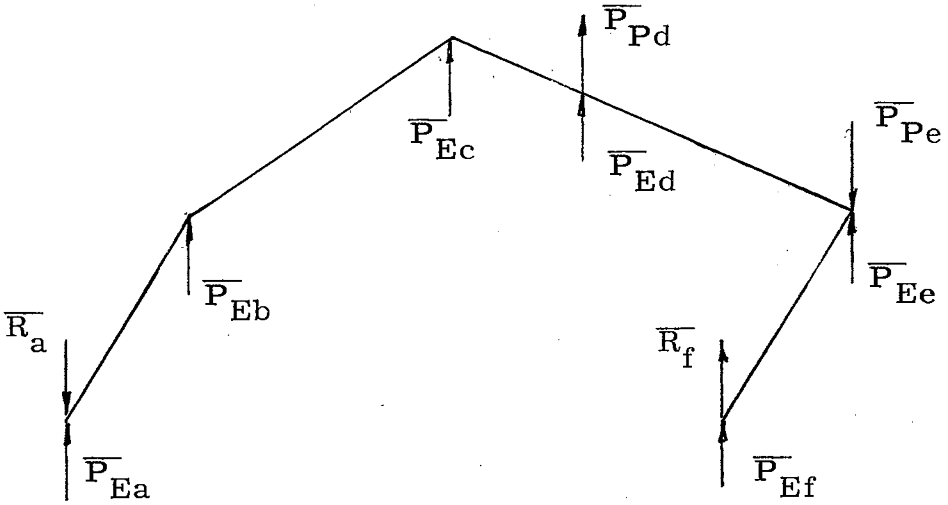


Fig. 4-3c

Conjugate Frame

Setting the sum of the moments about three sides equal to zero:

$$\sum M@_{\overline{af}} = 0$$

$$15\,\overline{P}_{Eb} + 25\,\overline{P}_{Ec} + 21.67\,(\overline{P}_{Ed} + \overline{P}_{Pd}) + 15\,(\overline{P}_{Ee} - \overline{P}_{Pe}) = 0$$

Substituting values of elastic weights:

$$21.67\,\overline{P}_{Pd} - 15\,\overline{P}_{Pe} = \frac{-42{,}666.4}{EI} \qquad \text{Eq. 4-3a}$$

$$\sum M@_{\overline{ab}} = 0$$

$$10\,\overline{P}_{Ec} + 20(\overline{P}_{Ed} + \overline{P}_{Pd}) + 40(\overline{P}_{Ee} - \overline{P}_{Pe} + \overline{P}_{Ef} + \overline{R}_f) = 0$$

Substituting values of elastic weights:

$$\overline{P}_{Pd} - 2\overline{P}_{Pe} + 2\,\overline{R}_f = \frac{+177.9}{EI} \qquad \text{Eq. 4-3b}$$

$$\sum M @_{ef} = 0$$

$$20(\overline{P}_{Ed} + \overline{P}_{Pd}) + 30\,\overline{P}_{Ec} + 40(\overline{P}_{Eb} + \overline{P}_{Ea} - \overline{R}_{a}) = 0$$

Substituting values of plastic weights:

$$\overline{P}_{Pd} - 2\,\overline{R}_{a} = -\frac{1,557.10}{EI} \qquad \text{Eq. 4-3c}$$

Inspection of the Eq.'s. 4-3a, b, c, shows that only $\overline{P}_{Pd}$ may be equated to zero, leaving all other redundants positive; therefore, $\overline{P}_{Pd}$ must be the last plastic hinge to form.

Setting $\overline{P}_{Pd}$ equal to zero, and solving Eq.'s. 4-3a, b, c, yields:

$$\overline{P}_{Pe} = \frac{2,844.4}{EI}\,(144 \text{ in.}^2/\text{ ft.}^2) = \frac{409,590}{EI}$$

$$\overline{R}_{a} = \frac{778.55}{EI}\,(144 \text{ in.}^2/\text{ ft.}^2) = \frac{112,110}{EI}$$

$$\overline{R}_{f} = \frac{2,933.35}{EI}\,(144 \text{ in.}^2/\text{ ft.}^2) = \frac{422,402}{EI}$$

Since the conjugate bending moment equals the deflection of the real structure:

$$\Delta_{bx} = \overline{M}_{bx} = 15\overline{P}_{Ea} + 15\,\overline{R}_{a} = \frac{15,113.}{EI}\,(1728 \text{ in.}^3/\text{ft.}^3)$$

$$= \frac{26.116 \times 10^6}{EI} \text{ inches.}$$

$$\Delta_{ex} = \overline{M}_{ex} = 15\,\overline{P}_{Ee} + 15\,\overline{R}_{f} = \frac{37,175}{EI}\,(1728 \text{ in.}^3/\text{ ft.}^3)$$

$$= \frac{64.239.\ 10^6}{EI}$$

Where E is in kips/ in.^{2} and I is in inches^4.

CHAPTER V

DEFORMATION ANALYSIS OF MULTIPLE PANEL FRAMES

5-1 GENERAL

Multiple panel frames may consist of any number of closed polygons. Each polygon must obey the principles of closed polygon geometry and may be treated as an individual unit, however each panel will involve the conjugate reactions and plastic weights as redundants. These redundants are common to adjacent polygons, and provide the necessary compatibility relationships.

✓ The deformation analysis of multiple panel frames, as in previous examples, begins with known loads, beam sections, the bending moment diagram, and the collapse mechanism. The angular functions are computed and the elastic weights are evaluated.

It is noted that elastic weights are evaluated by Eq. 2-4b which was derived for the case of only two members intersecting at the point of application of the elastic weight. In multiple panel frames, three or more members often intersect at a point, and the end moments of these members may have different values at the point of intersection. The following modified form of Eq. 2-4b is used in this case:

$$\overline{P_{Ej}} = M_{ij}G_{ij} + M_{ji}F_{ji} + M_{jk}F_{jk} + M_{kj}G_{kj} + \Sigma\, \tau_s$$

It should also be noted that elastic and plastic weights which are common to adjacent panels, according to the sign convention stated in Chapter II, have different signs, depending upon which panel is being considered. This convention provides for automatic compatibility between panels.

Selection of the last hinge to form is more involved in multiple panel frames than in single panel frames; however, the String Polygon Method offers reasonable advantage over the Slope Deflection Method. The selection of the last hinge to form before collapse is best explained in the following example.

5-2 EXAMPLE OF A MULTIPLE PANEL GABLE FRAME

The plastic and real hinge rotations and lateral deflections at the top of the columns are determined for the frame shown in Fig. 5-2a.

The assumed ultimate loads are shown on the frame. The beam and column segments are constant section between joints. The three sizes of beams are indicated by moments of inertia I_1, I_2, I_3. The assumed values of M_p, E, and I are as follows:

$M_{p_1} = 304$ ft. kips $\qquad I_1 = 13,824$ in.^{4}

$M_{p_2} = 530$ ft. kips $\qquad I_2 = 31,657$ in.^{4}

$M_{p_3} = 760$ ft. kips $\qquad I_3 = 49,628$ in.^{4}

$$E = 3 \times 10^3 \text{ kips/in.}^2$$

The corresponding flexural rigidity constants are denoted by:

$$D_1 = \frac{1}{EI_1} = 2.411 \times 10^{-8}$$

$$D_2 = \frac{1}{EI_2} = 1.053 \times 10^{-8}$$

$$D_3 = \frac{1}{EI_3} = .672 \times 10^{-8}$$

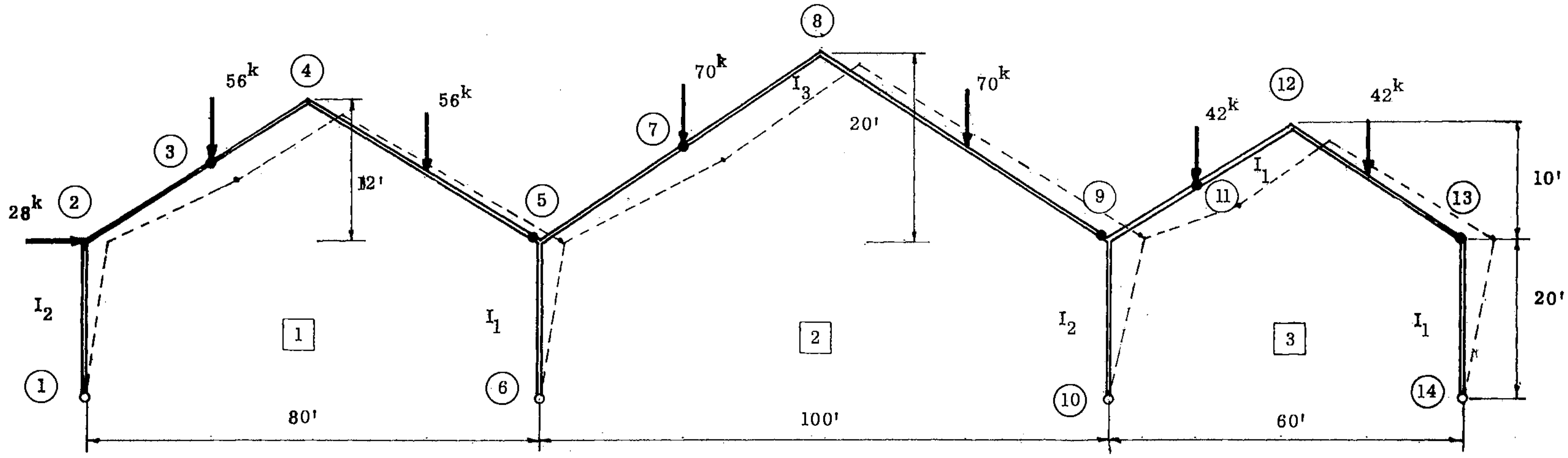


Fig. 5-2a

Three Panel Gable Frame

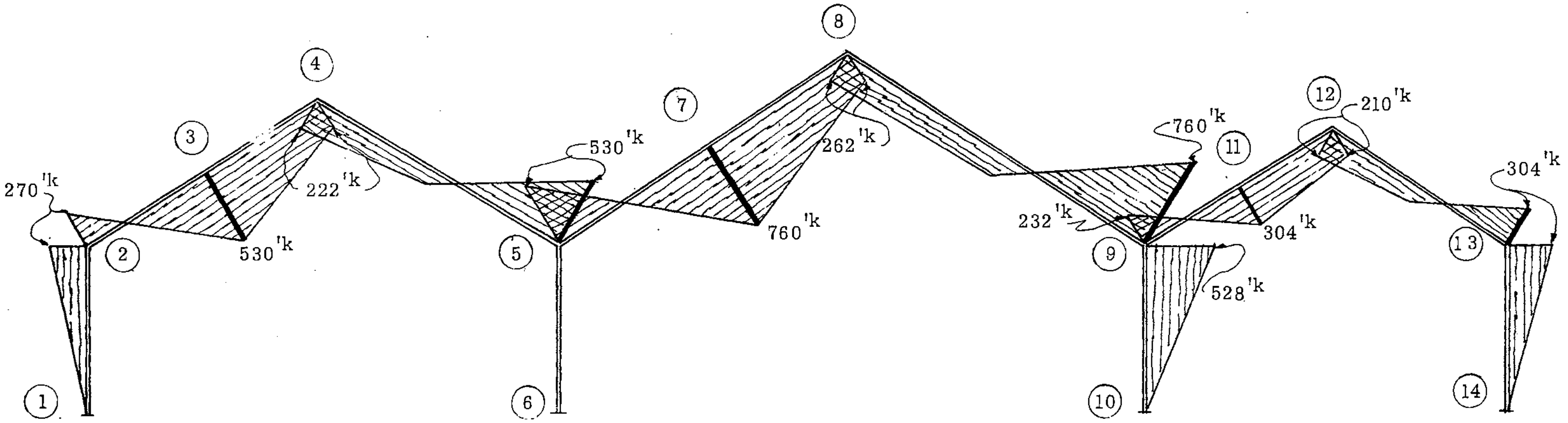


Fig. 5-2b

Bending Moment Diagram

The collapse mechanism is shown in Fig. 5-2a and the corresponding bending moment diagram is shown in Fig. 5-2b.

The angular functions F, G, and τ are tabulated in Table 5-1. For convenience all of the angular functions are written in terms of D_1.

TABLE 5-1 Angular Functions

Beam Segment ij	$\frac{F_{ij(ji)}}{D_1}$	$\frac{G_{ij(ji)}}{D_1}$	$\frac{\tau_{ij(ji)}}{D_1}$
1, 2	2.91	1.45	0
2, 3	3.04	1.52	0
3, 4	3.04	1.52	0
4, 5	6.08	3.04	+ 2665
5, 6	6.66	3.33	0
5, 7	2.50	1.25	0
7, 8	2.50	1.25	0
8, 9	5.00	2.50	+ 3530
9, 10	2.91	1.45	0
9, 11	5.27	2.64	0
11, 12	5.27	2.64	0
12, 13	10.54	5.27	+ 2625
13, 14	6.66	3.33	0

The elastic weights, for each of the points on the polygon, are tabulated for each panel in Tables 5-2, 3, 4. The value D_1 is common to all terms and is omitted from the tables.

TABLE 5-2 Elastic Weights for Panel 1

Point	M_iG_{ij}	$M_j\Sigma F_j$	M_kG_{kj}	$\Sigma\tau_j$	$\overline{P}_{Ej}$
1	0	0	- 392	0	- 392
2	0	- 1606	+ 805	0	- 799
3	- 410	+ 3222	+ 337	0	+ 3149
4	+ 806	+ 2024	- 1611	+ 2665	+ 3884
5	+ 675	- 3222	0	+ 2665	+ 118
6	0	0	0	0	0

TABLE 5-3 Elastic Weights for Panel 2

Point	M_iG_{ij}	$M_i\Sigma F_j$	M_kG_{ij}	$\Sigma\tau_j$	$\overline{P}_{Ej}$
6	0	0	0	0	0
5	0	- 1325	+ 950	0	- 375
7	- 663	+ 3800	+ 327	0	+ 3464
8	+ 950	+ 1965	- 1900	+ 3530	+ 4545
9	+ 655	- 3800 - 1536	0	+ 3530	- 1151
10	- 765	0	0	0	- 765

TABLE 5-4 Elastic Weights for Panel 3

Point	M_iG_{ij}	$M_j\Sigma F_j$	M_kG_{kj}	$\Sigma\tau_j$	$\overline{P}_{Ej}$
10	0	0	+ 766	0	+ 766
9	0	+ 1536 - 1223	+ 802	0	+ 1115
11	- 612	+ 3204	+ 554	0	+ 3146
12	+ 803	+ 3320	- 1602	+ 2625	+ 5146
13	+ 1106	- 5229	0	+ 2625	- 1498
14	- 1012	0	0	0	- 1012

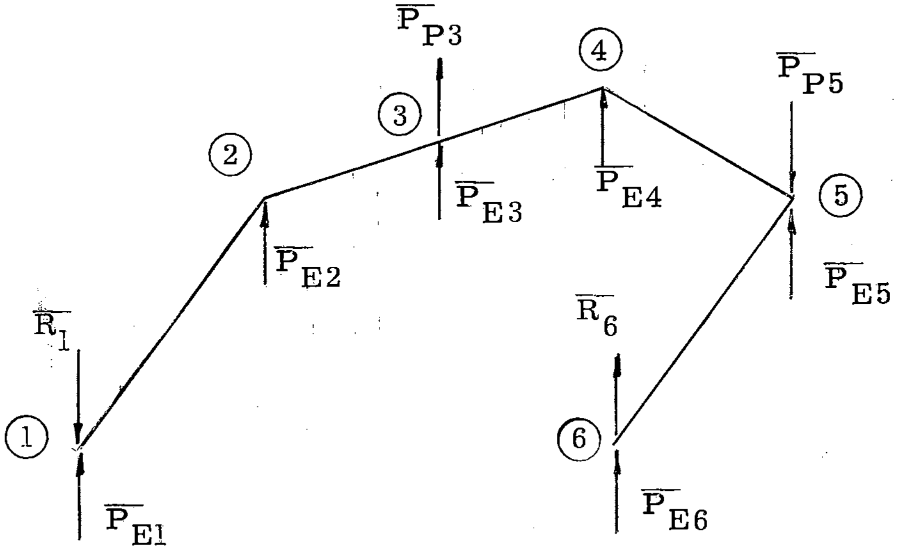


Fig. 5-2c

Conjugate Panel One

The equilibrium equations for Panel One are:

$$\sum M@_{\overline{1,6}} = 20(\overline{P}_{E2} + \overline{P}_{E5} - \overline{P}_{P5}) + 26(\overline{P}_{E3} + \overline{P}_{P3}) + 32\,\overline{P}_{E4} = 0$$

$$\overline{P}_{P5} - 1.3\,\overline{P}_{P3} = +9,627\ D_1 \qquad \text{Eq. 5-2a}$$

$$\sum M@_{\overline{56}} = 80(\overline{P}_{E1} + \overline{P}_{E2} - \overline{R}_1) + 60(\overline{P}_{E3} + \overline{P}_{P3}) + 40\,\overline{P}_{E4} = 0$$

$$4\,\overline{R}_1 - 3\,\overline{P}_{P3} = +12,451\ D_1 \qquad \text{Eq. 5-2b}$$

$$\sum M@_{\overline{12}} = 20(\overline{P}_{E3} + \overline{P}_{P3}) + 40\overline{P}_{E4} + 80(\overline{P}_{E5} - \overline{P}_{P5} + \overline{P}_{E6} + \overline{R}_6) = 0$$

$$4\,\overline{P}_{P5} - 4\,\overline{R}_6 - \overline{P}_{P3} = +11,389\ D_1 \qquad \text{Eq. 5-2c}$$

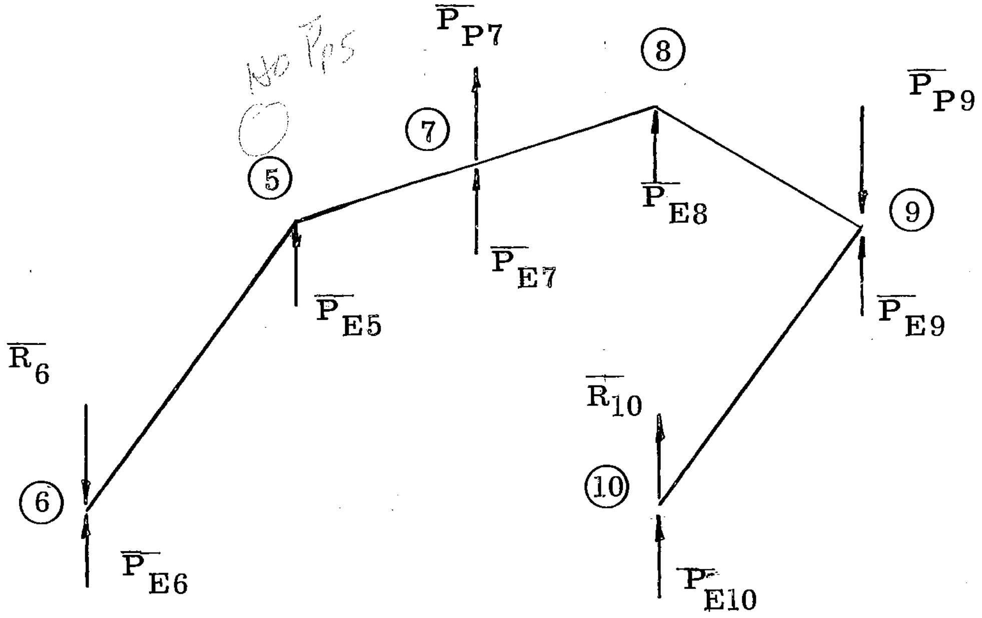


Fig. 5-2d

Conjugate Panel Two

The equilibrium equations for Panel Two are:

$$\sum M@_{\overline{6,10}} = 20(\overline{P}_{E5} + \overline{P}_{E9} - \overline{P}_{P9}) + 30(\overline{P}_{E7} + \overline{P}_{P7}) + 40\,\overline{P}_{E8} = 0$$

$$2\,\overline{P}_{P9} - 3\,\overline{P}_{P7} = +25{,}520\ D_1 \qquad \text{Eq. 5-2d}$$

$$\sum M@_{\overline{9,10}} = 100(\overline{P}_{E6} + \overline{P}_{E5} - \overline{R}_6) + 75(\overline{P}_{E7} + \overline{P}_{P7}) + 50\,\overline{P}_{E8} = 0$$

$$4\,\overline{R}_6 - 3\,\overline{P}_{P7} = +17{,}982\ D_1 \qquad \text{Eq. 5-2e}$$

$$\sum M@_{\overline{5,6}} = 25(\overline{P}_{E7} + \overline{P}_{P7}) + 50(\overline{P}_{E8}) + 100(\overline{P}_{E9} + \overline{P}_{E10} - \overline{P}_{P9} + \overline{R}_{10}) = 0$$

$$4\,\overline{P}_{P9} - 4\,\overline{R}_{10} - \overline{P}_{P7} = +4{,}890\ D_1 \qquad \text{Eq. 5-2f}$$

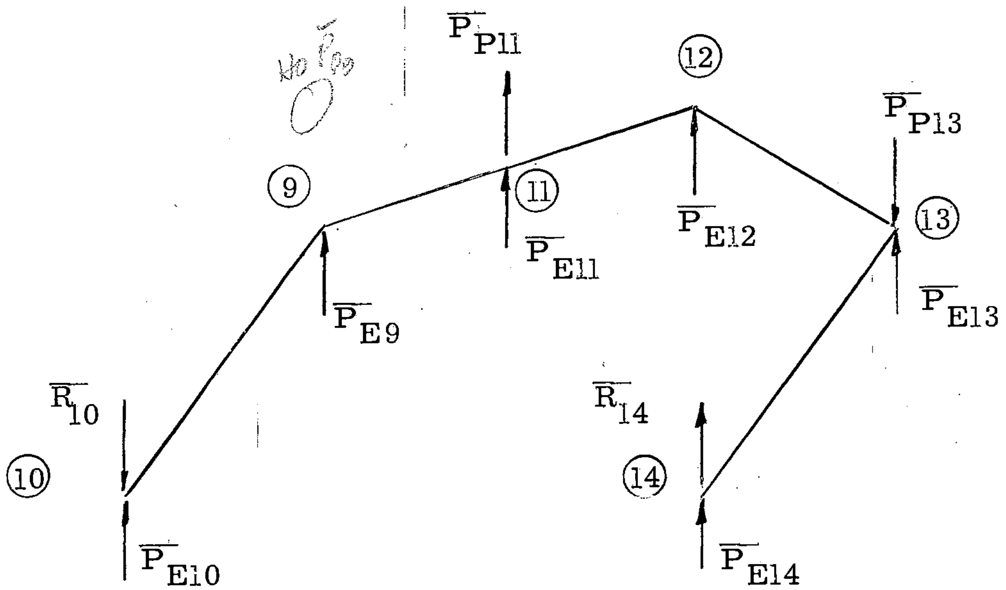


Fig. 5-2e

Conjugate Panel Three

The equilibrium equations for Panel Three are:

$$\sum M @_{\overline{10,14}} = 20(\overline{P}_{E9} + \overline{P}_{E13} - \overline{P}_{P13}) + 25(\overline{P}_{E11} + \overline{P}_{P11}) + 30\,\overline{P}_{E12} = 0$$

$$2\,\overline{P}_{P13} - 2.5\,\overline{P}_{P11} = +22{,}537\,D_1 \qquad \text{Eq. 5-2g}$$

$$\sum M @_{\overline{9,10}} = 15(\overline{P}_{E11} + \overline{P}_{P11}) + 30(\overline{P}_{E12}) + 60(\overline{P}_{E13} + \overline{P}_{E14} - \overline{P}_{P13} + \overline{R}_{14}) = 0$$

$$4\,\overline{P}_{P13} - 4\,\overline{R}_{14} - \overline{P}_{P11} = +3{,}398\,D_1 \qquad \text{Eq. 5-2h}$$

$$\sum M @_{\overline{13,14}} = 30(\overline{P}_{E12}) + 45(\overline{P}_{E11} + \overline{P}_{P11}) + 60(\overline{P}_{E9} + \overline{P}_{E10} - \overline{R}_{10}) = 0$$

$$4\,\overline{R}_{10} - 3\,\overline{P}_{P11} = +27{,}222\,D_1 \qquad \text{Eq. 5-2i}$$

The equations 5-2a through 5-2i are nine equilibrium equations which describe the relationship between all plastic and real hinges. Simultaneous solution cannot be performed , however, until a tenth relationship is obtained since ten unknown values are present.

The tenth relationship is obtained by determining the last plastic hinge to form, and equating the respective plastic weight to zero.

As in previous examples, the smallest value of rotation (plastic weight) in each panel is determined by inspection of the equilibrium equations for each respective panel.

Thus, in Panel 1, $\overline{P}_{P3}$ is least. In Panel 2, $\overline{P}_{P7}$ is least, and in Panel 3, $\overline{P}_{P11}$ is least.

By eliminating $\overline{P}_{P5}$ from Eq. 5-2a and 5-2c:

$$\overline{R}_6 = 6,780\ D_1 + 1.05\ \overline{P}_{P3}$$

and, from Eq. 5-2e:

$$\overline{R}_6 = 4,495\ D_1 + .75\ \overline{P}_{P7}$$

and by equating these two expressions:

$$1.05\ \overline{P}_{P3} = -2,285\ D_1 + .75\ \overline{P}_{P7} \qquad \text{Eq. 5-2j}$$

If $\overline{P}_{P7} = 0$, $\overline{P}_{P3}$ is negative; therefore, the value of $\overline{P}_{P3}$ is less than that of $\overline{P}_{P7}$.

Eliminating $\overline{P}_{P9}$ and $\overline{P}_{P7}$ from Eq. 5-2f, d, j, yields:

$$\overline{R}_{10} = 15,346\ D_1 + 1.75\ \overline{P}_{P3}$$

and from Eq. 5-2i:

$$\overline{R}_{10} = 6,805\ D_1 + .75\ \overline{P}_{P11}$$

and by equating these two expressions:

$$1.75\ \overline{P}_{P3} = -8,541\ D_1 + 3\ \overline{P}_{11} \qquad \text{Eq. 5-2k}$$

If $\overline{P}_{P11} = 0$, $\overline{P}_{P3}$ is negative; therefore, the value of $\overline{P}_{P3}$ is less than that of $\overline{P}_{P11}$.

It has been shown that $\overline{P}_{P3}$ has a value less than any of the other five plastic weights and, therefore, represents the last hinge to form before collapse.

Equating $\overline{P}_{P3}$ to zero, the numbered equations may be solved by substitution. Thus from:

Eq. 5-2a	$\overline{P}_{P5} = +9,627\ D_1$
Eq. 5-2b	$\overline{R}_1 = +3,113\ D_1$
Eq. 5-2c	$\overline{R}_6 = +6,780\ D_1$
Eq. 5-2j	$\overline{P}_{P7} = +3,047\ D_1$
Eq. 5-2d	$\overline{P}_{P9} = +17,330\ D_1$
Eq. 5-2f	$\overline{R}_{10} = +15,346\ D_1$
Eq. 5-2i	$\overline{P}_{P11} = +11,387\ D_1$
Eq. 5-2g	$\overline{P}_{P13} = +25,502\ D_1$
Eq. 5-2h	$\overline{R}_{14} = +21,806\ D_1$

The units of these values are not consistent and will be revised later.

The values computed for the plastic weights and conjugate reactions are all positive, which indicates that the last hinge to form was selected correctly.

An independent check on the values of the plastic weights and conjugate reactions is obtained by equating the algebraic sum of all conjugate weights in each panel to zero, and varifying the equality.

The lateral deflections at the points 2, 5, 9, 13, are as follows:

$$\Delta_{2x} = \overline{M}_{2x} = (\overline{P}_{E1} + \overline{R}_{1})(20) = 70{,}100\ D_1$$

$$\Delta_{5x} = \overline{M}_{5x} = (\overline{P}_{E6} + \overline{R}_{6})(20) = 135{,}600\ D_1$$

$$\Delta_{9x} = \overline{M}_{9x} = (\overline{P}_{E10} + \overline{R}_{10})(20) = 322{,}222\ D_1$$

$$\Delta_{13x} = \overline{M}_{13x} = (\overline{P}_{E14} + \overline{R}_{14})(20) = 415{,}883\ D_1$$

The units of these values are not consistent and will be revised later.

Multiplying, the plastic weight and conjugate reaction values which were computed, by 144 in.^{2}/ ft.^{2} to make the units consistent, and substituting the value of D_1, the final hinge rotations are:

$\overline{P}_{P5}$ = .0334 radians

$\overline{R}_{1}$ = .0108 radians

$\overline{R}_{6}$ = .0235 radians

$\overline{P}_{P7}$ = .0106 radians

$\overline{P}_{P9}$ = .0602 radians

$\overline{R}_{10}$ = .0533 radians

$\overline{P}_{P11}$ = .0395 radians

$\overline{P}_{P13}$ = .0885 radians

$\overline{R}_{14}$ = .0757 radians

Multiplying the deflection values which were computed by 1728 in.^{3}/ft.^{3} to make units consistent, and substituting the value of D_1, the final deflections are:

Δ_{2x} = 2.92 in.

Δ_{9x} = 13.42 in.

Δ_{5x} = 5.65 in.

Δ_{13x} = 17.33 in.

CHAPTER VI

SUMMARY AND CONCLUSIONS

The deformation analysis of planar frames under ultimate load by the String Polygon Method is presented in this thesis.

The points of major significance found in this study may be summarized as follows:

1. Closed polygons which undergo small deformation are the basic units of the analysis, and any planar frames may be considered as either one or a system of closed polygons.

2. The vertices of the polygons may be selected at convenent points on the frame, and all elastic deformations, plastic deformations and real hinge rotations are considered to act at these selected points.

3. A form of the three moment equation is used to transfer the effect of elastic deformation which occurs between the vertices, to the vertices of the polygon.

4. The angle changes are considered as vectors applied at the vertex where it occurs, and in a direction perpendicular to the plane of the frame.

5. Geometrical compatability is required by the conjugate equlibrium equations, which are written in terms of the redundants.

6. The last hinge to form in a single panel frame may always be determined by rational analysis of the equlibrium equations. In multiple panel frames the number of possibilities for the last hinge to form is reduced by rational analysis to correspond to the number of panels in the system. A process is provided to further determine which of the remaining hinges is actually the last to form.

7. Plastic and real hinge rotations are obtained directly from the solution of the conjugate equlibrium equations. The deflections of previously selected points are determined by computing the conjugate bending moment at that point. Intermediate deflections maybe determined by computing the conjugate bending moment at the point and adding the deflection of the simple beam segment due to loads at that same point.

The String Polygon Method makes available three conjugate equlibrium equations for each panel of the frame. These equations are written in terms of plastic and real hinge rotations.

Since the equations are free of deflection terms, the number of redundants; and therefore, the number of simultaneous equations needed for solution is greatly reduced from that required by the Slope Deflection Method.

The conjugate equlibrium equations may be written, such that, the last hinge to form in the system before final collapse, is determined by rational analysis and simple algebraic manipulation.

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