THE SLIPPED-BLOCK DESIGN

By

WILLIAM EDWARD TIMON, JR.

Bachelor of Science
Northwestern State College
Natchitoches, Louisiana
1950

Master of Science
Tulane University
New Orleans, Louisiana
1951

Submitted to the Faculty of the Graduate School of
the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
DOCTOR OF PHILOSOPHY
May, 1962

THE SLIPPED-BLOCK DESIGN

Thesis Approved:

David L. Weeks

Thesis Adviser

Carl E. Marshall

Herbert Scholz

George W. Newell

Robert D. Morrison

Robert MacVicar

Dean of the Graduate School

ACKNOWLEDGMENTS

The author is deeply indebted to Dr. David L. Weeks for suggesting the problem of this thesis and for the aid and inspiration he supplied during the period of preparation.

The author is also indebted to the National Science Foundation and Northwestern State College of Louisiana for the financial assistance they provided, and to his parents and sisters for the encouragement they provided over a period of many years.

An expression of gratitude is extended to Professors Robert D. Morrison, Carl E. Marshall, Herbert Scholz, George W. Newell, and Roy Gladstone for serving on the author's advisory committee, and to Mrs. Patty Darby for her work in typing this paper.

Considerable credit is due Professor C. G. Killen of Northwestern State College, who has remained a loyal friend, booster, and adviser ever since the author enrolled in his freshman algebra class. The failure of the author to perform as expected at times has not lessened his support.

The greatest expression of the author's gratitude must go to his wife, Katherine, who has shouldered most of the family responsibilities over the past two years, provided a never-ending source of encouragement, and who, along with our children, provided the inspiration for the author to pursue a more advanced degree.

TABLE OF CONTENTS

Chapter		Page
I.	THE GENERAL TWO-WAY CLASSIFICATION	1
	Theory From the General Linear Hypothesis Model.	1
	The Scalar Model for the General Two-Way Classification	6
	A Matrix Model for the General Two-Way Classification	8
	Test of the Hypothesis H_0: $\tau_1 = \tau_2 = \ldots = \tau_t$ . . .	22
	Confidence Intervals	26
II.	THE GENERAL SLIPPED-BLOCK DESIGN	30
	Definition and Notation	30
III.	THE SLIPPED-BLOCK DESIGN WITH TWO BLOCKS . .	34
	The Derivation of $A + t^{-1} J_t^t$ and $(A + t^{-1} J_t^t)^{-1}$ . . .	34
	The Variances of Estimates of Treatment Differences	39
	The Extension of Results	40
	Test of the Hypothesis $\tau_1 = \tau_2 = \ldots = \tau_t$	44
	A Computing Procedure for the Analysis of Variance	46
	Estimation of $\tau_i - \bar{\bar{\tau}}$	58
	Confidence Intervals	60
	An Example to Illustrate the Computing Procedure .	60
IV.	THE SLIPPED-BLOCK DESIGN WITH TWO OR MORE BLOCKS AND OVERLAP OF ONE	69
	The Derivation of the Matrices A, A**, and $(A^{**})^{-1}$	69
	The Variances of Estimates of Treatment Differences	84
	The Extension of Results	88
	Test of the Hypothesis $\tau_1 = \tau_2 = \ldots = \tau_t$	89

Chapter Page

A Computing Procedure for the Analysis of Variance. 89
Estimation of $\tau_i - \tau_t$ 93
Confidence Intervals 95
An Example To Illustrate the Computing Procedure . 96

V. SUMMARY OF RESULTS 104

BIBLIOGRAPHY 105

LIST OF TABLES

Table		Page
I.	AOV for the General Linear Hypothesis Model	5
II.	An Example of the General Two-Way Classification	10
III.	AOV for the General Two-Way Classification	25
IV.	AOV for the Slipped-Block Design	45
V.	AOV for Example 1	52
VI.	Alternate AOV for Example 1	53
VII.	AOV for Example 2	64
VIII.	AOV for Example 3	99
IX.	Estimates of Variances of Treatment Differences	100

LIST OF ILLUSTRATIONS

Illustration — Page

I. Matrices for the Example in Table II 11

II. Examples of Slipped-Block Designs 32

III. Layout for the Slipped-Block Design With Two Blocks 35

IV. The Matrix A for the Design in Illustration III 36

V. Layout for the Slipped-Block Design With r Replicates of Two Basic Blocks 42

VI. Tables for Computing the Analysis of Variance 55

VII. Layout for the Slipped-Block Design With Overlap One . . 70

VIII. The A Matrix for the Design in Illustration VII 72

IX. Some Examples of A^{**} and $(A^{**})^{-1}$ 74

CHAPTER I

THE GENERAL TWO-WAY CLASSIFICATION

Theory From the General Linear Hypothesis Model

The primary objective of this investigation is to develop the theory and methods necessary for the statistical analysis of a design which will be called the Slipped-Block Design. The basic model for this design is the same as for the general two-way classification without interaction, but it has some special properties which simplify the analysis. Since the Slipped-Block Design is a special case of the general two-way classification, the definition and description of the design will be delayed until the general theory underlying the analysis of the general two-way classification is developed.

Since the general two-way classification is itself a special case of the general linear hypothesis model of less than full rank, a few definitions will be given and some basic theorems stated without proof from Graybill (1). Then the general two-way classification will be discussed.

<u>DEFINITION 1.</u> The model $Y = X\beta + e$, where Y is an $n \times 1$ observed random vector, X is an $n \times p$ matrix of known fixed quantities, β is a $p \times 1$ vector of unknown parameters, and e is an $n \times 1$ random vector,

is called the general linear hypothesis model of less than full rank if each element of X is either zero or one, and the rank of X is $k < p$. The distributional properties of the vector e are somewhat arbitrary. In this paper, two cases will be considered:

(1) e distributed $N(\phi, \sigma^2 I)$

(2) $E(e) = \phi$, and $E(ee') = \sigma^2 I$.

<u>DEFINITION 2.</u> A parameter or a function of the parameters is said to be linearly estimable if there exists a linear combination of the observations whose expected value is equal to the parameter or function of the parameters. Unless otherwise specified, when an estimable function is mentioned, it will be a linearly estimable function.

<u>THEOREM 1.</u> In the model $Y = X\beta + e$, if $E(e) = \phi$ and $\text{Cov}(e) = \sigma^2 I$, then the linear combination $C'\beta$ is estimable if and only if there exists a solution for r in the matrix equation $X'Xr = C$.

<u>THEOREM 2.</u> In the model $Y = X\beta + e$, if $E(e) = \phi$ and $\text{Cov}(e) = \sigma^2 I$, then the best linear unbiased estimator of any estimable function $C'\beta$ is $r'X'Y$, where r satisfies the matrix equation $X'Xr = C$.

<u>DEFINITION 3.</u> If C is a matrix such that $C = (C_1, C_2, \ldots, C_m)$, where C_i is p x 1, then the matrix function $C'\beta$ is said to be estimable if each $C_i'\beta$ is estimable, for $i = 1, 2, \ldots, m$.

<u>THEOREM 3.</u> In the model $Y = X\beta + e$, $X\beta$ and $X'X\beta$ are estimable.

<u>THEOREM 4.</u> If $C_1'\beta$, $C_2'\beta$, ... , $C_q'\beta$ are estimable, then any linear combination of these quantities is estimable.

<u>THEOREM 5.</u> In the model $Y = X\beta + e$, if the rank of X is k, then there are exactly k linearly independent estimable functions. Furthermore, any estimable function must be a linear combination of the rows of $X\beta$.

<u>THEOREM 6.</u> If $C'\beta$ is an estimable function, and if r_1 and r_2 both satisfy $X'Xr = C$, then $r_1'X'Y = r_2'X'Y = \widehat{C'\beta} = C'\hat{\beta}$, where the symbol $\hat{\beta}$ denotes any solution of the normal equations $X'X\hat{\beta} = X'Y$.

<u>DEFINITION 4.</u> In the model $Y = X\beta + e$, the linear combination of parameters $\Sigma\, c_i\beta_i$ is called a contrast if $\Sigma\, c_i = 0$.

<u>DEFINITION 5.</u> A hypothesis H_0 is called estimable if there exists a set of linearly independent estimable functions $C_1'\beta$, $C_2'\beta$, ..., $C_s'\beta$, such that H_0 is true if and only if $C_1'\beta = C_2'\beta = \ldots = C_s'\beta = 0$.

<u>THEOREM 7.</u> In the model $Y = X\beta + e$, where X is of rank k, the quantity

$$\hat{\sigma}^2 = \frac{1}{n-k}(Y - X\hat{\beta})'(Y - X\hat{\beta}) = \frac{1}{n-k}(Y'Y - \hat{\beta}'X'Y)$$

is invariant for any $\hat{\beta}$ that is a solution of the normal equations $X'X\hat{\beta} = X'Y$. Furthermore, the quantity $\dfrac{(n-k)\hat{\sigma}^2}{\sigma^2}$ is distributed as chi-square with $n - k$ degrees of freedom, and $\hat{\sigma}^2$ is an unbiased estimate of σ^2.

<u>THEOREM 8.</u> If Y is distributed $N(\mu, \sigma^2 I)$, then $\frac{Y'BY}{\sigma^2}$ is distributed as a non-central chi-square with k degrees of freedom and non-centrality parameter λ, where $\lambda = \frac{\mu' B \mu}{2\sigma^2}$, if and only if B is a symmetric idempotent matrix of rank k.

<u>THEOREM 9.</u> In the model $Y = X\beta + e$, the test of the hypothesis H_0: $\beta_1 = \beta_2 = \ldots = \beta_q$ $(q \leq k)$, which is equivalent to testing the linearly independent estimable functions $C_1'\beta = C_2'\beta = \ldots = C_s'\beta = 0$, can be carried out as follows:

(1) Obtain any solution to the normal equations $X'X\hat{\beta} = X'Y$, and form $R(\beta) = Y'Y - \hat{\beta}'X'Y$, where $R(\beta)$ denotes the reduction in the sum of squares due to β.

(2) Impose the conditions of the hypothesis H_0 (that is, assume H_0 is true) on the model $Y = X\beta + e$ to obtain the reduced model $Y = Z\gamma + e$. Obtain any solution to the reduced normal equations $Z'Z\tilde{\gamma} = Z'Y$ and form $R(\gamma) = \tilde{\gamma}'Z'Y$.

(3) Then the quantity $\frac{\hat{\beta}'X'Y - \tilde{\gamma}'Z'Y}{Y'Y - \hat{\beta}'X'Y} \cdot \frac{n-k}{s}$ is distributed as a non-central F-variate with s and n - k degrees of freedom and non-centrality parameter λ.

As a consequence of the above theorem, the following analysis of variance table can be written for testing the hypothesis H_0: $\beta_1 = \beta_2 = \ldots = \beta_q$. The null hypothesis is rejected if $\frac{\alpha_{ss}}{E_{ss}} \cdot \frac{n-k}{s}$

exceeds the tabular value of the F-variate with s and n - k degrees of freedom.

TABLE I

AOV FOR THE GENERAL LINEAR HYPOTHESIS MODEL

Source		d.f.	S.S.	M.S.
Total		n	$Y'Y$	
$R(\beta)$		k	$\hat{\beta}'X'Y$	
$R(\gamma)$	k - s		$\tilde{\gamma}'Z'Y$	
$R(\beta \mid \gamma)$	s		$\hat{\beta}'X'Y - \tilde{\gamma}'Z'Y = a_{ss}$	$a_{ms} = \frac{a_{ss}}{s}$
Error		n - k	$Y'Y - \hat{\beta}'X'Y = E_{ss}$	$E_{ms} = \frac{E_{ss}}{n-k}$

The preceding theorem and the analysis of variance table, together with the definition of the non-central F-distribution, imply that $\frac{a_{ss}}{\sigma^2}$ is distributed as a non-central chi-square with s degrees of freedom and non-centrality parameter λ. Since the expected value of a non-central chi-square variable is the sum of the degrees of freedom and twice the non-centrality parameter, it follows that $E\left(\frac{a_{ss}}{\sigma^2}\right) = s + 2\lambda$.

This result leads to the following theorem:

THEOREM 10. Under the conditions of Theorem 9, the non-centrality parameter is given by

$$\lambda = \frac{s[E(a_{ms})]}{2\sigma^2} - \frac{s}{2} .$$

The Scalar Model for the General Two-Way Classification

Let the model for the general two-way classification without interaction be given by

$$(1) \qquad y_{ijk} = \mu + \tau_i + \beta_j + e_{ijk} \qquad \begin{array}{l} i = 1, 2, \ldots, t \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n_{ij} \end{array}$$

where y_{ijk} is observation number k in cell ij; μ, τ_i, and β_j are unknown parameters; and the e_{ijk} are random variables with mean zero and variance σ^2. The e_{ijk} will be assumed to be normally and independently distributed for the purposes of interval estimation and tests of hypotheses. Cell ij contains n_{ij} observations, and, if $n_{ij} = 0$, the cell contains no observations. That is, the observations y_{ij0} do not exist. The notation $N_{i.} = \sum_{j=1}^{b} n_{ij}$, $N_{.j} = \sum_{i=1}^{t} n_{ij}$, and $N_{..} = \sum_i \sum_j n_{ij}$ will be used.

Using this notation, the normal equations for the model in (1) can be written as

$$\mu: \quad N_{..}\hat{\mu} + \sum_i N_{i.}\hat{\tau}_i + \sum_j N_{.j}\hat{\beta}_j = Y_{...}$$

$$\tau_r: \quad N_{r.}\hat{\mu} + N_{r.}\hat{\tau}_r + \sum_j n_{rj}\hat{\beta}_j = Y_{r..} \qquad r = 1, 2, \ldots, t$$

$$\beta_s: \quad N_{.s}\hat{\mu} + \sum_i n_{is}\hat{\tau}_i + N_{.s}\hat{\beta}_s = Y_{.s.} \qquad s = 1, 2, \ldots, b.$$

It is assumed that the n_{ij} are values such that $\tau_i - \tau_j$ is estimable for every $i \neq j = 1, 2, \ldots, t$ and that $\beta_{i'} - \beta_{j'}$ is estimable for every $i' \neq j' = 1, 2, \ldots, b$. This assumption leads to the following theorem:

<u>THEOREM 11.</u> If the n_{ij} in the model (1) are such that $\tau_i - \tau_j$ and $\beta_{i'} - \beta_{j'}$ are estimable for all $i \neq j$ and all $i' \neq j'$, then

(a) there are exactly $b + t - 1$ linearly independent estimable functions, and

(b) $\Sigma\, c_i \tau_i$ and $\Sigma\, d_j \beta_j$ are estimable if $\Sigma\, c_i = \Sigma\, d_j = 0$.

<u>Proof:</u> It is clear that the $b + t - 1$ estimable functions $\tau_1 - \tau_2$, $\tau_1 - \tau_3, \ldots, \tau_1 - \tau_t, \beta_1 - \beta_2, \beta_1 - \beta_3, \ldots, \beta_1 - \beta_b$, and $N_{..}\mu + \Sigma\, N_{i.}\, \tau_i + \Sigma\, N_{.j} \beta_j$ are linearly independent. There are $b+t+1$ parameters and therefore $b + t + 1$ equations in the system of normal equations above. The sum of the t equations for τ_r is equal to the equation for μ. Also, the sum of the b equations for β_s is equal to the equation for μ. Hence there are at least two linearly dependent equations among the $b + t + 1$ normal equations. This, coupled with Theorem 5 and the fact that there are $b + t - 1$ linearly independent estimable functions, implies that the rank of the normal equations is exactly $b + t - 1$.

Since the $\tau_i - \tau_j$ are estimable for all $i \neq j$, every linear combination of the $\tau_i - \tau_j$ is estimable in view of Theorem 4. If the expression $\sum_{j=1}^{t} (\tau_i - \tau_j) \frac{1}{t}$, $j \neq i$, is considered, then it follows that

$\frac{t-1}{t}\tau_i - \frac{1}{t}\sum_{j \neq i}\tau_j = \frac{t-1}{t}\tau_i - \frac{1}{t}(\tau. - \tau_i) = (\tau_i - \bar{\tau}.)$. This shows that $\tau_i - \bar{\tau}.$ is estimable for all i. Therefore, by Theorem 4 again, $\Sigma c_i(\tau_i - \bar{\tau}.)$ is estimable, but this becomes $\Sigma c_i\tau_i$ if $\Sigma c_i = 0$. A similar result follows for $\Sigma d_j\beta_j$. This completes the proof.

A Matrix Model for the General Two-Way Classification

It may be desirable to use a matrix model rather than the scalar model given in (1), and such a model is defined by

$$Y = X\gamma + e, \tag{2}$$

where Y is an N.. x 1 vector of observations, X is an N.. x (b + t + 1) matrix, γ is a (b + t + 1) x 1 vector of parameters, and e is an N.. x 1 vector having multivariate normal distribution with mean ϕ and covariance matrix $\sigma^2 I$. Now partition γ so that $\gamma' = [\mu\ \beta'\ \tau']$, where μ is 1 x 1, β is b x 1, and τ is t x 1. Partition X into $[X_0\ X_1\ X_2]$, where X_0 is N.. x 1, X_1 is N.. x b, and X_2 is N.. x t. Then the model (2) becomes

$$Y = [X_0\ X_1\ X_2]\begin{bmatrix}\mu\\ \beta\\ \tau\end{bmatrix} + e$$

or

$$Y = X_0\mu + X_1\beta + X_2\tau + e. \tag{3}$$

It is worthwhile to note that X_0 is an N.. x 1 vector in which each element is unity, since μ is in every observation. Also, an m x n matrix all of whose elements are equal to unity will be denoted by J_n^m, or simply by J, if the dimensions are obvious. The X_1 matrix has a column corresponding to each block, and if the observations are ordered by blocks, then the first column of X_1 will have a one for each observation contained in block 1. That is, the first $\sum_{i=1}^{t} n_{i1}$ elements will be equal to one, and all other elements in the first column will be zeroes. The second column of X_1 will have zeroes for the first $\sum_{i=1}^{t} n_{i1}$ elements, then $\sum_{i=1}^{t} n_{i2}$ elements all equal to one, and the remaining elements will be zeroes. This arrangement continues so that all elements of column b are zero except the last $\sum_{i=1}^{t} n_{ib}$ elements, and these are all equal to one.

The X_2 matrix has a column corresponding to each treatment, and if the columns are in numerical order corresponding to the treatments, then the first $\sum_{i=1}^{t} n_{i1}$ elements of the first column will consist of n_{11} ones followed by zeroes; the next group of $\sum_{i=1}^{t} n_{i2}$ elements will consist of n_{12} ones followed by zeroes; and this pattern continues so that the last group of $\sum_{i=1}^{t} n_{ib}$ elements will have n_{1b} ones followed by zeroes. In general, for $r > 1$, column r of X_2 will have $\sum_{i=1}^{r-1} n_{i1}$ zeroes followed by n_{r1} unity elements, and then

$\sum_{i=r+1}^{t} n_{i1}$ zeroes for the first group of $\sum_{i=1}^{t} n_{i1}$ elements. The second group of $\sum_{i=1}^{t} n_{i2}$ elements will have $\sum_{i=1}^{r-1} n_{i2}$ zeroes, then n_{r2} unity elements, and finally $\sum_{i=r+1}^{t} n_{i2}$ zeroes. This same pattern continues so that the last group of $\sum_{i=1}^{t} n_{ib}$ elements has $\sum_{i=1}^{r-1} n_{ib}$ zeroes, then n_{rb} ones, and finally $\sum_{i=r+1}^{t} n_{rb}$ zeroes.

Each element of the vector μ is the scalar μ, the elements of the vector β are the b block constants, and the elements of τ are the t treatment constants. As an illustration to aid the preceding description, the twenty observations in Table II can be represented in matrix form as shown in Illustration I.

TABLE II

AN EXAMPLE OF THE GENERAL TWO-WAY CLASSIFICATION

	Blocks		
Treatments	6 3	4 1	
	3 1	4 3 1 5	4 3
	8 9	7	
	4 2		8 3 6

$$N = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

It is useful to note that in each row of X_1, there is one and only

ILLUSTRATION I

MATRICES FOR THE EXAMPLE IN TABLE II

$$
\underset{\mathbf{Y}}{\begin{bmatrix} 6\\3\\3\\1\\8\\9\\4\\2\\4\\1\\4\\1\\3\\5\\7\\4\\3\\8\\6\\3 \end{bmatrix}}
=
\underset{\mathbf{X}}{\overset{X_0 \qquad X_1 \qquad\qquad X_2}{\left[\begin{array}{c|ccc|cccc}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\
\hline
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right]}}
\underset{\gamma}{\begin{bmatrix} \mu\\ \beta_1\\ \beta_2\\ \beta_3\\ \tau_1\\ \tau_2\\ \tau_3\\ \tau_4 \end{bmatrix}}
+
\underset{e}{\begin{bmatrix} e_{111}\\ e_{112}\\ e_{211}\\ e_{212}\\ e_{311}\\ e_{312}\\ e_{411}\\ e_{412}\\ e_{121}\\ e_{122}\\ e_{221}\\ e_{222}\\ e_{223}\\ e_{224}\\ e_{321}\\ e_{231}\\ e_{232}\\ e_{431}\\ e_{432}\\ e_{433} \end{bmatrix}}
$$

one element equal to unity since no observation can appear in more than one block. The same thing is true about X_2 since no observation receives more than one treatment. Since each column of X_1 contains a one for each observation appearing in that block, it follows that $X_1'X_1$ is diagonal with diagonal elements equal to $N_{.j}$, the number of observations in each block. Similarly, $X_2'X_2$ is diagonal with diagonal elements $N_{i.}$, and the matrix $X_2'X_1 = N = [n_{ij}]$, where the n_{ij}'s are as defined in the previous section. Note further that in the layout of Table II, the matrix N can be written down directly by letting the columns of the N matrix correspond to the columns for blocks in the table, and the rows of the matrix would correspond to treatments. The number of observations in each cell then becomes an element of N as shown in Table II.

The following two relationships also must hold since there are N.. observations and each row of X_1 and X_2 has only one element equal to unity while the others are zeroes:

(4)

$$\text{(a)}\quad X_1 J_1^b = J_1^{N..} \quad \text{and} \quad J_b^1 X_1' = J_{N..}^1$$

$$\text{(b)}\quad X_2 J_1^t = J_1^{N..} \quad \text{and} \quad J_t^1 X_2' = J_{N..}^1$$

With these relations, (3) can be written as $Y = X_1(J_1^b \mu + \beta) + X_2\tau + e$, since $X_1 J_1^b \mu = X_0\mu$. Now if $\alpha = J_1^b \mu + \beta$, the model becomes

(5) $$Y = X_1\alpha + X_2\tau + e,$$

and the normal equations $X'X\hat{\gamma} = X'Y$ can be written as

$$\begin{bmatrix} X_1' \\ X_2' \end{bmatrix} [X_1 \quad X_2] \begin{bmatrix} \hat{\alpha} \\ \hat{\tau} \end{bmatrix} = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} Y.$$

Upon performing the indicated multiplication, the expression above becomes

(6) $$X_1'X_1\hat{\alpha} + X_1'X_2\hat{\tau} = X_1'Y$$

and

(7) $$X_2'X_1\hat{\alpha} + X_2'X_2\hat{\tau} = X_2'Y.$$

Note that $X_1'X_1$ has an inverse since it is diagonal with non-zero elements on the diagonal since every block contains at least one observation.

If equation (6) is solved for $\hat{\alpha}$ in terms of $\hat{\tau}$, the solution

$$\hat{\alpha} = (X_1'X_1)^{-1}(X_1'Y - X_1'X_2\hat{\tau})$$

is obtained. If this result is substituted in (7), then

$$X_2'X_1[(X_1'X_1)^{-1}(X_1'Y - X_1'X_2\hat{\tau})] + X_2'X_2\hat{\tau} = X_2'Y$$

or

(8) $$[X_2'X_2 - X_2'X_1(X_1'X_1)^{-1}X_1'X_2]\hat{\tau} = X_2'Y - X_2'X_1(X_1'X_1)^{-1}X_1'Y.$$

If it is now agreed to let $A = X_2'X_2 - X_2'X_1(X_1'X_1)^{-1}X_1'X_2$ and $q = [X_2' - X_2'X_1(X_1'X_1)^{-1}X_1']Y$, then (8) can be written as

(9) $$A\hat{\tau} = q,$$

and this is a system of t equations in t unknowns.

In order to solve the system (9), it will be helpful to determine its rank. The symbol r(B) will be used to denote the rank of a matrix B. First, the following lemma will be proved:

<u>LEMMA 1.</u> The sum of the t rows of A is ϕ_t^1, or equivalently, $J_t^1 A = \phi_t^1$. A similar result holds for the columns since A is symmetric.

<u>Proof:</u> Since $J_t^1 A = J_t^1 X_2' X_2 - J_t^1 X_2' X_1 (X_1' X_1)^{-1} X_1' X_2$, it follows from (4) that

$$\begin{aligned} J_t^1 A &= J_{N..}^1 X_2 - J_{N..}^1 X_1 (X_1' X_1)^{-1} X_1' X_2 \\ &= J_{N..}^1 X_2 - (J_b^1 X_1') X_1 (X_1' X_1)^{-1} X_1' X_2 \\ &= J_{N..}^1 X_2 - J_b^1 X_1' X_2 \\ &= J_{N..}^1 X_2 - J_{N..}^1 X_2 = \phi . \end{aligned}$$

This completes the proof of the lemma.

<u>THEOREM 12.</u> The rank of the matrix A is t - 1.

<u>Proof:</u> By Theorem 11 and the assumption immediately preceding it with regard to the estimability of $\tau_i - \tau_j$, there are t - 1 linearly independent estimable functions of the τ_i. These functions must come from the system $A\hat{\tau} = q$. Therefore r(A) is at least t - 1. In view of the preceding lemma, r(A) is at most t - 1. Hence it follows that r(A) is exactly t - 1. This completes the proof.

Before proceeding with the solution of the system $A\hat{\tau} = q$, some more useful relationships will be derived. First of all, referring back to equation (8), if G' is defined by $G' = X_2' - X_2' X_1 (X_1' X_1)^{-1} X_1'$, then (8) can be written as

(10) $$G' X_2 \hat{\tau} = G' Y .$$

However, note that $G'X_1 = X_2'X_1 - X_2'X_1(X_1'X_1)^{-1}X_1'X_1$, or

$$G'X_1 = X_2'X_1 - X_2'X_1 = \phi .$$

Hence it follows readily that

$$G'G = [X_2' - X_2'X_1(X_1'X_1)^{-1}X_1'][X_2 - X_1(X_1'X_1)^{-1}X_1'X_2] = G'X_2 = A,$$

and therefore (10) can be written as

(11) $$G'G\hat{\tau} = G'Y,$$

which is the same system as $A\hat{\tau} = q$.

Since the rank of the system $A\hat{\tau} = q$ is $t - 1$, one additional restriction can be imposed on the $\hat{\tau}_i$ in order to obtain a unique solution. One condition that is useful to impose is $\Sigma\hat{\tau}_i = 0$. This can be written $J_t^1\hat{\tau} = 0$, so that the systems $A\hat{\tau}=q$ and $(A + t^{-1}J_t^t)\hat{\tau} = q$ have exactly the same solution when this restriction is used. The restriction $J_t^1\hat{\tau} = 0$ and the equations $A\hat{\tau} = q$ can be combined in the form

(12) $$\begin{bmatrix} A & J_1^t \\ J_t^1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\tau} \\ Z \end{bmatrix} = \begin{bmatrix} q \\ 0 \end{bmatrix},$$

and if A^* is defined by

$$A^* = \begin{bmatrix} A & J_1^t \\ J_t^1 & 0 \end{bmatrix},$$

it is necessary to show that A^* is non-singular. However, a more general theorem will be proved to cover the use of any non-estimable function of the τ_i as an additional restriction to solve the system $A\hat{\tau} = q$. Let $\lambda'\hat{\tau} = 0$, where $\lambda'\tau$ is a non-estimable function of the τ_i. If the vector $\lambda' = (\lambda_1 \lambda_2 \ldots \lambda_t)$, then $\Sigma \lambda_i \neq 0$, for if $\Sigma \lambda_i = 0$, $\lambda'\tau$ is an estimable function by Theorem 11. If the matrix A_1 is defined by

$$A_1 = \begin{bmatrix} A & \lambda \\ \lambda' & 0 \end{bmatrix},$$

the following theorem can be proved:

<u>THEOREM 13</u>. The $(t + 1) \times (t + 1)$ matrix A_1 is non-singular.

<u>Proof:</u> If the matrix A is written as

$$A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where a_{11} is 1×1, A_{12} is $1 \times (t - 1)$, A_{21} is the transpose of A_{12}, and A_{22} is $(t - 1) \times (t - 1)$, then since the rows and columns of A sum to zero by Lemma 1, it follows that the sum of any $t - 1$ rows (columns) is equal to the remaining row (column). Therefore, the matrices below all have the same rank. This is indicated by the equivalence symbol.

$$\begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \sim \begin{bmatrix} 0 & \phi \\ A_{21} & A_{22} \end{bmatrix} \sim \begin{bmatrix} 0 & \phi \\ \phi & A_{22} \end{bmatrix}$$

Now A_{22} is non-singular since $r(A) = t - 1$, for if $r(A_{22})$ were less than $t - 1$, an additional row and column dependency could be found, and this would make $r(A)$ less than $t - 1$. This would contradict Theorem 12. Hence $r(A_{22}) = t - 1$, and $|A_{22}| \neq 0$.

The following matrices are equivalent with respect to rank for the same reasons used previously. Let λ^* denote the last $t - 1$ elements of λ. Then

$$A_1 = \begin{bmatrix} a_{11} & A_{12} & \lambda_1 \\ A_{21} & A_{22} & \lambda^* \\ \lambda_1 & \lambda^{*\prime} & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & \phi & \Sigma\lambda_i \\ A_{21} & A_{22} & \lambda^* \\ \lambda_1 & \lambda^{*\prime} & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & \phi & \Sigma\lambda_i \\ \phi & A_{22} & \lambda^* \\ \Sigma\lambda_i & \lambda^{*\prime} & 0 \end{bmatrix}.$$

Now let B denote the last matrix above, and if $|B|$ is expanded in terms of elements of the first row, then

$$|B| = (-1)^{t+2}\,\Sigma\lambda_i = \begin{vmatrix} \phi & A_{22} \\ \Sigma\lambda_i & \lambda^{*\prime} \end{vmatrix}.$$

If the reduced determinant is expanded in terms of the first column, then since $\Sigma\lambda_i$ is now in row t, the result is

$$|B| = (-1)^{2t+3}(\Sigma\lambda_i)^2 \;|A_{22}| \neq 0 \text{ since } |A_{22}| \neq 0.$$

Therefore A_1 is non-singular since B is non-singular and A_1 is equivalent to B with regard to rank. Hence the rank of A_1 is $t + 1$.

This completes the proof.

Suppose now that A_1^{-1} is written as

$$A_1^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where B_{11} is t x t, B_{12} is t x 1, B_{21} is the transpose of B_{12}, and B_{22} is 1 x 1. Then consider the system

$$\begin{bmatrix} A & \lambda \\ \lambda' & 0 \end{bmatrix} \begin{bmatrix} \hat{\tau} \\ Z \end{bmatrix} = \begin{bmatrix} q \\ 0 \end{bmatrix},$$

which can now be written as

$$\begin{bmatrix} \hat{\tau} \\ Z \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} q \\ 0 \end{bmatrix}.$$

Performing the indicated multiplication gives the solution $Z = 0$ and $\hat{\tau} = B_{11}q$, so that the following theorems can be proved:

<u>THEOREM 14</u>. (a) $B_{21} = B'_{12} = (\Sigma\lambda_i)^{-1} J_t^1$

(b) $B_{22} = 0$

(c) $\lambda' B_{11} = \phi$

(d) $B_{11} A B_{11} = B_{11}$

(e) $AB_{11} = I_t - (\Sigma\lambda_i)^{-1} \lambda J_t^1$, and AB_{11} is a symmetric idempotent matrix of rank t - 1.

Proof: Since $A_1 A_1^{-1} = I$, it follows that

$$\begin{bmatrix} A & \lambda \\ \lambda' & 0 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_t & \phi \\ \phi & 1 \end{bmatrix},$$

and if the multiplication is performed and the corresponding elements equated, the following equations must hold:

(1) $AB_{11} + \lambda B_{21} = I_t$

(2) $AB_{12} + \lambda B_{22} = \phi$

(3) $\lambda' B_{11} = B_{11}\lambda = \phi$

(4) $\lambda' B_{12} = 1$

Proof of (a): Multiply (1) above by J_t^1 to obtain $J_t^1 AB_{11} + J_t^1 \lambda B_{21} = J_t^1$, and since $J_t^1 A = \phi$ from Lemma 1, $B_{21} = (\Sigma \lambda_i)^{-1} J_t^1$. It follows that

$B_{12} = (\Sigma \lambda_i)^{-1} J_1^t$ and therefore $B_{21} = B_{12}' = (\Sigma \lambda_i)^{-1} J_t^1$.

Proof of (b): Multiply (2) by J_t^1 to obtain $J_t^1 AB_{12} + J_t^1 \lambda B_{22} = 0$, but since $J_t^1 A = 0$, and $\Sigma \lambda_i \neq 0$, it follows that $B_{22} = 0$.

Proof of (c): This is (3) above.

Proof of (d): Multiply (1) by B_{11} on the left to obtain $B_{11} AB_{11} +$

$B_{11}\lambda B_{11} = B_{11}$, but from (3), $B_{11}\lambda = \phi$, so that $B_{11}AB_{11} = B_{11}$.

Proof of (e): Since $B_{21} = (\Sigma \lambda_i)^{-1} J_t^1$, substituting for B_{21} in (1) gives

$AB_{11} + (\Sigma \lambda_i)^{-1} \lambda J_t^1 = I$, or $AB_{11} = I - (\Sigma \lambda_i)^{-1} \lambda J_t^1$. Hence it follows that $(AB_{11})(AB_{11}) = I - 2(\Sigma \lambda_i)^{-1} \lambda J_t^1 + (\Sigma \lambda_i)^{-1} \lambda J_t^1 = I - (\Sigma \lambda_i)^{-1} \lambda J_t^1$.

Therefore $AB_{11} = B_{11}A$ is a symmetric idempotent matrix. Since the rank of an idempotent matrix is equal to its trace, it follows that

$$r(AB_{11}) = \text{trace } [AB_{11}] = \text{trace } [I_t - (\Sigma \lambda_i)^{-1} \lambda J_t^1]$$

$$= \text{trace } [I_t] - (\Sigma \lambda_i)^{-1} \text{ trace } [J_t^1 \lambda]$$

$$= t - (\Sigma \lambda_i)^{-1} (\Sigma \lambda_i) = t - 1.$$

This completes the proof of the theorem.

<u>COROLLARY 14.1</u> The matrix $A + t^{-1} J_t^t$ is non-singular with inverse equal to $B_{11} + t^{-1} J_t^t$.

Proof: The system $(A + t^{-1} J_t^t) \hat{\tau} = q$ is equivalent to $A\hat{\tau} = q$ if the restriction $\Sigma \hat{\tau}_i = 0$ is imposed. This means $\lambda' \hat{\tau} = 0$, where $\lambda = J_1^t$. Therefore $AB_{11} = I - t^{-1} J_t^t$, and now it is noted that

$$(A + t^{-1} J_t^t)(B_{11} + t^{-1} J_t^t) = AB_{11} + t^{-1} AJ_t^t + t^{-1} J_t^t B_{11} + t^{-2} J_t^t J_t^t ,$$

but since $AJ_t^t = \phi$ and $\lambda' B_{11} = J_t^1 B_{11} = \phi$, it follows that

$$(A + t^{-1} J_t^t)(B_{11} + t^{-1} J_t^t) = I_t - t^{-1} J_t^t + \phi + \phi + t^{-1} J_t^t = I_t .$$

This completes the proof.

<u>THEOREM 15.</u> If the restriction $\lambda' \hat{\tau} = 0$ is used in order to solve the system $A\hat{\tau} = q$, then $E(\hat{\tau}) = \tau - (\Sigma \lambda_i)^{-1} \lambda (\tau.)$.

Proof: First note that $E(q) = E(G'Y) = E[G'(\mu J_1^{N..} + X_1\beta + X_2\tau + e)]$, or $E(q) = E(G'X_2\tau + G'e)$ since $G'J_1^{N..} = \phi$ and $G'X_1 = \phi$. Now since $G'X_2 = A$ and $E(G'e) = G'[E(e)] = \phi$, it follows that $E(q) = A\tau$. Hence $E(\hat{\tau}) = E(B_{11}q) = B_{11}E(q) = B_{11}A\tau$, but from Theorem 14, $B_{11}A$ equals $I - (\Sigma\lambda_i)^{-1}\lambda J_t^1$, so that $E(\hat{\tau}) = [I - (\Sigma\lambda_i)^{-1}\lambda J_t^1]\tau = \tau - (\Sigma\lambda_i)^{-1}\lambda(\tau.)$. This completes the proof.

It has now been shown that if the system $A\hat{\tau} = q$ is solved by imposing the restriction $\lambda'\hat{\tau} = 0$, where $\lambda'\tau$ is any non-estimable function of the τ_i, then the solution is always $\hat{\tau} = B_{11}q$, and $E(\hat{\tau})$ is always $\tau - (\Sigma\lambda_i)^{-1}\lambda(\tau.)$.

Since the covariance matrix of $\hat{\tau}$ is needed in order to determine confidence intervals on $\Sigma c_i\tau_i$, where $\Sigma c_i = 0$, the following theorem will now be proved:

THEOREM 16. Under the conditions of Theorem 15, $\text{Cov}(\hat{\tau}) = \sigma^2 B_{11}$.

Proof:

$$\begin{aligned}\text{Cov}(\hat{\tau}) &= E[\hat{\tau} - E(\hat{\tau})][\hat{\tau} - E(\hat{\tau})]' \\ &= E[B_{11}q - E(B_{11}q)][B_{11}q - E(B_{11}q)]' \\ &= E\left\{B_{11}[q - E(q)][q - E(q)]'B_{11}\right\} \\ &= B_{11}E[q - E(q)][q - E(q)]'B_{11} \\ &= B_{11}[\text{Cov}(q)]B_{11}.\end{aligned}$$

Since $q = G'Y$ from equation (10), it follows that

$$\text{Cov}(q) = \text{Cov}(G'Y) = E[G'Y - E(G'Y)][G'Y - E(G'Y)]'.$$

Now $E(G'Y) = E[G'(X_1\alpha + X_2\tau + e)]$, but $G'X_1 = \phi$, and $E(e) = \phi$, so that $E(G'Y) = G'X_2\tau$. This means that

$$G'Y - E(G'Y) = G'(X_1\alpha + X_2\tau + e) - G'X_2\tau = G'e,$$

and therefore it follows that

$$\begin{aligned}\text{Cov}(q) &= E[(G'e)(G'e)'] \\ &= G'[E(ee')]G \\ &= G'(\sigma^2 I)G \\ &= \sigma^2 G'G = \sigma^2 A.\end{aligned}$$

But this means

$$\text{Cov}(\hat{\tau}) = \sigma^2 B_{11} A B_{11} = \sigma^2 B_{11},$$

from (d) of Theorem 14. This completes the proof.

Note now that the covariance matrix of $\hat{\tau}$ is $\sigma^2 B_{11}$, and the inverse of $A + t^{-1}J_t^t$ is $B_{11} + t^{-1}J_t^t$. However, in determining the variances of the estimates of treatment differences, say $\text{Var}(\widehat{\tau_i - \tau_j})$, it does not matter if $\sigma^2 B_{11}$ or $\sigma^2(B_{11} + t^{-1}J_t^t)$ is used in the formula

$$\text{Var}(\widehat{\tau_i - \tau_j}) = \text{Var}(\hat{\tau}_i - \bar{\tau}.) + \text{Var}(\hat{\tau}_j - \bar{\tau}.) - 2\,\text{Cov}[(\hat{\tau}_i - \bar{\tau}.), (\hat{\tau}_j - \bar{\tau}.)],$$

because the contributions from $t^{-1}J_t^t$ add to zero.

Test of the Hypothesis H_0: $\tau_1 = \tau_2 = \ldots = \tau_t$

If it is desired to test the hypothesis H_0: $\tau_1 = \tau_2 = \ldots = \tau_t$

and construct an analysis of variance table from which this test can be performed, it is best to make use of Theorem 9. This theorem states that for the general linear hypothesis model $Y = X\beta + e$, $R(\beta) = \hat{\beta}'X'Y$, where $\hat{\beta}$ is any solution to the normal equations $X'X\hat{\beta} = X'Y$.

The original matrix model for the general two-way classification was $Y = X\gamma + e$, and this was rewritten as $Y = X_1\alpha + X_2\tau + e$, with the normal equations being

$$X_1'X_1\hat{\alpha} + X_1'X_2\hat{\tau} = X_1'Y$$

$$X_2'X_1\hat{\alpha} + X_2'X_2\hat{\tau} = X_2'Y.$$

It then follows that $R(\gamma) = R(\alpha, \tau) = \hat{\alpha}_0 X_1'Y + \hat{\tau}_0'X_2'Y$, where $\hat{\alpha}_0$ and $\hat{\tau}_0$ are any solution to the above normal equations. Since $(X_1'X_1)^{-1}$ exists, it follows that $\hat{\alpha}_0 = (X_1'X_1)^{-1}(X_1'Y - X_1'X_2\hat{\tau}_0)$ from the first normal equation, and therefore

$$\begin{aligned} R(\alpha, \tau) &= (Y'X_1 - \hat{\tau}_0'X_2'X_1)(X_1'X_1)^{-1}X_1'Y + \hat{\tau}_0'X_2'Y \\ &= Y'X_1(X_1'X_1)^{-1}X_1'Y - \hat{\tau}_0'X_2'X_1(X_1'X_1)^{-1}X_1'Y + \hat{\tau}_0'X_2'Y \\ &= Y'X_1(X_1'X_1)^{-1}X_1'Y + \hat{\tau}_0'[X_2'Y - X_2'X_1(X_1'X_1)^{-1}X_1'Y]. \end{aligned}$$

Since $G'Y = q$, the coefficient of $\hat{\tau}_0'$ above is q. It follows that

$$R(\alpha, \tau) = Y'X_1(X_1'X_1)^{-1}X_1'Y + \hat{\tau}_0'q. \tag{13}$$

Under the hypothesis H_0, let $\tau_1 = \tau_2 = \dots = \tau_t = \tau^*$, where τ^* is a scalar. Then the model $Y = X_1\alpha + X_2\tau + e$ becomes

$Y = X_1\alpha + X_2J_1^t\tau^* + e$, and since $X_2J_1^t = J_1^{N..}$, this can be written as $Y = X_1\alpha + \tau^*J_1^{N..} + e$. If it is recalled that $\alpha = \mu J_1^b + \beta$ and that $X_1J_1^b = J_1^{N..}$, it is then possible to write

$$Y = X_1(\mu + \tau^*)J_1^b + X_1\beta + e \text{ or } Y = X_1[(\mu + \tau^*)J_1^b + \beta] + e.$$

Now let $\alpha^* = (\mu + \tau^*)J_1^b + \beta$, so that the reduced model can be written as $Y = X_1\alpha^* + e$, and α^* and α are not the same unless $\tau_i = 0$ for all i.

For the model under H_0, $Y = X_1\alpha^* + e$, the normal equations are $X_1'X_1\hat{\alpha}^* = X_1'Y$, so that $\hat{\alpha}^* = (X_1'X_1)^{-1}X_1'Y$, and therefore

$$R(\alpha^*) = \hat{\alpha}^{*\prime}X_1'Y = Y'X_1(X_1'X_1)^{-1}X_1'Y.$$

From the general linear hypothesis, $R(\tau|\alpha) = R(\alpha, \tau) - R(\alpha^*)$, so that reference to (13) gives $R(\tau|\alpha) = \hat{\tau}_0'q$. Since $\hat{\tau}_0$ is any solution to the system $A\hat{\tau} = q$, it follows that $\hat{\tau}_0 = B_{11}q$. Then $R(\tau|\alpha) = q'B_{11}q$, so that the following analysis of variance table can be written in order to test H_0: $\tau_1 = \tau_2 = \ldots = \tau_t$.

TABLE III

AOV FOR THE GENERAL TWO-WAY CLASSIFICATION

Source	d.f.	S.S.
Total	N..	$Y'Y$
$R(\alpha, \tau)$	$b + t - 1$	$Y'X_1(X'_1X_1)^{-1}X'_1Y + q'B_{11}q$
$R(\alpha^*)$	b	$Y'X_1(X'_1X_1)^{-1}X'_1Y$
$R(\tau \mid \alpha)$	$t - 1$	$q'B_{11}q$
Error	$N.. - b - t + 1$	$Y'Y - q'B_{11}q - Y'X_1(X'_1X_1)^{-1}X'_1Y = E_{ss}$

Consider now that since $q = G'Y$, then $q'B_{11}q$ is $Y'GB_{11}G'Y$, and note that $(GB_{11}G')(GB_{11}G') = G'B_{11}G'GB_{11}G' = G'B_{11}AB_{11}G = G'B_{11}G$, so that $G'B_{11}G$ is idempotent. Since $Y \sim N(\mu^*, \sigma^2 I)$, where $\mu^* = \mu J_1^{N..} + X_1\beta + X_2\tau$, it follows from Theorem 8 that

$$\frac{q'B_{11}q}{\sigma^2} = \frac{Y'GB_{11}G'Y}{\sigma^2}$$

is distributed as a non-central chi-square with degrees of freedom equal to $r(GB_{11}G')$ and non-centrality parameter

$$\lambda = \frac{\mu^{*\prime}GB_{11}G'\mu^*}{2\sigma^2}.$$

Since $GB_{11}G'$ is idempotent, its rank is equal to its trace so that $r(GB_{11}G') = \text{trace } GB_{11}G' = \text{trace } G'GB_{11} = \text{trace } AB_{11}$, and AB_{11} has rank $t - 1$ by virtue of Theorem 14.

Now $2\sigma^2\lambda = (\mu J^1_{N..} + \beta' X_1' + \tau' X_2')GB_{11}G'(\mu J_1^{N..} + X_1\beta + X_2\tau)$, but $J^1_{N..}G = \phi$, $G'J_1^{N..} = \phi$, and also $G'X_1 = \phi$, so that $2\sigma^2\lambda = \tau' X_2 GB_{11}G'X_2\tau$. However, $G'X_2 = A$, and since A is symmetric, $X_2'G = A$ also. It then follows that $2\sigma^2\lambda = \tau' AB_{11}A\tau$, but since $AB_{11} = I - t^{-1}J^t_t$ when the restriction $\Sigma\hat{\tau}_i = 0$ is imposed, the expression for $2\sigma^2\lambda$ becomes $\tau'(I - t^{-1}J^t_t)A = \tau' A\tau$, since $J^1_t A = \phi$.

If H_0 is true, then $\tau_1 = \tau_2 = \ldots \tau_t = \tau^*$; $2\sigma^2\lambda = \tau^* J^1_t A J^t_1 \tau^* = \phi$, and $\frac{q'B_{11}q}{\sigma^2}$ has a central chi-square distribution with t - 1 degrees of freedom. From Theorem 9

$$\frac{q'B_{11}q}{E_{ss}} \cdot \frac{N.. - b - t + 1}{t - 1}$$

has a non-central F-distribution with t - 1 and N.. - b - t + 1 degrees of freedom and non-centrality parameter $\lambda = \frac{\tau' A\tau}{2\sigma^2}$. This becomes a central F-distribution if and only if H_0 is true.

Confidence Intervals

If reference is made to Theorem 15, and λ is chosen to be J^t_1, which is equivalent to solving the system $(A + t^{-1}J^t_t)\hat{\tau} = q$ under the restriction $\Sigma\hat{\tau}_i = 0$, then $E(\hat{\tau}) = \tau - t^{-1}(\tau.)J^t_1$ or $E(\hat{\tau}) = \tau - (\bar{\tau}.)J^t_1$.

It follows that $E(\hat{\tau}_i) = \tau_i - (\bar{\tau}.)$ if the i^{th} element of $E(\hat{\tau})$ is chosen. Reference to Theorem 16 gives the result that if b_{ij} is any element of B_{11} under the restriction $J_t^1 \hat{\tau} = 0$, then $\sigma^2 b_{ij} = \text{Cov}[(\widehat{\tau_i - \bar{\tau}.}), (\widehat{\tau_j - \bar{\tau}.})]$.

If the variance of the estimate of any two treatment differences is desired, it can be obtained as indicated in the section on the matrix model for the general two-way classification.

If it is desired to set a confidence interval on $\Sigma c_i \tau_i$, then it is noted that $\Sigma c_i(\tau_i - \bar{\tau}.) = \Sigma c_i \tau_i$ since $\Sigma c_i = 0$ for all estimable functions. Therefore $\Sigma c_i \hat{\tau}_i$ is distributed $N[\Sigma c_i \tau_i, \sigma^2 \Sigma c_i c_j b_{ij}]$, and

$$u = \frac{\Sigma c_i \hat{\tau}_i - \Sigma c_i \tau_i}{\sigma \sqrt{\Sigma c_i c_j b_{ij}}}$$

is distributed $N(0, 1)$.

If reference is made to Table III and the discussion following it, it is evident that $v = \frac{E_{ss}}{\sigma^2}$ is distributed as chi-square with $N.. - b - t + 1$ degrees of freedom and is independent of u. It follows that

$$\frac{u}{\sqrt{\frac{v}{N.. - b - t + 1}}}$$

has Student's t-distribution with $N.. - b - t + 1$ degrees of freedom. Since $\frac{E_{ss}}{N.. - b - t + 1}$ is the error mean square, it will be denoted by E_{ms}, and therefore a $1 - \alpha$ confidence interval on $\Sigma c_i \tau_i$

is given by

(14) $$\Sigma c_i(\widehat{\tau_i - \bar{\tau}.}) - C \leq \Sigma c_i\tau_i \leq \Sigma c_i(\widehat{\tau_i - \bar{\tau}.}) + C,$$

where $C = t_{\alpha/2}\sqrt{E_{ms}\ \Sigma c_i c_j b_{ij}}$, and $t_{\alpha/2}$ is the tabular value of Student's t with $N_{..} - b - t + 1$ degrees of freedom for the desired value of α.

Another value of λ which gives some very useful results in the solution of the system $A\hat{\tau} = q$ is $\lambda' = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the one occurs as the s^{th} element of λ'. This is imposing the restriction that $\hat{\tau}_s = 0$ on the system $A\hat{\tau} = q$. For this value of λ ,

Theorems 14 and 15 state that $B_{11}A = I - \lambda J_t^1$, so that $E(\hat{\tau}) = B_{11}A$ becomes $\tau - \lambda J_t^1 \tau$. Hence $E(\hat{\tau}_i) = \tau_i - \tau_s$.

It is evident that the restriction $\hat{\tau}_s = 0$ is equivalent in actual practice to deleting the s^{th} row and column of A, the s^{th} element of $\hat{\tau}$ and q in the system $A\hat{\tau} = q$, and then solving the system of $t - 1$ equations in $t - 1$ unknowns, $A^{**}\hat{\tau}^{**} = q^{**}$. However, the results of Theorems 14, 15, and 16 hold in general for any non-estimable condition on the τ_i . Therefore for the condition described above, $\text{Cov}(\hat{\tau}) = \sigma^2 B_{11}$ is such that if b_{ij}^{**} is any element of B_{11} under the restriction $\hat{\tau}_s = 0$, then $\sigma^2 b_{ij}^{**} = \text{Cov}[(\widehat{\tau_i - \tau_s}), (\widehat{\tau_j - \tau_s})]$. If the variance of the estimate of any two treatment differences is desired, then the same procedure described before can be used.

Setting confidence intervals on a linear combination of the τ_i, say $\Sigma c_i\tau_i$, follows as in the previous case. Since only contrasts are estimable, it follows that $\Sigma c_i\widehat{(\tau_i - \tau_s)} = \Sigma c_i\hat{\tau}_i$. Therefore $\Sigma c_i\hat{\tau}_i$ is distributed $N[\Sigma c_i\tau_i, \sigma^2 \Sigma c_i c_j b^{**}_{ij}]$, and

$$\mu = \frac{\Sigma c_i\hat{\tau}_i - \Sigma c_i\tau_i}{\sigma\sqrt{\Sigma c_i c_j b^{**}_{ij}}}$$

is distributed N(0, 1). Since the distributional properties remain the same regardless of the λ that is used, the desired $1 - \alpha$ confidence interval is given by

$$(15) \qquad \Sigma c_i\widehat{(\tau_i - \tau_s)} - D \leq \Sigma c_i\tau_i \leq \Sigma c_i\widehat{(\tau_i - \tau_s)} + D,$$

where $D = t_{\alpha/2}\sqrt{E_{ms}\Sigma c_i c_j b^{**}_{ij}}$.

CHAPTER II

THE GENERAL SLIPPED-BLOCK DESIGN

Definition and Notation

The General Slipped-Block Design will be defined as a special case of the general two-way classification without interaction. The correspondence between the notation used in Chapter I for the general two-way classification and the notation for the General Slipped-Block Design is given below along with some new notation:

(1) The number of blocks is b.

(2) The number of treatments is t.

(3) $N_{.j} = k_j$ is the number of observations in block j, where $j = 1, 2, \ldots, b$, and n_{ij} is 0 or 1 for all i and j.

(4) $N_{i.} = u_i$ is the number of observations on treatment i, where $i = 1, 2, \ldots, t$.

(5) The symbol w_{pq} denotes the number of treatments common to block p and block q. The number w_{pq} will be called the overlap, and it is defined only for $p < q$.

(6) The symbol s_{pq} denotes the positive difference between the number of the first treatment in block q and the number of

the first treatment in block p. The number s_{pq} will be called the slip, and it is defined only for $p < q$.

The preceding notation leads to the following useful relationships:

(7) $k_j = w_{j, j+1} + s_{j, j+1}$, for $j = 1, 2, \ldots, b - 1$

(8) $t = \sum_{j=1}^{b-1} s_{j, j+1} + k_b$

The following restrictions must hold in order for a general two-way classification to be called a Slipped-Block Design:

(9) $w_{j, j+1} \geq 1$ for $j = 1, 2, \ldots, b - 1$

(10) $s_{j, j+1} \geq 1$ for $j = 1, 2, \ldots, b - 1$

Property (9) above is necessary in order to satisfy the assumption immediately preceding Theorem 11. Property (10) is the feature which gives the Slipped-Block Design its name, since it requires the first treatment in block (j+1) to have a higher number than the first treatment in block j. This is equivalent to "slipping" block j down in the statistical layout in order to obtain block (j+1).

In order to clarify the definition of the Slipped-Block Design, examples of two-way classifications which are Slipped-Block Designs are given in Illustration II with the values of b, t, k_j, u_i, w_{pq}, and s_{pq} as indicated. An "x" denotes an observation in the statistical layout given in Illustration II and in all layouts of this type which follow.

ILLUSTRATION II

EXAMPLES OF SLIPPED-BLOCK DESIGNS

Example 1.

Blocks

Treatments	1	2	3	4
1	x			
2	x	x		
3	x	x		
4		x	x	
5		x	x	
6			x	x
7				x

$b = 4;\ t = 7$

$k_1 = k_3 = 3;\ k_2 = 4;\ k_4 = 2$

$u_1 = u_7 = 1;\ u_2 = u_3 = u_4 = u_5 = u_6 = 2$

$w_{12} = 2;\ w_{13} = 0;\ w_{14} = 0$

$w_{23} = 2;\ w_{24} = 0$

$w_{34} = 1$

$s_{12} = 1;\ s_{13} = 3;\ s_{14} = 5$

$s_{23} = 2;\ s_{24} = 4$

$s_{34} = 2$

Example 2.

Blocks

Treatments	1	2	3	4	5
1	x				
2	x	x			
3	x	x	x		
4	x	x	x	x	
5		x	x	x	x
6			x	x	x
7				x	x
8					x

$b = 5;\ t = 8$

$k_j = 4$ for all j

$u_1 = u_8 = 1;\ u_2 = u_7 = 2$

$u_3 = u_6 = 3;\ u_4 = u_5 = 4$

$w_{12} = 3;\ w_{13} = 2;\ w_{14} = 1;\ w_{15} = 0$

$w_{23} = 3;\ w_{24} = 2;\ w_{25} = 1$

$w_{34} = 3;\ w_{35} = 2$

$w_{45} = 3$

$s_{12} = 1;\ s_{13} = 2;\ s_{14} = 3;\ s_{15} = 4$

$s_{23} = 1;\ s_{24} = 2;\ s_{25} = 3$

$s_{34} = 1;\ s_{35} = 2$

$s_{45} = 1$

The analysis of the General Slipped-Block Design can be performed by considering it as a general two-way classification and proceeding in the manner described in Chapter I. No simpler analysis will be given in this paper. However, if some additional assumptions are made, a simpler analysis is possible for certain cases. These analyses will be given in the remainder of this paper.

The Slipped-Block Designs considered in this paper will be assumed to have $N_{.j} = k_j = k$ observations in each block. For any two adjacent blocks, the slip will always be $s_{j, j+1} = s$, and the overlap will be $w_{j, j+1} = n$. The slip and overlap for any two blocks which are not adjacent will not be of any importance.

If reference is made to the discussion following Table II, it becomes apparent that $X_1'X_1$ will always have diagonal elements $N_{.j} = k$, and therefore $(X_1'X_1)^{-1} = k^{-1}I_b$ when the above assumptions are made. The matrix $A = X_2'X_2 - X_2'X_1(X_1'X_1)^{-1}X_1'X_2$ defined for the general two-way classification now becomes $A = X_2'X_2 - k^{-1}X_2'X_1X_1'X_2$, but since N was defined to be $X_2'X_1$, it follows that

(1) $$A = X_2'X_2 - k^{-1}NN'.$$

The same substitutions used above to obtain (1) reduces the vector $q = X_2'Y - X_2'X_1(X_1'X_1)^{-1}X_1'Y$ to

(2) $$q = X_2'Y - k^{-1}NX_1'Y.$$

CHAPTER III

THE SLIPPED-BLOCK DESIGN WITH TWO BLOCKS

The Derivation of $A + t^{-1}J_t^t$ and $(A + t^{-1}J_t^t)^{-1}$

The statistical layout for the Slipped-Block Design with two blocks is given in Illustration III. With the assumptions given in Chapter II, it is seen that the relations $k_j = w_{j,j+1} + s_{j,j+1}$ and $t = \sum_{j=1}^{b-1} s_{j,j+1} + k_j$ from that chapter become $k = s + n$ and $t = s + k = 2s + n$. Therefore, if any two of the four quantities k, s, t, and n are known, the remaining two can be determined.

Observation of Illustration III indicates that if the method described in Chapter I for writing the matrix N is used, and if it is recalled that $X_2'X_2$ is diagonal with diagonal elements $N_{i.}$, then the matrix $A = X_2'X_2 - k^{-1}NN'$ can be written as shown in Illustration IV. Performing the multiplication indicated in Illustration IV to obtain NN' and partitioning the matrices as indicated in the illustration gives

$$A = \begin{bmatrix} I_s & \phi_n^s & \phi_s^s \\ \phi_s^n & 2I_n & \phi_s^n \\ \phi_s^s & \phi_n^s & I_s \end{bmatrix} - \frac{1}{k} \begin{bmatrix} J_s^s & J_n^s & \phi_s^s \\ J_s^n & 2J_n^n & J_s^n \\ \phi_s^s & J_n^s & J_s^s \end{bmatrix},$$

ILLUSTRATION III

LAYOUT FOR THE SLIPPED-BLOCK DESIGN WITH TWO BLOCKS

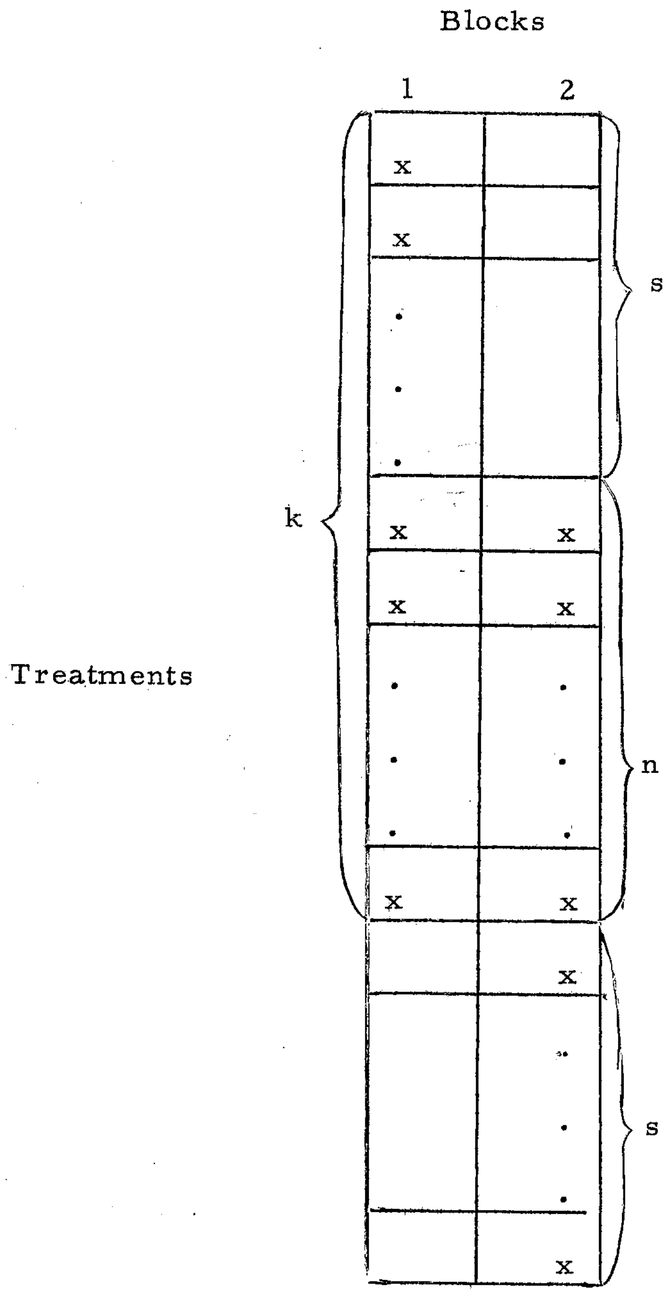


ILLUSTRATION IV

THE MATRIX A FOR THE DESIGN IN ILLUSTRATION III

$$A = \begin{bmatrix} 1 & & & & & & & & & \\ & 1 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 2 & & & & & \\ & & & & & 2 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & 2 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 1 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\ \cdot\\1\end{matrix}}\right\} \text{first } s \text{ rows} \\ \left.\vphantom{\begin{matrix}2\\2\\ \cdot\\2\end{matrix}}\right\} \text{next } n \text{ rows} \\ \left.\vphantom{\begin{matrix}1\\1\\ \cdot\\1\end{matrix}}\right\} \text{last } s \text{ rows} \end{matrix} - \frac{1}{k} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 1 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\ \cdot\\ \cdot\\ \cdot\\1\end{matrix}}\right\} s \\ \vphantom{\begin{matrix}1\\1\\ \cdot\\ \cdot\\ \cdot\\1\end{matrix}} \\ \left.\vphantom{\begin{matrix}1\\1\\ \cdot\\ \cdot\\ \cdot\\1\end{matrix}}\right\} s \end{matrix} \begin{bmatrix} \overbrace{1\,1 \ldots 1}^{s} & \overbrace{1\,1 \ldots 1}^{n} & \overbrace{0\,0 \ldots 0}^{s} \\ 0\,0 \ldots 0 & 1\,1 \ldots 1 & 1\,1 \ldots 1 \end{bmatrix}$$

All off-diagonal elements of X'_2X_2 are equal to zero.

and the final result for the matrix A is

$$A = \begin{bmatrix} I_s - \frac{1}{k} J_s^s & -\frac{1}{k} J_n^s & \phi_s^s \\ -\frac{1}{k} J_s^n & 2I_n - \frac{2}{k} J_n^n & -\frac{1}{k} J_s^n \\ \phi_s^s & -\frac{1}{k} J_n^s & I_s - \frac{1}{k} J_s^s \end{bmatrix} .$$

Since the dimensions of the identity submatrices are obvious from the dimensions on the other submatrices, the subscripts on them will be omitted. If the matrix

$$t^{-1} J_t^t = t^{-1} \begin{bmatrix} J_s^s & J_n^s & J_s^s \\ J_s^n & J_n^n & J_s^n \\ J_s^s & J_n^s & J_s^s \end{bmatrix}$$

is partitioned as indicated and added to the matrix A, the result is

$$A + t^{-1} J_t^t = \begin{bmatrix} I - \frac{t-k}{kt} J_s^s & \frac{k-t}{kt} J_n^s & \frac{1}{t} J_s^s \\ \frac{k-t}{kt} J_s^n & 2I - \frac{2t-k}{kt} J_n^n & \frac{k-t}{kt} J_s^n \\ \frac{1}{t} J_s^s & \frac{k-t}{kt} J_n^s & I - \frac{t-k}{kt} J_s^s \end{bmatrix} .$$

If the relations $k = s + n$ and $t = 2s + n$ are used to simplify only the numerators in the above matrix, it can be written as

$$(1)\quad A+t^{-1}J^t_t = \begin{bmatrix} I - \frac{s}{kt}J^s_s & -\frac{s}{kt}J^s_n & \frac{1}{t}J^s_s \\ -\frac{s}{kt}J^n_s & 2I - \frac{t+s}{kt}J^n_n & -\frac{s}{kt}J^n_s \\ \frac{1}{t}J^s_s & -\frac{s}{kt}J^s_n & I - \frac{s}{kt}J^s_s \end{bmatrix}.$$

If it is now assumed that $(A + t^{-1}J^t_t)^{-1}$ has the same form as $A + t^{-1}J^t_t$, then the inverse can be represented in the form

$$(A + t^{-1}J^t_t)^{-1} = \begin{bmatrix} aI + bJ^s_s & cJ^s_n & dJ^s_s \\ cJ^n_s & eI + fJ^n_n & cJ^n_s \\ dJ^s_s & cJ^s_n & aI + bJ^s_s \end{bmatrix}$$

If the product of the above matrix and the matrix in (1) is equated to $I_{2s+n} = I_t$, where the identity is partitioned in the same manner as the two matrices forming the product, it is readily seen that $a = 1$ and $e = 1/2$. The necessary algebra can then be carried out in order to complete the solution for b, c, d, and f. The matrix resulting from this procedure is

$$(2)\quad (A + t^{-1}J^t_t)^{-1} = \begin{bmatrix} I + \frac{2ks}{nt^2}J^s_s & \frac{s}{t^2}J^s_n & \frac{ns-kt}{nt^2}J^s_s \\ \frac{s}{t^2}J^n_s & \frac{1}{2}I + \frac{4s+n}{2t^2}J^n_n & \frac{s}{t^2}J^n_s \\ \frac{ns-kt}{nt^2}J^s_s & \frac{s}{t^2}J^s_n & I + \frac{2ks}{nt^2}J^s_s \end{bmatrix}.$$

It can be verified that the matrix in (2) is the inverse of the matrix in (1) by multiplying the two matrices together and substituting s + n for k and 2s + n for t in order to show that the product is $I_{2s+n} = I_t$.

The Variances of Estimates of Treatment Differences

If reference is made to Theorem 16 and the comment immediately following it, then from the inverse (2), it is observed that the variances of the estimates of treatment differences are as follows:

(a) If i = 1, 2, ..., s and j = 1, 2, ... , s, i ≠ j, then

$$\text{Var}\ (\widehat{\tau_i - \tau_j}) = \sigma^2[2(1 + \frac{2ks}{nt^2}) - 2(\frac{2ks}{nt^2})] = 2\sigma^2 .$$

(b) If i = 1, 2, ... , s and j = s + 1, s + 2, ... , s + n, then

$$\text{Var}\ (\widehat{\tau_i - \tau_j}) = \sigma^2[1 + \frac{2ks}{nt^2} + \frac{1}{2} + \frac{4s + n}{2t^2} - 2(\frac{s}{t^2})] .$$

If a common denominator is obtained, and the relationships k = s + n and t = 2s + n are employed, the above expression simplifies to

$$\text{Var}\ (\widehat{\tau_i - \tau_j}) = \sigma^2 (\frac{3}{2} + \frac{1}{2n}).$$

(c) If i = 1, 2, ... , s and j = s + n + 1, s + n + 2, ... , 2s + n = t, then

$$\text{Var}\ (\widehat{\tau_i - \tau_j}) = \sigma^2[1 + \frac{2ks}{nt^2} + 1 + \frac{2ks}{nt^2} - 2(\frac{ns - kt}{nt^2})] .$$

If this expression is simplified in the same manner described in (b), then

$$\text{Var}\ (\widehat{\tau_i - \tau_j}) = \sigma^2 (2 + \frac{2}{n}) .$$

(d) If i = s + 1, s + 2, ... , s + n, and j = s + n + 1, s + n + 2, ... , 2s + n = t, then

$$\text{Var}\ (\widehat{\tau_i - \tau_j}) = \sigma^2 [\frac{1}{2} + \frac{4s + n}{2t^2} + 1 + \frac{2ks}{nt^2} - 2(\frac{s}{t^2})]$$

$$= \sigma^2 (\frac{3}{2} + \frac{1}{2n});$$

the same result that was obtained in (b).

(e) If i = s + 1, s + 2, ... , s + n and j = s + 1, s + 2, ... , s + n, $i \neq j$, then

$$\text{Var}\ (\widehat{\tau_i - \tau_j}) = \sigma^2 [2 (\frac{1}{2} + \frac{4s + n}{2t^2}) - 2(\frac{4s + n}{2t^2})]$$

$$= \sigma^2 .$$

(f) If i = s + n + 1, s + n + 2, ... , 2s + n = t, and j = s + n + 1, s + n + 2, ... , 2s + n = t, $i \neq j$, the results are the same as obtained in (a).

It is worth noting that in (a), (e), and (f), the variance of $(\widehat{\tau_i - \tau_j})$ does not depend on n, but in (b), (c), and (d), the variance decreases as n increases.

The Extension of Results

The variances of $(\widehat{\tau_i - \tau_j})$ have been obtained in general for the case of two blocks where the number of treatments, the overlap, the slip, and the number of observations per block are not specified. Before proceeding with the discussion of confidence intervals and tests of hypotheses for this case, the results will be extended to permit r replications of each of the two basic blocks.

The statistical layout for the design under consideration is as shown in Illustration V. Suppose now that the matrix $X_2'X_2 - k^{-1}NN'$ for this case is designated by A_r in order to distinguish it from the matrix A obtained when the design consisted of only the two basic blocks. Then if $X_2'X_2$ and NN' are multiplied out in the same manner used for the case of two blocks, it is again possible to partition these product matrices to obtain

$$A_r = r\left\{\begin{bmatrix} I & \phi_n^s & \phi_s^s \\ \phi_s^n & 2I_n & \phi_s^n \\ \phi_s^s & \phi_n^s & I \end{bmatrix} - \frac{1}{k}\begin{bmatrix} J_s^s & J_n^s & \phi_s^s \\ J_s^n & 2J_n^n & J_s^n \\ \phi_s^s & J_n^s & J_s^s \end{bmatrix}\right\},$$

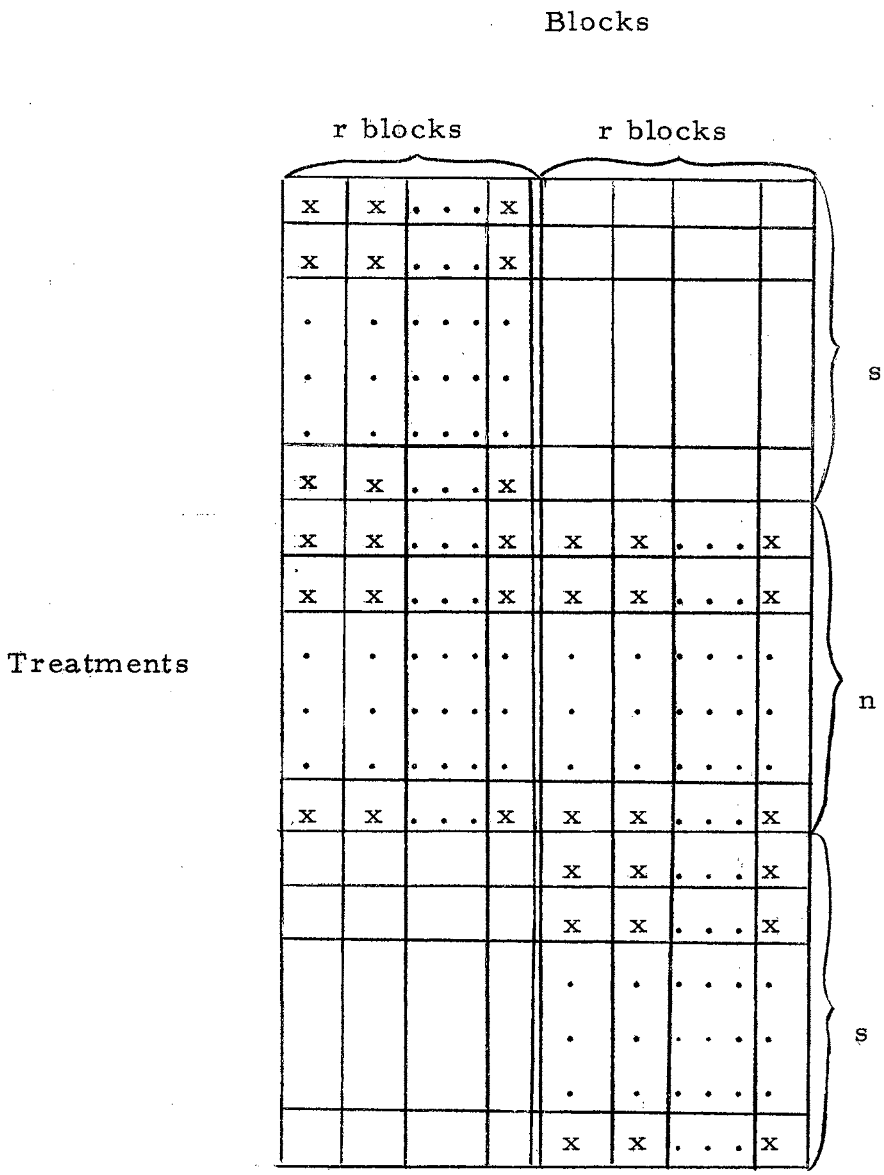
and performing the subtraction gives

$$A_r = r\begin{bmatrix} I - \frac{1}{k}J_s^s & -\frac{1}{k}J_n^s & \phi_s^s \\ -\frac{1}{k}J_s^n & 2I - \frac{2}{k}J_n^n & -\frac{1}{k}J_s^n \\ \phi_s^s & -\frac{1}{k}J_n^s & I - \frac{1}{k}J_s^s \end{bmatrix} = rA.$$

Now, if it is recalled that adding $t^{-1}J_t^t$ is just a means of imposing the condition that $\Sigma\hat{\tau}_i = 0$ on the system $A\hat{\tau} = q$, then adding

ILLUSTRATION V

LAYOUT FOR THE SLIPPED-BLOCK DESIGN WITH r REPLICATES OF TWO BASIC BLOCKS



$rt^{-1}J_t^t$ would impose the same condition on $A_r\hat{\tau} = q_r$ because $r\ \Sigma\ \hat{\tau}_i = 0$ if $\Sigma\ \hat{\tau}_i = 0$. The notation q_r serves the same purpose as A_r does. If $rt^{-1}J_t^t$ is added to A_r, then $A_r + rt^{-1}J_t^t = r(A + t^{-1}J_t^t)$, and this means that $r(A + t^{-1}J_t^t)^{-1} = r^{-1}(A + t^{-1}J_t^t)^{-1}$. The net result is that replicating each basic block r times divides all elements of the covariance matrix of $\hat{\tau}$ obtained for $r = 1$ by r.

The only difficulty arising here is showing that adding $rt^{-1}J_t^t$ to A_r in order to solve the system $A_r\hat{\tau} = q_r$ results in $\hat{\tau}$ having the same expected value as it does when $t^{-1}J_t^t$ is added to A in order to solve the system $A\hat{\tau} = q$. When the condition $\Sigma\ \hat{\tau}_i = 0$ is thus imposed on the latter system, the solution $\hat{\tau} = B_{11}q$ to that system is such that $E(\hat{\tau}) = B_{11}A\tau$. It is now necessary to show that imposing the same condition on the system $A_r\hat{\tau} = q_r$ by adding $rt^{-1}J_t^t$ to A_r does not change $E(\hat{\tau})$.

If the system $A_r\hat{\tau} = q_r$ is considered, then from the preceding discussion, this system is equivalent to the system $r(A + t^{-1}J_t^t)\hat{\tau} = q_r$. If this system is multiplied by the inverse matrix, then, by virtue of Corollary 14.1 and the above results, the solution

$$\hat{\tau} = (r^{-1}B_{11} + r^{-1}t^{-1}J_t^t)q_r = r^{-1}(B_{11} + t^{-1}J_t^t)q_r$$

is obtained. Since $J_t^1 q_r = 0$ just as $J_t^1 q = 0$, this solution becomes $\hat{\tau} = r^{-1}B_{11}q_r$, which corresponds to the solution $\hat{\tau} = B_{11}q$ obtained when

$r = 1$. Since $E(q) = A\tau$ and $E(\hat{\tau}) = B_{11}A\tau$ from Theorem 15, it follows that $E(q_r) = A_r\tau$. Therefore $E(\hat{\tau}) = r^{-1}B_{11}A_r\tau = r^{-1}B_{11}(rA)\tau = B_{11}A\tau$, and this is exactly the same result obtained for $r = 1$. Since the expected values remain the same as they were for $b = 2$, and the inverse (2) is the same except for the factor r^{-1}, it is not difficult to analyze the design when each of the two basic blocks is replicated r times.

Test of the Hypothesis $\tau_1 = \tau_2 = \ldots = \tau_t$

Reference to the preceding section and to the discussion of the test of this same hypothesis for the general two-way classification in the fourth section of Chapter I reveals that the analysis of variance given in Table III can now be written as given below since $(X_1'X_1)^{-1} = k^{-1}I_b$ for the Slipped-Block Design. Note that with the assumptions made at the end of Chapter II, the analysis of variance below is a general one for all the Slipped-Block Designs considered in this paper, and not necessarily just for a design with two blocks or r replicates of two basic blocks.

TABLE IV

AOV FOR THE SLIPPED-BLOCK DESIGN

Source	d. f.	S. S.
Total	bk	$Y'Y$
$R(\alpha, \tau)$	$b + t - 1$	$k^{-1}Y'X_1X_1'Y + q'B_{11}q$
$R(\alpha^*)$	b	$k^{-1}Y'X_1X_1'Y$
$R(\tau \mid \alpha)$	$t - 1$	$q'B_{11}q$
Error	$bk-b-t+1$	$Y'Y - q'B_{11}q - k^{-1}Y'X_1X_1'Y = E_{ss}$

Since the distributional properties of the vector Y are the same as given in the discussion following Table III, $\frac{q'B_{11}q}{\sigma^2}$ has a non-central chi-square distribution with $t - 1$ degrees of freedom and non-centrality parameter $\lambda = \frac{\tau'A\tau}{2\sigma^2}$. If the hypothesis $\tau_1 = \tau_2 = \ldots = \tau_t$ is true, then $\lambda = 0$, and $\frac{q'B_{11}q}{\sigma^2}$ has a central chi-square distribution with $t - 1$ degrees of freedom. From Theorem 9,

$$v = \frac{q'B_{11}q}{E_{ss}} \cdot \frac{bk - b - t + 1}{t - 1}$$

has a non-central F-distribution with $t - 1$ and $bk - b - t + 1$ degrees of freedom and non-centrality parameter $\frac{\tau'A\tau}{2\sigma^2}$. If the hypothesis is true, this reduces to a central F-distribution. Therefore v is compared with the tabular value of the F-variate with the appropriate degrees of freedom in order to test the hypothesis $\tau_1 = \tau_2 = \ldots = \tau_t$.

A Computing Procedure for the Analysis of Variance

In an experiment involving even a relatively small number of observations, the computation of the quantities represented by the quadratic forms in Table IV is quite tedious. In the special cases under consideration in this chapter, it is possible to derive a rather simple computing procedure.

The computing procedure will be derived by using the inverse given in equation (2) of this chapter to obtain the quadratic form for $R(\tau|\alpha)$, $q'B_{11}q$, in a different form. Since the sum of squares for $R(\alpha^*)$ is simply the uncorrected sum of squares for blocks and $Y'Y$ is the total uncorrected sum of squares, there is no difficulty experienced in computing these quantities in the ordinary manner. The difficult part of the computing would be inverting the t x t matrix $(A + t^{-1}J_t^t)$ in order to obtain B_{11} and then $q'B_{11}q$. However, if a different form for $q'B_{11}q$ is obtained as described above, and if the sum of squares for error is obtained by subtracting $q'B_{11}q$ and $k^{-1}Y'X_1X_1'Y$ from $Y'Y$, it can be shown that the sum of squares for error is easily obtained without the inverse (2). It then becomes more practical to compute $Y'Y$, $k^{-1}Y'X_1X_1'Y$, and the error sum of squares directly, and then obtain

$q'B_{11}q$ by subtraction.

If reference is made to the discussion of the X_1, X_2, and N matrices in the third section of Chapter I, and to the layout for the Slipped-Block Design with two blocks in Illustration III, it is evident that $X_1'Y$ is a b x 1 vector whose elements are the block totals, and $X_2'Y$ is a t x 1 vector whose elements are the treatment totals. Furthermore, note that

$$NX_1' = \begin{bmatrix} J_1^s & \phi_1^s \\ J_1^n & J_1^n \\ \phi_1^s & J_1^s \end{bmatrix} \begin{bmatrix} J_s^1 & J_n^1 & \phi_n^1 & \phi_s^1 \\ \phi_s^1 & \phi_n^1 & J_n^1 & J_s^1 \end{bmatrix} = \begin{bmatrix} J_s^s & J_n^s & \phi_n^s & \phi_s^s \\ J_s^n & J_n^n & J_n^n & J_s^n \\ \phi_s^s & \phi_n^s & J_n^s & J_s^s \end{bmatrix}$$

Now suppose that the t x 1 vector $q = X_2'Y - k^{-1}NX_1'Y$ is partitioned into sub-vectors q_1, q_2, and q_3, where q_1 is s x 1, q_2 is n x 1, and q_3 is s x 1. Also partition the bk x 1 or (2s + 2n) x 1 vector Y into sub-vectors V_1, V_2, V_3, and V_4, where V_1 and V_4 are s x 1, and V_2 and V_3 are n x 1. That is, the first s observations in block 1 are the elements of V_1, and the last n observations comprise V_2. The first n observations in block 2 are the elements of V_3, and the last s observations are the elements of V_4. This means

$$J_{s+n}^1 \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad \text{and} \quad J_{s+n}^1 \begin{bmatrix} V_3 \\ V_4 \end{bmatrix}$$

are the block totals.

If it is noted that the elements of V_1 and V_4 are the treatment totals for the first s and the last s treatments, respectively, and the elements of $(V_2 + V_3)$ are the treatment totals for the n intervening treatments, it follows that the elements of these vectors are the elements of X'_2Y. Therefore, the sub-vectors of $q = X_2'Y - k^{-1}NX_1'Y$ are as follows:

$$q_1 = V_1 - k^{-1}J^s_{s+n} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

(3) $$q_2 = (V_2 + V_3) - k^{-1}J^n_{2n+2s} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

$$q_3 = V_4 - k^{-1}J^s_{s+n} \begin{bmatrix} V_3 \\ V_4 \end{bmatrix} .$$

From $(A + t^{-1}J^t_t)^{-1}$ as given in (2), and from the fact that the matrix $B_{11} = (A + t^{-1}J^t_t)^{-1} - t^{-1}J^t_t$ by virtue of Corollary 14.1, it appears that it is necessary to subtract $t^{-1}J^t_t$ from the inverse matrix in order to obtain B_{11} and then $q'B_{11}q$. However, since $J^1_tq = q'J^t_1 = 0$, it is possible to use either $(A + t^{-1}J^t_t)^{-1}$ or B_{11} to obtain the same result. The inverse (2) is already partitioned in the same manner as q, so that by performing the multiplication and making use of the relations $k = s + n$ and $t = 2s + n$, the quantity $q'B_{11}q$ simplifies to

(4) $$V_1'V_1 - \frac{1}{k} V_1'J^s_sV_1 - \frac{2}{k} V_1'J^s_nV_2 + \frac{1}{2} V_2'V_2 + V_2'V_3 + \frac{s-n}{2kn} V_2'J^n_nV_2$$

$$- \frac{1}{n} V_2' J_n^n V_3 + \frac{1}{2} V_3' V_3 + \frac{s-n}{2kn} V_3' J_n^n V_3 - \frac{2}{k} V_3' J_s^n V_4 + V_4' V_4 - \frac{1}{k} V_4' J_s^s V_4.$$

Since $X_1'Y$ has the two block totals,

$$J_{s+n}^1 \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \text{ and } J_{s+n}^1 \begin{bmatrix} V_3 \\ V_4 \end{bmatrix},$$

for its elements, it follows that the uncorrected sum of squares for blocks, $R(\alpha^*) = k^{-1} Y'X_1X_1'Y$, is

(5) $$\frac{1}{k}(V_1'J_s^s V_1 + V_2'J_n^n V_2 + 2V_1'J_n^s V_2 + V_3'J_n^n V_3 + V_4'J_s^s V_4 + 2V_4'J_n^s V_3).$$

Subtracting the expressions in (4) and (5) from $Y'Y$ results in the error term for the analysis of variance. Thus

$Y'Y - k^{-1}Y'X_1X_1'Y - q'B_{11}q$ is equal to

(6) $$\frac{1}{2}V_2'V_2 + \frac{1}{2}V_3'V_3 - V_2'V_3 - \frac{1}{2n} V_2'J_n^n V_2 - \frac{1}{2n} V_3'J_n^n V_3 + \frac{1}{n} V_2'J_n^n V_3 .$$

However, the lemma below shows that this is exactly the expression for interaction in the n x 2 table composed of the vectors V_2 and V_3 .

<u>LEMMA 2.</u> The interaction in the n x 2 table composed of the two n x 1 vectors V_2 and V_3 is as given in (6) above.

<u>Proof:</u> The table and the necessary totals are as shown below:

			Totals
	V_2	V_3	$V_2 + V_3$
Totals	$J_n^1 V_2$	$J_n^1 V_3$	$J_n^1(V_2 + V_3)$

The uncorrected sums of squares are given in (a), (b), and (c).

(a) The total sum of squares for the table is $V_2'V_2 + V_3'V_3$.

(b) The sum of squares for the row totals is $\frac{(V_2' + V_3')(V_2 + V_3)}{2}$.

(c) The sum of squares for the column totals is

$$\frac{(V_2'J_1^n)(J_n^1V_2) + (V_3'J_1^n)(J_n^1V_3)}{n} .$$

(d) The correction factor for all three of the above is

$$\frac{(V_2'J_1^n + V_3'J_1^n)(J_n^1V_2 + J_n^1V_3)}{2n} .$$

If the multiplication indicated in (b), (c), and (d) is performed, it then follows that the interaction in the table is given by subtracting the sum of the expressions in (b) and (c) from the sum of the expressions in (a) and (d). The result is as shown in (6). This completes the proof.

The computation of the analysis of variance thus can be performed rather simply for the case in which there are only two blocks. The total uncorrected sum of squares, $Y'Y$, and the uncorrected sum of squares for blocks, $k^{-1}Y'X_1X_1'Y$, are computed in the ordinary manner. Then the error can be obtained by computing the interaction in the n x 2 table composed of V_2 and V_3. Then the adjusted sum of squares for treatments, $q'B_{11}q$, can be obtained by subtraction. This would complete the analysis of variance table. The computing procedure is illustrated in Example 1. In all the tables which follow in this paper, a double line will be used to separate the observations from totals, treatment numbers, and block numbers.

Example 1.

	Blocks 1	Blocks 2	Totals
Treatments 1	6		6
2	8		8
3	7	3	10
4	4	5	9
5	9	8	17
6		2	2
7		6	6
Totals	34	24	58

The total uncorrected sum of squares is $6^2 + 8^2 + \ldots + 6^2 = 384$.

The uncorrected sum of squares for blocks is $\frac{34^2 + 24^2}{5} = 346\frac{2}{5}$.

The n x 2 table composed of V_2 and V_3 is shown below:

		Totals
7	3	10
4	5	9
9	8	17
20	16	36

The total sum of squares for the table is

$7^2 + 3^2 + \ldots + 8^2 = 244.$

The sum of squares for rows is

$$\frac{10^2 + 9^2 + 17^2}{2} = 235.$$

The sum of squares for columns is

$$\frac{20^2 + 16^2}{3} = 218\frac{2}{3} .$$

The correction factor for the table is $\frac{36^2}{6} = 216$, and therefore the interaction is $244 + 216 - 235 - 218\frac{2}{3} = 6\frac{1}{3}$. The sum of squares for treatments is obtained by subtraction so the analysis of variance is as given in Table V. Note that a check could be obtained by using (4) to compute the sum of squares for treatments.

TABLE V

AOV FOR EXAMPLE 1

Source	d. f.			S. S.
Total		10		384
$R(\alpha, \tau)$		8		$377\frac{2}{3}$
$R(\alpha^*)$	2		$346\frac{2}{5}$	
$R(\tau \mid \alpha^*)$	6		$31\frac{4}{15}$	
Error		2		$6\frac{1}{3}$

Since the sum of squares for the mean is $\frac{58^2}{10} = 336\frac{2}{5}$, the more familiar terminology for the various components of the table could be used to write the following analysis of variance:

TABLE VI

ALTERNATE AOV FOR EXAMPLE 1

Source	d. f.	S. S.
Total	10	384
Mean	1	$336\frac{2}{5}$
Blocks (unadjusted)	1	10
Treatments (adjusted)	6	$31\frac{4}{15}$
Error	2	$6\frac{1}{3}$

In the case of r replications of the two basic blocks, the computing procedure extends without difficulty. The same procedure used for obtaining the computing technique for the two basic blocks will be employed. That is, it will be shown that the error sum of squares as given in Table IV can be obtained more easily than $q'B_{11}q$. Then $q'B_{11}q$ can be obtained by subtraction.

The layout for this case is given in Illustration V. It is necessary to extend the notation used for two blocks to this case. Let the vectors $V_{11}, V_{12}, \ldots, V_{1r}$ be composed of the first s observations in each of the first r blocks and let $V_{21}, V_{22}, \ldots, V_{2r}$ be composed of the last n observations in each of these same blocks. Let $V_{31}, V_{32}, \ldots, V_{3r}$ be composed of the first n observations in each member of the second group of r blocks, and let $V_{41}, V_{42}, \ldots, V_{4r}$ consist of the last s observations in each of these same blocks. Also let $V_j = \sum_{i=1}^{r} V_{ji}$. Note that this definition of V_1, V_2, V_3, and V_4 holds for the previous case in which r = 1.

With the above definitions for the V_j, it is seen that the expressions for q_1, q_2, and q_3 in this case are exactly the same as given in (3) for the case in which r = 1. In view of the result $A_r = rA$ obtained in the third section of this chapter, it follows that $R(\tau | \alpha^*) = q'B_{11}\, q$ is obtained by multiplying the expression in equation (4) by r^{-1} . Furthermore, the uncorrected sum of squares for blocks is readily seen to be

$$(7) \quad \frac{1}{k}\left[(V'_{11}\ V'_{21})\, J^{s+n}_{s+n} \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} + \ldots + (V'_{1r}\ V'_{2r})\, J^{s+n}_{s+n} \begin{bmatrix} V_{1r} \\ V_{2r} \end{bmatrix} + (V'_{31}\ V'_{41})\, J^{s+n}_{s+n} \begin{bmatrix} V_{31} \\ V_{41} \end{bmatrix} + \ldots + (V'_{3r}\ V'_{4r})\, J^{s+n}_{s+n} \begin{bmatrix} V_{3r} \\ V_{4r} \end{bmatrix} \right] .$$

Now consider the four tables shown in Illustration VI. The table in (a) is the layout for the design under consideration. This table can be compared with Illustration V to see the correspondence between the observations and the vectors V_{ji} . In (b) of Illustration VI is the (n + s) x r or k x r table composed of the observations in the first r blocks, and (c) is a similar table for the second group of r blocks. In (d) is an n x 2 table composed of the sums of the vectors

ILLUSTRATION VI

TABLES FOR COMPUTING THE ANALYSIS OF VARIANCE

(a)

V_{11}	V_{12}	. . .	V_{1r}				
V_{21}	V_{22}	. . .	V_{2r}	V_{31}	V_{32}	. . .	V_{3r}
				V_{41}	V_{42}	. . .	V_{4r}

(b)

				Totals
V_{11}	V_{12}	. . .	V_{1r}	V_1
V_{21}	V_{22}	. . .	V_{2r}	V_2
$J^1_{s+n}\begin{bmatrix}V_{11}\\V_{21}\end{bmatrix}$	$J^1_{s+n}\begin{bmatrix}V_{12}\\V_{22}\end{bmatrix}$	. . .	$J^1_{s+n}\begin{bmatrix}V_{1r}\\V_{2r}\end{bmatrix}$	$J^1_{s+n}\begin{bmatrix}V_1\\V_2\end{bmatrix}$

(c)

				Totals
V_{31}	V_{32}	. . .	V_{3r}	V_3
V_{41}	V_{42}	. . .	V_{4r}	V_4
$J^1_{s+n}\begin{bmatrix}V_{31}\\V_{41}\end{bmatrix}$	$J^1_{s+n}\begin{bmatrix}V_{32}\\V_{42}\end{bmatrix}$	. . .	$J^1_{s+n}\begin{bmatrix}V_{3r}\\V_{4r}\end{bmatrix}$	$J^1_{s+n}\begin{bmatrix}V_2\\V_3\end{bmatrix}$

(d)

		Totals
V_2	V_3	$V_2 + V_3$
$J^1_n V_2$	$J^1_n V_3$	$J^1_n [V_2 + V_3]$

V_{2i} and V_{3i} as indicated. In tables (b), (c), and (d), the last row in the table is composed of the totals of the corresponding columns.

If the interaction is computed for the tables in (b), (c), and (d), the following results are obtained:

(b) $$V_{11}'V_{11} + V_{12}'V_{12} + \dots + V_{1r}'V_{1r} + V_{21}'V_{21} + \dots + V_{2r}'V_{2r}$$

$$+ \frac{(V_1' \; V_2')J_{s+n}^{s+n}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}{rk} - \frac{V_1'V_1 + V_2'V_2}{r}$$

$$- \frac{(V_{11}' \; V_{21}')J_{s+n}^{s+n}\begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} + \dots + (V_{1r}' \; V_{2r}')J_{s+n}^{s+n}\begin{bmatrix} V_{1r} \\ V_{2r} \end{bmatrix}}{k}$$

(c) $$V_{31}'V_{31} + V_{32}'V_{32} + \dots + V_{3r}'V_{3r}' + V_{41}'V_{41} + \dots + V_{4r}'V_{4r}$$

$$+ \frac{(V_3' \; V_4')J_{s+n}^{s+n}\begin{bmatrix} V_3 \\ V_4 \end{bmatrix}}{rk} - \frac{V_3'V_3 + V_4'V_4}{r}$$

$$- \frac{(V_{31}' \; V_{41}')J_{s+n}^{s+n}\begin{bmatrix} V_{31} \\ V_{41} \end{bmatrix} + \dots + (V_{3r}' \; V_{4r}')J_{s+n}^{s+n}\begin{bmatrix} V_{3r} \\ V_{4r} \end{bmatrix}}{k}$$

(d) $$\frac{V_2'V_2 + V_3'V_3}{r} + \frac{(V_2'J_1^n + V_3'J_1^n)(J_n^1V_2 + J_n^1V_3)}{2nr}$$

$$- \frac{(V_2' + V_3')(V_2 + V_3)}{2r} - \frac{(V_2'J_1^n)(J_n^1V_2) + (V_3'J_1^n)(J_n^1V_3)}{nr}$$

If the expressions in (b), (c), and (d) above are combined with the expression in (7) and r^{-1} times the expression in (4), the final result is

$$(8)\quad V'_{11}V_{11} + V'_{12}V_{12} + \ldots + V'_{1r}V_{1r} + V'_{21}V_{21} + \ldots + V'_{2r}V_{2r} + V'_{31}V_{31} + \ldots + V'_{3r}V_{3r} + V'_{41}V_{41} + \ldots + V'_{4r}V_{4r} \quad .$$

The expression in (8) is the total uncorrected sum of squares, $Y'Y$, for the layout shown in (a) of Illustration VI, and, in slightly different form, in Illustration V. Thus it has been shown that the sum of the uncorrected sum of squares for blocks, the sum of squares for treatments, and the proposed sum of squares for error is $Y'Y$, where the proposed error is the sum of the expressions in (b), (c), and (d). Therefore the proposed error term must really be the error term as given in Table IV. Since this error term is the sum of the interactions from three two-way tables, it is easily computed, and the computing procedure for the analysis of variance is now apparent. The total uncorrected sum of squares and the uncorrected sum of squares for blocks can be computed in the ordinary manner. The error sum of squares can be computed as the sum of the interactions of the three two-way tables in the manner described, and then the sum of squares for treatments can be obtained by subtraction. The procedure is illustrated in Example 2, which will be given after the computing procedures for some more quantities are derived.

Estimation of $\tau_i - \bar{\tau}$.

In the discussion preceding Theorem 14 in Chapter I, it was determined that $\hat{\tau} = B_{11}q$. Since $J_t^1 q = 0$, it is also true that $\hat{\tau} = (A + t^{-1}J_t^t)^{-1}q$. Therefore, under the restriction that $\Sigma\hat{\tau}_i = 0$, the solution $\hat{\tau}$ to the system of equations $A\hat{\tau} = q$ can be expressed in a form that may lead to a computing procedure by forming the product of $(A + t^{-1}J_t^t)^{-1}$ as given in (2) and q as given in (3). Then, in order to permit r replications of the two basic blocks, this product is multiplied by r^{-1}. The results can be simplified by using the relationships $k = s + n$ and $t = 2s + n$ in order to obtain the components of the vector $\hat{\tau}$ as shown below in (9), (10), and (11).

(9) $\hat{\tau}_1^* = \frac{1}{r}\left[V_1 - \frac{1}{t}J_s^s(V_1 + V_4) - \frac{k}{nt} J_n^s V_2 + \frac{s}{nt} J_n^s V_3 \right]$, where $\hat{\tau}_1^*$ has elements $\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_s$.

(10) $\hat{\tau}_2^* = \frac{1}{r}\left[\frac{1}{2}(V_2 + V_3) - \frac{1}{2t}J_n^n(V_2 + V_3) - \frac{1}{t}J_s^n(V_1 + V_4)\right]$, where $\hat{\tau}_2^*$ has elements $\hat{\tau}_{s+1}, \hat{\tau}_{s+2}, \ldots, \hat{\tau}_{s+n}$.

(11) $\hat{\tau}_3^* = \frac{1}{r}\left[V_4 - \frac{1}{t}J_s^s(V_1 + V_4) - \frac{k}{nt} J_n^s V_3 + \frac{s}{nt} J_n^s V_2 \right]$, where $\hat{\tau}_3^*$ has elements $\hat{\tau}_{s+n+1}, \hat{\tau}_{s+n+2}, \ldots, \hat{\tau}_t$. Note that the symbol $\hat{\tau}_i$ merely indicates a solution to the system $A\hat{\tau} = q$. Under the restriction

$\Sigma \hat{\tau}_i = 0$ used to solve this system, it has been shown that $E(\hat{\tau}_i) = (\tau_i - \bar{\tau}.)$.

Since the J matrices in the above expressions simply add the elements of the corresponding V_j, the computing procedure is not difficult. Let $v_{11}, v_{12}, \ldots, v_{1s}$ denote the sums of the observations on treatments 1, 2, ..., s, respectively. Let $v_{2,s+1}, v_{2,s+2}, \ldots, v_{2,s+n}$ denote the sums of the observations on treatments s + 1 through s + n, respectively, in the first group of r blocks only; and let $v_{3,s+1}, v_{3,s+2}, \ldots, v_{3,s+n}$ denote the sums on the same group of treatments in the second group of r blocks. Also, let $v_{4,s+n+1}, v_{4,s+n+2}, \ldots, v_{4t}$ denote the sums of the observations on the last group of s treatments. It is apparent that the v_{1p}, $v_{2,s+q}$, $v_{3,s+q}$, and $v_{4,s+n+p}$, where $p = 1, 2, \ldots, s$ and $q = 1, 2, \ldots, n$, are the elements of the vectors V_1, V_2, V_3, and V_4, respectively. Let v_1, v_2, v_3, and v_4 denote the sums of the v_{1p}, $v_{2,s+q}$, $v_{3,s+q}$, and $v_{4,s+n+p}$, respectively. Then the equations (9), (10), and (11) can be written in a form more adaptable for computing as follows:

(9*) $\hat{\tau}_i = \frac{1}{r}\left[v_{1i} - \frac{1}{t}(v_1 + v_4) - \frac{k}{nt} v_2 + \frac{s}{nt} v_3 \right]$, for $i = 1, 2, \ldots, s$.

(10*) $\hat{\tau}_i = \frac{1}{r}\left[\frac{1}{2}(v_{2i} + v_{3i}) - \frac{1}{2t}(v_2 + v_3) - \frac{1}{t}(v_1 + v_4)\right]$, for $i = s+1, s+2, \ldots, s+n$.

(11*) $\hat{\tau}_i = \frac{1}{r}\left[v_{4i} - \frac{1}{t}(v_1 + v_4) - \frac{k}{nt} v_3 + \frac{s}{nt} v_4\right]$, for

$i = s+n+1, s+n+2, \ldots, s+2n = t$.

The computing procedure for obtaining $\hat{\tau}_i$ with the preceding formulas will also be illustrated in Example 2.

Confidence Intervals

If reference is made to the section on confidence intervals in Chapter I, it is evident that confidence intervals on linear combinations of the treatments can be computed just as they were for the general two-way classification. The only difference is that the elements b_{ij} of B_{11} in equation (14) of Chapter I could be replaced by the appropriate expression from equation (2) of this chapter. Since this would involve writing several different expressions for the b_{ij}, it is simpler to leave the confidence interval in the more general form of (14). The actual procedure for computing a confidence interval is illustrated in the following section.

An Example To Illustrate Computing Procedures

The following example will illustrate the procedure for computing the analysis of variance, testing the hypothesis that all treatments are equal, finding estimates of the $\tau_i - \bar{\tau}.$, standard errors of estimates of treatment differences, and confidence intervals on linear combinations of the treatments. Note that there are two basic blocks in the example and there are four replicates of each of these, but there is no difficulty caused by thinking of the design as having 8 blocks.

Example 2.

Treatments	Blocks 1	2	3	4	5	6	7	8	Totals
1	3	2	4	7					16
2	7	7	2	4					20
3	8	3	8	2	2	8	9	7	47
4	4	5	9	5	6	4	3	2	38
5	2	6	3	6	5	1	6	3	32
6					9	8	4	2	23
7					6	7	4	3	20
Totals	24	23	26	24	28	28	26	17	196

(1) Computation of the analysis of variance

The following tables and computations are labeled (b), (c), and (d) in order to correspond to (b), (c), and (d) in Illustration V and the results labeled (b), (c), and (d) immediately preceding the illustration. The numbers at the extreme left of each of these tables are treatment numbers, and the last row of each table consists of the totals of the corresponding columns.

(b)

	Blocks 1	2	3	4	Totals	
1	3	2	4	7	16	v_{11}
2	7	7	2	4	20	v_{12}
3	8	3	8	2	21	v_{23}
4	4	5	9	5	23	v_{24}
5	2	6	3	6	17	v_{25}
	24	23	26	24	97	

$$3^2 + 2^2 + \ldots + 6^2 = 569$$

$$\frac{16^2 + \ldots + 17^2}{4} = \frac{1915}{4} = 478\frac{3}{4}$$

$$\frac{24^2 + \ldots + 24^2}{5} = \frac{2357}{5} = 471\frac{2}{5}$$

$$\frac{97^2}{20} = \frac{9409}{20} = 470\frac{9}{20}$$

The interaction in the table is $569 + 470\frac{9}{20} - 478\frac{3}{4} - 471\frac{2}{5} = 89\frac{3}{10}$.

Note that the computations for each table proceed exactly as shown in Example 1.

(c) Blocks

	5	6	7	8	Totals	
3	2	8	9	7	26	v_{33}
4	6	4	3	2	15	v_{34}
5	5	1	6	3	15	v_{35}
6	9	8	4	2	23	v_{46}
7	6	7	4	3	20	v_{47}
	28	28	26	17	99	

$$2^2 + 8^2 + \ldots + 3^2 = 609$$

$$\frac{26^2 + \ldots 20^2}{4} = \frac{2055}{4} = 513\frac{3}{4}$$

$$\frac{28^2 + \ldots + 17^2}{5} = \frac{2533}{5} = 506\frac{3}{5}$$

$$\frac{99^2}{20} = \frac{9801}{20} = 490\frac{1}{20}$$

The interaction in the table is $609 + 490\frac{1}{20} - 513\frac{3}{4} - 506\frac{3}{5} = 78\frac{7}{10}$.

(d) (4) Totals

3	21	26	47
4	23	15	38
5	17	15	32
	61	56	117

The (4) at the top of the table is to indicate that each entry in the table is the sum of four observations. The 21, 23, and 17 are obtained by adding the observations on treatments 3, 4, and 5, respectively, over the first four blocks. The entries 26, 15, and 15 are obtained by adding the observations on the same treatments over the last four blocks. The computation of the interaction is given in the usual manner.

$$\frac{21^2 + 26^2 + \ldots + 15^2}{4} = \frac{2385}{4} = 596\ \frac{1}{4}$$

$$\frac{47^2 + 38^2 + 32^2}{8} = \frac{4677}{8} = 584\ \frac{5}{8}$$

$$\frac{61^2 + 56^2}{12} = \frac{6857}{12} = 571\ \frac{5}{12}$$

$$\frac{117^2}{24} = \frac{13689}{24} = 570\ \frac{3}{8}$$

The interaction in the table is $596\ \frac{1}{4} + 570\ \frac{3}{8} - 584\ \frac{5}{8} - 571\ \frac{5}{12} = 10\ \frac{7}{12}$.

The error is the sum of the interactions for the three tables. Hence the sum of squares for error is $89\ \frac{3}{10} + 78\ \frac{7}{10} + 10\ \frac{7}{12} = 178\ \frac{7}{12}$. The total uncorrected sum of squares is $3^2 + 2^2 + \ldots + 3^2 = 1178$, the total uncorrected sum of squares for blocks is $471\ \frac{2}{5} + 506\ \frac{3}{5} = 978$ from (b) and (c), and the correction factor is $\frac{196^2}{40} = 960\ \frac{2}{5}$. Therefore, the analysis of variance is as given in Table VII.

TABLE VII

AOV FOR EXAMPLE 2

Source	d.f.	S.S.	M.S.
Total	40	1178	
Mean	1	$960\frac{2}{5}$	
Blocks (unadjusted)	7	$17\frac{3}{5}$	
Treatments (adjusted)	6	$21\frac{5}{12}$	$1\frac{65}{72} = 1.90$
Error	26	$178\frac{7}{12}$	$6\frac{271}{312} = 6.87$

Note that the computing procedure for this case is consistent with that used in Example 1 for the case in which $r = 1$. If $r = 1$, the interactions computed for the tables in (b) and (c) are zero, and hence only table (d) would be used. The computation of the interaction for the table in (d) is exactly the same as in Example 1, so that the case in which $r = 1$ is simply a special example of the case now under consideration. The treatment sum of squares in the above table was obtained by subtraction. If a check is desired, r^{-1} times the expression in equation (4) affords a direct computation of the sum of squares for treatments.

(2) Test of H_0: $\tau_1 = \tau_2 = \ldots = \tau_t$

In order to test this hypothesis, the number $\frac{1.90}{6.87} = 0.277$ is compared with 2.47, the tabular value of the F-variate for 6 and 26 degrees of freedom at the 5 per cent level. Since 0.277 is less than 2.47, there is no evidence to reject H_0.

(3) Computation of the standard errors

Since the error mean square is an unbiased estimate of σ^2 by virtue of Theorem 7, the standard errors of the estimates of the treatment differences, $\widehat{\tau_i - \tau_j}$, are obtained by substituting the error mean square for σ^2 in (a), (b), (c), (d), (e), and (f) of the second section of this chapter, and then extracting the square root of the result. Since these expressions were for $r = 1$, it is necessary to multiply them by r^{-1}, which is $\frac{1}{4}$ for this example. Since $n = 3$, $k = 5$, $t = 7$, and $s = 2$, the following results are obtained for the squares of the standard errors:

(a) If $i = 1, 2$, and $j = 1, 2$, $i \neq j$, then the result is $2\hat{\sigma}^2 = 13.74$.

(b) If $i = 1, 2$, and $j = 3, 4, 5$, then the result is $\hat{\sigma}^2 \left(\frac{3}{2} + \frac{1}{2n}\right) = 11.45$.

(c) If $i = 1, 2$, and $j = 6, 7$, then the result is $\hat{\sigma}^2 \left(2 + \frac{2}{n}\right) = 14.88$.

(d) If $i = 3, 4, 5$, and $j = 6, 7$, then the result is the same as obtained in (b).

(e) If $i = 3, 4, 5$, and $j = 3, 4, 5$, $i \neq j$, then the result is $\hat{\sigma}^2 = 6.87$.

(f) If $i = 6, 7$, and $j = 6, 7$, $i \neq j$, then the result is the same as obtained in (a).

The standard errors are the square roots of the results obtained above.

(4) Estimation of $\tau_i - \bar{\tau}$.

If reference is made to the columns of totals in tables (b) and (c) of this example, it is seen that these row totals are the values of the v_{ij}'s needed to substitute in equations (9*), (10*), and (11*) of this chapter in order to estimate $(\tau_i - \bar{\tau}.)$. This is indicated by the v_{ij}'s at the right of these columns. From these values, $v_1 = 16 + 20 = 36$, $v_2 = 21 + 23 + 17 = 61$, $v_3 = 26 + 15 + 15 = 56$, and $v_4 = 23 + 20 = 43$. Substituting these values and the values for k, s, n, t, and r gives the following estimates:

$$\hat{\tau}_1 = \frac{1}{4}[\,16 - \frac{1}{7}(79) - \frac{5}{21}(61) + \frac{2}{21}(56)] = \frac{1}{4}[\,16 - \frac{430}{21}\,] = -\frac{47}{42}$$

$$\hat{\tau}_2 = \frac{1}{4}[\,20 - \frac{430}{21}\,] = -\frac{5}{42}$$

$$\hat{\tau}_3 = \frac{1}{4}[\frac{1}{2}(47) - \frac{1}{14}(117) - \frac{1}{7}(79)] = \frac{1}{4}[\,\frac{47}{2} - \frac{275}{14}\,] = \frac{27}{28}$$

$$\hat{\tau}_4 = \frac{1}{4}[\,\frac{1}{4}(38) - \frac{275}{14}\,] = -\frac{9}{56}$$

$$\hat{\tau}_5 = \frac{1}{4}[\,\frac{1}{2}(32) - \frac{275}{14}\,] = -\frac{51}{56}$$

$$\hat{\tau}_6 = \frac{1}{4}[\,23 - \frac{1}{7}(79) - \frac{5}{21}(56) + \frac{2}{21}(61)] = \frac{1}{4}[\,23 - \frac{395}{21}\,] = \frac{22}{21}$$

$$\hat{\tau}_7 = \frac{1}{4}[\,20 - \frac{395}{21}\,] = \frac{25}{84}$$

Recall now that under the assumption that $\Sigma \tau_i = 0$, which was used to solve the system $A\hat{\tau} = q$, $E(\hat{\tau}_i) = (\tau_i - \bar{\tau}.)$, so that the above results are unbiased estimates of $(\tau_i - \bar{\tau}.)$. The assumption used in solving $A\hat{\tau} = q$ also affords a check of the above computation, since the sum of the above estimates must be zero. If an estimate of a linear combination of the τ_i is desired, say $\Sigma c_i \tau_i$, then $\Sigma c_i \hat{\tau}_i$ is used as an estimate. Also note that $\hat{\tau}'q = q'B_{11}q = R(\tau|\alpha^*)$. Therefore, if the $\hat{\tau}_i$ are computed, this is another means of obtaining the sum of squares for treatments without finding the inverse of $A + t^{-1}J_t^t$.

(5) Confidence Intervals

Suppose that a 95 per cent confidence interval is desired on $\tau_2 - \tau_5$. Reference to equation (14) of Chapter I shows that this means $c_1 = 1$ and $c_2 = -1$, so that $\Sigma c_i = 0$ as required by Theorem 11. The standard error $E_{ms}\sqrt{\Sigma c_i c_j b_{ij}}$ for a treatment difference involving one of the first s treatments and one of the treatments numbered from s + 1 to s + n is the square root of the result given in (b) of the computation of standard errors. Hence, for this case, it is $\sqrt{11.45} = 3.39$. Also

$$\widehat{\tau_2 - \tau_5} = \hat{\tau}_2 - \hat{\tau}_5 = -\frac{5}{42} + \frac{51}{56} = \frac{19}{24} = 0.79,$$

and since the tabular value of the t-variate with 26 degrees of freedom at the 5 per cent level is 2.056, the desired confidence interval is as follows:

$$0.79 - 2.056\sqrt{11.45} \leq \tau_2 - \tau_5 \leq 0.79 + 2.056\sqrt{11.45}$$

$$0.79 - 6.95 \leq \tau_2 - \tau_5 \leq 0.79 + 6.95$$

$$-6.16 \leq \tau_2 - \tau_5 \leq 7.74$$

This completes the example illustrating the computing procedure for the analysis of the case involving r replicates of two basic blocks for $r \geq 1$.

CHAPTER IV

THE SLIPPED-BLOCK DESIGN WITH TWO OR MORE BLOCKS AND OVERLAP OF ONE

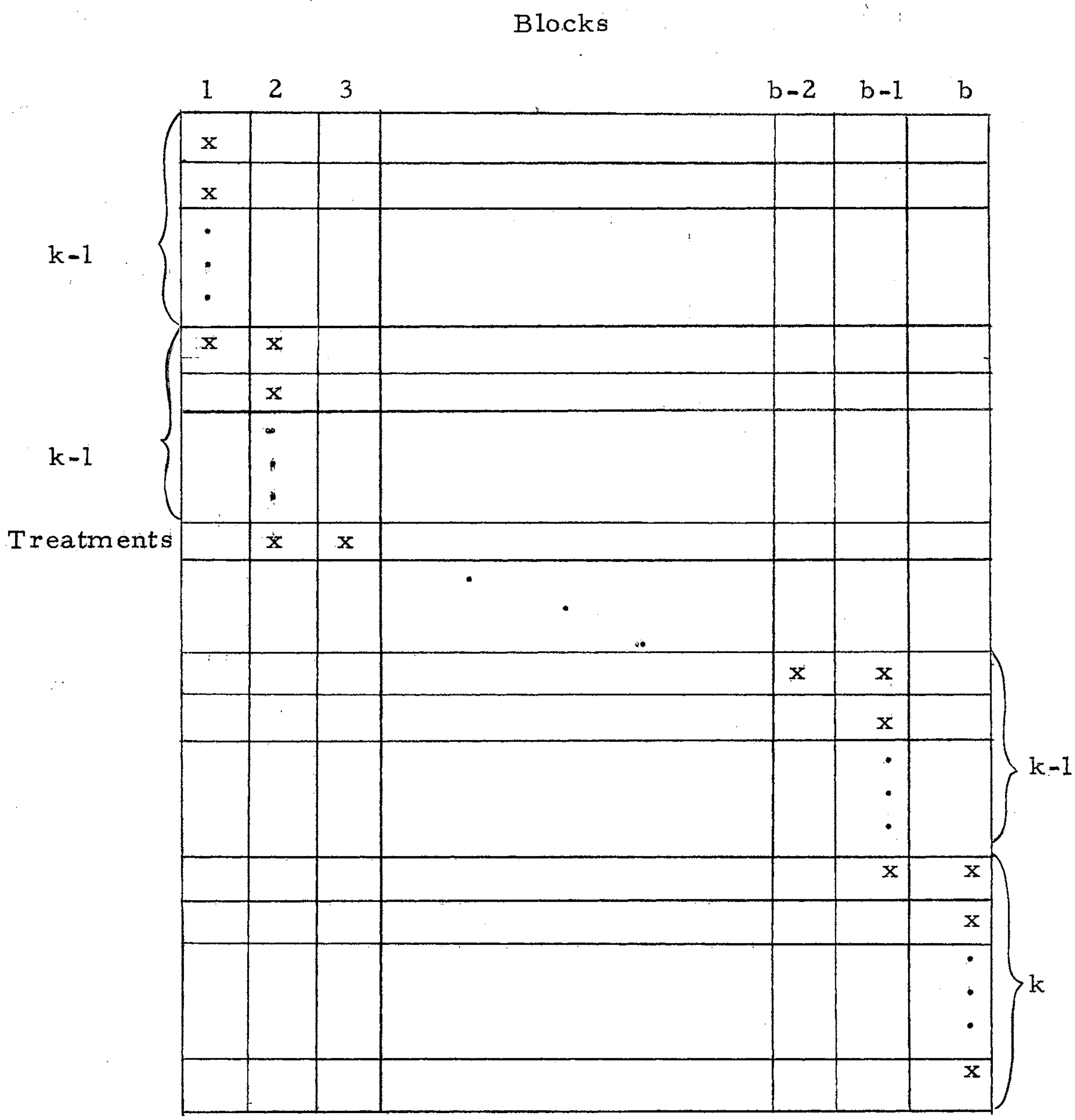
The Derivation of the Matrices A, A**, and $(A^{**})^{-1}$

When the General Slipped-Block Design was defined in Chapter II, it was stated that no simpler analysis of it would be given in this paper other than the analysis available by treating it as a general two-way classification. It was also stated in Chapter II that all the Slipped-Block Designs considered in this paper would have $k_j = k$ observations in each block. In addition, it was assumed that for any two adjacent blocks, the slip would always be s, and the overlap would always be n.

In Chapter III, an analysis of the Slipped-Block Design for two basic blocks and r replications of the two basic blocks was given. This analysis is valid for any values of s, n, k, and t. Naturally, it is desirable to extend the analysis to designs having more than two basic blocks. In this chapter, the designs under consideration will have two or more basic blocks, but the overlap will always be one. By referring to properties (7) and (8) in Chapter II or observing the statistical layout of this design as shown in Illustration VII, it is evident that the relations $k = s + 1$ and $t = (b - 1)(k - 1) + k = bk - b + 1$ must hold for

ILLUSTRATION VII

LAYOUT FOR THE SLIPPED-BLOCK DESIGN WITH OVERLAP ONE



this case.

The procedure for determining the matrix $A = X_2'X_2 - k^{-1}NN'$ is the same as described and carried out in detail in Chapter III. If this procedure is used again, the final result for A is as shown in Illustration VIII. Note that if A is partitioned in the manner indicated in this illustration, then all except the last row and column of submatrices have dimensions $(k - 1) \times (k - 1)$, and the submatrices in the last row and column have dimensions $k \times k$.

Now suppose that the last row and column of A are deleted. In view of the results given in Chapter I, this is equivalent to using the vector $\lambda' = (0, 0, \ldots, 0, 1)$ to impose the condition $\hat{\tau}_t = 0$ on the system $A\hat{\tau} = q$. If the last component of the vectors $\hat{\tau}$ and q is eliminated, then the system of $t - 1$ equations in $t - 1$ unknowns, $A^{**}\hat{\tau}^{**} = q^{**}$, is obtained. The elimination of the last row and column of A to obtain A^{**} means A^{**} consists of b^2 submatrices, and each of these submatrices has dimensions $(k - 1) \times (k - 1)$. If these submatrices are denoted by A^{**}_{ij}, where the i and j denote the row and column of submatrices in which A^{**}_{ij} appears, then A^{**} can be represented as shown below:

(1) (a) $A^{**}_{11} = k^{-1}[\, kI - J^{k-1}_{k-1} \,]$

(b) $$A^{**}_{ii} = k^{-1}\begin{bmatrix} (2k - 2)J^1_1 & -J^1_{k-2} \\ -J^{k-2}_1 & kI - J^{k-2}_{k-2} \end{bmatrix} \quad \text{for } i = 2, 3, \ldots, b$$

(c) $A^{**}_{i,i+1} = k^{-1}[\, -J^{k-1}_1 \quad \phi^{k-1}_{k-2} \,]$ for $i = 1, 2, \ldots, b - 1$

ILLUSTRATION VIII

THE A MATRIX FOR THE DESIGN IN ILLUSTRATION VII

$\frac{1}{k}$ ×

k-1 -1 -1 . . . -1 -1 k-1 -1 . . . -1 (k-1) x . . (k-1) -1 -1 -1 . . . -1 k-1	-1 0 0. . . 0 -1 0 0. . . 0 . . (k-1)x(k-1) -1 0 0. . . 0		Ø	
-1 -1. . . -1 0 0. . . 0 . . (k-1)x(k-1) 0 0. . . 0	2k-2 -1 -1. . .-1 -1 k-1 -1. . . -1 (k-1)x(k-1) . . -1 -1 -1. . . -1 k-1	-1 -1 . . . -1		
	-1 -1. . . -1	2k-2 . . .		
				-1 0 0. . . 0 -1 0 0. . . 0 . . (k-1) x k -1 0 0. . . 0
Ø			-1 -1. . . -1 0 0. . . 0 0 0. . . 0 . . k x (k-1) 0 0. . . 0	2k-2 -1 -1. . . -1 -1 k-1 -1. . . -1 . . k x k -1 -1 -1 . . . -1 k-1

(d) $A^{**}_{i,i-1} = k^{-1} \begin{bmatrix} -J^{1}_{k-1} \\ \phi^{k-2}_{k-1} \end{bmatrix}$ for $i = 2, 3, \ldots, b$

(e) $A^{**}_{ij} = \phi^{k-1}_{k-1}$, except for the preceding cases.

In order to determine $(A^{**})^{-1}$, it was assumed that the inverse was of the same general form as A^{**}. Observation of several examples indicated that the $(k - 1) \times (k - 1)$ submatrices of $(A^{**})^{-1}$ could be represented as shown in equation (2) which follows. Note that throughout this discussion, b denotes the number of basic blocks in the design, and i and j denote the row and column of submatrices, not the individual rows and columns of A^{**} or $(A^{**})^{-1}$.

(2) (a) $(A^{**}_{ii})^{-1} = [\, I + (2b - 2i + 1) J^{k-1}_{k-1} \,]$

(b) $(A^{**}_{ij})^{-1} = [\, (2b - 2j + 2) J^{k-1}_{1} \quad (2b - 2j + 1) J^{k-1}_{k-2} \,]$ for $i < j$

(c) $(A^{**}_{ij})^{-1} = \begin{bmatrix} (2b - 2i + 2) J^{1}_{k-2} \\ (2b - 2i + 1) J^{k-2}_{k-1} \end{bmatrix} = [\, (A^{**}_{ji})^{-1}]'$ for $i > j$

In order to clarify the notation used for A^{**} and $(A^{**})^{-1}$, three examples are given in Illustration IX. The same basic block size, $k = 4$, was used for all three examples so that it would be possible to see that $(A^{**})^{-1}$ for $b = 2$ is a submatrix of $(A^{**})^{-1}$ for $b = 3$, and that the latter matrix is itself a submatrix of $(A^{**})^{-1}$ for $b = 4$. Note that the scalar

ILLUSTRATION IX

SOME EXAMPLES OF A** AND $(A^{**})^{-1}$

A** and $(A^{**})^{-1}$

b = 2
k = 4
t = 7

$$
A^{**} = \left[\begin{array}{rrr|rrr}
3 & -1 & -1 & -1 & 0 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 \\
-1 & -1 & 3 & -1 & 0 & 0 \\
\hline
-1 & -1 & -1 & 6 & -1 & -1 \\
0 & 0 & 0 & -1 & 3 & -1 \\
0 & 0 & 0 & -1 & -1 & 3
\end{array}\right]
\qquad
(A^{**})^{-1} = \left[\begin{array}{rrr|rrr}
4 & 3 & 3 & 2 & 1 & 1 \\
3 & 4 & 3 & 2 & 1 & 1 \\
3 & 3 & 4 & 2 & 1 & 1 \\
\hline
2 & 2 & 2 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 2
\end{array}\right]
$$

$$
\left[\begin{array}{rrr|rrr|rrr}
3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & -1 & -1 & 6 & -1 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 3 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & -1 & -1 & 6 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 3
\end{array}\right]
\qquad
\left[\begin{array}{rrr|rrr|rrr}
6 & 5 & 5 & 4 & 3 & 3 & 2 & 1 & 1 \\
5 & 6 & 5 & 4 & 3 & 3 & 2 & 1 & 1 \\
5 & 5 & 6 & 4 & 3 & 3 & 2 & 1 & 1 \\
\hline
4 & 4 & 4 & 4 & 3 & 3 & 2 & 1 & 1 \\
3 & 3 & 3 & 3 & 4 & 3 & 2 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 & 4 & 2 & 1 & 1 \\
\hline
2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2
\end{array}\right]
$$

$$
\left[\begin{array}{rrr|rrr|rrr|rrr}
3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & -1 & -1 & 6 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & -1 & -1 & 6 & -1 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 6 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 3
\end{array}\right]
\qquad
\left[\begin{array}{rrr|rrr|rrr|rrr}
8 & 7 & 7 & 6 & 5 & 5 & 4 & 3 & 3 & 2 & 1 & 1 \\
7 & 8 & 7 & 6 & 5 & 5 & 4 & 3 & 3 & 2 & 1 & 1 \\
7 & 7 & 8 & 6 & 5 & 5 & 4 & 3 & 3 & 2 & 1 & 1 \\
\hline
6 & 6 & 6 & 6 & 5 & 5 & 4 & 3 & 3 & 2 & 1 & 1 \\
5 & 5 & 5 & 5 & 6 & 5 & 4 & 3 & 3 & 2 & 1 & 1 \\
5 & 5 & 5 & 5 & 5 & 6 & 4 & 3 & 3 & 2 & 1 & 1 \\
\hline
4 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 2 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 3 & 2 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 2 & 1 & 1 \\
\hline
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2
\end{array}\right]
$$

There is a coefficient of $\frac{1}{k} = \frac{1}{4}$ on each example of the A** matrix. If the design included r replications of each of the b basic blocks, then each element of A** would be multiplied by r and each element of $(A^{**})^{-1}$ would be multiplied by r^{-1}.

element in the first row and first column is 2b in each case. With this element as the starting point, it is quite simple to write out $(A^{**})^{-1}$ for any block size, any number of basic blocks, and any number of replications of the b basic blocks. This will prove to be of some use in later developments.

It will now be shown that $(A^{**})^{-1}$ as given in (2) is the inverse of A^{**}. The procedure will be to show that the product of the two matrices is $I_{bk-b} = I_{t-1}$. The multiplication is divided into four cases as follows:

<u>Case I.</u> The multiplication of the first column of $(A^{**})^{-1}$ by the first row of A^{**}.

(a) The multiplication of the first column of $(A^{**})^{-1}$ by the first row of A^{**}.

This product is $\sum_{j=1}^{b} (A^{**}_{ij})(A^{**}_{ji})^{-1}$, but since all the A^{**}_{ij} are null except the first two, the result is

$$k^{-1}[kI - J^{k-1}_{k-1}][I + (2b - 1)\, J^{k-1}_{k-1}] + k^{-1}[\,-J^{k-1}_{1} \quad \phi^{k-1}_{k-2}] \begin{bmatrix} (2b - 2)\, J^{1}_{k-1} \\ (2b - 3)\, J^{k-2}_{k-1} \end{bmatrix},$$

and this product simplifies to

$$k^{-1}[kI + k(2b - 1)\, J^{k-1}_{k-1} - J^{k-1}_{k-1} - (k - 1)(2b - 1)\, J^{k-1}_{k-1}] + k^{-1}[-(2b - 2)\, J^{k-1}_{k-1}].$$

The combined coefficient of J^{k-1}_{k-1} is zero, so that the above expression becomes $k^{-1}[\,kI\,] = I_{k-1}$, which is the desired result.

(b) The multiplication of the second column of $(A^{**})^{-1}$ by the first row of A**.

In the remainder of this proof, it is best to partition the submatrices to a greater degree. The partitioning will always be as shown in the matrices below. In the future, no dimensions will appear on the J matrices. In this case, the product is

$$k^{-1}\begin{bmatrix}(k-1)\,J_1^1 & -J_{k-2}^1\\ -J_1^{k-2} & kI - J_{k-2}^{k-2}\end{bmatrix}\begin{bmatrix}(2b-2)\,J_1^1 & (2b-3)\,J_{k-2}^1\\ (2b-2)J_1^{k-2} & (2b-3)\,J_{k-2}^{k-2}\end{bmatrix}$$

$$+\,k^{-1}\begin{bmatrix}-J_1^1 & \phi_{k-2}^1\\ -J_1^{k-2} & \phi_{k-2}^{k-2}\end{bmatrix}\begin{bmatrix}(2b-2)\,J_1^1 & (2b-3)\,J_{k-2}^1\\ (2b-3)\,J_1^{k-2} & I+(2b-3)\,J_{k-2}^{k-2}\end{bmatrix}.$$

The coefficients of J_1^1, J_{k-2}^1, J_1^{k-2}, and J_{k-2}^{k-2} are as listed in (1), (2), (3), and (4) below, except that the factor k^{-1} is omitted. Throughout this proof, the coefficients of these same matrices will be given in this order. Any other coefficients listed will be identified as they occur.

(1) $(k-1)(2b-2) - (k-2)(2b-2) - (2b-2) = 0.$

(2) This is the same as (1) with $2b-2$ replaced by $2b-3$.

(3) $-(2b-2) + k(2b-2) - (k-2)(2b-2) - (2b-2) = 0.$

(4) This is the same as (3) with $2b-2$ replaced by $2b-3$.

(c) The multiplication of the q^{th} column of $(A^{**})^{-1}$ by the first row of A^{**}, where $q = 3, 4, \ldots, b-1$.

Here the product is

$$k^{-1}\begin{bmatrix}(k-1)J & -J \\ -J & kI-J\end{bmatrix}\begin{bmatrix}(2b-2q+2)J & (2b-2q+1)J \\ (2b-2q+2)J & (2b-2q+1)J\end{bmatrix}$$

$$+k^{-1}\begin{bmatrix}-J & \phi \\ -J & \phi\end{bmatrix}\begin{bmatrix}(2b-2q+2)J & (2b-2q+1)J \\ (2b-2q+2)J & (2b-2q+1)J\end{bmatrix},$$

and the coefficients are as follows:

(1) $(k-1)(2b-2q+2)-(k-2)(2b-2q+2)-(2b-2q+2)=0$

(2) This is the same as (1) with $2b-2q+1$ replacing $2b-2q+2$.

(3) $-(2b-2q+2)+k(2b-2q+2)-(k-2)(2b-2q+2)-(2b-2q+2)=0.$

(4) This is the same as (3) with $2b-2q+1$ replacing $2b-2q+2$.

Therefore the multiplication of each column of $(A^{**})^{-1}$ by the first row of A^{**} does give the first $k-1$ rows of $I_{bk-b}=I_{t-1}$. This is the desired result.

<u>Case II.</u> The multiplication of the p^{th} column of $(A^{**})^{-1}$ by the p^{th} row of A^{**}, where $p = 2, 3, \ldots, b-1$.

Here the product is $\sum_{q=1}^{b}(A^{**}_{pq})(A^{**}_{qp})^{-1}$, but since all but three of the submatrices of A^{**} are null, the only values that q assumes are $p-1$, p, and $p+1$. Hence the product is

$$k^{-1}\begin{bmatrix}-J & -J \\ \phi & \phi\end{bmatrix}\begin{bmatrix}(2b-2p+2)J & (2b-2p+1)J \\ (2b-2p+2)J & (2b-2p+1)J\end{bmatrix}$$

$$+ k^{-1} \begin{bmatrix} (2k-2)J & -J \\ -J & kI-J \end{bmatrix} \begin{bmatrix} (2b-2p+2)J & (2b-2p+1)J \\ (2b-2p+1)J & I+(2b-2p+1)J \end{bmatrix}$$

$$+ k^{-1} \begin{bmatrix} -J & \phi \\ -J & \phi \end{bmatrix} \begin{bmatrix} (2b-2p)J & (2b-2p)J \\ (2b-2p-1)J & (2b-2p-1)J \end{bmatrix},$$

and the coefficients are shown below with the coefficient of I_{k-2} given in (5).

(1) $-(2b - 2p + 2) - (k - 2)(2b - 2p + 2) + (2k - 2)(2b - 2p + 2)$

$-(k - 2)(2b - 2p + 1) - (2b - 2p) = k$

(2) $-(2b - 2p + 1) - (k - 2)(2b - 2p + 1) + (2k - 2)(2b - 2p + 1) - 1$

$-(k - 2)(2b - 2p + 1) - (2b - 2p) = 0.$

(3) $-(2b - 2p + 2) + k(2b - 2p + 1) - (k - 2)(2b - 2p + 1) - (2b - 2p) = 0$

(4) $-(2b - 2p + 1) + k(2b - 2p + 1) - 1 - (k - 2)(2b - 2p + 1) - (2b - 2p) = 0$

(5) k

Since the coefficient k^{-1} is being omitted in the above, if the combination of the results in (1) and (5) is multiplied by k^{-1}, the result is I_{k-1}. This is the desired result.

Case III. The multiplication of column q of $(A^{**})^{-1}$ by row p of A^{**}, where $q = 2, 3, \ldots, b - 1$, and $p = 2, 3, \ldots, b - 1$, but $p \neq q$.

(a) The multiplication in which the first non-null submatrix in row p of A^{**} matches with a submatrix on the diagonal of $(A^{**})^{-1}$, i.e., $p=q+1$.

This product is of the form

$$k^{-1}\begin{bmatrix} -J & -J \\ \phi & \phi \end{bmatrix}\begin{bmatrix} (2b-2q+2)J & (2b-2q+1)J \\ (2b-2q+1)J & I+(2b-2q+1)J \end{bmatrix}$$

$$+k^{-1}\begin{bmatrix} (2k-2)J & -J \\ -J & kI-J \end{bmatrix}\begin{bmatrix} [2b-2(q+1)+2]J & [2b-2(q+1)+2]J \\ [2b-2(q+1)+1]J & [2b-2(q+1)+1]J \end{bmatrix}$$

$$+k^{-1}\begin{bmatrix} -J & \phi \\ -J & \phi \end{bmatrix}\begin{bmatrix} [2b-2(q+2)+2]J & [2b-2(q+2)+2]J \\ [2b-2(q+2)+1]J & [2b-2(q+2)+1]J \end{bmatrix},$$

and the coefficients are as follows:

(1) $-(2b-2q+2)-(k-2)(2b-2q+1)+(2k-2)(2b-2q)$

$-(k-2)(2b-2q-1)-(2b-2q-2)=0$

(2) $-(2b-2q+1)-1-(k-2)(2b-2q+1)+(2k-2)(2b-2q)$

$-(k-2)(2b-2q-1)-(2b-2q-2)=0$

(3) $-(2b-2q)+k(2b-2q-1)-(k-2)(2b-2q-1)-(2b-2q-2)=0$

(4) This is the same as (3).

(b) The multiplication in which the second non-null submatrix in row p of A** matches with a submatrix on the diagonal of $(A^{**})^{-1}$. In this case $p = q$, and this is Case II.

(c) The multiplication in which the third non-null submatrix in row p of A** matches with a submatrix on the diagonal of $(A^{**})^{-1}$, i.e., $p = q - 1$.

This product has the form

$$
k^{-1}\begin{bmatrix} -J & -J \\ \phi & \phi \end{bmatrix}\begin{bmatrix} (2b-2q+2)J & (2b-2q+1)J \\ (2b-2q+2)J & (2b-2q+1)J \end{bmatrix}
$$

$$
+k^{-1}\begin{bmatrix} (2k-2)J & -J \\ -J & kI-J \end{bmatrix}\begin{bmatrix} (2b-2q+2)J & (2b-2q+1)J \\ (2b-2q+2)J & (2b-2q+1)J \end{bmatrix}
$$

$$
+k^{-1}\begin{bmatrix} -J & \phi \\ -J & \phi \end{bmatrix}\begin{bmatrix} (2b-2q+2)J & (2b-2q+1)J \\ (2b-2q+1)J & I+(2b-2q+1)J \end{bmatrix},
$$

and the coefficients are as follows:

(1) $-(2b-2q+2)-(k-2)(2b-2q+2)+(2k-2)(2b-2q+2)$

$-(k-2)(2b-2q+2)-(2b-2q+2)=0$

(2) This is the same as (1) if $2b-2q+2$ is replaced by $2b-2q+1$.

(3) $-(2b-2q+2)+k(2b-2q+2)-(k-2)(2b-2q+2)-(2b-2q+2)=0.$

(4) This is the same as (3) except that $2b-2q+1$ replaces $2b-2q+2$.

(d) The multiplication in which all three non-null submatrices in row p of A** match with submatrices above the diagonal of $(A^{**})^{-1}$, i.e., $p \leq q-2$.

In this case, all three submatrices of $(A^{**})^{-1}$ are the same so that it is possible to add the non-null matrices of A** to obtain the product

$$k^{-1}\begin{bmatrix}(2k-4)J & -2J\\ -2J & kI-J\end{bmatrix}\begin{bmatrix}(2b-2q+2)J & (2b-2q+1)J\\ (2b-2q+2)J & (2b-2q+1)J\end{bmatrix},$$

and the coefficients are as follows:

(1) $(2k-4)(2b-2q+2)-2(k-2)(2b-2q+2)=0$

(2) This is (1) with $2b-2q+1$ replacing $2b-2q+2$.

(3) $-2(2b-2q+2)+k(2b-2q+2)-(k-2)(2b-2q+2)=0$

(4) This is (3) with $2b-2q+2$ replaced by $2b-2q+1$.

(e) The multiplication in which all three non-null submatrices on row p of A** match with submatrices below the diagonal of $(A^{**})^{-1}$, i.e., $p \geq q+2$. Let m be the column of A** which contains the first non-null matrix.

Then the product has the form

$$k^{-1}\begin{bmatrix}-J & -J\\ \phi & \phi\end{bmatrix}\begin{bmatrix}(2b-2m+2)J & (2b-2m+2)J\\ (2b-2m+1)J & (2b-2m+1)J\end{bmatrix}$$

$$+\,k^{-1}\begin{bmatrix}(2k-2)J & -J\\ -J & kI-J\end{bmatrix}\begin{bmatrix}[2b-2(m+1)+2]J & [2b-2(m+1)+2]J\\ [2b-2(m+1)+1]J & [2b-2(m+1)+1]J\end{bmatrix}$$

$$+\,k^{-1}\begin{bmatrix}-J & \phi\\ -J & \phi\end{bmatrix}\begin{bmatrix}[2b-2(m+2)+2]J & [2b-2(m+2)+2]J\\ [2b-2(m+2)+1]J & [2b-2(m+2)+1]J\end{bmatrix},$$

and the coefficients are as follows:

(1) $-(2b - 2m + 2) - (k - 2)(2b - 2m + 1) + (2k - 2)(2b - 2m)$

$-(k - 2)(2b - 2m - 1) - (2b - 2m - 2) = 0$

(2) This is the same as (1).

(3) $-(2b - 2m) + k(2b - 2m - 1) - (k - 2)(2b - 2m - 1) - (2b - 2m - 2) = 0$

(4) This is the same as (3).

<u>Case IV</u>. The multiplication of each column of $(A^{**})^{-1}$ by the last row of A^{**}.

(a) The multiplication of the q^{th} column of $(A^{**})^{-1}$ by the last row of A^{**}, where $q = 1, 2, \ldots, b - 2$.

Since all the submatrices of A^{**} for this case are null except those in the last two columns, these non-null submatrices of A^{**} will always match with the last two submatrices in each column of $(A^{**})^{-1}$. For the first $b - 2$ columns of $(A^{**})^{-1}$, the submatrix in row $b - 1$ is the same in every column, and so is the submatrix in each column of row b. From part (c) of equation (2), or from Illustration IX, it is seen that $2b - 2(b - 1) + 2 = 4$ and $2b - 2b + 2 = 2$. Therefore the product is of the form

$$k^{-1}\begin{bmatrix} -J & -J \\ \phi & \phi \end{bmatrix}\begin{bmatrix} 4J & 4J \\ 3J & 3J \end{bmatrix} + k^{-1}\begin{bmatrix} (2k - 2)J & -J \\ -J & kI - J \end{bmatrix}\begin{bmatrix} 2 & 2J \\ J & J \end{bmatrix},$$

and the coefficients are as follows:

(1) $- 4 - 3(k - 2) + 2(2k - 2) - (k - 2) = 0$

(2) This is the same as (1).

(3) $- 2 + k - (k - 2) = 0$

(4) This is the same as (3).

(b) The multiplication of column b - 1 of $(A^{**})^{-1}$ by the last row of A^{**}.

Since $2b - 2(b - 1) + 2 = 4$ and $2b - 2b + 2 = 2$ as in (a), the product is of the form

$$k^{-1}\begin{bmatrix} -J & -J \\ \phi & \phi \end{bmatrix}\begin{bmatrix} 4J & 3J \\ 3J & I + 3J \end{bmatrix} + k^{-1}\begin{bmatrix} (2k-2)J & -J \\ -J & kI - J \end{bmatrix}\begin{bmatrix} 2J & 2J \\ J & J \end{bmatrix},$$

and the coefficients are as follows:

(1) $-4 - 3(k - 2) + 2(2k - 2) - (k - 2) = 0$

(2) $-3 - 1 - 3(k - 2) + 2(2k - 2) - (k - 2) = 0$

(3) $-2 + k - (k - 2) = 0$

(4) $-2 + k - (k - 2) = 0$

(c) The multiplication of the last column of $(A^{**})^{-1}$ by the last row of A^{**}.

The product is of the form

$$k^{-1}\begin{bmatrix} -J & -J \\ \phi & \phi \end{bmatrix}\begin{bmatrix} 2J & J \\ 2J & J \end{bmatrix} + k^{-1}\begin{bmatrix} (2k-2)J & -J \\ -J & kI - J \end{bmatrix}\begin{bmatrix} 2J & J \\ J & I + J \end{bmatrix},$$

and the coefficients are shown below. The coefficient of I_{k-2} is given in (5).

(1) $-2 - 2(k - 2) + 2(2k - 2) - (k - 2) = k$

(2) $-1 - (k - 2) + (2k - 2) - 1 - (k - 2) = 0$

(3) $-2 + k - (k - 2) = 0$

(4) $-1 + k - 1 - (k - 2) = 0$

(5) k

Since the factor k^{-1} has been omitted in each of the above, combining the results of (1) and (5) with this factor and the fact that the coefficients in (2), (3), and (4) are all zero gives the product I_{k-1}, which is the desired result.

Consideration of part (a) of Case I, Case II, and part (c) of Case IV shows that when column p of $(A^{**})^{-1}$ is multiplied by row p of A^{**}, the $(k - 1) \times (k - 1)$ identity, I_{k-1}, is the result. The combination of all remaining cases and parts of cases show that when column q of $(A^{**})^{-1}$ is multiplied by row p of A^{**}, where $p \neq q$, then the product is the $(k - 1) \times (k - 1)$ null matrix, ϕ_{k-1}. This shows that the matrix $(A^{**})^{-1}$ defined by equation (2) is the inverse of the matrix A^{**} defined by equation (1). This completes the proof that $(A^{**})(A^{**})^{-1} = I_{bk-b} = I_{t-1}$.

The Variances of Estimates of Treatment Differences

From Theorem 15, it follows that the solution $\hat{\tau}$ to the system of equations $A\hat{\tau} = q$ is such that $E(\hat{\tau}) = \tau_i - \tau_t$ if the restriction $\hat{\tau}_t = 0$ is imposed on the system in order to obtain a unique solution. A theorem similar to Theorem 16 shows that if $B_{11}^{**} = (A^{**})^{-1}$, then Cov $(\hat{\tau}) = \sigma^2 B_{11}^{**}$, so that σ^2 times the diagonal elements of B_{11}^{**} gives the variances of $\widehat{\tau_i - \tau_t}$, and σ^2 times the off-diagonal elements gives the covariances of $(\widehat{\tau_i - \tau_t})$ and $(\widehat{\tau_j - \tau_t})$, where $i \neq j$.

The above results will now be used to find the variances of the estimates of treatment differences. Note first of all that the b - 1 treatments with numbers k, 2k - 1, 3k - 2, ... , [(b - 1)(k - 1) + 1] all appear twice since they appear in both the basic block numbered a and the basic block numbered a + 1, respectively, for a = 1, 2, ... , b - 1. All other treatments appear in only one basic block. Also note that those treatments that appear twice correspond to the first element in a submatrix on the diagonal of $(A^{**})^{-1}$, except that no treatment which appears in two blocks corresponds to the first element of $(A_{11}^{**})^{-1}$. There are b of these $(k - 1) \times (k - 1)$ submatrices in B_{11}^{**}, and there are b basic blocks in the design. With these remarks and those of the preceding paragraph, the variances of the estimates of treatment differences can be determined by considering two cases:

<u>Case I.</u> Two treatments appearing in the same basic block.

Suppose the two treatments are in block number p. Throughout the remainder of this section, block will refer to basic block. Then

$\sigma^2(A^{**}_{pp})^{-1}$ is the covariance matrix for treatments numbered from $p(k-1)-(k-2)$ through $p(k-1)+1$, which is a total of $k-1$ treatments. As noted above, the remaining treatment in block p also appears in block $p+1$, and this treatment corresponds to the first diagonal element in $(A^{**}_{p+1,p+1})^{-1}$. Hence, for the first $k-1$ treatments in block p, if $i \neq j$,

$$\mathrm{Var}\,(\widehat{\tau_i - \tau_j}) = \sigma^2[\,2(2b-2p+2) - 2(2b-2p+1)\,] = 2\sigma^2.$$

For treatment number k in block p, i.e., $j = pk - p + 1$,

$$\mathrm{Var}\,(\widehat{\tau_i - \tau_j}) = \sigma^2[\,2\,\{2b - 2(p+1) + 2\} - 2\,\{2b - 2(p+1) + 1\}] = 2\sigma^2.$$

Therefore it makes no difference in the variance if a treatment appears in two blocks, for the variance of $(\widehat{\tau_i - \tau_j})$ is $2\sigma^2$ for any two treatments which appear in the same block.

Case II. Two treatments appearing in different basic blocks.

Suppose τ_i is in block p and τ_j is in block q, where $p < q$, and consider the following:

(a) If τ_i is one of the first $k-1$ treatments in block p, then $\mathrm{Var}\,(\widehat{\tau_i - \tau_t}) = \sigma^2(2b-2p+2)$. If τ_j is the first treatment in block q, then it is also the last treatment in block $q-1$. Then $\mathrm{Var}\,(\widehat{\tau_j - \tau_t}) = \sigma^2(2b-2q+2)$, $\mathrm{Cov}\,[\,\widehat{\tau_i - \tau_t}, \widehat{\tau_j - \tau_t}\,] = \sigma^2(2b-2q+2)$, and therefore

$$\mathrm{Var}\,(\widehat{\tau_i - \tau_j}) = \sigma^2[\,2b-2p+2+2b-2q+2-2(2b-2q+2)\,]$$

$$= 2\sigma^2(q-p).$$

(b) If τ_i is one of the first k - 1 treatments in block p, and τ_j is one of the middle k - 2 treatments in block q, then

$$\text{Var}\,(\widehat{\tau_i - \tau_j}) = \sigma^2 [\, 2b - 2p + 2 + 2b - 2q + 2 - 2(2b - 2q + 1)]$$

$$= 2\sigma^2 (q - p + 1).$$

(c) If τ_i is one of the first k - 1 treatments in block p, and τ_j is the last treatment in block q, and hence the first treatment in block q + 1, then

$$\text{Var}\,(\widehat{\tau_i - \tau_j}) = \sigma^2 [\, 2b - 2p + 2 + 2b - 2(q+1) + 2 - 2 \{2b - 2(q+1) + 2\}\,]$$

$$= 2\sigma^2 (q - p + 1).$$

(d) If τ_i is the last treatment in block p, it is the first treatment in block p + 1, and therefore

$$\text{Var}\,(\widehat{\tau_i - \tau_t}) = \sigma^2 [\, 2b - 2(p + 1) + 2\,] = 2\sigma^2 (b - p).$$

Therefore if τ_j is as was considered in (a), (b), and (c), the results are as given below. These results are labeled to correspond with the preceding (a), (b), and (c).

(a) $$\text{Var}\,(\widehat{\tau_i - \tau_j}) = \sigma^2 [\, 2b - 2p + 2b - 2q + 2 - 2(2b - 2q + 2)]$$

$$= 2\sigma^2 (q - p - 1).$$

(b) $$\text{Var}\,(\widehat{\tau_i - \tau_j}) = \sigma^2 [\, 2b - 2p + 2b - 2q + 2 - 2(2b - 2q + 1]$$

$$= 2\sigma^2 (q - p).$$

(c) $\text{Var}\,(\widehat{\tau_i - \tau_j}) = \sigma^2 [\, 2b - 2p + 2b - 2(q + 1) + 2 - 2 \left\{ 2b - 2(q+1) + 2 \right\}]$

$$= 2\sigma^2 (q - p).$$

The results of Case II indicate that if τ_i appears in block p, but not in block p + 1, and if τ_j appears in block q, but not in block q - 1, then $\text{Var}\,(\widehat{\tau_i - \tau_j}) = 2\sigma^2 (q - p + 1)$. Since Case I shows that $\text{Var}\,(\widehat{\tau_i - \tau_j})$ for two treatments in the same block is $2\sigma^2$, the only values that $\text{Var}\,(\widehat{\tau_i - \tau_j})$ assumes are $2\sigma^2$, $4\sigma^2$, ..., $2b\sigma^2$, where b is the number of basic blocks in the design, and there are no replications of the basic blocks.

The Extension of Results

Since there are no degrees of freedom for error in the design given in Illustration VII, it is useful to note that if each of the b basic blocks is replicated r times, then each element of the matrix A is multiplied by r, which is the same result obtained for the design in Chapter III. Hence, if the matrices are denoted by A_r and A_r^{**} when $r > 1$, it follows that $A_r = rA$, $A_r^{**} = rA^{**}$, and $(A_r^{**})^{-1} = r^{-1}(A^{**})^{-1}$. This results in the variances obtained in the preceding section being divided by r if $r > 1$. This result was noted in Illustration IX. A similar procedure to that used in the third section of Chapter III would show that $E(\hat{\tau})$ does not change when each basic block is replicated r times.

Test of the Hypothesis $\tau_1 = \tau_2 = \ldots = \tau_t$

The analysis of variance table for testing this hypothesis in the case now being considered is exactly the same as shown in Table IV except that $R(\tau|\alpha) = q^{**\prime}B_{11}^{**}q^{**}$ instead of $q'B_{11}q$. All the distributional properties remain the same also, so that the test of the hypothesis is carried out in exactly the same manner described previously.

A Computing Procedure for the Analysis of Variance

It was mentioned in the preceding section that the only difference in the analysis of variance table for this case and the one given in Table IV was that $R(\tau|\alpha) = q^{**\prime}B_{11}^{**}\ q^{**}$. For any specific example, it is not as difficult to obtain B_{11}^{**} as it is to obtain the matrix B_{11} used in Chapter III. However, it is desirable to derive a computing procedure for the analysis of variance which does not require the inverse or the q^{**} vector, for there may be instances in which a test of the hypothesis that all the treatments are equal is the only result desired. Since it happens that a simple computing procedure for the analysis of variance can be developed, it will be given. However, if estimates of the treatments are desired, it is easier to obtain B_{11}^{**} and q^{**} to compute such estimates.

The derivation of a general expression for $q^{**\prime}B_{11}^{**}q^{**}$ corresponding to the expression for $q'B_{11}q$ in equation (3) of Chapter III is extremely long. Therefore a different approach in obtaining a computing procedure for the analysis of variance is preferred. This computing procedure will involve treating each of the r replicates of the b basic blocks as a separate randomized block. The sum of squares for error and for treatments in each of these randomized blocks can then be computed in the usual manner. The sums of squares for treatments from these b randomized blocks will be pooled in order to obtain the sum of squares for treatments in the Slipped-Block Design. The same procedure gives the sum of squares for error. If the r replicates of the basic blocks are considered as being br regular blocks, then the sum of squares for the br regular blocks and the total sum of squares can be obtained in the regular way. It will be shown that such a procedure for computing the analysis of variance leads to an F-test for testing the hypothesis that all the treatments are equal, just as the one given in Table IV does. Suppose that r replications of each basic block is considered as a randomized block with r blocks and k treatments. Then there are b of these randomized blocks in the design. It has already been noted that the b - 1 treatments with numbers k, 2k - 1, 3k - 2, ... , [(b - 1)(k - 1) + 1], appear in two basic blocks, and all other treatments appear in only one basic block. Suppose that y_{mnj} represents one of the r observations containing treatment m in the j^{th} randomized block, where $m = j(k - 1) - (k - 2), \ldots, j(k - 1) + 1$; $n = 1, 2, \ldots, r$; and $j = 1, 2, \ldots, b$. Note that m ranges over k - 1 consecutive integers. Then let $y_{m.j}$ be the sum of the r observations on treatment m in the j^{th} randomized block, and let $\bar{y}_{..j}$ denote the mean of the observations in the j^{th} randomized block. Then the sum of squares for treatments in the j^{th} randomized block is

$$(3) \qquad r\left[\sum_{m=c}^{d} (\bar{y}_{m.j} - \bar{y}_{..j})^2\right],$$

where $c = j(k - 1) - (k - 2)$, and $d = j(k - 1) + 1$.

It is known that if each of the b expressions of the form shown in (3) is divided by σ_j^2, then the resulting expression has a non-central chi-square distribution with $k - 1$ degrees of freedom and non-centrality parameter λ_j. The symbols σ_j^2 and λ_j denote the variance and non-centrality parameter associated with the j^{th} randomized block.

If τ_{mj} is the constant for treatment m in the j^{th} randomized block, then the expected mean square for treatments in that block is

$$(4) \qquad \sigma_j^2 + \frac{r}{k - 1}\left[\sum_{m=c}^{d} (\tau_{mj} - \bar{\tau}_{.j})^2\right],$$

and from Theorem 10, it follows that the non-centrality parameter λ_j is

$$(5) \qquad r\left[\frac{\sum_{m=c}^{d} (\tau_{mj} - \bar{\tau}_{.j})^2}{2\sigma_j^2}\right].$$

Since (5) is the sum of non-negative quantities, it follows that λ_j is zero if and only if the k treatments in each randomized block are equal. Therefore the distribution of the treatment sum of squares becomes a central chi-square if and only if the k treatments in any individual randomized block are equal.

Since the τ_i in the model for the general two-way classification are fixed, the individual chi-square distributions for the b randomized blocks are independent even though there is one treatment common to each adjacent pair of randomized blocks. Hence the distribution of the sum of these b chi-square variates is either a central or non-central chi-square distribution with b(k - 1) degrees of freedom.

The error for the entire design is the sum of the errors for the b randomized blocks, and therefore

$$v = r \sum_{j=1}^{b} \left[\frac{\sum_{m=c}^{d} (\bar{y}_{m.j} - \bar{y}_{..j})^2}{\sigma_j^2} \right] \tag{6}$$

is distributed as a non-central chi-square with b(k - 1) degrees of freedom and non-centrality parameter λ. It is necessary to show that $\lambda = 0$ if and only if $\tau_1 = \tau_2 = \ldots = \tau_t$. If it is assumed that $\sigma_j^2 = \sigma^2$ for all j, then the sum of squares for error, when divided by σ^2, has a central chi-square distribution.

Since the proposed sum of squares for treatments in the design is obtained by pooling the sums of squares for treatments in the b independent randomized blocks, the expected mean square for treatments is the sum of the expected values of the individual components. Applying Theorem 10 once more means that the non-centrality λ will involve, apart from multiplicative constants, the sum of terms of the form $(\tau_{mj} - \bar{\tau}_{.j})^2$. Since these quantities are all non-negative, the sum of any number of them is zero if and only if each individual term is zero. Hence λ is zero if and only if $\tau_1 = \tau_2 = \ldots = \tau_t$. Therefore, the analysis of variance computed in the manner described provides an F-test for the hypothesis being considered. The computing procedure and test will be illustrated in Example 3, which appears after some more computing procedures are developed.

Estimation of $\tau_i - \tau_t$

It was pointed out at the beginning of the second section of this chapter that $\hat{\tau} = B_{11}^{**}q^{**}$, and $E(\hat{\tau}_i) = \tau_i - \tau_t$, for the case under consideration in this chapter. In this section, a computing procedure for estimating $\tau_i - \tau_t$ will be obtained. The procedure will require finding $B_{11}^{**} = (A^{**})^{-1}$, but since this is easily done for any particular example of the design now being considered, the computing procedure is not difficult. The computing procedure will be valid for both $r = 1$ and $r > 1$. If the examples of $(A^{**})^{-1} = B_{11}^{**}$ given in Illustration IX and $(A_{ij}^{**})^{-1}$ as given in equation (2) are considered, it becomes apparent that when r replications of another basic block of size k is added to a design with r replications of b basic blocks of the same size, the b^2 submatrices of $(A^{**})^{-1}$ for the design with b basic blocks are exactly the same as the b x b array of (k - 1) x (k - 1) submatrices in the lower right-hand portion of $(A^{**})^{-1}$ for the design with b + 1 blocks.

Note also from Illustration IX and equation (2) that the scalar element in the first row and column of $(A^{**})^{-1}$ is always 2b, if the factor r^{-1} is written as a coefficient of the entire matrix $(A^{**})^{-1}$, and b is the number of basic blocks in the design. Since $(b + 1)^2 = b^2 + 2b + 1$, there are 2b + 1 additional submatrices in the inverse for b + 1 basic blocks as compared to the inverse for b basic blocks. However, since

$$(A^{**}_{11})^{-1} = [I + (2b - 2 + 1) J^{k-1}_{k-1}] = [I + (2b - 1) J^{k-1}_{k-1}] ,$$

for the inverse with b blocks, equation (2) and the examples in Illustration IX indicate that for b + 1 basic blocks, the following results hold:

$$(A^{**}_{11})^{-1} = [I + \{2(b + 1) - 2 + 1\} J^{k-1}_{k-1}] = [I + (2b + 1) J^{k-1}_{k-1}]$$

$$(A^{**}_{12})^{-1} = [2 \{(b + 1) - 1\} J^{k-1}_{1} \quad (2 \{(b + 1) - 1\} - 1) J^{k-1}_{k-2}]$$

$$= [2b J^{k-1}_{1} \quad (2b - 1) J^{k-1}_{k-2}]$$

$$(A^{**}_{12})^{-1} = [(A^{**}_{12})^{-1}]'$$

$(A^{**}_{1j})^{-1}$ for b + 1 basic blocks = $(A^{**}_{1j})^{-1}$ for b basic blocks, and

$(A^{**}_{j1})^{-1}$ for b + 1 basic blocks = $(A^{**}_{j1})^{-1}$ for b basic blocks if $j > 2$.

Since the derivation of a general expression for $\hat{\tau}$ in terms of $(A^{**})^{-1}$, such as was derived and listed in equations (9), (10), and (11) of Chapter III for the design considered in that chapter, would involve a very long proof, the procedure for computing $\hat{\tau}$ for the case now being considered will just be illustrated by an example. In view of the general pattern of $(A^{**})^{-1}$ that has been indicated, it should then be apparent that the computing procedure would hold in general for any size basic block, any number of basic blocks, and any number of replications of the b basic blocks.

The vector q is given by $X_2'Y - k^{-1} NX_1'Y$ for all the Slipped-Block Designs under consideration in this paper, and reference to the fifth section of Chapter III indicates that $X_2'Y$ is a vector in which the i^{th} component is the total of all observations receiving treatment i. Also $NX_1'Y$ is a vector in which the i^{th} component is the sum of all blocks containing treatment i. The elimination of the last component of q to obtain q** does not alter this fact. Therefore the computation of any component of the vector q** is straightforward. The procedure for obtaining q** and $\hat{\tau} = B_{11}^{**} q^{**}$ is illustrated in Example 3. Note also that $\hat{\tau}'q = q'B_{11}q = R(\tau|\alpha)$ just as for the case in Chapter III, since $\hat{\tau}_t = 0$.

Confidence Intervals

If reference is made to the section on confidence intervals in Chapter I, it is evident that confidence intervals on linear combinations of the τ_i can be computed just as they were for the general two-way classification. The only difference is that the elements b_{ij}^{**} of B_{11}^{**} in equation (15) of Chapter I could be replaced by the appropriate expression for b^{**}_{ij} obtained from equation (2) of this chapter. Since this would involve writing a confidence interval for each of several different cases, it is simpler to leave the confidence interval in the more general form of (15). The actual procedure for computing a confidence interval is illustrated in Example 3.

An Example To Illustrate the Computing Procedure

The example which follows illustrates the computing procedure for obtaining the analysis of variance, testing the hypothesis that all the treatments are equal, estimating treatments, computing standard errors of the estimates of treatment differences, and determining confidence intervals.

Example 3.

	Basic Blocks	1		2		3		
		1	2	3	4	5	6	Totals
Treatments	1	3	4					7
	2	5	2					7
	3	8	9	8	7			32
	4			2	8			10
	5			3	6	4	8	21
	6					2	7	9
	7					9	6	15
Totals		16	15	13	21	15	21	101

(1) Computation of the analysis of variance

If the two replications of the three basic blocks are considered as six individual or regular blocks as indicated by the numbering in the table, then the sum of squares for blocks is computed in the usual way. Of course, the total sum of squares is also computed in the usual way. Therefore, the following results are readily obtained:

(a) The total uncorrected sum of squares is $3^2 + 5^2 + \ldots + 6^2 = 675$.

(b) The sum of squares for the mean is $\frac{101^2}{18} = 566\ \frac{13}{18}$.

(c) The sum of squares for the individual blocks, corrected for the mean, is $\frac{16^2 + 15^2 + \ldots + 21^2}{3} - \frac{101^2}{18} = 18\ \frac{17}{18}$.

However, in order to compute the sum of squares for treatments and error, the design is considered as three independent randomized blocks. The treatment numbers and individual block numbers are retained in the tables below in order to help clarify the procedure.

	1	2	Totals
1	3	4	7
2	5	2	7
3	8	9	17
	16	15	31

The total uncorrected sum of squares for the table is $3^2 + 5^2 + \ldots + 9^2 = 199$.

The sum of squares for the row totals is

$$\frac{7^2 + 7^2 + 17^2}{2} = 193\ \frac{1}{2} .$$

The sum of squares for the column totals is $\frac{16^2 + 15^2}{3} = 160\ \frac{1}{3}$.

The correction factor for the table is $\frac{31^2}{6} = 160\ \frac{1}{6}$.

The interaction for the table, which is the error sum of squares for a randomized block design, is $199 + 160\ \frac{1}{6} - 193\ \frac{1}{2} - 160\ \frac{1}{3} = 5\ \frac{1}{3}$. If the corrected sum of squares for treatments is desired, it is $193\ \frac{1}{2} - 160\ \frac{1}{6} = 33\ \frac{1}{3}$.

	3	4	Totals
3	8	7	15
4	2	8	10
5	3	6	9
	13	21	34

The same procedure used before results in the sums of squares as shown below:

Error sum of squares = $12\ \frac{1}{3}$.

Treatment sum of squares = $10\ \frac{1}{3}$

	5	6	Totals
5	4	8	12
6	2	7	9
7	9	6	15
	15	21	36

The same procedure used before results in the sums of squares as shown below:

Error sum of squares = 19

Treatment sum of squares = 9

If the sums of squares for treatments in the three tables is pooled, the result is $52\ \frac{2}{3}$, and the same procedure gives $36\ \frac{2}{3}$ as the error sum of squares for the Slipped-Block Design. Combining these results with those obtained in (a), (b), and (c) results in the analysis of variance given in the following table:

TABLE VIII

AOV FOR EXAMPLE 3

Source	d.f.	S.S.	M.S.
Total	18	675	
Mean	1	$566\frac{13}{18}$	
Blocks (unadjusted)	5	$18\frac{17}{18}$	
Treatments (adjusted)	6	$52\frac{2}{3}$	$8\frac{7}{9} = 8.78$
Error	6	$36\frac{2}{3}$	$6\frac{1}{9} = 6.11$

(2) Test of $H_0: \tau_1 = \tau_2 = \ldots = \tau_t$

In order to test this hypothesis, the number $\frac{8.78}{6.11} = 1.44$ is compared with 4.28, the value of the F-variate with 6 and 6 degrees of freedom. Since 1.44 is less than 4.28, there is no evidence to reject H_0.

(3) Computation of the standard errors

Since the error mean square, given in Table VIII, 6.11, is an unbiased estimate of σ^2 by virtue of Theorem 7, the standard errors of the treatment differences are obtained by substituting this number for σ^2 in the expressions worked out in Cases I and II of the second section of this chapter, and then extracting the square root of the result. Since the results of Cases I and II were for r = 1, it is necessary to divide by r = 2 for this particular example. Since there are three basic blocks, the results of Cases I and II give the results shown in Table IX for the estimates of the variances of $(\widehat{\tau_i - \tau_j})$, where i and j assume the values indicated in the table. The standard errors are the square roots of the entries in the table.

TABLE IX

ESTIMATES OF VARIANCES OF TREATMENT DIFFERENCES

i \ j	2	3	4	5	6	7
1	6.11	6.11	12.22	12.22	18.33	18.33
2		6.11	12.22	12.22	18.33	18.33
3			6.11	6.11	12.22	12.22
4				6.11	12.22	12.22
5					6.11	6.11
6						6.11

(4) Estimation of $\tau_i - \tau_t$

If reference is made to Illustration IX and equations (1) and (2) of this chapter, it is easy to write A** and $(A^{**})^{-1}$ for Example 3. Since k = 3, b = 3, r = 2, and t = 7, the results are as follows:

$$A^{**} = \frac{1}{3}\begin{bmatrix} 4 & -2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 \\ -2 & -2 & 8 & -2 & -2 & 0 \\ 0 & 0 & -2 & 4 & -2 & 0 \\ 0 & 0 & -2 & -2 & 8 & -2 \\ 0 & 0 & 0 & 0 & -2 & 4 \end{bmatrix} \qquad (A^{**})^{-1} = \frac{1}{2}\begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 6 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 4 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

Note that the two replications of the three basic blocks results in each element of A^{**} for $r = 1$ being multiplied by two, and hence each element of $(A^{**})^{-1}$ for $r = 1$ is divided by two.

Since a component q_i of the vector $q = X_2'Y - k^{-1}NX_1'Y$ is computed by subtracting k^{-1} times the total of all basic blocks containing treatment i from the total of all observations receiving treatment i, the following results are obtained for the q_i:

$$q_1 = 7 - \frac{1}{3}(31) = -\frac{10}{3}$$

$$q_2 = 7 - \frac{1}{3}(31) = -\frac{10}{3}$$

$$q_3 = 32 - \frac{1}{3}(31 + 34) = \frac{31}{3}$$

$$q_4 = 10 - \frac{1}{3}(34) = -\frac{4}{3}$$

$$q_5 = 21 - \frac{1}{3}(34 + 36) = -\frac{7}{3}$$

$$q_6 = 9 - \frac{1}{3}(36) = -3$$

$$q_7 = 15 - \frac{1}{3}(36) = 3$$

As a partial check, the sum of the q_i must be zero, and it is seen that this is the case. However, since the last row and column of A were deleted in order to obtain A**, which is equivalent to setting $\hat{\tau}_7 = 0$ in this example, there is no need to compute q_7 except as a check. The result desired in this section is $\hat{\tau} = B^{**}_{11}\, q^{**}$, where $\hat{\tau}$ again indicates a solution to the system $A\hat{\tau} = q$. If the product $B^{**}_{11} q^{**}$ is formed, the results are as follows:

$$\hat{\tau}_1 = -\frac{7}{2} \qquad \hat{\tau}_2 = -\frac{7}{2} \qquad \hat{\tau}_3 = \frac{3}{2}$$

$$\hat{\tau}_4 = -1 \qquad \hat{\tau}_5 = -\frac{3}{2} \qquad \hat{\tau}_6 = -3$$

It is possible to arrive at several schemes for simplifying the multiplication involved in finding the product $B^{**}_{11} q^{**}$ because of the pattern of B^{**}_{11}. Such procedures will not be discussed in this paper.

Note also that $\hat{\tau}'q = \hat{\tau}^{**'} q^{**} = 52\ 2/3 = R(\tau_r | \alpha)$, since $\hat{\tau}_7 = 0$.

(5) Confidence intervals

Suppose that a 95 per cent confidence interval is desired on $\tau_3 - \tau_6$. Reference to equation (15) of Chapter I shows that this means $c_1 = 1$ and $c_2 = -1$, so that $\Sigma c_i = 0$ as required by Theorem 11. The standard error $\sqrt{E_{ms}\ \Sigma\, c_i c_j b^{**}_{ij}}$ for $\widehat{(\tau_3 - \tau_6)}$ is $\sqrt{12.22} = 3.50$ from Table IX. From the estimates of the τ_i, $\widehat{(\tau_3 - \tau_6)} = \hat{\tau}_3 - \hat{\tau}_6 = \frac{3}{2} - (-3) = 4.50$, and since the tabular value of the t-variate with 6 degrees of freedom at the 5 per cent level is 2.447, the desired confidence interval is as follows:

$$4.50 - 2.447 \sqrt{12.22} \leq \tau_3 - \tau_6 \leq 4.50 + 2.447 \sqrt{12.22}$$

$$4.50 - 8.56 \leq \tau_3 - \tau_6 \leq 4.50 + 8.56$$

$$-4.06 \leq \tau_3 - \tau_6 \leq 13.06$$

This completes the example illustrating the computing procedure for the analysis of the case involving r replicates of b basic blocks and an overlap of one.

CHAPTER V

SUMMARY OF RESULTS

Under the assumptions given in Chapter II, the Slipped-Block Design has been analyzed for the case of two basic blocks and r replications of two basic blocks in which the overlap between the two basic blocks is greater than or equal to one. An analysis has also been given for a design with two or more basic blocks if the overlap is one. In both of these cases, the theory leading to an analysis of variance, testing of hypotheses, estimating treatments, finding standard errors of treatment differences, and determining confidence intervals was developed. A computing procedure for obtaining each of these quantities was developed, and examples were worked out in order to illustrate the computing procedure. The block size and number of treatments could be as desired by the experimenter in both cases.

An analysis has not been worked out for the case in which the number of blocks is greater than two when the overlap is greater than one. Neither has any particular application of the Slipped-Block Designs to actual experimentation been given in this thesis, but it is believed that they may prove useful in survey-type experiments in which some information on a large number of treatments is desired, but only a small number of observations on each treatment is available.

BIBLIOGRAPHY

(1) Graybill, F. A. An Introduction to Linear Statistical Models, Volume I. New York: McGraw-Hill Book Company, Inc., 1961.

VITA

William Edward Timon, Jr.

Candidate for the Degree of

Doctor of Philosophy

Thesis: THE SLIPPED-BLOCK DESIGN

Major Field: Mathematical Statistics

Biographical:

Personal Data: Born at Natchitoches, Louisiana, January 13, 1924, the son of William E. and Arvena Ola Timon.

Education: Attended grade school in Natchitoches, Louisiana; graduated from Natchitoches High School in 1940; received the Bachelor of Science degree from Northwestern State College of Louisiana, with a major in Mathematics, in January, 1950; received the Master of Science degree from Tulane University, with a major in Mathematics, in September, 1951; completed requirements for the Doctor of Philosophy degree in May, 1962.

Professional experience: Entered the United States Army in 1943; was a graduate assistant at Tulane University 1950-51; worked as a mathematician and programmer at Esso Standard Oil Company 1951-53; taught as an Instructor of Mathematics at Louisiana State University 1951-54; was an Assistant Professor of Mathematics at the University of Southwestern Louisiana 1956-57; at Northwestern State College of Louisiana 1954-56, and from 1957 to the present time.