ANALYSIS OF PLATE-BEAM STRUCTURES

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PREFACE

The flexibility approach to the structural analysis of systems of thin plates and elastic beams is presented in this dissertation. Deformations of plate edges and supporting beams are developed in the form of Fourier series which have coefficients in terms of redundant forces and moments. Exact expressions are reduced to a form that allows term by term solutions; and compatibility between plate and beam elements is obtained through the use of "Edge-Deflection" and "Edge-Slope" equations.

This research is the outgrowth of ideas expressed by Professor Jan J. Tuma in the summer of 1961. At that time, Professor Tuma suggested that the method of flexibilities used in the analysis of frames could be extended to the analysis of structural systems of plates and beams.

In completing the final phase of his graduate study, the writer wishes to express his sincere appreciation to the following individuals and organizations:

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NOMENCLATURE

a . . . . Length of Plate.

$a_{11} \cdots a_{ij} \cdots a_{kk}$

$\bar{a}_{11} \cdots \bar{a}_{ij} \cdots \bar{a}_{kk}$ . . . Elements of Matrices $\left[A_m^I\right]$, $\left[A_m^{II}\right]$ Respectively.

b . . . . Width of Plate.

$b_{11} \cdots b_{ij} \cdots b_{kk}$ . . . . Elements of an Array.

c_m . . . . A Constant, $\dfrac{(3+\mu)\text{sh}\,\beta_m a}{\beta_m a(1-\mu)}$.

$e_{mn}^{1.)}$, $e_{mn}^{2.)}$ . . . . . . Fourier Series Coefficients.

$f_{mr}^{1.)}$, $f_{mr}^{2.)}$ . . . . . . Fourier Series Coefficients.

g_{mr} . . . . Fourier Series Coefficient.

h . . . . Thickness of Plate.

j . . . . Summation Index.

k . . . . Number of Terms Used in the Series,

$\sum_n S_n^I \sin \alpha_n x$ and $\sum_n S_n^{II} \sin \alpha_n x$

k_n . . . . A Constant, $\dfrac{(3+\mu)\text{sh}\,\alpha_n b}{\alpha_n b(1-\mu)}$.

m, n . . . . Summation Indices.

p . . . . Intensity of Normal Load Per Unit Area.

p_o . . . . Maximum Intensity of One-Directional Load Variation.

q_m . . . . Coefficient of Series for q(y).

q_{mr}	Fourier Series Coefficient.
$q(y)$	Applied Loading on Edge Beams.
r	Summation Index.
$t(y)$	Applied Torque Variation.
t_m	Coefficient of Series for t(y).
u, v	Coordinates of Concentrated Load.
u_1	u - a.
w	Deflection of Plate.
w_{11}, w_{12}, w_1, w_o, $w_{10} \cdots w_L$	Deflection Surfaces.
x_1	x - a.
x, y, z	Coordinate Axes.
$\left[A_m^I\right]$, $\left[A_m^{II}\right]$, $\left[A_m\right]$	Coefficient Matrices.
A_n, B_n, C_n, D_n, A_m, B_m, C_m, D_m, C_1, C_2, C_3, C_4	Constants of Integration.
$\left[B\right]$	A Diagonal Matrix.
BR_{yzm}	Coefficient of Series for Wide Beam Reaction Due to Loads.
Bz_i	Deflection Due to Loads of Beam Supporting Plate's ith Edge.
$B\psi_i$	Twist Due to Loads of Beam Supporting Plate's ith Edge.
C	Eccentricity of Edge Forces.
D	Flexural Rigidity of Plate.
E	Young's Modulus of Elasticity.
E_m	Fourier Series Coefficient.
$\left[E_1\right]$, $\left[E_2\right]$	Matrices with Elements $e_{mn}^{1.)}$ and $e_{mn}^{2.)}$, Respectively.

$E_1(\beta_m x)$, $E_2(\beta_m x)$	Reaction Functions.
$[F_1]$, $[F_2]$	Matrices with Elements $f_{mr}^{1.)}$ and $f_{mr}^{2.)}$, Respectively.
$F_1(\beta_m x)$, $F_2(\beta_m x)$	Reaction Functions.
$[G]$	Matrices with Elements g_{mr}.
G	Modulus of Rigidity.
$G(\beta_m x)$	Reaction Function.
H_m	Coefficient of Series for $B\psi_i$.
$[I]$	Identity Matrix.
I_i	Moment of Inertia of Beam Supporting ith Edge of Plate.
J_i	Torsional Constant of Beam Supporting ith Edge of Plate.
M_{ABx}, M_{BAx}, M_{ABy}, M_{BAy}	End Moments of Edge Beam, AB.
M_x, M_y	Plate Bending Moments Per Unit Length.
M'_{xp}, M''_{xp}, M'_{yp}, M''_{yp} . .	Edge Moments Per Unit Length of pth Plate.
$(M'_{xp})_m$, $(M''_{xp})_m$, $(M'_{yp})_m$, $(M''_{yp})_m$ . . .	Coefficients of Series for Edge Moments.
$[M'_{xm} g_m]$, $[M'_{xm}(-1)^m g_m]$	Reactive Force Matrices Due to M'_x.
M_{xy}, M_{yx}	Plate Twisting Moments Per Unit Length.
P	Concentrated Load.
P_m	Coefficient of One-Directional Load Function Series.
P_{mj}	Coefficient of Two-Directional Load Function Series.

$\left[Q\right]$ . . . . Matrix with Elements q_{mr} .

$Q(\beta_m x)$ . . . . Reaction Function.

R, R_L . . . . Corner Reactions.

R_{xz}, R_{yz} . . . . Plate Reactive Forces Per Unit Length.

R'_{xzp}, R''_{xzp}

R'_{yzp}, R''_{yzp} . . . . . . Edge Forces Per Unit Length of pth Plate.

$(R'_{xzp})_m, (R''_{xzp})_m$,

$(R'_{yzp})_m, (R''_{yzp})_m$ . . . Coefficients of Series for Edge Forces.

$\left[R'_{xzm} f_m^{1.)}\right]$,

$\left[R'_{xzm} (-1)^m f_m^{1.)}\right]$ . . . . Reactive Force Matrices Due to R'_{xz} .

$\left[S\right]$ . . . . Column Vector of Coefficients S_n^I and S_n^{II}.

$S_n^I, S_n^{II}, T_m, U_m$ . . . . Coefficients of Series for Arbitrary Edge Forces.

V_{xz}, V_{yz} . . . . Plate Shearing Forces Per Unit Length.

X, Y . . . . Functions in Product Solution to $\nabla^4 w=0$.

$Z(\alpha_n y), Z(\beta_m x)$ . . . . . Deflection Functions.

$\alpha_n, \alpha_j, \alpha_r$ . . . . $\frac{n\pi}{a}, \frac{j\pi}{a}, \frac{r\pi}{a}$, Respectively.

β . . . . A Constant.

β_m, β_r . . . . $\frac{m\pi}{b}, \frac{r\pi}{b}$, Respectively.

$\gamma_{mn}^I, \gamma_{mn}^{II}$ . . . . Solution Coefficients for Constants S_n^I, S_n^{II}.

$\left[\gamma^I\right], \left[\gamma^{II}\right]$ . . . . Matrices with Elements γ_{mn}^I and γ_{mn}^{II}, Respectively.

$\zeta_{mn}^I, \zeta_{mn}^{II}$ . . . . Solution Coefficients for Constants S_n^I, S_n^{II} .

$\left[\zeta^I\right], \left[\zeta^{II}\right]$ . . . . Matrices with Elements ζ_{mn}^I and ζ_{mn}^{II}, Respectively.

$(\eta_{AB})_m$ . . . . Component of Twist Flexibility of Edge Beam, AB.

θ_{pi} . . . . Slope of Edge i of p th Plate.

$(\theta_i^L)_m$ . . . . Component of Angular Load Function at i.

$(\theta_{ij}^R)_m$, $(\theta_{ij}^M)_m$ . . . . . Component of Angular Flexibilities at i Due to Forces and Moments at j, Respectively.

λ_{mn}^I, λ_{mn}^{II} . . . . Solution Coefficients for Constants S_n^I , S_n^{II} .

$\left[\lambda^I\right]$, $\left[\lambda^{II}\right]$ . . . . Matrices with Elements λ_{mn}^I and λ_{mn}^{II} , Respectively.

μ . . . . Poisson's Ratio.

ξ_{mn}^I, ξ_{mn}^{II} . . . . Solution Coefficients for Constants S_n^I, S_n^{II}.

ρ_m^u . . . . A Constant, $1+\beta_m a \text{ cth } \beta_m a - \beta_m u \text{ cth } \beta_m u$.

ρ_m^{u1} . . . . A Constant, $1+\beta_m a \text{ cth } \beta_m a - \beta_m u_1 \text{ cth } \beta_m u_1$.

σ_x , σ_y . . . . Normal Stresses.

τ_{xy}, τ_{yx} . . . . Shearing Stresses.

ϕ_{mn}^{o}, $\phi_{mn}^{1.)}$,

$\phi_{mn}^{2.)}$, $\phi_{mn}^{3.)}$,

$\phi_{mn}^{4.)}$, $\phi_{mn}^{5.)}$ . . . . . Fourier Series Coefficients.

$(\chi_{AB})_m$, $(\chi_{BA})_m$ . . . . Components of Linear Flexibilities of Edge Beam, AB.

ψ_i . . . . Twist of Beam Supporting Plate's ith Edge.

$(\psi_i)_A$ . . . . Twist of Beam AB at A.

Δ_{oo}, Δ_{ao}, Δ_{ob}, Δ_{ab} . . . Corner Displacements.

Δ_{pi} . . . . Deflection of Edge i of p th Plate.

Δ_{Az}, Δ_{Bz} . . . . Displacements of Supports of Edge Beam, AB.

$(\Delta_i^L)_m$ . . . . Component of Linear Load Function at i .

$(\Delta^R_{ij})_m$, $(\Delta^M_{ij})_m$ . . . . . Component of Linear Flexibilities at i Due to Forces and Moments at j, Respectively.

$\phi_o(\beta_m x)$, $\phi_1(\alpha_n y)$,

$\phi_2(\alpha_n y)$, $\phi_3(\beta_m x)$,

$\phi_4(\beta_m x)$, $\phi_5(\beta_m x)$ . . . Deflection Functions.

$\left[\phi_{10}\right]$, $\left[\phi_{1A}\right]$ . . . . . . Diagonal Matrices.

∇^2 . . . . The Laplacian Operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

CHAPTER I

INTRODUCTION

1-1. Statement of the Problem

This study is concerned with the analysis of plate structures, and is confined to those with plate elements whose behavior can be satisfactorily described by the small deflection theory of thin plates.

Plate structures complying to these restrictions may be classified as follows:

1. Single span plates
2. Continuous plates
3. Complex plate systems
4. Plate-frame structures
5. Plate-beam structures

This investigation is restricted to the analysis of plate-beam structures, though the approach is sufficiently general to include the first, second, and the fourth classifications. The third classification is intended to include plates of any shape and material, acted upon by both in-plane and out-of-plane loading.

In the ensuing discussion, only thin rectangular plates are considered. Loading may consist of any general system of forces and couples applied in such a manner that they are primarily resisted by bending.

Plate-beam structures are structural systems composed of both thin plates and elastic edge beams. More specifically;

> a plate-beam structure is a system of beams and thin plates arranged in such a manner that it is stable and capable of resisting applied loads.

The study of such structures centers around the problem of finding a method that will permit compatibility to be established between the deformations of the plates and the supporting members. This problem may be approached by considering the entire structure as a unit, or by studying the deformations of each part of the structure separately. For each approach, the final analysis may be accomplished by using

1. Exact or approximate classical methods
2. The method of stiffnesses
3. The method of flexibilities
4. Finite differences.

In this study, individual parts of the structure are considered. The method of flexibilities is adopted and flexibilities are obtained in the form of Fourier series.

Plate-beam analysis is of practical interest, since this type of structure occurs quite frequently in several areas of engineering design. Floor slabs in buildings, retaining walls, rectangular tanks, and certain types of dams and locks are a few examples. Parts of the hulls of ships and submarines and the fuselages of the latest aircraft and missiles are no more than thin sheets of metal supported by elastic beams, and hence, in many cases, fall into the realm of plate-beam structures.

Often, due to the lack of more precise analytical techniques, the analysis of plate-beam systems is accomplished through a multitude of simplifying assumptions, and the results may sometimes only vaguely represent the true behavior of the structure.

The need for research in this area has not gone unnoticed. Korenev[1], in his survey of Russian developments in the theory of plates, said of future work in this area,

> A series of new papers in the theory of plate bending must be awaited, in particular, studies on ... stiffened plates considered as systems of plates and bars acting together, etc.

It is the purpose of this thesis to show that, through the use of Fourier series, the flexibility approach may be successfully applied to the analysis of plate-beam structures.

1-2. Scope and Procedure of Investigation

The theory developed in this work is subject to the limitations due to the usual assumptions of thin plate theory. These assumptions are;

1. The material is linearly elastic, homogeneous, isotropic, and continuous.
2. The thickness of the plate is constant and is small in comparison with the other dimensions.
3. Stresses normal to the plate's middle surface are small and can be neglected.
4. Planes normal to the plate's middle surface before deformation remain normal after deformation.
5. The plate undergoes small elastic deformations.
6. Loads are applied normally to the middle surface, and in-plane forces have a negligible influence on the final deflection of the plate.

In the development of an analytical approach to the problem under consideration, the basic structure for plate elements is taken to be a rectangular plate supported at four corners. Flexibilities due to a general system of normal loads and arbitrary edge moments and forces are developed algebraically by using a Levy solution to the governing differential equation of thin plates. These are reduced to a form that allows a term by term solution of the problem.

Flexibilities of a simple bar due to end moments and a general system of applied loads are then formulated.

Using these flexibilities, compatibility between plate and beam elements is established. Compatibility relationships lead to a set of equations for each term in the Fourier series for flexibilities. Solutions to the equations resulting from a finite number of terms of the series are evaluated and the final redundants are obtained by superposition.

1-3. Historical Study

Most of the important work in the area of plate structures and associated boundary value problems has been critically evaluated and is listed in the bibliography under the numbers 2 through 75. The contributions which are directly related to the study are presented in this article.

In this study, plates are analyzed by the method first proposed by Levy(2) in 1899. The so called Levy solution is discussed in Article 1-6.

Several papers published during the last half century contributed much to the present knowledge of methods dealing with plates with mixed or unusual boundary conditions. In this area the work of Nádai(3, 4), Fletcher and Thorne(5), Holl(6, 7), Hencky(8) and Galerkin(9) should be mentioned.

The papers by Nádai, Hencky, and Galerkin, in particular, threw light on methods to attack the problem of a plate supported only at its corners, which is the basic structure for plates in plate-beam systems.

Insight to the problem of plate-beam structures is gained through reviewing some of the past work in continuous plates. An exact method for the solution of plates continuous in one direction over rigid supports was first developed by Galerkin[10] in 1933. More work on this problem was done by Habel[11], Bleich[12] and others[13, 14, 15]. Marcus[16] also considered this problem. In addition, he successfully analyzed plates continuous in one direction over rigid supports by means of a three-moment equation similar to that used to study continuous beams. Using the method of finite differences, Marcus was also able to study certain plate-beam structures.

Consideration was given to the problem of rectangular plates continuous in two directions over rigid supports by Girkmann[17], who presented the approximate method for the analysis of this type of structure first formulated by Bittner[18]. More recently Engelbreth[19], Maugh and Pan[20], and Siess and Newmark[21] studied this problem.

The complex problem of plates continuous in one direction over elastic beams was not solved until 1937 by Weber[22]. Only a year later, Jensen[23] obtained more general results for this problem. In that same year Newmark[24] presented a solution to this problem based on the stiffness approach. Fisher[25] considered this problem in his study of bridge slabs.

Sutherland, Goodman, and Newmark[26] successfully analyzed some special cases of plates continuous in two directions supported by elastic beams. More recently, Ang[27] developed a numerical procedure for analyzing this type of structure and, with Newmark[28] and then Prescott[29], extended this approach to the analysis of some types of plate-beam structures.

These researchers, however, were not the first to obtain exact solutions for elastically supported plates continuous in two directions. This was done by Kalmanok(30) in his book on the structural mechanics of plates which appeared in 1950. In the same publication Kalmanok solves for the moments and deflections in a rectangular tank, a problem solved earlier by Young(31), and develops a seven and then a thirteen moment equation for continuous plates.

Steady-state vibrations of such structures were considered by Dill(32) and Dill and Pister(33).

Tekinalp(34), in his doctoral thesis in 1952, approached the problem of beam and plate systems using Green's functions and integral solutions. He managed to solve, along with some other important problems, the problem of a uniformly loaded plate, infinite in length in one direction, and supported by identical symmetrical frames spaced at equal intervals.

Recently, Wood (35, 36, 37, 38) analyzed some types of plate-beam structures by finite differences and considered the ultimate strength of such structures. He suggested composite beam-plate structures as a topic for future research.

1-4. The Plate Equation

The equations governing the equilibrium and bending of bars are well known and recorded to a greater extent than those of their more complex counterpart, the thin plate. For this reason, and also for the purpose of establishing certain relationships and sign conventions for future reference, the governing equations of the theory of thin plates will be presented in this article.

Consider an elastic element cut from a thin plate and in equilibrium under the action of a general system of loads (Fig. 1-1). The intensity of the load per unit area acting on the plate is denoted by p. The stress resultants per unit length developed on the faces of the element are the shearing forces V_{xz}, V_{xy}; the bending moments M_x, M_y; and the twisting moments M_{xy} and M_{yx} These are shown acting in the positive sense in the figure.

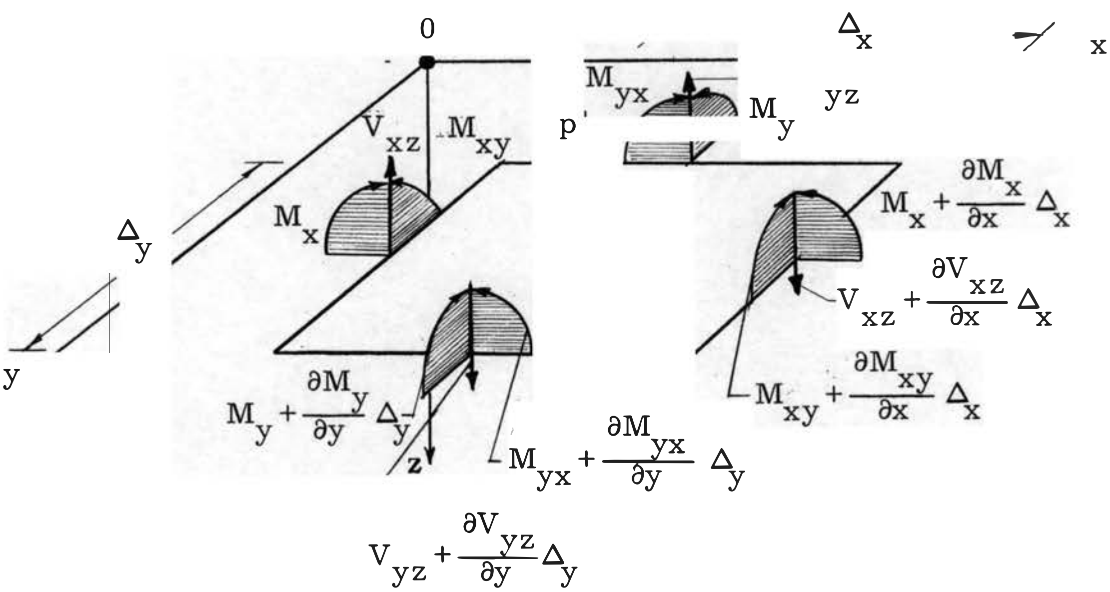


Fig. 1-1
Typical Plate Element

The six stress resultants are functions of the derivatives of w, the deflection of the plate, and are given in terms of these derivatives by the relationships

$$M_x = - D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \qquad (1\text{-}1a)$$

$$M_y = - D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right) \qquad (1\text{-}1b)$$

$$M_{xy} = M_{yx} = - D(1 - \mu) \frac{\partial^2 w}{\partial x \partial y} \qquad (1\text{-}1c)$$

$$V_{xz} = - D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right) \qquad (1\text{-}1d)$$

$$V_{yz} = - D\left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y}\right) , \qquad (1\text{-}1e)$$

where D is the flexural rigidity of the plate and μ is Poisson's ratio. It may be shown that

$$D = \frac{Eh^3}{12(1 - \mu^2)} ,$$

in which E is the modulus of elasticity of the plate material and h is the thickness of the plate.

The normal stresses in the x and y direction, σ_x and σ_y respectively, and the shearing stress τ_{xy} are related to the deflection in accordance with the expressions

$$\left.\begin{aligned} \sigma_x &= \frac{Ez}{1 - \mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) \\ \sigma_y &= - \frac{Ez}{1 - \mu^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) \\ \tau_{xy} &= \tau_{yx} = - \frac{Ez}{1 - \mu^2} (1 - \mu) \frac{\partial^2 w}{\partial x \partial y} , \end{aligned}\right\} \qquad (1\text{-}2)$$

in which z is the coordinate of the point in question measured from the plate's middle surface. Through the use of Equations (1-1) the maximum fiber stresses at a point may be expressed in terms of the moments as follows:

$$\left.\begin{aligned} (\sigma_x)_{max} &= \frac{6\,M_x}{h^2} \\ (\sigma_y)_{max} &= \frac{6\,M_y}{h^2} \\ (\tau_{xy})_{max} &= \frac{6\,M_{xy}}{h^2} \quad . \end{aligned}\right\} \qquad (1\text{-}3)$$

Substituting the moments and shears in terms of the derivatives of the deflection into the equations of equilibrium of a typical plate element, results in the governing partial differential equation of the theory of thin plates,

$$\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} \quad . \qquad (1\text{-}4)$$

Introducing the Laplacian differential operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, the above equation becomes

$$\nabla^4 w = \frac{p}{D} \; . \qquad (1\text{-}5)$$

Equation (1-4) is an elliptic fourth order, linear partial differential equation with constant coefficients, in the two independent variables x and y. The solution surface is uniquely determined by given boundary conditions; and any function, w(x, y), that simultaneously satisfies both the partial differential equation and the boundary conditions is the desired solution. Once the solution surface is known, the six stress resultants along with the individual stresses may be evaluated.

1-5. Boundary Conditions

At this point it will be beneficial to record, for later use, three of the more common boundary conditions that arise in thin plate problems. They are listed below for an edge, $x = 0$, parallel to the y-axis. For the other edges it is merely necessary to interchange x and y in the expressions given.

a.) Clamped Edge - The deflection along a clamped edge and the slope in a direction perpendicular to that edge is zero. Thus,

$$(w)_{x=0} = 0, \qquad \left(\frac{\partial w}{\partial x}\right)_{x=0} = 0 . \tag{1-6}$$

b.) Simply Supported Edge - The bending moment in a plane perpendicular to a simply supported edge is zero. Along a simply supported edge the curvature and the deflection are zero. Hence,

$$(w)_{x=0} = 0, \qquad \left(\frac{\partial^2 w}{\partial x^2}\right)_{x=0} = 0 . \tag{1-7}$$

c.) Free Edge - No bending moments, twisting moments, or shearing forces can exist along a free edge. Therefore,

$$(M_x)_{x=0} = 0, \qquad (V_{xz})_{x=0} = 0, \qquad (M_{xy})_{x=0} = 0. \tag{1-8}$$

It may be shown that the last two of these relationships can be combined and only two conditions, instead of three, are needed to completely determine the deflection surface. This is accomplished by introducing the edge reaction given by

$$(R_{xz})_{x=0} = (V_{xz})_{x=0} + \left(\frac{\partial M_{xy}}{\partial y}\right)_{x=0} . \tag{1-9}$$

This reactive force may be expressed in terms of the deflection by substituting Equations (1-1c) and (1-1d) into Equation (1-9). If this is done it is found that

$$(R_{xz})_{x=0} = -D\left[\left(\frac{\partial^3 w}{\partial x^3}\right)_{x=0} + (2-\mu)\left(\frac{\partial^3 w}{\partial x\,\partial y^2}\right)_{x=0}\right] . \quad (1\text{-}10)$$

Thus, the boundary conditions for a free edge are

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_{x=0} + \mu\left(\frac{\partial^2 w}{\partial y^2}\right)_{x=0} = 0 ,$$

$$\left(\frac{\partial^3 w}{\partial x^3}\right)_{x=0} + (2-\mu)\left(\frac{\partial^3 w}{\partial x\,\partial y^2}\right)_{x=0} = 0 . \quad (1\text{-}11)$$

Similar considerations show that the corner reaction at the origin is

$$(R)_{x=0,\,y=0} = 2(M_{xy})_{x=0,\,y=0} = -2D(1-\mu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)_{x=0,\,y=0} . \quad (1\text{-}12)$$

1-6. General Solution of the Plate Equation

The expression

$$w = F_1(x+iy) + xF_2(x+iy) + F_3(x-iy) + xF_4(x-iy) , \quad (1\text{-}13)$$

where F_1, F_2, F_3, and F_4 are arbitrary functions of the indicated arguments, is the general form of the complementary solution to the plate equation (Equation 1-4). From the theory of ordinary differential equations it is recalled that general solutions to linear equations of order i contain i arbitrary constants which must be evaluated using i boundary or initial conditions. In the case of linear partial differential equations of order i, the general solution contains i arbitrary functions of the independent variables. Little is gained, however, by trying to fit the above form of the solution to any boundary conditions.

For this reason it is the practice of analysts to seek solutions to many linear partial differential equations by the method of separation of variables. In this approach, a form of the solution is assumed that, on substitution into the equation, reduces it to a system of ordinary differential equations. The problem then is one of evaluating arbitrary constants rather than arbitrary functions by using the auxiliary conditions. When the complementary solution is found, then the sum of it and any particular solution form the general solution of the equation.

If the complementary solution is assumed to be of the form

$$w = XY ,$$

where X is a function of x only and Y is a function of y only, then substituting in Equation (1-4) gives, for p equal to zero,

$$X^{IV} Y + 2X^{II}Y^{II} + XY^{IV} = 0.$$

The Roman numerals indicate differentiations with respect to the functions' arguments.

Dividing this relationship by XY yields

$$\frac{X^{IV}}{X} + 2\,\frac{X^{II}Y^{II}}{XY} + \frac{Y^{IV}}{Y} = 0 . \qquad (1\text{-}14)$$

This form is inseparable unless the form of one of these functions is assumed. Assume, for example, that

$$Y^{II} = -\beta^2 Y , \qquad (1\text{-}15)$$

where β^2 is some constant. Substituting this into Equation (1-14) gives

$$\frac{X^{IV} - 2\beta^2 X^{II}}{X} = -\frac{Y^{IV}}{Y} . \qquad (1\text{-}16)$$

The solution to Equation (1-15) is

$$Y = C_1 \cos \beta y + C_2 \sin \beta y , \qquad (1\text{-}17)$$

where C_1 and C_2 are arbitrary constants. Thus, the right-hand side of Equation (1-16) reduces to $-\beta^4$. Multiplying both sides of this equation by X and then transposing, gives the fourth order linear differential equation

$$X^{IV} - 2\beta^2 X^{II} + \beta^4 X = 0 . \qquad (1\text{-}18)$$

The general solution of this equation is

$$X = C_3 \cosh \beta x + C_4 \sinh \beta x + C_5 x \cosh \beta x + C_6 x \sinh \beta x , \qquad (1\text{-}19)$$

in which C_3, C_4, C_5, and C_6 are arbitrary constants.

Now Equations (1-17) and (1-19) are solutions for any and all real values of β ; and the sum of all of these solutions is also a solution. There is an infinity of values of β ; and if for the nth value, β_n, the corresponding solutions are denoted X_n and Y_n , then the general solution to this so-called biharmonic equation is

$$w = \sum_{n=1}^{\infty} X_n Y_n \quad . \tag{1-20}$$

The procedure just outlined is a slightly generalized form of the Levy[2] solution. This method is adopted in the discussion to follow.

CHAPTER II

FUNCTIONS OF THE BASIC PLATE

2-1. General

A complete analysis of plate-beam systems is accomplished when all deformations, stresses, and stress resultants are determined for each element in the structure. Since the stress resultants must be such that each element is in equilibrium and that compatible deformations are produced, all of these quantities are related; and, once the deflection curves or surfaces of all elements are found, any other function may be calculated.

A complementary approach may also be taken. It is possible to express these deformations and stress resultants in terms of the unknown forces and moments which act at the boundaries of each structural element. If these forces and moments are determined, then all other quantities may be evaluated.

These well known concepts define two basic approaches that have evolved in structural analysis, the stiffness approach, in which moments and forces are expressed in terms of unknown deformations, and the flexibility approach, in which deformations are expressed in terms of unknown forces and moments. The latter approach is adopted in this investigation since it is more direct for the type of structure being analyzed, and is sufficiently basic that, with modifications, the stiffness approach may be formulated if desired.

It will be necessary, therefore, to develop expressions for the deflection of a typical plate element bending under the action of applied moments and forces along each edge and loading normal to its plane. Edge deflections and slopes are dependent upon the nature of the loading and the applied moments and forces. A basic structure which provides complete generality in edge conditions must be selected, and a provision must be made for distributed edge forces and moments defined by functions of as general a nature as possible.

The basic structure for plate elements is a single span rectangular plate, free along all edges with point supports at each corner. Once its deflection is determined in terms of the arbitrary edge forces and moments, its deformations must be made compatible with those of the other structural elements that comprise the system. Thus, it is necessary to study some basic problems of plates supported only at their corners.

2-2. Plat With Arbitrary Edge Forces

In this article, the problem of a plate supported only at its corners and subjected to arbitrary forces applied along two edges is considered. The deflection surface of this plate is to be denoted w_1. It is shown that, once w_1 is determined, the deformations of the basic plate structure can be found rather easily.

It is necessary to superimpose two deflection surfaces to obtain w_1. Algebraically,

$$w_1 = w_{11} + w_{12} ,$$

in which;

1. w_{11} is the deflection surface of a rectangular plate simply sup-

ported along two opposite edges and free along the remaining edges. This plate is under the action of applied edge forces, S^I and S^{II}, distributed along the free edges, where

$$S^I = S_1^I \sin\frac{\pi x}{a} + S_2^I \sin\frac{2\pi x}{a} + \cdots + S_k^I \sin\frac{k\pi x}{a} = \sum_n^k S_n^I \sin\frac{n\pi x}{a} ,$$

and

$$S^{II} = S_1^{II} \sin\frac{\pi x}{a} + S_2^{II} \sin\frac{2\pi x}{a} + \cdots + S_k^{II} \sin\frac{k\pi x}{a} = \sum_n^k S_n^{II} \sin\frac{n\pi x}{a} .$$

In these series, S_1^I , $\cdots$, S_k^I, S_1^{II}, $\cdots$, S_k^{II} are arbitrary coefficients, k is some positive interger, and

$$n = 1, 2, 3, \cdots, k-1, k .$$

It is emphasized that S^I and S^{II} are not the final arbitrary edge forces which act on surface w_1. The final edge forces are functions of these coefficients.

2. w_{12} is the deflection surface of a rectangular plate simply supported along the edges corresponding to the free edges of surface w_{11}, and free along the edges corresponding to the simply supported edges of surface w_{11}. Surface w_{12} is developed by forces distributed along its free edges which are equal and opposite to those created along the simply supported edges of surface w_{11}.

The mechanics of this superposition procedure are illustrated in Fig. 2-1. As is indicated in the figure, reactive forces are developed along the simply supported edges of surface w_{11} which are functions, d and f, of S^I and S^{II}. Forces equal and opposite to these reactions are applied on the free edges of surface w_{12}. These, in turn, develop reactions along the simply supported edges of this plate which are also functions of S^I and S^{II}, denoted g and h. Thus, when w_{11} and w_{12} are superimposed, the resulting deflection surface is that of a plate supported only at the corners with the forces,

$$S^I - g(S^I, S^{II})$$

and

$$S^{II} + h(S^I, S^{II}) ,$$

acting along two opposite, free edges.

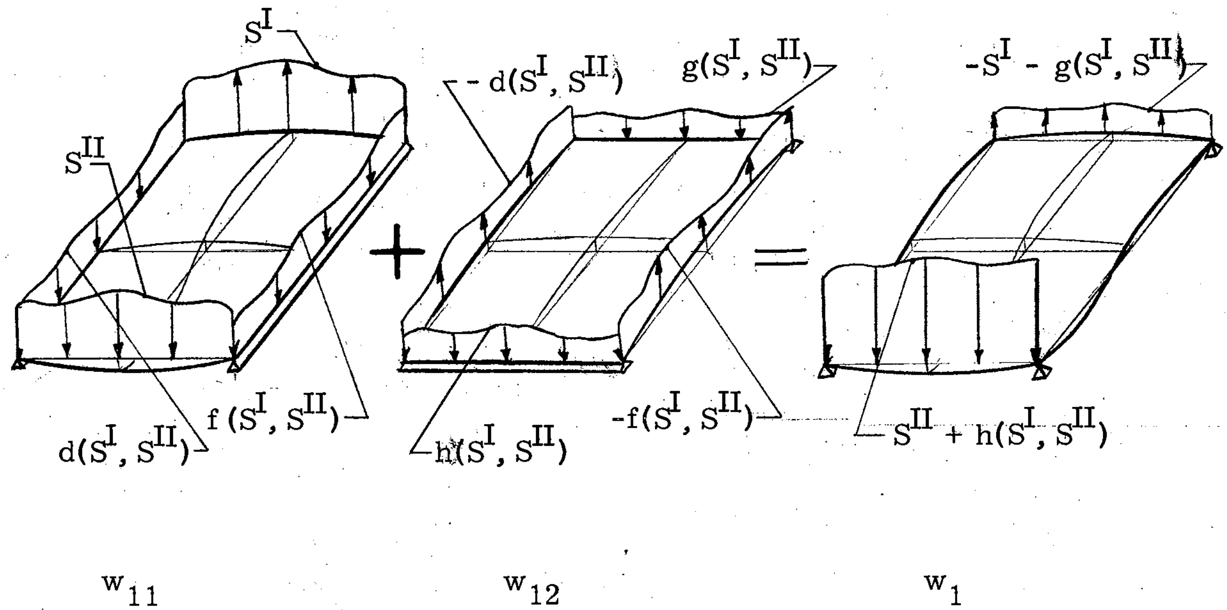


Fig. 2-1

Superposition Procedure for Obtaining

Deflection Surface w_1

Since S^{I} and S^{II} have not been specified, these edge forces are completely arbitrary in nature. As was mentioned earlier, the deflection surfaces of various plates supported only at their corners is needed. The formulation of these surfaces is accomplished by first calculating the deflection surface, w_2' . This surface is that of a plate free along two opposite edges and simply supported on the edges corresponding to those of surface w_1 on which the arbitrary forces $S^{I} - g(S^{I}, S^{II})$ and $S^{II} + h(S^{I}, S^{II})$ act. Surface w_2' may be caused by any general type of loading, including forces and moments applied along its free edges. The calculation of this surface is straight forward and presents no unusual difficulties.

Known reactive forces must be developed along the simply supported edges of surface w_2'. If the arbitrary forces along the edges of deflection surface w_1 are adjusted so that they are equal and opposite to those of surface w_2', the sum of these gives the deflection surface of a plate supported only at its corners and under the action of the general loading of w_2'. This procedure is illustrated in Fig. 2-2.

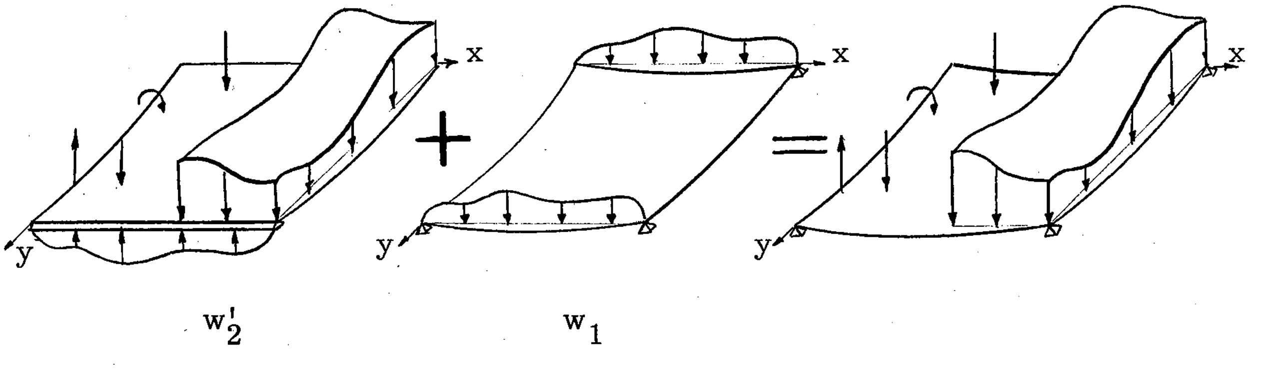


Fig. 2-2

Development of Free-Edged Plate

Under General Loading

To obtain the expression for the deflection surface w_{11}, consider a rectangular plate simply supported along the edges parallel to the x-axis and free along those parallel to the y-axis (Fig. 2-3). Let the length of the edges parallel to the x-axis be a and those parallel to the y-axis be b .

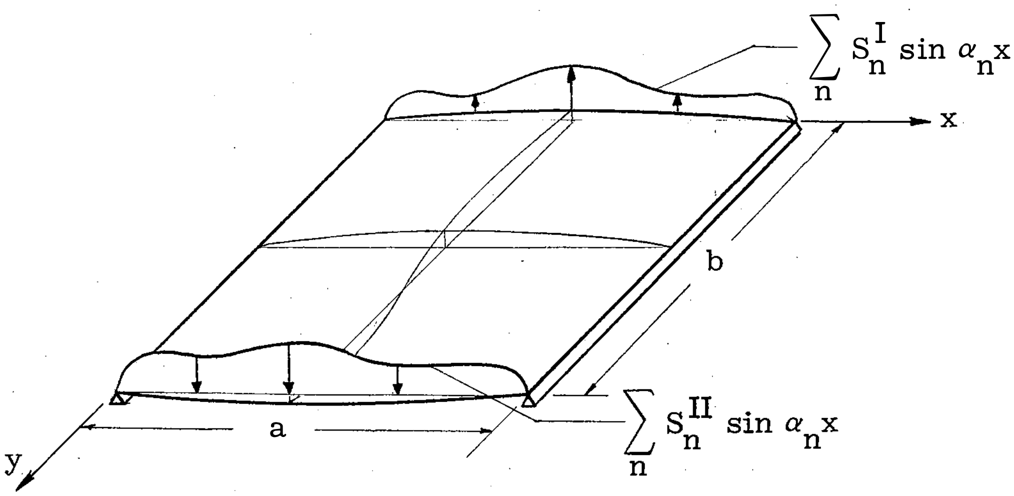


Fig. 2-3

Deflection Surface w_{11}

The resulting deflection surface must satisfy the homogeneous plate equation, $\nabla^4 w_{11} = 0$, and the boundary conditions,

$$\left(\frac{\partial^2 w_{11}}{\partial y^2} + \mu \frac{\partial^2 w_{11}}{\partial x^2}\right)_{y=0,\,b} = 0 \qquad (2\text{-}1a)$$

$$-D\left(\frac{\partial^3 w_{11}}{\partial y^3} + (2-\mu)\frac{\partial^3 w_{11}}{\partial x^2 \partial y}\right)_{y=0} = \sum_{n}^{k} S_n^I \sin \alpha_n x \qquad (2\text{-}1b)$$

$$- D\left(\frac{\partial^3 w_{11}}{\partial y^3} + (2 - \mu)\frac{\partial^3 w_{11}}{\partial x^2 \partial y}\right)_{y=b} = \sum_{n}^{k} S_n^{II} \sin \alpha_n x \qquad (2\text{-}1c)$$

$$(w_{11})_{x=o,\, a} = 0 \qquad (2\text{-}1d)$$

$$\left(\frac{\partial^2 w_{11}}{\partial x^2}\right)_{x=o,\, a} = 0\,, \qquad (2\text{-}1e)$$

in which

$$\alpha_n = \frac{n\pi}{a}\ .$$

These requirements may be fulfilled if w_{11} is taken to be of the form

$$w_{11} = \sum_{n=1}^{k} w_{11n}\ , \qquad (2\text{-}2)$$

where

$$w_{11n} = Z(\alpha_n y) \sin \alpha_n x\ , \qquad (2\text{-}3)$$

and $Z(\alpha_n y)$ is some function of $\alpha_n y$.

The form given by Equation (2-3) automatically satisfies Equations (2-1d, e), and when substituted into Equation (1-4) results in the ordinary differential equation

$$\frac{d^4 Z(\alpha_n y)}{dy^4} - 2\alpha_n^2 \frac{d^2 Z(\alpha_n y)}{dy^2} + \alpha_n^4 Z(\alpha_n y) = 0\ . \qquad (2\text{-}4)$$

From Equation (1-19) it follows that the solution of this equation is

$$Z(\alpha_n y) = A_n \operatorname{ch} \alpha_n y + B_n \operatorname{sh} \alpha_n y + C_n \alpha_n y \operatorname{ch} \alpha_n y + D_n \alpha_n y \operatorname{sh} \alpha_n y, \tag{2-5}$$

where A_n, B_n, C_n, and D_n are constants and, for simplicity in notation,

$$\operatorname{ch} \alpha_n y = \cosh \alpha_n y ,$$

$$\operatorname{sh} \alpha_n y = \sinh \alpha_n y , \text{ etc.}$$

Substituting the resulting equation for w_{11} into Equations (2-1) gives four equations from which the constants A_n, B_n, C_n, and D_n may be evaluated. Performing such calculations, it is found that the final deflection surface may be written in the form

$$w_{11n} = \frac{S_n^I a^3}{D} \phi_1 (\alpha_n y) + \frac{S_n^{II} a^3}{D} \phi_2 (\alpha_n y) \sin \alpha_n x . \tag{2-6}$$

The functions ϕ_1 and ϕ_2 are given in Table 2-1.

By means of Equation (1-10), it is found that the reactive forces developed along the simply supported edges are

$$(R_{xzn})_{x=0} = (-1)^n (R_{xzn})_{x=a} = -(S_n^I E_1 (\alpha_n y) + S_n^{II} E_2 (\alpha_n y)) , \tag{2-7}$$

where the functions E_1 and E_2 are given in Table 2-2.

The functions $\phi_1 (\alpha_n y)$, $\phi_2 (\alpha_n y)$, $E_1 (\alpha_n y)$, and $E_2 (\alpha_n y)$ may be represented in the Fourier series

TABLE 2-1

DEFLECTION FUNCTIONS

$$k_n = \frac{(3+\mu)\,\text{sh}\,\alpha_n b}{\alpha_n b(1-\mu)} \qquad\qquad c_m = \frac{(3+\mu)\,\text{sh}\,\beta_m a}{\beta_m a(1-\mu)}$$

$$\phi_o(\beta_m x) = \frac{\mu}{\beta_m^5 a^5 (1-\mu)^2 (c_m - 1)} \Big[((1+\mu)\,\text{sh}\,\beta_m a - \beta_m a(1-\mu))\,\text{ch}\,\beta_m x + \beta_m(1-\mu)\,\text{sh}\,\beta_m a\,x\,\text{sh}\,\beta_m x + (1 - \text{ch}\,\beta_m a)\,((1+\mu)\,\text{sh}\,\beta_m x + \beta_m(1-\mu)\,x\,\text{ch}\,\beta_m x)\Big]$$

$$\phi_1(\alpha_n y) = \frac{1}{n^3 \pi^3 \alpha_n b(1-\mu)^2(k_n^2 - 1)} \Big[(k_n\,\text{ch}\,\alpha_n b + 1)(-2\,\text{ch}\,\alpha_n y + \alpha_n(1-\mu)\,y\,\text{sh}\,\alpha_n y) + (2k_n\,\text{sh}\,\alpha_n b - \alpha_n b(1-\mu))\,\text{sh}\,\alpha_n y - \alpha_n(1-\mu)\,k_n\,\text{sh}\,\alpha_n b\,y\,\text{ch}\,\alpha_n y$$

$$\phi_2(\alpha_n y) = \frac{1}{n^3 \pi^3 \alpha_n b(1-\mu)^2(k_n^2 - 1)} \Big[(k_n + \text{ch}\,\alpha_n b)(2\,\text{ch}\,\alpha_n y - \alpha_n(1-\mu)\,y\,\text{sh}\,\alpha_n y) + \text{sh}\,\alpha_n b\,((1+\mu)\,\text{sh}\,\alpha_n y + \alpha_n(1-\mu)\,y\,\text{ch}\,\alpha_n y)\Big]$$

$$\phi_3(\beta_m x) = \frac{1}{\beta_m^4 a^4 (1-\mu)^2 (c_m^2 - 1)} \Big[(c_m\,\text{ch}\,\beta_m a + 1)(-2\,\text{ch}\,\beta_m x + \beta_m(1-\mu)\,x\,\text{sh}\,\beta_m x) + (2c_m\,\text{sh}\,\beta_m a - \beta_m a(1-\mu))\,\text{sh}\,\beta_m x - \beta_m(1-\mu)\,c_m\,\text{sh}\,\beta_m a\,x\,\text{ch}\,\beta_m x\Big]$$

$$\phi_4(\beta_m x) = \frac{1}{\beta_m^4 a^4 (1-\mu)^2 (c_m^2 - 1)} \Big[(c_m + \text{ch}\,\beta_m a)(2\,\text{ch}\,\beta_m x - \beta_m(1-\mu)\,x\,\text{sh}\,\beta_m x) + \text{sh}\,\beta_m a\,((1+\mu)\,\text{sh}\,\beta_m x + \beta_m(1-\mu)\,x\,\text{ch}\,\beta_m x)\Big]$$

$$\phi_5(\beta_m x) = \frac{1}{\beta_m^3 a^3 (1-\mu)^2 (c_m^2 - 1)} \Big[-((1+\mu)c_m\,\text{sh}\,\beta_m a - \beta_m a(1-\mu))\,\text{ch}\,\beta_m x - \beta_m(1-\mu)\,c_m\,\text{sh}\,\beta_m a\,x\,\text{sh}\,\beta_m x + \frac{((c_m^2 - 1)\,\text{ch}\,\beta_m a + c_m\,\text{sh}^2\,\beta_m a)}{c_m + \text{ch}\,\beta_m a}((1+\mu)\,\text{sh}\,\beta_m x + \beta_m(1-\mu)\,x\,\text{ch}\,\beta_m x)\Big]$$

TABLE 2-2

REACTION FUNCTIONS

$$k_n = \frac{(3+\mu)\,\text{sh}\,\alpha_n b}{\alpha_n b\,(1-\mu)} \qquad\qquad c_m = \frac{(3+\mu)\,\text{sh}\,\beta_m a}{\beta_m a\,(1-\mu)}$$

$$E_1(\alpha_n y) = \frac{1}{\alpha_n b\,(k_n^2 - 1)}\Big[(k_n\,\text{ch}\,\alpha_n b + 1)(2\,\text{ch}\,\alpha_n y + \alpha_n y\,\text{sh}\,\alpha_n y) - k_n\,\text{sh}\,\alpha_n b\,(2\,\text{sh}\,\alpha_n y + \alpha_n y\,\text{ch}\,\alpha_n y) - \alpha_n b\,\text{sh}\,\alpha_n y\Big]$$

$$E_2(\alpha_n y) = \frac{1}{\alpha_n b\,(k_n^2 - 1)}\Big[(k_n + \text{ch}\,\alpha_n b)(-2\,\text{ch}\,\alpha_n y - \alpha_n y\,\text{sh}\,\alpha_n y) + \text{sh}\,\alpha_n b\,\Big(\Big(\frac{5-\mu}{1-\mu}\Big)\text{sh}\,\alpha_n y + \alpha_n y\,\text{ch}\,\alpha_n y\Big)\Big]$$

$$F_1(\beta_m x) = \frac{1}{\beta_m a\,(c_m^2 - 1)}\Big[(c_m\,\text{ch}\,\beta_m a + 1)(2\,\text{ch}\,\beta_m x + \beta_m x\,\text{sh}\,\beta_m x) - c_m\,\text{sh}\,\beta_m a\,(2\,\text{sh}\,\beta_m x + \beta_m x\,\text{ch}\,\beta_m x) - \beta_m a\,\text{sh}\,\beta_m x\Big]$$

$$F_2(\beta_m x) = \frac{1}{\beta_m a\,(c_m^2 - 1)}\Big[(c_m + \text{ch}\,\beta_m a)(-2\,\text{ch}\,\beta_m x - \beta_m x\,\text{sh}\,\beta_m x) + \text{sh}\,\beta_m a\,\Big(\Big(\frac{5-\mu}{1-\mu}\Big)\text{sh}\,\beta_m x + \beta_m x\,\text{ch}\,\beta_m x\Big)\Big]$$

$$G(\beta_m x) = \frac{1}{(c_m^2 - 1)}\Big[\beta_m a\,\text{ch}\,\beta_m x - c_m\,\text{sh}\,\beta_m a\,\Big(\Big(\frac{5-\mu}{1-\mu}\Big)\text{ch}\,\beta_m x + \beta_m x\,\text{sh}\,\beta_m x\Big) + \frac{(c_m^2 - 1)\,\text{ch}\,\beta_m a + c_m\,\text{sh}^2\,\beta_m a}{c_m + \text{ch}\,\beta_m a}\Big(\Big(\frac{5-\mu}{1-\mu}\Big)\text{sh}\,\beta_m x + \beta_m x\,\text{ch}\,\beta_m x\Big)\Big]$$

$$Q(\beta_m x) = \frac{\mu}{\beta_m^2 a^2\,(c_m - 1)}\Big[\Big(\Big(\frac{5-\mu}{1-\mu}\Big)\text{sh}\,\beta_m a - \beta_m a\Big)\text{ch}\,\beta_m x + \beta_m\,\text{sh}\,\beta_m a\,x\,\text{sh}\,\beta_m x + (1 - \text{ch}\,\beta_m a)\Big(\Big(\frac{5-\mu}{1-\mu}\Big)\text{sh}\,\beta_m x + \beta_m x\,\text{ch}\,\beta_m x\Big)\Big]$$

TABLE 2-3A

FOURIER SINE SERIES COEFFICIENTS

Function	Symbol	Coefficient
$\cosh \beta_m x$	$h_{mn}^{1.)}$	$\dfrac{2\left(1 - (-1)^n \cosh \beta_m a\right)}{n\pi\left(1 + (\beta_m/\alpha_n)^2\right)}$
$\sinh \beta_m x$	$h_{mn}^{2.)}$	$\dfrac{-2(-1)^n \sinh \beta_m a}{n\pi\left(1 + (\beta_m/\alpha_n)^2\right)}$
$x \cosh \beta_m x$	$h_{mn}^{3.)}$	$\dfrac{2\alpha_n(-1)^n\left(2\beta_m \sinh \beta_m a - a(\alpha_n^2 + \beta_m^2)\text{ch}\, \beta_m a\right)}{a(\alpha_n^2 + \beta_m^2)}$
$x \sinh \beta_m x$	$h_{mn}^{4.)}$	$\dfrac{4\alpha_n \beta_m\left((-1)^n \cosh \beta_m a - 1\right) - 2a\, \alpha_n (-1)^n (\alpha_n^2 + \beta_m^2)\sinh \beta_m a}{a(\alpha_n^2 + \beta_m^2)}$
$\Phi_o(\beta_m x)$	ϕ°_{mr}	$\dfrac{\mu}{\beta_m^5\, \alpha^5 (1-\mu)(c_m - 1)}\left[\text{sh}\, \beta_m a \left(\frac{1+\mu}{1-\mu} h_{mr}^{1.)} + \beta_m h_{mr}^{4.)}\right) - \beta_m a\, h_{mr}^{1.)} + (1-\text{ch}\, \beta_m a)\left(\frac{1+\mu}{1-\mu} h_{mr}^{2.)} + \beta_m h_{mr}^{3.)}\right)\right]$
$\Phi_1(\alpha_n y)$	$\phi_{nr}^{1.)}$	$\dfrac{(k_n\, \text{ch}\, \alpha_n b + 1)\left(-2h_{nr}^{1.)} + \alpha_n(1-\mu)h_{nr}^{4.)}\right) + k_n\, \text{sh}\, \alpha_n b\left(2h_{nr}^{2.)} - \alpha_n(1-\mu)h_{nr}^{3.)}\right) - \alpha_n b(1-\mu)h_{nr}^{2.)}}{n^3 \pi^3 \alpha_n b(1-\mu)^2 (k_n^2 - 1)}$
$\Phi_2(\alpha_n y)$	$\phi_{nr}^{2.)}$	$\dfrac{(k_n + \text{ch}\, \alpha_n b)\left(2h_{nr}^{1.)} - \alpha_n(1-\mu)h_{nr}^{4.)}\right) + \text{sh}\, \alpha_n b\left((1+\mu)h_{nr}^{2.)} + \alpha_n(1-\mu)h_{nr}^{4.)}\right)}{n^3 \pi^3 \alpha_n b(1-\mu)^2 (k_n^2 - 1)}$
$\Phi_3(\beta_m x)$	$\phi_{mr}^{3.)}$	$\dfrac{b^3}{a^3}\, \phi_{mr}^{1.)}$

TABLE 2-3B

FOURIER SINE SERIES COEFFICIENTS

Function	Symbol	Coefficient
$\Phi_4(\beta_m x)$	$\phi_{mr}^{4.)}$	$\frac{b^3}{a^3}\,\phi_{mr}^{2.)}$
$\Phi_5(\beta_m x)$	$\phi_{mr}^{5.)}$	$\frac{-c_m \,\mathrm{sh}\,\beta_m a\left((1+\mu)h_{mr}^{1.)} + \beta_m(1-\mu)h_{mr}^{4.)}\right) + \beta_m a(1-\mu)h_{mr}^{1.)} + \frac{(c_m^2-1)\,\mathrm{ch}\,\beta_m a + c_m\,\mathrm{sh}^2\,\beta_m a}{c_m + \mathrm{ch}\,\beta_m a}\left((1+\mu)h_{mr}^{2.)} + \beta_m(1-\mu)h_{mr}^{3.)}\right)}{\beta_m^3 a^3 (1-\mu)^2 (c_m^2 - 1)}$
$E_1(\alpha_n y)$	$e_{mn}^{1.)}$	$\frac{(k_n\,\mathrm{ch}\,\alpha_n b + 1)(2h_{nm}^{1.)} + \alpha_n h_{nm}^{4.)}) - k_n\,\mathrm{sh}\,\alpha_n b(2h_{nm}^{2.)} + \alpha_n h_{nm}^{3.)}) - \alpha_n b\,h_{nm}^{4.)}}{\alpha_n b(k_n^2 - 1)}$
$E_2(\alpha_n y)$	$e_{mn}^{2.)}$	$\frac{-(k_n + \mathrm{ch}\,\alpha_n b)(2h_{nm}^{1.)} + \alpha_n h_{nm}^{4.)}) + \mathrm{sh}\,\alpha_n b(\frac{5-\mu}{1-\mu}h_{nm}^{2.)} + \alpha_n h_{nm}^{3.)})}{\alpha_n b(k_n^2 - 1)}$
$F_1(\beta_m x)$	$f_{mr}^{1.)}$	$e_{rm}^{1.)}$
$F_2(\beta_m x)$	$f_{mr}^{2.)}$	$e_{rm}^{2.)}$
$G(\beta_m x)$	g_{mr}	$\frac{\beta_m a\,h_{mr}^{1.)} - c_m\,\mathrm{sh}\,\beta_m a(\frac{5-\mu}{1-\mu}h_{mr}^{1.)} + \beta_m h_{mr}^{4.)}) + \frac{(c_m^2-1)\mathrm{ch}\,\beta_m a + c_m\,\mathrm{sh}^2\,\beta_m a}{c_m + \mathrm{ch}\,\beta_m a}(\frac{5-\mu}{1-\mu}h_{mr}^{2.)} + \beta_m h_{mr}^{3})}{c_m^2 - 1}$
$Q(\beta_m x)$	q_{mr}	$\frac{\mu\left[\mathrm{sh}\,\beta_m a(\frac{5-\mu}{1-\mu}h_{mr}^{1.)} + \beta_m h_{mr}^{4.)}) - \beta_m a\,h_{mr}^{1.)} + (1 - \mathrm{ch}\,\beta_m a)(\frac{5-\mu}{1-\mu}h_{mr}^{2.)} + \beta_m h_{mr}^{3.)})\right]}{\beta_m^2 a^2 (c_m - 1)}$

$$\varnothing_1(\alpha_n y) = \sum_r \phi_{nr}^{1.)} \sin \beta_r y \qquad \bigg\| \qquad \varnothing_2(\alpha_n y) = \sum_r \phi_{nr}^{2.)} \sin \beta_r y \tag{2-8a}$$

$$E_1(\alpha_n y) = \sum_m e_{mn}^{1.)} \sin \beta_m y \qquad \bigg\| \qquad E_2(\alpha_n y) = \sum_m e_{mn}^{2.)} \sin \beta_m y \ , \tag{2-8b}$$

in which

$$\beta_m = \frac{m\pi}{b} \ , \quad m = 1, 2, 3, \cdots$$

$$\beta_r = \frac{r\pi}{b} \ , \quad r = 1, 2, 3, \cdots \ , \tag{2-8c}$$

and the coefficients $\phi_{nr}^{1.)}$, $\phi_{nr}^{2.)}$, $e_{mn}^{1.)}$, and $e_{mn}^{2.)}$ are listed along with those for some other functions of interest in Tables 2-3A and 2-3B.

Thus, the reactions along the simply supported edges become

$$(R_{xzn})_{x=0} = (-1)^n (R_{xzn})_{x=a}$$

$$= S_n^I \sum_m e_{mn}^{1.)} \sin \beta_m y - S_n^{II} \sum_m e_{mn}^{2.)} \sin \beta_m y \ . \tag{2-9}$$

The deflection surface, w_{12}, of a plate simply supported along the edges parallel to the x-axis, free along those parallel to the y-axis, and bent by arbitrary forces distributed along $x = 0$ of the form

$$\sum_m T_m \sin \beta_m y \ ,$$

and along x = a of the form

$$\sum_m U_m \sin \beta_m y \ ,$$

must now be formulated (Fig. 2-4). Following a procedure similar to that used in calculating surface w_{11}, it is found that w_{12} is defined by the equations

$$w_{12} = \sum_m w_{12m} \ , \qquad (2\text{-}10)$$

and

$$w_{12m} = \left(\frac{T_m a^3}{D} \phi_3 (\beta_m x) + \frac{U_m a^3}{D} \phi_4 (\beta_m x) \right) \sin \beta_m y \ . \qquad (2\text{-}11)$$

Again, the deflection functions ϕ_3 and ϕ_4 are given in Table 2-1.

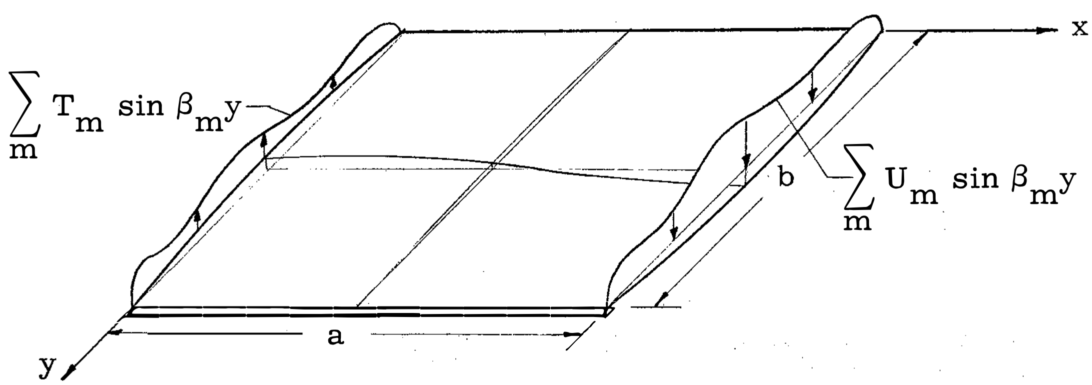


Fig. 2-4

Deflection Surface w_{12}

The reactive forces developed along the simply supported edges are

$$(R_{yzm})_{y=0} = (-1)^m (R_{yzm})_{y=b} = -\left(T_m F_1(\beta_m x) + U_m F_2(\beta_m x)\right) . \tag{2-12}$$

The reaction functions, F_1 and F_2, are recorded with others of interest in Table 2-2.

In a manner similar to those of surface w_{11}, the functions $\phi_3(\beta_m x)$, $\phi_4(\beta_m x)$, $F_1(\beta_m x)$, and $F_2(\beta_m x)$ may be expressed by Fourier series whose coefficients are tabulated in Tables 2-3A, B. The Fourier representations are

$$\phi_3(\beta_m x) = \sum_r \phi_{mr}^{3.)} \sin \alpha_r x \qquad \Bigg\| \qquad \phi_4(\beta_m x) = \sum_r \phi_{mr}^{4.)} \sin \alpha_r x \tag{2-13a}$$

$$F_1(\beta_m x) = \sum_r f_{mr}^{1.)} \sin \alpha_r x \qquad \Bigg\| \qquad F_2(\beta_m x) = \sum_r f_{mr}^{2.)} \sin \alpha_r x , \tag{2-13b}$$

where, again,

$$\alpha_r = \frac{r\pi}{a} , \; r = 1, 2, 3, \cdots . \tag{2-13c}$$

An examination of Equations (2-10) through (2-13) shows that the relations defining surface w_{12} are identical in form to the corresponding relations of surface w_{11}. In fact, with the exception of the coefficients S_n^I, S_n^{II}, T_m, and U_m, the expressions for deflections and edge reactions of surface w_{12} result from merely interchanging x with y, m with n, and a with b. Mathematically,

$$\frac{b^3}{a^3}\,\phi_1(\beta_m x) = \phi_3(\beta_m x) \qquad \Big\| \qquad \frac{b^3}{a^3}\,\phi_2(\beta_m x) = \phi_4(\beta_m x) \quad (2\text{-}14)$$

and

$$E_1(\beta_m x) = F_1(\beta_m x) \qquad \Big\| \qquad E_2(\beta_m x) = F_2(\beta_m x)\ . \quad (2\text{-}15)$$

If the amplitudes of the force distributions along the free edges of w_{12} are chosen so that

$$\left.\begin{aligned} T_m &= \sum_n \left(S_n^I e_{mn}^{1.)} + S_n^{II} e_{mn}^{2.)}\right) \\ U_m &= \sum_n (-1)^n \left(S_n^I e_{mn}^{1.)} + S_n^{II} e_{mn}^{2.)}\right)\ , \end{aligned}\right\} \quad (2\text{-}16)$$

then, by adding deflection surfaces w_{11} and w_{12}, the deflection surface w_1 is obtained.

Substituting Equations (2-16) into Equation (2-11) leaves surface w_{12} in terms of the coefficients S_n^I and S_n^{II}. If this result is added to Equation (2-6), the final deflection surface becomes

$$\begin{aligned} w_1 = &\sum_n \frac{a^3}{D}\left(S_n^I\,\phi_1(\alpha_n y) + S_n^{II}\,\phi_2(\alpha_n y)\right)\sin\alpha_n x\ + \\ &+ \sum_m\sum_n \frac{a^3}{D}\left(S_n^I e_{mn}^{1.)} + S_n^{II} e_{mn}^{2.)}\right)\left(\phi_3(\beta_m x)\ +\right. \\ &\left.+\ (-1)^n\,\phi_4(\beta_m x)\right)\sin\beta_m y\ . \end{aligned} \quad (2\text{-}17)$$

The final force distributions on the edges parallel to the x-axis are

$$(R_{yz})_{y=0} = \sum_n S_n^I \sin \alpha_n x - \sum_m \sum_n (S_n^I e_{mn}^{1.)} + S_n^{II} e_{mn}^{2.)})$$

$$\times \left(\sum_r f_{mr}^{1.)} \sin \alpha_r x + (-1)^n \sum_r f_{mr}^{2.)} \sin \alpha_r x \right) \qquad (2\text{-}18)$$

and

$$(R_{yz})_{y=b} = \sum_n S_n^{II} \sin \alpha_n x - \sum_m \sum_n (-1)^m (S_n^I e_{mn}^{1.)} + S_n^{II} e_{mn}^{2.)})$$

$$\times \left(\sum_r f_{mr}^{1.)} \sin \alpha_r x + (-1)^n \sum_r f_{mr}^{2.)} \sin \alpha_r x \right) . \qquad (2\text{-}19)$$

As was mentioned earlier, the deflection surfaces of corner supported plates may now be obtained by a proper adjustment of S_n^I and S_n^{II} in Equations (2-18) and (2-19). This process presents some difficulties, however, since these coefficients cannot be solved for on a term by term basis. In other words, since the reactions of surface w_2' are in terms of infinite series, this adjustment leads to an infinite number of equations and an infinite number of unknowns. The inversion of such an infinite order matrix is, of course, impossible, and only a finite number of terms, say k terms, can be taken. This is the reason that the applied edge forces for surface w_{11} are represented by k terms rather than a complete infinite series, and hence, 2k equations are needed to solve for the coefficients, S_n^I and S_n^{II}.

The procedure by which this set of equations is formulated for specific loading cases is somewhat involved. It will be discussed in the following two articles of this chapter.

2-3. Edge Forces and Moments on Free-Edged Plates

The deflection surface of a plate simply supported along the edges parallel to the x-axis, free on the other edges, and subjected to the action of a distributed force along the edge $x = 0$ of the form

$$R'_{xz} = \sum_m R'_{xzm} \sin \beta_m y \ , \tag{2-20}$$

can be obtained from the calculations made previously. Such a surface (Fig. 2-5) results from merely setting

$$T_m = R'_{xzm} \tag{2-21a}$$

and

$$U_m = 0 \ , \tag{2-21b}$$

in Equation (2-11). Hence, this deflection surface, denoted w_2, is given by

$$w_2 = \sum_m \frac{R'_{xzm} \, a^3}{D} \, \phi_3 (\beta_m x) \sin \beta_m y \ , \tag{2-22}$$

where $\phi_3 (\beta_m x)$ is defined in Table 2-1.

The reactions along the simply supported edges are

$$(R_{yzm})_{y=0} = (-1)^m (R_{yzm})_{y=b} = - F_1 (\beta_m x) R'_{xzm} \ . \tag{2-23}$$

Using Equation (2-13b), the reaction along $y = 0$ may be written

$$(R_{yzm})_{y=0} = - R'_{xzm} \sum_r f^{1.)}_{mr} \sin \alpha_r x \ . \tag{2-24}$$

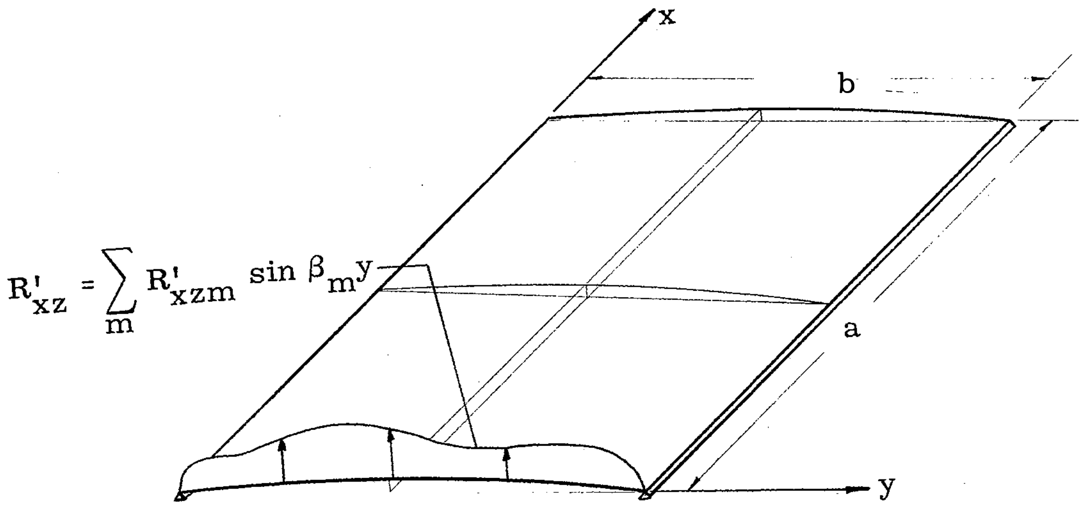


Fig. 2-5

Deflection Surface w_2

In order to obtain the expression for the deflection surface of a plate free along all edges and bending under the action of R'_{xz} along $x = 0$, it is now necessary to add surface w_1 of Equation (2-17) to surface w_2 of Equation (2-22), and adjust S_n^I and S_n^{II} so that the forces along $x = 0$, $y = b$ are practically zero. Let this sum of deflection surfaces be denoted w_R (Fig. 2-6). Then, if the surface w_1 with coefficients S_n^I and S_n^{II} in terms of R'_{xzm} is called w_{1R},

$$w_R = w_{1R} + w_2 \quad . \tag{2-25}$$

The set of 2k equations needed to solve for the coefficients is obtained by equating to zero the total reactions developed along the edges $y = 0$ and $y = b$. k equations result from each edge, giving a total of 2k equations. Thus, along the edge $y = 0$,

$$\sum_n^k S_n^I \sin \alpha_n x - \sum_m \sum_n^k (S_n^I e_{mn}^{1.)} + S_n^{II} e_{mn}^{2.)}) \sum_r (f_{mr}^{1.)} + $$

$$+ (-1)^n f_{mr}^{2.)}) \sin \alpha_r x - \sum_m \sum_r R'_{xzm} f_{mr}^{1.)} \sin \alpha_r x = 0 , \quad (2\text{-}26)$$

and along $y = b$,

$$\sum_n^k S_n^{II} \sin \alpha_n x - \sum_m \sum_n^k (-1)^m (S_n^I e_{mn}^{1.)} + S_n^{II} e_{mn}^{2.)}) \sum_r (f_{mr}^{1.)} + $$

$$+ (-1)^n f_{mr}^{2.)}) \sin \alpha_r x - \sum_m \sum_r (-1)^m R'_{xzm} f_{mr}^{1.)} \sin \alpha_r x = 0 .$$

(2-27)

A k term approximation leads to the set of equations for the edge $y=0$:

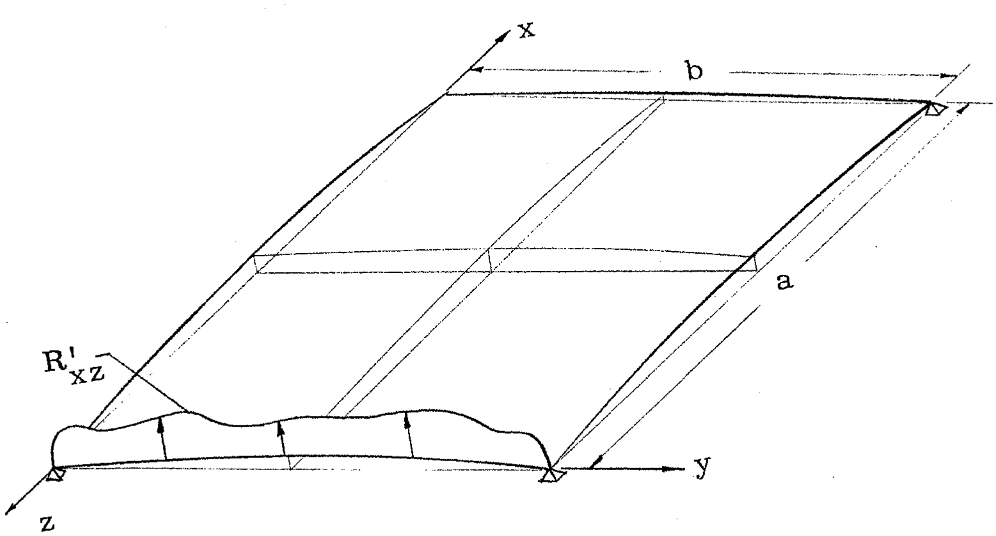


Fig. 2-6

Deflection Surface w_R

$$
\begin{bmatrix}
\left[1-\sum_{m}^{k} e_{m1}^{1.)}\left(f_{m1}^{1.)}-f_{m1}^{2.)}\right)\right] & \left[-\sum_{m}^{k} e_{m2}^{1.)}\left(f_{m1}^{1.)}+f_{m1}^{2.)}\right)\right] & \cdots & \left[-\sum_{m}^{k} e_{mk}^{2.)}\left(f_{m1}^{1.)}+(-1)^{k} f_{m1}^{2.)}\right)\right] \\
\left[-\sum_{m}^{k} e_{m1}^{1.)}\left(f_{m2}^{1.)}-f_{m2}^{2.)}\right)\right] & \left[1-\sum_{m}^{k} e_{m2}^{1.)}\left(f_{m2}^{1.)}+f_{m2}^{2.)}\right)\right] & \cdots & \left[-\sum_{m}^{k} e_{mk}^{2.)}\left(f_{m2}^{1.)}+(-1)^{k} f_{m2}^{2.)}\right)\right] \\
\cdots & \cdots & \cdots & \cdots \\
\left[-\sum_{m}^{k} e_{m1}^{1.)}\left(f_{mk}^{1.)}-f_{mk}^{2.)}\right)\right] & \left[-\sum_{m}^{k} e_{m2}^{1.)}\left(f_{mk}^{1.)}+f_{mk}^{2.)}\right)\right] & \cdots & \left[1-\sum_{m}^{k} e_{mk}^{2.)}\left(f_{mk}^{1.)}+(-1)^{k} f_{mk}^{2.)}\right)\right]
\end{bmatrix}
\begin{bmatrix}
S_{1}^{I} \\ S_{2}^{I} \\ \cdot \\ \cdot \\ \cdot \\ S_{k}^{II}
\end{bmatrix}
=
\begin{bmatrix}
\sum_{m}^{k} R'_{xzm} f_{m1}^{1.)} \\ \sum_{m}^{k} R'_{xzm} f_{m2}^{1.)} \\ \cdot \\ \cdot \\ \cdot \\ \sum_{m}^{k} R'_{xzm} f_{mk}^{1.)}
\end{bmatrix} .
\qquad (2\text{-}28)
$$

Results may be obtained from more or less than k terms. Less than k terms, however, will likely result in more error in the coefficients S_n^I and S_n^{II}; and more than k terms will eliminate the diagonal element in each additional equation. This will not necessarily enhance any additional accuracy and calls for a greater than k term approximation for the edge forces of surface w_{11}. Hence, taking k terms of these series seems entirely permissible and, in fact, logical. Doing this amounts to approximating the functions $E_1(\alpha_n y)$, $E_2(\alpha_n y)$, $F_1(\beta_m x)$, and $F_2(\beta_m x)$ by the first k terms of their Fourier series expansions.

Equations(2-28) result from expanding Equation (2-26) and then collecting coefficients corresponding to each of the k sine terms, $\sin \frac{\pi x}{a}$ through $\sin \frac{k\pi x}{a}$, for only one side of the plate. Since the resulting equation must vanish for all values of x, it is necessary that the amplitudes of each mode vanish independently. The k equations (Equations 2-28) contain 2k unknowns, $S_1^I, \cdots, S_k^I, S_1^{II}, \cdots, S_k^{II}$, which must ultimately be expressed in terms of R'_{xzm} in order to allow the deflection surface w_R to be expressed explicitly in terms of the components of this edge force. The additional equations needed are obtained from considering the edge $y = b$.

Equations(2-28) may be written in the form

$$\left[A_m^I\right]\left[S\right] = \left[R'_{xzm} f_m^{1.)}\right] , \qquad (2\text{-}29)$$

where $\left[A_m^I\right]$ is the coefficient matrix of Equations(2-28), $\left[S\right]$ is the column vector of unknown coefficients, and $\left[R'_{xzm} f_m^{1.)}\right]$ is the column vector on the right-hand side of the equation.

The matrix $\left[A_m^I\right]$ is of order k by 2k. The elements of this matrix obey a definite pattern. For this reason it is possible to write

$$\left[A_m^I\right] = \begin{bmatrix} (1+\Sigma a_{11}) & \Sigma a_{12} & \cdots & \Sigma a_{1,2k} \\ \Sigma a_{21} & (1+\Sigma a_{22}) & \cdots & \Sigma a_{2,2k} \\ \cdot & \cdot & \cdots & \cdot \\ \Sigma a_{k1} & \Sigma a_{k2} & \cdots & (1+\Sigma a_{k,2k}) \end{bmatrix} . \quad (2\text{-}30)$$

The element corresponding to the ith row and the jth column has been denoted Σa_{ij} where

$$a_{ij} = - e_{mj}^{1.)} (f_{mi}^{1.)} + (-1)^j f_{mi}^{2.)}) , \; j \leq k \quad (2\text{-}31a)$$

and

$$a_{ij} = - e_{mj}^{2.)} (f_{mi}^{1.)} + (-1)^{j-k} f_{mi}^{2.)}), \; j > k \; . \quad (2\text{-}31b)$$

As is indicated, unity must be added to this general term to obtain correct diagonal elements.

Similarly, for the edge $y = b$, Equation (2-27) may be expanded to give the equations

$$\left[A_m^{II}\right]\left[S\right] = \left[R'_{xzm} (-1)^m f_m^{1.)}\right] \; , \quad (2\text{-}32)$$

in which $\left[R'_{xzm} (-1)^m f_m^{1.)}\right]$ is identical to the right-hand side of Equation (2-28) with the exception that each element is multiplied by $(-1)^m$, and the coefficient matrix $\left[A_m^{II}\right]$ is given by

$$\left[A_m^{II}\right] = \begin{bmatrix} \Sigma \bar{a}_{11} & \Sigma \bar{a}_{12} & \cdots & \Sigma \bar{a}_{1,2k} \\ \Sigma \bar{a}_{21} & \Sigma \bar{a}_{22} & \cdots & \Sigma \bar{a}_{2,2k} \\ \cdot & \cdot & \cdots & \cdot \\ \Sigma \bar{a}_{k1} & \Sigma \bar{a}_{k2} & \cdots & (1+\Sigma \bar{a}_{k,2k}) \end{bmatrix} , \quad (2\text{-}33)$$

in which

$$\bar{a}_{ij} = (-1)^m a_{ij} = -(-1)^m e_{mj}^{1.)} (f_{mi}^{1.)} + (-1)^j f_{mi}^{2.)}), \; j \leq k \qquad (2\text{-}34a)$$

and

$$\bar{a}_{ij} = (-1)^m a_{ij} = -(-1)^m e_{mj}^{2.)} (f_{mi}^{1.)} + (-1)^{j-k} f_{mi}^{2.)}) \; j > k \, . \qquad (2\text{-}34b)$$

Equations (2-29) and (2-32) comprise the set of 2k equations, and may be written in the form

$$\begin{bmatrix} (1 + \Sigma a_{11}) & \Sigma a_{12} & \cdots & \Sigma a_{1,2k} \\ \Sigma a_{21} & (1 + \Sigma a_{22}) & \cdots & \Sigma a_{2,2k} \\ \cdot & \cdot & \cdot & \cdot \\ \Sigma a_{k1} & \Sigma a_{k2} & \cdots & \Sigma a_{k,2k} \\ \Sigma \bar{a}_{11} & \Sigma \bar{a}_{12} & \cdots & \Sigma \bar{a}_{1,2k} \\ \cdot & \cdot & \cdot & \cdot \\ \Sigma \bar{a}_{k1} & \Sigma \bar{a}_{k2} & \cdots & (1 + \Sigma \bar{a}_{k,2k}) \end{bmatrix} \begin{bmatrix} S_1^I \\ S_2^I \\ \vdots \\ S_k^I \\ S_1^{II} \\ \vdots \\ S_k^{II} \end{bmatrix} = \begin{bmatrix} \Sigma R'_{xzm} f_{m1}^{1.)} \\ \Sigma R'_{xzm} f_{m2}^{1.)} \\ \vdots \\ \Sigma R'_{xzm} f_{mk}^{1.)} \\ \Sigma R'_{xzm} (-1)^m f_{m1}^{1.)} \\ \vdots \\ \Sigma R'_{xzm} (-1)^m f_{mk}^{1.)} \end{bmatrix} . \qquad (2\text{-}35)$$

If the coefficient matrix in Equation (2-35) is denoted $\left[A_m\right]$, it is evident that

$$\left[A_m\right] = \begin{bmatrix} A_m^I \\ \hline A_m^{II} \end{bmatrix} . \qquad (2\text{-}36)$$

Using this nomenclature, Equation (2-35) may be written

$$\left[A_m\right]\left[S\right] = \begin{bmatrix} R'_{xzm} f_m^{1.)} \\ \hline R'_{xzm} (-1)^m f_m^{1.)} \end{bmatrix} . \qquad (2\text{-}37)$$

The solution of Equation (2-37) is

$$\begin{bmatrix} S_1^I \\ S_2^I \\ \vdots \\ S_k^I \\ S_1^{II} \\ \vdots \\ S_k^{II} \end{bmatrix} = \begin{bmatrix} \sum_m \gamma_{m1}^I R'_{xzm} \\ \sum_m \gamma_{m2}^I R'_{xzm} \\ \vdots \\ \sum_m \gamma_{mk}^I R'_{xzm} \\ \sum_m \gamma_{m1}^{II} R'_{xzm} \\ \vdots \\ \sum_m \gamma_{mk}^{II} R'_{xzm} \end{bmatrix} . \tag{2-38}$$

The coefficients, γ_{mn}^I and γ_{mn}^{II}, in this equation are merely the coefficients of the terms R'_{xzm} which result from premultiplying the right-hand side of Equation (2-37) by the inverse of $\left[A_m\right]$. From Equation (2-38), it is seen that

$$\left.\begin{aligned} \sum_n S_n^I &= \sum_n \sum_m \gamma_{mn}^I R'_{xzm} , \\ \sum_n S_n^{II} &= \sum_n \sum_m \gamma_{mn}^{II} R'_{xzm} . \end{aligned}\right\} \tag{2-39}$$

Thus, Equations (2-16) become

$$T_m = \sum_n \sum_r (e_{mn}^{1.)} \gamma_{rn}^I + e_{mn}^{2.)} \gamma_{rn}^{II}) R'_{xzr} , \tag{2-40a}$$

and

$$U_m = \sum_n \sum_r (-1)^n (e_{mn}^{1.)} \gamma_{rn}^{I} + e_{mn}^{2.)} \gamma_{rn}^{II}) R'_{xzr} \quad . \qquad (2\text{-}40b)$$

It should be noted that the use of an additional summation index, r, is necessary in Equations (2-40).

For large values of k it becomes necessary to solve numerically for the coefficients γ_{mn}^{I} and γ_{mn}^{II}. By means of Equations (2-39) and (2-40), the deflection surface w_1 can be written in terms of these coefficients and the components of the applied force, R'_{xz}. After substituting Equations(2-39) into Equation (2-17) and then using the relation given by Equation (2-25), the final deflection surface, w_R, becomes

$$w_R = \sum_m \frac{R'_{xzm} a^3}{D} \phi_3 (\beta_m x) \sin \beta_m y + \sum_m \sum_n \frac{R'_{xzm} a^3}{D} (\gamma_{mn}^{I} \phi_1 (\alpha_n y) +$$

$$+ \gamma_{mn}^{II} \phi_2 (\alpha_n y)) \sin \alpha_n x + \sum_m \sum_n \sum_r \frac{R'_{xzr} a^3}{D} (e_{mn}^{1.)} \gamma_{rn}^{II} +$$

$$+ e_{mn}^{2.)} \gamma_{rn}^{II})(\phi_3(\beta_m x) + (-1)^n \phi_4 (\beta_m x)) \sin \beta_m y \ . \qquad (2\text{-}41)$$

It is possible to assign to each of the terms of this equation a definite physical interpretation. The first term represents the deflection surface of a plate subjected to the edge force, R'_{xz}, with the edges parallel to the x-axis simply supported. The second term is the additional deflection due to cancelling the reactions along the simply supported edges of the plate. This term, it should be noted, has no influence on the deflection along the edges $x = 0$ and $x = a$. The third term in this equation is smaller than the first. It represents the correction to the edge deflections due to releasing the edges of the plate. Such a correction must also influence deflections elsewhere in the plate, as is confirmed by an inspection of the equation.

These interpretations allow the relative magnitude of various terms in this equation to be speculated. Intuitively, the first two terms in Equation (2-41) have a dominating influence on the final deflection for common length to width ratios. The last term, as was mentioned, is merely a correction term for the edge deflections. Its contribution to these deflections is illustrated in Fig. (2-7). The solid line represents the final deflection along the edge $x = 0$; the dashed line the deflection along that edge due to the first term in Equation (2-41).

The following array results if k terms of this correction are expanded along $x = 0$:

$$\left[R'_{xz1}\sum_n b_{11} + R'_{xz2}\sum_n b_{12} + \cdots R'_{xzk}\sum_n b_{1k}\right]\frac{a^3}{D}\sin\frac{\pi y}{b} +$$

$$\left[R'_{xz1}\sum_n b_{21} + R'_{xz2}\sum_n b_{22} + \cdots R'_{xzk}\sum_n b_{2k}\right]\frac{a^3}{D}\sin\frac{2\pi y}{b} +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots +$$

$$\left[R'_{xz1}\sum_n b_{k1} + R'_{xz2}\sum_n b_{k2} + \cdots R'_{xzk}\sum_n b_{kk}\right]\frac{a^3}{D}\sin\frac{k\pi y}{b} ,$$

(2-42a)

where

$$b_{ij} = (e^{1.)}_{in}\,\gamma^{I}_{jn} + e^{2.)}_{in}\,\gamma^{II}_{jn})(\phi_3(0,i) + (-1)^n\,\phi_4(0,i)) , \quad (2\text{-}42b)$$

and $\phi_3(0, i)$, $\phi_4(0, i)$ represent the functions $\phi_3(\beta_m x)$ and $\phi_4(\beta_m x)$, of Equation (2-11), evaluated at $x = 0$ for $m = i$.

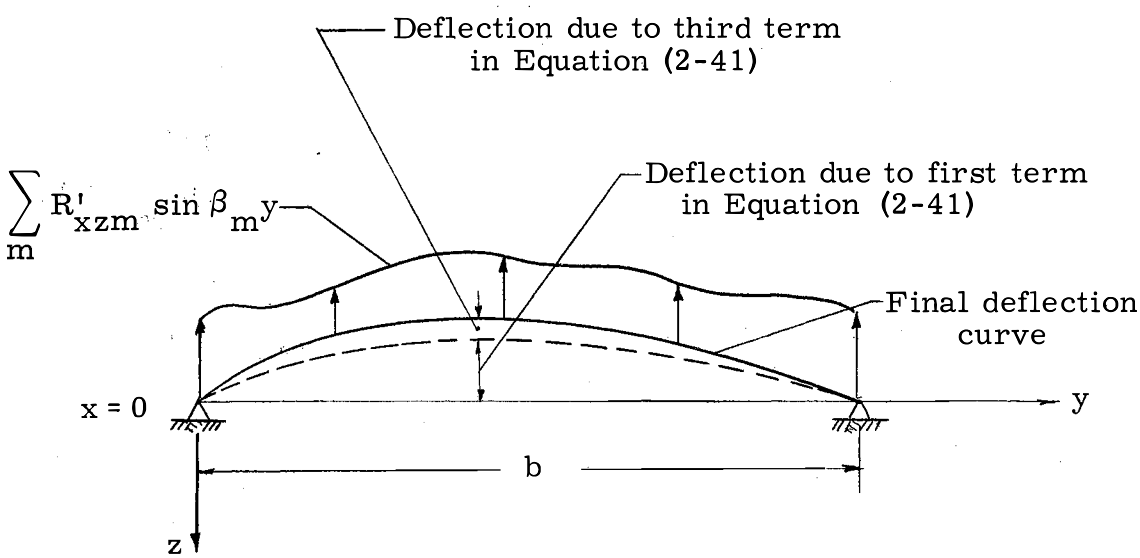


Fig. 2-7

Physical Interpretation of Terms of the

Deflection Surface w_R

Since the elements of this array are composed of products of terms of convergent series, a strong diagonal can be expected. The magnitude of these diagonal elements is greatly increased when the influence of the first term of Equation (2-41) is introduced. If this is done, Equation (2-42a) becomes

$$\left[R'_{xz1}\left(\phi_3(0,1)+\sum_n b_{11}\right)+R'_{xz2}\sum_n b_{12}+\cdots+R'_{xzk}\sum_n b_{1k}\right]\frac{a^3}{D}\sin\frac{\pi y}{b}+$$

$$\left[R'_{xz1}\sum_n b_{12}+R'_{xz2}\left(\phi_3(0,2)+\sum_n b_{22}\right)+\cdots+R'_{xzk}\sum_n b_{1k}\right]\frac{a^3}{D}\sin\frac{2\pi y}{b}+$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\left[R'_{xz1}\sum_n b_{1k}+R'_{xz2}\sum_n b_{2k}+\cdots+R'_{xzk}\left(\phi_3(0,k)+\sum_n b_{kk}\right)\right]\frac{a^3}{D}\sin\frac{k\pi y}{b}\ .$$

(2-43)

As a consequence of this strong diagonal, an approximate form of Equation (2-43) may be formulated by neglecting elements not appearing on the diagonal in comparison to the diagonal elements. The limitations and validity of this approximation will be investigated in a later chapter; but, it is worthwhile to mention that, in the related problem of a plate with clamped edges, similar arrays occur and the diagonal elements are often as much as twenty times larger than the largest non-diagonal element occuring in a given row. With this approximation, Equation (2-43) may be reduced to the series

$$\sum_m \frac{R'_{xzm}\,a^3}{D}\left[\phi_3(0,m)+\sum_n\left(e^{1.)}_{mn}\,\gamma^{I}_{mn}+e^{2.)}_{mn}\,\gamma^{II}_{mn}\right)\left(\phi_3(0,m)+\right.\right.$$

$$\left.\left.+\ (-1)^n\,\phi_4(0,m)\right)\right]\sin\beta_m y\ . \qquad (2\text{-}44)$$

If this modification is extended to incorporate all values of x, Equation (2-41) reduces to

$$w_R = \sum_m \frac{R'_{xzm} a^3}{D} \Big[\phi_3(\beta_m x) + \sum_n \Big(e^{1.)}_{mn} \gamma^{I}_{mn} + e^{2.)}_{mn} \gamma^{II}_{mn} \Big) \Big(\phi_3(\beta_m x) +$$

$$+ (-1)^n \phi_4(\beta_m x) \Big) \Big] \sin \beta_m y + \sum_m \sum_n \frac{R'_{xzm} a^3}{D} \Big(\gamma^{I}_{mn} \phi_1(\alpha_n y) +$$

$$+ \gamma^{II}_{mn} \phi_2 (\alpha_n y) \Big) \sin \alpha_n x \quad . \qquad (2\text{-}45)$$

By using a procedure similar to that just completed, it is possible to find the deflection surface of a plate supported only at its corners with an applied moment along the edge $x = 0$. This is accomplished by first obtaining the deflection surface of a plate simply supported along the edges parallel to the x-axis, free along the remaining edges, and subjected to the action of an applied moment distributed along the edge $x = 0$ of the form

$$M'_x = \sum_m M'_{xm} \sin \beta_m y \quad . \qquad (2\text{-}46)$$

This deflection surface, denoted w_3 and shown in Fig. 2-8, may be written

$$w_3 = \sum_m \frac{M'_{xm} a^2}{D} \phi_5 (\beta_m x) \sin \beta_m y \quad , \qquad (2\text{-}47)$$

where ϕ_5 is tabulated in Table 2-1.

The reactions developed along the simply supported edges are

$$(R_{yzm})_{y=0} = (-1)^m (R_{yzm})_{y=b} = - \frac{G(\beta_m x)}{a} M'_{xm} \quad . \qquad (2\text{-}48)$$

Refer, as before, to Table 2-2 for the deflection function $G(\beta_m x)$.

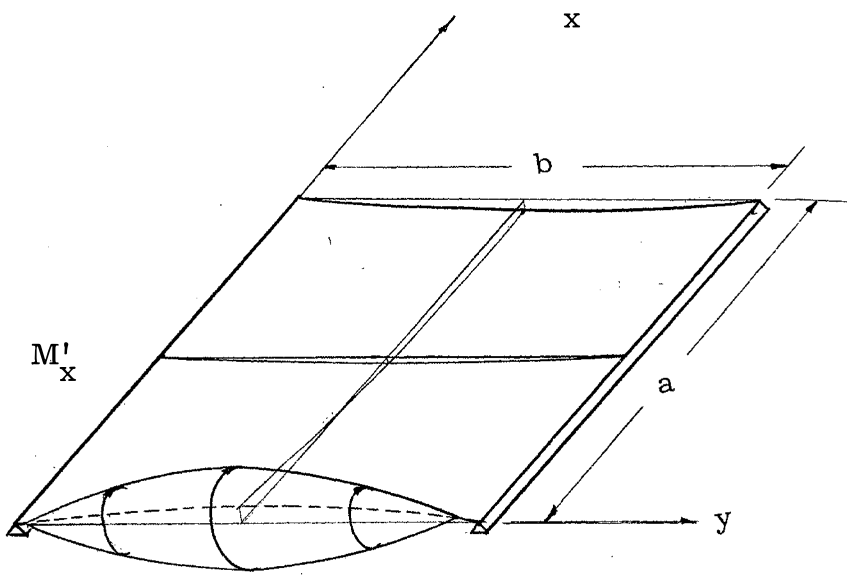


Fig. 2-8

Deflection Surface w_3

The functions $\phi_5(\beta_m x)$ and $G(\beta_m x)$ may be represented in the Fourier series

$$\phi_5 (\beta_m x) = \sum_r \phi_{mr}^{5.)} \sin \alpha_r x \quad ,$$

$$G(\beta_m x) = \sum_r g_{mr} \sin \alpha_r x \tag{2-49}$$

The coefficients $\phi_{mr}^{5.)}$ and g_{mn} are given in Tables 2-3B.

Thus, the reactive force distribution along the edge $y = 0$ may be written

$$(R_{yzm})_{y=0} = -\frac{1}{a} M'_{xm} \sum_{r} g_{mr} \sin \alpha_r x \qquad (2\text{-}50)$$

In order to eliminate this reaction, it is necessary to adjust the coefficients, S_n^I and S_n^{II}, of deflection surface w_1 and obtain some new deflection surface, w_{1M}, which has reactions along $y = 0, b$ equal and opposite to those given by Equation (2-48). When w_{1M} is added to the deflection surface w_3 of Equation (2-47), the deflection surface of a plate supported only at its corners and acted upon by M'_x along the edge $x = 0$ is obtained. This deflection surface will be called w_M. (Fig. 2-9). Therefore,

$$w_M = w_{1M} + w_3 . \qquad (2\text{-}51)$$

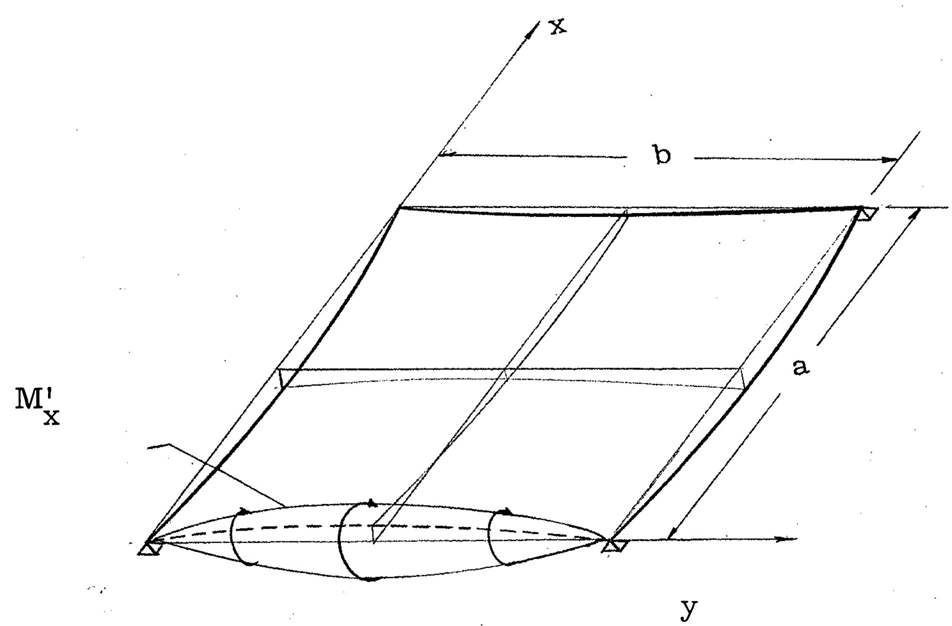


Fig. 2-9

Deflection Surface w_M

The set of 2k equations needed to solve for the coefficients S_n^I and S_n^{II} arises from equating to zero the sum of the reactions given by Equations (2-18) and (2-19) and those of Equation (2-48). Thus, for all values of x along y=0 ,

$$\sum_n S_n^I \sin \alpha_n x - \sum_m \sum_n \left(S_n^I e_{mn}^{1.)} + S_n^{II} e_{mn}^{2.)} \right) \sum_r \left(f_{mr}^{1.)} + \right.$$

$$\left. + (-1)^n f_{mr}^{2.)} \right) \sin \alpha_r x - \sum_m \sum_r \frac{M'_{xm}}{a} g_{mr} \sin \alpha_r x = 0 \qquad (2\text{-}52a)$$

$$\sum_n S_n^{II} \sin \alpha_n x - \sum_m \sum_n (-1)^m \left(S_n^I e_{mn}^{1.)} + S_n^{II} e_{mn}^{2.)} \right) \sum_r \left(f_{mr}^{1.)} + \right.$$

$$\left. + (-1)^n f_{mr}^{2.)} \right) \sin \alpha_r x - \sum_m \sum_r \frac{(-1)^m M'_{xm}}{a} g_{mr} \sin \alpha_r x = 0 .$$

(2-52b)

As before, these expressions may be expanded for a k-term approximation to yield the set of equations

$$\left[A_{mi} \right] \left[S \right] = \frac{1}{a} \left[M'_{xm}\, g_m \right] = \frac{1}{a} \begin{bmatrix} \sum_m M'_{xm}\, g_{m1} \\ \sum_m M'_{xm}\, g_{m2} \\ \cdot \\ \cdot \\ \cdot \\ \sum_m M'_{xm} (-1)^m g_{mk} \end{bmatrix} . \qquad (2\text{-}53)$$

The matrices on the left-hand side of Equation (2-53) are identical to those appearing in Equation (2-37). The solution of Equation (2-53) is

$$\begin{bmatrix} S_1^I \\ S_2^I \\ \vdots \\ S_k^I \\ S_1^{II} \\ S_2^{II} \\ \vdots \\ S_k^{II} \end{bmatrix} = \frac{1}{a} \begin{bmatrix} \sum_m \lambda_{m1}^I \ M'_{xm} \\ \sum_m \lambda_{m2}^I \ M'_{xm} \\ \vdots \\ \sum_m \lambda_{mk}^I \ M'_{xm} \\ \sum_m \lambda_{m1}^{II} \ M'_{xm} \\ \sum_m \lambda_{m2}^{II} \ M'_{xm} \\ \vdots \\ \sum_m \lambda_{mk}^{II} \ M'_{xm} \end{bmatrix} , \tag{2-54}$$

where λ_{mn}^I and λ_{mn}^{II} are the coefficients of the terms M'_{xm} which result from premultiplying the right-hand side of Equation (2-53) by the inverse of $\left[A_m\right]$. From Equation (2-54) it follows that

$$\left.\begin{aligned} \sum_n S_n^I &= \sum_m \sum_n \lambda_{mn}^I \frac{M'_{xm}}{a} , \\ \sum_n S_n^{II} &= \sum_m \sum_n \lambda_{mn}^{II} \frac{M'_{xm}}{a} . \end{aligned}\right\} \tag{2-55}$$

Substituting these expressions into Equation (2-17) defines surface w_1 in terms of the components of the applied moment, M'_x. If this result, along with deflection surface w_3 of Equation (2-47), is introduced into Equation (2-51), it is found that the final deflection surface, w_M, takes the form

$$w_M = \sum_m \frac{M'_{xm} a^2}{D} \phi_5(\beta_m x) \sin \beta_m y + \sum_m \sum_n \frac{a^2 M'_{xm}}{D} \Big(\lambda^I_{mn} \phi_1 (\alpha_n y) +$$

$$+ \lambda^{II}_{mn} \phi_2 (\alpha_n y) \Big) \sin \alpha_n x + \sum_m \sum_n \sum_r \frac{M'_{xr} a^2}{D} \Big(e^{1.)}_{mn} \lambda^I_{rn} +$$

$$+ e^{2.)}_{mn} \lambda^{II}_{rn} \Big) \Big(\phi_3 (\beta_m x) + (-1)^n \phi_4 (\beta_m x) \Big) \sin \beta_m y \ . \qquad (2\text{-}56)$$

The first term in this equation represents the deflection of a plate simply supported along the edges parallel to the x-axis and bent by the moment M'_x distributed along the free edge, $x = 0$. The second term represents the additional deflection due to cancelling the reactions developed along the supports and releasing the edges $y = 0$, b. This term does not effect the deflection along the free edges $x = 0$ and $x = a$. Finally, the third term yields the corrections to the edge deflections of term one due to releasing the simply supported edges. It also influences deflections elsewhere in the plate.

Following an argument similar to that used for the deflection surface w_R, it is possible to express Equation (2-56) in the approximate form

$$w_M = \sum_m \frac{M'_{xm} a^2}{D} \Big[\emptyset_5 (\beta_m x) + \sum_n \left(e^{1.)}_{mn} \lambda^{I}_{mn} + e^{2.)}_{mn} \lambda^{II}_{mn} \right) \Big(\emptyset_3 (\beta_m x) +$$

$$+ (-1)^n \emptyset_4 (\beta_m x) \Big) \Big] \sin \beta_m y + \sum_m \sum_n \frac{M'_{xm} a^2}{D} \Big(\lambda^{I}_{mn} \emptyset_1 (\alpha_n y) +$$

$$+ \lambda^{II}_{mn} \emptyset_2 (\alpha_n y) \Big) \sin \alpha_n x \quad . \qquad (2\text{-}57)$$

The equations (2-41, 45, 56, and 57) define the deflection surfaces of plates free on all edges, supported at the corners, and bending under the action of applied forces or moments distributed arbitrarily along one edge. A typical plate in a plate beam structure may be subjected not only to edge loading but also to loads normal to its plane. The cases of free-edged plates under this type of loading will be discussed in the next article.

2-4. Normally Loaded Plates Supported at Their Corners

The deflection surfaces of plates with free edges, supported only at their corners, and bending under the action of any of the standard types of normal loading may be formulated according to the same philosophy used in the previous article. The general procedure may be outlined as follows:

(1) Obtain a Levy solution for the deflection surface, w_o, of a plate simply supported along the edges parallel to the x-axis, free along the remaining edges, and subjected to normal loads of any desired variation.

(2) Compute the reactions along the simply supported edges and represent them in Fourier sine series.

(3) Formulate a set of equations for S_n^I and S_n^{II} of deflection surface w_1 that will adjust these coefficients so that on superimposing w_o and the surface w_1 the reactions along the simply supported edges will be cancelled out.

(4) Solve this set of equations and record for the final deflection surface the sum of w_{10} and w_o, where w_{10} is the surface w_1 after the coefficients S_n^I and S_n^{II} have been determined.

If a plate supported along $y = 0, b$, free along $x = 0, a$, and bending under the action of some load $p(x, y)$ is considered, the governing differential equation of the plate no longer is homogeneous and a particular solution must be found.

In general, any load may be expressed as a trigonometric series of the form

$$p(x, y) = \sum_m P(x) \sin \beta_m y , \tag{2-58}$$

where $P(x)$ is some function of x. If the load varies only with respect to y, it is called a one directional load variation and may be represented by

$$p(y) = \sum_m P_m \sin \beta_m y , \tag{2-59}$$

where P_m is some coefficient independent of x and y determined by the integral

$$P_m = \frac{2}{b} \int_0^b p(y) \sin \beta_m y \, dy . \tag{2-60}$$

Substituting into Equation (1-4) gives

$$\nabla^4 w_o = \frac{P(y)}{D} = \frac{1}{D}\sum_m P_m \sin \beta_m y \,, \tag{2-61}$$

where, again, w_o is the deflection surface of a plate supported only along $y = 0$, b.

The solution to this equation may be taken to be of the form

$$w_o = w_{oc} + w_{op} \,, \tag{2-62a}$$

in which w_{oc} is the complementary solution,

$$w_{oc} = \sum Z(\beta_m x) \sin \beta_m y = \sum [A_m \operatorname{ch} \beta_m x + B_m \operatorname{sh} \beta_m x + $$

$$+ C_m \beta_m x \operatorname{ch} \beta_m x + D_m \beta_m x \operatorname{sh} \beta_m x] \sin \beta_m y, \tag{2-62b}$$

of the homogeneous equation

$$\nabla^4 w_{oc} = 0 \,, \tag{2-62c}$$

and w_{op} is any particular solution of the equation

$$\nabla^4 w_{op} = \frac{1}{D}\sum_m P_m \sin \beta_m y \,. \tag{2-62d}$$

The coefficients A_m, B_m, C_m, and D_m of Equation (2-62b) are arbitrary constants.

The deflection curve of a simply supported wide beam must satisfy the differential equation (Equation 2-62d) and therefore may be taken as a particular solution. Hence, if

$$w_{op} = w_{op}(y) = \sum_m E_m \sin \beta_m y \quad , \tag{2-63a}$$

then

$$\nabla^4 w_{op} = \sum_m \beta_m^4 E_m \sin \beta_m y = \sum_m \frac{P_m}{D} \sin \beta_m y \quad , \tag{2-63b}$$

and therefore

$$E_m = \frac{P_m}{\beta_m^4 D} \quad . \tag{2-63b}$$

Deflection surface w_o becomes

$$w_o = w_{op}(y) + \sum_m Z(\beta_m x) \sin \beta_m y = \sum_m \left(\frac{P_m}{\beta_m^4 D} + Z(\beta_m x) \right) \sin \beta_m y \quad . \tag{2-64}$$

This function must satisfy the conditions

$$\left(\frac{\partial^2 w_o}{\partial x^2} + \mu \frac{\partial^2 w_o}{\partial y^2} \right)_{x=0,\, a} = 0 \quad , \tag{2-65a}$$

$$\left(\frac{\partial^3 w_o}{\partial x^3} + (2 - \mu) \frac{\partial^3 w_o}{\partial x \, \partial y^2} \right)_{x=0,\, a} = 0 \quad . \tag{2-65b}$$

By using these conditions, the coefficients of Equation (2-62b) may be evaluated. After performing the necessary calculations, it is found that the final deflection surface (Fig. 2-10) is

$$w_o = \sum_m \frac{P_m a^4}{D} \left(\frac{1}{(\beta_m a)^4} + \phi_o (\beta_m x) \right) \sin \beta_m y \quad , \tag{2-66}$$

where ϕ_o is given in Table 2-1.

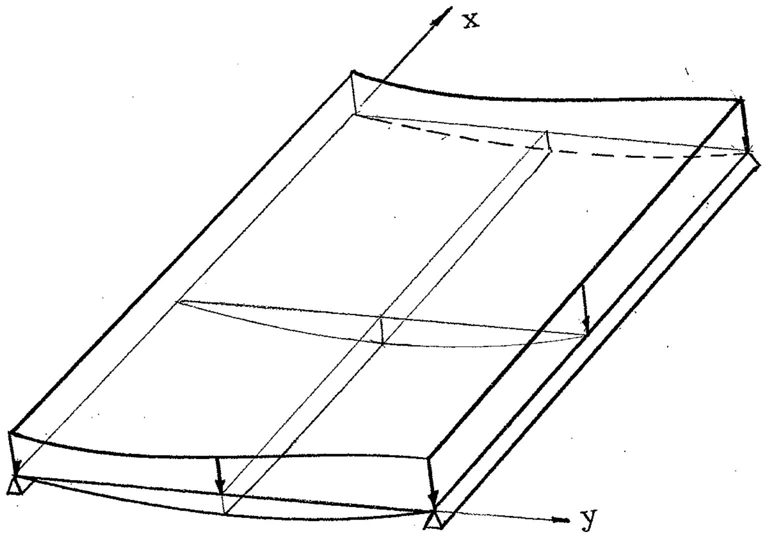


Fig. 2-10

Deflection Surface w_o Due to a One-Directional

Load Variation

Fourier coefficients for some typical types of loading are given in Table 2-4. It should be noted that the function $\phi_o(\beta_m x)$ vanishes for $\mu = 0$, and the deflection surface reduces to that of a simply supported wide beam.

The reactions along the simply supported edges are

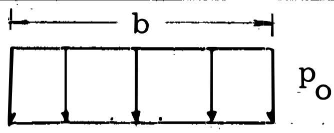
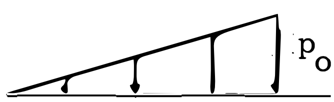
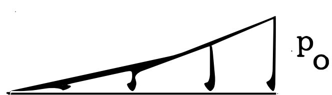
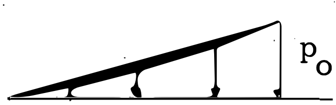
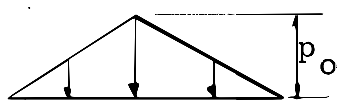
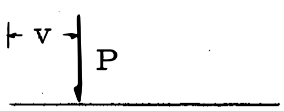
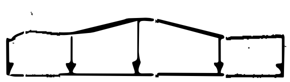
$$(R_{yzm})_{y=o} = (-1)^m (R_{yzm})_{y=b} = - P_m a\, Q(\beta_m x) + (BR_{yzm}) , \quad (2\text{-}67)$$

where (BR_{yzm}) is the coefficient of the Fourier series for the reaction of a simply supported wide beam and $Q(\beta_m x)$ is a deflection function recorded in Table 2-2.

The functions, $\phi_o(\beta_m x)$ and $Q(\beta_m x)$, may be represented by the

TABLE 2-4

COEFFICIENTS P_m FOR ONE DIRECTIONAL LOAD VARIATIONS

Case	p (y)	Sketch	P_m
1.*	p_o		$\frac{4 p_o}{m\pi}$
2.	$\frac{p_o y}{b}$		$\frac{-2(-1)^m p_o}{m\pi}$
3.	$\frac{p_o y^2}{b^2}$		$\frac{[2(-1)^m (2-m^2\pi^2)-4]p_o}{m\pi\, \beta_m^3}$
4.	$p_o y^n$		$\frac{2p_o}{b^{n+1}} \int_o^b y^n \sin \beta_m y\, dy$
5.*	Triangular		$\frac{8 p_o (-1)^{\frac{m-1}{2}}}{m^2\pi^2}$
6.	Line		$\frac{P \sin v}{b}$
7.	p(y)		$\frac{2}{b} \int_o^b p(y) \sin \beta_m y\, dy$

* m = 1, 3, 5 · · ·

Fourier series

$$\phi_o(\beta_m x) = \sum_r \phi^\circ_{mr} \sin \alpha_r x \, , \qquad Q(\beta_m x) = \sum_r q_{mr} \sin \alpha_r x \, . \quad (2\text{-}68)$$

The coefficients, ϕ°_{mr} and q_{mr}, are defined in Tables 2-3A and 2-3B. Thus Equation (2-67) can be written

$$(R_{yzm})_{y=0} = (R_{yzm})_{y=b}(-1)^m = -P_m a \sum_r q_{mr} \sin \alpha_r x + (BR_{yzm})_{y=0} \quad (2\text{-}69)$$

According to step (3.) of the procedure outlined at the beginning of this article, it is now necessary to superimpose surfaces w_o and w_1 and eliminate the reactions given by Equation (2-69). This leads to the set of equations

$$\sum_n S^I_n \sin \alpha_n x - \sum_m \sum_n \left(S^I_n e^{1.)}_{mn} + S^{II}_n e^{2.)}_{mn} \right) \sum_r \left(f^{1.)}_{mr} + (-1)^n f^{2.)}_{mr} \right)$$

$$\times \sin \alpha_r x - \sum_m \sum_r P_m a \, q_{mr} \sin \alpha_r x + (BR_{yzm})_{y=0} = 0 \, ,$$

$$(2\text{-}70)$$

$$\sum_n S^{II}_n \sin \alpha_n x - \sum_m \sum_n \left((-1)^m \; S^I_n e^{1.)}_{mn} + S^{II}_n e^{2.)}_{mn} \right) \sum_r \left(f^{1.)}_{mr} + (-1)^n f^{2.)}_{mr} \right)$$

$$\times \sin \alpha_r x - \sum_m \sum_r (-1)^m P_m a \, q_{mr} \sin \alpha_r x + (BR_{yzm})_{y=b} = 0 \, .$$

The Equations (2-70) may be written in matrix notation as follows:

$$\left[A_m\right]\left[S\right] = a\left[P_m\, q_m\right] = a\begin{bmatrix} \sum_m P_m\, q_{m1} + (BR_{yz1})_{y=0} \\ \sum_m P_m\, q_{m2} + (BR_{yz2})_{y=0} \\ \cdot \\ \cdot \\ \cdot \\ \sum_m (-1)^m\, P_m\, q_{mk} + (BR_{yzk})_{y=b} \end{bmatrix}, \qquad (2\text{-}71)$$

where the matrices on the left hand side of this expression are the same as those of Equations (2-37) and (2-53).

If the solution to Equation (2-71) is such that

$$\sum_n S_n^I = \sum_m \sum_n \varsigma_{mn}^I\, a\, P_m$$

and (2-72)

$$\sum_n S_n^{II} = \sum_m \sum_n \varsigma_{mn}^{II}\, a\, P_m \quad ,$$

the final deflection surface may be written (Fig. 2-11)

$$w_L = \sum_m \frac{P_m a^4}{D}\left(\frac{1}{\beta_m^4\, a^4} + \phi_o(\beta_m y)\right)\sin\beta_m y + \sum_m \sum_n \frac{P_m a^4}{D}\left(\varsigma_{mn}^I\; \phi_1(\alpha_n y) + \right.$$

$$\left. + \varsigma_{mn}^{II}\; \phi_2(\alpha_n y)\right).\sin\alpha_n x + \sum_m \sum_n \sum_r \frac{P_r a^4}{D}\left(e_{mn}^{1.)}\; \varsigma_{rn}^I + \right.$$

$$\left. + e_{mn}^{2.)}\; \varsigma_{rn}^{II}\right)\left(\phi_3(\beta_m x) + (-1)^n\; \phi_4(\beta_m x)\right)\sin\beta_m y \quad . \qquad (2\text{-}73)$$

As before, the first term represents the deflection surface if edges parallel to the x-axis are simply supported, and the last two terms the correction due to releasing these edges.

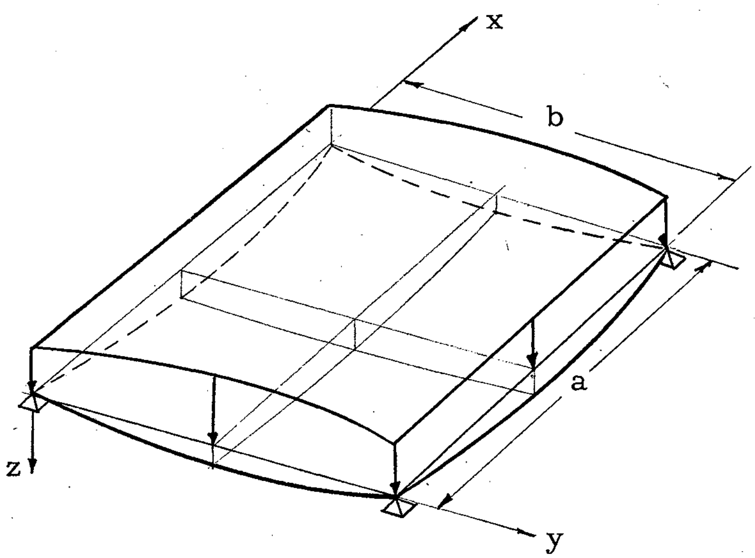


Fig. 2-11

Deflection Surface w_L Due to a One-Directional Load Variation

Introducing the same approximation used to obtain Equations (2-45 and 57), Equation (2-73) reduces to

$$w_L = \sum_m \frac{P_m a^4}{D}\left[\frac{1}{\beta_m^4 a^4} + \phi_o(\beta_m x) + \sum_n \left(e_{mn}^{1.)}\, \zeta_{mn}^{I} + e_{mn}^{2.)}\, \zeta_{mn}^{II}\right)\left(\phi_3(\beta_m x) + (-1)^n\, \phi_4(\beta_m x)\right)\right] \sin \beta_m y + \sum_m \sum_n \frac{P_m a^4}{D}\left(\zeta_{mn}^{I}\, \phi_1(\alpha_n y) + \zeta_{mn}^{II}\, \phi_2(\alpha_n y)\right) \sin \alpha_n x \, . \qquad (2\text{-}74)$$

If the applied loading varies both with respect to x and y, the problem of obtaining the deflection surface becomes more complicated. For such loading cases Equation (2-58) becomes

$$p(x, y) = \sum_m \sum_j P_{mj} \sin \alpha_j x \sin \beta_m y \ ,$$

where

$$\alpha_j = \frac{j\pi}{a} , \quad j = 1, 2, 3, \cdots ,$$

and

$$P_{mj} = \frac{4}{ab} \int_0^a \int_0^b p(x, y) \sin \alpha_j x \sin \beta_m y \, dx \, dy \ .$$

Timoshenko(39) gives for the deflection of a plate simply supported on all edges and loaded by the above loading the Navier solution

$$w = \frac{1}{\pi^4 D} \sum_m \sum_j \frac{P_{mj}}{\left(\frac{j^2}{a^2} + \frac{m^2}{b^2}\right)^2} \sin \alpha_j x \sin \beta_m y \ , \qquad (2\text{-}75)$$

for which the reactions along $x = 0, a$ are

$$\sum_j (R_{xzm})_{x=0} = \sum_j (-1)^j (R_{xzm})_{x=a} = \sum_j \frac{P_{mj}(\alpha_j^3 + (2-\mu)\alpha_j \beta_m^2)}{\pi^4\left((j/a)^2 + (m/b)^2\right)^2} \sin \beta_m y \ . \qquad (2\text{-}76)$$

By choosing T_m and U_m of Equation (2-11) to be equal and opposite to the corresponding coefficient above, and then superimposing surface w_{12} and the surface of Equation (2-75), the final deflection surface, w_o, is obtained. The remaining steps in the solution are identical to those discussed previously.

A similar procedure may be used to obtain deflection surfaces of plates that experience non-uniform temperature changes or other special effects.

Fortunately, the most important of these unsymmetrical loading cases, that of a concentrated load, may be formulated in a simpler manner. This may be accomplished by first considering a plate simply supported on all edges and subjected to a concentrated load, P, at point (u, v). Timoshenko[(39)] gives for the deflection surface of this plate the single series

$$w = \sum_{m} \frac{Pa^2}{D} \frac{\text{sh}\, \beta_m u \sin \beta_m v}{m\pi\, \beta_m^2\, a^2\, \text{sh}\, \beta_m a} \left(\rho_m^u\, \text{sh}\, \beta_m x_1 - \beta_m x_1\, \text{ch}\, \beta_m x_1 \right) \sin \beta_m y \ , \tag{2-77a}$$

where

$$x_1 = a - x \ , \tag{2-77b}$$

and

$$\rho_m^u = 1 + \beta_m a \,\text{cth}\, \beta_m a - \beta_m u \,\text{cth}\, \beta_m u \ . \tag{2-77c}$$

This equation is valid for $x \geq u$. For $x < u$, it is necessary to replace x_1 with x, u with $u_1 = a - u$, and ρ_m^u with

$$\rho_m^{u1} = 1 + \beta_m a \,\text{cth}\, \beta_m a - \beta_m u_1 \,\text{cth}\, \beta_m u_1 \ .$$

Referring to Equation (2-11), if T_m is made equal and opposite to the reactions of the plate of Equation (2-77a) developed along $x = 0$, and U_m equal and opposite to those developed along $x = a$, then by adding deflection surface w_2 to that of Equation (2-77a), the deflection surface of a plate supported only along $y = 0, b$ and subjected to the concentrated load, P, is obtained (Fig. 2-12). Such calculations yield

$$w_o = w_4 + \sum_m \frac{P(1-\mu)a^3 \sin \beta_m v}{D b \, sh \, \beta_m a} \left[sh \, \beta_m u \left(\frac{1+\mu}{1-\mu} + \rho_m^u \right) \phi_4(\beta_m x) + \right.$$

$$\left. + \; sh \, \beta_m u_1 \left(\frac{1+\mu}{1-\mu} + \rho_m^{u1} \right) \phi_3(\beta_m x) \right] \sin \beta_m y \;, \qquad (2\text{-}78)$$

in which w_4 is the surface defined by Equation (2-77a).

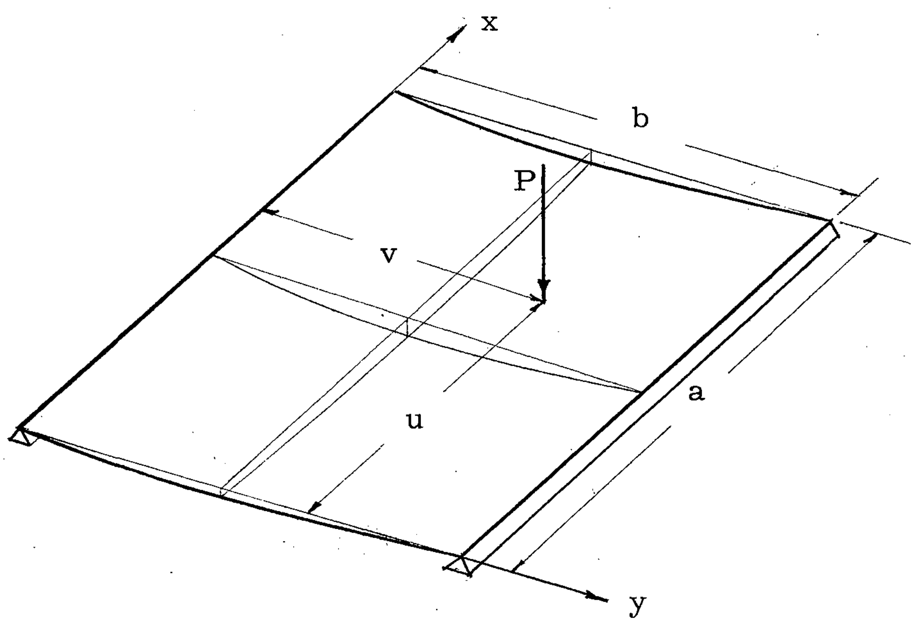


Fig. 2-12

Deflection Surface w_o Due to a

Concentrated Load

The remaining calculations are similar to those previously discussed. If the coefficients in Equation (2-17) are such that

$$\left.\begin{aligned}\sum_n S_n^{I} &= \sum_m \sum_n \xi_{mn}^{I}\,P \\ \sum_n S_n^{II} &= \sum_m \sum_n \xi_{mn}^{II}\,P \quad ,\end{aligned}\right\} \qquad (2\text{-}79)$$

the final deflection surface, w_L', for this loading case is (Fig. 2-13)

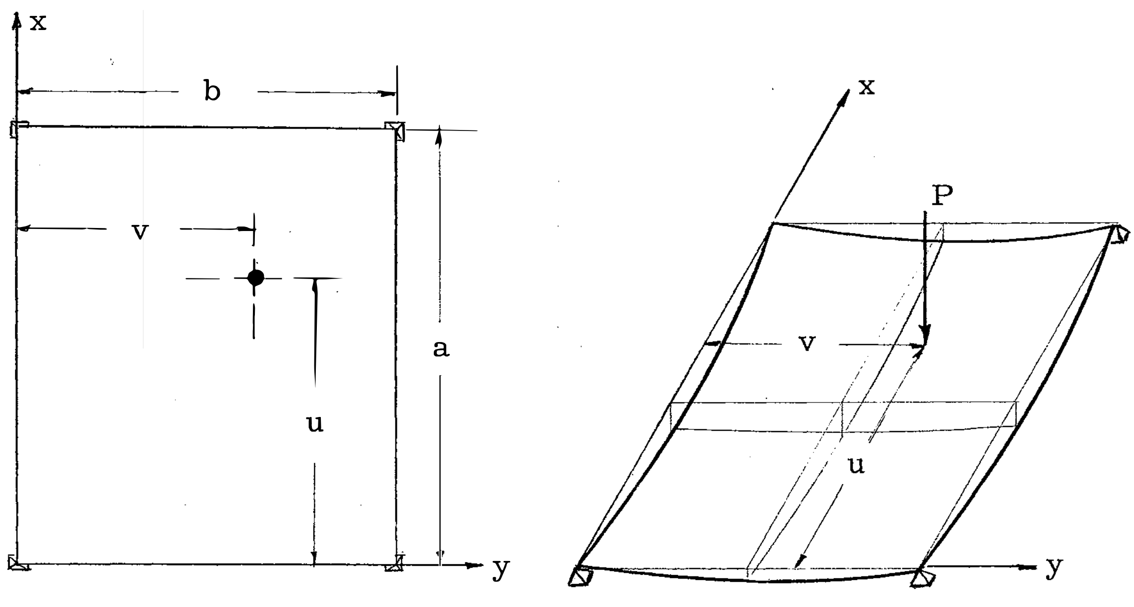
$$w_L' = w_o + \sum_m \sum_n \frac{a^3}{D}\left(\xi_{mn}^{I}\, \phi_1(\alpha_n y) + \xi_{mn}^{II}\, \phi_2(\alpha_n y)\right) \sin \alpha_n x \; +$$

$$+ \sum_m \sum_n \sum_r \frac{Pa^3}{D}\left(e_{mn}^{1.)}\, \xi_{rn}^{I} + e_{mn}^{2.)}\, \xi_{rn}^{II}\right)\left(\phi_3(\beta_m x) \; +\right.$$

$$\left. + \; (-1)^n\, \phi_4(\beta_m x)\right) \sin \beta_m y \quad . \qquad (2\text{-}80)$$

The term, w_o, in this equation is given by Equation (2-78). It should be noted that ξ_{mn}^{I} and ξ_{mn}^{II} are functions of u and v, the coordinates of the load.

Equation (2-80) can be simplified by writing it in an approximate form similar to that of Equations (2-45, 57, and 74). This approximation is not necessary, however, for the deflection surfaces developed by applied loads, since all terms are known quantities.



(a) (b)

Fig. 2-13

Deflection Surface w_L Due to a

Concentrated Load

2-5. Displacement of Supports of Free-Edged Plates

In addition to edge effects and applied normal loads, plates in plate-beam structures may experience corner displacements. The solution of a plate supported only at its corners with one or more corners undergoing an arbitrary displacement will be discussed in this article.

It is well known that the solution for a plate subjected to concentrated corner loads, P, (Fig. 2-14) is the hyperbolic surface

$$w = \frac{-4P(x^2 - y^2)}{D(1 - \mu)} .$$

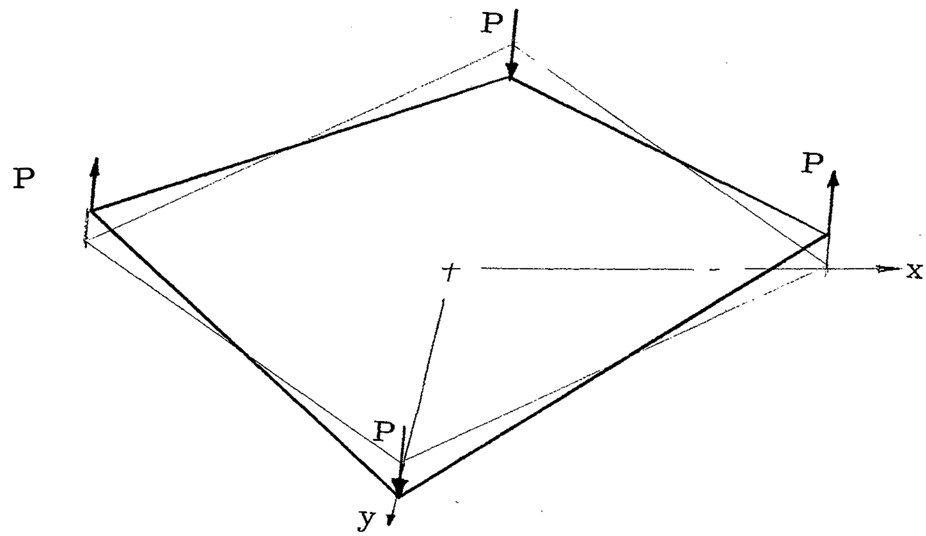


Fig. 2-14

Plate Bent by Concentrated

Corner Loads

If a solution is sought for a plate having its corners displaced an amount Δ, it is merely necessary to consider the more general form

$$w = C_1xy + C_2x + C_2y + C_3 .$$

C_1, C_2, C_3, and C_4 are constants to be determined through the use of Equation (1-12) and the corner conditions

$$w_{x=o,\,y=o} = \Delta_{oo}$$

$$w_{x=o,\,y=b} = \Delta_{ob}$$

$$w_{x=a,\,y=o} = \Delta_{ao}$$

with the origin now at the plates corner. The final deflection surface for this case is

$$w = \frac{-R}{2D(1-\mu)} xy + \frac{\Delta_{ao} - \Delta_{oo}}{a} x + \frac{\Delta_{ob} - \Delta_{oo}}{b} y + \Delta_{oo} \,, \quad (2\text{-}81)$$

in which R is the concentrated corner reaction. This plate is in a state of pure twist.

If only the corner (a, b) undergoes a displacement, say Δ_{ab}, the deflection surface is (Fig. 2-15)

$$w = \frac{-Rxy}{2D(1-\mu)} = \frac{\Delta_{ab}xy}{ab} \,. \quad (2\text{-}82)$$

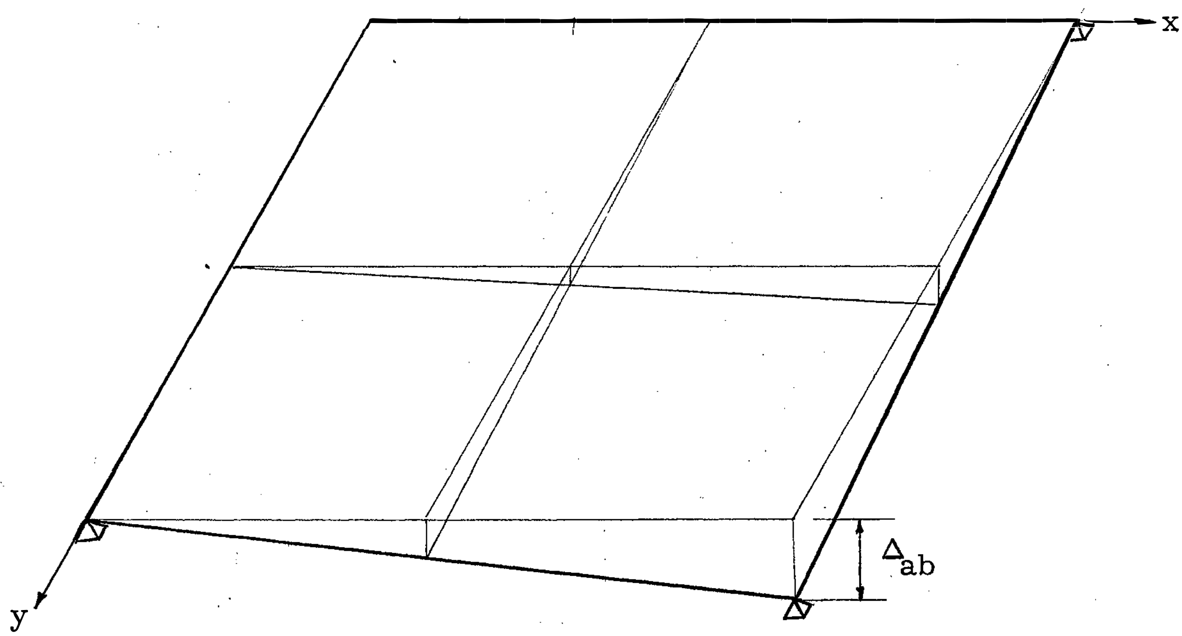


Fig. 2-15

Plate With Corner Displacement

Several interesting plate problems may be easily formulated now that these results have been established. For example, if R in Equation (2-82) is chosen so that

$$R = R_L = 2D(1-\mu)\left(\frac{\partial^2 w_L}{\partial x\,\partial y}\right)_{x=a,\,y=b},$$

where w_L is the deflection surface defined by Equation (2-73), the deflection surface of a plate supported at only three corners is obtained (Fig. 2-16a). Denoting this surface w_a, it is seen that

$$w_a = w_L - \frac{R_L xy}{2D(1-\mu)} . \qquad (2\text{-}83)$$

For one-directional load variations which are symmetrical with respect to the line $y = b/2$, the reaction at the origin of surface w_a vanishes and this function acquires the form

$$w_b = w_L - \frac{R_L}{2D(1-\mu)}(xy + ab) . \qquad (2\text{-}84)$$

This expression represents the deflection surface of a plate supported only at two corners (Fig. 2-16b). Nádai[3] obtained this surface in a different form for the case of a uniform load.

If w_L is replaced by w_M or w_L' in the above equations and R_L by the corner reactions corresponding to these surfaces, several unusual deflection surfaces may be obtained. Two of these are illustrated in Fig. 2-16c and d.

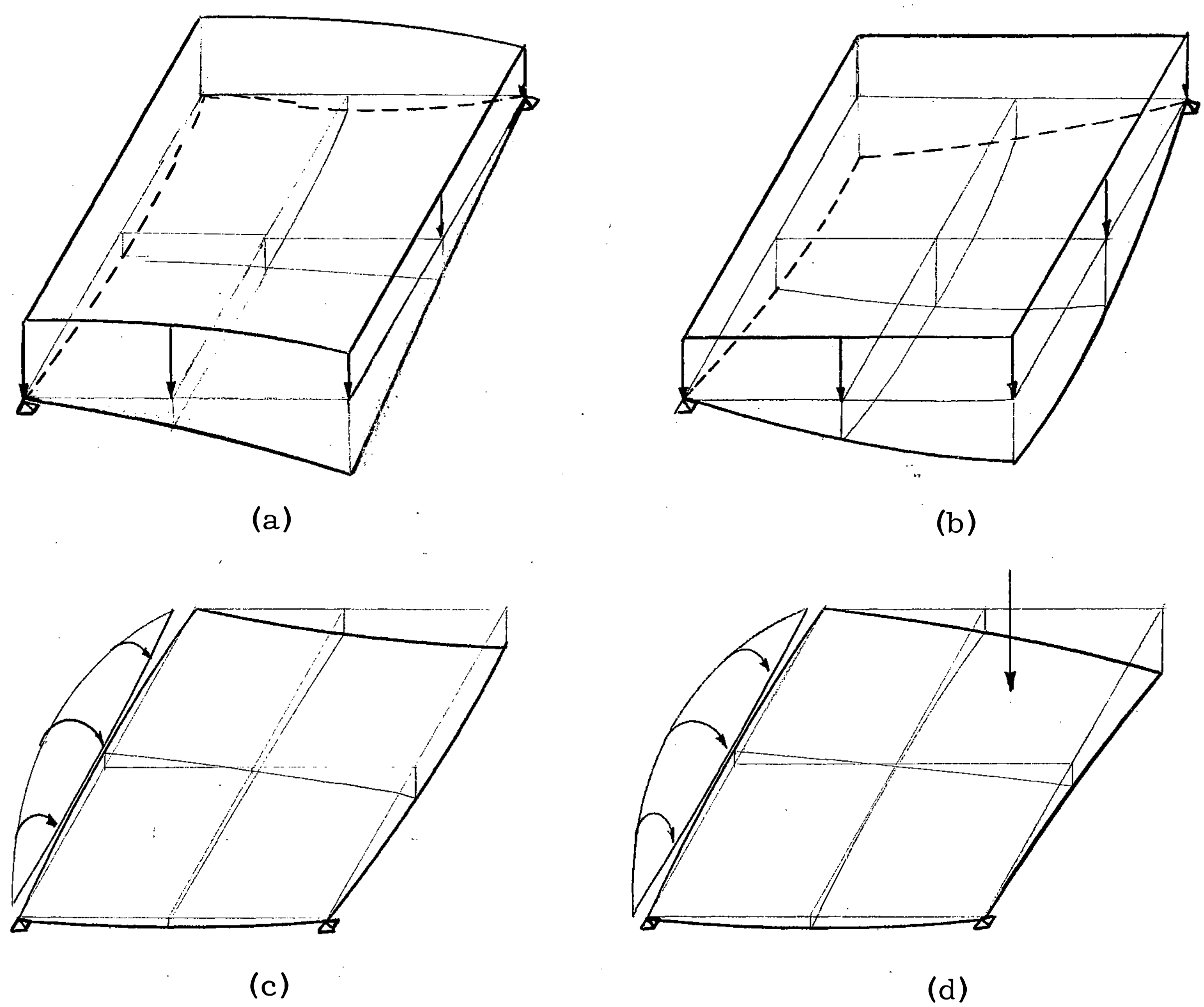


Fig. 2-16

Corner Supported Plates

Equations (2-41, 56, 73 and 80) define the deflection surfaces of plates under various loading conditions supported at their corners. The expressions in this article represent certain cases in which these corner supports are removed. Since the only similar solutions that exist in the literature are those for the simpler case of a symmetrical uniform load, it is believed that these expressions are presented for the first time in this dissertation.

CHAPTER III

COMPATIBILITY RELATIONSHIPS

3-1. Plate Flexibilities

If a typical plate in a plate-beam structure is isolated from the other structural elements, in general it will be acted upon by applied loads and by moments and forces distributed along each edge (Fig. 3-1).

Moment and force components corresponding to the edges $x = 0$ and $y = 0$ will be identified by the prime (') superscript and those corresponding to the remaining edges by the double-prime ('') superscript. The reactive edge quantities acting on the pth plate in a given structural system are the edge moments

$$M'_{xp} = \sum_m (M'_{xp})_m \sin \beta_m y \tag{3-1a}$$

$$M''_{xp} = \sum_m (M''_{xp})_m \sin \beta_m y \tag{3-1b}$$

$$M'_{yp} = \sum_n (M'_{yp})_n \sin \alpha_n x \tag{3-1c}$$

$$M''_{yp} = \sum_n (M''_{yp})_n \sin \alpha_n x \quad , \tag{3-1d}$$

and the edge forces

$$R'_{xzp} = \sum_m (R'_{xzp})_m \sin \beta_m y \tag{3-2a}$$

$$R''_{xzp} = \sum_m (R''_{xzp})_m \sin \beta_m y \tag{3-2b}$$

$$R'_{yzp} = \sum_n (R'_{yzp})_n \sin \alpha_n x \tag{3-2c}$$

$$R''_{yzp} = \sum_n (R''_{yzp})_n \sin \alpha_n x \quad . \tag{3-2d}$$

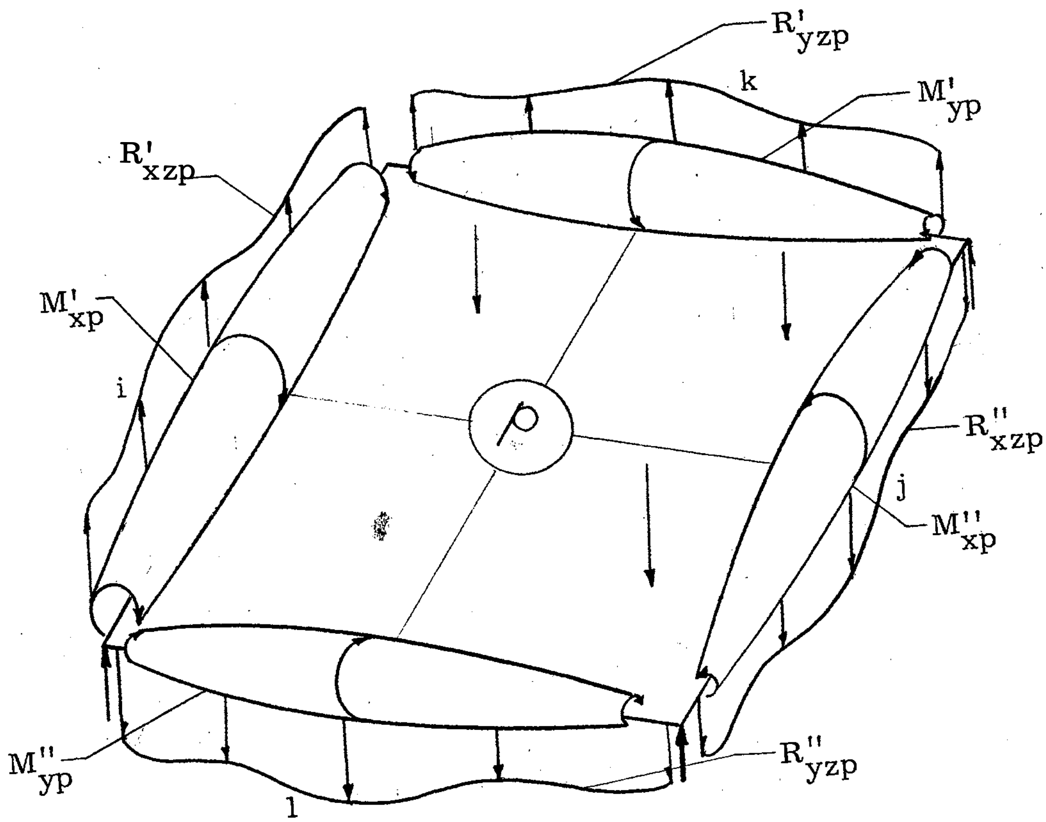


Fig. 3-1

Free Body of Typical Plate Element

In general, the coefficients of the above series will be unknown and the eight functions may be considered as redundants for the structure being studied. Relationships needed to solve for these unknowns are furnished by statics and by conditions requiring the accumulative edge deformations developed by applied loads and by all redundants to be compatible with those of adjacent structural members. Thus, edge deflections and slopes due to loads and each of the redundants must be evaluated.

Edge deformations of the corner supported plates discussed in the previous chapter may be obtained by substituting the appropriate values of x and y into the particular expression for the deflection surface or its first derivatives. Edge deflections and slopes of the surfaces w_R, w_M, and w_L of Equations (2-41, 56, and 73), respectively, are recorded in Tables 3-1A, B and 3-2A, B.

Caution should be taken in calculating edge slopes by differentiating the wide beam term in surface w_L, since it is not always possible to obtain series for the derivatives of functions by differentiating the series for the function. This difficulty is overcome by first finding the slope of the wide beam and then representing it in a Fourier series.

By using the functions defined in these tables, it is possible to write the final deflection or slope of any edge of a plate in terms of the eight redundants given in Equations (3-1 and 3-2). Deformations due to redundants acting along edges other than $x = 0$ are obtained by simply making the obvious replacement of x with y, a with b, etc. Given a set of boundary conditions, it is also possible to expand these series, collect coefficients of like terms, and obtain a sufficient number of equations to solve for all of the unknowns.

TABLE 3-1A

EDGE DEFLECTIONS OF THE BASIC PLATE

1. Along the edge x = 0 :

$$(w_L)_{x=0} = \sum_m \frac{a^4}{D}\left[P_m\left(\frac{1}{\beta_m^4 a^4} + \emptyset_o(0)\right) + \sum_n \sum_r P_r\,(e^{1.)}_{mn}\,\zeta^{I}_{rn} + e^{2.)}_{mn}\,\zeta^{II}_{rn})(\emptyset_3(0) + (-1)^n\,\emptyset_4(0))\right] \sin\beta_m y$$

$$(w_R)_{x=0} = \sum_m \frac{a^3}{D}\left[R'_{xzm}\,\emptyset_3(0) + \sum_n \sum_r R'_{xzr}\,(e^{1.)}_{mn}\,\gamma^{I}_{rn} + e^{2.)}_{mn}\,\gamma^{II}_{rn})(\emptyset_3(0) + (-1)^n\,\emptyset_4(0))\right] \sin\beta_m y$$

$$(w_M)_{x=0} = \sum_m \frac{a^2}{D}\left[M'_{xm}\,\emptyset_5(0) + \sum_n \sum_r M'_{xr}\,(e^{1.)}_{mn}\,\lambda^{I}_{rn} + e^{2.)}_{mn}\,\lambda^{II}_{rn})(\emptyset_3(0) + (-1)^n\,\emptyset_4(0))\right] \sin\beta_m y$$

2. Along the edge y = 0 :

$$(w_L)_{y=0} = \sum_n \sum_m \frac{a^4}{D}\left[P_m\,(\zeta^{I}_{mn}\,\emptyset_1(0) + \zeta^{II}_{mn}\,\emptyset_2(0))\right] \sin\alpha_n x$$

$$(w_R)_{y=0} = \sum_n \sum_m \frac{a^3}{D}\left[R'_{xzm}\,(\gamma^{I}_{mn}\,\emptyset_1(0) + \gamma^{II}_{mn}\,\emptyset_2(0))\right] \sin\alpha_n x$$

$$(w_M)_{y=0} = \sum_n \sum_m \frac{a^2}{D}\left[M'_{xm}\,(\lambda^{I}_{mn}\,\emptyset_1(0) + \lambda^{II}_{mn}\,\emptyset_2(0))\right] \sin\alpha_n x$$

TABLE 3-1B

EDGE DEFLECTIONS OF THE BASIC PLATE

1. Along the edge x = a :

$$(w_L)_{x=a} = \sum_m \frac{a^4}{D} \left[P_m \left(\frac{1}{\beta_m^4 a^4} + \phi_o(\beta_m a) \right) + \sum_n \sum_r P_r \left(e_{mn}^{1.)} \zeta_{rn}^{I} + e_{mn}^{2.)} \zeta_{rn}^{II} \right) \left(\phi_3(\beta_m a) + (-1)^n \phi_4(\beta_m a) \right) \right] \sin \beta_m y$$

$$(w_R)_{x=a} = \sum_m \frac{a^3}{D} \left[R'_{xzm} \phi_3(\beta_m a) + \sum_n \sum_r R'_{xzr} \left(e_{mn}^{1.)} \gamma_{rn}^{I} + e_{mn}^{2.)} \gamma_{rn}^{II} \right) \left(\phi_3(\beta_m a) + (-1)^n \phi_4(\beta_m a) \right) \right] \sin \beta_m y$$

$$(w_M)_{x=a} = \sum_m \frac{a^2}{D} \left[M'_{xm} \phi_5(\beta_m a) + \sum_n \sum_r M'_{xr} \left(e_{mn}^{1.)} \lambda_{rn}^{I} + e_{mn}^{2.)} \lambda_{rn}^{II} \right) \left(\phi_3(\beta_m a) + (-1)^n \phi_4(\beta_m a) \right) \right] \sin \beta_m y$$

2. Along the edge y = b :

$$(w_L)_{y=b} = \sum_n \sum_m \frac{a^4}{D} \left[P_m \left(\zeta_{mn}^{I} \phi_1(\alpha_n b) + \zeta_{mn}^{II} \phi_2(\alpha_n b) \right) \right] \sin \alpha_n x$$

$$(w_R)_{y=b} = \sum_n \sum_m \frac{a^3}{D} \left[R'_{xzm} \left(\gamma_{mn}^{I} \phi_1(\alpha_n b) + \gamma_{mn}^{II} \phi_2(\alpha_n b) \right) \right] \sin \alpha_n x$$

$$(w_M)_{y=b} = \sum_n \sum_m \frac{a^2}{D} \left[M'_{xm} \left(\lambda_{mn}^{I} \phi_1(\alpha_n b) + \lambda_{mn}^{II} \phi_2(\alpha_n b) \right) \right] \sin \alpha_n x$$

TABLE 3-2A

EDGE SLOPES OF THE BASIC PLATE

1. Along the edge x = 0 :

$$\left(\frac{\partial w_L}{\partial x}\right)_{x=0} = \sum_m \frac{a^4}{D} \left\{ \left[P_m \frac{d\,\emptyset_o(0)}{dx} + \sum_n \sum_r P_r \left(e_{mn}^{1.)} \zeta_{rn}^{I} + e_{mn}^{2.)} \zeta_{rn}^{II}\right)\left(\frac{d\,\emptyset_3(0)}{dx} + (-1)^n \frac{d\,\emptyset_4(0)}{dx}\right) \right] \sin \beta_m y + \sum_n \sum_r \alpha_n P_m \left(\zeta_{mn}^{I} \phi_{nr}^{1.)} + \zeta_{mn}^{II} \phi_{nr}^{2.)}\right) \sin \beta_r y \right\}$$

$$\left(\frac{\partial w_R}{\partial x}\right)_{x=0} = \sum_m \frac{a^3}{D} \left\{ \left[R'_{xzm} \frac{d\,\emptyset_3(0)}{dx} + \sum_n \sum_r R'_{xzr} \left(e_{mn}^{1.)} \gamma_{mn}^{I} + e_{mn}^{2.)} \gamma_{rn}^{II}\right)\left(\frac{d\,\emptyset_3(0)}{dx} + (-1)^n \frac{d\,\emptyset_4(0)}{dx}\right) \right] \sin \beta_m y + \sum_n \sum_r \alpha_n R'_{xzm} \left(\gamma_{mn}^{I} \phi_{nr}^{1.)} + \gamma_{mn}^{II} \phi_{nr}^{2.)}\right) \sin \beta_r y \right\}$$

$$\left(\frac{\partial w_M}{\partial x}\right)_{x=0} = \sum_m \frac{a^2}{D} \left\{ \left[M'_{xm} \frac{d\,\emptyset_5(0)}{dx} + \sum_n \sum_r M'_{xr} \left(e_{mn}^{1.)} \lambda_{rn}^{I} + e_{mn}^{2.)} \lambda_{rn}^{II}\right)\left(\frac{d\,\emptyset_3(0)}{dx} + (-1)^n \frac{d\,\emptyset_4(0)}{dx}\right) \right] \sin \beta_m y + \sum_n \sum_r \alpha_n M'_{xm} \left(\lambda_{mn}^{I} \phi_{nr}^{1.)} + \lambda_{mn}^{II} \phi_{nr}^{2.)}\right) \sin \beta_r y \right\}$$

2. Along the edge y = 0 :

$$\left(\frac{\partial w_L}{\partial y}\right)_{y=0} = \sum_m \frac{a^4}{D} \left\{ P_m \left[\frac{1}{\beta_m^3 a^4} + \sum_n \left(\beta_m \phi^{\circ}_{mn} + \zeta_{mn}^{I} \frac{d\,\emptyset_1(0)}{dy} + \zeta_{mn}^{II} \frac{d\,\emptyset_2(0)}{dy}\right) \sin \alpha_n x \right] + \sum_n \sum_r \sum_j \beta_m P_r \left(e_{mn}^{1.)} \zeta_{rn}^{I} + e_{mn}^{2.)} \zeta_{rn}^{II}\right)\left(\phi_{mj}^{3.)} + (-1)^n \phi_{mj}^{4.)}\right) \sin \alpha_j x \right\}$$

$$\left(\frac{\partial w_R}{\partial y}\right)_{y=0} = \sum_m \frac{a^3}{D} \left\{ R'_{xzm} \sum_n \left(\beta_m \phi_{mn}^{3.)} + \gamma_{mn}^{I} \frac{d\,\emptyset_1(0)}{dy} + \gamma_{mn}^{II} \frac{d\,\emptyset_2(0)}{dy}\right) \sin \alpha_n x + \sum_n \sum_r \sum_j \beta_m R'_{xzr} \left(e_{mn}^{1.)} \gamma_{rn}^{I} + e_{mn}^{2.)} \gamma_{rn}^{II}\right)\left(\phi_{mj}^{3.)} + (-1)^n \phi_{mj}^{4.)}\right) \sin \alpha_j x \right\}$$

$$\left(\frac{\partial w_M}{\partial y}\right)_{y=0} = \sum_m \frac{a^2}{D} \left\{ M'_{xm} \sum_n \left(\beta_m \phi_{mn}^{5.)} + \lambda_{mn}^{I} \frac{d\,\emptyset_1(0)}{dy} + \lambda_{mn}^{II} \frac{d\,\emptyset_2(0)}{dy}\right) \sin \alpha_n x + \sum_n \sum_r \sum_j \beta_m M'_{xr} \left(e_{mn}^{1.)} \lambda_{rn}^{I} + e_{mn}^{2.)} \lambda_{rn}^{II}\right)\left(\phi_{mj}^{3.)} + (-1)^n \phi_{mj}^{4.)}\right) \sin \alpha_j x \right\}$$

TABLE 3-2B

EDGE SLOPES OF THE BASIC PLATE

1. Along the edge x = a :

$$\left(\frac{\partial w_L}{\partial x}\right)_{x=a} = \sum_m \frac{a^4}{D}\left\{\left[P_m \frac{d\Phi_o(\beta_m a)}{dx} + \sum_n\sum_r P_r \left(e_{mn}^{1.)} \zeta_{rn}^{I} + e_{mn}^{2.)} \zeta_{rn}^{II}\right)\left(\frac{d\Phi_3(\beta_m a)}{dx} + (-1)^n \frac{d\Phi_4(\beta_m a)}{dx}\right)\right] \sin \beta_m y + \sum_n\sum_r (-1)^n \alpha_n P_m \left(\zeta_{mn}^{I} \phi_{nr}^{1.)} + \zeta_{mn}^{II} \phi_{nr}^{2.)}\right) \sin \beta_r y\right\}$$

$$\left(\frac{\partial w_R}{\partial x}\right)_{x=a} = \sum_m \frac{a^3}{D}\left\{\left[R'_{xzm} \frac{d\Phi_3(\beta_m a)}{dx} + \sum_n\sum_r R'_{xzr} \left(e_{mn}^{1.)} \gamma_{mn}^{I} + e_{mn}^{2.)} \gamma_{mn}^{II}\right)\left(\frac{d\Phi_3(\beta_m a)}{dx} + (-1)^n \frac{d\Phi_4(\beta_m a)}{dx}\right)\right] \sin \beta_m y + \sum_n\sum_r (-1)^n \alpha_n R'_{xzm} \left(\gamma_{mn}^{I} \phi_{nr}^{1.)} + \gamma_{mn}^{II} \phi_{nr}^{2.)}\right) \sin \beta_r y\right\}$$

$$\left(\frac{\partial w_M}{\partial x}\right)_{x=a} = \sum_m \frac{a^2}{D}\left\{\left[M'_{xm} \frac{d\Phi_5(\beta_m a)}{dx} + \sum_n\sum_r M'_{xr} \left(e_{mn}^{1.)} \lambda_{rn}^{I} + e_{mn}^{2.)} \lambda_{rn}^{II}\right)\left(\frac{d\Phi_3(\beta_m a)}{dx} + (-1)^n \frac{d\Phi_4(\beta_m a)}{dx}\right)\right] \sin \beta_m y + \sum_n\sum_r (-1)^n \alpha_n M'_{xm} \left(\lambda_{mn}^{I.} \phi_{nr}^{1.)} + \lambda_{mn}^{II} \phi_{nr}^{2.)}\right) \sin \beta_r y\right\}$$

2. Along the edge y = b :

$$\left(\frac{\partial w_L}{\partial y}\right)_{y=b} = \sum_m \frac{a^4}{D}\left\{P_m\left[\frac{(-1)^m}{\beta_m^3 a^4} + \sum_n \left((-1)^n \beta_m \phi_{mn}^{\circ} + \zeta_{mn}^{I} \frac{d\Phi_1(\alpha_n b)}{dy} + \zeta_{mn}^{II} \frac{d\Phi_2(\alpha_n b)}{dy}\right) \sin \alpha_n x\right] + \sum_n\sum_r\sum_j (-1)^m \beta_m P_r \left(e_{mn}^{1.)} \zeta_{rn}^{I} + e_{mn}^{2.)} \zeta_{rn}^{II}\right)\left(\phi_{mj}^{3.)} + (-1)^n \phi_{mj}^{4.)}\right) \sin \alpha_j x\right\}$$

$$\left(\frac{\partial w_R}{\partial y}\right)_{y=b} = \sum_m \frac{a^3}{D}\left\{R'_{xzm} \sum_n \left((-1)^m \beta_m \phi_{mn}^{3.)} + \gamma_{mn}^{I} \frac{d\Phi_1(\alpha_n b)}{dy} + \gamma_{mn}^{II} \frac{d\Phi_2(\alpha_n b)}{dy}\right) \sin \alpha_n x + \sum_n\sum_r\sum_j (-1)^m \beta_m R'_{xzr} \left(e_{mn}^{1.)} \gamma_{rn}^{I} + e_{mn}^{2.)} \gamma_{rn}^{II}\right)\left(\phi_{mj}^{3.)} + (-1)^n \phi_{mj}^{4.)}\right) \sin \alpha_j x\right\}$$

$$\left(\frac{\partial w_M}{\partial y}\right)_{y=b} = \sum_m \frac{a^2}{D}\left\{M'_{xm} \sum_n \left((-1)^m \beta_m \phi_{mn}^{5.)} + \lambda_{mn}^{I} \frac{d\Phi_1(\alpha_n b)}{dy} + \lambda_{mn}^{II} \frac{d\Phi_2(\alpha_n b)}{dy}\right) \sin \alpha_n x + \sum_n\sum_r\sum_j (-1)^m \beta_m M'_{xr} \left(e_{mn}^{1.)} \lambda_{rn}^{I} + e_{mn}^{2.)} \lambda_{rn}^{II}\right)\left(\phi_{mj}^{3.)} + (-1)^n \phi_{mj}^{4}\right) \sin \alpha_j x\right\}$$

This procedure, however, is extremely complex and impractical. It does not permit the structure under consideration to be analyzed on a term by term basis, and a finite number of an infinite number of equations must be used. Therefore, a simplified approach is highly desirable even though the resulting functions may not be exact.

Such an approach can be developed if the expressions given in Tables 3-1 and 3-2 are written so that each component of any of the redundant forces or moments produces edge deformations that correspond to the same term in the series. In other words, a given harmonic of an applied redundant causes deformations in terms of the same harmonic.

For purposes of identification, the edges $x = 0$, $x = a$, $y = 0$, and $y = b$ are to be denoted by the symbols i, j, k, and l, respectively. If the above simplication is adopted, the deformations developed by R'_{xz} and M'_x can then be written in the approximate form:

1. Deflections:

$$(w_R)_{x=0} \doteq \sum_m R'_{xzm}(\Delta^R_{ii})_m \sin \beta_m y \qquad (3\text{-}3a)$$

$$(w_R)_{x=a} \doteq \sum_m R'_{xzm}(\Delta^R_{ji})_m \sin \beta_m y \qquad (3\text{-}3b)$$

$$(w_R)_{y=0} \doteq \sum_n R'_{xzn}(\Delta^R_{ki})_n \sin \alpha_n x \qquad (3\text{-}3c)$$

$$(w_R)_{y=b} \doteq \sum_n R'_{xzn}(\Delta^R_{li})_n \sin \alpha_n x \qquad (3\text{-}3d)$$

$$(w_M)_{x=0} \doteq \sum_m M'_{xm}(\Delta^M_{ii})_m \sin \beta_m y \qquad (3\text{-}4a)$$

$$(w_M)_{x=a} \doteq \sum_m M'_{xm} (\Delta^M_{ji})_m \sin \beta_m y \qquad (3\text{-}4b)$$

$$(w_M)_{y=0} \doteq \sum_n M'_{xn} (\Delta^M_{ki})_n \sin \alpha_n x \qquad (3\text{-}4c)$$

$$(w_M)_{y=b} \doteq \sum_n M'_{xn} (\Delta^M_{li})_n \sin \alpha_n x \ . \qquad (3\text{-}4d)$$

2. Slopes:

$$\left(\frac{\partial w_R}{\partial x}\right)_{x=0} \doteq \sum_m R'_{xzm} (\theta^R_{ii})_m \sin \beta_m y \qquad (3\text{-}5a)$$

$$\left(\frac{\partial w_R}{\partial x}\right)_{x=0} \doteq \sum_m R'_{xzm} (\theta^R_{ji})_m \sin \beta_m y \qquad (3\text{-}5b)$$

$$\left(\frac{\partial w_R}{\partial y}\right)_{y=0} \doteq \sum_n R'_{xzn} (\theta^R_{ki})_n \sin \alpha_n x \qquad (3\text{-}5c)$$

$$\left(\frac{\partial w_R}{\partial y}\right)_{y=b} \doteq \sum_n R'_{xzn} (\theta^R_{li})_n \sin \alpha_n x \qquad 3\text{-}5d)$$

$$\left(\frac{\partial w_M}{\partial x}\right)_{x=0} \doteq \sum_m M'_{xm} (\theta^M_{ii})_m \sin \beta_m y \qquad (3\text{-}6a)$$

$$\left(\frac{\partial w_M}{\partial x}\right)_{x=a} \doteq \sum_m M'_{xm} (\theta^M_{ji})_m \sin \beta_m y \qquad (3\text{-}6b)$$

$$\left(\frac{\partial w_M}{\partial y}\right)_{y=0} \doteq \sum_n M'_{xn} \, (\theta^M_{ki})_n \sin \alpha_n x \tag{3-6c}$$

$$\left(\frac{\partial w_M}{\partial y}\right)_{y=b} \doteq \sum_n M'_{xn} \, (\theta^M_{li})_n \sin \alpha_n x \quad . \tag{3-6d}$$

The coefficients Δ and θ are the components of the linear and angular flexibilities of the plate and are defined in Tables 3-3 and 3-4. The subscripts m and n are, as before, summation indices. The double subscripts of i, j, k, or l refer to specific edges of the plate; the first indicating the edge at which the deformation takes place and the second, the edge along which the force or moment was applied that caused the deformation.

The coefficients, Δ, of Equations (3-3 and 3-4) may be interpreted physically as follows (Fig. 3-2):

$(\Delta^R_{ii})_m$, $\left((\Delta^M_{ii})_m\right)$ is the maximum deflection per unit length of the edge i of a plate supported only at its corners due to a unit-amplitude sinusoidal force (moment) at edge i.

$(\Delta^R_{ji})_m$, $\left((\Delta^M_{ji})_m\right)$ is the maximum deflection per unit length of the edge j of a plate supported only at its corners due to a unit-amplitude sinusoidal force (moment) at edge i.

$(\Delta^R_{ki})_n$, $\left((\Delta^M_{ki})_n\right)$ is the maximum deflection per unit length of the edge k of a plate supported only at its corners due to a unit-amplitude sinusoidal force (moment) at edge i.

TABLE 3-3

COMPONENTS OF LINEAR FLEXIBILITIES

$$(\Delta^R_{ii})_m = \frac{a^3}{D}\left[\phi_3(0) + \sum_n (e^{1.}_{mn}\gamma^I_{mn} + e^{2.}_{mn}\gamma^{II}_{mn})(\phi_3(0) + (-1)^n \phi_4(0))\right]$$

$$(\Delta^R_{ji})_m = \frac{a^3}{D}\left[\phi_3(\beta_m a) + \sum_n (e^{1.}_{mn}\gamma^I_{mn} + e^{2.}_{mn}\gamma^{II}_{mn})(\phi_3(\beta_m a) + (-1)^n \phi_4(\beta_m a)\right]$$

$$(\Delta^R_{ki})_n = \frac{a^3}{D}\left[\gamma^I_{nn}\phi_1(0) + \gamma^{II}_{nn}\phi_2(0)\right]$$

$$(\Delta^R_{li})_n = \frac{a^3}{D}\left[\gamma^I_{nn}\phi_1(\alpha_n b) + \gamma^{II}_{nn}\phi_2(\alpha_n b)\right]$$

$$(\Delta^M_{ii})_m = \frac{a^2}{D}\left[\phi_5(0) + \sum_n (e^{1.}_{mn}\lambda^I_{mn} + e^{2.}_{mn}\lambda^{II}_{mn})(\phi_3(0) + (-1)^n \phi_4(0))\right]$$

$$(\Delta^M_{ji})_m = \frac{a^2}{D}\left[\phi_5(\beta_m a) + \sum_n (e^{1.}_{mn}\lambda^I_{mn} + e^{2.}_{mn}\lambda^{II}_{mn})(\phi_3(\beta_m a) + (-1)^n \phi_4(\beta_m a))\right]$$

$$(\Delta^M_{ki})_n = \frac{a^2}{D}\left[\lambda^I_{nn}\phi_1(0) + \lambda^{II}_{nn}\phi_2(0)\right]$$

$$(\Delta^M_{li})_n = \frac{a^2}{D}\left[\lambda^I_{nn}\phi_1(\alpha_n b) + \lambda^{II}_{nn}\phi_2(\alpha_n b)\right]$$

TABLE 3-4

COMPONENTS OF ANGULAR FLEXIBILITIES

$$(\theta^R_{ii})_m = \frac{a^3}{D}\left[\frac{d\Phi_3(0)}{dx} + \sum_n\left[\frac{n\pi}{a}(\gamma^I_{mn}\phi^{1.)}_{nm} + \gamma^{II}_{mn}\phi^{2.)}_{nm}) + (e^{1.)}_{mn}\gamma^I_{mn} + e^{2.)}_{mn}\gamma^{II}_{mn})(\frac{d\Phi_3(0)}{dx} + (-1)^n\frac{d\Phi_4(0)}{dx})\right]\right]$$

$$(\theta^R_{ji})_m = \frac{a^3}{D}\left[\frac{d\Phi_3(\beta_m a)}{dx} + \sum_n\left[\frac{(-1)^n n\pi}{a}(\gamma^I_{mn}\phi^{1.)}_{nm} + \gamma^{II}_{mn}\phi^{2.)}_{nm}) + (e^{1.)}_{mn}\gamma^I_{mn} + e^{2.)}_{mn}\gamma^{II}_{mn})(\frac{d\Phi_3(\beta_m a)}{dx} + (-1)^n\frac{d\Phi_4(\beta_m a)}{dx})\right]\right]$$

$$(\theta^R_{ki})_n = \frac{a^3}{D}\left[\frac{n\pi}{b}\phi^{3.)}_{nn} + (\gamma^I_{nn}\frac{d\Phi_1(0)}{dy} + \gamma^{II}_{nn}\frac{d\Phi_2(0)}{dy}) + \frac{n\pi}{b}\sum_m(e^{1.)}_{nm}\gamma^I_{nm} + e^{2.)}_{nm}\gamma^{II}_{nm})(\phi^{3.)}_{nn} + (-1)^m\phi^{4.)}_{nn})\right]$$

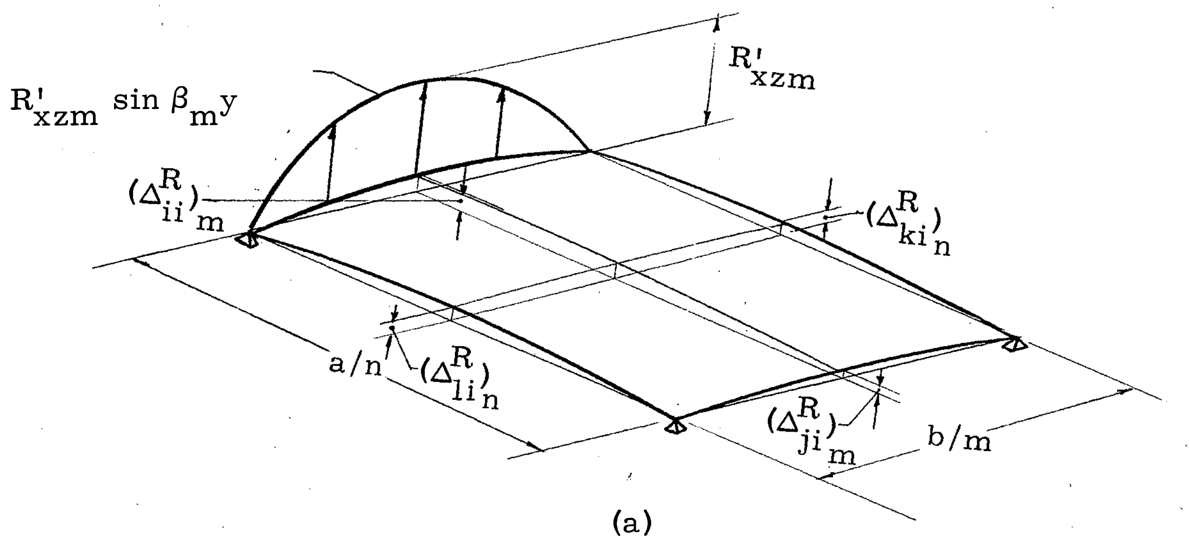
$$(\theta^R_{li})_n = \frac{a^3}{D}\left[\frac{(-1)^n n\pi}{b}\phi^{3.)}_{nn} + (\gamma^I_{nn}\frac{d\Phi_1(\alpha_n b)}{dy} + \gamma^{II}_{nn}\frac{d\Phi_2(\alpha_n b)}{dy}) + \frac{(-1)^n n\pi}{b}\sum_m(e^{1.)}_{nm}\gamma^I_{nm} + e^{2.)}_{nm}\gamma^{II}_{nm})(\phi^{3.)}_{nn} + (-1)^m\phi^{4.)}_{nn})\right]$$

$$(\theta^M_{ii})_m = \frac{a^2}{D}\left[\frac{d\Phi_5(0)}{dx} + \sum_n\left[\frac{n\pi}{a}(\lambda^I_{mn}\phi^{1.)}_{nm} + \lambda^{II}_{mn}\phi^{2.)}_{nm}) + (e^{1.)}_{mn}\lambda^I_{mn} + e^{2.)}_{mn}\lambda^{II}_{mn})(\frac{d\Phi_3(0)}{dx} + (-1)^n\frac{d\Phi_4(0)}{dx})\right]\right]$$

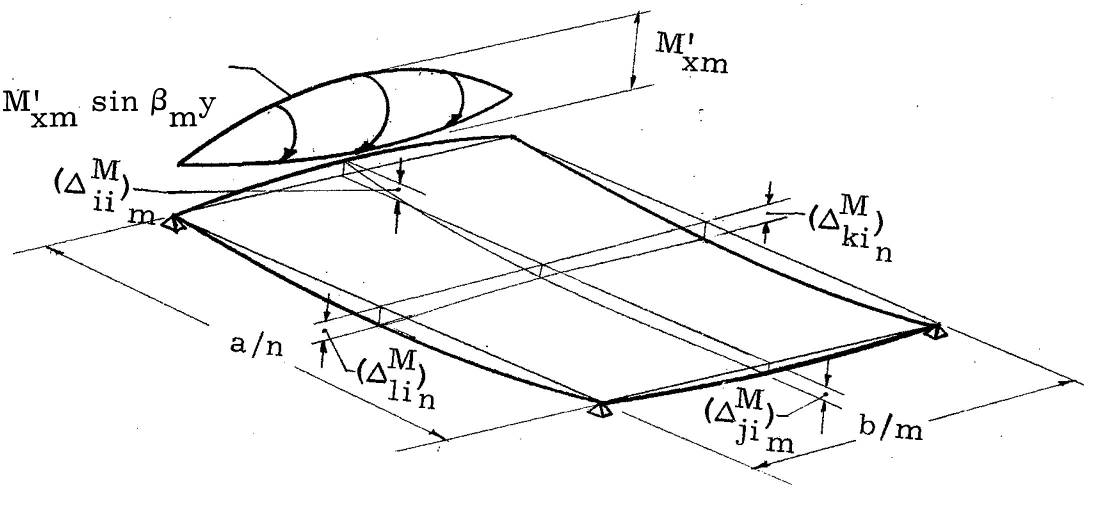
$$(\theta^M_{ji})_m = \frac{a^2}{D}\left[\frac{d\Phi_5(\beta_m a)}{dx} + \sum_n\left[\frac{(-1)^n n\pi}{a}(\lambda^I_{mn}\phi^{1.)}_{nm} + \lambda^{II}_{mn}\phi^{2.)}_{nm}) + (e^{1.)}_{mn}\lambda^I_{mn} + e^{2.)}_{mn}\lambda^{II}_{mn})(\frac{d\Phi_3(\beta_m a)}{dx} + (-1)^n\frac{d\Phi_4(\beta_m a)}{dx})\right]\right]$$

$$(\theta^M_{ki})_n = \frac{a^2}{D}\left[\frac{n\pi}{b}\phi^{5.)}_{nn} + (\lambda^I_{nn}\frac{d\Phi_1(0)}{dy} + \lambda^{II}_{nn}\frac{d\Phi_2(0)}{dy}) + \frac{n\pi}{b}\sum_m(e^{1.)}_{nm}\lambda^I_{nm} + e^{2.)}_{nm}\lambda^{II}_{nm})(\phi^{3.)}_{nn} + (-1)^m\phi^{4.)}_{nn})\right]$$

$$(\theta^M_{li})_n = \frac{a^2}{D}\left[\frac{(-1)^n n\pi}{b}\phi^{5.)}_{nn} + (\lambda^I_{nn}\frac{d\Phi_1(\alpha_n b)}{dy} + \lambda^{II}_{nn}\frac{d\Phi_2(\alpha_n b)}{dy}) + \frac{(-1)^n n\pi}{b}\sum_m(e^{1.)}_{nm}\lambda^I_{nm} + e^{2.)}_{nm}\lambda^{II}_{nm})(\phi^{3.)}_{nn} + (-1)^m\phi^{4.)}_{nn})\right]$$



(a)



(b)

Fig. 3-2

Physical Interpretation of

Linear Flexibilities

$(\Delta^R_{li})_n$, $\left((\Delta^M_{li})_n\right)$ is the maximum deflection per unit length of the edge l of a plate supported only at its corners due to a unit-amplitude sinusoidal force (moment) at edge i.

Similarly, the coefficients, θ, of Equations (3-5 and 3-6) may be given the following physical interpretation (Fig. 3-3):

$(\theta^R_{ii})_m$, $\left((\theta^M_{ii})_m\right)$ is the maximum slope per unit length of the edge i of a plate supported only at its corners due to a unit-amplitude sinusoidal force (moment) at edge i.

$(\theta^R_{ji})_m$, $\left((\theta^M_{ji})_m\right)$ is the maximum slope per unit length of the edge j of a plate supported only at its corners due to a unit-amplitude sinusoidal force (moment) at edge i.

$(\theta^R_{ki})_n$, $\left((\theta^M_{ki})_n\right)$ is the maximum slope per unit length of the edge k of a plate supported only at its corners due to a unit-amplitude sinusoidal force (moment) at edge i.

$(\theta^R_{li})_n$, $\left((\theta^M_{li})_n\right)$ is the maximum slope per unit length of the edge l of a plate supported only at its corners due to a unit-amplitude sinusoidal force (moment) at edge i.

It should be noted that positive redundants may cause negative deformations and that some of these flexibility components, the Δ^R's for example, will be negative quantities. Also it is noted that, as a consequence of the approximation used in arriving at these flexibilities, any sinusoidal disturbance applied to a single edge of a corner-supported plate produces deformations along adjacent edges that are also sinusoidal. This leads to incompatibilities in adjacent modes of deformation except at node points.

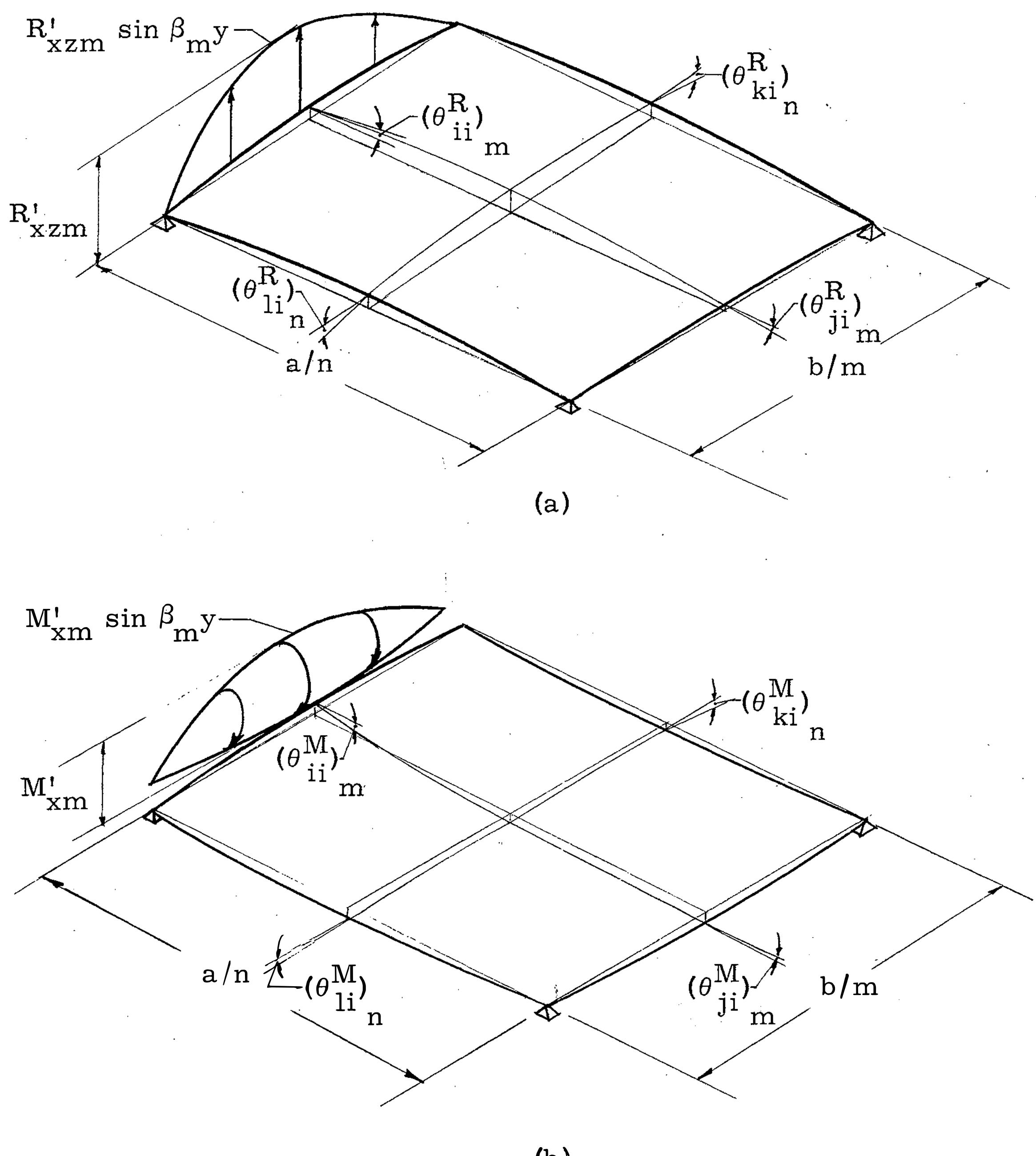


Fig. 3-3

Physical Interpretation of

Angular Flexibilities

This inconsistancy is remedied, however, when it is observed that the values used for the flexibility components of adjacent edges (the edge k and l of a plate loaded along i) are only slightly better than one-term approximations of the series representing the true deformations along those edges. An inspection of the deflection surfaces of plates subjected to edge forces and moments indicates that deflections along the adjacent edges resemble very closely one or two superimposed harmonics. Thus, these deformations will, in most cases, be represented by one or two terms of the series with sufficient accuracy.

Edge deformations of plates subjected to normal loads may be written in a similar manner. For example,

$$(w_L)_{x=0} = \sum_m (\Delta_i^L)_m \sin \beta_m y \qquad (3\text{-}7a)$$

$$\left(\frac{\partial w_L}{\partial x}\right)_{x=0} = \sum_m (\theta_i^L)_m \sin \beta_m y \,, \qquad (3\text{-}7b)$$

etc. Since the coefficient, P_m, of Equation (2-73), is a known quantity, no approximation is necessary for these coefficients. They merely represent the total coefficient of a given sine term when the series given in Tables 3-1A and 3-2A for w_L are expanded and like terms are collected.

$(\Delta_i^L)_m$ and $(\theta_i^L)_m$ are components of the linear and angular load functions of the plate, and may be interpreted physically as follows (Fig. 3-4):

$(\Delta_i^L)_m$ is the maximum deflection per unit length of the edge i of a plate supported only at its corners due to the m th component of the applied load.

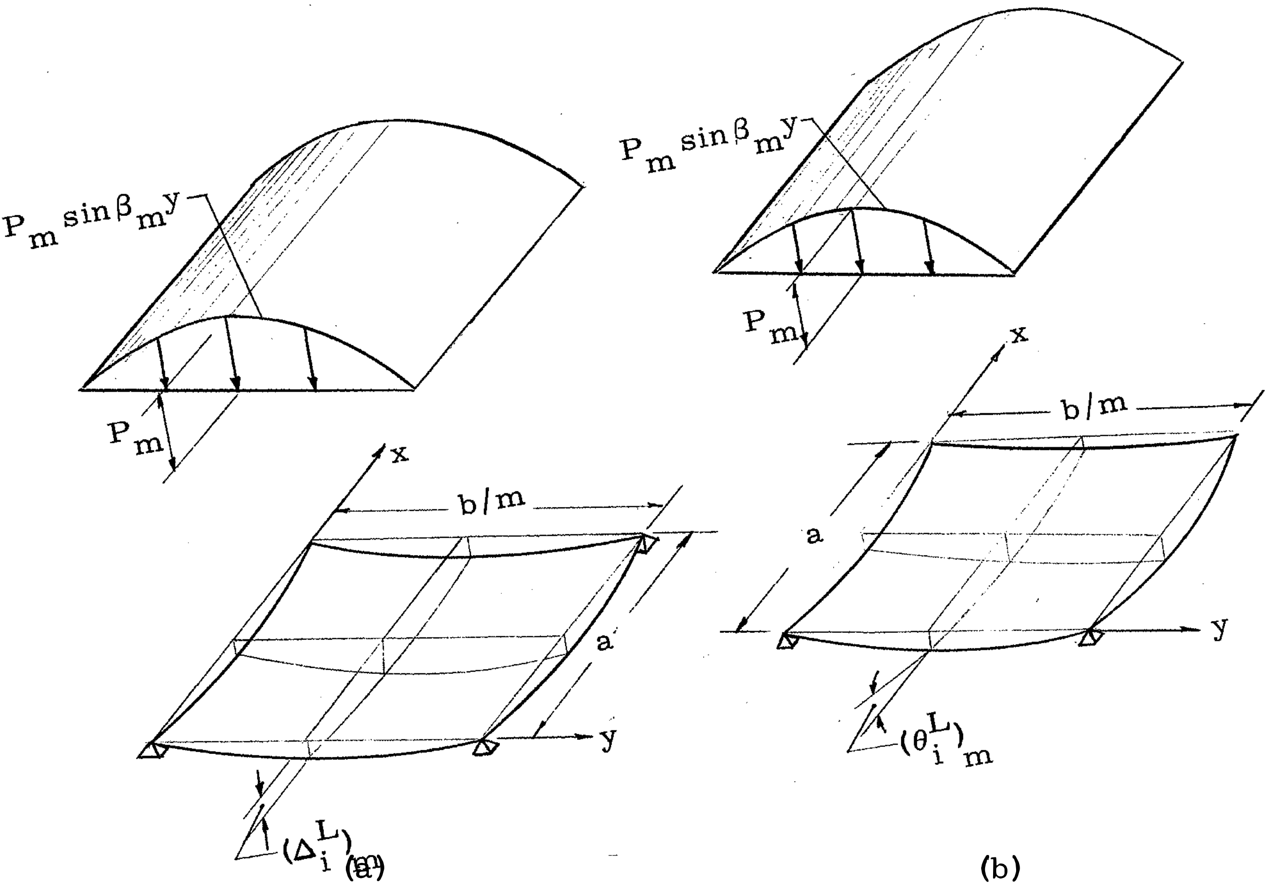


Fig. 3-4

Physical Interpretation of

Linear and Angular Load Functions

$(\theta_i^L)_m$ is the maximum slope per unit length of the edge i of a plate supported only at its corners due to the m th component of the applied load.

Now that the components of the flexibilities and load functions of a plate have been defined, it is possible to write the total deflection or slope of any edge of a given plate subjected to completely general loading conditions. If, again, the p th plate of a plate-beam structure is considered, the total deflection and slope along edge i due to applied loads and the eight redundants given by Equations (3-1 and 3-2) are

$$\Delta_{pi} = \sum_{m} \Big[(R'_{xzp})_m (\Delta^R_{ii})_m + (R''_{xzp})_m (\Delta^R_{ij})_m + (R'_{yzp})_m (\Delta^R_{ik})_m +$$

$$+ (R''_{yzp})_m (\Delta^R_{il})_m + (M'_{xp})_m (\Delta^M_{ii})_m + (M''_{xp})_m (\Delta^M_{ij})_m +$$

$$+ (M''_{yp})_m (\Delta^M_{ik})_m + (M''_{yp})_m (\Delta^M_{il})_m + (\Delta^L_{i})_m \Big] \sin \beta_m y \, , \qquad (3\text{-}8)$$

and

$$\theta_{pi} = \sum_{m} \Big[(R'_{xzp})_m (\theta^R_{ii})_m + (R''_{xzp})_m (\theta^R_{ij})_m + (R'_{yzp})_m (\theta^R_{ik})_m +$$

$$+ (R''_{yzp})_m (\theta^R_{il})_m + (M'_{xp})_m (\theta^M_{ii})_m + (M''_{xp})_m (\theta^M_{ij})_m +$$

$$+ (M'_{yp})_m (\theta^M_{ik})_m + (M''_{yp})_m (\theta^M_{il})_m + (\theta^L_{i})_m \Big] \sin \beta_m y \; . \qquad (3\text{-}9)$$

Thus, by using simplified forms of the plate's flexibilities, it is possible to compute edge deformations on a term by term basis. That such an approach will lead to only small errors for common length to width ratios is implied by the following observations:

1. Fourier series in Levy-type solutions often converge very rapidly and functions arising in such solutions can usually be represented with sufficient accuracy by only a few terms of their series.

Timoshenko[39], in solving the closely related problem of clamped-edged plates, uses only four terms of the series and obtains answers less than one percent in error.

2. Boundary conditions along two edges are very closely satisfied, the only source of error being an approximation in the correction term mentioned in the previous chapter. Conditions along the remaining edges are sufficiently satisfied by one or two terms of the series, as is indicated by the nature of the deflection along those edges.

In fact, Galerkin[9], in solving the problem of a uniformly-loaded square plate with free edges, satisfied all boundary conditions by taking many terms in the series and found that the center deflection was

$$w_{x=a/2,\ y=b/2} = 0.0257 \frac{pa^4}{D} \quad .$$

Nadai[3], in considering the same problem, partially satisfied boundary conditions along two edges by taking only one term in the series. He obtained for the center deflection

$$w_{x=a/2,\ y=b/2} = 0.0253 \frac{pa^4}{D} \quad ,$$

a value in error one and one-half percent.

3. If a small error exists in edges adjacent to the ith edge, its effect must be "carried over" to the edge under consideration. Therefore, the influence of errors existing in deformations of these edges is very small on the edges used to obtain the final solution of the problem.

That this is true can also be seen in Bittner's[18] seemingly crude analysis of plates continuous in two directions over rigid supports. In this analysis only one term of a similar series was used; and yet, the final average moments differed from those computed by Siess and Newmark[21] by a maximum of ten percent.

Some modifications must be made for cases involving symmetrically loaded plates supported by beams with unsymmetrical end conditions. Otherwise, compatibility equations for even values of m are homogeneous and possess only trivial solutions. To overcome this difficulty, the exact expressions given in Tables 3-1A, B and 3-2A, B may be used; or the influence of redundants corresponding to $m = 2$ on those for $m = 1$ may be accounted for by using only one term in their exact expressions.

3-2. Deflection of Supporting Members

In general, plate elements in plate-beam structures will be elastically supported along their edges by elastic beams. Consider, as a typical example, the beam AB elastically connected to the p th plate in a plate-beam structure along the edge i (Fig. 3-5).

Other than the reactive moments and forces transmitted by the plate, the beam will be acted upon by applied loading; the moments M_{ABx}, M_{ABy}, M_{BAx}, and M_{BAy}; and the shearing forces V_{ABz} and V_{BAz}, as shown in the figure. These moments need not be considered as redundants. They are merely the bending and twisting moments at A and B and may be in terms of the applied loading and redundant forces or moments acting elsewhere in the system.

No moments act about the z-axis at A and B. The stiffness afforded by the plate along the edge i prevents such moments from being developed. Deformations due to bending about these axes would require the middle surface of the plate to stretch, an effect taken as negligible in the thin plate theory.

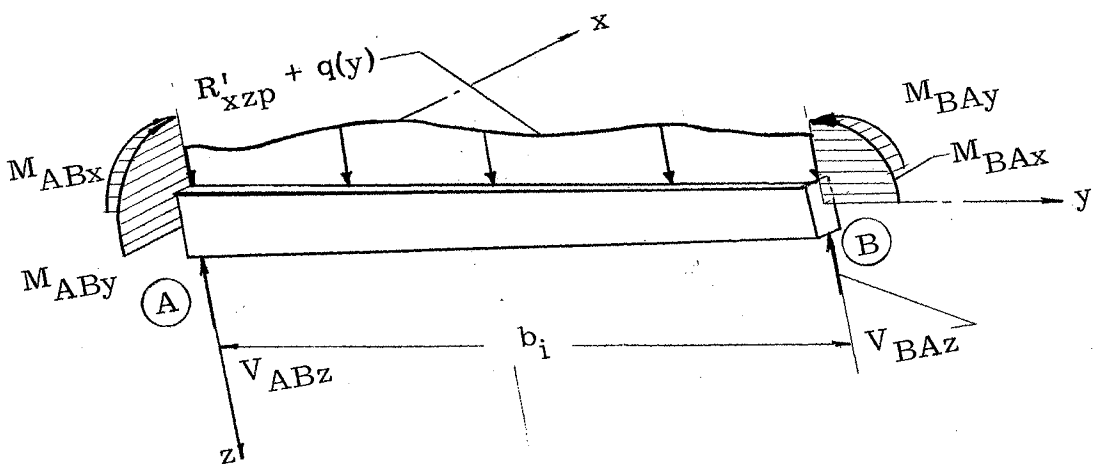


Fig. 3-5

Typical Edge Beam AB

The reactive forces, R'_{xzp}, distributed along the edge i of the plate act as an additional applied load on the beam. Thus, for any point along the beam, AB,

$$\frac{d^4 z_i}{dy^4} = \sum_m \frac{(R'_{xzp})_m}{EI_i(y)} \sin \beta_m y + \frac{q(y)}{EI_i(y)} , \qquad (3-10)$$

in which

z_i is the deflection of the beam along the edge i of the plate.

$EI_i(y)$ is the flexural rigidity of the beam.

$q(y)$ is the applied loading acting on the beam.

If EI_i is constant, the bending moment at any point is

$$M_x = -EI_i \frac{d^2 z_i}{dy^2} = \sum_m \frac{(R'_{xzp})_m}{\beta_m^2} \sin \beta_m y - \frac{M_{ABx} - M_{BAx}}{b} y + M_{ABx}$$

(3-11)

Dividing by EI_i, integrating twice, and evaluating the constants of integration from the conditions that points A and B do not displace, gives for the deflection curve of the beam

$$z_i = \sum_m \frac{(R'_{xzp})_m}{{}^{4}_{m} EI_i} \sin \beta_m y + \frac{y^3 - 3by^3 + 2b^2 y}{6\,bEI_i} M_{ABx} + \frac{b^2 y - y^3}{6\,bEI_i} M_{BAx} + Bz_i \ ,$$

(3-12)

where Bz_i is the deflection due to the applied load, q(y).

The functions in parentheses which appear as coefficients of the moments, M_{ABx} and M_{BAx}, may be represented by the Fourier series

$$\frac{y^3 - 3\,by^2 + 2\,b^2 y}{6\,bEI_i} = \sum_m (\chi_{AB})_m \sin \beta_m y \ , \qquad (3\text{-}13a)$$

$$\frac{b^2 y - y^3}{6\,bEI_i} = \sum_m (\chi_{BA})_m \sin \beta_m y \ , \qquad (3\text{-}13b)$$

in which

$$(\chi_{AB})_m = \frac{2b^2}{m^3 \pi^3 EI_i} \ , \qquad (3\text{-}14a)$$

$$(\chi_{BA})_m = \frac{-2(-1)^m b^2}{m^3 \pi^3 EI_i} \ . \qquad (3\text{-}14b)$$

These quantities may be interpreted physically as follows:

$(\chi_{AB})_m$ is the maximum ordinate of the m th term of the deflection curve of member AB due to a unit end moment, $M_{ABx} = 1$.

$(\chi_{BA})_m$ is the maximum ordinate of the m th term of the deflection curve of member AB due to a unit end moment, $M_{BAx} = 1$.

The function Bz_i, physically, is the deflection curve of the simply supported beam AB due to q(y). Thus, to account for this loading, it is merely necessary to add to Equation (3-12) the Fourier series representation of the deflection curve of a simple beam due to a given q(y). In general,

$$Bz_i = \sum_m \frac{q_m}{\beta_m^4 EI_i} \sin \beta_m y \ , \tag{3-15a}$$

where

$$q_m = \frac{2}{b} \int_0^b q(y) \sin \beta_m y \, dy \ . \tag{3-15b}$$

The coefficients q_m are identical to those used for one-directional load variations for plate elements, and, hence, may be taken from Table 2-4.

Substituting Equations (3-13) and (3-15a) into Equation (3-12) gives

$$z_i = \sum_m \Big[(R'_{xzp})_m \left(\frac{1}{\beta_m^4 EI_i}\right) + M_{ABx} (\chi_{AB})_m + M_{BAx} (\chi_{BA})_m + \left(\frac{q_m}{\beta_m^4 EI_i}\right) \Big] \sin \beta_m y \ . \tag{3-16}$$

An inspection of the coefficients $(\chi_{AB})_m$ and $(\chi_{BA})_m$ shows that if M_{ABx} and M_{BAx} are equal, the deflection due to these moments vanishes for even values of m, and, as should be expected, a deflection curve which is symmetrical with respect to the center line of the beam is produced.

If the flexural rigidity of the beam is not constant, it is necessary to represent its variation as a function of y. This function must be then substituted into Equation (3-10). The final deflection curve is obtained by successive integrations of the resulting equation.

If the ends A and B are displaced Δ_{Az} and Δ_{Bz}, respectively, Equation (3-12) assumes the form

$$z_i = \sum_m \frac{(R'_{xzp})}{\beta_m^4 EI_i} \sin \beta_m y + \left(\frac{y^3 - 3by^2 + 2b^2 y}{6bEI_i}\right) M_{ABx} + $$

$$+ \left(\frac{b^2 y - y^3}{6bEI_i}\right) M_{BAx} + Bz_i + \left(\frac{b-y}{b}\right)\Delta_{Az} + \Delta_{Bz} \, . \qquad (3\text{-}17)$$

These deflections may be taken as unknowns; or they may be eliminated from the above equation by using the equations of statics for the structure under consideration, and expressing them in terms of redundant forces or moments. Equation (3-17) may be written in terms of a single series by following the same procedure used to obtain Equation (3-16).

3-3. Twist of Supporting Members

The twist of beams supporting plate elements of plate-beam structures will be considered in this article. Calculations to follow are subject to the limitations arising from the usual assumptions of linear frame analysis. It will also be temporarily assumed that the edge of the plate coincides with the axis of the center of twist of the member under consideration; and, hence, that the force R'_{xzp} produces no torque.

In general, a typical beam, AB, elastically connected to the ith edge of a plate element will undergo torsional deformations due to end moments M_{ABy} and M_{BAy}, the moment M'_{xp} transmitted by the plate, and due to applied couples (Fig. 3-6). The total torque at any section is

$$M_y = M_{ABy} + \sum_m \frac{(M'_{xp})_m}{\beta_m} (\cos \beta_m y - 1) + t(y) , \qquad (3\text{-}18)$$

where t(y) is the applied torque variation.

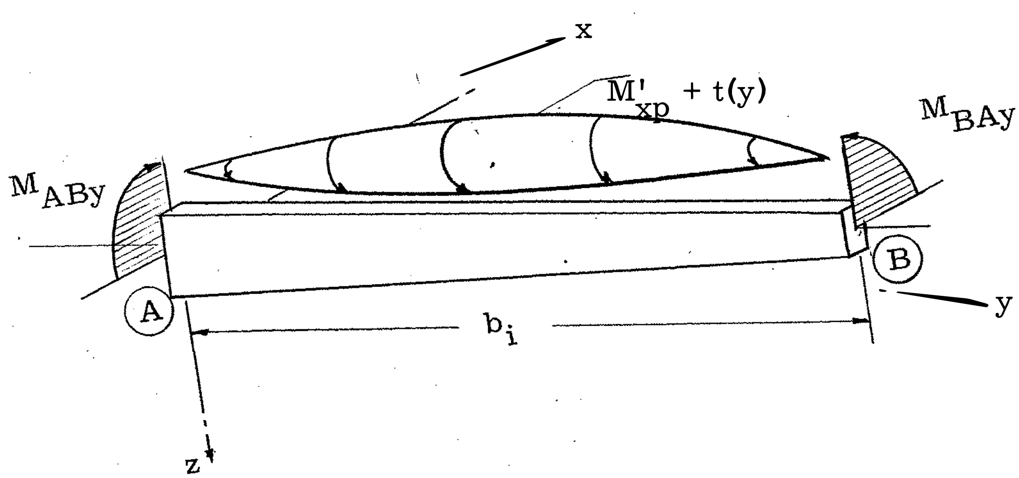


Fig. 3-6

Torsional Moments Acting On

Edge Beam AB

If the angle of twist at a given cross section is denoted ψ_i, it follows that

$$- GJ_i(y) \frac{d\psi_i}{dy} = M_y = M_{ABy} + \sum_m \frac{(M'_{xp})_m}{\beta_m} (\cos \beta_m y - 1) + t(y) ,$$

(3-19)

in which $GJ_i(y)$ is the torsional stiffness of the beam. A positive angle of twist, ψ_i, is taken as one in the direction of a positive end moment, M_{ABy}.

To evaluate ψ_i, it is necessary to divide Equation (3-19) by $GJ_i(y)$ and then to integrate the resulting equation. If $GJ_i(y)$ is constant, the final angle of twist becomes

$$\psi_i = (\frac{-y}{GJ_i}) M_{ABy} - \sum_m \frac{(M'_{xp})_m}{\beta_m^2 GJ_i} \sin \beta_m y + \sum_m \frac{(M'_{xp})_m}{\beta_m GJ_i} y + B\psi_i + (\psi_i)_A ,$$

(3-20)

in which

$B\psi_i$ is the angle of twist due to the applied torque loading, $t(y)$.

$(\psi_i)_A$ is the angle that joint A has rotated in the y-direction.

The function in parentheses which appears as a coefficient of M_{ABy} in Equation (3-20) may be represented in the Fourier series

$$(\frac{-y}{GJ_i}) = \sum_m (\eta_{AB})_m \sin \beta_m y , \qquad (3\text{-}21a)$$

where

$$(\eta_{AB})_m = \frac{2(-1)^m b}{m \pi GJ_i} . \qquad (3\text{-}21b)$$

The third term on the right-hand side of Equation (3-20) must be approximated by the series

$$- \sum_m \frac{2(-1)^m (M'_{xp})_m}{\beta_m^2 \, GJ_i} \sin \beta_m y \quad . \tag{3-22}$$

The function $B\psi_i$ may also be represented in a Fourier series. First, it is assumed that

$$B\psi_i = \sum_m H_m \sin \beta_m y \quad , \tag{3-23}$$

where H_m is some coefficient yet to be determined.

The applied torque, t(y), may be represented by the series

$$t(y) = \sum_m t_m \cos \beta_m y \quad , \tag{3-24a}$$

where

$$t_m = \frac{2}{b} \int_o^b t(y) \cos \beta_m y \, dy \quad . \tag{3-24b}$$

Differentiating Equation (3-23) and substituting the resulting expression into Equation (3-19) gives

$$- GJ_i \sum_m H_m \beta_m \cos \beta_m y = t(y) = \sum_m t_m \cos \beta_m y \quad . \tag{3-25a}$$

Thus,

$$H_m = - \frac{t_m}{\beta_m \, GJ_i} \quad , \tag{3-25b}$$

and Equation (3-23) becomes

$$B\psi_i = -\sum_m \frac{t_m}{\beta_m GJ_i} \sin \beta_m y \quad . \tag{3-26}$$

Finally, the constant, $(\psi_i)_A$, may be either taken as an unknown or expressed in terms of moments acting elsewhere in the structure. This may be done in a variety of ways. To illustrate one possibility, suppose that the edge k of the p th plate being considered is elastically supported by a beam, AC, of the constant flexural rigidity, EI_k. Since the rigid joint A has undergone a rotation $(\psi_i)_A$ in the y-direction, it follows from the familiar slope-deflection equations that

$$(\psi_i)_A = \frac{-a}{6\,EI_k}\left[2\,M_{ACy} + M_{CAy} + 2\,FM^L_{AC} - FM^L_{CA}\right], \tag{3-27}$$

in which

FM^L_{AC} is the fixed-end moment of member AC at A due to loads and R'_{yzp};

FM^L_{CA} is the fixed-end moment of member CA at C due to loads and R'_{yzp}.

This constant may be represented in the Fourier series

$$(\psi_i)_A = \sum_m \frac{-2[(-1)^m - 1](\psi_i)_A}{m\pi} \sin \beta_m y \quad . \tag{3-28}$$

Substituting Equations (3-21a, 22, 26, and 28) into Equation (3-20) gives for the angle of twist the single series

$$\psi_i = \sum_m \left[(\eta_{AB})_m M_{ABy} - \left(\frac{1+2(-1)^m}{\beta_m^2 GJ_i}\right)(M'_{xp})_m - \frac{t_m}{\beta_m GJ_i} - \right.$$

$$\left. - \frac{2((-1)^m - 1)(\psi_i)}{m\pi} \right] \sin \beta_m y . \qquad (3\text{-}29)$$

If, instead of passing through the center of twist of the beam AB, the line of action of the reactive force R'_{xzp} is found to have an eccentricity, C, Equation (3-29) may be modified by replacing $(M'_{xp})_m$ with the quantity

$$(M'_{xp})_m - C(R'_{xzp})_m .$$

The coefficients of terms in Equation (3-29) may be interpreted physically as follows:

$(\eta_{AB})_m$ is the maximum angle of twist of the m th term of the twist series of member AB due to a unit end twisting moment at A $(M_{ABy} = 1)$.

$-\dfrac{1+2(-1)^m}{\beta_m^2 GJ_i}$ is the maximum angle of twist of the member AB due to a unit-amplitude sinusoidal moment applied along the member's longitudinal axis.

$-\dfrac{t_m}{\beta_m GJ_i}$ is the maximum angle of twist of the member AB due to the m th component of the applied couple loading.

The last term in this equation is the rotation of joint A, as was defined earlier. The first two coefficients are the components of the twist flexibilities of the beam; and the third coefficient represents the components of the twist-load function of the beam.

3-4. The Edge-Slope and Edge-Deflection Equations

The details of the procedure that must be adopted to use the expressions for plate and beam deformations derived in the preceding articles in plate-beam analysis are quite varied; and depend largely on the nature of the structure under consideration.

In general, as many as eight unknown force and moment variations must be added to the number of redundants of a plate-beam structure for each plate element. Hence, eight additional equations must be formulated.

Consider, for example, a portion of a typical plate-beam structure acted upon by a general system of loads (Fig. 3-7). The member, AB, supports the edge i of the p th plate of this structural system. Since this member is connected integrally with the edge of the plate, its deflection must be the same as that of the plate's edge i. Therefore, from Equations (3-8) and (3-16), it follows that

$$\Delta_{pi} - z_i = 0 , \tag{3-30}$$

or

$$\sum_m \Bigg[\left((\Delta^R_{ii})_m - \frac{1}{\beta_m^4 EI_i} \right) (R'_{xzp})_m + (\Delta^R_{ij})_m (R''_{xzp})_m + (\Delta^R_{ik})_m (R'_{yzp})_m +$$

$$+ (\Delta^R_{il})_m (R''_{yzp})_m + (\Delta^M_{ii})_m (M'_{xp})_m + (\Delta^M_{ij})_m (M''_{xp})_m + (\Delta^M_{ik})_m (M'_{yp})_m +$$

$$+ (\Delta^M_{il})_m (M''_{yp})_m - (\chi_{AB})_m (M_{ABx}) - (\chi_{BA})_m (M_{BAx}) + (\Delta^L_i)_m -$$

$$- \frac{q_m}{\beta_m^4 EI_i} \Big] \sin \beta_m y = 0 \ . \qquad (3\text{-}31)$$

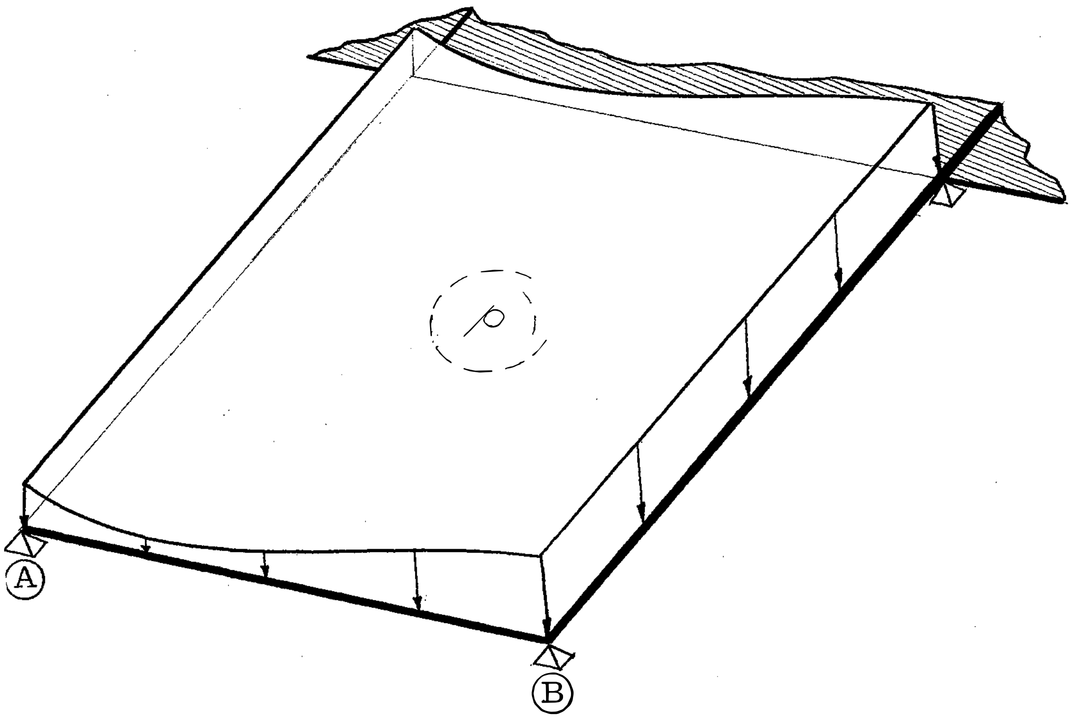


Fig. 3-7

Portion of a Typical Plate-Beam

Structure

If Equation (3-31) is to be valid for all values of y, the coefficients of each sine term of the expanded series must vanish independently. Thus, for any value of m ,

$$\left((\Delta^R_{ii})_m - \frac{1}{\beta_m^4 EI_i}\right)(R'_{xzp})_m + (\Delta^R_{ij})_m (R''_{xzp})_m + (\Delta^R_{ik})_m (R'_{yzp})_m +$$

$$+ (\Delta^R_{il})_m (R''_{yzp})_m + (\Delta^M_{ii})_m (M'_{xp})_m + (\Delta^M_{ij})_m (M''_{xp})_m +$$

$$+ (\Delta^M_{ik})_m (M'_{yp})_m + (\Delta^M_{il})_m (M''_{yp})_m - (\chi_{AB})_m (M_{ABx}) -$$

$$- (\chi_{BA})_m (M_{BAx}) + (\Delta^L_i)_m - \frac{q_m}{\beta_m^4 EI_i} = 0 . \qquad (3\text{-}32)$$

This equation is to be called an Edge-Deflection equation.

Also, the slope of the plate element along the edge i must be the same as the twist of the member connected to that edge. Thus, from Equations (3-9) and 3-29) it follows that

$$\theta_{pi} - \psi_i = 0 . \qquad (3\text{-}33)$$

Hence, for all values of y and m,

$$\left((\theta^M_{ii})_m + (\frac{1+2(-1)^m}{\beta_m^2 GJ_i})\right)(M'_{xp})_m + (\theta^M_{ij})_m (M''_{xp})_m + (\theta^M_{ik})_m (M'_{yp})_m +$$

$$+ (\theta^M_{il})_m (M''_{yp})_m + (\theta^R_{ii})_m (R'_{xzp})_m + (\theta^R_{ij})_m (R''_{xzp})_m + (\theta^R_{ik})_m (R'_{yzp})_m +$$

$$+ (\theta^R_{il})_m (R''_{yzp})_m - (\eta_{AB})_m (M_{ABy}) + \frac{2((-1)^m - 1)}{m\pi}(\psi_i)_A + (\theta^L_i)_m + \frac{t_m}{\beta_m GJ_i} = 0$$

(3-34)

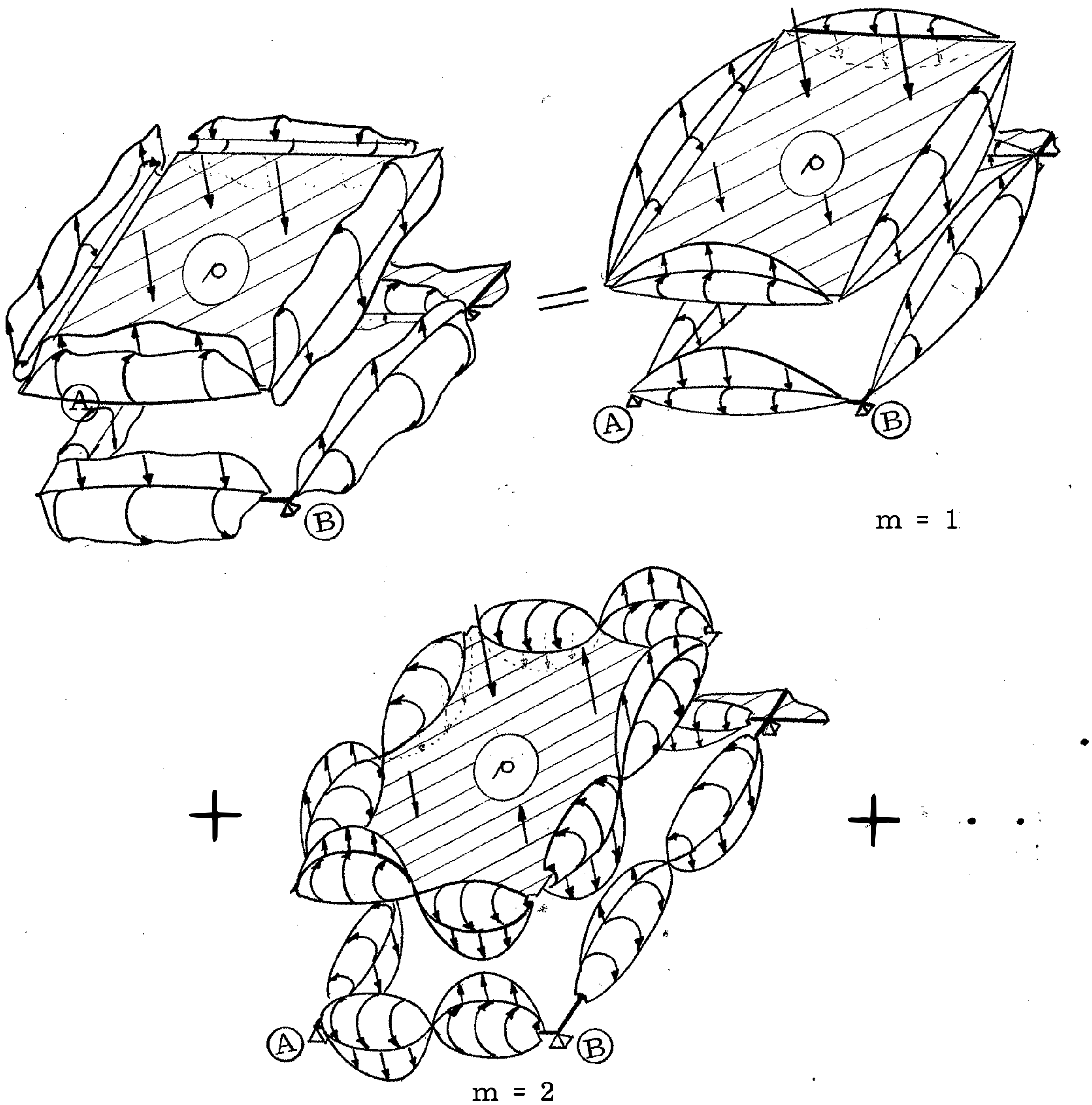


Fig. 3-8

Solution by Superposition

of Harmonics

This equation shall be called an Edge-Slope equation.

The fact that Equations (3-32) and (3-34) are valid for all values of the summation index indicates that numerical values of all of the redundants of a given plate-beam structure may be found for each value of m. Since the principle of superposition is assumed to hold for the structures under consideration, final redundants may be obtained by simply adding the results obtained for $m = 1, 2, 3, \cdots$, etc. Therefore, a given plate-beam structure is first analyzed as if it were loaded by only the first harmonic of each of the plate's unknown force and moment variations. These results are recorded and the procedure is repeated for $m = 2, 3, \cdots$, until the desired accuracy is obtained. For each case the structure is in equilibrium, but the resulting deformations are not compatible. Sufficient compatibility is achieved only after several solutions have been superimposed. This procedure is illustrated in Fig. 3-8.

Equations (3-32) and (3-34) are applicable only to the case in which a single supporting member is connected to the edge of a single plate. Several other cases are possible. For example, consider the case in which a plate is continuous over the member AB, and the deformations of two plates must be considered (Fig. 3-9). The Edge-Slope and Edge-Deflection equations follow from the relationships

$$\Delta_{pi} + \Delta_{p+1,j} - z_i = 0 \qquad (3\text{-}35)$$

and

$$\theta_{pi} + \theta_{p+1,j} - \psi_i = 0\,, \qquad (3\text{-}36)$$

where $\Delta_{p+1,j}$ and $\theta_{p+1,j}$ are the edge deformations of the p+1th plate along its edge j.

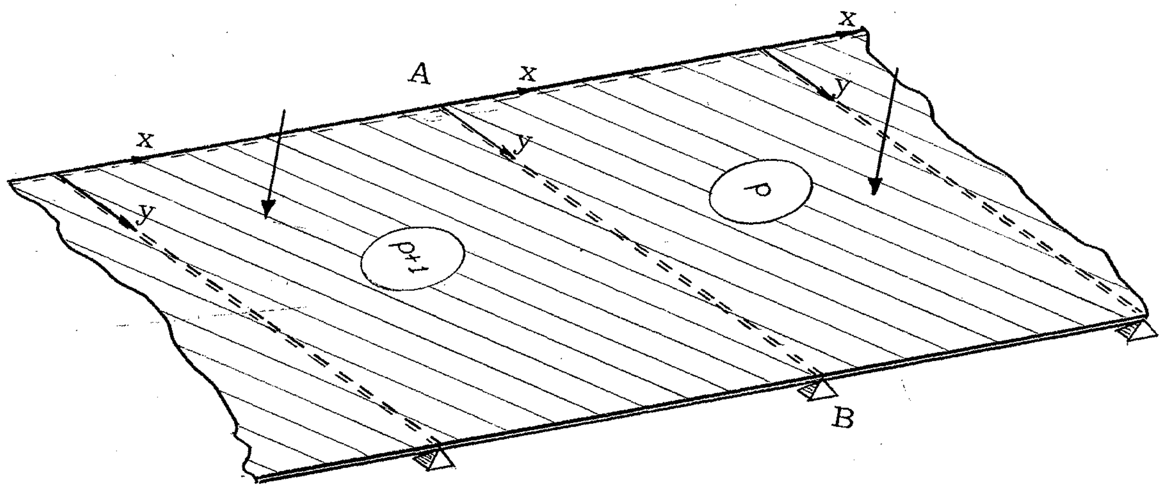


Fig. 3-9

Continuous Plate Supported By

Elastic Beams

Following procedures similar to those mentioned, the necessary compatibility equations for many plate and beam combinations can be formulated.

CHAPTER IV

NUMERICAL APPLICATION OF THE THEORY

4-1. Calculation of Flexibilities

The numerical calculation of the constants needed for plate-beam analysis is extremely laborious and requires the use of an electronic computer. The method for calculating these constants, however, can be reduced to a series of matrix operations. This is accomplished by expanding the coefficients of the Fourier series for the deflection and reaction functions into the arrays:

$$\left[E_1\right],\ \left[E_2\right],\ \left[F_1\right],\ \left[F_2\right],\ \left[G\right],\ \left[Q\right],\ \left[\gamma^{I}\right],\ \left[\gamma^{II}\right],\ \left[\lambda^{I}\right],\ \left[\lambda^{II}\right],\ \left[\xi^{I}\right],\ \left[\xi^{II}\right].$$

For example,

$$\left[E_1\right] = \begin{bmatrix} e^{1.)}_{11} & e^{1.)}_{12} & \cdots & e^{1.)}_{1,k} \\ e^{1.)}_{21} & e^{1.)}_{22} & \cdots & e^{1.)}_{2,k} \\ \cdot & \cdot & \cdot & \cdot \\ e^{1.)}_{k1} & e^{1.)}_{k2} & \cdots & e^{1.)}_{k,k} \end{bmatrix}, \cdots, \left[\gamma^{I}\right] = \begin{bmatrix} \gamma^{I}_{11} & \gamma^{I}_{12} & \cdots & \gamma^{I}_{1,k} \\ \gamma^{I}_{21} & \gamma^{I}_{22} & \cdots & \gamma^{I}_{2,k} \\ \cdot & \cdot & \cdot & \cdot \\ \gamma^{I}_{k1} & \gamma^{I}_{k2} & \cdots & \gamma^{I}_{k,k} \end{bmatrix}, \text{ etc.}$$

The matrix, $\left[A_m\right]$, of Equation (2-36) can be written in terms of submatrices resulting from products and sums of the above arrays as;

$$[A_m] = [I] - \left[\begin{array}{c:c} [F_1]^T[E_1] - [F_2]^T[E_1][B] & [F_1]^T[E_2] - [F_2]^T[E_2][B] \\ \hdashline [F_1]^T[B][E_1] - [F_2]^T[B][E_1][B] & [F_1]^T[B][F_2] - [F_1]^T[B][E_2][B] \end{array}\right],$$

in which $[I]$ is the identity matrix of order 2k; the superscript, T, denotes the transpose of the matrix; and

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & (-1)^{k+1} \end{bmatrix} .$$

The matrix $[A_m]$ for a square plate ($\mu = 0$) corresponding to a five term approximation is recorded in Table 4-1.

The matrices $[\gamma^I]$, $[\gamma^{II}]$, $[\lambda^I]$, $[\lambda^{II}]$, $[\varsigma^I]$, and $[\varsigma^{II}]$ are obtained by the following operations:

$$\left[\begin{array}{c} [\gamma^I]_{k \times k} \\ \hdashline [\gamma^I]_{k \times k} \end{array}\right] = [A_m]^{-1}_{2k \times 2k} \left[\begin{array}{c} [F_1]^T \\ \hdashline -[F_1]^T[B] \end{array}\right], \quad \left[\begin{array}{c} [\lambda^I]_{k \times k} \\ \hdashline [\lambda^{II}]_{k \times k} \end{array}\right] = [A_m]^{-1}_{2k \times 2k} \left[\begin{array}{c} [G]^T \\ \hdashline -[G]^T[B] \end{array}\right]$$

$$\left[\begin{array}{c} [\varsigma^I]_{k \times k} \\ \hdashline [\varsigma^{II}]_{k \times k} \end{array}\right] = [A_m]^{-1}_{2k \times 2k} \left[\begin{array}{c} [Q]^T \\ \hdashline -[Q]^T[B] \end{array}\right],$$

TABLE 4-1

ELEMENTS OF THE MATRIX $\left[A_m\right]$

a/b = 1, k = 5, μ = 0

Row \ Col.	1	2	3	4	5	6	7	8	9	10
1	1.002275	-.006652	.000095	-.000951	.000014	.002722	-.006725	.000095	-.000951	.000014
2	———	.985781	———	-.002995	———	———	-.014333	———	-.002995	———
3	.002549	-.021272	1.000120	-.005339	.000019	.003033	-.021441	.000120	-.005339	.000019
4	———	-.023770	———	.992879	———	———	-.023950	———	-.007121	———
5	.001778	-.026566	.000088	-.008639	1.000014	.002110	-.026773	.000088	-.008639	.000014
6	-.002214	.002587	-.000085	.000156	-.000012	.997339	.002660	-.000085	.000156	-.000012
7	———	.000272	———	-.000113	———	———	1.000387	———	-.000113	———
8	-.002306	.000239	-.000077	-.000098	-.000010	-.002790	.000409	.999923	-.000098	-.000010
9	———	-.000812	———	-.000166	———	———	-.000631	———	.999834	———
10	-.001552	-.000193	-.000048	-.000131	-.000006	-.001884	.000015	-.000048	-.000131	.999994

where $\left[A_m\right]^{-1}$ is the inverse of the matrix $\left[A_m\right]$. Matrices for a square plate with zero μ are given in Table 4-2.

Additional matrices containing elements corresponding to terms of the deflection functions $\phi_0, \phi_1, \cdots, \phi_5$ and their derivatives must also be formulated. This process is simplified if advantage is taken of Maxwell's Theorem. For example, for a square plate it may be shown that

$$\phi_1(0) = \phi_3(0) = -\phi_4(n\pi), \quad \phi_1(n\pi) = \phi_3(n\pi) = -\phi_4(0)$$

$$\phi_5(0) = -\frac{d\phi_1(0)}{dy} = -\frac{d\phi_2(n\pi)}{dy} = -\frac{d\phi_3(0)}{dx} = -\frac{d\phi_4(n\pi)}{dx}$$

$$\phi_5(n\pi) = \frac{d\phi_1(n\pi)}{dy} = \frac{d\phi_2(0)}{dy} = \frac{d\phi_3(n\pi)}{dx} = \frac{d\phi_4(0)}{dx}.$$

These quantities may be expanded to form diagonal matrices in which each element along the diagonal corresponds to a value of n or m. These must in turn be multiplied by the arrays previously mentioned to obtain final flexibilities.

For example, the column vector $\left[\Delta_{ii}^{R}\right]$, whose elements correspond to linear flexibilities for each value of m, is obtained by taking only the diagonal elements of the array resulting from the operations

$$\left[\phi_{10}\right] + \left[\left[\gamma^{I}\right]\left[E_1\right]^T + \left[\gamma^{II}\right]\left[E_2\right]^T\right]\left[\phi_{10}\right] + \left[\left[\gamma^{I}\right]\left[B\right]\left[E_1\right]^T + \left[\gamma^{II}\right]\left[B\right]\left[E_2\right]^T\right]\left[\phi_{1A}\right],$$

where $\left[\phi_{10}\right]$ and $\left[\phi_{1A}\right]$ are diagonal matrices with elements $\phi_1(0)$ and $\phi_1(\alpha_n b)$, respectively. Following similar procedures, the remaining plate flexibilities and load functions may be evaluated.

Constants corresponding to a five term approximation for a uniformly loaded square plate ($\mu = 0$) are recorded in Tables 4-3 and 4-4.

TABLE 4-2

MATRICES FOR A UNIFORMLY LOADED SQUARE PLATE

$\mu = 0, \quad k = 5$

$[E_1] = [F_1]^T$					$[\gamma^I]$					$a[\lambda^I]$				
.106070	.016908	.004243	.001469	.000628	.106109	.017699	.004243	.001653	.000628	-1.367148	-.507028	-.240583	-.136010	-.087134
.114205	.052444	.020077	.008488	.004037	.114192	.054562	.020075	.009012	.004037	-.795564	-.600543	-.366860	-.238919	-.165736
.114555	.067531	.035355	.018334	.009913	.114587	.070794	.035354	.019208	.009912	-.596123	-.581488	-.444690	-.310296	-.214610
.079034	.067128	.043431	.026525	.016158	.079014	.071192	.043427	.027664	.016158	-.394414	-.543392	-.515765	-.429156	-.339134
.078454	.062826	.045877	.031559	.021221	.078465	.067477	.045874	.032920	.021220	-.348179	-.466822	-.491512	-.445288	-.373352
$[E_2] = [F_2]^T$					$[\gamma^{II}]$					$a[\lambda^{II}]$				
.127224	.017174	.004247	.001469	.000628	-.106109	.016698	-.004243	.001423	-.000628	1.367148	-.507028	.240583	-.136010	.087134
.114205	.052444	.020077	.008488	.004037	-.114192	.052430	-.020075	.008492	-.004037	.791950	-.581706	.346938	-.207614	.135336
.137402	.068596	.035391	.018335	.009913	-.114587	.067616	-.035354	.018350	-.009912	.596123	-.581488	.444690	-.310296	.214610
.079034	.067128	.043431	.026525	.016158	-.079014	.067240	-.043427	.026554	-.016158	.387420	-.524768	.491366	-.396784	.304255
.094101	.063817	.045923	.031561	.021221	-.078465	.062937	-.045874	.031586	-.021220	.348179	-.466822	.491512	-.445288	.373352

$[G]$					$\frac{1}{a}[\xi^I P_m]$	$\frac{1}{a}[\xi^{II} P_m]$	$\phi_1(0)$	$\phi_1(\alpha_n b)$	$\phi_5(0)$	$\phi_5(\alpha_n b)$	$\frac{1}{a}\frac{d\phi_5(0)}{dx}$	$\frac{1}{a}\frac{d\phi_5(\beta_m a)}{dx}$
-1.366572	-.791195	-.595899	-.388370	-.348057	-.634986	.634986	-.021931	-.003850	-.033214	.006176	.213087	-.000833
-.507086	-.582035	-.581554	-.525004	-.466875	.000000	.000000	-.002688	-.000031	-.008442	.000132	.106107	-.000434
-.240007	-.343163	-.444501	-.489531	-.491423	-.210372	.210372	-.000796	-.000000	-.003753	.000004	.070736	-.000024
-.136102	-.206666	-.310402	-.395831	-.445373	.000000	.000000	-.000336	-.000000	-.002111	.000000	.053051	-.000001
-.086785	-.132382	-.214533	-.302596	-.373333	-.126043	.126043	-.000172	-.000000	-.001351	.000000	.042441	-.000000

TABLE 4-3

LINEAR AND ANGULAR FLEXIBILITIES OF A SQUARE PLATE

(k = 5, μ = 0)

	Linear Flexibilities							
m	$\frac{D}{a^3}(\Delta^R_{ii})_m$	$\frac{D}{a^3}(\Delta^R_{ji})_m$	$\frac{D}{a^3}(\Delta^R_{ki})_m$	$\frac{D}{a^3}(\Delta^R_{li})_m$	$\frac{D}{a^2}(\Delta^M_{ii})_m$	$\frac{D}{a^2}(\Delta^M_{ji})_m$	$\frac{D}{a^2}(\Delta^M_{ki})_m$	$\frac{D}{a^2}(\Delta^M_{li})_m$
1	-.02188	-.00378	-.00274	-.00274	-.03364	+.00511	+.03525	+.03525
2	-.00270	-.00002	-.00015	+.00015	-.00826	-.00004	+.00160	-.00160
3	-.00080	.00000	-.00003	-.00003	-.00369	-.00008	+.00035	+.00035
4	-.00034	.00000	-.00001	+.00001	-.00208	-.00003	+.00014	-.00014
5	-.00017	.00000	.00000	.00000	-.00134	-.00002	+.00006	+.00006
	Angular Flexibilities							
m	$\frac{D}{a^2}(\theta^R_{ii})_m$	$\frac{D}{a^2}(\theta^R_{ji})_m$	$\frac{D}{a^2}(\theta^R_{ki})_m$	$\frac{D}{a^2}(\theta^R_{li})_m$	$\frac{D}{a}(\theta^M_{ii})_m$	$\frac{D}{a}(\theta^M_{ji})_m$	$\frac{D}{a}(\theta^M_{ki})_m$	$\frac{D}{a}(\theta^M_{li})_m$
1	+.02467	+.01459	-.03563	+.03563	+.32889	-.11803	-.02549	+.02549
2	+.00796	-.00030	-.00366	-.00366	+.11167	+.00485	-.00494	-.00494
3	+.00342	+.00159	-.00106	+.00106	+.07298	-.00294	-.00144	+.00144
4	+.00517	-.00007	-.00045	-.00045	+.05381	+.00071	-.00090	-.00090
5	+.00129	+.00007	-.00043	+.00043	+.04287	-.00066	-.00046	+.00046

TABLE 4-4

LINEAR AND ANGULAR LOAD FUNCTIONS FOR A UNIFORMLY LOADED SQUARE PLATE

(a/b = 1, μ = 0, k = 5)

m	$(\theta_i^L)_m = -(\theta_j^L)_m = (\theta_k^L)_m = -(\theta_l^L)_m$ $\frac{D}{p_o a^3}(\theta_i^L)_m$	$(\Delta_i^L)_m = (\Delta_j^L)_m = (\Delta_k^L)_m = (\Delta_l^L)_m$ $\frac{D}{p_o a^4}(\Delta_i^L)_m$
1	+.05923	+.01637
2	.00000	.00000
3	.00203	.00017
4	.00000	.00000
5	.00037	.00002

An inspection of the flexibilities for a square plate (Table 4-3) shows that slopes and deflections along edge k are $(-1)^m$ times those along the edge l. It is also seen that positive edge forces and moments cause negative (upward) deflections along the edge i.

For square plates, positive edge moments along i, which are symmetrical with respect to the center line of the plate, cause negative slopes along edge l. The fast convergence and relative magnitude of deformations occuring along edges k and l indicate that the approximate forms of these quantities are sufficiently accurate.

Load functions for uniformly loaded square plates converge very rapidly and could be adequately represented by only the first term of their series.

From Table 4-4 it is seen that the maximum edge deflection of a uniformly loaded square plate is

$$(w_L)_{x=0,\, y=\frac{b}{2}} = .01656 \frac{a^4 p_o}{D} .$$

This value compares favorably with that obtained by Nádai[3],

$$(w_L)_{x=0,\, y=\frac{b}{2}} = .01717 \frac{a^4 p_o}{D} ,$$

the difference being 3.6 percent. This slight discrepancy occurs because Nádai did not neglect Poisson's ratio.

4-2. Numerical Example

To illustrate the procedure of analysis for plate-beam structures, the structure shown in Fig. 4-1 is considered.

All values are given in pounds, feet, or pound-feet per unit length, and properties of individual elements are recorded in Table 4-5.

TABLE 4-5

PHYSICAL PROPERTIES OF THE STRUCTURAL ELEMENTS

Plate ABCD	Beam AB
a = b = 10.00 ft.	Length = a = 10.00 ft.
$D = 7.5 \times 10^6$ lb.-ft.	$EI = 150 \times 10^6$ lb.-ft.^{2}
$\mu = 0$	$GJ = 50 \times 10^6$ lb.-ft.^{2}

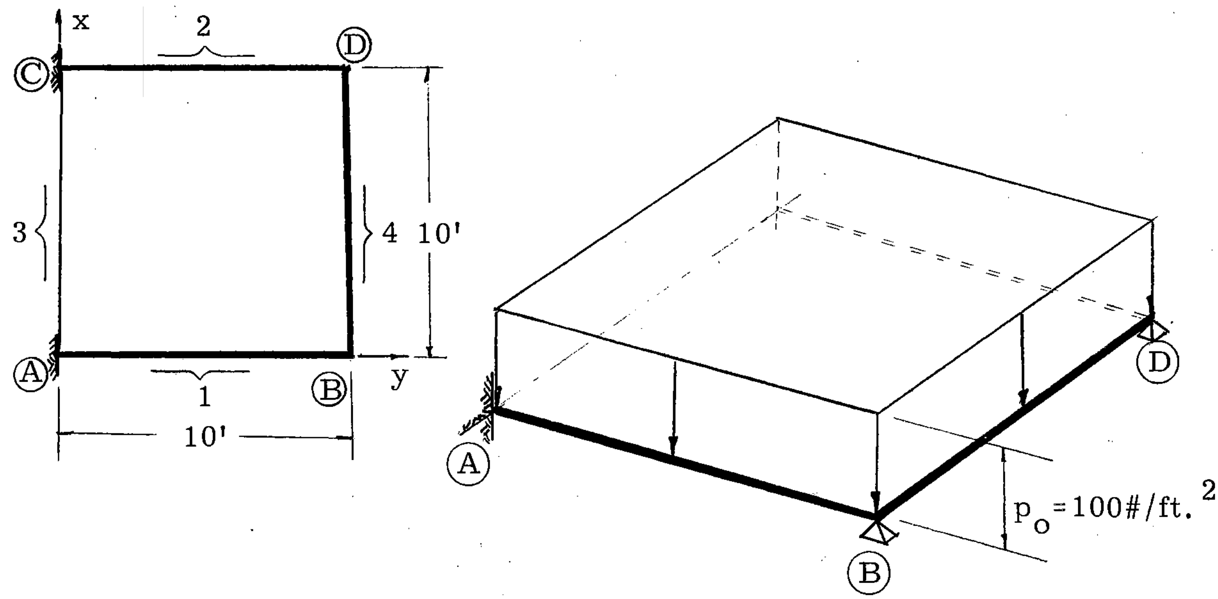


Fig. 4-1

Square Plate Supported by Elastic Beams

It is required to evaluate all edge forces and moments acting on the plate shown, and end moments acting on the supporting beams. Beam supports A and C are fixed and the edge 3 (line AC) of the plate is a free edge. The edge identified by the number 4 is elastically supported by the beam BD which is connected to members AB and CD. The shapes and elastic properties of all edge beams are identical with the exception of member BD, which is free to rotate on the point supports and is assumed to have negligible torsional rigidity. The analysis is accomplished as follows:

A.) Redundants

For the structure considered it is seen that

$$M_{BAx} = M_{DCx} = M'_y = R'_{yz} = M''_y = (\psi_1)_A = (\psi_2)_C = 0 .$$

Thus, the remaining unknown edge forces and moments acting on the plate are

$$M'_x , \; M''_x , \; R'_{xz} , \; R''_{xz} , \text{ and } R''_{yz} \; .$$

Due to symmetry, components of $(R''_{yz})_m$ corresponding to even values of m are zero, and

$$R'_{xz} = - R''_{xz} , \quad M'_x = M''_x \; .$$

Unknown end moments acting on the edge beams are

$$M_{ABx} , \; M_{CDx} , \; M_{ABy} , \; M_{CDy} , \; M_{BDy} , \text{ and } M_{DBy} \; .$$

The remaining moments are found by statics. Also, due to symmetry

$$M_{ABx} = - M_{CDx} , \quad M_{ABy} = - M_{CDy} , \quad M_{BDy} = M_{DBy} \; .$$

B.) Flexibilities and Load Functions

Plate ABCD:

Linear and angular flexibilities of the plate are given in Table 4-3. Load functions are given in Table 4-4.

Member AB:

Linear and twist flexibilities of the beam AB are recorded in Table 4-6. All load functions are zero for the supporting beams.

TABLE 4-6

LINEAR AND TWIST FLEXIBILITIES

OF EDGE BEAMS

m	$\frac{EI}{b^2}(\chi_{AB})_m$	$\frac{EI}{b^2}(\chi_{BA})_m$	$\frac{GJ}{b}(\eta_{AB})_m$	$\left(\frac{1+2(-1)^m}{m^2\pi^2}\right)$
1	+.06450	+.06450	-.63662	-.10132
2	+.00806	-.00806	+.31831	+.07599
3	+.00239	+.00239	-.21221	-.01126
4	+.00101	-.00101	+.15916	+.01900
5	+.00052	+.00052	-.12732	-.00405

C.) Edge-Deflection Equations (Equation 3-32)

Computations are shown in detail for m = 1, and the results are given for m = 2, 3, and 5. Equations for m = 4 have trivial solutions since the effects of m = 1, 2, 3, and 5 have negligible influences on these quantities.

Substituting the plate and beam flexibilities and load functions for m = 1 into Equation (3-32) results in the Edge-Deflection equations for members AB and BD. If these are multiplied by EI/a^4 the following expressions result:

Member AB

$$-.06159(R'_{xz})_1 + .00548(R''_{yz})_1 - .00571(M'_x)_1 - .00065(M_{ABx})_1 +$$

$$+ 32.7400 + .00010(R'_{xz})_2 + .00020(M'_x)_2 = 0 .$$

Member BD

$$.03349\,(R''_{yz})_1 - .01096\,(R'_{xz})_1 + .01410\,(M'_x)_1 - .00129\,(M_{BDy})_1 + $$

$$+ 32.7400 = 0 .$$

D.) Edge-Slope Equation (Equation 3-34)

Substituting the proper flexibilities into Equation (3-34) and multiplying by $\frac{GJ}{a^2}$ gives for member AB,

$$.19663\,(M'_x)_1 + .26173\,(R'_{xz})_1 + .23753\,(R''_{yz})_1 + .06366\,(M_{ABy})_1 +$$

$$+ 394.8667 - .06393\,(R'_{xz})_2 + .05182\,(M'_x)_2 = 0 .$$

The last two terms in the compatibility equations for member AB are taken from the exact edge deformation expressions given in Tables 3-1A, B and 3-2A, B. They are included to account for the fact that member AB has unsymmetrical end conditions.

It can be shown that the influence of terms corresponding to m = 1 on those for even values of m is very small. Similarly, the influence of $(R'_{xz})_2$ and $(M'_x)_2$ on the deformations occuring along BD due to $(R'_{xz})_1$, $(M'_x)_1$, $(R''_{yz})_1$, etc. are very small and assumed negligible.

E.) Deformation Equations for Supporting Beams

The condition that the slope is zero at A leads to the relationship

$$- .22156\,(R'_{xz})_1 + .31831\,(R''_{yz})_1 + .0100\,(M_{ABx})_1 +$$

$$+ .17125\,(R'_{xz})_2 = - 202.6423.$$

Equating the twist of member AB at B to the slope of member BD at that point gives

$$- (M_{ABy})_1 + 3.18309\,(M'_x)_1 - .16667\,(M_{BDy})_1 +$$

$$+ 1.07505\,(R''_{yz})_1 + 1.59155\,(M'_x)_2 = 0 .$$

F.) Equation of Statics

Isolating member AB and equating to zero the sum of moments about its longitudinal axis gives

$$(M_{BDy})_1 - (M_{ABy})_1 + 6.36620\,(M'_x)_1 = 0 .$$

G.) Final Forces and Moments

Following the procedure just outlined for successive values of m, a sufficient number of equations can be generated to solve for all unknowns. Results for a five term approximation are given in Table 4-7.

Moments and deflections at points in the plate may be found though the superposition of surfaces given by Equations (2-41, 56, and 73) after the substitution of the appropriate values for the components of the edge forces and moments.

TABLE 4-7

FINAL FORCES AND MOMENTS

m	$(R'_{xz})_m$	$(M'_x)_m$	$(R''_{yz})_m$	M_{ABx}	M_{ABy}	M_{BDy}
1	391.43	-498.14	-646.81	9986.56	-3353.82	-182.55
2	- 57.81	-693.20	———	———	———	———
3	311.12	-348.20	-162.15	2658.49	- 427.76	311.13
4	———	———	———	———	———	———
5	163.54	-105.47	-116.64	960.43	- 77.60	56.69

CHAPTER V

SUMMARY AND CONCLUSIONS

5-1. Summary

The analysis of plate-beam structural systems by the flexibility approach is presented in this study. The results of the analysis are limited to thin rectangular plates of constant thickness and are obtained in the form of Fourier series.

Edge forces and moments acting on the plates are chosen as redundants and are represented by sine series.

A thin rectangular plate supported at its corners is selected as the basic structure for plate members. Levy solutions are obtained for edge deflections and slopes of the basic plate due to arbitrary edge forces, edge moments, and applied loading. Final deflection surfaces are in the form of single, double, and triple summations whose coefficients are in terms of the components of the series for edge forces and moments. These series are reduced to an approximate form in which there exists a one to one correspondence between components of edge redundants and individual sine terms.

Linear and twist flexibilities of the elastic beams supporting plate members are developed in the form of trigonometric series. The coefficients of these series are in terms of applied loading, edge forces and moments of the plate, and end moments of the beam.

Compatibility of plate and beam deformations is accomplished by means of "Edge-Deflection" and "Edge-Slope" equations. A sufficient number of equations are formulated to solve for all unknowns for each term in the series. Final answers are obtained by superposition of the solutions corresponding to each term.

A systematic sequence of matrix operations is presented that provides an adequate method for calculating plate flexibilities; and a numerical example is included to illustrate the application of the theory developed to the analysis of a typical plate-beam structure.

5-2. Conclusions

Levy solutions for the deflection surfaces of corner supported plates under the action of any type of static loading may be obtained by superimposing the following surfaces:

1.) The deflection surface of a plate simply supported on two opposite edges, free along the remaining edges, and subjected to a general system of applied loads.

2.) The deflection surface of a plate supported at its corners and bent by arbitrary forces distributed along the edges corresponding to the simply supported edges of the surface of part 1.).

The series coefficients defining these arbitrary edge forces may be adjusted so that on superimposing the two surfaces, all edges of the plate become free. This adjustment leads to an infinite number of equations and an infinite number of unknowns. Results may be obtained by taking a finite number of these equations. The resulting coefficient matrix of these equations has a very strong diagonal and differs little from an identity matrix.

Exact expressions for edge deformations of the basic structure for plate elements, a corner supported plate, are very complicated, and all terms of the series for edge redundants influence each term of the deformation series. An approximate form of these deformations, which introduces only small errors, is obtained by retaining only the strong diagonal terms in the exact arrays and reducing these expressions to a form in which there is only one component of an edge redundant series for each sine term. This approximation allows the analyst to solve plate-beam systems on a term by term basis, and to then superimpose the solutions to find final results.

Modifications in this procedure are necessary in cases involving symmetrically loaded plates supported by beams with unsymmetrical end conditions; as equations corresponding to even terms in the series have trivial solutions. To remedy this difficulty, one additional term is retained from the exact expressions which accounts for the influences of the first and second harmonics of edge redundants on one another. Numerical calculations show that these influences are small and indicate that additional refinements are unwarranted. Load functions were found to be in good agreement with existing data.

5-3. Extension

The analysis of plate-beam structures is a broad area of structural engineering, and one in which much work remains to be done. With the basic philosophy of flexibility analysis by trigonometric series established, the theory developed in this study may be extended to cover more general cases. An immediate extension of this research would be to develop analytical expressions for plate flexibilities for the case in which the basic plate is supported by elastic springs. The method of analysis of the basic plate structure should then be extended to include the influences of in-plane forces, antisotropy of plate materials, elastic foundations, and combinations of these special effects.

After a thorough investigation of plates supported at their corners, the extension of this research should be directed toward the analysis of general plate-frame structures.

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