SUBADDITIVE FUNCTIONS OF ONE REAL VARIABLE

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PREFACE

This paper is concerned with certain problems in the theory of subadditive functions of a real variable. The basic definitions appear on page 1 and the entire first chapter serves as an introduction and orientation to the remaining material. Chapter II contains a basic rotation theorem and some lemmas on continuity and boundedness which will be used in later chapters. Chapters III and IV deal with several kinds of extensions of functions which yield or preserve subadditivity; in particular, Chapter III is devoted to the maximal subadditive extension to $E = [0,\infty)$ of a subadditive function on $[0,a]$. Contrary to previous work on this topic, no assumptions of continuity are made.

The last three chapters are devoted to sets of subadditive functions. Chapter V discusses convergence, especially uniform convergence, of subadditive functions -- motivated by a theorem of Bruckner -- and gives an example (the Cantor function) of a monotone subadditive function with unusual properties. In Chapters VI and VII convex cones of subadditive functions are discussed and the extremal element problems considered. Chapter VI contains a complete solution of these problems in a simple case, and

Chapter VII discusses partial solutions in other cases, applications of the results of previous chapters, and some unsolved problems. An index to numbered propositions, theorems, lemmas, and remarks is provided in Appendix A to facilitate the many cross-references made in the body of the paper. Appendix B contains a list of special or unusual notations used in the paper.

I am deeply indebted to Dr. L. Wayne Johnson, Chairman of the Department of Mathematics, and to his entire staff for providing a climate in which the study of mathematics can be pleasantly and profitably undertaken. The members of my advisory committee have been particularly generous with their time and encouragement. The comradeship and encouragement of my fellow students -- especially John Allen, F. W. Ashley, Jr., David Cecil, and Glen Haddock, who helped me to organize my thoughts by listening patiently to explanations of my problems -- have contributed markedly to the result. I am especially grateful for the friendly counsel and challenging questions offered by Professor E. K. McLachlan which have contributed a great deal to the construction of this paper.

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CHAPTER I

SUBADDITIVE FUNCTIONS

This thesis is concerned with certain real-valued functions of one real variable; in particular, the word "function" will always mean "finite-valued, Lebesgue measurable function." The definitions to be given will be given, whenever appropriate, in the context of their applications although, for example, one can study subadditive functions on an arbitrary additive semi-group and concave functions on any convex set.

Definition 1: A function f defined on a set H of real numbers and with range contained in the set R of all real numbers, is subadditive on H if, for all elements x and y of H such that x+y is an element of H,

$$f(x+y) \leq f(x)+f(y).$$

If equality holds, f is called additive; if the inequality is reversed, f is superadditive. A function g is concave on the (possibly infinite) interval D if, for all x and y in D and all t which satisfy $0 \leq t \leq 1$,

$$f(tx+(1-t)y) \geq tf(x)+(1-t)f(y).$$

If this inequality is reversed, f is convex on D. A measurable concave function is continuous except possibly at boundary points of D. [1, p. 96][1].

The principal work on the general theory of subadditive functions is that of Hille and Phillips [2]. This reference also includes a part of the work of Rosenbaum [3] on subadditive functions of several variables.

The use of some sets H as the domains of subadditive functions is somewhat "dishonest." For example, every function defined on the interval $[2,3]$ is subadditive there. In most of what follows H will be the closed half-line $E = [0,\infty)$, the set $J = \{0,1,2,\ldots\}$ of all non-negative integers, or some initial portion of either.

Some basic properties of subadditive functions f on R -- noted in the works of Hille and Phillips [2] and Rosenbaum [3] are:

1) $f(0) \geq 0$ and $f(-x) \geq -f(x)$;
2) if g is also subadditive, then f+g is subadditive;
3) if $t \geq 0$, then tf is subadditive;
4) if $\{f_s\}$ is any family of subadditive functions and if $\sup_s\{f_s(x)\}$ is finite for every x, then $g: x \to \sup_s\{f_s(x)\}$ is subadditive;
5) if g is subadditive and non-decreasing, then the composite $h: x \to g(f(x))$ is subadditive; and
6) if f is also non-negative on E and if g is a positive non-decreasing function on E, then the product $h: x \to f(x)g(x)$ is subadditive on E.

[1]The symbol "[ ]" indicates a reference to the Bibliography.

A large class of examples of subadditive functions on E can be obtained from the following proposition.[2]

Proposition 1 [1, p. 83]: If f, defined on $(0,\infty)$, is such that, for each $t > 0$,

$$f(x) \leq \frac{x}{t} f(t) \text{ for all } x \geq t,$$

then f is subadditive on $(0,\infty)$. (In other words, f is subadditive if the function f^*, defined by $f^*(x) = f(x)/x$, is non-increasing.)

Many examples to follow will show that the condition of this proposition is not necessary for subadditivity. Functions which do satisfy Proposition 1 include any function which is non-negative and non-increasing on E, any concave function f on E with $f(0) \geq 0$ (in particular, any non-negative constant function), and the function defined by $g(x) = \sqrt{a^2 + x^2}$, $x \geq 0$, which is convex. Convex subadditive functions form an easily characterized class since the converse of Proposition 1 holds if f is convex. [2, p. 239]. If one insists that $f(0) = 0$, the class becomes trivial.

Remark 1: If f is convex and subadditive on E and if $f(0) = 0$, then f is additive on E.

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[2]No connotation of merit or importance is attached to the usage of "proposition" and "theorem." A result called a "proposition" is due to another author. Results labeled "theorem," "lemma," or "remark" are believed by this author to be new.

Proof: Let $x \in (0,\infty)$ and $t \in (0,1]$ be given. Then

$$f(tx) = f(tx+(1-t)0) \leq tf(x)+(1-t)f(0) = tf(x),$$

so that division by tx yields $f(tx)/tx \leq f(x)/x$. Thus $f(x)/x$ is non-decreasing on $(0,\infty)$, and, by the result mentioned above, must be constant. Therefore, $f(x)/x = c$, a constant, on $(0,\infty)$. Since $f(0) = 0$, $f(x) = cx$ for all $x \in E$.

Another remark which serves to develop the intuitive aspects of subadditivity follows. It implies, roughly, that the function f spends at least half its time on [0,1] on or above the line $y = f(1)x$. The third remark implies that the set [0,1] is general enough to be useful.

Remark 2: If f is a continuous subadditive function on [0,1], then

$$f(1)/2 \leq \int_0^1 f(x)dx.$$

Proof: Let $x \in [0,1]$. Then $f(1) \leq f(x) + f(1-x)$, so that

$$f(1) = \int_0^1 f(1)dx \leq \int_0^1 f(x)dx + \int_0^1 f(1-x)dx.$$

By the change of variable $v = 1-x$ one obtains

$$\int_0^1 f(1-x)dx = \int_1^0 f(v)(-dv) = \int_0^1 f(v)dv = \int_0^1 f(x)dx.$$

Thus

$$f(1) \leq 2\int_0^1 f(x)dx.$$

Remark 3: If f is subadditive on R and $k \in R$, then the function g such that $g(x) = f(kx)$ is subadditive on R.

Proof: By direct calculation, $g(x+y) = f(k(x+y))$ $= f(kx+ky) \leq f(kx)+f(ky) = g(x)+g(y)$.

Thus the sets [0,1] and J are just as general domains of subadditive functions as the sets [0,a], $a > 0$, and $\{0, a, 2a, 3a, \ldots\}$. It also follows from Remark 3, using $k = -1$, that $g: x \to f(-x)$ is subadditive on R; that is, "reflection in the y-axis" preserves subadditivity.

Another large class of functions which are subadditive -- again including the non-negative constant functions -- appears in the following result.

Proposition 2 [1, p.83]: Any function f such that

$$\sup\{f(x) : x \in H\} \leq 2(\inf\{f(x) : x \in H\})$$

is subadditive on H.

Hille and Phillips [2, p.246] discuss functions f defined by $f(x) = a$ if $x \in A$ and $f(x) = b$ if $x \in cA$, where A is closed under addition and cA is its set-theoretic complement. They note that f is subadditive if $0 \leq a \leq 2b$ and that, if $b \leq 2a$, the hypothesis that A is closed under addition can be dropped since f then satisfies Proposition 2. Under certain conditions a converse is possible and is proved below.

Definition 2: Let $A \subset B \subset R$, where R is the set of all real numbers. The set A is closed under addition with respect to B if $x, y \in A$ and $x+y \in B$ imply $x+y \in A$.

Theorem 1: Let $0 \leq 2a < b$ in R and let $A \subset B \subset R$. Define f on B by $f(x) = a$ if $x \in A$ and by $f(x) = b$ if $x \in B \cap cA$. Then f is subadditive on B if, and only if, A is closed under addition with respect to B.

Proof: To show that the relation between A and B implies subadditivity, the various possible cases will be considered for $x, y, x+y \in B$.

1) If $x, y \in A$, then $x+y \in A$ and $f(x+y) = a \leq f(x)+f(y) = 2a$.

2) If $x \in A$ and $y \in B \cap cA$, then $f(x+y) = a$ or b, while $f(x)+f(y) = a+b$.

3) If $x, y \in B \cap cA$, then $f(x+y) = a$ or b, and $f(x)+f(y) = 2b$. In this case $f(x+y) < f(x)+f(y)$.

Thus f is subadditive on B.

Conversely, if there are elements $x, y \in A$ such that $x+y \in B \cap cA$, then $f(x+y) = b > f(x)+f(y) = 2a$, which means that f is not subadditive on B.

In particular, the characteristic function of the irrational numbers is subadditive on R.[3] This is an example of a subadditive function which is discontinuous at every point and an example which negates the converse of Proposition 2 in every interval. The above theorem implies that the characteristic function of a set $A \subset R$ is subadditive if,

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[3]The characteristic function of $A \subset R$ is the function defined by $X(A;x) = 1$ if $x \in A$, $= 0$ if $x \in cA$.

and only if, cA is an additive semi-group in R. An example of a continuous, non-negative, subadditive function with infinitely many separated zeros is given by $f(x) = |\sin x|$ (since $|\sin(x+y)| \le |(\sin x)(\cos y)| + |(\cos x)(\sin y)| \le |\sin x| + |\sin y|$, $|\cos x|$ being ≤ 1). This example will be generalized in Chapter IV.

CHAPTER II

SOME GEOMETRY OF SUBADDITIVITY

Several properties of subadditive functions which have a graphical interpretation will be developed in this chapter. Two of these, the lemmas, will be of value in proving theorems and validating examples in later chapters.

The function $f^*: x \to f(x)/x$ associated with a subadditive function f has already appeared in Proposition 1 and will appear later in this work, notably in Chapter III.[1] The properties of f^*, the properties of additive functions, several examples of subadditive functions, and especially Theorem 5 suggest that subadditivity may be preserved under rotation of coordinate axes; as the following theorem shows, subadditivity is preserved if the function concept is.

<u>Theorem 2:</u> Let f be subadditive on R and let θ, $0 < \theta < \frac{\pi}{2}$ (or $-\frac{\pi}{2} < \theta < 0$), be a rotation of axes such that the graph $v = f(u)$ in the rectangular Cartesian (u,v)-system is the graph of a single-valued real function for all rotations α of the axes such that $0 \leq \alpha \leq \theta$ (respectively, $\theta \leq \alpha \leq 0$). Then the function g obtained by referring the

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[1]See also [2] and [4].

graph of f to the (u',v')-system obtained by the rotation θ is subadditive on R.

<u>Proof:</u> The theorem will be proved by the method of contradiction. It will be shown that, if g is not subadditive, then there exist points of the graph which determine a line perpendicular to a rotated position of the u-axis for some rotation α included in the statement of the theorem.

Let x', y', and z' be values of u' such that $x'+y' = z'$. Let x, y, and z be the u-coordinates of $(x',g(x'))$, $(y',g(y'))$, and $(z',g(z'))$, respectively. Suppose that $g(z') > g(x')+g(y')$. Let $G = g(z')-g(x')-g(y')$. From the rotation formula $f(u) = u'\sin(\theta)+g(u')\cos(\theta)$ it follows that

$$f(z)-f(x)-f(y) = (z'-x'-y')\sin(\theta)+G\cos(\theta).$$

Since $z'-x'-y' = 0$,

$$f(z)-f(x)-f(y) = G\cos(\theta). \qquad (A)$$

In the same way, the formula $u = u'\cos(\theta)-g(u')\sin(\theta)$ yields

$$x+y-z = G\sin(\theta). \qquad (B)$$

In the case $\theta > 0$ (see Figure 1), equation (B) yields $x+y = z+G\sin(\theta) > z$, which implies that the point $(x+y,f(x+y))$ is to the right of the line $u = z$. Using equation (A), $f(x+y) \leq f(x)+f(y) = f(z)-G\cos(\theta)$, which implies that $(x+y,f(x+y))$ is on or below the line $u' = z'$. Therefore, the points $(z,f(z))$ and $(x+y,f(x+y))$ determine a line which makes an angle α, $0 < \alpha \leq \theta$, with the vertical line $u = z$. For this rotation α of the (u,v)-axes these two distinct (since $x+y > z$) points have the same abscissa.

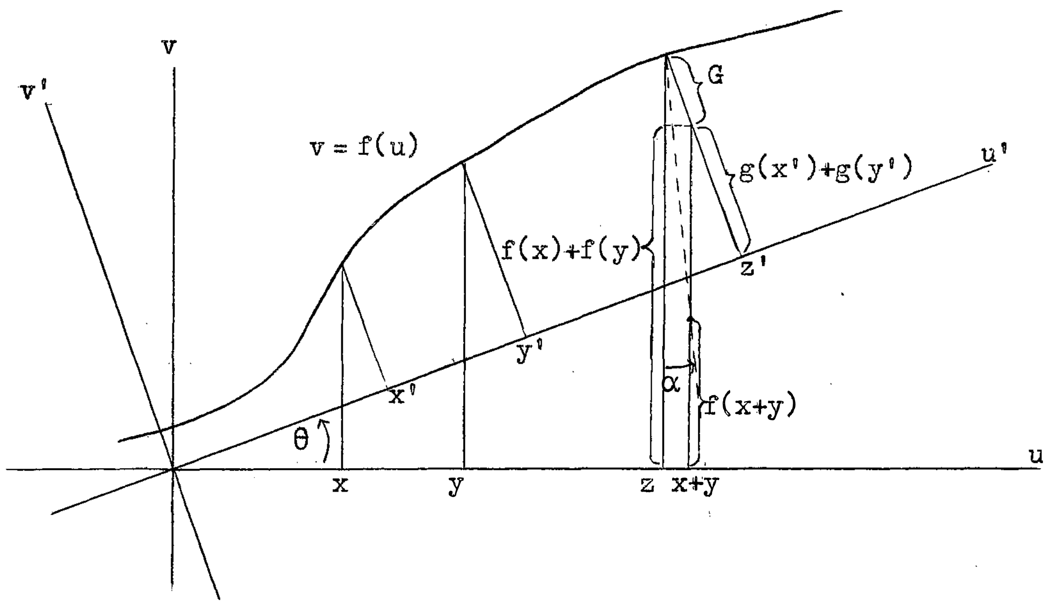


Figure 1. Subadditivity and Positive Rotations

In the case $\theta < 0$ (see Figure 2), $\sin(\theta) < 0$, which means that equation (B) implies $x+y < z$. The inequality $f(x+y) \leq f(x)+f(y) = f(z)-G\cos(\theta)$ still holds -- which means that the rotation again exists for which $(z,f(z))$ and $(x+y,f(x+y))$ have the same abscissa.

The restriction $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ can be removed by repeated rotations $\theta' < \frac{\pi}{2}$, although the property $\cos(\theta) > 0$ was necessary to the proof given.

The class of continuous subadditive functions is quite large. For example, the following curve-fitting problem has the indicated solution.

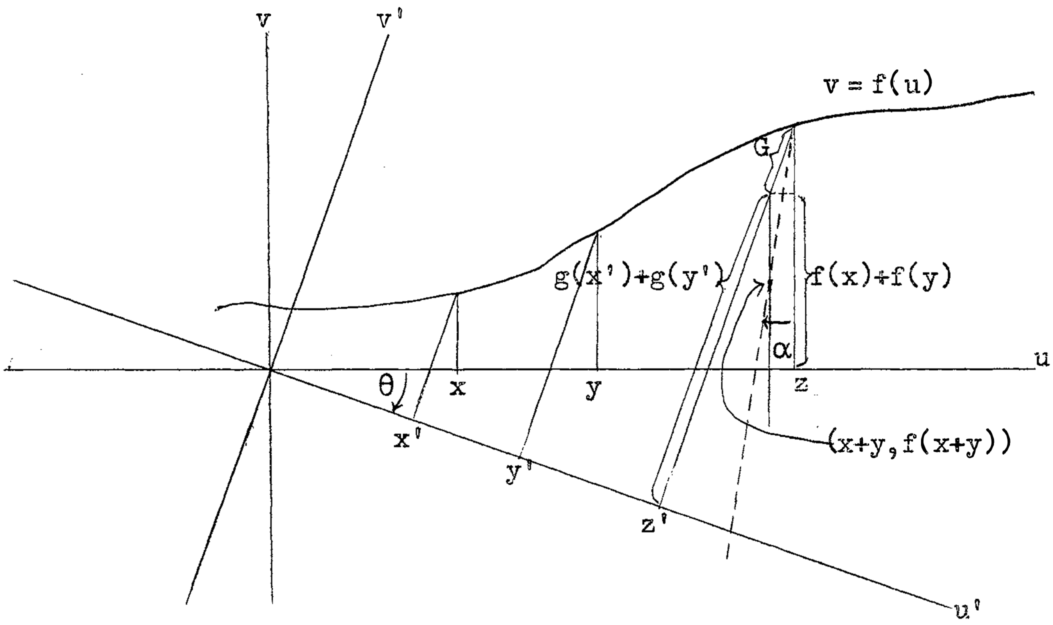


Figure 2. Subadditivity and Negative Rotations

Remark 4: Let x, u, y, and v be non-negative numbers with $x < u$. Except for the case $u = nx$ and $v > ny$, n a positive integer, there exists a continuous subadditive function f on E with $y = f(x)$ and $v = f(u)$.

Proof: If $v \leq y$, let $f(t) = y$ for $t \in [0, x]$, $f(t) = (y-v)(t-x)/(x-u) + y$ for $t \in [x, u]$, and $f(t) = v$ for $x \in [u, \infty)$. This "polygonal" function is non-increasing and non-negative so it is subadditive by Proposition 1.

If $v > y$ and $u \neq nx \neq 0$, then there exists $b > 0$ such that $f(t) = y + b|\sin(\pi t/x)|$ is the desired function. The function f is here the sum of two subadditive functions.

In each case the function f is bounded above on E and is non-negative.

In the case $u = nx$, if f is subadditive, $v = f(nx) \leq nf(x) = ny$, and no subadditive function is possible for which $v > ny$.

The following lemmas deal with continuity and boundedness of subadditive functions on E and will be useful later in verifying examples and proving a convergence theorem.

Lemma 1: If g is a subadditive function on $E = [0,\infty)$ and if $g(x) \leq mx$ for all $x \in E$ and some $m \in R$, then $u < v$ in E implies that $g(v)-g(u) \leq m(v-u)$.

Proof: Since $v-u \in E$, $g(v-u+u) \leq g(v-u)+g(u)$, implying that $g(v)-g(u) \leq g(v-u) \leq m(v-u)$ by hypothesis.

For subadditive functions defined on R, Hille and Phillips [2, p. 247] show that, if f is continuous at $t = 0$ but $f(0) > 0$, the discontinuities of f may be everywhere dense in R; if, however, f is continuous at $t = 0$ and $f(0) = 0$, then f is continuous everywhere. For functions defined only on E, the following simpler results are all that is required.

Lemma 2: Let f be a subadditive function defined on E with at most a finite number of discontinuities on any compact subinterval of E. If $f(x) \to 0$ as $x \to 0$, then the oscillation of f at a point of discontinuity y

-- $\lim_{x \to y+} f(x) - \lim_{x \to y-} f(x)$ -- is negative. If f is also non-decreasing on E, then f is continuous on E.

Proof: Let $0 < h \leq x$. Then $f(x+h) \leq f(x)+f(h)$, which implies that $\lim_{h \to 0+} f(x+h) \leq f(x)$. Also $f(x-h+h) \leq f(x-h)+f(h)$, or $f(x)-f(h) \leq f(x-h)$, so $\lim_{h \to 0+} f(x-h) \geq f(x)$. Thus at a point of discontinuity y, where at least one of the limits is not equal to f(y), $\lim_{h \to 0+} f(y+h) < \lim_{h \to 0+} f(y-h)$.

If f is non-decreasing, then $f(x+h) \geq f(x) \geq f(x-h)$ for $h > 0$, which means that $\lim_{h \to 0+} f(x+h) = f(x) = \lim_{h \to 0+} f(x-h)$.

The principal result of this chapter has been the solution given in Theorem 2 to the rotation problem. A discussion of the frequently associated problem of translation of axes has been rather glaringly omitted. The problem has been studied, however. In particular, an unpublished article written by P. C. Hammer of the University of Wisconsin, "Subadditivity in General," contains the following result, which is stated here in less general form than that in which it appears in that article.

Proposition: If $\sup_{x,y}\{f(x+y)-f(x)-f(y)\}$ exists and equals b and if $c \geq b$, $b, c \in R$, then c+f is subadditive (and conversely).

Hammer then applies this result and some calculus to the statement of Rosenbaum [3] that 3+sin(t) is subadditive on R, but 3 is not the smallest value for which this is true. Hammer shows that $3\sqrt{3}/2$ is the smallest value.

CHAPTER III

THE MAXIMAL SUBADDITIVE EXTENSION

Definition 3: Let f be a subadditive function on the interval $[0,a]$, $a > 0$. The function Sf defined at each $x \epsilon E = [0,\infty)$ by

$$Sf(x) = \inf \Sigma f(x_i),$$

where the infimum is taken over all finite sets $\{x_1, x_2, \ldots, x_n\}$ (where x_i may equal x_j for $i \neq j$) such that $x_1 + x_2 + \ldots + x_n = x$ and $x_i \leq a$ $(i = 1, 2, \ldots, n)$, is called the maximal subadditive extension of f to E. Each set $\{x_1, x_2, \ldots, x_n\}$ is called an a-partition of x.

The function Sf was defined and investigated (using some different notations) by Bruckner [4;5] in the analogous case of superadditive functions. Most of his results pertain to continuous and non-negative superadditive functions and are not immediately applicable here. More will be said about this situation in Chapter V, but a few of his general results follow.

Proposition 3 [5]: a) The function Sf is subadditive on E, and, if g is any subadditive function on E which is an extension of f, then $Sf(x) \geq g(x)$ for all $x \epsilon E$.

b) If f and g are subadditive on $[0,a]$ and if $f(x) \geq g(x)$ for all $x \in [0,a]$, then $Sf(x) \geq Sg(x)$ for all $x \in E$. c) If $c \geq 0$, then $c(Sf(x)) = S(cf(x))$ for all $x \in E$.

<u>Proposition 4</u> [4]: If $x \in E$ and $x \leq Ma$, M a positive integer, if $\{x_1, \ldots, x_n\}$ is an a-partition of x, and if f is subadditive on $[0,a]$, then there exists an a-partition $\{y_1, y_2, \ldots, y_r\}$ of x such that $r \leq 2M+1$ and

$$Sf(x) \leq \sum_{i=1}^{r} f(y_i) \leq \sum_{i=1}^{n} f(x_i). \qquad (A)$$

<u>Proof</u>: If $u, v \in \{x_1, \ldots, x_n\}$ and $u+v \leq a$, then $f(u+v) \leq f(u)+f(v)$. Thus replacement of u and v by u+v yields an a-partition which is at least as good (in the sense of (A)) as the original. Repetition of this procedure yields the desired a-partition.

<u>Definition 4</u>: Any a-partition D such that $u, v \in D$ implies $u+v > a$ is called a <u>refined</u> a-partition.

By the above proof, a refined a-partition can be obtained as a "refinement" of any given a-partition without loss of accuracy of approximation to Sf. A refined a-partition of x does not contain 0 if $x > 0$ and does contain at most one element of $(0, \frac{a}{2}]$. A refinement of a given a-partition is not necessarily unique. For example, if $a = 1$, $x = 2$, and the a-partition of x is $\{1/3, 1/3, 1/3, 1/2, 1/2\}$, then possible refinements are $\{1, 1\}$, $\{1/3, 5/6, 5/6\}$, and $\{2/3, 5/6, 1/2\}$.

<u>Proposition 5</u> [4]: a) If f is continuous and subadditive on [0,a] and $x \in E$, then there exists a refined a-partition $\{x_1, \ldots, x_n\}$ of x such that $Sf(x) = f(x_1)+f(x_2)+\ldots+f(x_n)$. b) If f is subadditive and continuous on [0,a] and $f(0) = 0$, then Sf is uniformly continuous on E.

<u>Proposition 6</u> [4]: If f is a subadditive function on the set $\{0, b, 2b, 3b, \ldots\}$, $b > 0$, and if F is the function whose graph is obtained by joining by straight line segments the points $(0,f(0))$, $(b,f(b))$, $(2b,f(2b))$, ... in that order, then F is subadditive on E.

Proposition 6 provides a convenient way of constructing examples of subadditive functions on E. The behavior of such examples -- obtained by applying the definition of Sf to f defined on the finite set $\{0, b, \ldots, nb\}$, where the infimum always exists -- is further amplified by Theorem 6 and Corollary 6a, which appear later in this chapter. It should be noted, however, that, if f is defined on $\{0, b, 2b, \ldots, nb\}$ and if Pf denotes the "polygonal extension" of f (in the sense of Proposition 6) to [0,nb], then $P(Sf(x)) \neq S(Pf(x))$ in general. For example, if f is defined on $\{0,1,2,3\}$ by $f(0) = f(3) = 0$, $f(1) = 4$, and $f(2) = 1$, then $Sf(4) = 2$ and $P(Sf)(3\frac{1}{2}) = 1$; but if one considers the half-integers also, then it follows that $S(Pf)(3\frac{1}{2}) = 2$.

The next few theorems exhibit some of the properties of the maximal subadditive extension.

Theorem 3: Let f be subadditive on [0,a]. Then f is non-decreasing on [0,a] if, and only if, Sf is non-decreasing on E.

Proof: Since f is the restriction of Sf to [0,a] (written, "$f = Sf|[0,a]$"), the monotonicity of f follows from that of Sf. Conversely, if Sf decreases, then there exist $x,y \in E$ such that $y > x$ and $Sf(y) < Sf(x)$.(*) Also, take $y-x < \frac{a}{2}$. (This can be done since the interval [x,y] can be decomposed into subintervals of length less than $\frac{a}{2}$ by a partition $x = x_0 < x_1 < \ldots < x_p = y$. Then $Sf(x_{i-1}) \leq Sf(x_i)$ for all $i = 1,2,\ldots,p$ implies that $Sf(x_0) \leq Sf(x_p)$ -- a contradiction.)

If $y \in [0,a]$, then f decreases on [0,a] and the contraposition argument is complete. In the case that $y > a$, let $\varepsilon > 0$ be given. Then there exists a refined a-partition $\{y_1,\ldots,y_n\}$ of y such that $y_1 > \frac{a}{2}$ and $Sf(y)+\varepsilon > f(y_1)+\ldots+f(y_n)$. (See pp. 15,16.) Let $z = y_1-(y-x)$. Then $\{z,y_2,y_3,\ldots,y_n\}$ is an a-partition of x, so that $Sf(x) \leq f(z)+f(y_2)+f(y_3)+\ldots+f(y_n)$. Then subtraction of this result from the preceding inequality yields $Sf(y)-Sf(x)+\varepsilon > f(y_1)-f(z)$. Since ε is arbitrary, it follows that $f(y_1)-f(z) \leq Sf(y)-Sf(x) < 0$ (*), which means that in this case, too, f decreases on [0,a].

Corollary 3a: Let f be subadditive on [0,a]. Then f is strictly increasing on [0,a] if, and only if, Sf is strictly increasing on E.

Proof: Repeat the proof of Theorem 3, replacing "<" by "≤" in the inequalities (*).

Theorem 4: If f is subadditive on $[0,a]$, if $0 < c < a$, and if $g = f|[0,c]$,[1] then $Sg(x) \geq Sf(x)$ for all $x \epsilon E$. Also, $Sg = Sf$ if, and only if, $Sg|[0,a] = f$.

Proof: If $x \epsilon E$ and C is the collection of all c-partitions of x and A the collection of all a-partitions of x, then $C \subset A$ and $Sg(x) = \inf\{\Sigma f(x_i) : \{x_i\} \epsilon C\}$ $\geq \inf\{\Sigma f(x_i) : \{x_i\} \epsilon A\} = Sf(x)$.

To prove the second part, if there is an $x \epsilon (c, a]$ such that $Sg|[0, a](x) > f(x)$ ("<" has just been ruled out), then, since $Sf = f$ on $(c, a]$, $Sg(x) > Sf(x)$. Thus $Sg = Sf$ implies $Sg|[0,a] = f$. Conversely, if $Sg|[0,a] = f$, let $\varepsilon > 0$ be given. Then there is an a-partition $\{x_1, \ldots, x_n\}$ for $x \epsilon E$ such that $Sf(x) > f(x_1)+\ldots+f(x_n)-\frac{\varepsilon}{2}$. For each x_i there exists a c-partition $\{y_i^1, y_i^2, \ldots, y_i^{M(i)}\}$ such that

$$Sg(x_i) > \sum_{r=1}^{M(i)} g(y_i^r)-\varepsilon/2n.$$

Thus $$Sf(x) > \sum_{i=1}^{n} f(x_i)-\frac{\varepsilon}{2} = \sum_{i=1}^{n} Sg(x_i)-\frac{\varepsilon}{2} > \sum_{i=1}^{n}\left(\sum_{r=1}^{M(i)} g(y_i^r)-\varepsilon/2n\right)-\frac{\varepsilon}{2}$$

$$= \sum_{i=1}^{n}\left(\sum_{r=1}^{M(i)} g(y_i^r)\right)-\varepsilon.$$

Since $\sum_{i=1}^{n}\left(\sum_{r=1}^{M(i)} y_i^r\right) = x$, it follows that $\sum_{i=1}^{n}\left(\sum_{r=1}^{M(i)} g(y_i^r)\right) \geq Sg(x)$.

---

[1] If f is a function defined on a set B and if $D \subset B$, then $f|D$ denotes the function g defined on D by $g(x) = f(x)$ for all $x \epsilon D$.

Therefore, $Sf(x) > Sg(x) - \varepsilon$. Since ε is arbitrary, $Sf(x) \geq Sg(x)$. Since $Sg(x) \geq Sf(x)$ by the first part of this theorem, $Sf(x) = Sg(x)$.

It is possible to obtain sharp bounds on the function Sf by proving the general case of a theorem for which Bruckner invokes a hypothesis of continuity [4, pp. 1159-60].

<u>Theorem 5</u>: If f is a bounded subadditive function on $(0,a]$, then the graph of Sf is bounded between parallel lines on $(0,\infty)$. More precisely, if $m = \inf\{f(x)/x : x \in (0,a]\}$ and $b = \sup\{f(x) - mx : x \in (0,a]\}$, then $mx \leq Sf(x) \leq mx + b$ for all $x \in (0,\infty)$.

<u>Proof:</u> Note than, if $x \in (0,\frac{a}{2}]$, then $f(2x) \leq 2f(x)$ and $f(2x)/2x \leq f(x)/x$, so that only those values of x in $(\frac{a}{2},a]$ need to be considered in finding a lower bound of $f(x)/x$. By hypothesis, $|f(x)| \leq M$ on $(0,a]$ for some positive $M \in R$. Thus $|f(x)/x| \leq 2M/a$ for all $x \in (\frac{a}{2},a]$. Therefore $\{f(x)/x : x \in (0,a]\}$ is bounded below, which means that there exists a real number $m = \inf\{f(x)/x : x \in (0,a]\}$. Similarly, since $\{f(x) - mx : x \in (0,a]\}$ is bounded above by $M + |ma|$, let $b = \sup\{f(x) - mx : x \in (0,a]\}$.

Let $\varepsilon > 0$ be given and consider $y \in (a,\infty)$. Let $\{x_1, \ldots, x_n\}$ be a refined a-partition for y such that $Sf(y) + \varepsilon \geq f(x_1) + \ldots + f(x_n)$. Since $x_i \neq 0$ and $m \leq f(x_i)/x_i$ $(i = 1, 2, \ldots, n)$,

$$m \leq \left(\sum_{i=1}^{n} f(x_i)\right) / \sum_{i=1}^{n} x_i \leq (Sf(y) + \varepsilon)/y,$$

or $my \leq Sf(y)+\varepsilon$. Since ε is arbitrary, $my \leq Sf(y)$.

There exists a unique integer p such that $y = \frac{ap}{2}+z$, where $0 \leq z < \frac{a}{2}$. Let $t \in (\frac{a}{2},a]$ such that $f(t)/t < m+\varepsilon/ap$. Then the integer r is uniquely determined such that $y = rt+z'$, where $0 \leq z' < t$, and $r \leq p$ since $t > \frac{a}{2}$. It follows that $Sf(y) \leq rf(t)+f(z')$. Since $f(t) < tm+t\varepsilon/ap$ and $f(z') \leq mz'+b$ (by definition of b), $Sf(y) < r(tm+t\varepsilon/ap)+mz'+b = m(rt+z')+b+(t/a)(r/p)\varepsilon \leq my+b+\varepsilon$. Thus $Sf(y) \leq my+b$ since ε is arbitrary.

Corollary 5a: Every subadditive function f on E which is bounded on (0,a) and negative at a is such that there exists $M > 0$ for which $x > M$ implies $f(x) < 0$.

Proof: Let $g = f|[0,a]$. Then, for m and b defined in Theorem 5, $m \leq f(a)/a < 0$. Thus for x beyond the point at which the line $y = mx+b$ crosses the x-axis (namely, for $x > -b/m$), $f(x) \leq Sg(x) \leq mx+b < 0$.

A slightly more general form of this corollary is true. It is true if "finite-valued" is substituted for "bounded," but a finite-valued subadditive function on E can be unbounded on a bounded interval only in a neighborhood of the origin. [2, p. 241 and p.243]. An example is given by $f(0) = 0$ and $f(x) = 1/x$ if $x > 0$.

In constructing examples of subadditive functions and maximal subadditive extensions of them using the technique of Proposition 6 (p. 17), one encounters situations of which the following examples are representative.

1) Let f be defined on the set $J_4 = \{0,1,2,3,4\}$ by $f(0) = f(4) = 0$, $f(1) = f(3) = 2$, and $f(2) = 1$. Then the next few values of $Sf(n)$, found by taking $\min\{Sf(1)+Sf(n-1), Sf(2)+Sf(n-2),\ldots,Sf([\frac{n}{2}])+Sf(n-[\frac{n}{2}])\}$,[2] are

n:	1	2	3	4	5	6	7	8	9	10	11	12
Sf(n):	2	1	2	0	2	1	2	0	2	1	2	0

It appears that Sf is periodic with period 4 -- a conjecture which will be verified in the theorem to follow. (See Figure 3.)

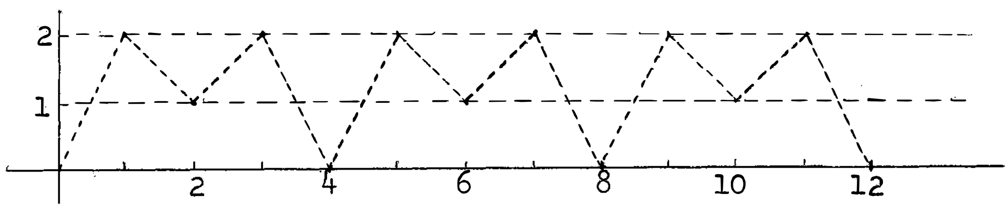


Figure 3. A Repeating Maximal Extension

2) If g is defined on J_4 by $g(0) = g(4) = 0$, $g(1) = 4$, $g(2) = 3$, and $g(3) = 1$, then the first several values of $Sg(n)$ are

n:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Sg(n):	4	3	1	0	4	2	1	0	3	2	1	0	3	2	1	0

It will be shown to be the case that Sg is periodic with period 4 on the set $\{8,9,10,11,\ldots\}$ -- a fact which is

[2]The symbol [x] denotes the unique integer such that $x-1 < [x] \le x$.

again more plausible after additional computation. (See Figure 4.)

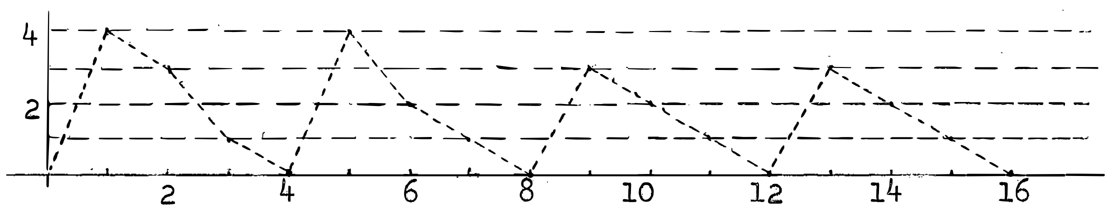


Figure 4. An Eventually Repeating Maximal Extension

The pertinent theorem will be stated and proved for the set $E = [0,\infty)$, but it is just as valid for, and more easily applied to, the set $J = \{0,1,2,\ldots\}$. The theorem is more general than the examples indicate since it is not necessary that f take on the value 0 at the right end-point of its interval of definition.

Theorem 6: Let f be a subadditive function on $[0,a]$ such that, for some $n \in J$, $Sf((n+1)a+x) = f(a)+Sf(na+x)$ for all $x \in (0,a]$. Then $Sf(ma+x) = (m-n)f(a)+Sf(na+x)$ for all $m \geq n$ and all $x \in (0,a]$. (Essentially, if $Sf|((n+1)a,(n+2)a]$ is a "copy" of $Sf|(na,(n+1)a]$, then Sf continues to copy itself *ad infinitum*.)

Proof: Let $y \in ((n+2)a,\infty)$; then y has a representation of the form $y = ma+x$, where $m \in J$ and $x \in (0,a]$. Let

$\{y_1, y_2, \ldots, y_r\}$ be an a-partition for y. Since each $y_i \le a$, there exists $k \in J$ such that $y_1+y_2+\ldots+y_k = (n+1)a+x_1$, $x_1 \in (0,a]$. Replace $y_1, y_2, \ldots, y_k$ by $na+x_1$ and a. By hypothesis, $f(a)+Sf(na+x_1) = Sf((n+1)a+x_1) \le f(y_1)+\ldots+f(y_k)$ (since $\{y_1, \ldots, y_k\}$ is an a-partition for $(n+1)a+x_1$).

There exists $k' > k$ such that $(na+x_1)+y_{k+1}+\ldots+y_{k'} = (n+1)a+x_2$, $x_2 \in (0,a]$. Replace $na+x_1, y_{k+1}, \ldots, y_{k'}$ by $na+x_2$ and a. By Theorem 4, $Sf = S(Sf|[0,(n+1)a])$, and both these partitions of $(n+1)a+x_2$ are $(n+1)a$-partitions, so that

$$Sf((n+1)a+x_2) = f(a)+Sf(na+x_2) \le Sf(na+x_1)+f(y_{k+1})+\ldots+f(y_{k'}).$$

Repeat this procedure until an $(n+1)a$-partition for y of the form $Y = \{na+x, a, a, \ldots, a\}$, with m−n a's, is obtained. This yields the inequality

$$\sum_{i=1}^{r} f(y_i) \ge f(a)+Sf(na+x_1)+\sum_{i=k+1}^{r} f(y_i)$$

$$\ge 2f(a)+Sf(na+x_2)+\sum_{i=k'+1}^{r} f(y_i) \ge \ldots$$

$$\ge (m-n)f(a)+Sf(na+x),$$

which holds for every a-partition of y.

Therefore, $(m-n)f(a)+Sf(na+x) \le Sf(y)$, but, since Y is an $(n+1)a$-partition for y, $(m-n)f(a)+Sf(na+x) \ge Sf(y)$.

<u>Corollary 6a</u>: Let f be a subadditive function on $[0,a]$ such that $Sf(a+x) = f(a)+f(x)$ for all $x \in (0,a]$. Then $Sf(ma+x) = mf(a)+f(x)$ for all $m \in J$ and all $x \in (0,a]$.

<u>Proof</u>: Take $n = 0$ in Theorem 6.

Bruckner has shown [4; 5, p. 2] that a function f which is concave on [0,a] with $f(0) \geq 0$ has a maximal subadditive extension Sf which behaves according to the rule of Corollary 6a. Such a function is given by $f(x) = |\sin x|$, which was shown earlier to be subadditive. This function will now be used to provide two examples.

1) The inequalities $mx \leq Sf(x) \leq mx+b$ of Theorem 5 are the best possible on every interval in E of length a. If $f(x) = |\sin x|$ and $a = \pi$, then $m = f(\pi)/\pi = 0$ and $b = f(\frac{\pi}{2}) = 1$. By Bruckner's result just mentioned, $Sf(x) = |\sin x|$ on E, and the bounds $0 \leq |\sin x| \leq 1$ are realized on every closed interval of length π.

2) The infimum of a family of subadditive functions is not necessarily subadditive. For example, if $f(x) = |\sin x|$, $g(x) = 2x/3\pi$, and $h(x) = \inf\{f(x), g(x)\}$, then $h(3\pi/2) = 1 > h(\frac{\pi}{2}) + h(\pi) = 1/3 + 0$.

The concluding portion of this chapter is devoted to a discussion of the additivity of the operator S. If f and g are functions defined on the same set D, then write $f \geq g$ if, and only if, $f(x) \geq g(x)$ for all $x \in D$. This is a partial ordering of the set of all functions on D. In the usual terminology, Proposition 3, parts b and c (p. 16), state that S is a monotone, positive-homogeneous operator on the set of all subadditive functions defined on [0,a] into the set of all subadditive functions on E. The following theorem states that S is also superadditive.

Theorem 7: If f and g are subadditive on [0,a], then $S(f+g) \geq Sf+Sg$.

Proof: Using the definition of S (p. 15), $S(f+g)(x) = \inf\Sigma(f(x_i)+g(x_i)) \geq \inf\Sigma f(x_i)+\inf\Sigma g(x_i) = Sf(x)+Sg(x)$. Equality may not hold. As an example in a finite case, let $f(0) = g(0) = 0$, $f(1) = g(1) = 2$, $f(2) = g(2) = f(3) = g(3) = g(4) = 3$, and $f(4) = 5$ on $J_4 = \{0,1,2,3,4\}$. Then $Sf(5) = 6$, $Sg(5) = 5$, and $S(f+g)(5) = 12$.

A couple of conditions under which equality holds can be mentioned. First, if $f = cg$, c a non-negative constant, then Proposition 3c implies that $S(cg+g) = (c+1)Sg = c(Sg)+Sg = S(cg)+Sg$. Second, if f and g are both concave on [0,a], then f+g is concave and $S(f+g)(na+x) = n(f+g)(a)+(f+g)(x) = nf(a)+f(x)+ng(a)+g(x) = Sf(na+x)+Sg(na+x)$ for all $x\epsilon(0,a]$ -- by Corollary 6a and Bruckner's result noted there. The set of all functions having the properties of f in that corollary is also closed under addition, so that these functions provide a generalization of the notion of concave functions non-negative at 0.

Theorem 8: Let f and g be subadditive on [0,a] with the property that $Sf(a+x) = f(a)+f(x)$ and $Sg(a+x) = g(a)+g(x)$ for all $x\epsilon(0,a]$. If $h = f+g$, then $Sh(a+x) = h(a)+h(x)$ for all $x\epsilon(0,a]$.

Proof: Assume that there exists an a-partition $\{x_1,x_2,\ldots,x_n\}$ for a+x such that $h(x_1)+\ldots+h(x_n)<h(a)+h(x)$.

Then $h(a)+h(x) > h(x_1)+\ldots+h(x_n) \geq Sh(a+x)$
$= S(f+g)(a+x) \geq Sf(a+x)+Sg(a+x) = f(a)+f(x)+g(a)+g(x)$
$= (f+g)(a)+(f+g)(x) = h(a)+h(x)$ -- a contradiction.

Thus $h(a)+h(x) \leq h(x_1)+\ldots+h(x_n)$ for every a-partition of a+x; hence $Sh(a+x) = h(a)+h(x)$.

Corollary 8a: The functions f and g of Theorem 8 satisfy $S(f+g) = Sf+Sg$.

Proof: By Theorem 8, $Sh(a+x) = h(a)+h(x)$. By Corollary 6a, $Sh(ma+x) = mh(a)+h(x) = m(f+g)(a)+(f+g)(x)$ $= mf(a)+f(x)+mg(a)+g(x) = Sf(ma+x)+Sg(ma+x)$, for all $m \in J$ and all $x \in (0,a]$. Since every $y \in E$ has the form $ma+x$, $Sh(y) = Sf(y)+Sg(y)$ at every $y \in E$.

CHAPTER IV

OTHER EXTENSIONS

This chapter deals with extensions of subadditive functions from E to R, from J to E, and from [0,a] to E. The first consideration is that of the way in which the behavior of a subadditive function f for positive x affects its behavior for negative x -- a consideration which sheds some light on the existence of an extension of a function g on E to R as an even function. Hille and Phillips [2, pp. 244-5] show that a finite-valued subadditive function defined on $(0,\infty)$ has no finite subadditive extension to R if either $f(x) \to \infty$ as $x \to 0$ or $f(x)/x \to -\infty$ as $x \to \infty$. This supplies some idea of what not to expect of subadditive even functions. (An even function f is one which satisfies the relation $f(-x) = f(x)$ for all $x \in R$. If $f(-x) = -f(x)$, then f is called an odd function.)

<u>Proposition 7</u> [6]: Every even subadditive function is nowhere negative. Every measurable odd subadditive function is of the form $f(x) = mx$, m a constant.

<u>Remark 5</u>: Every function f on E such that $\sup\{f(x) : x \in E\} \leq 2(\inf\{f(x) : x \in E\})$ can be extended to R as an even subadditive function. If f is non-increasing

on E, this condition is also necessary.

Proof: That the function f is subadditive is a consequence of Proposition 2 (p. 5). The supremum and infimum of the extension on R will be the same as those on E, so the extension is subadditive by the same proposition.

If f is non-increasing and F is its even extension to R, then $\sup\{f(x) : x \in E\} = f(0)$. If there is a point $y \in E$ such that $2f(y) < f(0)$, then $F(0) = F(y-y) > F(y)+F(-y) = 2f(y)$, and F is not subadditive on R.

Theorem 9: If f is non-decreasing and subadditive on $[0,\infty)$ and non-increasing and subadditive on $(-\infty,0)$, then f is subadditive on R.

Proof: If $xy > 0$, then $f(x+y) \le f(x)+f(y)$ by hypothesis. If $x > 0$, $y < 0$, and $x+y \ge 0$, then $f(x+y) \le f(x) \le f(x)+f(y)$ since $f(y) \ge 0$; if $x+y < 0$, then $f(x+y) \le f(y) \le f(x)+f(y)$. If $x = 0$, then $f(x+y) = f(y) \le f(x)+f(y)$.

Corollary 9a: Every non-decreasing subadditive function defined on E can be extended to R as an even subadditive function.

Corollary 9b: Every non-decreasing subadditive function f on E can be extended to R by $f(x) = 0$ for all $x < 0$, or by $f(x) = f(0)$ for all $x < 0$.

Theorem 10: Let f be subadditive on E. Then f can be extended to a subadditive even function F on R if, and only if, $f(x-y) \le f(x)+f(y)$ for all $x \ge y$ in E.

Proof: If f is subadditive on E, and if F is the even function on R which is an extension of f, then $F(x+y) \le F(x)+F(y)$ whenever $xy \ge 0$. Thus assume $f(x-y) \le f(x)+f(y)$ and let $u > 0$ and $v < 0$ be given. If $u \ge |v|$, then $F(u+v) = F(u-|v|) = f(u-|v|) \le f(u)+f(|v|)$ $= F(u)+F(v)$. If $u < |v|$, then $F(u+v) = F(u-|v|) =$ $f(|v|-u) \le f(|v|)+f(u) = F(v)+F(u)$.

Conversely, if there exist $u,v \in E$ such that $u \ge v$ and $f(u-v) > f(u)+f(v)$, then $F(u-v) = f(u-v) > f(u)+f(v)$ $= F(u)+F(-v)$, and the subadditivity inequality fails for the even function F at the pair $u,-v \in R$.

The following lemmas will be useful in Chapters VII and V, respectively. In each lemma a non-decreasing subadditive function on $[0,a]$ is extended to a larger set as a subadditive function.

Lemma 3: Let f be a non-decreasing subadditive function on $[0,a]$ (or on $J_k = \{0,1,2,\ldots,k\}$). Extend f by $F(x) = f(a)$, $x > a$ (respectively, by $F(n) = f(k)$, $n > k$). If $F = f$ on the original domain, then F is subadditive on E (on J).

Proof (for the case of $[0,a]$): The function F is non-decreasing since $x < a \le y$ implies $F(x) = f(x) \le f(a) = F(y)$. Let $u+v = x > a$, $u,v \in E$. If $u,v > a$, then $F(u)+F(v) = 2F(x)$. If $u \le a$, then $v > a-u$ and $F(u)+F(v) \ge F(u)+F(a-u) \ge F(a)$ $= F(x)$.

Lemma 4: Let f be non-decreasing and subadditive on $[0,a]$, $a>0$. Let g be defined by $g(x)=f(x)$ if $x\epsilon[0,a]$, $g(x)=f(a)$ if $x\epsilon(a,2a]$, and $g(x)=f(a)+f(x-2a)$ if $x\epsilon(2a,3a]$. Then g is subadditive on $[0,3a]$.

Proof: If $x,y,x+y\epsilon[0,2a]$, then $g(x+y)\leq g(x)+g(y)$ by Lemma 3. For $x+y\epsilon(2a,3a]$ consider the various cases.

1) If $x,y\epsilon(a,2a]$, then $g(x+y)\leq g(3a)=2f(a)$ $=g(x)+g(y)$.

2) If $x\epsilon[0,a]$ and $y\epsilon(a,2a]$, then $y-2a\leq 0$ and $g(x+y)=f(a)+f(x+y-2a)\leq f(a)+f(x)=g(y)+g(x)$.

3) If $x\epsilon[0,a]$ and $y\epsilon(2a,3a]$, then $g(x+y)$ $=f(a)+f(x+y-2a)\leq f(a)+f(x)+f(y-2a)\leq g(x)+g(y)$.

(Since g is non-decreasing on $[0,3a]$, this construction may be repeated as often as necessary.)

Attention turns now to some theorems involving interpolation of subadditive functions by concave functions. More precisely, if f is a subadditive function defined on the set J of all non-negative integers (see Remark 3), how can f be extended to $E=[0,\infty)$ by interpolation of values while retaining the subadditivity property? A simple answer follows, but it is only a special case of the general theorem.

Theorem 11: If f is a non-decreasing subadditive function on J and F is defined on E by $F(0)=f(0)$ and $F(x)=f(n)$ for all $x\epsilon(n-1,n]$ $(n=1,2,3,\ldots)$, then F is subadditive and non-decreasing on E.

Proof: For each $x \in E$ let $n(x)$ be the unique integer such that $n(x)-1 < x \le n(x)$. The function F is non-decreasing since $x < y$ implies $n(x) \le n(y)$ and $F(x) = f(n(x)) \le f(n(y)) = F(y)$. If $x, y \in E$, then $F(x)+F(y) = f(n(x))+f(n(y)) \ge f(n(x)+n(y)) = F(n(x)+n(y)) \ge F(x+y)$ since $n(x)+n(y) \ge x+y$. Thus F is subadditive on E.

This extension of f as a left-continuous step function and the extension of f as a polygonal function (in the sense of Proposition 6) are both subadditive whenever f is non-decreasing and subadditive. The interpolating functions on $[0,1]$ -- $g(0) = 0$, $g(x) = f(1)$ if $0 < x \le 1$ in the first case, and $g(x) = f(1)x$ in the second case -- are both monotone subadditive functions, but subadditivity is not, in general, a strong enough hypothesis. A general theorem is true if g is a monotone concave function with $g(1) = f(1)$. The g-functions of these two cases then are the extremes of this class of interpolating functions.

Theorem 12: Let f be a non-decreasing subadditive function on J. Let g be a non-decreasing concave function on $[0,1]$ with $g(0) = 0$ and $g(1) = 1$. The function F defined on E by

$$F(x) = f([x])+\{f([x+1])-f([x])\}g(x-[x]),$$

where $[x]$ is the integer $x-1 < [x] \le x$, is subadditive and non-decreasing on E.

Proof: By the definition of F and monotonicity of g, F is non-decreasing on any interval $[n,n+1]$, $n \in J$. Since the intervals $[n,n+1]$ and $[n+1,n+2]$ have a point in common for each n and since f is non-decreasing, F is non-decreasing on E. To show subadditivity, let $x = m+u$ and $y = n+v$, where $m,n \in J$ and $u,v \in [0,1)$.

1) In the case $u+v = h \leq 1$ and $g(u)+g(v) \leq 1$, note that g is subadditive on [0,1] by Proposition 1. Thus $g(h) \leq g(u)+g(v)$. The calculation proceeds as follows:

$$\begin{aligned} F(x+y) &= f(m+n)+\{f(m+n+1)-f(m+n)\}g(h) \\ &\leq f(m+n)+\{f(m+n+1)-f(m+n)\}\{g(u)+g(v)\} \\ &= f(m+n)\{1-g(u)-g(v)\}+f(m+n+1)\{g(u)+g(v)\} \\ &\leq \{f(m)+f(n)\}\{1-g(u)-g(v)\}+\{f(m+1)+f(n)\}g(u) \\ &\quad +\{f(m)+f(n+1)\}g(v) \\ &= f(m)+\{f(m+1)-f(m)\}g(u)+f(n)+\{f(n+1)-f(n)\}g(v) \\ &= F(x)+F(y). \end{aligned}$$

2) If $u+v = h \leq 1$ but $g(u)+g(v) > 1$, assume that the notation has been selected so that $f(n+1)-f(n) \leq f(m+1)-f(m)$. Thus y denotes the point in the unit interval of smaller increase in f. Since $g(u)+g(v) > 1$,

$$\{f(n+1)-f(n)\}\{g(u)+g(v)\}+f(n) \geq \{f(n+1)-f(n)\}+f(n) = f(n+1),$$

so that
$$\begin{aligned} F(x+y) &\leq f(m+n+1) \leq f(m)+f(n+1) \leq f(m) \\ &\quad +\{f(n+1)-f(n)\}\{g(u)+g(v)\}+f(n) \\ &\leq f(m)+\{f(n+1)-f(n)\}g(u)+\{f(n+1)-f(n)\}g(v)+f(n) \\ &\leq f(m)+\{f(m+1)-f(m)\}g(u)+f(n)+\{f(n+1)-f(n)\}g(v) \\ &= F(x)+F(y). \end{aligned}$$

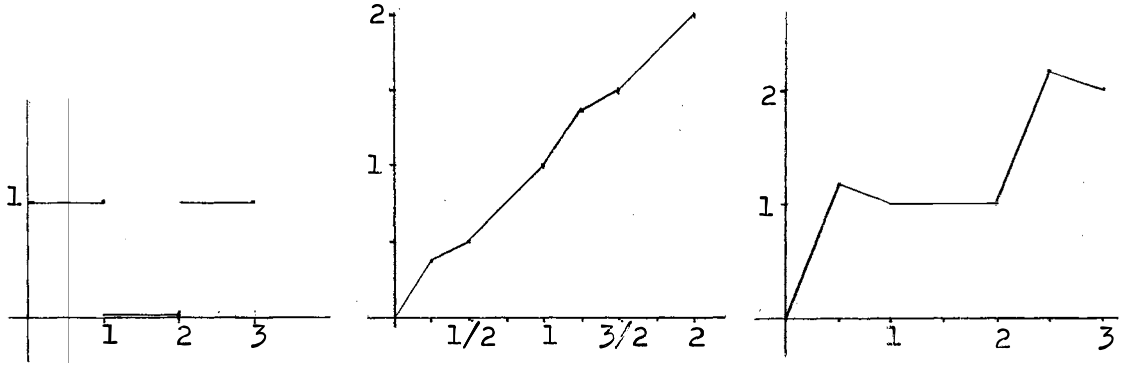
3) If $u+v = 1+h > 1$, the proof requires more information about the function g. Define $G(x) = 1-g(1-x)$. (In effect, turn the graph of g upside-down and let (1,1) be

the origin.) $G(0) = 0$ and G is convex on $[0,1]$. Thus $G(tx+(1-t)y) \leq tG(x)+(1-t)G(y)$ for all $t,x,y \in [0,1]$. Let $y = 0$, $x = a+b$, $t = a/(a+b)$; then $G(a) \leq aG(a+b)/(a+b)$. Let $y = 0$, $x = a+b$, $t = b/(a+b)$; then $G(b) \leq bG(a+b)/(a+b)$. Adding these two results yields $G(a)+G(b) \leq G(a+b)$. Letting $a = 1-u$, $b = 1-v$ -- so that $a+b = 2-(u+v) = 2-(1+h) = 1-h$ -- the result is $G(1-u)+G(1-v) \leq G(1-h)$, which translates to g as $1-g(u)+1-g(v) \leq 1-g(h)$ or as $g(h) \leq g(u)+g(v)-1$. Then:

$$\begin{aligned}
F(x+y) &= f(m+n+1)+\{f(m+n+2)-f(m+n+1)\}g(h) \\
&\leq f(m+n+1)+\{f(m+n+2)-f(m+n+1)\}\{g(u)+g(v)-1\} \\
&\leq f(m+n+1)\{1-g(u)\}+f(m+n+1)\{1-g(v)\} \\
&\quad +f(m+n+2)\{g(u)+g(v)-1\} \\
&\leq \{f(m)+f(n+1)\}\{1-g(u)\}+\{f(m+1)+f(n)\}\{1-g(v)\} \\
&\quad +\{f(m+1)+f(n+1)\}\{g(u)+g(v)-1\} \\
&= f(m)+\{f(m+1)-f(m)\}g(u)+f(n)+\{f(n+1)-f(n)\}g(v) \\
&= F(x)+F(y).
\end{aligned}$$

Theorems 11 and 12 may fail if f is not monotone. For example, the extension in Theorem 11 when applied to $f(n) = 1$ if n is odd and $f(n) = 0$ if n is even gives a function F which is not subadditive inasmuch as $F(3) = 1 > 2F(1.5) = 0$. (Figure 5)

Theorem 12 may fail if g is subadditive but not concave on $[0,1]$. As an example let g be defined polygonally by $g(0) = 0$, $g(1/4) = 3/8$, $g(\frac{1}{2}) = \frac{1}{2}$, and $g(1) = 1$. Then g is subadditive by Proposition 1. If $f(n) = n$ for all $n \in J$, then $F(5/4) = 11/8 > F(3/4)+F(1/2) = 3/4+1/2 = 5/4$. (Figure 6)



Figures 5, 6, 7. The Failure of Weakened Forms of Theorem 12

Theorem 12 may fail if g is not monotone. For example, let $f(0) = 0$, $f(1) = f(2) = 1$, and $f(3) = 2$; let g be defined polygonally by $g(0) = 0$, $g(\frac{1}{2}) = 1+\varepsilon$ $(\varepsilon > 0)$, and $g(1) = 1$. Then $F(5/2) = 2+\varepsilon > F(1)+F(3/2) = 2$. (Figure 7)

Finally, Theorem 12 <u>must</u> fail if f is not subadditive since $F(n) = f(n)$ if $n \in J$. A proof similar to that of the previous theorem, but a much shorter proof, is given for the following theorem, which was suggested by the subadditivity of $|\sin x|$, and which generalizes that subadditivity.

<u>Theorem 13:</u> Let g be concave and non-negative on $[0,1)$. The extension F of g as a periodic function to $E = [0,\infty)$, defined by $F(x) = g(x-[x])$, is subadditive on E.

<u>Proof:</u> Let $x = m+u$ and $y = n+v$, where $m,n \in J$ and $u,v \in [0,1)$.

1) If $u+v < 1$, the subadditivity of g on $[0,1)$ yields

the inequality $F(x+y) = F(u+v) = g(u+v) \leq g(u)+g(v) = F(x)+F(y)$.

2) If $u+v = 1+h \geq 1$, then $h < u$ and $h < v$. Since g is concave and non-negative, let $g(1) = 0$ and g will be concave on $[0,1]$. Now $g(u) = g(\frac{1-u}{1-h}h+\frac{u-h}{1-h}1) \geq \frac{1-u}{1-h}g(h)+\frac{u-h}{1-h}g(1)$; that is, $g(u)/(1-u) \geq g(h)/(1-h)$. Similarly, $g(v)/(1-v) \geq g(h)/(1-h)$. Therefore,

$$\frac{g(h)}{1-h} \leq \frac{g(u)+g(v)}{1-u+1-v} = \frac{g(u)+g(v)}{2-(1+h)} = \frac{g(u)+g(v)}{1-h}.$$

Multiplication by $1-h$ gives $g(h) \leq g(u)+g(v)$, so $F(x+y) = g(h) \leq g(u)+g(v) = F(x)+F(y)$.

It is essential that g be non-negative because Theorem 5 and Corollary 5a prevent any bounded subadditive function which takes on negative values from being periodic. However, the condition that g be concave is not a necessary condition. The polygonal extension G of the function Sf of Figure 3 (p. 22) is not concave, but Proposition 6 guarantees that G is subadditive.

These last two theorems have served to emphasize the close relationship between concavity and subadditivity hinted at in earlier results such as Proposition 1.

CHAPTER V

CONVERGENCE

This chapter is devoted to the study of sequences of subadditive functions. This study was motivated by consideration of a theorem of Bruckner on the convergence of extensions of functions. This motivation is the failure of the analogue of his theorem when subadditive functions are used. The basic result pertaining to sequences of subadditive functions will be stated first, and then this failure will be discussed.

<u>Proposition 8</u> [2, p. 238]: If $\{f_n\}$ is a pointwise convergent sequence of subadditive functions, then the function $f: x \to \lim f_n(x)$ is subadditive.

Bruckner's theorem, mentioned above, is the following [4, p. 1157]:

> Let $\{f_n\}$ be a sequence of continuous non-negative superadditive functions converging to the continuous function f on [0,a]. Let F_n denote the minimal superadditive extension of f_n. Then f is a continuous non-negative superadditive function, and the minimal superadditive extension of f is $\lim F_n$.

The proof of this theorem makes use of the uniform convergence of $\{f_n\}$ to f implied by the monotonicity of each f_i.

No such monotonicity is available in the subadditive case, and the statement obtained from Bruckner's theorem by replacing "superadditive" by "subadditive" and "minimal" by "maximal" is false. (It is, of course, still true if "non-negative" is also replaced by "non-positive.") To verify the failing case with an example, let $f(x) = x$ on $[0,1]$; then $Sf(x) = x$ on $E = [0,\infty)$. Let f_n $(n = 1,2,3,\ldots)$ be defined polygonally on $[0,1]$ by $f_n(1/2^n) = f_n(1-1/2^n) = 1/2$ and by $f_n(z) = z$ if $z = k/2^n$ $(k = 0,2,3,4,\ldots,2^n-2,2^n)$. An application of Theorem 5 -- with $m = 2^{n-1}/(2^n-1)$ and $b = \frac{1}{2}(1-1/(2^n-1))$ -- to f_n yields the information that the sequence $\{Sf_n\}$ is approaching boundedness above by the line $y = \frac{1}{2}x+\frac{1}{2}$. Specifically, for each $x \in E$ and $\varepsilon > 0$, there exists $N \in J$ such that $n > N$ implies $Sf_n(x) < \frac{1}{2}x+\frac{1}{2}+\varepsilon$. Each f_n is a continuous non-negative subadditive function on $[0,1]$ by Proposition 6, and $f_n \to f$ (non-uniformly) on $[0,1]$, but $\{Sf_n\}$ is not approaching Sf. (See Figure 8.)

Solid line is graph of $y = f_3(x)$

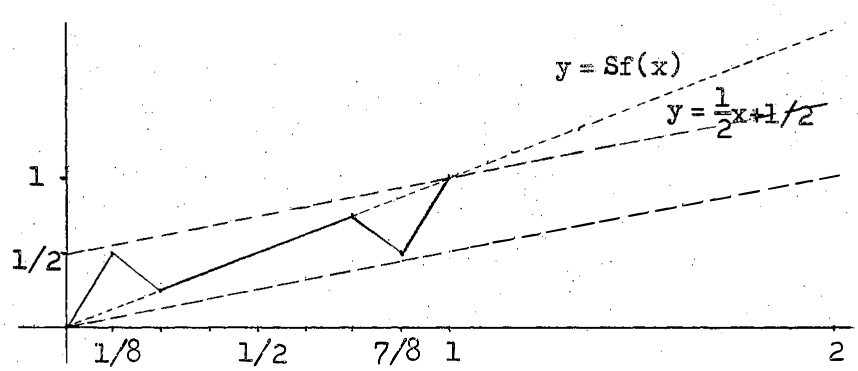


Figure 8. Non-Uniform Convergence of Maximal Extensions

The appropriate positive result, to which Bruckner's proof applies, is the following one.

<u>Proposition 9</u>: Let $\{f_n\}$ be a sequence of continuous subadditive functions converging uniformly to the function f on [0,a]. Then $\lim Sf_n = Sf$. (The function f is, of course, continuous and subadditive.)

The hypotheses of continuity and uniform convergence may be traded for a different kind of restriction in the following manner.

<u>Theorem 14</u>: If $\{f_n\}$ is a sequence of subadditive functions on [0,a] converging to f there, and if $f_n \geq f$ for all n, then $Sf_n \to Sf$ on E.

<u>Proof</u>: Let $x \in E$ and $\varepsilon > 0$ be given, and let $\{x_1, x_2, \ldots, x_r\}$ be a refined a-partition of x such that $Sf(x) \geq f(x_1) + \ldots + f(x_r) - \frac{\varepsilon}{2}$. (The function f is subadditive by Proposition 8.) Note that r is bounded above by some integer $M = M(x)$. There exists $m_i \in J$ $(i = 1, 2, \ldots, r)$ such that $n > m_i$ implies $f_n(x_i) - f(x_i) < \varepsilon/2M$. If $n > \max \{m_i\}$, then $Sf_n(x) \leq f_n(x_1) + \ldots + f_n(x_r) < f(x_1) + \ldots + f(x_r) + r\varepsilon/2M \leq Sf(x) + \varepsilon/2 + \varepsilon/2$. But $f_n \geq f$ on [0,a] implies $Sf_n(x) \geq Sf(x)$ by Proposition 3b. Thus $|Sf_n(x) - Sf(x)| < \varepsilon$.

An attempt to implement Proposition 9 by inserting a condition implying uniform convergence might proceed in the direction of one of the usual conditions [7, p. 86] such as:

1) The functions f_i are continuous, $f_1 \le f_2 \le f_3 \le \dots$, and f is continuous, or

2) The functions f_i are non-decreasing and f is continuous.

However, a different kind of restriction, a bound on the rate of increase at the origin, is possible in the subadditive case.

Theorem 15: Let $\{f_n\}$ be a sequence of subadditive functions (not necessarily continuous) converging to the continuous function f on $[0,a]$ and such that there exists a real number $m > 0$ such that $f_n(x) \le mx$ for all n and all $x \in [0,a]$. Then the convergence $f_n \to f$ is uniform on $[0,a]$.

Proof: The limit function f is subadditive by Proposition 8. Also, since $f_n(x) \le mx$ and $f_n(x) \to f(x)$, $f(x) \le mx$ at each $x \in [0,a]$. By Lemma 1 (p. 12), if $x > y$ in $[0,a]$, then $f(x)-f(y) \le m(x-y)$ and $f_n(x)-f_n(y) \le m(x-y)$

Let $\varepsilon > 0$ be given. Since f is continuous on $[0,a]$, there exists $\rho > 0$ such that $|f(x)-f(y)| < \varepsilon/4$ whenever $|x-y| \le \rho$. Let $\delta = \min\{\varepsilon/4m, \rho\}$. Let $0 = x_0 < x_1 < \dots < x_r = a$ be a partition of $[0,a]$ with $x_i - x_{i-1} \le \delta$ $(i = 1,2,\dots,r)$. Since $f_n(x_i) \to f(x_i)$, there exists $N_i \in J$ such that $|f_n(x_i)-f(x_i)| < \varepsilon/4$ whenever $n > N_i$ $(i = 1,2,\dots,r)$. Let $x \in [0,a]$ with $x_{k-1} \le x \le x_k$, and let $n > \max\{N_i\}$. Then

$$f(x)-f_n(x) = (f(x)-f(x_{k-1}))+(f(x_{k-1})-f(x_k))+(f(x_k)-f_n(x_k))$$
$$+(f_n(x_k)-f_n(x)) < m(x-x_{k-1})+\varepsilon/4+\varepsilon/4+m(x_k-x)$$
$$< m\varepsilon/4m+\varepsilon/2+m\varepsilon/4m = \varepsilon, \text{ and}$$

$$f_n(x)-f(x) = (f_n(x)-f_n(x_{k-1}))+(f_n(x_{k-1})-f(x_{k-1}))$$
$$+(f(x_{k-1})-f(x_k))+(f(x_k)-f(x)) < m(x-x_{k-1})+\varepsilon/4+\varepsilon/4+m(x_k-x) < \varepsilon.$$

Since these inequalities hold independent of the choice of x, the convergence is uniform.

That this theorem may fail without the m-condition is a consequence of the example on page 38. The condition, however, is not necessary since $f_n(x) \equiv 1/n$ gives a uniformly convergent sequence of subadditive functions. Converses of theorems of the above types on convergence of subadditive functions are, in general, not true. For example, even under conditions of monotone convergence of continuous functions a sequence of functions, no one of which is subadditive, may have a subadditive limit function. To show this, let f be defined polygonally on $[0,2]$ by $f(0) = 0$ and $f(1) = f(2) = 1$. Let each f_n be defined polygonally by $f_n(0) = 0$, $f_n(1) = f_n(2) = f_n(3/2 \pm 1/2^{n+1}) = 1$, and $f_n(3/2) = 1+1/2^{n-1}$. Then $f_n(3/2)-f_n(3/2-1/2^{n+1}) = 2/2^{n+1}$, and f_n is not subadditive by Lemma 1. (See Figure 9.)

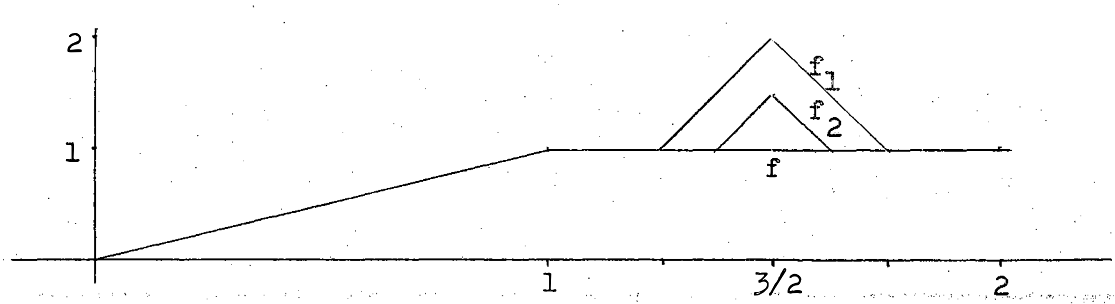


Figure 9. A Sequence of Non-Subadditive Functions

The following remarks concern a subadditive function with certain pathological properties of continuity and for which a couple of the earlier lemmas have been introduced. It is introduced in the theory of functions of a real variable [8, p. 193] as an example of a function which is uniformly continuous but not absolutely continuous. The function, which is called the Cantor function, is constructed as follows:

Let D be the Cantor "middle third" set constructed by deleting from the closed interval [0,1] the open intervals $I(1,1) = (1/3, 2/3)$; $I(2,1) = (1/9, 2/9)$ and $I(2,2) = (7/9, 8/9)$; $I(3,1) = (1/27, 2/27)$, $I(3,2) = (7/27, 8/27)$, $I(3,3) = (19/27, 20/27)$, and $I(3,4) = (25/27, 26/27)$; ... . Define $K(x) = (2k-1)/2^n$ if $x \in I(n,k)$ and define $K(x) = \lim_{t \to x} K(t)$ if $x \in D = c\bigcup_{n,k} I(n,k)$. The continuous non-decreasing function K thus defined on [0,1] is the Cantor function.

<u>Remark 6:</u> The Cantor function K is non-decreasing and subadditive on [0,1].

<u>Proof:</u> Define K_n $(n = 1, 2, 3, \ldots)$ on [0,1] to be the function obtained by joining polygonally consecutive endpoints of the graph of K restricted to all $I(r,k)$ $(r = 1, 2, \ldots, n)$ and the points (0,0) and (1,1). (See Figure 10.)

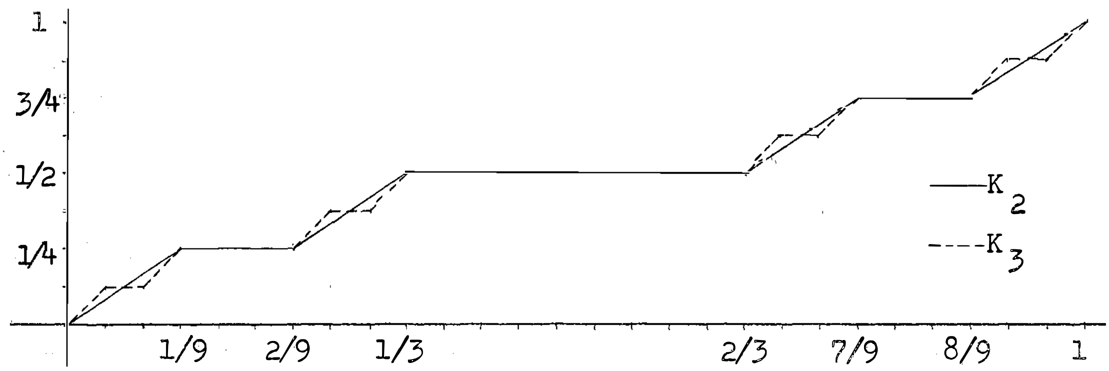


Figure 10. Approximations to the Cantor Function

Each function K_n is additive and non-decreasing on $[0,1/3^n]$ and satisfies the conditions of Lemma 4 when extended to $[0,3/3^n]$, so it is subadditive on $[0,3/3^n]$. Repeating the extension n times by Lemma 4 yields K_n, which is, therefore, subadditive on $[0,1]$ $(n = 1,2,\ldots)$. Since $K_n \to K$ uniformly on $[0,1]$, K is subadditive by Proposition 8.

Note that the function K, usually defined in the above manner, might alternatively be defined as $\lim K_n$ -- the uniform convergence then giving the continuity of K. It will be shown in Chapter VII that K has additional properties in relation to a convex cone of subadditive functions -- properties to which the remaining chapters are devoted.

CHAPTER VI

CONES OF SUBADDITIVE FUNCTIONS

The remaining material concerns certain subsets of a linear space over the field R of real numbers. Certain standard notational devices which will be used are defined below, and then the sets and elements of particular interest are defined.

Definition 5: Let A and B be subsets of a real linear (vector) space L, and let $t \in R$. Then $A+B = \{x+y : x \in A \text{ and } y \in B\}$, $-A = \{x : -x \in A\}$, $A-B = A+(-B)$, and $tA = \{tx : x \in A\}$.

Definition 6: A set C in a real linear space L is a cone if 1) C is convex, 2) $tC \subset C$ for all $t \geq 0$ in R, and 3) $C \cap (-C) = \{\theta\}$ where θ is the origin in L. Condition 1 can be replaced by 1') $C+C \subset C$.

Definition 7: Let C be a cone in L. An element $x \in C$ is called an extremal element of C if $x_1, x_2 \in C$ and $x_1 + x_2 = x$ imply that x_1 and x_2 are scalar multiples of x.

In an appropriate linear space (space of all functions defined on a given set, space of bounded functions on a set, etc.) certain sets of subadditive functions are cones.

Let D denote one of the sets $J = \{0,1,2,\ldots\}$, $J_k = \{0,1,2,\ldots,k\}$ $(k > 0)$, $[0,a]$, or $E = [0,\infty)$. Then $C(D)$ will denote the cone of all non-decreasing subadditive functions defined on D. Some problems of characterizing the extremal elements of cones of functions have been considered by Choquet [9] and McLachlan [10; 11]. A motivation for this study is provided by the following theorem of Choquet.

<u>Proposition 10</u> [9, p. 237]: If the vector space L is a locally convex Hausdorff space, and if A is a convex compact subset of L, then, for every $x_0 \in A$, there exists a (Radon) measure $u_0 \geq 0$ on the closure of $e(A)$, the set of extreme points of A, whose center of gravity is x_0.

This theorem applies to a cone C if there exists a hyperplane in L cutting the cone C in such a set A which has the added property that every ray of C intersects it in exactly one point, since the extremal elements of C are the non-negative scalar multiples of the extreme points of A. [12, p. 82].

McLachlan has observed [11] that the most difficult aspect of such problems is that of finding "non-proportional" decompositions of the non-extremal elements as the sum of elements which are not scalar multiples. He also noted that the relation $C_1 \subset C_2$ can be used only in the following way: If $C_1 \subset C_2$ and $x \in C_1$ is an extremal element of C_2, then x is an extremal element of C_1.

Proposition 11 [10]: Let C be a cone of functions on one of the sets D mentioned above, and let $f, f_1, f_2 \in C$ such that $f_1 + f_2 = f$. a) If C is a cone of non-negative functions, $x \in D$, and $f(x) = 0$, then $f_1(x) = f_2(x) = 0$. b) If C is a cone of non-decreasing functions, $x, y \in D$, and $f(x) = f(y)$, then $f_1(x) = f_1(y)$ and $f_2(x) = f_2(y)$. c) If C is a cone of subadditive functions, $x, y, x+y \in D$, and $f(x+y) = f(x) + f(y)$, then $f_i(x+y) = f_i(x) + f_i(y)$ $(i = 1, 2)$.

Corollary: If C is a cone of subadditive functions on D, then every function f which is additive on D is an extremal element of C.

The additive functions may be the only extremal elements of a cone, as in the case of the cone $C'(J_k)$ consisting of the zero function and all strictly increasing subadditive functions on J_k.

Theorem 16: A function $f \in C'(J_k)$ is an extremal element of $C'(J_k)$ if, and only if, f is additive on J_k.

Proof: An additive f is extremal by the corollary above. If $f \in C'(J_k)$, let $r = \min\{f(m) - f(m-1) : m = 1, 2, \ldots, k\} > 0$. Let $f_1(n) = \frac{rn}{2}$ and $f_2(n) = f(n) - f_1(n)$. Since f_1 is additive, f_2 is subadditive. Both f_1 and f_2 are strictly increasing with $f_i(m+1) - f_i(m) \geq \frac{r}{2}$ $(i = 1, 2;\ m = 0, 1, \ldots, k-1)$. If f is not additive, $f_1 + f_2$ is a non-proportional decomposition of f.

Two other examples of cones of subadditive functions of types already considered here can be mentioned, namely, the cone of all subadditive, non-negative, periodic functions of period p defined on E and the cone of all functions on [0,a] satisfying the condition of Theorem 8 (p. 26). This last example can be verified by appealing to Theorem 8 and Proposition 3c.

It is easily shown that, if C is a cone in L, then C-C is a subspace of L. The remainder of this chapter is devoted to a solution of the problems of identifying the extremal elements of C, showing the existence of an integral representation in the sense of Proposition 10, and determining C-C for the cone $C = C(J_k)$, the cone of all non-decreasing subadditive functions on $J_k = \{0,1,2,\ldots,k\}$. It will occasionally be convenient to think of $C(J_k)$ as a subset of Euclidean (k+1)-space, R^{k+1}.

In connection with this problem there is a conjecture due to Choquet that the extremal elements of C(J) are those functions $f \in C(J)$ for which $f(m+1)-f(m)$ is equal to 0 or to f(1) for every $m \in J$. That all such functions are extremal was proved by McLachlan [11]. However, there are other extremal elements and an example of one will be given once the extremal elements of $C(J_k)$ have been found.

The first things to note about the elements $f \in C(J_k)$ are that $f(n) \geq 0$ for all $n \in J_k$ since $f(0) \geq 0$ and $f(n) \geq f(n-1)$, and that $f(n) = 0$ $(n > 1)$ if, and only if, $f(1) = 0$.

Lemma 5: If f is an extremal element of $C(J_k)$, then $f(0) = 0$ or $f(0) = f(1)$.

Proof (by contraposition): Without loss of generality assume that $f(1) = 1$ (see property 3, p. 2). Then assume that the above conclusion fails, so that $0 < f(0) < 1$. Let $f_1(n) = f_2(n) = \frac{f(n)}{2}$ if $n > 0$; let $f_1(0) = 0$ or $\frac{1}{2}$ according as $f(0) < \frac{1}{2}$ or $\geq \frac{1}{2}$, respectively; let $f_2(0) = f(0) - f_1(0)$. Since $f_1(0) \neq \frac{f(0)}{2}$, the decomposition is non-proportional. Since $f_1(0)$ and $f_2(0)$ are non-negative and each is no larger than $\frac{1}{2}$, $f_1, f_2 \in C(J_k)$.

If $f \not\equiv 0$ in $C(J_k)$, let f be normalized by $f(1) = 1$ (that is, consider the proportional function $f' : n \to f(n)/f(1)$). Consider all equations $f(n)+f(m) = f(n+m)$ ($1 \leq n,m$ and $n+m \leq k$) and $f(n+1) = f(n)$ ($1 \leq n < k$) which are true for this function f. Replace $f(n)$ by x_n ($n = 1,2,\ldots,k$) to obtain a system $L(f)$ of linear equations, which has at least one solution, namely $x_n = f(n)$ for all x_n which appear in $L(f)$.

Theorem 17: Let $f \in C(J_k)$ with $f(1) = 1$, and $f(0) = 0$ or $f(0) = 1$. Then f is an extremal element of $C(J_k)$ if, and only if, x_i occurs in at least one equation of $L(f)$ for every $i = 1,2,\ldots,k$ and the system $L(f)$ has a unique solution when $x_1 = 1$.

Proof: This proof is closely patterned after the proof of a similar result given by McLachlan [10]. If f is not extremal but every x_i appears in $L(f)$, then there exist $f_1, f_2 \in C(J_k)$ such that $f_1+f_2 = f$ and $f_1 \neq tf$, $t \in R$.

Then $X = (1, f(2), \ldots, f(k))$ is a solution of $L(f)$ and $Z = (1, z_2, \ldots, z_k)$, with $z_i = f_1(i)/f_1(1)$ is a different solution which satisfies all the equations by virtue of Proposition 11b,c.

Conversely, if x_p does not appear in $L(f)$, then the minimum, u, of the set $A_1 \cup A_2 \cup A_3$ is positive, where
$A_1 = \{f(m)+f(n)-f(p) : mn \neq 0 \text{ and } m+n = p\}$,
$A_2 = \{f(p)+f(n)-f(p+n) : n > 0 \text{ and } p+n \leq k\}$, and
$A_3 = \{f(p+1)-f(p),\ f(p)-f(p-1)\}$. Let $f_i(n) = \frac{f(n)}{2}$ $(n \neq p)$, and let $f_i(p) = \frac{f(p)}{2}+(-1)^i u/4$ $(i = 1,2)$. Then $f = f_1+f_2$ is a non-proportional decomposition in $C(J_k)$.

If each x_i appears in $L(f)$ but there exists a solution $Y = (1, y_2, \ldots, y_k) \neq X$, the equation $x_1 = 1$ guarantees that $L(f)$ is not a homogeneous system, so $Y \neq tX$, $t \in R$. Also, for any $t \in R$, $Z = tX+(1-t)Y$ is a solution of $L(f)$ -- a fact which enables the specification that each $y_i > 0$ to be added since each $x_i > 0$ and there must then be a neighborhood of X in R^k in which the line $tX+(1-t)Y$ contains only k-tuples of positive numbers.

Let $u > 0$ be the minimum of the set $B_1 \cup B_2$, where
$B_1 = \{f(n+1)-f(n) : f(n+1) > f(n) \text{ and } n < k\}$ and
$B_2 = \{f(m)+f(n)-f(m+n) : m+n \leq k \text{ and } f(m)+f(n) > f(m+n)\}$.
Let $M = \max\{y_i\}$, let $r = 1+u/M$, and consider
$Z = (z_1, \ldots, z_k) = rX+(1-r)Y$.

The function $f' : i \to z_i$ is in $C(J_k)$ with $z_i = (1+u/M)x_i - uy_i/M$, and the function $f_1 : i \to f(i)-(u/2k)f'(i)$ gives the non-proportional decomposition

$f = f_1 + (u/2k)f'$ in $C(J_k)$ since, to form z_i, a number less than u was subtracted from a number bigger than x_i, and since $z_i = f'(i) \le k$.

Remark 7: If $x_{n-1} = x_n$ and $x_m + x_{n-1} = x_{m+n-1}$ are equations in L(f), then $x_{m-1} + x_n = x_{m+n-1}$ and $x_{m-1} = x_m$ are in L(f).

Proof: If $x_{m-1} + x_n > x_{m+n-1}$, then subtraction of $x_m + x_{n-1} = x_{m+n-1}$ and use of $x_n = x_{n-1}$ imply that $x_{m-1} - x_m > 0$, a contradiction of the non-decreasing property of f. Thus equality holds in $x_m + x_{n-1} = x_{m+n-1}$ and also, by subtraction, must hold in $x_{m-1} = x_m$.

Remark 8: If $x_{n-1} = x_n$ and $x_p + x_m = x_n$, where $p+m = n$, are equations in L(f), then $x_{p-1} + x_m = x_{n-1}$, $x_p + x_{m-1} = x_{n-1}$, $x_p = x_{p-1}$, and $x_m = x_{m-1}$ are in L(f).

Proof: If $x_{p-1} + x_m > x_{n-1}$, then subtraction of $x_p + x_m = x_n$ and use of $x_{n-1} = x_n$ imply that $x_{p-1} - x_p > 0$, a contradiction. Also $x_{p-1} = x_p$ as in the previous proof. The remaining equations follow by symmetry of hypotheses in p and m.

Theorem 17 may also be interpreted as a characterization of the extremal elements of the subcone C_1 in C(J) consisting of all non-decreasing subadditive functions f on J which are constant for all $n \ge k = k(f)$. Since no non-proportional decomposition exists for such an f if $n \le k$ and the proportionality must be preserved for $n > k$ by Proposition 11b, the extremal elements of C_1 are also extremal elements of C(J). This fact and the following example yield a counter-example to the Choquet conjecture.

Consider the function $f \in C(J_6)$ defined by $f(0) = 0$, $f(1) = 1$, and $f(n) = \frac{n}{2}$ $(n = 2,3,4,5,6)$. (See Figure 11.)

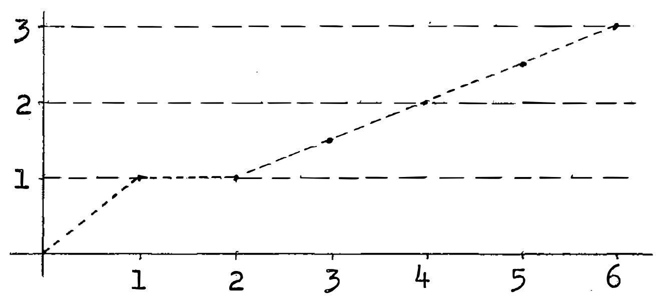


Figure 11. Example on the Choquet Conjecture

The system L(f) consists of $x_1 = 1$, $x_1 = x_2$, $2x_2 = x_4$, $x_2+x_3 = x_5$, $x_2+x_4 = x_6$, and $2x_3 = x_6$. The determinant of this system,

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & -1 \end{vmatrix},$$

is equal to 2, so the solution of the system is unique. Therefore, f is an extremal element of $C(J_6)$, and if f is extended to J by $f(n) = 3$ $(n > 6)$, then f is extremal in C_1 and C(J) although f is not of the form conjectured by Choquet.

The cone $C(J_k)$ is a convex subset of R^{k+1} with the usual topology of pointwise convergence (Euclidean topology). Let $B = \{f : f \in J_k \text{ and } f(1) = 1\}$. The origin $(0,0,\ldots,0) \notin B$; and, if $f \in C(J_k)$ and f is not the origin, then $f(1) = r > 0$ so that $g = (1/r)f$ is in B. Thus B intersects each ray of $C(J_k)$ in exactly one point. Also B is convex since $f,g \in B$ and $0 \leq t \leq 1$ imply $tf(1)+(1-t)g(1) = t+(1-t) = 1$.

The set B will be shown to be compact by showing that it is closed and bounded. Since B is a subset of the rectangle $\{(x_0,x_1,\ldots,x_k) : 0 \leq x_0 \leq 1,\ x_1 = 1, \text{ and } 1 \leq x_n \leq n\ (n = 2,\ldots,k)\}$, B is bounded. If f_n is a sequence of distinct elements of B with limit $f \in R^{k+1}$, then f is subadditive by Proposition 8, $f(1) = \lim f_n(1) = \lim 1 = 1$, and f is non-decreasing since $f(i) = \lim f_n(i) \leq \lim f_n(i+1) = f(i+1)$. Thus $f \in B$ and B is closed.

Since the hypotheses of Proposition 10 are satisfied by $B = A$ and $R^{k+1} = L$, the desired Radon measure exists for a multiple of each $f_0 \in C(J_k)$.

<u>Theorem 18</u>: The cone $C(J_k)$ generates all of R^{k+1}; that is, $C(J_k)-C(J_k) = R^{k+1}$.

<u>Proof</u>: Define vectors $v_i \in R^{k+1}$ by $v_0 = (1,1,\ldots,1,1)$ and $v_i = (0,1,2,3,\ldots i-1,i,i,\ldots,i)$ $(i = 1,2,\ldots,k)$. These vectors represent functions in $C(J_k)$ (of the Choquet type) and are subadditive by Proposition 1. The k+1 vectors v_i form a basis for R^{k+1} since the determinant

$$V_k = \begin{vmatrix} v_0 \\ v_1 \\ \vdots \\ v_k \end{vmatrix} = 1 \text{ for every } k = 1,2,\ldots .$$

To show this by induction, note that $V_1 = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$, and assume that

$$V_n = \begin{vmatrix} 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & \ldots & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 & \ldots & 3 & 3 & 3 \\ & \ldots & & & \ldots & & \ldots & \\ 0 & 1 & 2 & 3 & \ldots & n-2 & n-2 & n-2 \\ 0 & 1 & 2 & 3 & \ldots & n-2 & n-1 & n-1 \\ 0 & 1 & 2 & 3 & \ldots & n-2 & n-1 & n \end{vmatrix} = 1.$$

Consider

$$V_{n+1} = \begin{vmatrix} 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & \ldots & 2 & 2 & 2 \\ & \ldots & & & \ldots & & \ldots & \\ 0 & 1 & 2 & 3 & \ldots & n-1 & n-1 & n-1 \\ 0 & 1 & 2 & 3 & \ldots & n-1 & n & n \\ 0 & 1 & 2 & 3 & \ldots & n-1 & n & n+1 \end{vmatrix} .$$

Subtracting row 2 of V_{n+1} from rows $3,4,5,\ldots,n+2$ and expanding by minors of column 1 yields V_n as the only non-zero cofactor. Thus $V_{n+1} = V_n = 1$.

Since $H = \{v_0, v_1, \ldots, v_k\} \subset C(J_k)$ and H is a basis, every element $y \in R^{k+1}$ has the form

$$y = \sum_{i=0}^{k} a_i v_i = \sum_{a_i \geq 0} a_i v_i - \sum_{a_i < 0} |a_i| v_i ,$$

the difference of two elements of $C(J_k)$.

The following chapter discusses some relations between extremal elements of $C(J_k)$ and those of $C(J)$ and $C([0,1])$. The Cantor function reappears in the second case, and some unsolved problems are mentioned and discussed.

CHAPTER VII

EXTENSIONS OF EXTREMAL ELEMENTS

There are several ways in which extremal elements of the cone C(D) can be extended to extremal elements of a cone C(B), where $D \subset B$, and these will be discussed now. This discussion will lead to a discussion of the extremal element problems in the cone C([0,1]). The first of the extensions to be discussed is that resulting from use of the operator S, defined either from [0,a] to E or from J_k to J. The theorem is stated below for the case of [0,a], but its analogue for the case of J_k is proved in the same way.

<u>Theorem 19</u>: Let f be an extremal element of C([0,a]). Then Sf is an extremal element of C(E).

<u>Proof</u>: The function Sf is in C(E) by Proposition 3a and Theorem 3. Let $Sf = G+H$ on E, where $G, H \in C(E)$ and $Sf \neq G \neq 0$; let $g = G|[0,a]$ and $h = H|[0,a]$. Then $g, h \in C([0,a])$ and $g+h = f$. Since f is extremal, $g = tf$ and $h = (1-t)f$ on [0,a] for some $t \in (0,1)$.

If $x \in (a,\infty)$, then $G(x) \leq Sg(x) = S(tf)(x) = t(Sf)(x)$ by Proposition 3c (p. 16). Similarly, $H(x) \leq (1-t)Sf(x)$. Therefore, $Sf(x) = G(x)+H(x) \leq t(Sf)(x)+(1-t)Sf(x) = Sf(x)$, and equality must hold. Thus the decomposition is in the proportion $G(x):H(x) = t:(1-t)$ at every $x \in E$, and Sf is extremal in C(E).

In connection with this theorem, it should be noted that the sequence $\{f(n) : n \in J\}$ $= (0,1,1,2,2,3,3,3,3,4,4,4,4,4,4,4,4,5,5,\ldots)$, where there are 2^{n-1} entries of each integer $n > 1$, is an extremal element of C(J) and its restriction to J_k is an extremal element of $C(J_k)$ for every $k > 0$; however, f is not a maximal extension of any of its initial portions, nor is it a minimal extension in the sense of being constant for $n \geq k$. That f is extremal in C(J) is a consequence of McLachlan's result (p. 47). That the extension of $f|J_k$ by holding it constant for $n > k$ is in C(J) is a consequence of Lemma 3; its extremal character comes from Proposition 11b. It is not true, however, that an extremal element of C(J) must be extremal in $C(J_k)$, when restricted to J_k, for every $k > 0$. Infinitely many k will do.

<u>Remark 9</u>: If there exist infinitely many $k \in J$ such that $f|J_k$ is extremal in $C(J_k)$, then f is extremal in C(J).

<u>Proof</u>: If f is not an extremal element of C(J), then there exist $n \in J$ and $g, h \in C(J)$ such that $g(n):h(n) \neq g(1):h(1)$. Thus $f|J_k$ is not extremal in $C(J_k)$ for every $k \geq n$ since $g|J_k + h|J_k$ is a non-proportional decomposition.

The emphasis now turns to those functions which are extremal elements of $C([0,1])$. The "1" is frequently a convenience, but the theory, because of Remark 3 (p. 5), is equivalent to that of $C([0,a])$ for any $a > 0$. The first consideration will be the extent to which the extensions defined in Proposition 6 and Theorem 11 from $C(J_k)$ to $C([0,k])$ yield extremal elements.

Theorem 20: Let $f \in C(J_k)$ and let F be defined on $[0,k]$ by $F(0) = f(0)$ and $F(x) = f(n)$, $x \in (n-1,n]$ $(n = 1,2,\ldots,k)$. Then F is an extremal element of $C([0,k])$ if, and only if, f is an extremal element of $C(J_k)$.

Proof: By Theorem 11, $F \in C([0,k])$ if, and only if, $f \in C(J_k)$. Since F is constant on $(n-1,n]$ $(1 \le n \le k)$, the decomposition is constant there by Proposition 11b. Thus any non-proportional decomposition of F must yield a non-proportional decomposition on J_k of f, and conversely.

Theorem 21: Let $f \in C(J_k)$ and let P be its polygonal extension to $[0,k]$ (in the sense of Proposition 6). If P is an extremal element of $C([0,k])$, then f is an extremal element of $C(J_k)$.

Proof: If f is not extremal, then there is a non-proportional decomposition $f = f_1 + f_2$, and the corresponding polygonal extensions P_1 and P_2 then are a non-proportional (at least on J_k) decomposition $P = P_1 + P_2$ of P.

The converse of Theorem 21 is not true. For example, if $f \in C(J_2)$ is defined by $f(0) = f(1) = \frac{f(2)}{2} = 1$, then f is extremal, but the polygonal extension, P, of f (Figure 12) is not extremal in C([0,2]) since it can be decomposed on the half-integers.

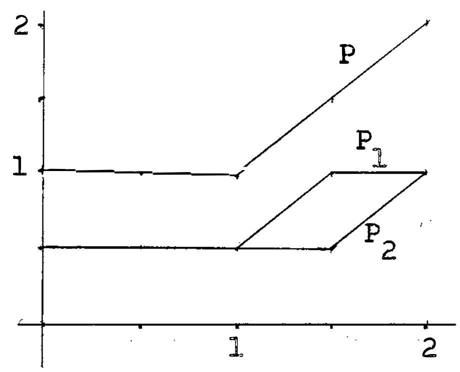


Figure 12. Polygonal Extension of an Extremal Element

Some other examples of extremal elements of C([0,1]) can be found. In particular, the Cantor function is one such example which seems to cloud the general problem of finding <u>all</u> the extremal elements.

<u>Remark 10:</u> The Cantor function K (p. 42) is an extremal element of C([0,1]).

<u>Proof:</u> Let $g,h \in C([0,1])$ and $g+h = K$. Since $\lim_{x\to 0}K(x) = 0$ and since g and h are non-negative on [0,1], $\lim_{x\to 0}g(x) = 0$ and $\lim_{x\to 0}h(x) = 0$. Thus g and h are continuous by Lemma 2. By Proposition 11b, g and h are constant

on $I(2,1)$ -- using the notation of page 42. Let $g(1/3) = r$ and $h(1/3) = t$, $r+t = 1/2$.

If $g(1/9) = g(2/9) < r/2$ or $h(1/9) = h(2/9) < t/2$, then $g(1/3)+h(1/3) \leq g(1/9)+g(2/9)+h(1/9)+h(2/9) < r+t = 1/2$ -- a contradiction since $K(1/3) = 1/2$. Thus $g(1/9) = r/2$ and $h(1/9) = t/2$. This argument can be repeated to show that $g(1/3^n) = r/2^{n-1}$ and $h(1/3^n) = t/2^{n-1}$ $(n = 1,2,3,\ldots)$.

Since the left endpoint of any interval $I(n,k)$ is the sum of endpoints of intervals $I(m,1)$ and the function value there is the sum of corresponding function values, the decomposition is proportional on $cD = \bigcup_{n,k} I(n,k)$, a set everywhere dense in $[0,1]$. Since $g = 2rK$ and $h = 2tK$ on cD, and since g, h, $2rK$, and $2tK$ are continuous on $[0,1]$, $g = 2rK$ and $h = 2tK$ on $[0,1]$. Thus the decomposition $K = g+h$ is proportional.

Another large class of extremal elements of $C([0,1])$, which includes the functions K_n used in Remark 6 to approximate K, is obtained in the following theorem.

<u>Theorem 22</u>: Let $f \in C([0,1])$ such that f is continuous on $[0,1]$, $f(0) = 0$, and the graph of f consists of a finite number of line segments with slopes $m > 0$ and 0 only. Then f is an extremal element of $C([0,1])$.

Proof: If $f = g+h$, $g,h \in C([0,1])$, then the hypotheses force g and h to be continuous on $[0,1]$ as in Remark 10. If $f \equiv 0$ on $[0,x_1]$, $x_1 > 0$, then $f \equiv 0$ on $[0,1]$, and the theorem holds by Proposition 11a. Let f have slope $m > 0$

on $[0,x_1]$, $0 < x_1 \le 1$; then f is additive there and $g(x) = tmx$ and $h(x) = (1-t)mx$, $0 \le t \le 1$, by Proposition 11c. In particular, $g(x_1) = tmx_1$ and $h(x_1) = (1-t)mx_1$. Then let f be constant on $[x_1,x_2]$, $x_1 < x_2 \le 1$, so that $f(x) = g(x)+h(x) = tmx_1+(1-t)mx_1$ on $[x_1,x_2]$ by Proposition 11b. If f has slope m on $[x_2,x_3]$, $x_2 < x_3 \le 1$, then, by Lemma 1, $g(x)-g(x_2) \le tm(x-x_2)$ and $h(x)-h(x_2) \le (1-t)m(x-x_2)$ for all $x \in (x_2,x_3]$.

Thus $f(x) = g(x)+h(x) \le g(x_2)+h(x_2)+m(x-x_2) = f(x_2)+m(x-x_2)$, and equality holds, implying that $g(x) = tm(x-x_2)+g(x_2)$ and $h(x) = (1-t)m(x-x_2)+h(x_2)$. Thus the proportionality is maintained. Repetition of these arguments establishes the proportionality of the decomposition on $[0,1]$.

Thus the functions K_n $(n = 1,2,\ldots)$ and K are extremal elements of $C([0,1])$, and $K_n \to K$; but it is not the case that the limit of a convergent sequence of extremal elements must be an extremal element. An example follows, taken from this cone. Similar examples can be constructed in Euclidean space.

As an example from $C([0,3])$, which is a closed set in the Cartesian product topology since subadditivity and monotonicity are both preserved by pointwise convergence, consider the function f of Figure 13 defined polygonally by $f(0) = 0$, $f(2) = 2$, and $f(3) = 5/2$.

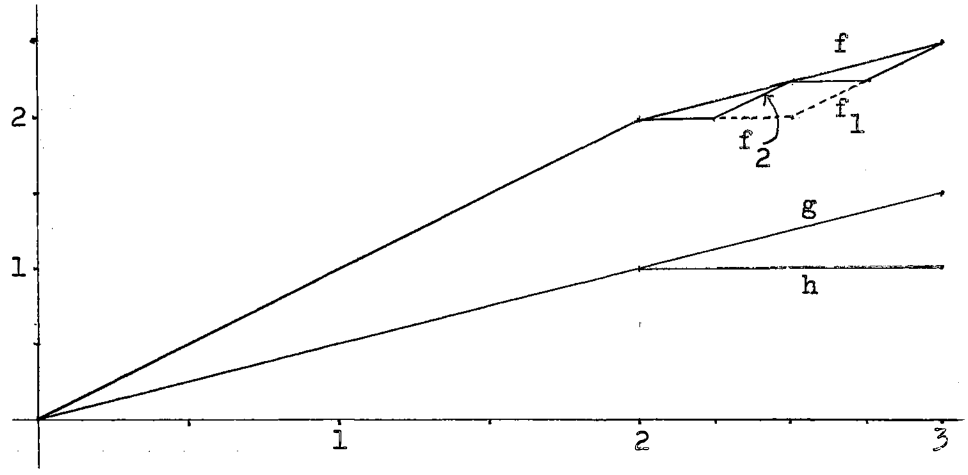


Figure 13. A Sequence of Extremal Elements

Let f_n be defined polygonally by $f(0) = 0$, $f(2) = 2$, and, on the points $2+k/2^n$ $(k = 1,2,\ldots,2^n)$, by $f_n(2+k/2^n) = f_n(2+(k-1)/2^n)$ if k is odd and $f_n(2+k/2^n) = f(2+k/2^n)$ if k is even. Then $f_n \to f$ (uniformly, in fact) on $[0,3]$, f_n is an extremal element of $C([0,3])$ by Theorem 22, and $f = g+h$ is not extremal since it has the decomposition $g(x) = x/2$ on $[0,3]$ and $h(x) = x/2$ on $[0,2]$, $= 1$ on $[2,3]$.

In $C([0,1])$, then, the set of extremal elements is not closed and includes at least the following elements of $C([0,1])$: 1) any non-negative constant function (Proposition 11b), 2) any additive function (Proposition 11c), 3) any continuous polygonal function with slopes of $m > 0$ and 0 only (Theorem 22), 4) the Cantor function (Remark 10), 5) any left-continuous step function extension of an extremal element of $C(\{0,1/k,2/k,\ldots,1\})$ (Theorem 20), and 6) any step function f with $f(x^+)-f(x^-)=f(0^+)$ or 0. To show this last, let $g(0^+)=tf(0^+)$ and $h(0^+)=(1-t)f(0^+)$. Then, where f jumps, g and h must jump by at least these amounts and cannot jump by more, so the proportionality is maintained.

The Choquet integral representation for elements of $C([0,1])$ (Proposition 10 and following) exists in the product topology.

<u>Theorem 23</u>: The set $B=\{f : f\in C([0,1]) \text{ and } f(1)=1\}$ is a compact convex set in the locally convex Hausdorff space R^I, where $I=[0,1]$, and B intersects each ray of $C(I)$ in exactly one point.

<u>Proof</u>: Consider $C(I)=C([0,1])$ as a subset of R^I, the set of all functions on I to R with the product topology. The function g defined on R^I by $g(f)=f(1)$ is a linear functional on R^I, so that $H=\{f: f(1)=1\}$ is a hyperplane in R^I. The set H is closed since $f_n(1)=1$ and $f_n\to f$ imply $f(1)=1$. The set $B=H\cap C(I)$ is closed and convex since it is the intersection of two closed, convex sets.

If $f\in C(I)$ and $f\not\equiv 0$ on I, then $f(1)=r>0$. Thus $(1/r)f\in B$ and is the only element of B of the form tf, $t\geq 0$. By Tychonoff's theorem, the set I^I is compact; therefore, B, a closed subset of I^I, is compact.

Thus the set B satisfies the Choquet theorem, and the integral representation of elements of C(I) by a Radon measure on the closure of the set of extremal elements exists.

The set B is not compact in the space m([0,1]) of all bounded functions on [0,1] with norm $\|f\| = \sup\{f(x) : x \in [0,1]\}$. To show this, let f_r be defined for each rational number $r \in (0,1)$ by $f_r(x) = \frac{1}{2}$ if $x \in [0,r]$ and $f_r(x) = 1$ if $x \in (r,1]$. Then $\{f_r\} \subset B$ by Proposition 2, but, if $r \neq s$, then $\|f_r - f_s\| = \frac{1}{2}$. Thus the set $\{f_r\}$ has no limit point.

Having thus surveyed and extended the work of a selected sample of two generations of mathematical progress in the theory of subadditive functions, and having observed that the characterization of the extremal elements of C(I) is an unsolved problem, this exposition concludes with a mention of some other unsolved problems. With respect to the present chapter, it is not known how the space C(I)-C(I) can be characterized. That is, what functions on the unit interval can be expressed as the difference of two subadditive functions? With regard to an earlier result, the analogue of Remark 4 in the case of more than two points may, or may not, be true. Also, a necessary and sufficient condition that $S(f+g) = Sf+Sg$ would be a welcome addition to Chapter III. The biggest question, however, is far more general. Since Rosenbaum [3] and Bruckner [4] have led the way in considering subadditivity in several dimensions, the project of extending some of the results of this paper to n-dimensional and other spaces would be an interesting and (perhaps) rewarding next step in the theory of subadditive functions.

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APPENDIX A

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APPENDIX B

SPECIAL NOTATIONS

cA	Set-theoretic complement of set A (p. 5)
C(D)	Cone of all non-decreasing subadditive functions defined on set D (p. 45)
E	The non-negative real line $\{x : 0 \le x < \infty\}$ (p. 2)
$f\|D$	Restriction of function f to set D (pp. 18, 19)
I(a,b)	Intervals used to define the Cantor function K (p. 42)
J	The set of all non-negative integers (p. 2)
J_k	The set $\{0,1,2,3,\ldots,k\}$, $k > 0$ (p. 30)
K	The Cantor function (p. 42)
K_n	Approximating function to the Cantor function (p. 42)
R	The set of all real numbers (p. 1)
R_n	Euclidean n-dimensional space (p. 47)
Sf	Maximal subadditive extension to E of function f (p. 15)
[x]	The greatest integer less than or equal to x (p. 22)
[N]	Bibliographical reference (p. 2)

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Thesis: SUBADDITIVE FUNCTIONS OF ONE REAL VARIABLE

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