INFLUENCE AREAS FOR CONTINUOUS BEAMS IN SPACE .

By

MOHAMMAD ESMET KAMEL

Bachelor of Science
Cairo University
Cairo, Egypt
1952

Master of Science
Cairo University
Cairo, Egypt, U. A. R.
1958

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of the Oklahoma State University
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the requirements for
the degree of
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BEAMS IN SPACE

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Thesis Adviser

Dean of the Graduate School

PREFACE

This research is the outgrowth of study on the analysis of space structures at Oklahoma State University for a number of years by several graduate students under the direction of Professor Jan J. Tuma. The possibility of developing the general concept of influence areas for the analysis of complex space structures was suggested to the writer in the summer of 1961 during a series of lectures by Professor Tuma on space structures.

This author wishes to take this opportunity to express his gratitude to the following organizations and individuals who have so graciously aided and encouraged him during his graduate study at Oklahoma State University and in the writing of this dissertation:

To Professor Jan J. Tuma, who has been an intellectual inspiration and who has offered much patient guidance in the clear expression of the ideas contained within this dissertation;

To Professors Roger L. Flanders, David M. MacAlpine, James V. Parcher, Kerry S. Havner and James W. Gillespie, who have encouraged and aided the writer during his graduate study;

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NOMENCLATURE

a_i, b_i . . . . Distance Between Two Lines in Space.

d_{ij} . . . . Length of Member ij.

i . . . . Notation for Joint.

m . . . . Notation for Cross-Section of Member ij.

n . . . . Number of Unknowns.

o . . . . Notation for Reference System.

p . . . . Point of Application for Loads.

s . . . . Notation for Cross-Section of Member jk.

A . . . . Cross-Sectional Area.

C . . . . Influence Coefficient for Angular Load Functions.

F_{ij} . . . . Angular Flexibility at End i of Member ij.

G_{ij} . . . . Angular Carry-Over Flexibility at End j of Member ij.

I . . . . Identity Matrix.

L . . . . Load.

M . . . . Moment.

N . . . . Force on Cross-Section.

P . . . . External Force.

$\overline{P}$ . . . . Joint Elastic Weight.

Q . . . . External Moment.

R . . . . Reaction.

S . . . . Vector in Space.

X, Y, Z . . . . Reference Axes.

α, β, γ . . . . Direction Parameters.

η . . . . Subtended Angle.

λ^n . . . . Linear Flexibility.

ϕ . . . . Angle Changes.

Δ . . . . Displacement.

Σ . . . . Summation.

CHAPTER I

INTRODUCTION

This thesis presents the analysis of continuous beams in space, loaded by stationary and moving loads.

The space continuous beam, as defined in this study, is a three-dimensional bent member of variable or constant cross-sections, resting on spherical hinges; thus it is capable of resisting forces and moments applied in any direction. The geometry of the continuous beam considered is general, and the loading may be forces and moments applied in any direction.

1-1. Historical Notes

The analysis of continuous beams attracted the attention of civil engineers during the last century. The first analytical approach to the analysis of continuous straight beams lying in plane and loaded by a coplanar system was presented by Clapeyron(1). The application of the Three Moment Equation to the analysis of other problems has been demonstrated by Müller-Breslau(2). The graphical analysis of continuous beams in connection with three moment equations and fixed points has been developed by Culman(3), and considerably extended by Ritter(4). The introduction of slope deflection equations to the analysis of coplanar continuous beams can be found in the work of Bendixen(5).

The analysis of continuous space beams by slope deflection was recorded by Bažant(6). The extension of the moment distribution method to the analysis of continuous space beams was recorded by Michalos[7].

The application of the joint carry-over moment procedure to space beams was derived by Tuma[8] in his lectures and recorded by Childress in his M. S. Thesis[9]. The flexibility approach to continuous beams lying in plane and loaded by forces perpendicular to the plane was developed by Tuma[8] and recorded by Patel in his M. S. Thesis[10].

The author's contribution is the generalization of the flexibility approach to any type of continuous member in space.

1-2. Statement of the Problem

A continuous beam of variable cross-section loaded by a general system of loads and supported by spherical hinges at unequal levels is considered (Fig. 1-1). The geometry of the beam, of the supports, and of the loads is known. The forces and moments at any section and at points of supports are required.

1-3. Assumptions

In the analysis of this problem, the following assumptions are being made:

(a) The material is homogeneous and isotropic.

(b) Deformations are small and elastic.

(c) Plane sections remain plane after deformation.

(d) Deformations due to shears are small and can be neglected.

1-4. Procedure of Investigation

In the development of the method of analysis, the following steps of investigation are considered:

(1) The geometry of the structure is defined in terms of coordinate and transformation matrices.

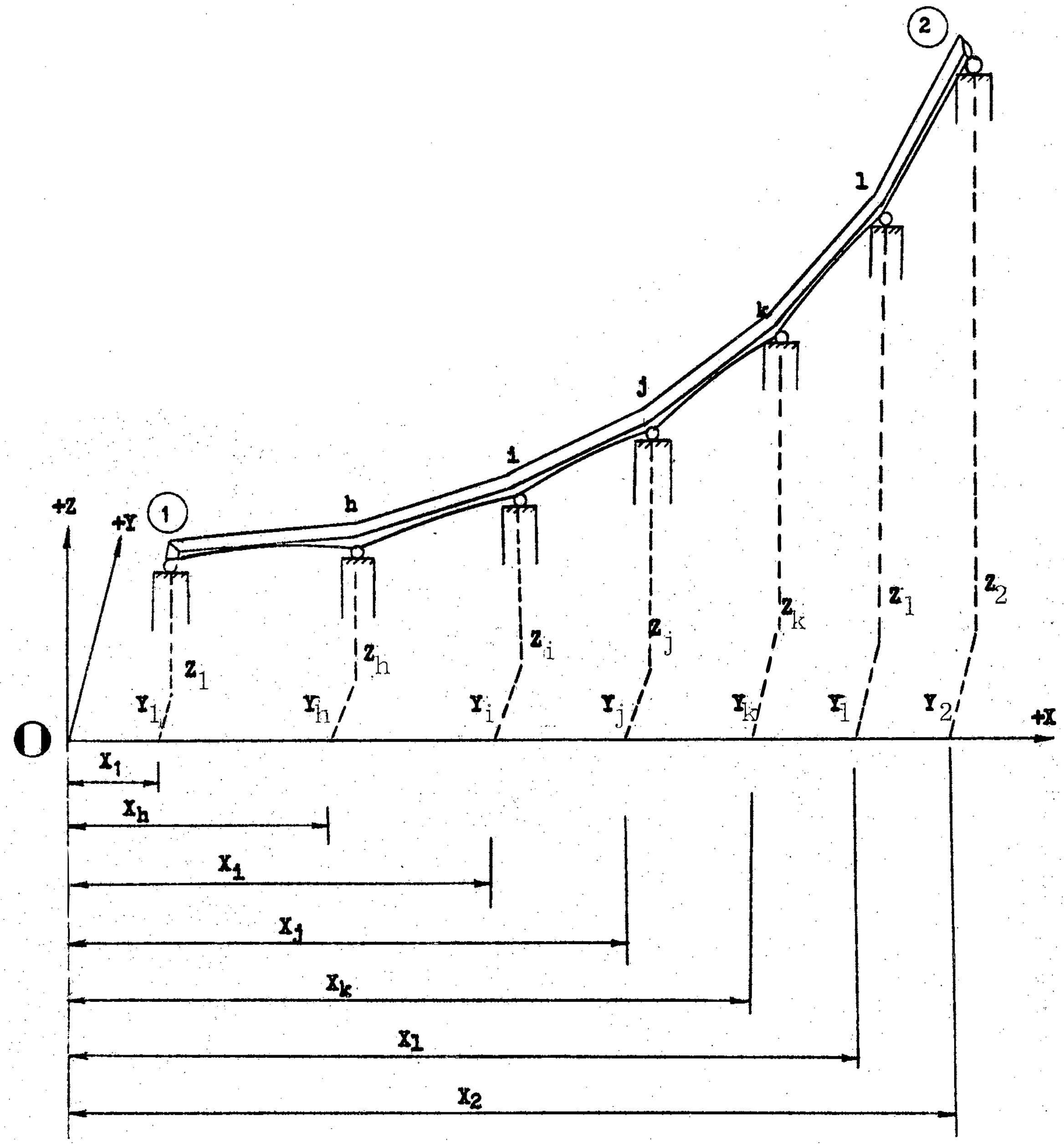


Figure 1-1
Continuous Beam in Space

(2) A basic structure (Fig. 1-2) is selected, and the forces and moments at the ends of the basic structure are introduced as unknowns.

(3) Through the equations of stereo-static equilibrium and special equations, certain unknowns are eliminated, and the number of unknowns is reduced to the number of redundant forces and moments.

(4) Elastic constants related to the action of redundants are developed in matric form and deflections of the basic structure expressed by the corresponding elastic weights.

(5) The compatibility equations are obtained by requiring the equilibrium of elastic weights in terms of flexibilities, load functions, and redundants.

(6) Compatibility equations are solved and the numerical values of redundants are substituted into the equations of cross-sectional and reactive elements.

(7) The case of moving loads is considered, and the equations obtained for stationary loading are extended to the influence areas of the beam functions.

(8) The procedure developed is illustrated by a numerical example.

1-5. Sign Convention and Notation

Signs of all analytical quantities are governed by the following sign convention:

(a) Loads, reactions, joint moments, and joint deformations represented by vectors acting in the direction of coordinate axes are positive (Fig. 1-3).

(b) Cross-sectional forces, cross-sectional moments, and cross-sectional deformations of the far end represented by vectors

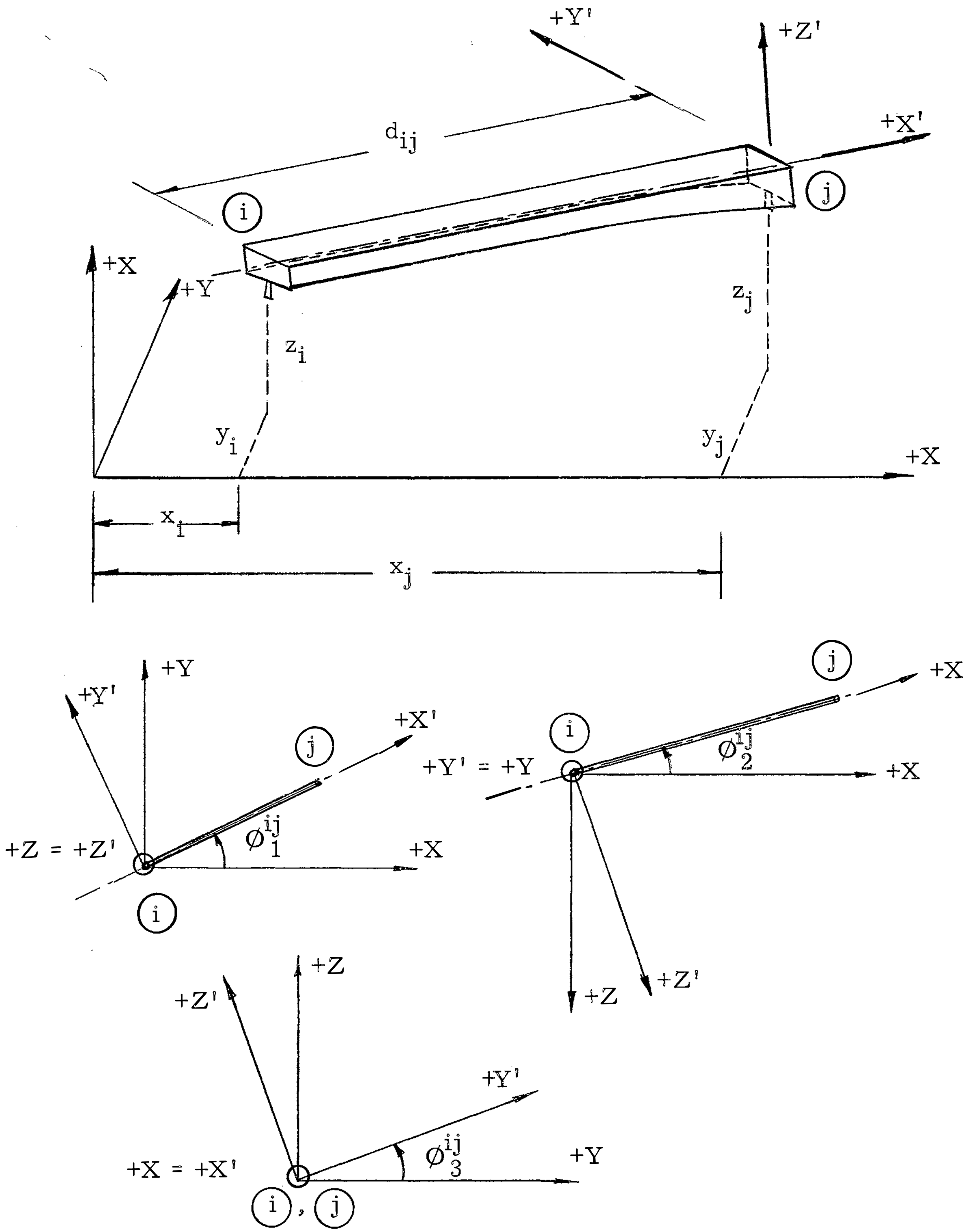


Figure 1-2
Basic Structure and its Geometry

acting in the direction of coordinate axes are positive (Fig. 1-4).

Force-vectors are represented by a line with a single arrow designating the sense; moment-vectors are represented by a line with a double arrow assigning the sense.

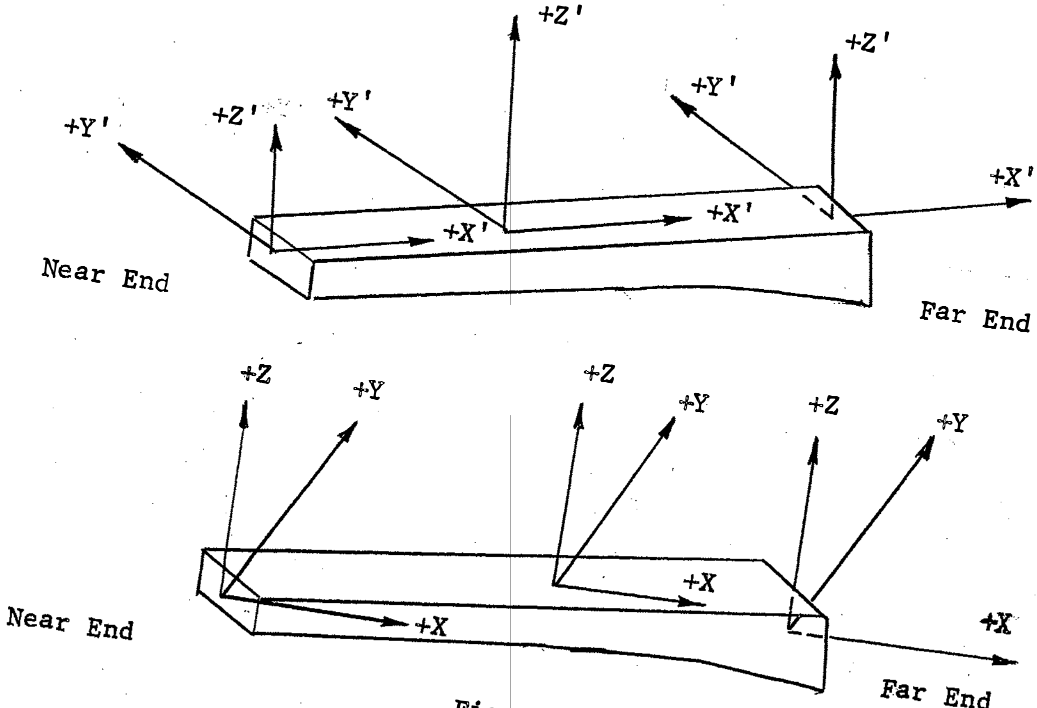


Figure 1 - 3
Sign Convention for Joint and
Reactive Vectors

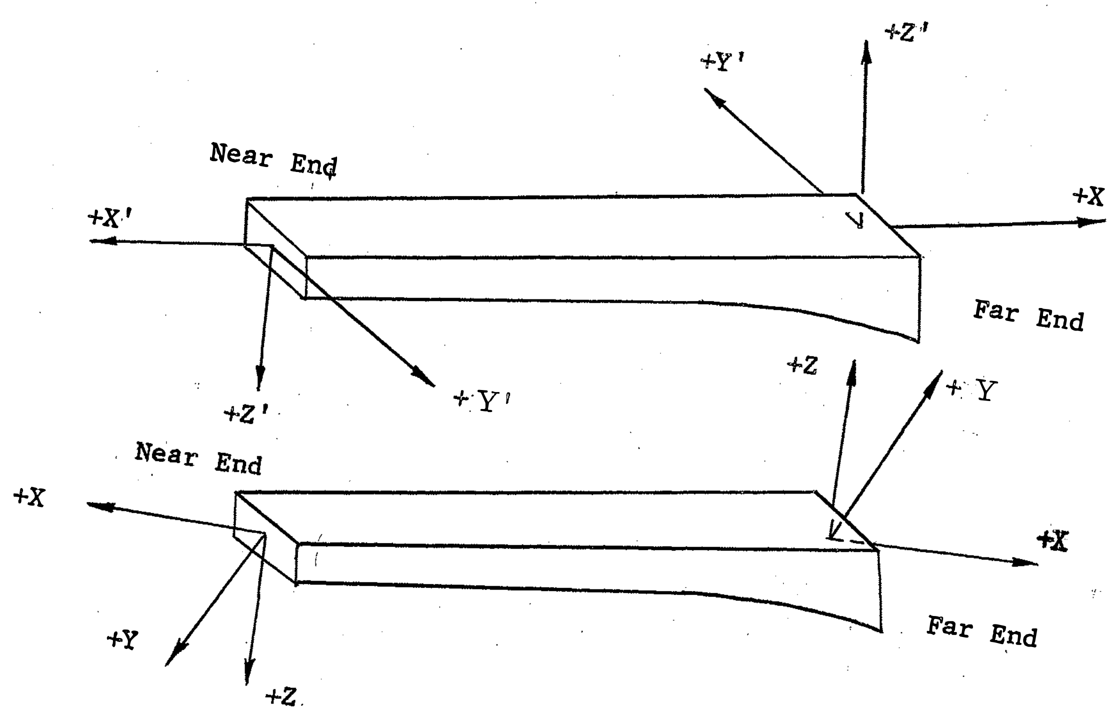


Figure 1 - 4
Sign Convention for
Cross-Sectional Vectors

CHAPTER II

GEOMETRY

The geometry of a continuous beam in space, simply supported at 1, h, i, j, k, l, 2 (Fig. 2-1), is investigated in this chapter.

Each span of this beam is a straight bar of variable or constant cross-section, and the supports 1, h, i, j, k, l, 2 are assumed to be spherical hinges.

In the study of the geometry of this beam, two systems of coordinates are considered. The first system, called the initial system, is related to the principal axes of each bar. The second set of coordinates, known as the reference system, is given by an arbitrarily selected set of orthogonal axes. There are as many initial systems as spans; but there is only one reference system. The study of geometric quantities, of loads, and cross-sectional elements in relationship to these two systems follows.

2-1. Geometry of Bars

If bar ij is isolated from the continuous beam 1 h i j k l 2 (Fig. 2-1), and related to the reference system X, Y, Z (Fig. 2-2), the coordinates of the end points i and j become x_{oi}, y_{oi}, z_{oi}, and x_{oj}, y_{oj}, z_{oj}, and the angles of this bar with the reference axes are then ω_{ijx}, ω_{ijy}, ω_{ijz}.

The length of the bar ij

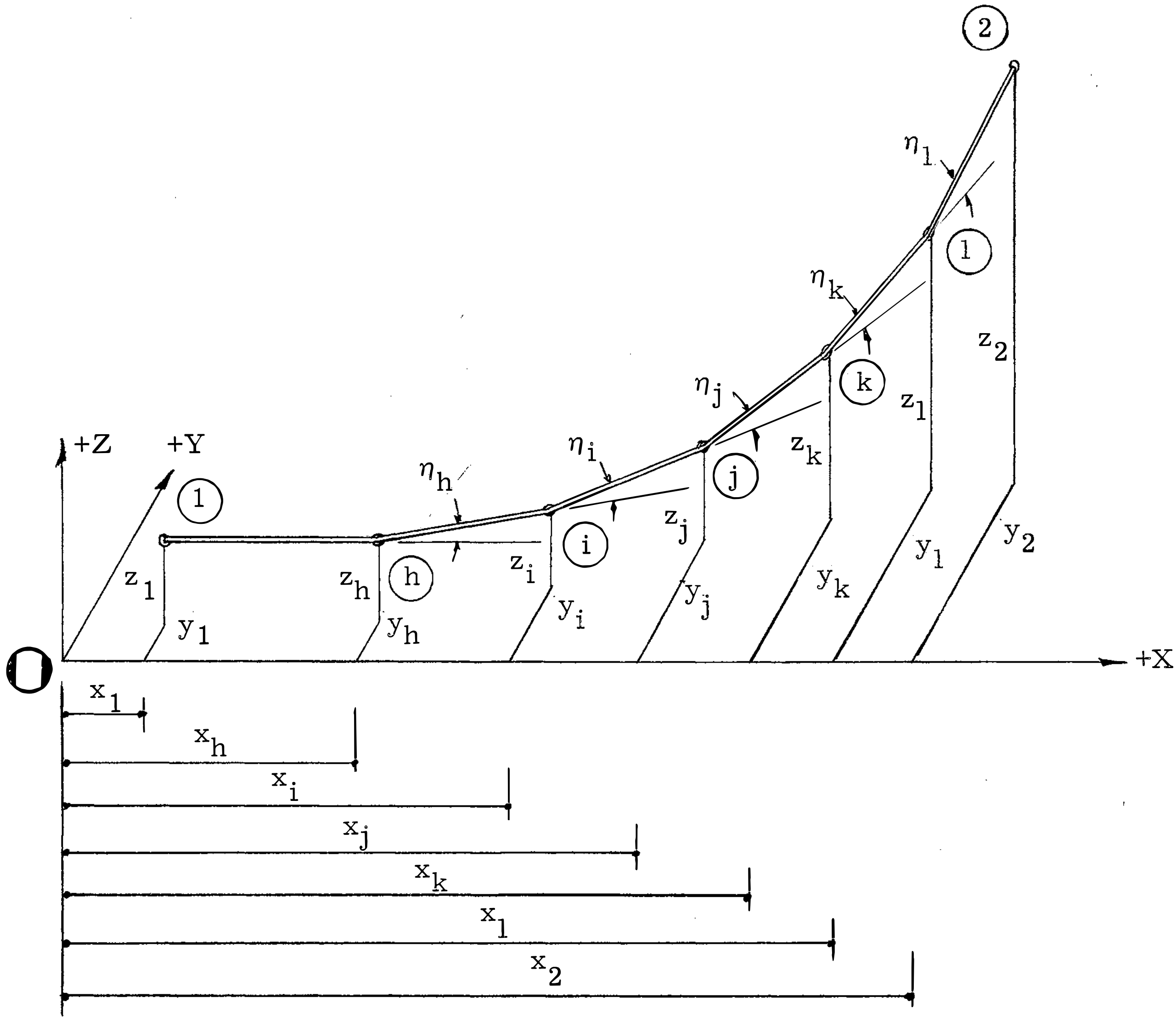


Figure 2-1
Geometry of Space Continuous Beam

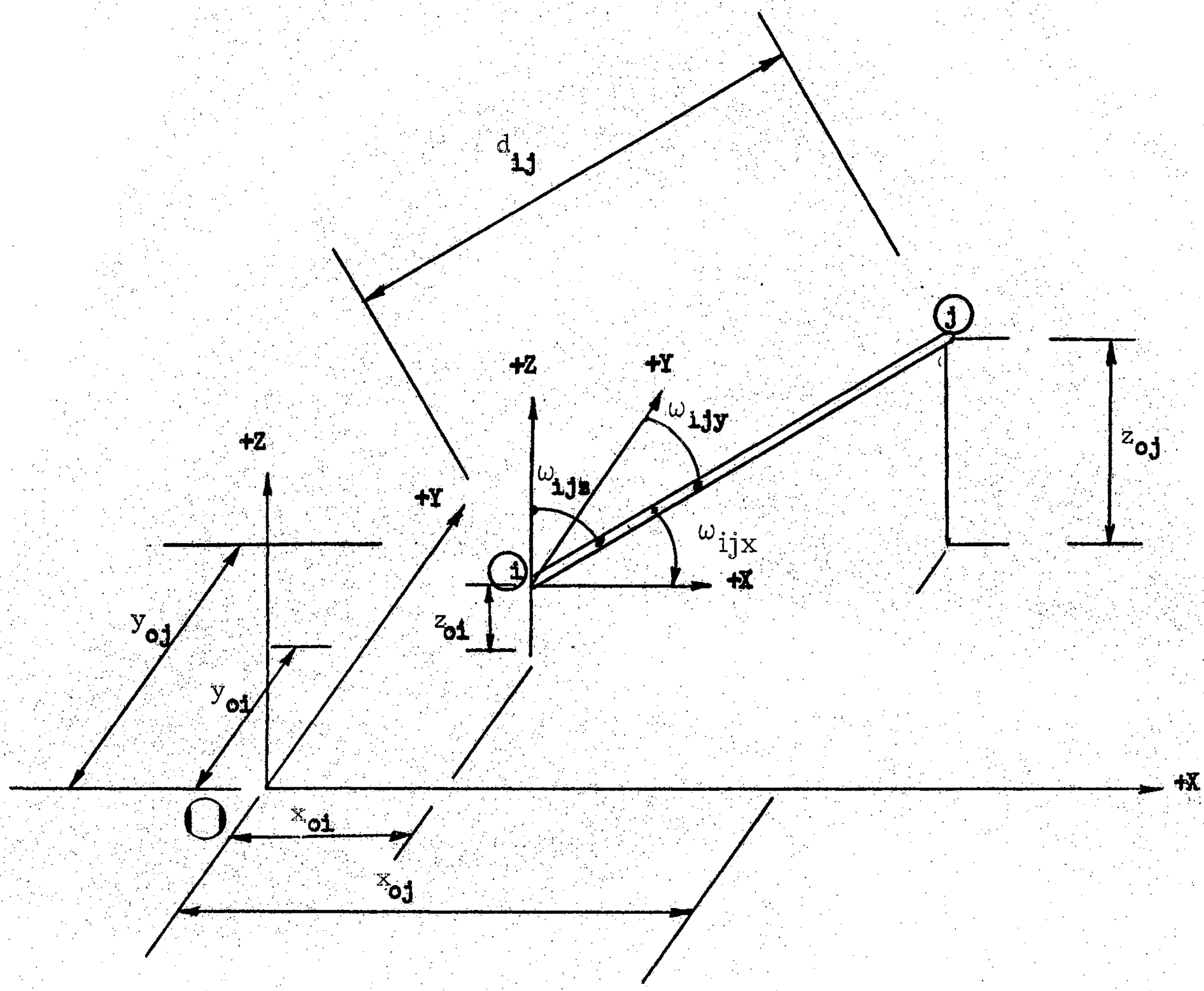


Figure 2-2
Geometry of Bar ij

$$d_{ij} = \sqrt{(x_{oj} - x_{oi})^2 + (y_{oj} - y_{oi})^2 + (z_{oj} - z_{oi})^2} \, . \tag{2-1}$$

The components of d_{ij} in terms of the reference coordinates are

$$\begin{aligned} d_{ijx} &= x_{oj} - x_{oi} \\ d_{ijy} &= y_{oj} - y_{oi} \\ d_{ijz} &= z_{oj} - z_{oi} \, , \end{aligned} \tag{2-2}$$

or in terms of the direction parameters

$$\cos \omega_{ijx} = \alpha_x^{ij} \qquad \cos \omega_{ijy} = \beta_x^{ij} \qquad \cos \omega_{ijz} = \gamma_x^{ij} \tag{2-3}$$

are

$$\begin{aligned} d_{ijx} &= d_{ij} \, \alpha_x^{ij} \\ d_{ijy} &= d_{ij} \, \beta_x^{ij} \\ d_{ijz} &= d_{ij} \, \gamma_x^{ij} \, . \end{aligned} \tag{2-4}$$

These components represent a column matrix,

$$\left[d_{ij} \right] = \begin{bmatrix} d_{ijx} \\ d_{ijy} \\ d_{ijz} \end{bmatrix} , \tag{2-5}$$

which is being used extensively in this study.

The next problem is to find the relationship between the coordinates of the reference system and those of the initial system. For this purpose an arbitrary point "A" lying off the line ij is selected and related to the reference system X_i^o, Y_i^o, Z_i^o, and to the initial system X_i', Y_i', Z_i' as shown in Figure 2-3. The superscripts indicate the system, and the subscripts the origin. The reference coordinates of this point are

$$S_{iA}^{o} = \begin{bmatrix} x_{iA}^{o} \\ y_{iA}^{o} \\ z_{iA}^{o} \end{bmatrix} \qquad (2\text{-}6)$$

and the initial coordinates are

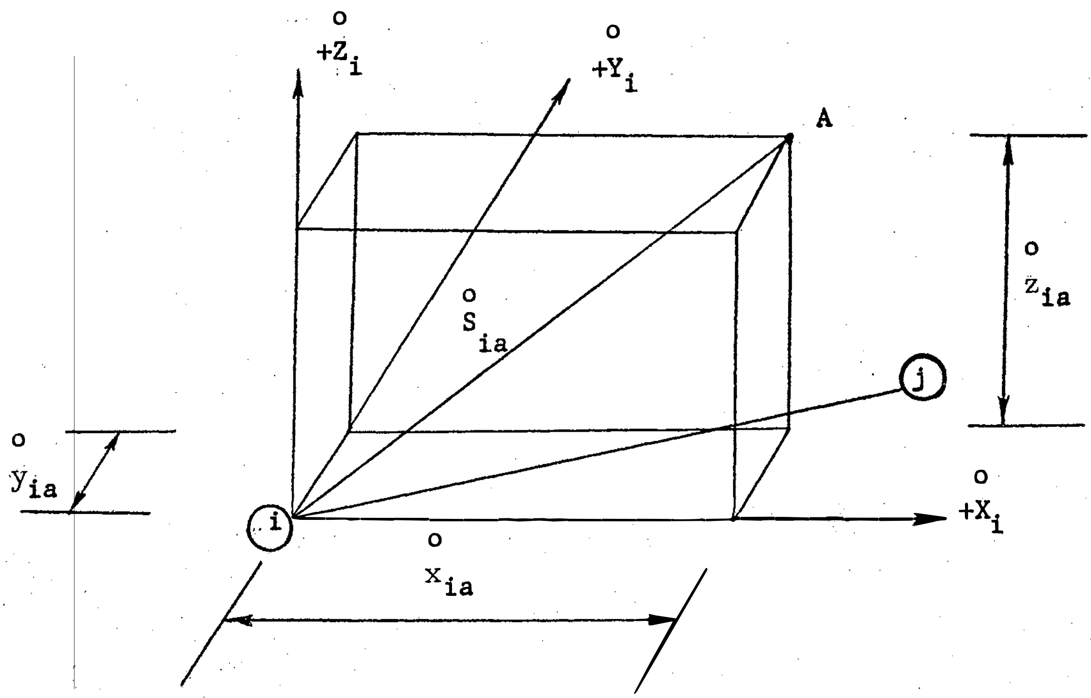
$$S_{iA}' = \begin{bmatrix} x'_{iA} \\ y'_{iA} \\ z'_{iA} \end{bmatrix} . \qquad (2\text{-}7)$$

The relationship between these coordinates is well known from space geometry and is restated here for completeness only:

$$\left[S_{ij}^{o}\right] = \left[\pi_{o1}^{ij}\right]\left[S'_{ij}\right] \qquad (2\text{-}8)$$

$$\left[S'_{ij}\right] = \left[\pi_{1o}^{ij}\right]\left[S_{ij}^{o}\right] . \qquad (2\text{-}9)$$

The transformation matrix, $\left[\pi_{o1}^{ij}\right]$, and its transpose, $\left[\pi_{1o}^{ij}\right]$, are functions of the direction parameters as shown below:



Reference Coordinates of Point A

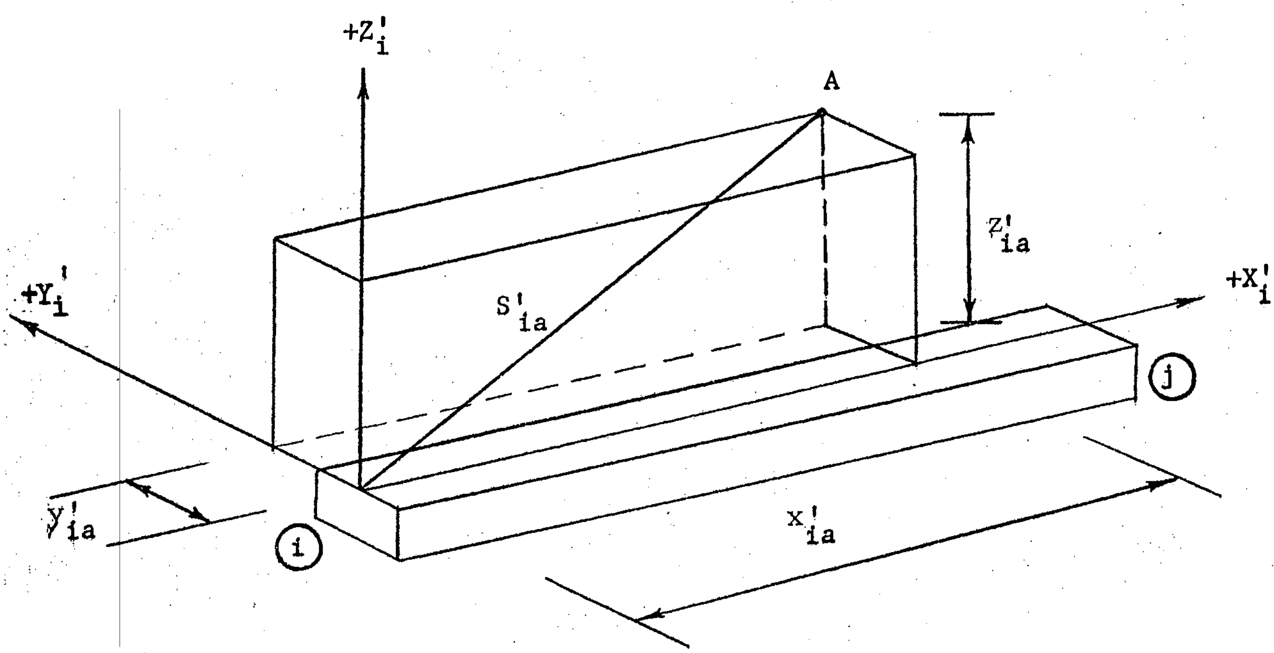


Figure 2 - 3
Initial Coordinates of Point A

$$\left[\pi_{o1}^{ij}\right] = \begin{bmatrix} \alpha_x^{ij} & \alpha_y^{ij} & \alpha_z^{ij} \\ \beta_x^{ij} & \beta_y^{ij} & \beta_z^{ij} \\ \gamma_x^{ij} & \gamma_y^{ij} & \gamma_z^{ij} \end{bmatrix} \qquad (2-10)$$

$$\left[\pi_{1o}^{ij}\right] = \begin{bmatrix} \alpha_x^{ij} & \beta_x^{ij} & \gamma_x^{ij} \\ \alpha_y^{ij} & \beta_y^{ij} & \gamma_y^{ij} \\ \alpha_z^{ij} & \beta_z^{ij} & \gamma_z^{ij} \end{bmatrix} . \qquad (2-11)$$

The meaning of the functions α, β, γ is explained in the Appendix Table A-1. In many important cases instead of the total matrix a column or a row submatrix must be used. In those instances, the following nomenclature is introduced:

$$\left[\pi_{o1x}^{ij}\right] = \begin{bmatrix} \alpha_x^{ij} \\ \beta_x^{ij} \\ \gamma_x^{ij} \end{bmatrix}, \left[\pi_{o1y}^{ij}\right] = \begin{bmatrix} \alpha_y^{ij} \\ \beta_y^{ij} \\ \gamma_y^{ij} \end{bmatrix}, \left[\pi_{o1z}^{ij}\right] = \begin{bmatrix} \alpha_z^{ij} \\ \beta_z^{ij} \\ \gamma_z^{ij} \end{bmatrix} \qquad (2-12)$$

$$\left[\pi_{o1\alpha}^{ij}\right] = \left[\alpha_x^{ij}\ \alpha_y^{ij}\ \alpha_z^{ij}\right] \qquad (a)$$

$$\left[\pi_{o1\beta}^{ij}\right] = \left[\beta_x^{ij}\ \beta_y^{ij}\ \beta_z^{ij}\right] \qquad (b) \qquad (2-13)$$

$$\left[\pi_{o1\gamma}^{ij}\right] = \left[\gamma_x^{ij}\ \gamma_y^{ij}\ \gamma_z^{ij}\right] \qquad (c)$$

and

$$\left[\pi^{ij}_{1ox}\right] = \left[\pi^{ij}_{o1\alpha}\right]' = \begin{bmatrix} \alpha^{ij}_{x} \\ \alpha^{ij}_{y} \\ \alpha^{ij}_{z} \end{bmatrix} \tag{2-14}$$

$$\left[\pi^{ij}_{1o\alpha}\right] = \left[\pi^{ij}_{o1x}\right]' = \left[\alpha^{ij}_{x}\ \beta^{ij}_{x}\ \gamma^{ij}_{x}\right] \tag{2-15}$$

$$\left[\pi^{ij}_{1oy}\right] = \left[\pi^{ij}_{o1\beta}\right]' = \begin{bmatrix} \beta^{ij}_{x} \\ \beta^{ij}_{y} \\ \beta^{ij}_{z} \end{bmatrix} \tag{2-16}$$

$$\left[\pi^{ij}_{1o\beta}\right] = \left[\pi^{ij}_{o1y}\right]' = \left[\alpha^{ij}_{y}\ \beta^{ij}_{y}\ \gamma^{ij}_{y}\right] \tag{2-17}$$

$$\left[\pi^{ij}_{1oz}\right] = \left[\pi^{ij}_{o1\gamma}\right]' = \begin{bmatrix} \gamma^{ij}_{x} \\ \gamma^{ij}_{y} \\ \gamma^{ij}_{z} \end{bmatrix} \tag{2-18}$$

$$\left[\pi^{ij}_{1o\gamma}\right] = \left[\pi^{ij}_{o1z}\right]' = \left[\alpha^{ij}_{z}\ \beta^{ij}_{z}\ \gamma^{ij}_{z}\right] . \tag{2-19}$$

2-2. Geometry of Loads

If bar ij is acted upon by a single concentrated load P applied at point p between i and j (Fig. 2-4), the load P can be conveniently resolved into three components related to the initial axes of the bar or to the reference axes of the system. Assuming that the transformation matrices, $\left[\pi^{P}_{1p}\right]$ and $\left[\pi^{P}_{op}\right]$, are known and given by direction parameters, the components of P_p with respect to the initial system are

$$\left[P'_p\right] = \left[\pi^{P}_{1px}\right] P_p \,. \tag{2-20}$$

Similarly, the components of P_p with respect to the reference system are

$$\left[P^o_p\right] = \left[\pi^{P}_{opx}\right] P_p \,. \tag{2-21}$$

If, instead of a force, a moment Q_q is applied on the bar ij, the same resolutions in matrix form can be performed:

$$\left[Q'_q\right] = \left[\pi^{Q}_{1qx}\right] Q_q \tag{2-22}$$

$$\left[Q^o_q\right] = \left[\pi^{Q}_{oqx}\right] Q_q \,. \tag{2-23}$$

The resolution of the applied load into one of the major systems of coordinates is of great importance, and a reference will be made to these resolutions at several places of this study.

2-3. Geometry of Cross-Sectional Elements

The bar ij loaded by a general force P_p and a general moment Q_q representing the resultant of loads and moments has six cross-sectional elements acting at each end. The end forces are designated by "N" and the end moments by the symbol "M". These end forces and end moments are listed in their corresponding matrices:

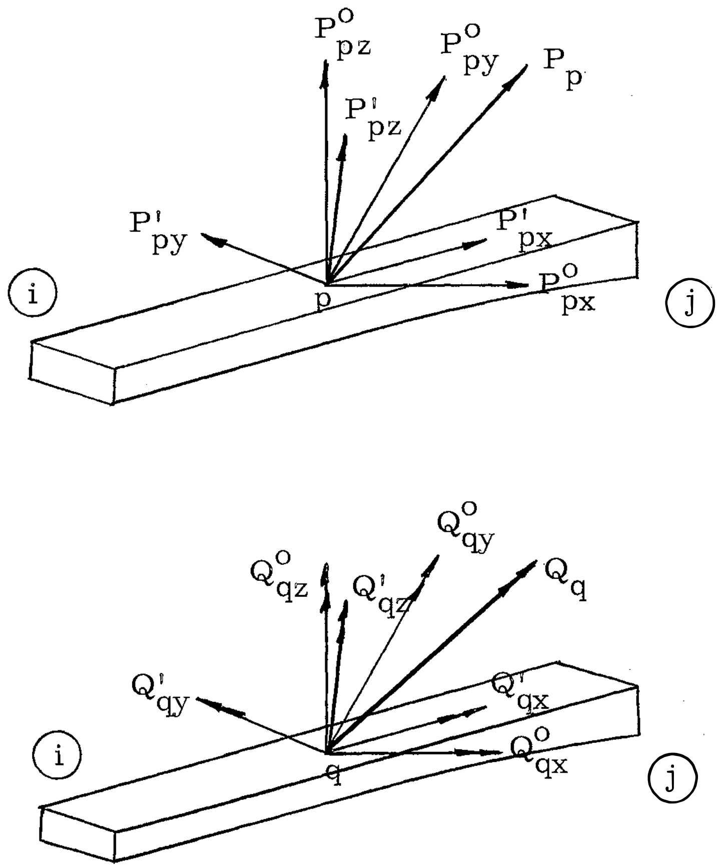


Figure 2-4
Components of Load and Moment Vectors along the Initial and Reference Systems

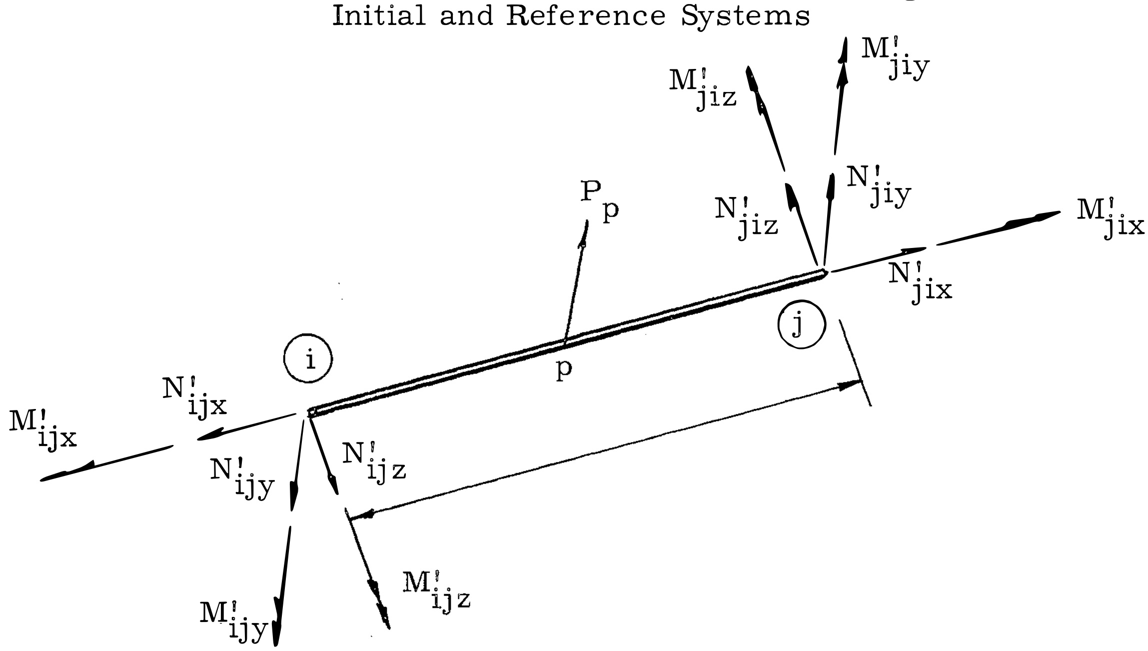


Figure 2-5
Cross-Sectional Elements at Ends of Bar ij along the Initial System

$$\left[N'_{ij} \right] = \begin{bmatrix} N'_{ijx} \\ N'_{ijy} \\ N'_{ijz} \end{bmatrix} \qquad (2\text{-}24a)$$

$$\left[N'_{ji} \right] = \begin{bmatrix} N'_{jix} \\ N'_{jiy} \\ N'_{jiz} \end{bmatrix} \qquad (2\text{-}24b)$$

$$\left[M'_{ij} \right] = \begin{bmatrix} M'_{ijx} \\ M'_{ijy} \\ M'_{ijz} \end{bmatrix} \qquad (2\text{-}25a)$$

$$\left[M'_{ji} \right] = \begin{bmatrix} M'_{jix} \\ M'_{jiy} \\ M'_{jiz} \end{bmatrix} . \qquad (2\text{-}25b)$$

The first subscript indicates the point of application of the cross-sectional element, the second denotes the far end, and the third one the direction of the cross-sectional vector.

These cross-sectional elements must be in many cases transformed from the initial system to the reference system or to another initial system. Whatever this transformation is, it can always be performed by means of the transformation matrices discussed in the first part of this chapter (Eq.'s 2-10, 11).

The end forces at i transformed from the initial system to the reference system are

$$\left[N^{o}_{ij}\right] = \left[\pi^{ij}_{o1}\right]\left[N'_{ij}\right] . \tag{2-26}$$

The end moments at the same point are

$$\left[M^{o}_{ij}\right] = \left[\pi^{ij}_{o1}\right]\left[M'_{ij}\right] . \tag{2-27}$$

The same equations can be written for the forces and moments of the opposite end, and similar equations can be written for all the members of the continuous beam.

CHAPTER III

STEREO-STATICS

Stereo-statics deal with end-conditioning elements and loads applied to the structure. The end-conditioning elements and the loads can be related by means of equations of static equilibrium. Because there are six end-conditioning elements at each end (a total of twelve), six end elements can be related to loads in terms of the remaining six. There is a large variety of choices possible; in this particular case the end bending moments, one torsional moment and one normal force, are assumed to be temporarily known, and the remaining end-conditioning elements (end shears, the other torsional moment, and the other normal force) can be easily obtained from statics.

3-1. Relationship Between End Forces, Moments and Loads

A free body ij is isolated from the continuous beam 1 h i j k l 2 . The loads and end-conditioning elements are related to the initial system ij (Fig. 3-1). Because the temporarily assumed elements are N'_{jix}, M'_{jix}, M'_{jiy}, M'_{jiz}, M'_{ijy}, and M'_{ijz}, it may serve to an advantage to use the following equations of stereo-static equilibrium:

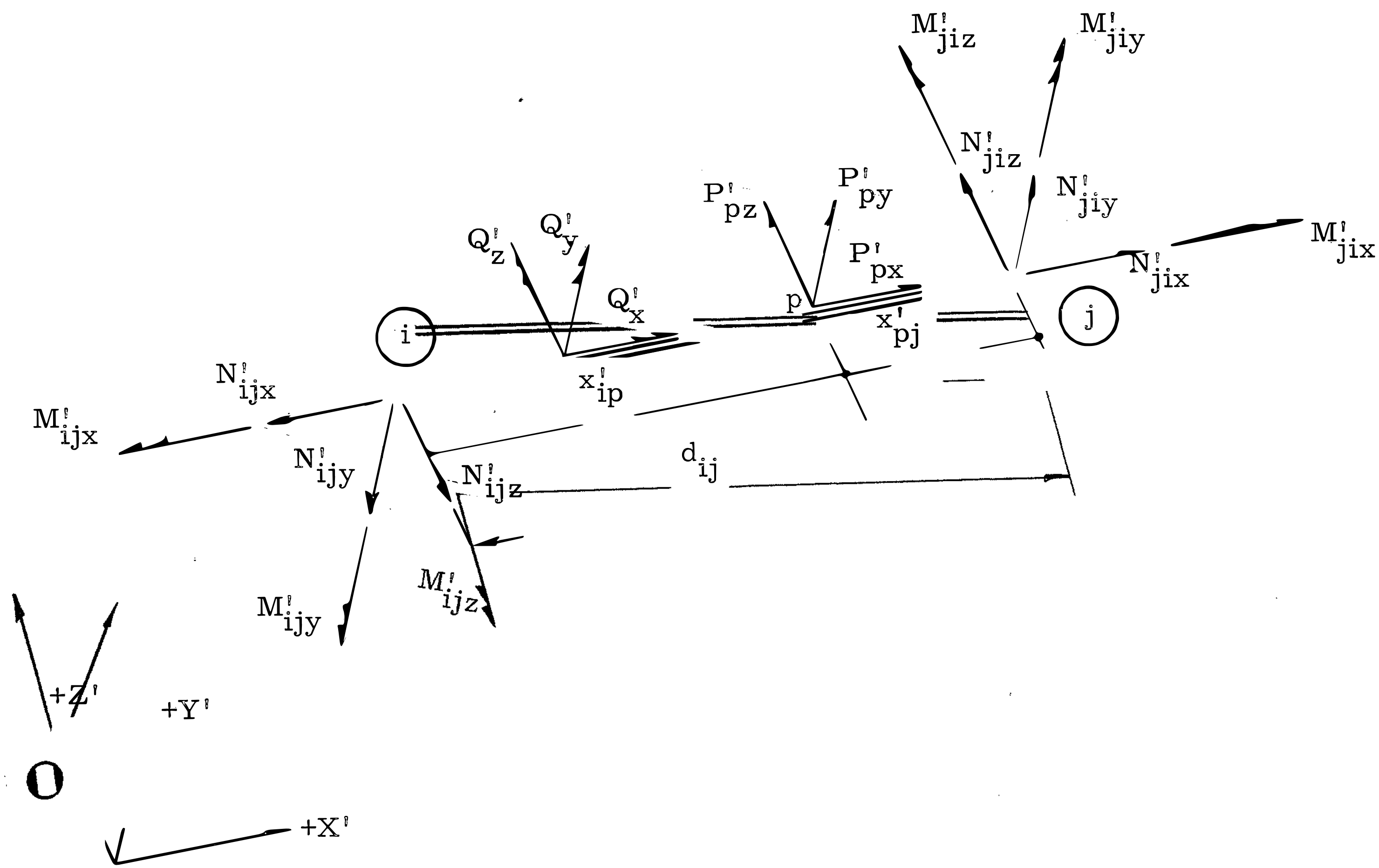


Figure 3-1

End Moments and Forces for Bar ij

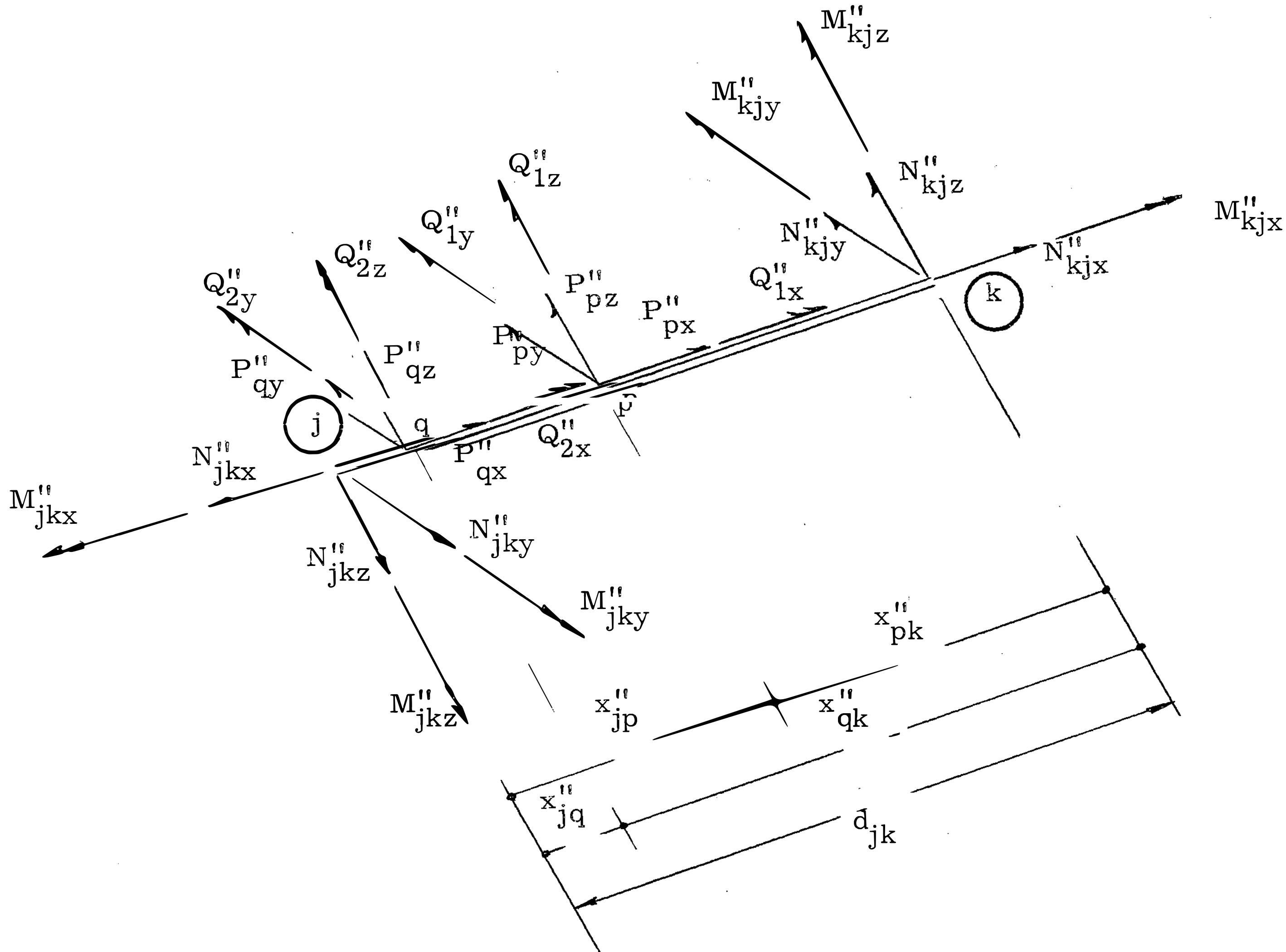


Figure 3-2

End Cross-Sectional Elements for Member jk along the Initial Axes

$$\begin{bmatrix} \Sigma N'_x \\ \Sigma M'_{ix} \\ \Sigma M'_{iy} \\ \Sigma M'_{iz} \\ \Sigma M'_{jy} \\ \Sigma M'_{jz} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{3-1}$$

Equation 3-1 expresses, in terms of nomenclature introduced in Table 3-1a after a rearrangement of the terms, that

$$\left[r^{ij}_{DD}\right]\left[D^{ij}\right] = \left[r^{ij}_{DS}\right]\left[S^{ij}\right] + \left[r^{ij}_{DW}\right]\left[W^{ij}\right] . \tag{3-2}$$

If more than one system of loads is applied, the principle of superposition must be used.

The same equation for the second span is obtained by cyclosymmetric substitution (Table 3-1b).

3-2. Cross-Sectional Elements

A cross-section m of bar ij is considered (Fig. 3-3). Cross-sectional elements are shown along the initial system. The six cross-sectional elements at m are functions of the end-conditioning elements of ij; they are related through equations of stereo-statics. Cross-sectional forces at m are

TABLE 3-1a | STEREO-STATICS | MEMBER ij

$$\left[S^{ij}\right] = \begin{bmatrix} N'_{jix} \\ M'_{jix} \\ -M'_{ijy} \\ M'_{jiy} \\ M'_{ijz} \\ -M'_{jiz} \end{bmatrix}, \qquad \left[D^{ij}\right] = \begin{bmatrix} N'_{ijx} \\ M'_{ijx} \\ N'_{ijz} \\ N'_{jiz} \\ N'_{ijy} \\ N'_{jiy} \end{bmatrix}, \qquad \left[W^{ij}\right] = \begin{bmatrix} P'_{px} \\ Q'_{x} \\ P'_{pz} \\ Q'_{y} \\ P'_{py} \\ -Q'_{z} \end{bmatrix}$$

$$\left[r^{ij}_{Ds}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d_{ij}} & \frac{1}{d_{ij}} & 0 & 0 \\ 0 & 0 & \frac{1}{d_{ij}} & \frac{1}{d_{ij}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{d_{ij}} & \frac{1}{d_{ij}} \\ 0 & 0 & 0 & 0 & \frac{1}{d_{ij}} & \frac{1}{d_{ij}} \end{bmatrix}, \qquad \left[r^{ij}_{DW}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{x'_{pj}}{d_{ij}} & \frac{1}{d_{ij}} & 0 & 0 \\ 0 & 0 & -\frac{x'_{pi}}{d_{ij}} & \frac{1}{d_{ij}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{x'_{pj}}{d_{ij}} & \frac{1}{d_{ij}} \\ 0 & 0 & 0 & 0 & -\frac{x'_{pi}}{d_{ij}} & \frac{1}{d_{ij}} \end{bmatrix}, \qquad \left[r^{ij}_{DD}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

TABLE 3-1b — STEREO-STATICS — MEMBER jk

$$\left[S^{jk}\right] = \begin{bmatrix} N'_{kjx} \\ M''_{kjx} \\ \hline -M''_{jky} \\ M''_{kjy} \\ \hline M''_{jkz} \\ -M''_{kjz} \end{bmatrix}, \quad \left[D^{jk}\right] = \begin{bmatrix} N''_{jkx} \\ M''_{jkx} \\ \hline N''_{jkz} \\ N''_{kjz} \\ \hline N''_{jky} \\ N''_{kjy} \end{bmatrix}, \quad \left[W_1^{jk}\right] = \begin{bmatrix} P''_{px} \\ Q''_{1x} \\ \hline P''_{pz} \\ Q''_{1y} \\ \hline P''_{py} \\ -Q''_{1z} \end{bmatrix}, \quad \left[W_2^{jk}\right] = \begin{bmatrix} P''_{qx} \\ Q''_{2x} \\ \hline P''_{qz} \\ Q''_{2y} \\ \hline P''_{qy} \\ -Q''_{2z} \end{bmatrix}$$

$$\left[r_{DS}^{jk}\right] = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{d_{jk}} & \frac{1}{d_{jk}} & 0 & 0 \\ 0 & 0 & \frac{1}{d_{jk}} & \frac{1}{d_{jk}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{1}{d_{jk}} & \frac{1}{d_{jk}} \\ 0 & 0 & 0 & 0 & \frac{1}{d_{jk}} & \frac{1}{d_{jk}} \end{array}\right], \quad \left[r_{DW1}^{jk}\right] = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{x''_{pk}}{d_{jk}} & \frac{1}{d_{jk}} & 0 & 0 \\ 0 & 0 & \frac{-x''_{pj}}{d_{jk}} & \frac{1}{d_{jk}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{x''_{pk}}{d_{jk}} & \frac{+1}{d_{jk}} \\ 0 & 0 & 0 & 0 & \frac{-x''_{pj}}{d_{jk}} & \frac{+1}{d_{jk}} \end{array}\right], \quad \left[r_{DW2}^{jk}\right] = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{x''_{qk}}{d_{jk}} & \frac{1}{d_{jk}} & 0 & 0 \\ 0 & 0 & \frac{-x''_{qj}}{d_{jk}} & \frac{1}{d_{jk}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{x''_{qk}}{d_{jk}} & \frac{+1}{d_{jk}} \\ 0 & 0 & 0 & 0 & \frac{-x''_{qj}}{d_{jk}} & \frac{+1}{d_{jk}} \end{array}\right]$$

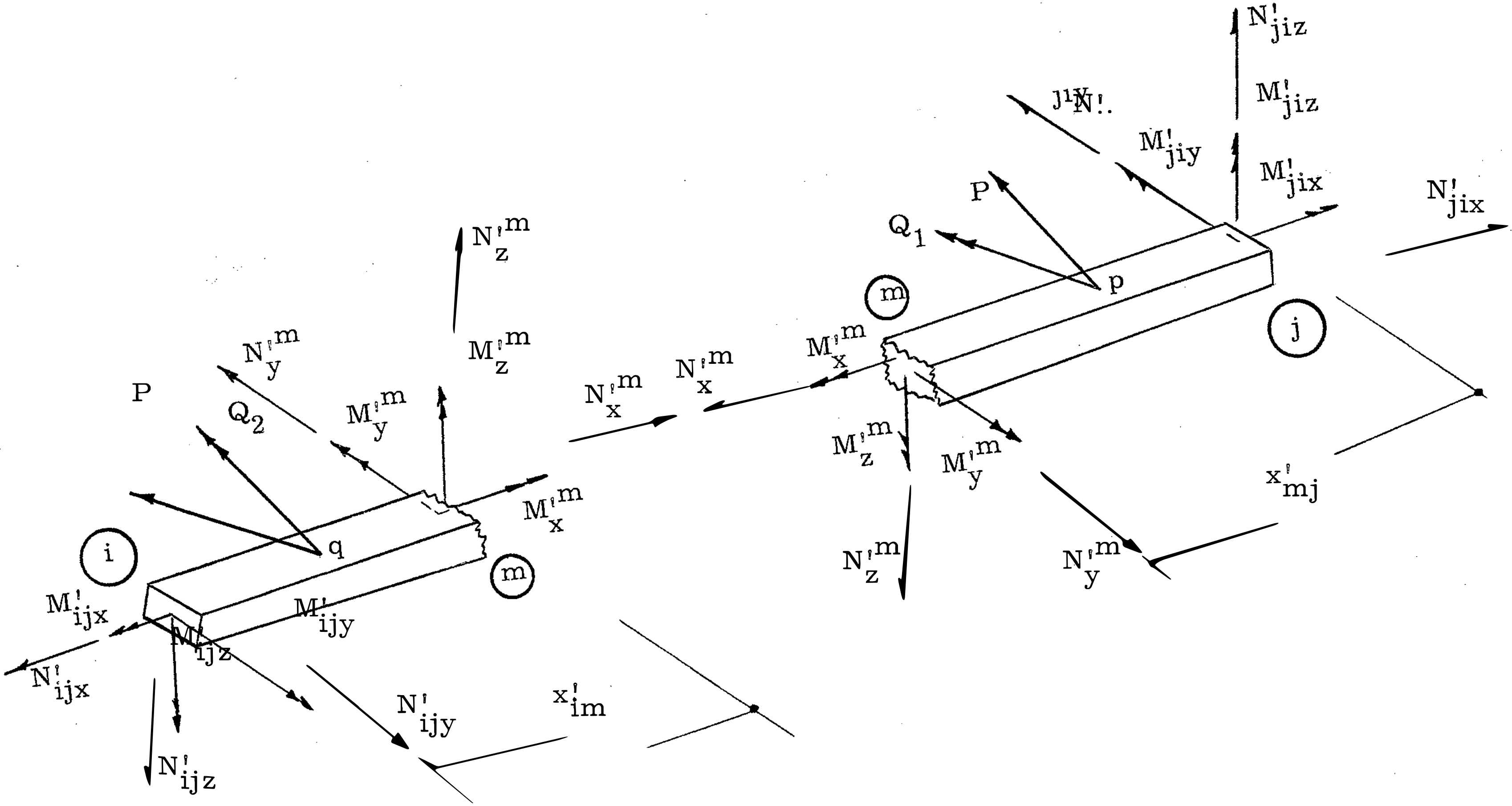


Figure 3-3

Cross-Sectional Elements, Member ij

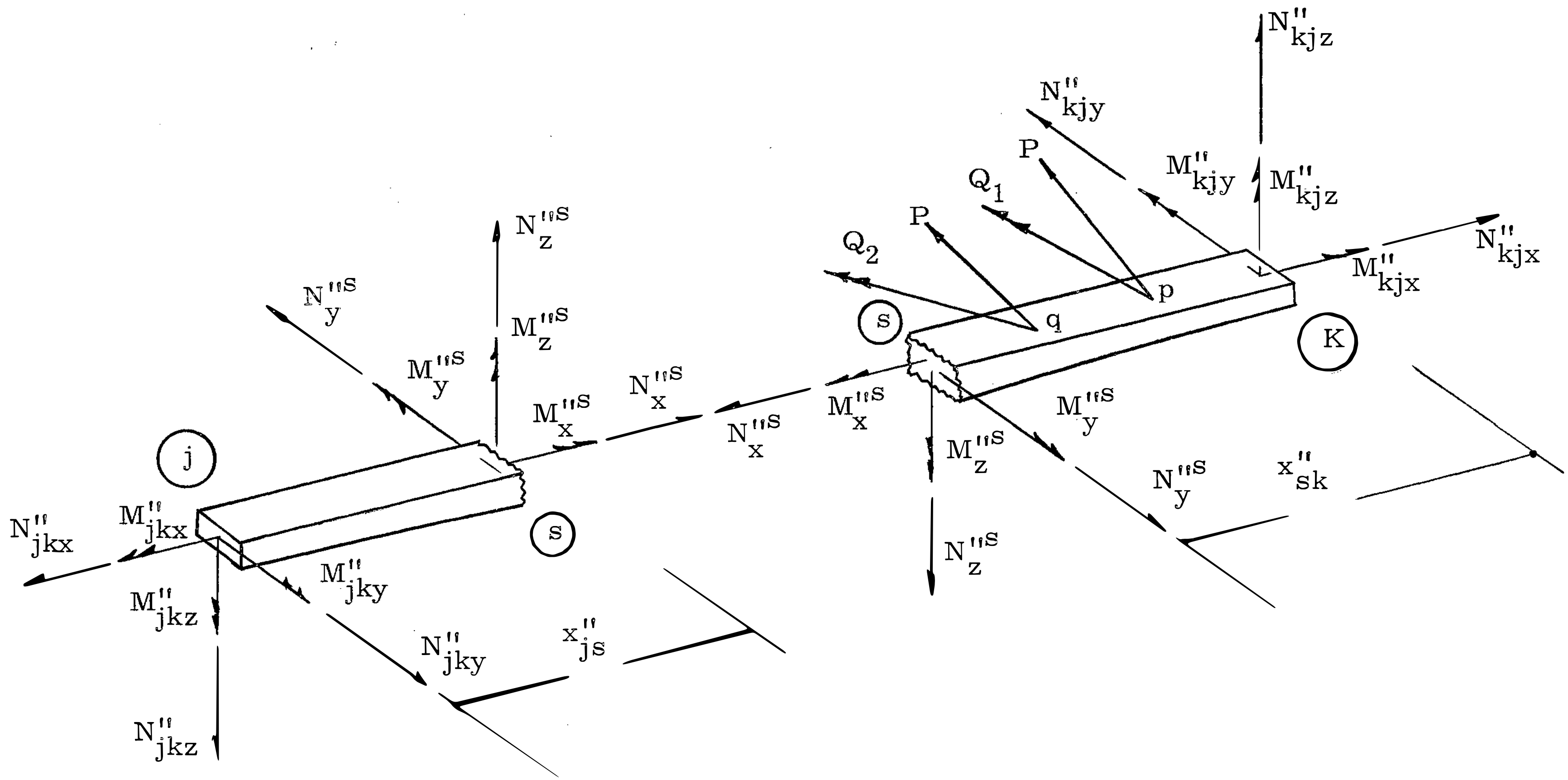


Figure 3-4

Cross-Sectional Elements, Member jk

$$\begin{bmatrix} N'^{m}_{x} \\ N'^{m}_{y} \\ N'^{m}_{z} \end{bmatrix} = \begin{bmatrix} N'_{jix} \\ N'_{jiy} \\ N'_{jiz} \end{bmatrix} + \begin{bmatrix} P'_{px} \\ P'_{py} \\ P'_{pz} \end{bmatrix} = \begin{bmatrix} N'_{ijx} \\ N'_{ijy} \\ N'_{ijz} \end{bmatrix} + \begin{bmatrix} P'_{qx} \\ P'_{qx} \\ P'_{qz} \end{bmatrix} , \tag{3-3}$$

and the cross-sectional moments at m in terms of nomenclature introduced in Table 3-2a are

$$\begin{bmatrix} M'^{m} \end{bmatrix} = \begin{bmatrix} U^{m} \end{bmatrix}\begin{bmatrix} M'_{ji} \end{bmatrix} + \begin{bmatrix} V^{m} \end{bmatrix}\begin{bmatrix} M'_{ij} \end{bmatrix} + \begin{bmatrix} BM'^{m} \end{bmatrix} . \tag{3-4}$$

By the use of transformation matrices previously outlined (Art. 2-1),

$$\begin{bmatrix} M'^{m} \end{bmatrix} = \begin{bmatrix} U^{m} \end{bmatrix}\begin{bmatrix} \pi^{ij}_{1o} \end{bmatrix}\begin{bmatrix} M^{o}_{ji} \end{bmatrix} + \begin{bmatrix} V^{m} \end{bmatrix}\begin{bmatrix} \pi^{ij}_{1o} \end{bmatrix}\begin{bmatrix} M^{o}_{ij} \end{bmatrix} + \begin{bmatrix} BM'^{m} \end{bmatrix} . \tag{3-5}$$

Moments on section s of member jk (Fig. 3-4),

$$\begin{bmatrix} M''^{s} \end{bmatrix} = \begin{bmatrix} U^{s} \end{bmatrix}\begin{bmatrix} M''_{kj} \end{bmatrix} + \begin{bmatrix} V^{s} \end{bmatrix}\begin{bmatrix} M''_{jk} \end{bmatrix} + \begin{bmatrix} BM''^{s} \end{bmatrix} , \tag{3-6}$$

and matrices appearing in Equation 3-6 are explained by Table 3-2b.

3-3. Number of Unknowns

The total number of unknowns is 12n, and they are calculated by means of three types of equations:

a -	Member equilibrium equations	-	6n
b -	Joint equilibrium equations	-	3(n + 1)
c -	Deformation equations	-	3(n - 1) .

TABLE 3-2a SECTION ELEMENTS MEMBER ij

$$\left[M'^m\right] = \begin{bmatrix} M'^m_x \\ M'^m_y \\ M'^m_z \end{bmatrix}, \quad \left[M'_{ji}\right] = \begin{bmatrix} M'_{jix} \\ M'_{jiy} \\ M'_{jiz} \end{bmatrix}, \quad \left[M'_{ij}\right] = \begin{bmatrix} M'_{ijx} \\ M'_{ijy} \\ M'_{ijz} \end{bmatrix}$$

$$\left[U^m\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{x'_{im}}{d_{ij}} & 0 \\ 0 & 0 & \frac{x'_{im}}{d_{ij}} \end{bmatrix}, \quad \left[V^m\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{x'_{mj}}{d_{ij}} & 0 \\ 0 & 0 & \frac{x'_{mj}}{d_{ij}} \end{bmatrix}$$

$$\left[BM'^m\right] = \begin{bmatrix} Q'_{1x} \\ Q'_{1y} \\ Q'_{1z} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x'_{mp} \\ 0 & -x'_{mj} & 0 \end{bmatrix} \begin{bmatrix} P'_{px} \\ P'_{py} \\ P'_{pz} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x'_{mj} \\ 0 & -x'_{mj} & 0 \end{bmatrix} \begin{bmatrix} N'_{jix} \\ N'_{jiy} \\ N'_{jiz} \end{bmatrix} .$$

TABLE 3-2b SECTION ELEMENTS MEMBER jk

$$\left[M''^{s} \right] = \begin{bmatrix} M''^{s}_{x} \\ M''^{s}_{y} \\ M''^{s}_{z} \end{bmatrix}, \quad \left[M''_{kj} \right] = \begin{bmatrix} M''_{kjx} \\ M''_{kjy} \\ M''_{kjz} \end{bmatrix}, \quad \left[M''_{kj} \right] = \begin{bmatrix} M''_{jkx} \\ M''_{jky} \\ M''_{jkz} \end{bmatrix}$$

$$\left[U^{s} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{x''_{js}}{d_{jk}} & 0 \\ 0 & 0 & \frac{x''_{js}}{d_{jk}} \end{bmatrix} \quad \left[V^{s} \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{x''_{sk}}{d_{jk}} & 0 \\ 0 & 0 & \frac{x''_{sk}}{d_{jk}} \end{bmatrix}$$

$$\left[BM''^{s} \right] \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x''_{sq} \\ 0 & -x''_{sq} & 0 \end{bmatrix} \begin{bmatrix} P''_{qz} \\ P''_{qy} \\ P''_{qz} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x''_{sk} \\ 0 & -x''_{sk} & 0 \end{bmatrix} \begin{bmatrix} N''_{kjx} \\ N''_{kjy} \\ N''_{kjz} \end{bmatrix}$$

$$+ \begin{bmatrix} Q''_{1x} \\ Q''_{1y} \\ Q''_{1z} \end{bmatrix} + \begin{bmatrix} Q''_{2x} \\ Q''_{2y} \\ Q''_{2z} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x''_{sp} \\ 0 & -x''_{sp} & 0 \end{bmatrix} \begin{bmatrix} P''_{px} \\ P''_{py} \\ P''_{pz} \end{bmatrix} .$$

It can be observed that the total sum of these equations is 12n and that there are as many equations as there are unknowns for the continuous beam.

3-4. Joint Equations

In addition to the member equilibrium equations previously discussed (Art. 3-1), there are three equations of moment equilibrium for each joint. If the equilibrium of joint j along three reference axes is considered (Fig. 3-5),

$$\begin{bmatrix} \Sigma M_{jx} \\ \Sigma M_{jy} \\ \Sigma M_{jz} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} . \qquad (3-7)$$

3-5. Selection of Redundants

There are three classes of unknown values left:

a - Forces

b - Bending moments

c - Torsion moments.

Forces are calculated from Hooke's law of axial deformation; they can be removed because they do not influence the moment equations. The moments can be selected as dependent or independent systems of redundants. Bending moments lie in the plane perpendicular to the torsion. Thus, at an intermediate joint there is only one line for moment which does not influence the torsions in the two members. This line is the intersection of the two respective planes of bending. The selection of the moment along this line as the redundant bending moment and the two torsions in the members as unknown torsional moments makes possible the separation of redundants. Thus, there are three unknown moments at each intermediate joint, two torsional and one bending (Fig. 3-6). Unknown moments at joint j are arranged in the form

$$\left[M'_{jj} \right] = \begin{bmatrix} M'_{jix} \\ M'_{jjz} \\ M'_{jkx} \end{bmatrix} . \tag{3-8}$$

From the previous discussion (Art. 3-1), the end torques of a panel ij are related through stereo-statics

$$M'_{ijx} = M'_{jix} + \Sigma Q'_x , \tag{3-9}$$

where

$\Sigma Q'_x$ = sum of twisting moments applied to the member ij.

The unknown bending moment and one unknown torque at each intermediate joint are selected as redundant moments (Fig. 3-7); for joint j the redundant matrix is written in the form

$$\left[M^R_j \right] = \begin{bmatrix} M'_{ijx} \\ M'_{jjz} \\ M'_{jkx} \end{bmatrix} \tag{3-10}$$

and at k

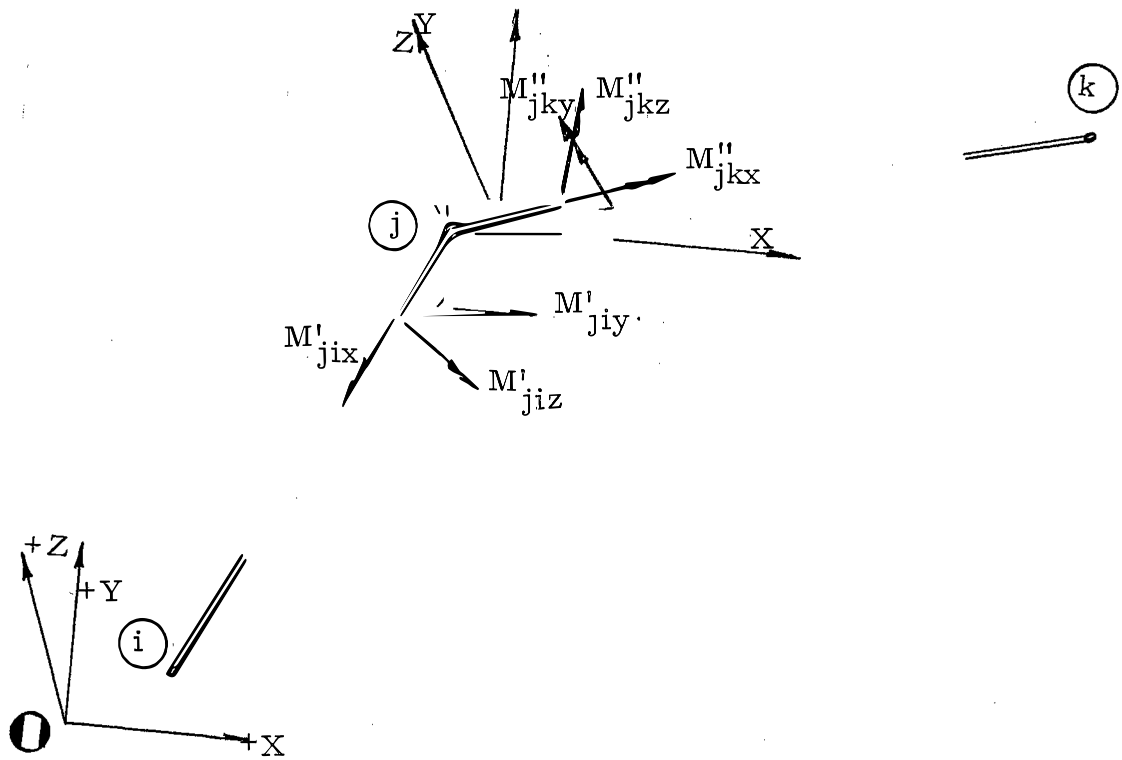


Figure 3-5
Equilibrium of Joint Moments

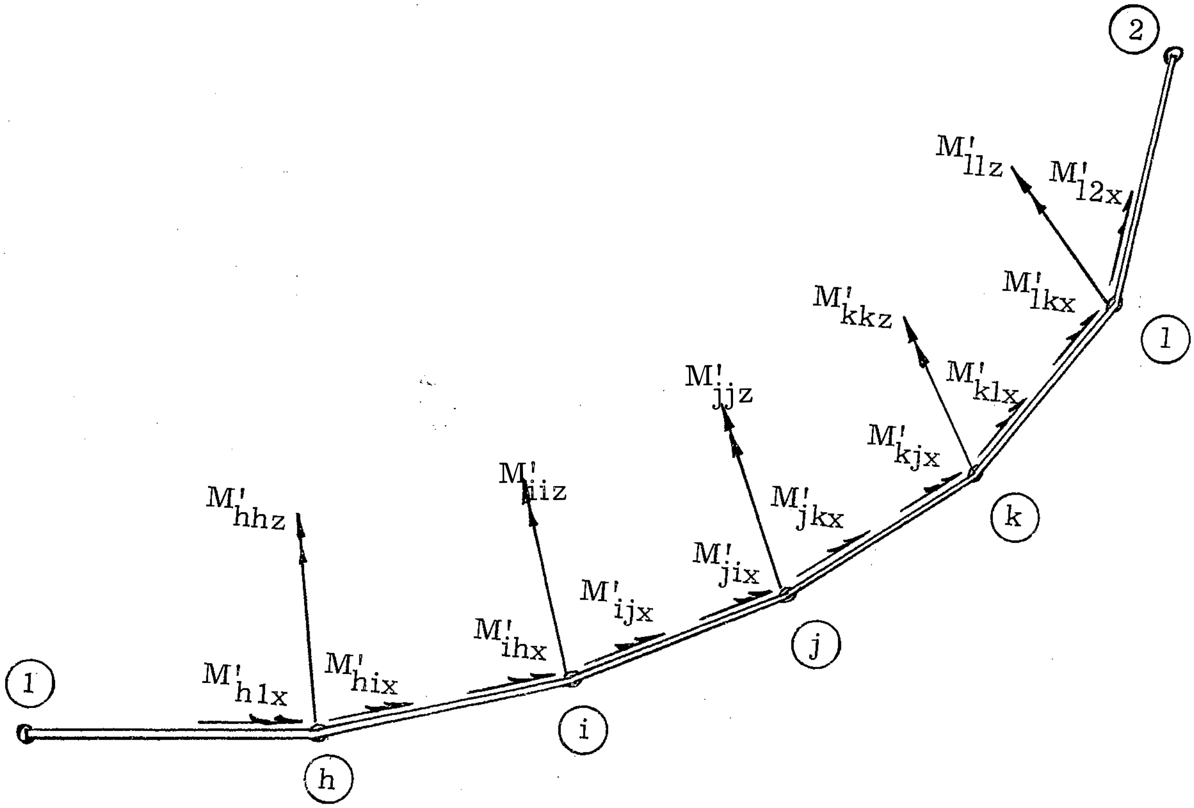


Figure 3-6
Unknown Moments

$$\begin{bmatrix} M_k^R \end{bmatrix} = \begin{bmatrix} M''_{jkx} \\ M''_{kkz} \\ M''_{klx} \end{bmatrix} . \tag{3-11}$$

This shows that there is a redundant bending moment at each intermediate joint and a redundant torsion in each intermediate span. For the whole continuous beam (Fig's. 3-7, 3-8) there are

n	independent redundant forces,
n - 1	independent redundant bending moments at intermediate hinges,
n - 2	independent redundant torsional moments in intermediate members.

3-6. Geometry of Redundants

It is important to relate redundant moments to the reference system; jointal and redundant transformation matrices are introduced in this study as powerful means for the analysis of moments and deformations. A system related to each joint of the continuous beam is termed the jointal system. It includes the longitudinal axes of the members and the normal to their plane. X'_{ij}, X'_{jk} and X'_{jj} constitute the jointal system at j (Fig. 3-9). Two of the three directions are known from previous discussion (Art. 2-1); the third is normal to them. The direction vectors of X'_{ij}, X'_{jk} are

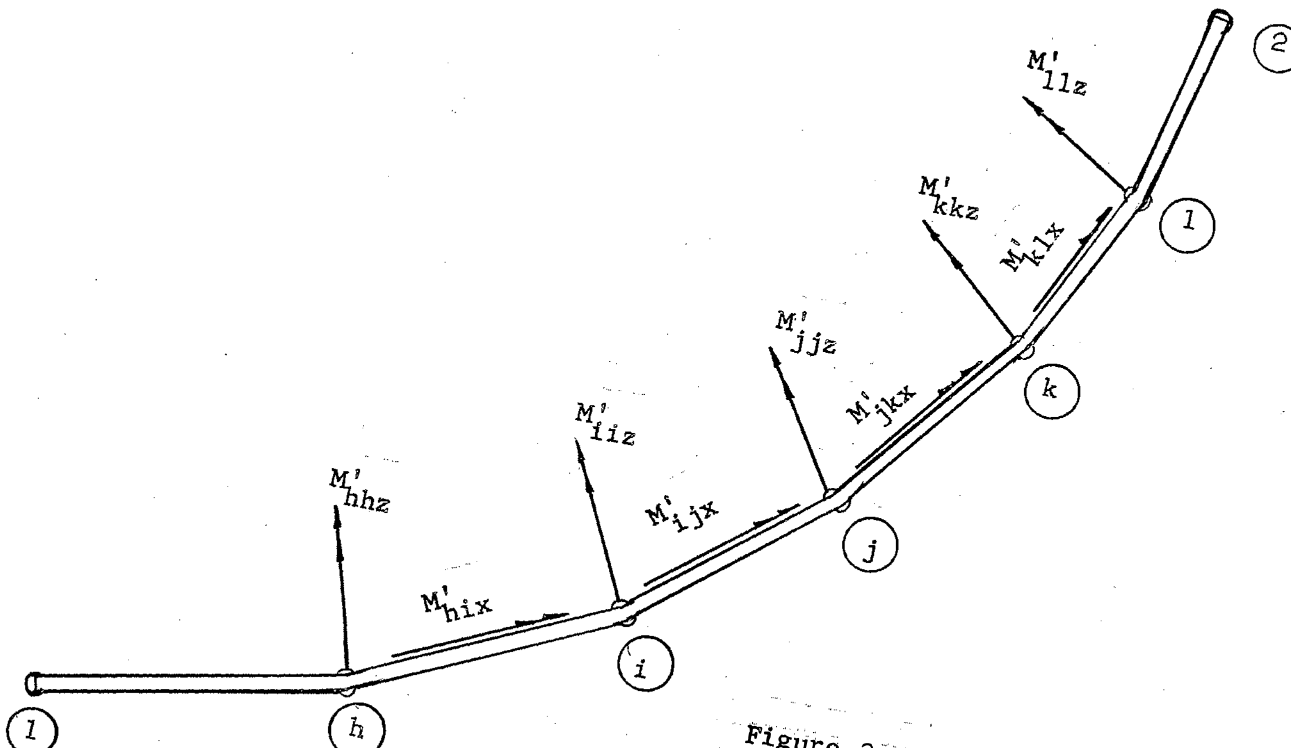


Figure 3 - 7
Redundant Moments

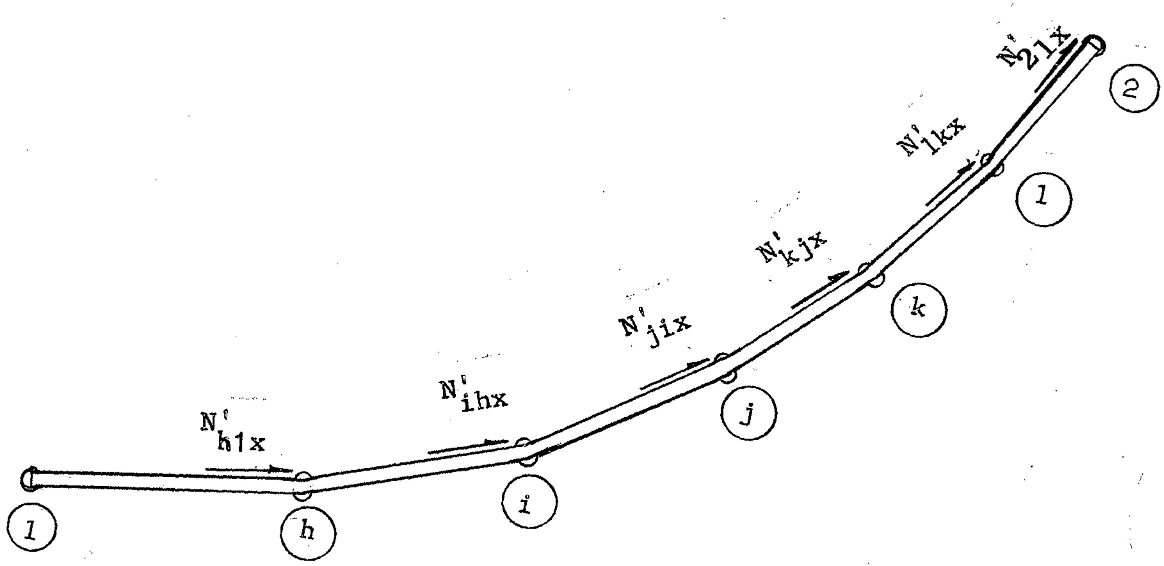


Figure 3 - 8
Redundant Forces

$$\left[\pi_{o1x}^{ij}\right] = \begin{bmatrix} \alpha_x^{ij} \\ \beta_x^{ij} \\ \gamma_x^{ij} \end{bmatrix} \tag{3-12}$$

and

$$\left[\pi_{o1x}^{jk}\right] = \begin{bmatrix} \alpha_x^{jk} \\ \beta_x^{jk} \\ \gamma_x^{jk} \end{bmatrix} . \tag{3-13}$$

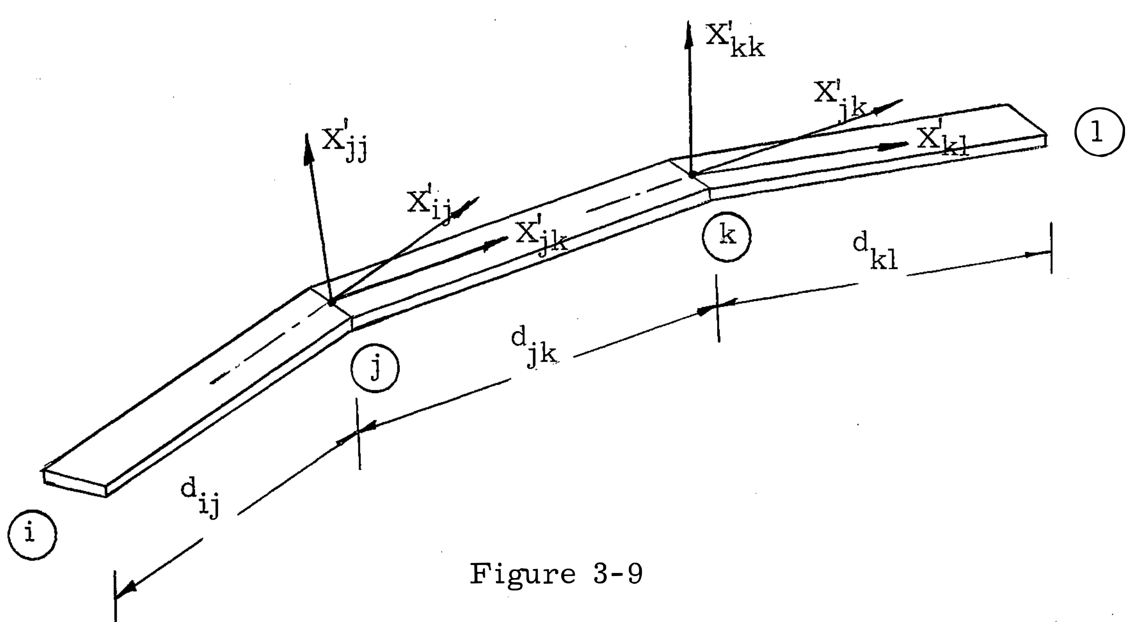


Figure 3-9

Jointal Systems

The direction parameters of X'_{jj} are denoted by α_x^{jj}, β_x^{jj}, γ_x^{jj} ; they are related through the equations

$$\alpha_x^{ij}\,\alpha_x^{jj} + \beta_x^{ij}\,\beta_x^{jj} + \gamma_x^{ij}\,\gamma_x^{ij} = 0 \qquad (3\text{-}14a)$$

$$\alpha_x^{jk}\,\alpha_x^{jj} + \beta_x^{jk}\,\beta_x^{jj} + \gamma_x^{jk}\,\gamma_x^{jj} = 0 \qquad (3\text{-}14b)$$

and

$$(\alpha_x^{jj})^2 + (\beta_x^{jj})^2 + (\gamma_x^{jj})^2 = 1 \; . \qquad (3\text{-}14c)$$

Equations 3-14a, 3-14b are written matrically as

$$\begin{bmatrix} \alpha_x^{ij} & \beta_x^{ij} \\ \alpha_x^{jk} & \beta_x^{jk} \end{bmatrix} \begin{bmatrix} \alpha_x^{jj} \\ \beta_x^{jj} \end{bmatrix} = - \begin{bmatrix} \gamma_x^{ij} \\ \gamma_x^{jk} \end{bmatrix} \gamma_x^{jj} \; . \qquad (3\text{-}15)$$

It should be noted that the left coefficient matrix must be non-singular. If the $\left[\alpha,\ \beta\right]$ matrix is found to be singular, the $\left[\alpha, \gamma\right]$ or $\left[\beta,\ \gamma\right]$ matrix would be used instead. Assigning a value of -1 for γ_x^{jj}, Equation 3-15 becomes

$$\begin{bmatrix} \alpha_x^{ij} & \beta_x^{ij} \\ \alpha_x^{jk} & \beta_x^{jk} \end{bmatrix} \begin{bmatrix} \alpha_x'^{jj} \\ \beta_x'^{jj} \end{bmatrix} = \begin{bmatrix} \gamma_x^{ij} \\ \gamma_x^{jk} \end{bmatrix} \qquad (3\text{-}16)$$

and

$$\begin{bmatrix} \alpha_x'^{jj} \\ \beta_x'^{jj} \end{bmatrix} = \begin{bmatrix} \alpha_x^{ij} & \beta_x^{ij} \\ \alpha_x^{jk} & \beta_x^{jk} \end{bmatrix}^{-1} \begin{bmatrix} \gamma_x^{ij} \\ \gamma_x^{jk} \end{bmatrix} . \qquad (3\text{-}17)$$

Thus, the direction parameters of X'_{jj} are

$$\alpha_x^{jj} = \frac{\alpha'^{jj}_x}{\sqrt{(\alpha'^{jj}_x)^2 + (\beta'^{jj}_x)^2 + (-1)^2}} , \tag{3-18a}$$

$$\beta_x^{jj} = \frac{\beta'^{jj}_x}{\sqrt{(\alpha'^{jj}_x)^2 + (\beta'^{jj}_x)^2 + (-1)^2}} , \tag{3-18b}$$

$$\gamma_x^{jj} = \frac{-1}{\sqrt{(\alpha'^{jj}_x)^2 + (\beta'^{jj}_x)^2 + (-1)^2}} , \tag{3-18c}$$

and jointal transformation matrix at j is

$$\left[\rho_{oj}\right] = \begin{bmatrix} \alpha_x^{ij} & \alpha_x^{jj} & \alpha_x^{jk} \\ \beta_x^{ij} & \beta_x^{jj} & \beta_x^{jk} \\ \gamma_x^{ij} & \gamma_x^{jj} & \gamma_x^{jk} \end{bmatrix} . \tag{3-19}$$

If the two members of the joint are perpendicular, the jointal matrix becomes orthogonal.

The inverse of the jointal matrix is also very important. Both matrices are used extensively in this study. The inverse is

$$\left[\rho_{jo}\right] = \left[\rho_{oj}\right]^{-1} . \tag{3-20}$$

From the geometry of the normal vector and the theory of matrices,

$$\left[\rho_{jo\beta}\right] = \left[\rho_{ojy}\right] . \tag{3-21}$$

Matrices involved in Equation 3-20 and their submatrices are explained in Table 3-3a.

TABLE 3-3a JOINTAL MATRICES JOINT j

$$\left[\rho_{ojx}\right] = \begin{bmatrix} \alpha_x^{ij} \\ \beta_x^{ij} \\ \gamma_x^{ij} \end{bmatrix}, \quad \left[\rho_{ojy}\right] = \begin{bmatrix} \alpha_x^{jj} \\ \beta_x^{jj} \\ \gamma_x^{jj} \end{bmatrix}, \quad \left[\rho_{ojz}\right] = \begin{bmatrix} \alpha_x^{jk} \\ \beta_x^{jk} \\ \gamma_x^{jk} \end{bmatrix}$$

$$\left[\rho_{oj\alpha}\right] = \begin{bmatrix} \alpha_x^{ij} & \alpha_x^{jj} & \alpha_x^{jk} \end{bmatrix}, \qquad \left[\rho_{ojx}\right] = \left[\pi_{o1x}^{ij}\right]$$

$$\left[\rho_{oj\beta}\right] = \begin{bmatrix} \beta_x^{ij} & \beta_x^{jj} & \beta_x^{jk} \end{bmatrix}, \qquad \left[\rho_{ojy}\right] = \left[\pi_{o1x}^{jj}\right]$$

$$\left[\rho_{oj\gamma}\right] = \begin{bmatrix} \gamma_x^{ij} & \gamma_x^{jj} & \gamma_x^{jk} \end{bmatrix}, \qquad \left[\rho_{ojz}\right] = \left[\pi_{o1x}^{jk}\right]$$

$$\left[\rho_{jox}\right] = \begin{bmatrix} \rho_{j\alpha x} \\ \rho_{j\beta x} \\ \rho_{j\gamma x} \end{bmatrix}, \quad \left[\rho_{joy}\right] = \begin{bmatrix} \rho_{j\alpha y} \\ \rho_{j\beta y} \\ \rho_{j\gamma y} \end{bmatrix}, \quad \left[\rho_{joz}\right] = \begin{bmatrix} \rho_{j\alpha z} \\ \rho_{j\beta z} \\ \rho_{j\gamma z} \end{bmatrix}$$

$$\left[\rho_{jo\alpha}\right] = \begin{bmatrix} \rho_{j\alpha x} & \rho_{j\alpha y} & \rho_{j\alpha z} \end{bmatrix}$$

$$\left[\rho_{jo\beta}\right] = \begin{bmatrix} \rho_{j\beta x} & \rho_{j\beta y} & \rho_{j\beta z} \end{bmatrix}, \quad \left[\rho_{jo}\right] = \begin{bmatrix} \rho_{j\alpha x} & \rho_{j\alpha y} & \rho_{j\alpha z} \\ \alpha_x^{jj} & \beta_x^{jj} & \gamma_x^{jj} \\ \rho_{j\gamma x} & \rho_{j\gamma y} & \rho_{j\gamma z} \end{bmatrix}$$

$$\left[\rho_{jo\gamma}\right] = \begin{bmatrix} \rho_{j\gamma x} & \rho_{j\gamma y} & \rho_{j\gamma z} \end{bmatrix}$$

Unknown moments at j related to reference components of the joint moments in terms of nomenclature introduced in Table 3-3b are

$$\left[M'_{jj}\right] = \left[\Gamma_{jo}\right]\left[M^{o}_{j}\right] \tag{3-22}$$

and

$$\left[M^{o}_{j}\right] = \left[\Gamma_{oj}\right]\left[M'_{jj}\right] . \tag{3-23}$$

Substituting Equations 3-9 and 3-10 into Equation 3-23,

$$\left[M^{o}_{j}\right] = \left[\Gamma_{oj}\right]\left[M^{R}_{j}\right] + \left[\Gamma_{ojx}\right] \Sigma Q'_{x} . \tag{3-24}$$

From the geometry of the normal vector,

$$\left[\Gamma_{oj}\right] = \left[\rho_{jo}\right]' . \tag{3-25}$$

TABLE 3-3b REDUNDANT MATRICES JOINT j

$$\begin{bmatrix} M'_{jj} \end{bmatrix} = \begin{bmatrix} M'_{jix} \\ M'_{jjz} \\ M'_{jkx} \end{bmatrix} \qquad \begin{bmatrix} \Gamma_{jo} \end{bmatrix} = \begin{bmatrix} \alpha_x^{ij} & \beta_x^{ij} & \gamma_x^{ij} \\ \alpha_x^{jj} & \beta_x^{jj} & \gamma_x^{jj} \\ \alpha_x^{jk} & \beta_x^{jk} & \gamma_x^{jk} \end{bmatrix}$$

$$\begin{bmatrix} \Gamma_{jox} \end{bmatrix} = \begin{bmatrix} \alpha_x^{ij} \\ \alpha_x^{jj} \\ \alpha_x^{jk} \end{bmatrix} \qquad \begin{bmatrix} \Gamma_{joy} \end{bmatrix} = \begin{bmatrix} \beta_x^{ij} \\ \beta_x^{jj} \\ \beta_x^{jk} \end{bmatrix} \qquad \begin{bmatrix} \Gamma_{joz} \end{bmatrix} = \begin{bmatrix} \gamma_x^{ij} \\ \gamma_x^{jj} \\ \gamma_x^{jk} \end{bmatrix}$$

$$\begin{bmatrix} \Gamma_{ojx} \end{bmatrix} = \begin{bmatrix} \Gamma_{o\alpha x} \\ \Gamma_{o\beta x} \\ \Gamma_{o\gamma x} \end{bmatrix} \qquad \begin{bmatrix} \Gamma_{ojy} \end{bmatrix} = \begin{bmatrix} \Gamma_{o\alpha y} \\ \Gamma_{o\beta y} \\ \Gamma_{o\gamma y} \end{bmatrix} \qquad \begin{bmatrix} \Gamma_{ojz} \end{bmatrix} = \begin{bmatrix} \Gamma_{o\alpha z} \\ \Gamma_{o\beta z} \\ \Gamma_{o\gamma z} \end{bmatrix}$$

$$\begin{bmatrix} \Gamma_{oj\alpha} \end{bmatrix} = \begin{bmatrix} \Gamma_{o\alpha x} & \Gamma_{o\alpha y} & \Gamma_{o\alpha z} \end{bmatrix} \qquad \begin{bmatrix} \Gamma_{oj\beta} \end{bmatrix} \begin{bmatrix} \Gamma_{joy} \end{bmatrix}'$$

$$\begin{bmatrix} \Gamma_{oj\beta} \end{bmatrix} = \begin{bmatrix} \Gamma_{o\beta x} & \Gamma_{o\beta y} & \Gamma_{o\beta z} \end{bmatrix} \qquad \begin{bmatrix} \Gamma_{oj} \end{bmatrix} \begin{bmatrix} \Gamma_{jo} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} \Gamma_{oj\gamma} \end{bmatrix} = \begin{bmatrix} \Gamma_{o\gamma x} & \Gamma_{o\gamma y} & \Gamma_{o\gamma z} \end{bmatrix} \qquad \begin{bmatrix} \Gamma_{jo} \end{bmatrix} \begin{bmatrix} \rho_{oj} \end{bmatrix}'$$

CHAPTER IV

ELASTO-STATICS

The deformation of a space continuous beam is investigated in this chapter. Methods for analyzing the deformations of space structures are

a - Energy Methods

b - Elastic Curve Methods

c - Analogies.

One of the most convenient of the analogies applied to space structures is the method of elastic weights. This method applied to space continuous beams is presented in this chapter. Deformation constants are outlined and the elasto-static equations developed. Each joint of the space beam in this study is considered to have three angular deformation components, and consequently three elastic weights are needed.

4-1. Angular Deformation Constants

A bar ij discussed in the second chapter and represented by Figure 2-5 is investigated for angular deformation constants at each end; they are

(1) Angular and Carry-over Flexibilities

(2) Angular Load Functions.

The angular flexibility is the end slope of the elastic curve of the basic structure at a given point due to unit moment applied at the same point about the same axis. Two types of angular flexibilities must be recognized, the flexure flexibility and the torsion flexibility.

The angular carry-over flexibility is the end slope of the elastic curve at a given point due to unit moment applied at another point about the same axis.

The end slope of the elastic curve due to applied loads only (the member being considered simply supported) is the angular load function at that point.

Angular deformation constants are illustrated in Tables 4-1a, b.

Matrices of Deformation Constants

The angular and carry-over flexibility matrices, referred to the initial system, are diagonal matrices with flexibilities displayed along the diagonal. The matrices for member ij at its end i are

$$\left[F'_{ij}\right] = \begin{bmatrix} F'_{ijx} & 0 & 0 \\ 0 & F'_{ijy} & 0 \\ 0 & 0 & F'_{ijz} \end{bmatrix} ; \qquad (4\text{-}1)$$

$$\left[G'_{ij}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & G'_{ijy} & 0 \\ 0 & 0 & G'_{ijz} \end{bmatrix} , \qquad (4\text{-}2)$$

and at the end j

$$\left[F'_{ji}\right] = \begin{bmatrix} F'_{jix} & 0 & 0 \\ 0 & F'_{jiy} & 0 \\ 0 & 0 & F'_{jiz} \end{bmatrix} , \qquad (4\text{-}3)$$

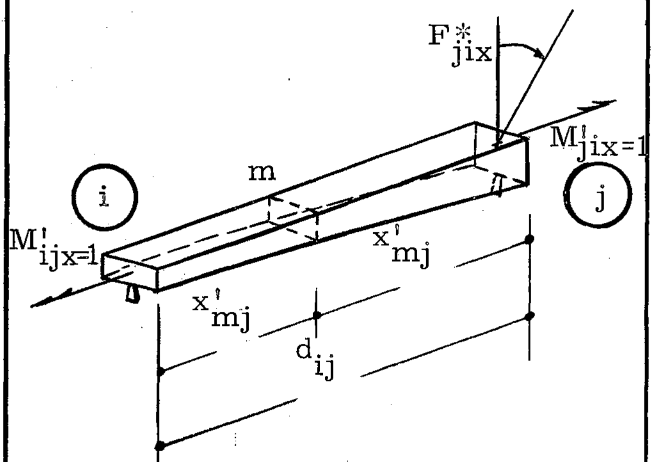
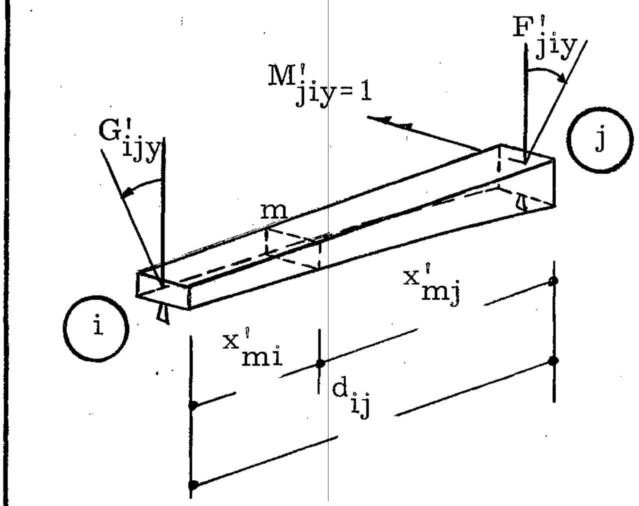
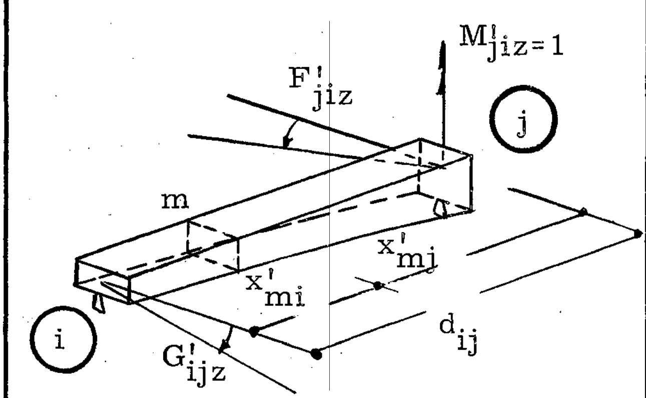
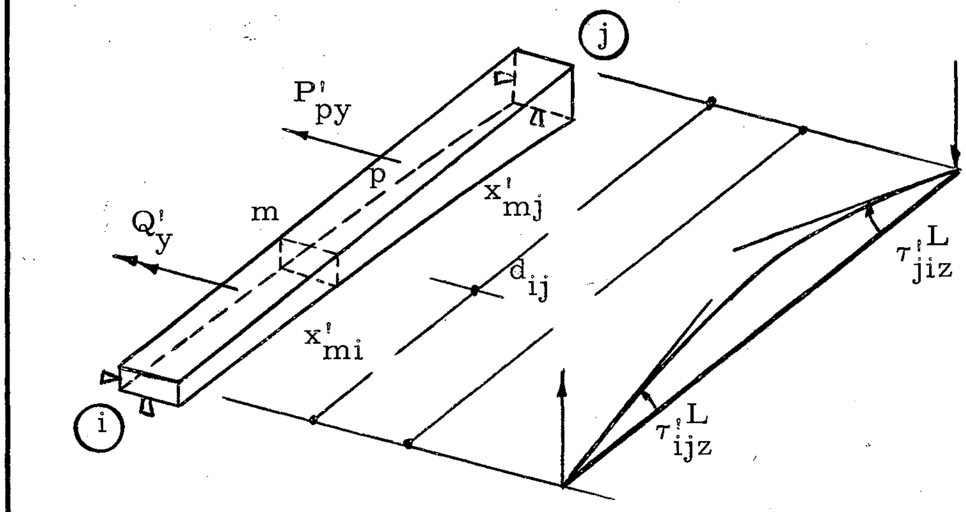
TABLE 4-1a ANGULAR AND CARRY-OVER FLEXIBILITIES	MEMBER ij
	$F^*_{jix} = \int_i^j \frac{dx'}{GJ'_x}$ = angular flexibility at j in the x' direction.
	$F'_{jiy} = \int_i^j \frac{(x'_{im})^2}{(d_{ij})^2} \frac{dx'}{EI'_y}$ = angular flexibility at j in the Y' direction $G'_{ijy} = \int_i^j \frac{(x'_{im})(x'_{mj})}{(d_{ij})^2} \frac{dx'}{EI'_y}$ = angular carry-over flexibility at i in the Y' direction
	$F'_{jiz} = \int_i^j \frac{(x'_{im})^2}{(d_{ij})^2} \frac{dx'}{EI'_z}$ = angular flexibility at j in the Z' direction $G'_{ijz} = \int_i^j \frac{(x'_{mj})(x'_{mi})}{(d_{ij})^2} \frac{dx'}{EI'_z}$ = angular carry-over at i in the Z' direction

TABLE 4-1b ANGULAR LOAD FUNCTIONS MEMBER ij

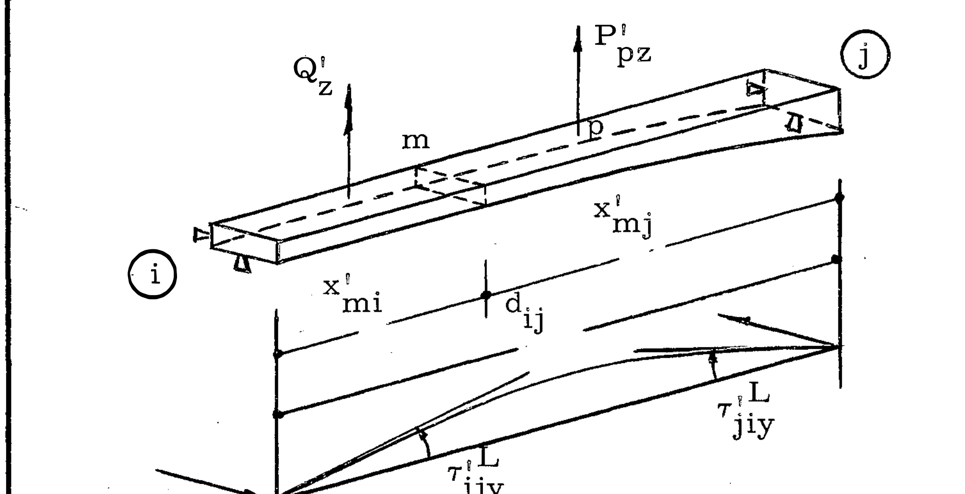


$$\tau'^L_{ijz} = \int_i^j BM'^m_z \; x'_{mj} \; \frac{dx'}{EI'_z}$$

= angular load function at i in the Z' direction

$$\tau'^L_{jiz} = \int_i^j BM'^m_z \; \frac{x'_{im}}{d_{ij}} \; \frac{dx'}{EI'_z}$$

= angular load function at j in the Z' direction



$$\tau'^L_{ijy} = \int_i^j BM'^m_y \; \frac{x'_{mj}}{d_{ij}} \; \frac{dx'}{EI'_y}$$

= angular load function at i in the Y' direction

$$\tau'^L_{jiy} = \int_i^j BM'^m_y \; \frac{x'_{im}}{d_{ij}} \; \frac{dx'}{EI'_y}$$

= angular load function at j in the Y' direction

$$\begin{bmatrix} G'_{ji} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & G'_{jiy} & 0 \\ 0 & 0 & G'_{jiz} \end{bmatrix} \tag{4-4}$$

where

$$F'_{ijx} = \tfrac{1}{2} F^*_{ijx} , \qquad F'_{jix} = \tfrac{1}{2} F^*_{jix} .$$

The matrix of angular load functions is a column matrix, the elements of which are the rotations along the initial axes X', Y', Z', respectively. As an illustration, the matrices of angular load functions at ends i, j of bar ij are

$$\begin{bmatrix} \tau'^{L}_{ij} \end{bmatrix} = \begin{bmatrix} \tau'^{L}_{ijx} \\ \tau'^{L}_{ijy} \\ \tau'^{L}_{ijz} \end{bmatrix} \tag{4-5}$$

and

$$\begin{bmatrix} \tau'^{L}_{ji} \end{bmatrix} = \begin{bmatrix} \tau'^{L}_{jix} \\ \tau'^{L}_{jiy} \\ \tau'^{L}_{jiz} \end{bmatrix} . \tag{4-6}$$

Deformation constants related to the reference system instead of the initial are to be used in many cases. They may be derived by the use of Equation 3-5 following the same procedure outlined before. It is much easier to transform the deformation constants from the initial system to the reference system by the use of the transformation matrices previously discussed (Eq's. 2-10, 11).

The angular and carry-over flexibilities for bar ij, referred to the reference system, are

$$\left[F^{o}_{ij}\right] = \left[\pi^{ij}_{1o}\right]\left[F'_{ij}\right]\left[\pi^{ij}_{o1}\right] , \qquad (4\text{-}7)$$

$$\left[F^{o}_{ji}\right] = \left[\pi^{ij}_{1o}\right]\left[F'_{ji}\right]\left[\pi^{ij}_{o1}\right] , \qquad (4\text{-}8)$$

$$\left[G^{o}_{ij}\right] = \left[\pi^{ij}_{1o}\right]\left[G'_{ij}\right]\left[\pi^{ij}_{o1}\right] , \qquad (4\text{-}9)$$

$$\left[G^{o}_{ji}\right] = \left[\pi^{ij}_{1o}\right]\left[G'_{ji}\right]\left[\pi^{ij}_{o1}\right] . \qquad (4\text{-}10)$$

The reference angular load functions for the same member are

$$\left[\tau^{Lo}_{ij}\right] = \left[\pi^{ij}_{1o}\right]\left[\tau'^{L}_{ij}\right] \qquad (4\text{-}11)$$

$$\left[\tau^{Lo}_{ji}\right] = \left[\pi^{ij}_{1o}\right]\left[\tau'^{L}_{ji}\right] . \qquad (4\text{-}12)$$

It is clear from Equations 4-9 and 4-10, since $\left[G'_{ij}\right] = \left[G'_{ji}\right]$, that

$$\left[G^{o}_{ij}\right] = \left[G^{o}_{ji}\right] . \qquad (4\text{-}13)$$

The matrices of angular and carry-over flexibilities referred to the reference system are symmetrical, in the general form

$$\left[F^{o}_{ij}\right] = \begin{bmatrix} F_{ijxx} & F_{ijxy} & F_{ijxz} \\ F_{ijxy} & F_{ijyy} & F_{ijyz} \\ F_{ijxz} & F_{ijyz} & F_{ijzz} \end{bmatrix} \qquad (4\text{-}14)$$

and

$$\left[G^{o}_{ij}\right] = \begin{bmatrix} G_{ijxx} & G_{ijxy} & G_{ijxz} \\ G_{ijxy} & G_{ijyy} & G_{ijyz} \\ G_{ijxz} & G_{ijyz} & G_{ijzz} \end{bmatrix} . \qquad (4\text{-}15)$$

The matrix for the reference angular load functions is a column matrix,

$$\left[\tau^{Lo}_{ij}\right] = \begin{bmatrix} \tau^{L}_{ijx} \\ \tau^{L}_{ijy} \\ \tau^{L}_{ijz} \end{bmatrix} . \qquad (4\text{-}16)$$

The angular flexibilities (Eq. 4-14) are defined by:

F_{ijxx} = rotation at i in the X direction due to a unit moment $M_{ijx} = 1$

F_{ijxy} = rotation at i in the Y direction due to a unit moment $M_{ijx} = 1$

= rotation at i in the X direction due to a unit moment $M_{ijy} = 1$

F_{ijxz} = rotation at i in the Z direction due to a unit moment $M_{ijx} = 1$

F_{ijzx} = rotation at i in the X direction due to a unit moment $M_{ijz} = 1$

F_{ijyy} = rotation at i in the Y direction due to a unit moment $M_{ijy} = 1$

F_{ijyz} = rotation at i in the Z direction due to a unit moment $M_{ijy} = 1$

= rotation at i in the Y direction due to a unit moment $M_{ijz} = 1$

F_{ijzz} = rotation at i in the Z direction due to a unit moment $M_{ijz} = 1$.

The angular carry-over flexibilities (Eq. 4-15) are defined below:

G_{ijxx} = rotation at j in the X direction due to a unit moment $M_{ijx} = 1$

G_{ijxy} = rotation at j in the Y direction due to a unit moment $M_{ijx} = 1$

= rotation at j in the X direction due to a unit moment $M_{ijy} = 1$

G_{ijxz} = rotation at j in the Z direction due to a unit moment $M_{ijx} = 1$

= rotation at j in the X direction due to a unit moment $M_{ijz} = 1$

G_{ijyz} = rotation at j in the Z direction due to a unit moment $M_{ijy} = 1$

= rotation at j in the Y direction due to a unit moment $M_{ijz} = 1$

G_{ijyy} = rotation at j in the Y direction due to a unit moment $M_{ijy} = 1$

G_{ijyz} = rotation at j in the Z direction due to a unit moment $M_{ijy} = 1$

= rotation at j in the Y direction due to a unit moment $M_{ijz} = 1$

G_{ijzz} = rotation at j in the Z direction due to a unit moment $M_{ijz} = 1$.

4-2. Elastic Weights

An elastic weight is defined as the change in slope $d\phi$ between two points on the elastic curve.

(a) Elemental Elastic Weights - $\bar{p}$

An elemental elastic weight is the change in slope in an element ds of the elastic curve. Elemental elastic weights in the initial system (Fig's. 4-1, 2, 3) are

$$\begin{bmatrix} \bar{p}'_x \\ \bar{p}'_y \\ \bar{p}'_z \end{bmatrix} = \begin{bmatrix} d\phi'_x \\ d\phi'_y \\ d\phi'_z \end{bmatrix} = \begin{bmatrix} \lambda'_x & 0 & 0 \\ 0 & \lambda'_y & 0 \\ 0 & 0 & \lambda'_z \end{bmatrix} \begin{bmatrix} M'_x \\ M'_y \\ M'_z \end{bmatrix}, \qquad (4\text{-}17)$$

and elemental elastic weights along the reference system (Fig's. 4-4, 5, 6) are

$$\begin{bmatrix} \bar{p}_x \\ \bar{p}_y \\ \bar{p}_z \end{bmatrix} = \begin{bmatrix} d\phi_x \\ d\phi_y \\ d\phi_z \end{bmatrix} = \begin{bmatrix} \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \lambda_{xy} & \lambda_{yy} & \lambda_{yz} \\ \lambda_{xz} & \lambda_{yz} & \lambda_{zz} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} \qquad (4\text{-}18)$$

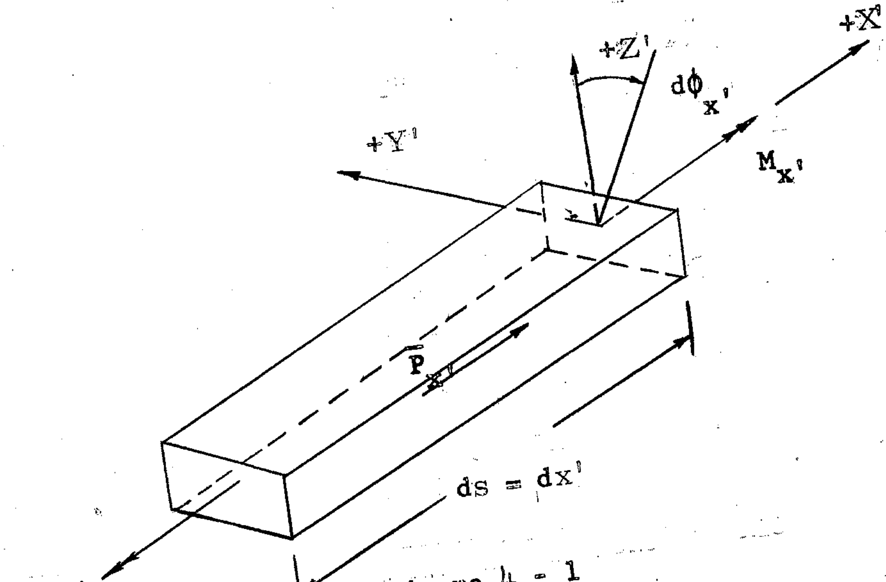


Figure 4 - 1
Elemental Elastic Weight Along X' Axis

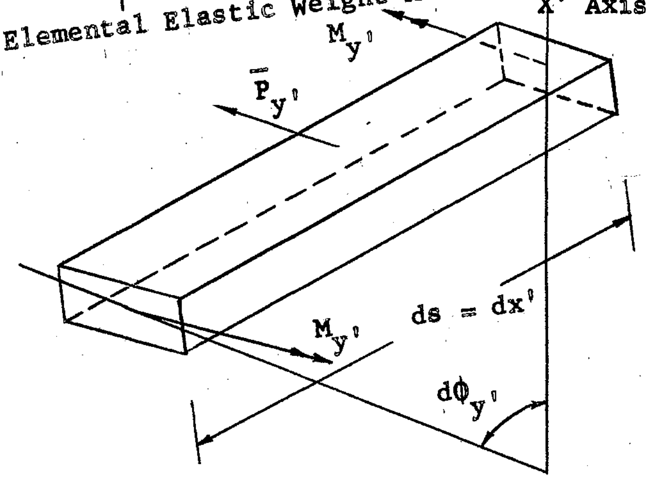


Figure 4 - 2
Elemental Elastic Weight Along Y' Axis

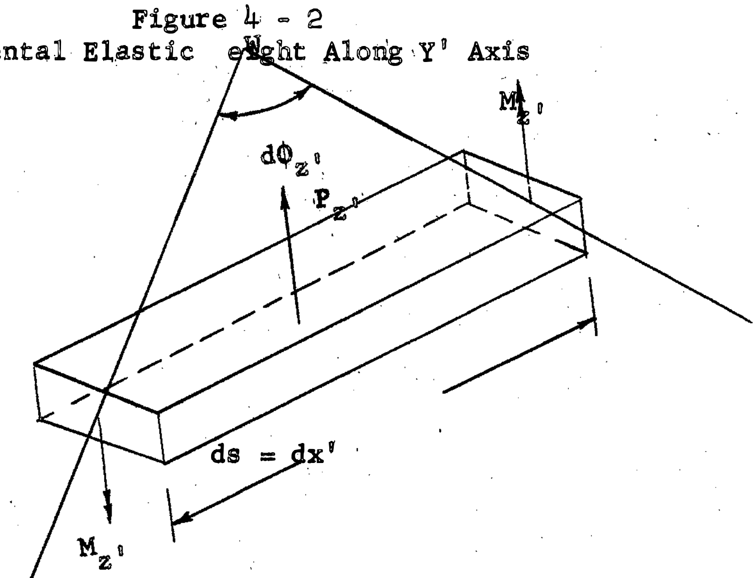


Figure 4 - 3
Elemental Elastic Weight Along Z' Axis

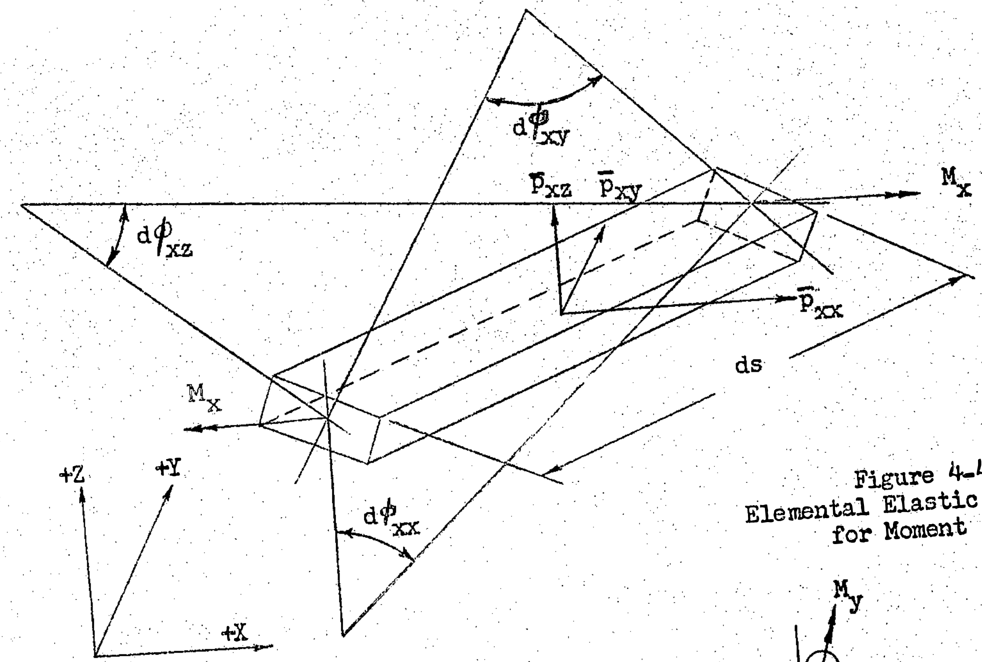


Figure 4-4
Elemental Elastic Weights
for Moment M_x

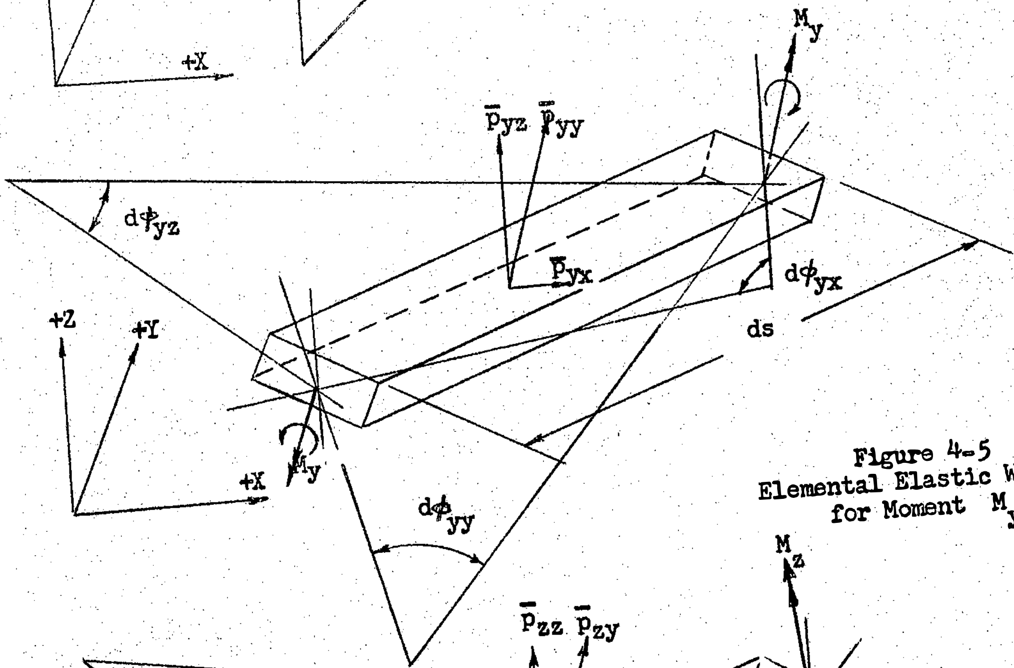


Figure 4-5
Elemental Elastic Weights
for Moment M_y

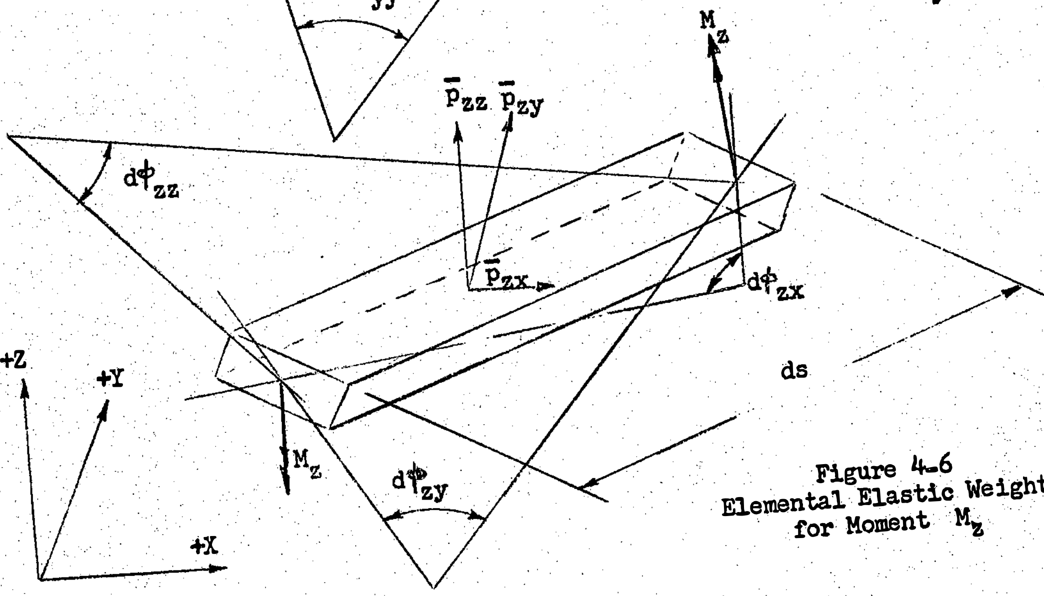


Figure 4-6
Elemental Elastic Weights
for Moment M_z

where

$$\begin{bmatrix} \overline{p}_x \\ \overline{p}_y \\ \overline{p}_z \end{bmatrix} = \begin{bmatrix} \overline{p}_{xx} + \overline{p}_{yx} + \overline{p}_{zx} \\ \overline{p}_{xy} + \overline{p}_{yy} + \overline{p}_{zy} \\ \overline{p}_{xz} + \overline{p}_{yz} + \overline{p}_{zz} \end{bmatrix} , \quad \begin{bmatrix} d\phi_x \\ d\phi_y \\ d\phi_z \end{bmatrix} = \begin{bmatrix} d\phi_{xx} + d\phi_{yx} + d\phi_{zx} \\ d\phi_{xy} + d\phi_{yy} + d\phi_{zy} \\ d\phi_{xz} + d\phi_{yz} + d\phi_{zz} \end{bmatrix} . \tag{4-19}$$

The coefficient matrices are denoted as follows:

$$\left[\lambda'\right] = \begin{bmatrix} \lambda'_x & 0 & 0 \\ 0 & \lambda'_y & 0 \\ 0 & 0 & \lambda'_z \end{bmatrix} , \quad \left[\lambda^o\right] = \begin{bmatrix} \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \lambda_{xy} & \lambda_{yy} & \lambda_{yz} \\ \lambda_{xz} & \lambda_{yz} & \lambda_{zz} \end{bmatrix}$$

and

$$\left[\lambda^o\right] = \left[\pi_{1o}\right]\left[\lambda'\right]\left[\pi_{o1}\right] . \tag{4-20}$$

(b) Panel Elastic Weights - $\overline{P}_{ij}$

The panel elastic weight is the change in slope for the whole panel. A bar ij has three panel elastic weights at each end (Fig. 4-7). Panel elastic weights are designated by $\overline{P}$ with the first subscript indicating the point of application, the second denoting the far end, and the third the direction of the vector. It must be noted that the torsional elastic weight is split in two, each half being applied at an end (Fig. 4-7). The panel elastic weights at the end i of member ij in the initial system are

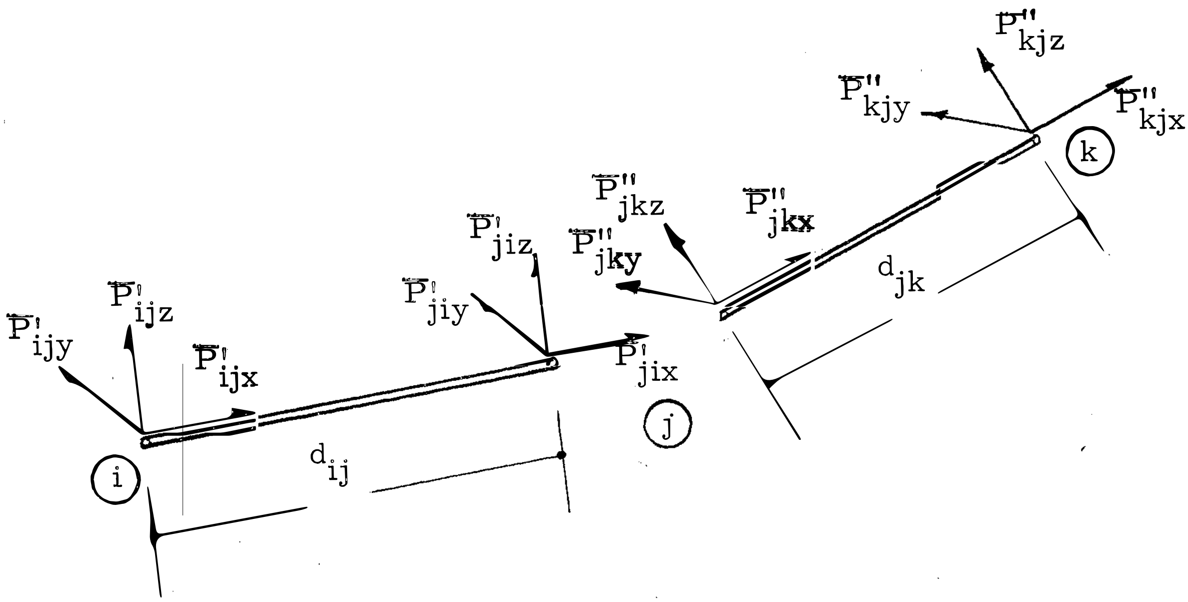


Figure 4-7

Panel Elastic Weights along Initial Systems

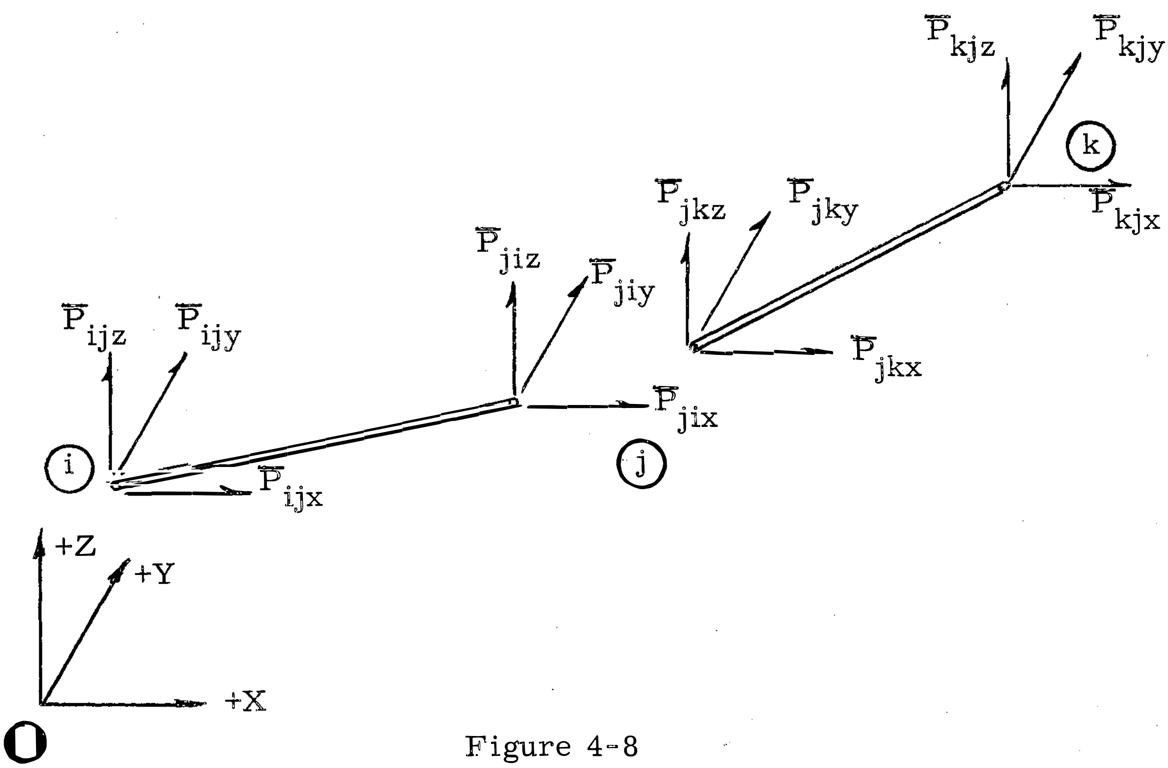


Figure 4-8

Panel Elastic Weights along the Reference System

$$\begin{bmatrix} \bar{P}'_{ijx} \\ \bar{P}'_{ijy} \\ \bar{P}'_{ijz} \end{bmatrix} = \begin{bmatrix} F'_{ijx} & 0 & 0 \\ 0 & F'_{ijy} & 0 \\ 0 & 0 & F'_{ijz} \end{bmatrix} \begin{bmatrix} M'_{ijx} \\ M'_{ijy} \\ M'_{ijz} \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & G'_{jiy} & 0 \\ 0 & 0 & G'_{jiz} \end{bmatrix} \begin{bmatrix} M'_{jix} \\ M'_{jiy} \\ M'_{jiz} \end{bmatrix} + \begin{bmatrix} \tau'^{(L)}_{ijx} \\ \tau'^{(L)}_{ijy} \\ \tau'^{(L)}_{ijz} \end{bmatrix} , \qquad (4\text{-}21)$$

and along the reference system

$$\begin{bmatrix} \bar{P}_{ijx} \\ \bar{P}_{ijy} \\ \bar{P}_{ijz} \end{bmatrix} = \begin{bmatrix} F_{ijxx} & F_{ijxy} & F_{ijxz} \\ F_{ijxy} & F_{ijyy} & F_{ijyz} \\ F_{ijxz} & F_{ijyz} & F_{ijzz} \end{bmatrix} \begin{bmatrix} M_{ijx} \\ M_{ijy} \\ M_{ijz} \end{bmatrix} +$$

$$\begin{bmatrix} G_{jixx} & G_{jixy} & G_{jixz} \\ G_{jixy} & G_{jiyy} & G_{jiyz} \\ G_{jixz} & G_{jiyz} & G_{jizz} \end{bmatrix} \begin{bmatrix} M_{jix} \\ M_{jiy} \\ M_{jiz} \end{bmatrix} + \begin{bmatrix} \tau^{(L)}_{ijx} \\ \tau^{(L)}_{ijy} \\ \tau^{(L)}_{ijz} \end{bmatrix} . \qquad (4\text{-}22)$$

Denoting

$$\begin{bmatrix} \bar{P}^{o}_{ij} \end{bmatrix} = \begin{bmatrix} \bar{P}_{ijx} \\ \bar{P}_{ijy} \\ \bar{P}_{ijz} \end{bmatrix} ,$$

the matrix Equation 4-22 becomes

$$\left[\bar{P}^{o}_{ij}\right] = \left[F^{o}_{ij}\right]\left[M^{o}_{ij}\right] + \left[G^{o}_{ji}\right]\left[M^{o}_{ji}\right] + \left[\tau^{Lo}_{ij}\right]. \qquad (4\text{-}23)$$

The components along the reference system for panel elastic weights at j are

$$\left[\bar{P}^{o}_{ji}\right] = \left[F^{o}_{ji}\right]\left[M^{o}_{ji}\right] + \left[G^{o}_{ij}\right]\left[M^{o}_{ij}\right] + \left[\tau^{Lo}_{ji}\right]. \qquad (4\text{-}24)$$

If bar jk is considered, its panel elastic weights in the reference system at j, k, respectively (Fig. 4-8), are

$$\left[P^{o}_{jk}\right] = \left[F^{o}_{jk}\right]\left[M^{o}_{jk}\right] + \left[G^{o}_{kj}\right]\left[M^{o}_{kj}\right] + \left[\tau^{Lo}_{jk}\right]. \qquad (4\text{-}25)$$

$$\left[\bar{P}^{o}_{kj}\right] = \left[F^{o}_{kj}\right]\left[M^{o}_{kj}\right] + \left[G^{o}_{jk}\right]\left[M^{o}_{jk}\right] + \left[\tau^{Lo}_{kj}\right]. \qquad (4\text{-}26)$$

(c) Joint Elastic Weights - $\bar{P}_j$

A joint elastic weight $\bar{P}_{jx}$ is the change in slope at the joint j in the X direction. Similarly, $\bar{P}_{jy}$, $\bar{P}_{jz}$ are the changes in the Y and Z directions, respectively. Thus the joint elastic weights at j are written in the matrix form as

$$\begin{bmatrix} \bar{P}_{jx} \\ \bar{P}_{jy} \\ \bar{P}_{jz} \end{bmatrix} = \begin{bmatrix} \bar{P}_{jix} \\ \bar{P}_{jiy} \\ \bar{P}_{jiz} \end{bmatrix} + \begin{bmatrix} \bar{P}_{jkx} \\ \bar{P}_{jky} \\ \bar{P}_{jkz} \end{bmatrix}. \qquad (4\text{-}27)$$

Substitution of Equations 4-24, 4-25 into Equation 4-27 gives

$$\left[\bar{P}_j^o\right] = \left[G_{ij}^o\right]\left[M_i^o\right] + \left[F_{ji}^o\right]\left[M_j^o\right] + \left[F_{jk}^o\right]\left[M_j^o\right] + \left[G_{kj}^o\right]\left[M_k^o\right] + \left[\tau_{ji}^{Lo}\right] + \left[\tau_{jk}^{Lo}\right] , \tag{4-28}$$

and after rearranging the terms,

$$\left[\bar{P}_j^o\right] = \left[G_{ij}^o\right]\left[M_i^o\right] + \left[\Sigma F_j^o\right]\left[M_j^o\right] + \left[G_{kj}^o\right]\left[M_k^o\right] + \left[\Sigma\tau_j^{Lo}\right] \tag{4-29}$$

where

$$\left[\bar{P}_j^o\right] = \begin{bmatrix} \bar{P}_{jx} \\ \bar{P}_{jy} \\ \bar{P}_{jz} \end{bmatrix} , \tag{4-30}$$

$$\left[\Sigma F_j^o\right] = \left[F_{ji}^o\right] + \left[F_{jk}^o\right] , \tag{4-31}$$

and

$$\left[\Sigma\tau_j^{Lo}\right] = \left[\tau_{ji}^{Lo}\right] + \left[\tau_{jk}^{Lo}\right] . \tag{4-32}$$

(d) Jointal Elastic Weights

Jointal elastic weights are the components of the joint elastic weights along the jointal system previously discussed (Art. 3-6). The respective matric equations relating jointal elastic weights at j to the reference system, and the reverse (Figs.4-9 and 10) are

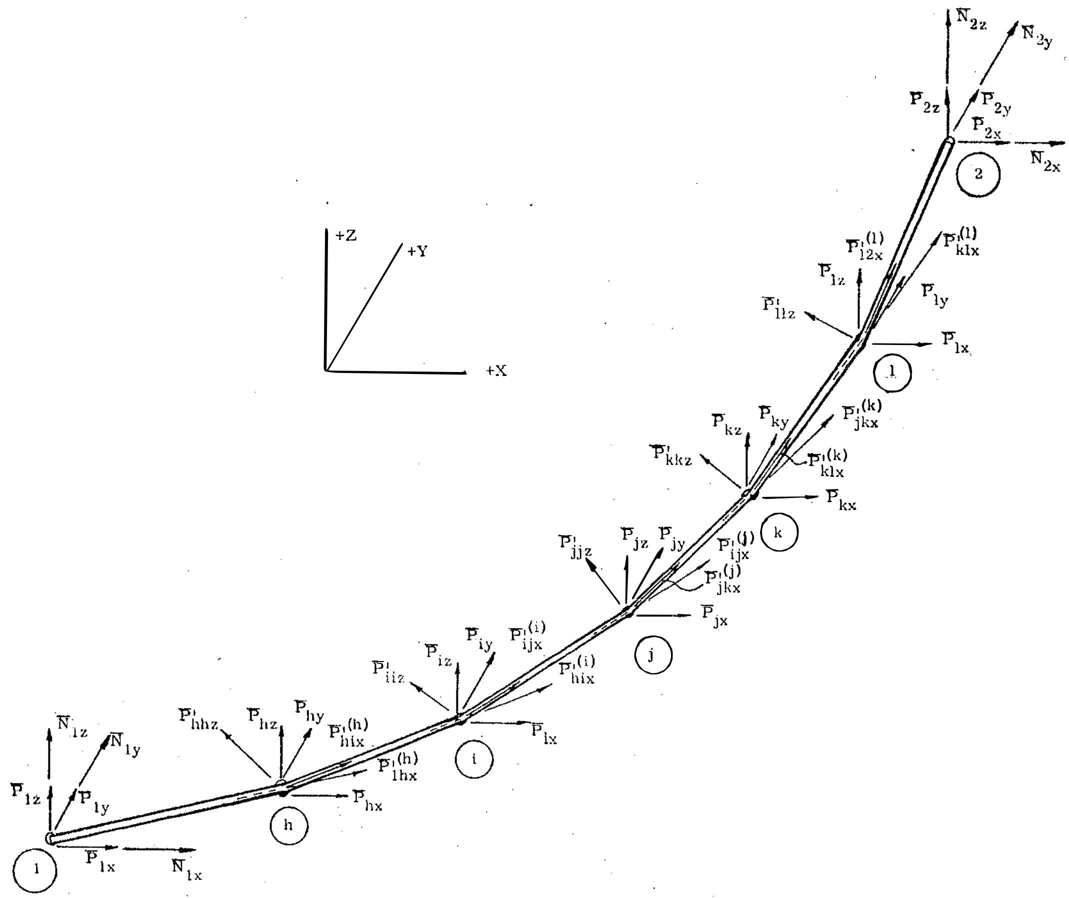


Figure 4-9
Jointal Elastic Weights
Continuous Beam

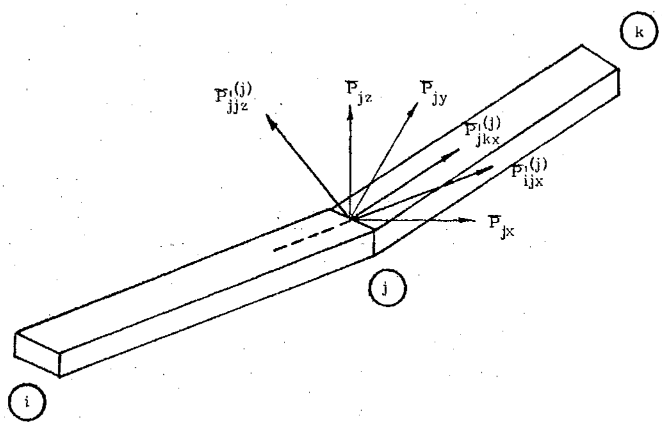


Figure 4-10
Jointal Elastic Weights
Joint j

$$\begin{bmatrix} \alpha_x^{ij} & \alpha_x^{jj} & \alpha_x^{jk} \\ \beta_x^{ij} & \beta_x^{jj} & \beta_x^{jk} \\ \gamma_x^{ij} & \gamma_x^{jj} & \gamma_x^{jk} \end{bmatrix} \begin{bmatrix} \overline{P}'_{jix} \\ \overline{P}'_{jjz} \\ \overline{P}'_{jkx} \end{bmatrix} = \begin{bmatrix} \overline{P}_{jx} \\ \overline{P}_{jy} \\ \overline{P}_{jz} \end{bmatrix} \qquad (4\text{-}33)$$

and

$$\begin{bmatrix} \rho_{j\alpha x} & \rho_{j\alpha y} & \rho_{j\alpha z} \\ \rho_{j\beta x} & \rho_{j\beta y} & \rho_{j\beta z} \\ \rho_{j\gamma x} & \rho_{j\gamma y} & \rho_{j\gamma z} \end{bmatrix} \begin{bmatrix} \overline{P}_{jx} \\ \overline{P}_{jy} \\ \overline{P}_{jz} \end{bmatrix} = \begin{bmatrix} \overline{P}'_{jix} \\ \overline{P}'_{jjz} \\ \overline{P}'_{jkx} \end{bmatrix} . \qquad (4\text{-}34)$$

4-3. Elasto-Static Equations

An elasto-static equation is a static equation of elastic functions. That is, it is a deformation equation for the real structure.

It has been shown (Art. 3-5) that the number of redundants is 3n - 3, where n is the number of spans in the continuous beam considered. As previously outlined, there are

- n - 1 redundant moments at intermediate joints,
- n - 2 redundant torsions in intermediate spans,
- n redundant forces along the members.

To analyze these redundants, 3n - 3 elasto-static equations are developed. They are classified as

- n - 1 elasto-static equations for compatibility of angular deformations at intermediate joints,
- n - 2 elasto-static equations for compatibility of angular defor-

mations in intermediate spans,

and

n elasto-static equations for compatibility of linear deformations along the members.

(a) Compatibility of Angular Deformations

The conjugate structure of the continuous beam 1 h i j k l 2 (Fig. 4-11) is a link mechanism in space, hinged at the respective joints. From the supporting conditions, the real structure has zero displacements at its joints. The conjugate structure with typical elastic weights and reactions is illustrated in Figure 4-9. The deformation of the continuous beam is defined by

hij = system notation,

Δ_{hj} = relative displacements of joints h, j along the joining line hj ,

Δ'_{jjz} = displacement in the X'_{jj} direction.

The relationship used between the functions of the conjugate structure and the deformations of the real structure is: the displacement of the real structure along a certain line is the bending moment of the conjugate structure about that line.

a-1. Compatibility Equations for Joints

A free body hij is isolated from the conjugate structure 1 h i j k l 2 (Fig. 4-12). Elastic weights are shown along the jointal system. Previous discussion (Art. 3-6) has shown that the elastic weight $\overline{P}'_{iiz}$ is normal to the plane hij and consequently normal to the line hj . The conjugate moment about hj is

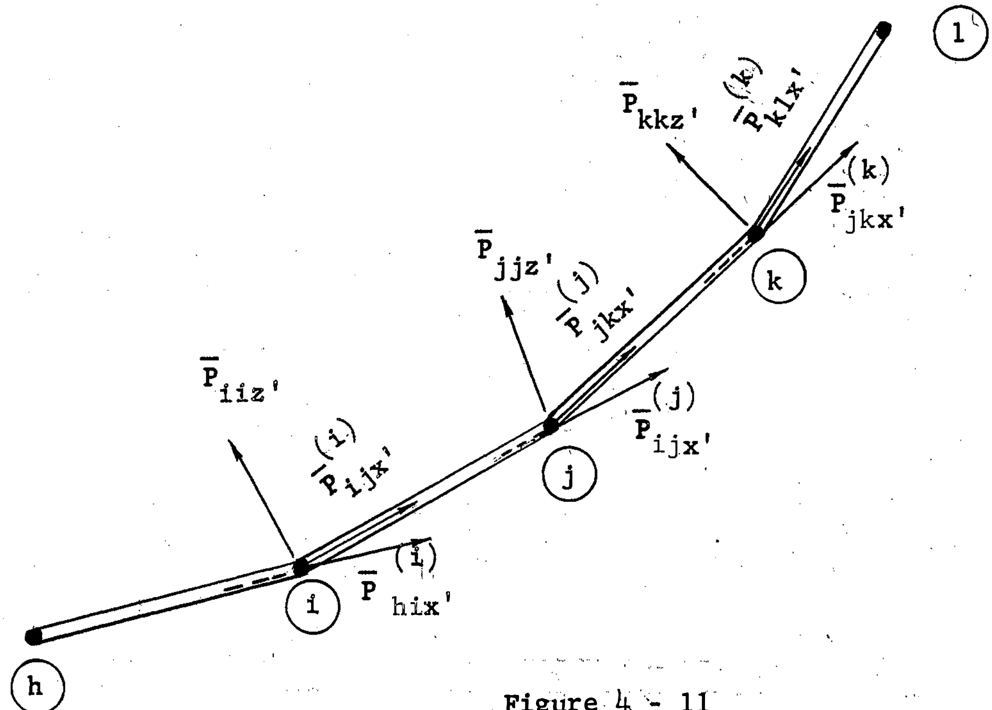


Figure 4 - 11
Conjugate Structure for a Continuous Beam
in Space on Rigid Spherical Hinges

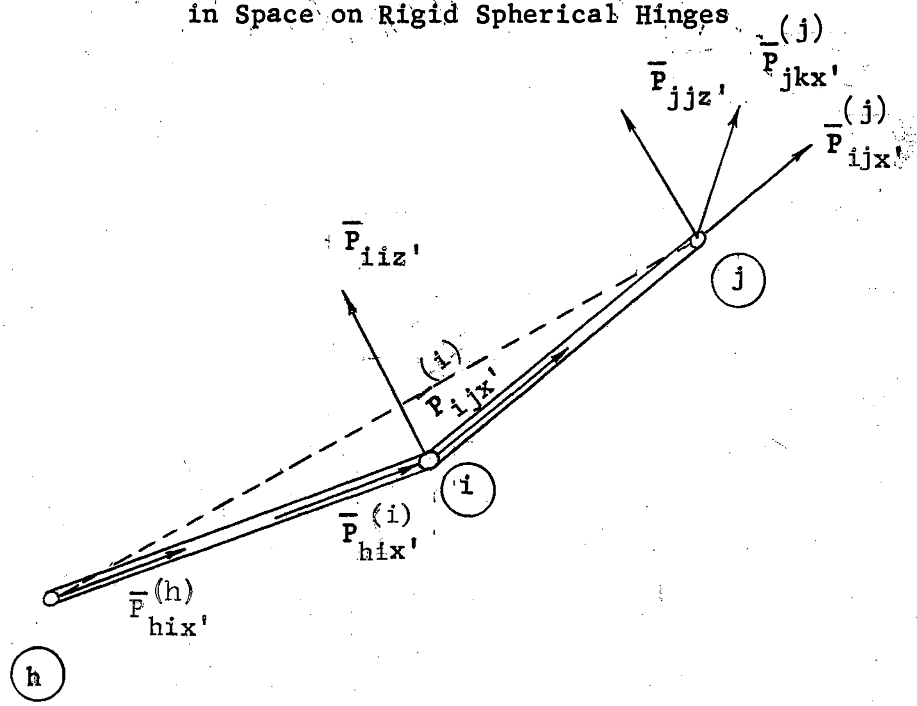


Figure 4 - 12
Free Body Sketch for System
hij of the Conjugate Structure

$$\overline{M}_{hj} = \Delta_{hj} = \overline{P}'_{iiz}\,(\xi) \qquad (4\text{-}35)$$

where

ξ = length of the perpendicular from i on hj .

The elasto-static condition gives

$$\Delta_{hj} = 0 = \overline{P}'_{iiz}\,(\xi)\ . \qquad (4\text{-}36)$$

Since ξ is not zero except when hij is a straight line (Fig. 4-12),

$$\overline{P}'_{iiz} = 0\ . \qquad (4\text{-}37)$$

Similarly, for other free bodies $\overline{ijk}$, $\overline{jkl}$, . . .

$$\overline{P}'_{jjz} = 0 \qquad (a)$$

$$\underline{\overline{P}'_{kkz}} = 0 \qquad (b) \qquad (4\text{-}38)$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\cdots \qquad \cdots \qquad \cdots$$

This indicates that the change in slope at an intermediate joint, normal to its plane, is zero.

a-2. Compatibility Equations of Members

The non-zero jointal elastic weights applied to the conjugate structure of the continuous beam are shown in Figure 4-13. The conjugate end forces at 1 are related to the initial system 1h. The conjugate moments about h in the Z', Y' directions are

$$\overline{M}'_{hz} = \overline{N}'_{1y}\,(d_{ij}) = 0\ ,$$

$$\overline{M}'_{hy} = \overline{N}'_{1z}\,(d_{ij}) = 0\ .$$

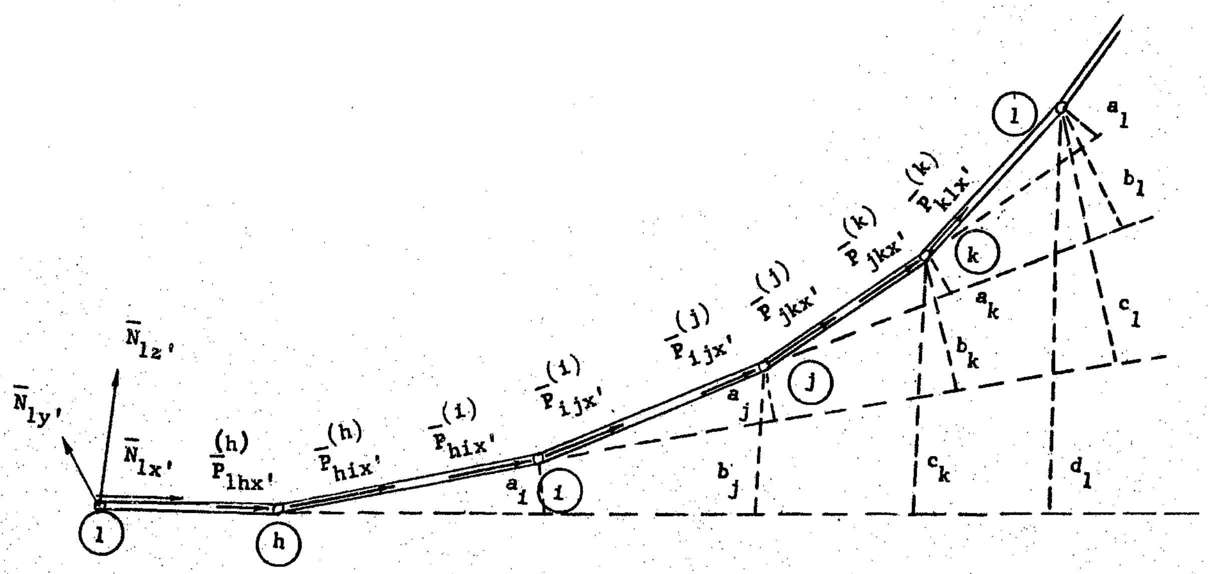


Figure 4-13
Conjugate Structure with the
Non-Zero Elastic Weights Applied

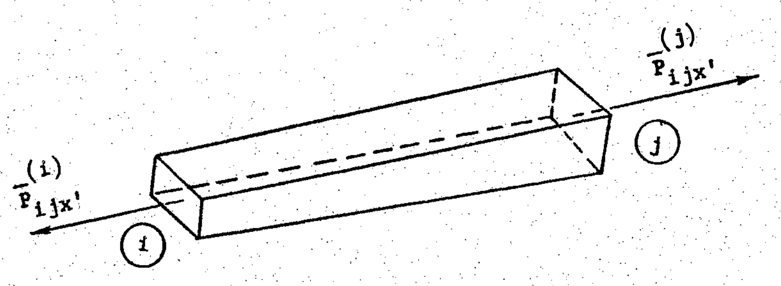


Figure 4-14
Elastic Weights for Member
ij Along its Longitudinal Axis

Thus,

$$\overline{N}'_{1y} = 0 = \overline{N}'_{1z} , \tag{4-39}$$

and the only existing function at 1 is the conjugate end thrust $\overline{N}'_{1x}$. By taking conjugate moments about i in the X'_{hh} direction,

$$\overline{M}'_{i} = (\overline{N}'_{1x} + \overline{P}'_{1hx}) a_i = 0 , \tag{4-40}$$

where

a_i = the distance between the two lines X'_{hh} and X'_{1h} ,

and since $a_i \neq 0$,

$$\overline{N}'_{1x} + \overline{P}'_{1hx} = 0 . \tag{4-41}$$

If the conjugate moments are taken about j in the X'_{ii} direction,

$$\overline{M}'_{j} = (\overline{N}'_{1} + \overline{P}'_{1hx}) b_j + (\overline{P}'^{(h)}_{hix} + \overline{P}'^{(i)}_{hix}) a_j = 0 \tag{4-42}$$

where

b_j = distance between the two lines X'^{j}_{ii} and X'_{1h}, and

a_j = distance between the two lines X'^{j}_{ii} and X'_{hi} .

By substituting Equation 4-41 into Equation 4-42 and since $a_j \neq 0$,

$$\overline{P}'^{(h)}_{hix} + \overline{P}'^{(i)}_{hix} = 0 . \tag{4-43}$$

The equation of conjugate moments about k along X'_{jj} gives

$$(\overline{N}'_{1x} + \overline{P}'_{1hx})a_k + (\overline{P}'^{(h)}_{hix} + \overline{P}'^{(i)}_{hix})b_k +$$

$$(\overline{P}'^{(i)}_{ijx} + \overline{P}'^{(j)}_{ijx})\,c_k = 0 \qquad (4\text{-}44)$$

and by substitutions from Equations 4-41, 43 into Equation 4-44,

$$\overline{P}'^{(i)}_{ijx} + \overline{P}'^{(j)}_{ijx} = 0\ . \qquad (4\text{-}45)$$

It should be noted that the equations of the end spans do not contribute to the elasto-static conditions since they are functions of the unknown conjugate thrusts ($\overline{N}'_{1hx}$ or $\overline{N}'_{21x}$) .

Thus,it can be stated that the sum of the jointal elastic weights along the axis of each bar is zero. This indicates that the change in slope at one end of a member along its axis is equal and opposite to the change at the other end (Fig. 4-14).

(b) Compatibility of Linear Deformations

A member ij is isolated from the continuous beam and acted upon by a force P'_{px} along its axis at point p (Fig. 4-15) . From the edge conditions, points i, j remain still, and point p displaces Δ_p to the right. There is an expansion in the portion ip equal to the contraction in pj. From the theorem of virtual work,

$$\Delta_p = \int_i^p N'_{ijx}\ \lambda'^N_x = \int_p^j N'_{jix}\ \lambda'^N_x \qquad (4\text{-}46)$$

where

$\lambda'^N_x = \dfrac{dx'}{EA'_x}$, elemental linear extensibility.

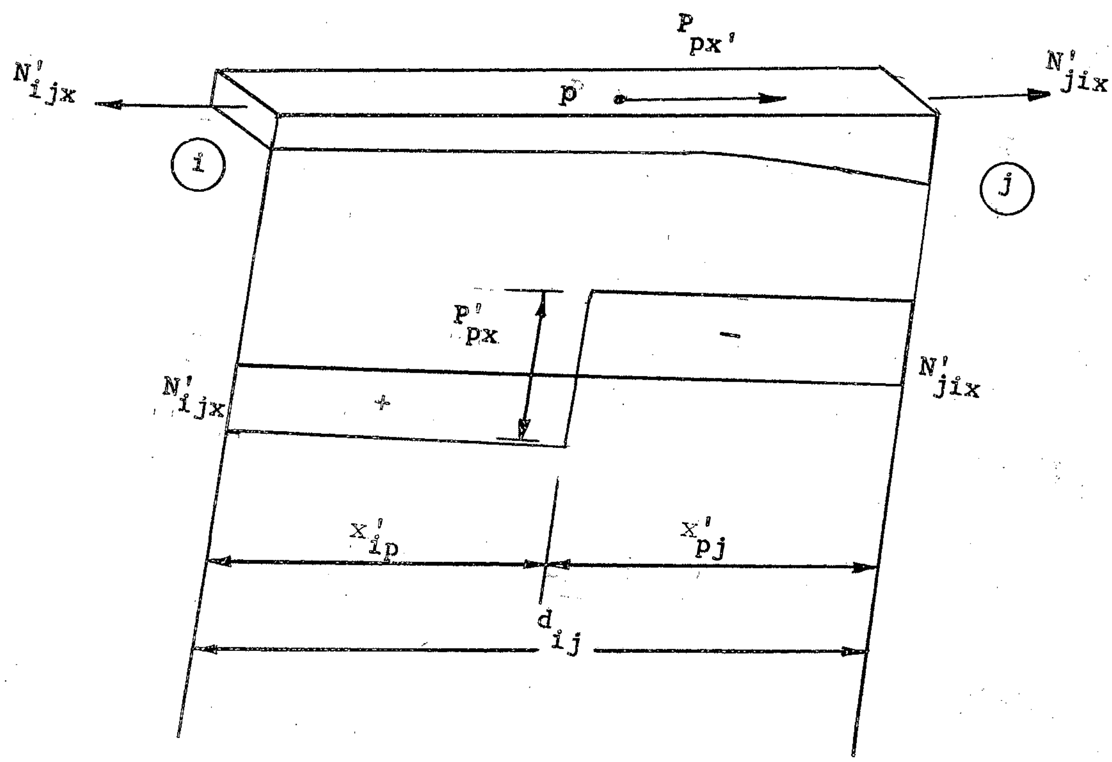


Figure 4-15
Thrust Diagram for Member ij

Denoting

$$\int_i^p \lambda'^N_x = \psi'_{ip} \tag{4-47}$$

and

$$\int_p^j \lambda'^N_x = \psi'_{jp} \quad , \tag{4-48}$$

Equation 4-46 becomes

$$N'_{ijx} \psi'_{ip} = N'_{jix} \psi'_{jp} \tag{4-49}$$

and

$$N'_{ijx} = N'_{jix} \frac{\psi'_{jp}}{\psi'_{ip}} \tag{4-50}$$

or

$$N'_{jix} = N'_{ijx} \frac{\psi'_{ip}}{\psi'_{jp}} \quad . \tag{4-51}$$

The equilibrium of forces gives

$$P'_{px} + N'_{ijx} + N'_{jix} = 0 \quad . \tag{4-52}$$

By substitution from Equation 4-51 into Equation 4-52

$$P'_{px} + N'_{ijx} + N'_{ijx} \frac{\psi'_{ip}}{\psi'_{jp}} = 0 \quad ,$$

and

$$N'_{ijx} = - P'_{px} \frac{\psi'_{jp}}{\psi'_{ip} + \psi'_{jp}} \quad . \tag{4-53}$$

Denoting

$$\psi'_{ip} + \psi'_{jp} = \int_i^j \lambda'^N_x = \psi_{ij} \quad , \tag{4-54}$$

Equation 4-53 becomes

$$N'_{ijx} = -P'_{px} \frac{\psi'_{jp}}{\psi_{ij}} \quad . \tag{4-55}$$

Equations 4-50, 52 give

$$P'_{px} + N'_{jix} \frac{\psi'_{jp}}{\psi'_{ip}} + N'_{jix} = 0$$

and

$$N'_{jix} = -P'_{px} \frac{\psi'_{ip}}{\psi'_{ij}} \quad . \tag{4-56}$$

For constant EA'_x , Equations 4-55, 56 are

$$N'_{ijx} = -P'_{px} \frac{x'_{pj}}{d_{ij}} \quad , \tag{4-57}$$

$$N'_{jix} = -P'_{px} \frac{x'_{ip}}{d_{ij}} \quad . \tag{4-58}$$

The same procedure is to be carried out for the n members of the continuous beam, and thus redundant forces are determined.

CHAPTER V

MOMENT EQUATIONS

From the previous discussion (Art. 4-3a) elasto-static equations for joints and elasto-static equations for members have been established.

The redundants of a continuous beam of order one are moments over supports, and the solution of this beam is completed when the moment matrix is inverted. In a general case of a continuous beam in space, similar equations to the three moment equations are being prepared, but these equations contain more than three unknown moments. The compatibility equations for joints are seven moment equations, while the equations for members include nine moments. In this chapter the derivation of these moment equations is outlined in the following sequence:

a - The compatibility equations of angular deformations are set in terms of elastic weights in the reference system and respective jointal transformation matrices.

b - Elastic weights for the whole structure in the reference system are expressed in terms of moments and angular load functions along the same system.

c - Redundant moments of the continuous beam are pointed out, and their matrix set-up is outlined.

d - By the use of the redundant transformation matrices, joint moments in the reference system are related to the beam redundant moments.

e - By substitution (from d), the elastic weights along the reference system are expressed in terms of redundant moments and reference components of angular load functions.

f - The elastic weights obtained (in e) are substituted into the elasto-static compatibility equations; thus the final moment equations are developed.

g - By rearranging terms, the moment equations are written in the proper matrix form, and the solution is attained by inverting the moment matrix.

h - Joint moments are obtained in terms of determined redundant moments by the use of suitable transformations.

i - End moments of members in their initial systems are determined as functions of reference joint moments using transformation matrices.

j - From previous discussion (Art. 4-3b) and stereo-static conditions, joint reactions are developed in terms of joint moments and applied loading.

5-1. Elasto-Static Equations for the Continuous Beam in Space

On the basis of previous discussion (Art. 4-3a-1) the zero change in slope normal to the plane of the two adjacent members ij, jk at the respective joint j leads to

$$\left[\rho_{jo\beta}\right]\left[\bar{\bar{P}}_j^o\right] = 0 \quad , \qquad (5\text{-}1)$$

where

$\left[\rho_{jo\beta}\right]$ = jointal transformation submatrix (Table 3-3a),

$\left[\overline{P}_j^o\right]$ = reference components of joint elastic weights at j.

Similarly, the jointal component of the elastic weights at k normal to the plane jkl leads to the relationship

$$\left[\rho_{ko\beta}\right]\left[\overline{P}_k^o\right] = 0 \qquad (5\text{-}2)$$

and

$\left[\rho_{ko\beta}\right]$ = 1 × 3 jointal transformation submatrix corresponding to the intermediate joint k.

For the whole continuous beam 1 h i j k l 2 the respective equations are

$$\left[\rho_\beta\right]\left[\overline{P}^o\right] = \left[0\right], \qquad (5\text{-}3)$$

and matrices appearing in Equation 5-3 are explained in Table 5-1.

The change in slope along the axis X'_{ij} of member ij at the end i is

$$\left[\rho_{io\gamma}\right]\left[\overline{P}_i^o\right], \qquad (5\text{-}4)$$

and at the end j is

$$\left[\rho_{jo\alpha}\right]\left[\overline{P}_j^o\right] \qquad (5\text{-}5)$$

where

$\left[\rho_{io\gamma}\right]$ = jointal transformation submatrix relating axis X'_{ij} to the reference system at i,

$\left[\rho_{jo\alpha}\right]$ = jointal transformation submatrix relating axis X'_{ij} to the reference system at j (Table 3-3a).

TABLE 5-1 ELASTO STATIC EQUATIONS CONTINUOUS BEAM

$$\left[\overline{P}^{o}\right] = \begin{bmatrix} \left[\overline{P}^{o}_{h}\right] \\ \left[\overline{P}^{o}_{i}\right] \\ \left[\overline{P}^{o}_{j}\right] \\ \left[\overline{P}^{o}_{k}\right] \\ \left[\overline{P}^{o}_{l}\right] \end{bmatrix}, \quad \left[M'_{ij}\right] = \begin{bmatrix} \left[M'_{ijx}\right] \\ \left[M'_{ijy}\right] \\ \left[M'_{ijz}\right] \end{bmatrix}, \quad \left[\rho_{\beta}\right] = \begin{bmatrix} \left[\rho_{ho\beta}\right] & & & & \\ & \left[\rho_{io\beta}\right] & & & \\ & & \left[\rho_{jo\beta}\right] & & \\ & & & \left[\rho_{ko\beta}\right] & \\ & & & & \left[\rho_{lo\beta}\right] \end{bmatrix}$$

$$\left[\rho_{\gamma\alpha}\right] = \begin{bmatrix} \left[\rho_{ho\gamma}\right] & \left[\rho_{io\alpha}\right] & & & & \\ & \left[\rho_{io\gamma}\right] & \left[\rho_{jo\alpha}\right] & & & \\ & & \left[\rho_{jo\gamma}\right] & \left[\rho_{ko\alpha}\right] & & \\ & & & \left[\rho_{ko\gamma}\right] & \left[\rho_{lo\alpha}\right] & \end{bmatrix}, \quad \left[\rho\right] = \begin{bmatrix} \begin{bmatrix} \rho_{ho\beta} \\ \rho_{ho\gamma} \end{bmatrix} & & & & \\ & \begin{bmatrix} \rho_{io\alpha} \\ \rho_{io\beta} \\ \rho_{io\gamma} \end{bmatrix} & & & \\ & & \begin{bmatrix} \rho_{jo\alpha} \\ \rho_{jo\beta} \\ \rho_{jo\gamma} \end{bmatrix} & & \\ & & & \begin{bmatrix} \rho_{ko\alpha} \\ \rho_{ko\beta} \\ \rho_{ko\gamma} \end{bmatrix} & \\ & & & & \begin{bmatrix} \rho_{lo\alpha} \\ \rho_{lo\beta} \end{bmatrix} \end{bmatrix}$$

As outlined in previous discussion (Eq. 4-45), the compatibility of angular deformations in member ij gives

$$\left[\rho_{io\gamma}\right]\left[\bar{P}_i^o\right] + \left[\rho_{jo\alpha}\right]\left[\bar{P}_j^o\right] = 0 \quad . \tag{5-6}$$

If the elasto-static condition of member jk is considered, by cyclosymmetry

$$\left[\rho_{jo\gamma}\right]\left[\bar{P}_j^o\right] + \left[\rho_{ko\alpha}\right]\left[\bar{P}_k^o\right] = 0 \quad , \tag{5-7}$$

and the system of equations for compatibility of angular deformations in different intermediate members of the continuous beam are

$$\left[\rho_{\gamma\alpha}\right]\left[\bar{P}^o\right] = \left[0\right] . \tag{5-8}$$

Equation 5-8 is in terms of nomenclature introduced in Table 5-1. Matric Equations 5-3, 8 can be written in one form, and the whole elasto-static equations for angular deformations in terms of nomenclature introduced in Table 5-1 are

$$\left[\rho\right]\left[\bar{P}\right] = \left[0\right] . \tag{5-9}$$

5-2. Elastic Weights Along the Reference System in Terms of Joint Moments in the Same System

From previous discussion (Art. 4-2c), the joint elastic weights at i, j and k are

$$\left[\bar{P}_i^o\right] = \left[G_{hi}^o\right]\left[M_h^o\right] + \left[\Sigma F_i^o\right]\left[M_i^o\right] + \left[G_{ji}^o\right]\left[M_j^o\right] + \left[\Sigma \tau_i^{Lo}\right] \tag{5-10}$$

$$\left[\bar{P}^o_j\right] = \left[G^o_{ij}\right]\left[M^o_i\right] + \left[\Sigma F^o_j\right]\left[M^o_j\right] + \left[G^o_{kj}\right]\left[M^o_k\right] + \left[\Sigma \tau^{Lo}_j\right] \qquad (5\text{-}11)$$

$$\left[\bar{P}^o_k\right] = \left[G^o_{jk}\right]\left[M^o_j\right] + \left[\Sigma F^o_k\right]\left[M^o_k\right] + \left[G^o_{lk}\right]\left[M^o_l\right] + \left[\Sigma \tau^{Lo}_k\right] , \qquad (5\text{-}12)$$

and for the whole continuous beam

$$\left[\bar{P}^o\right] = \left[FG^o\right]\left[M^o\right] + \left[\tau^{Lo}\right] . \qquad (5\text{-}13)$$

Table 5-2 explains the nomenclature in Equation 5-13.

5-3. Redundant Matrices

It was outlined in previous discussion (Art. 3-5) that the redundant moments are the bendings at intermediate hinges and the torsions in intermediate spans.

The redundant moments for the beam 1 h i j k l 2 are rearranged in the form

$$\left[M^R\right] = \begin{bmatrix} M'_{hhz} \\ M'_{hix} \\ M'_{iiz} \\ M'_{ijx} \\ M'_{jjz} \\ M'_{jkx} \\ M'_{kkz} \\ M'_{klx} \\ M'_{llz} \end{bmatrix} . \qquad (5\text{-}14)$$

TABLE 5-2 ELASTIC WEIGHTS CONTINUOUS BEAM

$$
\begin{bmatrix} M^{o} \end{bmatrix} = \begin{bmatrix} M_h^{o} \\ M_i^{o} \\ M_j^{o} \\ M_k^{o} \\ M_l^{o} \end{bmatrix}
\qquad
\begin{bmatrix} \tau^{Lo} \end{bmatrix} = \begin{bmatrix} \Sigma \tau_h^{Lo} \\ \Sigma \tau_i^{Lo} \\ \Sigma \tau_j^{Lo} \\ \Sigma \tau_k^{Lo} \\ \Sigma \tau_l^{Lo} \end{bmatrix}
\qquad
\begin{bmatrix} FG^{o} \end{bmatrix} = \begin{bmatrix}
\Sigma F_h^{o} & G_{ih}^{o} & & & \\
G_{hi}^{o} & \Sigma F_i^{o} & G_{ji}^{o} & & \\
 & G_{ij}^{o} & \Sigma F_j^{o} & G_{lk}^{o} & \\
 & & G_{jk}^{o} & \Sigma F_k^{o} & G_{lk}^{o} \\
 & & & G_{kl}^{o} & \Sigma F_l^{o}
\end{bmatrix}
$$

$$
\begin{bmatrix} \Gamma^{R} \end{bmatrix} = \begin{bmatrix}
\Gamma_{ohy} & \Gamma_{ohz} & & & & & & & \\
 & \Gamma_{oix} & \Gamma_{oiy} & \Gamma_{oiz} & & & & & \\
 & & & \Gamma_{ojx} & \Gamma_{ojy} & \Gamma_{ojz} & & & \\
 & & & & & \Gamma_{okx} & \Gamma_{oky} & \Gamma_{okz} & \\
 & & & & & & & \Gamma_{olx} & \Gamma_{oly}
\end{bmatrix} \cdot \begin{bmatrix} Q \end{bmatrix}
\qquad
\begin{bmatrix}
\Gamma_{ohx} & \Sigma Q_{x'}^{ih} \\
\Gamma_{oix} & \Sigma Q_{x'}^{hi} \\
\Gamma_{ojx} & \Sigma Q_{x'}^{ij} \\
\Gamma_{okx} & \Sigma Q_{x'}^{jk} \\
\Gamma_{olx} & \Sigma Q_{x'}^{kl}
\end{bmatrix}
$$

5-4. Relationship Between Redundant and Joint Moments

The relationship between reference components of joint moments at j and respective redundant moments as outlined in previous discussion (Art. 3-6) is

$$\left[M_j^o\right] = \left[\Gamma_{oj}\right]\left[M_j^R\right] + \left[\Gamma_{ojx}\right] \Sigma Q_x' \quad . \qquad (5\text{-}15)$$

Similar relationships hold for other joints of the continuous beam, and are expressed in terms of nomenclature introduced in Equation 5-14 and Table 5-2 as

$$\left[M^o\right] = \left[\Gamma^R\right]\left[M^R\right] + \left[Q\right] \quad . \qquad (5\text{-}16)$$

It can be pointed out from the geometric relationship previously outlined (Eq. 3-25) that

$$\left[\Gamma^R\right] = \left[\rho\right]' \quad . \qquad (5\text{-}17)$$

In case no twisting moments are applied to the structure, the end torsion moments of the panels are equal and Equation 5-16 reduces to

$$\left[M^o\right] = \left[\rho\right]'\left[M^R\right] \quad . \qquad (5\text{-}18)$$

5-5. Beam Elastic Weights Along the Reference System in Terms of Redundant Moments

Because there are as many elasto-static equations as there are redundants, it is very important to express the equations of compatibility for angular deformations in terms of respective redundant moments. Thus, elastic weights along the reference system involved in the elasto-static equations must be expressed as functions of independent redundant moments and applied loads by transformations. Substitution of Equations 5-16, 17, into Equation 5-13 gives

$$\left[\bar{P}^{o}\right] = \left[FG^{o}\right]\left[\rho\right]\left[M^{R}\right] + \left[\tau^{LQ}\right] \qquad (5\text{-}19)$$

where

$$\left[\tau^{LQ}\right] = \left[\tau^{Lo}\right] + \left[FG^{o}\right]\left[Q\right] . \qquad (5\text{-}20)$$

5-6. Compatibility Equations in Terms of Redundant Moments

An elasto-static equation for angular deformations at joint j (Eq. 5-1), expressed in terms of respective redundant moments and nomenclature in Table 5-3, is

$$\left[G^{\beta}_{ij}\right]\left[M^{R}_{i}\right] + \left[\Sigma F^{\beta}_{jj}\right]\left[M^{R}_{j}\right] + \left[G^{\beta}_{kj}\right]\left[M^{R}_{k}\right] + \left[\Sigma\tau^{\beta}_{j}\right] = 0 . \qquad (5\text{-}21)$$

This is a seven moment equation in terms of redundant moments M'_{ihx}, M'_{iiz}, M'_{ijx}, M'_{jjz}, M'_{jkx}, M'_{kkz} and M'_{klx} . Similar equations can be written for other intermediate joints.

The compatibility equation for angular deformations in an intermediate panel ij (Eq. 5-6) after similar substitutions is

$$\left[G^{\gamma\alpha}_{hi}\right]\left[M^{R}_{h}\right] + \left[\Sigma F^{\gamma}_{i} G^{\alpha}_{i}\right]\left[M^{R}_{i}\right] + \left[\Sigma F^{\alpha}_{j} G^{\gamma}_{j}\right]\left[M^{R}_{j}\right] + \left[G^{\gamma\alpha}_{kj}\right]\left[M^{R}_{k}\right] + \left[\Sigma\tau^{\gamma\alpha L}_{i,j}\right] = 0 . \qquad (5\text{-}22)$$

Equation 5-22 is a nine moment equation; it involves the redundant moments M'_{1hx}, M'_{hhz}, M'_{hix}, M'_{iiz}, M'_{ijx}, M'_{jjz}, M'_{jkx}, M'_{kkz} and M'_{klx} .

TABLE 5-3 REDUNDANT MOMENT EQUATIONS

$$\left[M_h^R\right] = \begin{bmatrix} M'_{ihx} \\ M'_{hhz} \\ M'_{hix} \end{bmatrix}, \quad \left[M_i^R\right] = \begin{bmatrix} M'_{hiz} \\ M'_{iiz} \\ M'_{ijx} \end{bmatrix}, \quad \left[M_j^R\right] = \begin{bmatrix} M'_{ijx} \\ M'_{jjz} \\ M'_{jkx} \end{bmatrix}, \quad \left[M_k^R\right] = \begin{bmatrix} M'_{jkx} \\ M'_{kkz} \\ M'_{klx} \end{bmatrix}$$

$$\left[G_{ij}^{\beta}\right] = \left[\rho_{jo\beta}\right]\left[G_{ij}^{o}\right]\left[\Gamma_{oi}\right], \qquad \left[G_{hi}^{\gamma\alpha}\right] = \left[\rho_{io\gamma}\right]\left[G_{hi}^{o}\right]\left[\Gamma_{oh}\right]$$

$$\left[\Sigma F_{jj}^{\beta}\right] = \left[\rho_{jo\beta}\right]\left[\Sigma F_{j}^{o}\right]\left[\Gamma_{oj}\right], \qquad \left[\Sigma F_{i}^{\gamma} G_{i}^{\alpha}\right] = \left[\rho_{io\gamma}\right]\left[\Sigma F_{i}^{o}\right]\left[\Gamma_{oi}\right] + \left[\rho_{jo\alpha}\right]\left[G_{ij}^{o}\right]\left[\Gamma_{oi}\right]$$

$$\left[G_{kj}^{\beta}\right] = \left[\rho_{jo\beta}\right]\left[G_{kj}^{o}\right]\left[\Gamma_{ok}\right], \qquad \left[\Sigma F_{j}^{\alpha} G_{j}^{\gamma}\right] = \left[\rho_{io\gamma}\right]\left[G_{ji}^{o}\right]\left[\Gamma_{oj}\right] + \left[\rho_{jo\alpha}\right]\left[\Sigma F_{j}^{o}\right]\left[\Gamma_{oj}\right]$$

$$\left[G_{jk}^{\beta}\right] = \left[\rho_{ko\beta}\right]\left[G_{jk}\right]\left[\Gamma_{oj}\right], \qquad \left[G_{kj}^{\gamma\alpha}\right] = \left[\rho_{jo\alpha}\right]\left[G_{kj}^{o}\right]\left[\Gamma_{ok}\right]$$

$$\left[\Sigma F_{kk}^{\beta}\right] = \left[\rho_{ko\beta}\right]\left[\Sigma F_{k}^{o}\right]\left[\Gamma_{ok}\right], \qquad \left[G_{ij}^{\gamma\alpha}\right] = \left[\rho_{jo\gamma}\right]\left[G_{ij}^{o}\right]\left[\Gamma_{oi}\right]$$

$$\left[G_{lk}^{\beta}\right] = \left[\rho_{ko\beta}\right]\left[G_{lk}^{o}\right]\left[\Gamma_{ol}\right], \qquad \left[\Sigma F_{j}^{\gamma} G_{j}^{\alpha}\right] = \left[\rho_{jo\gamma}\right]\left[\Sigma F_{j}^{o}\right]\left[\Gamma_{oj}\right] + \left[\rho_{ko\alpha}\right]\left[G_{jk}^{o}\right]\left[\Gamma_{oj}\right]$$

$$\left[\Sigma \tau_{j}^{\beta}\right] = \left[\rho_{jo\beta}\right]\Big[\left[G_{ij}^{o}\right]\left[\Gamma_{oix}\right]\Sigma Q'^{hi}_{x} + \left[\Sigma F_{j}^{o}\right]\left[\Gamma_{ojx}\right]\Sigma Q'^{ij}_{x} + \left[G_{kj}^{o}\right]\left[\Gamma_{okx}\right]\Sigma Q'^{jk}_{x} + \left[\Sigma \tau_{j}^{Lo}\right]\Big]$$

$$\left[\Sigma \tau_{ij}^{L\gamma\alpha}\right] = \left[\rho_{io\gamma}\right]\Big[\left[G_{hi}^{o}\right]\left[\Gamma_{ohx}\right]\Sigma Q'^{ih}_{x} + \left[\Sigma F_{i}^{o}\right]\left[\Gamma_{oix}\right]\Sigma Q'^{hi}_{x} + \left[\Sigma \tau_{i}^{Lo}\right]\Big] + \left[\rho_{jo\alpha}\right]\Big[\left[\Sigma F_{j}^{o}\right]\left[\Gamma_{ojx}\right]\Sigma Q'^{ij}_{x} + \left[G_{kj}^{o}\right]\left[\Gamma_{okx}\right]\Sigma Q'^{jk}_{x} + \left[\Sigma \tau_{j}^{Lo}\right]\Big]$$

Substituting the general equation of elastic weights (Eq. 5-19) into the matric equation of elasto-static compatibility for angular deformation of the continuous beam (Eq. 5-9),

$$\left[\rho\right]\left[\left[FG^{o}\right]\left[\rho\right]'\left[M^{R}\right] + \left[\tau^{LQ}\right]\right] = \left[0\right]. \qquad (5\text{-}23)$$

5-7. Solution of Final Compatibility Equations

The compatibility matric Equation 5-23 can be written as

$$\left[\rho\right]\left[FG^{o}\right]\left[\rho\right]'\left[M^{R}\right] + \left[\rho\right]\left[\tau^{LQ}\right] = \left[0\right], \qquad (5\text{-}24)$$

and after rearranging the terms,

$$\left[\rho\right]\left[FG^{o}\right]\left[\rho\right]'\left[M^{R}\right] = -\left[\rho\right]\left[\tau^{LQ}\right]. \qquad (5\text{-}25)$$

Denoting

$$\left[A^{L}\right] = \left[\rho\right]\left[FG^{o}\right]\left[\rho\right]', \qquad (5\text{-}26)$$

Equation 5-25 becomes

$$\left[A^{L}\right]\left[M^{R}\right] = -\left[\rho\right]\left[\tau^{LQ}\right]. \qquad (5\text{-}27)$$

From the theory of matrices, $\left[A^{L}\right]$ is a square nonsingular symmetric matrix; it has an inverse; thus

$$\left[M^{R}\right] = -\left[A^{L}\right]^{-1}\left[\rho\right]\left[\tau^{LQ}\right]. \qquad (5\text{-}28)$$

Equation 5-28 gives the redundant moments of the space beam 1 h i j k l 2 in terms of known functions; thus the solution for redundants is completed.

For a highly redundant space continuous beam, the dimension of the coefficient matrix $\left[A^L\right]$ becomes quite large, and solution of Equation 5-27 may be carried out by the use of electronic computers. If computer facilities are not available to the structural analyst, iteration or relaxation methods can be employed.

5-8. Determination of Reference Components of Joint Moments

Once the redundants of the continuous beam are obtained, other functions can be calculated in terms of them.

Joint moments in the reference system are determined by the use of Equations 5-16, 17:

$$\left[M^O\right] = \left[\rho\right]'\left[M^R\right] + \left[Q\right] . \qquad (5\text{-}29)$$

5-9. Determination of Initial Components of Joint Moments

The end cross-sectional moments for the members of the continuous beam in their respective initial systems are determined in terms of joint moments in the reference system as

$$
\begin{bmatrix}
[M'_{hl}] \\ [M'_{hi}] \\ [M'_{ih}] \\ [M'_{ij}] \\ [M'_{ji}] \\ [M'_{jk}] \\ [M'_{kj}] \\ [M'_{kl}] \\ [M'_{lk}] \\ [M'_{l2}]
\end{bmatrix}
=
\begin{bmatrix}
[\pi^{lh}_{lo}] & & & & \\
[\pi^{hi}_{lo}] & & & & \\
 & [\pi^{hi}_{lo}] & & & \\
 & [\pi^{ij}_{lo}] & & & \\
 & & [\pi^{ij}_{lo}] & & \\
 & & [\pi^{jk}_{lo}] & & \\
 & & & [\pi^{jk}_{lo}] & \\
 & & & [\pi^{kl}_{lo}] & \\
 & & & & [\pi^{kl}_{lo}] \\
 & & & & [\pi^{l2}_{lo}]
\end{bmatrix}
\begin{bmatrix}
[M^o_h] \\ [M^o_i] \\ [M^o_j] \\ [M^o_k] \\ [M^o_l]
\end{bmatrix}
\tag{5-30}
$$

5-10. Determination of Joint Reactions

From the previous discussion (Art. 3-1), end shearing forces are functions of joint moments and applied loads. End thrusts are evaluated in terms of loads as previously outlined (Art. 4-3b). By taking moments about the end j of member ij generally loaded (Fig. 3-1), and using Equation 4-55, the end forces $\left[N'_{ij}\right]$ are expressed as

$$
-\left[r'_{ij}\right]\left[N'_{ij}\right] + \left[I_{23}\right]\left[M'_{ji} - M'_{ij}\right] - \left[SW'^{ij}_{pj}\right] = \left[0\right],
\tag{5-31}
$$

and nomenclature in Equation 5-31 is explained by Table 5-4.

If moments are taken about i instead of j and deformation Equation 4-56 is used, the end forces at j in terms of bending moments, applied loads, and nomenclature introduced in Table 5-4 are

$$\left[-r'_{ij}\right]\left[N'_{ji}\right] + \left[I_{23}\right]\left[M'_{ji} - M'_{ij}\right] + \left[SW'^{ij}_{pi}\right] = \left[0\right] . \tag{5-32}$$

Transforming the end cross-sectional elements to the reference system, Equations 5-31, 32 become

$$\left[r'_{ij}\right]\left[\pi^{ij}_{1o}\right]\left[N^{o}_{ij}\right] = \left[I_{23}\right]\left[\pi^{ij}_{1o}\right]\left[M^{o}_{ji} - M^{o}_{ij}\right] - \left[SW'^{ij}_{pj}\right] , \tag{5-33}$$

$$\left[r'_{ij}\right]\left[\pi^{ij}_{1o}\right]\left[N^{o}_{ji}\right] = \left[I_{23}\right]\left[\pi^{ij}_{1o}\right]\left[M^{o}_{ji} - M^{o}_{ij}\right] + \left[SW'^{ij}_{pi}\right] , \tag{5-34}$$

from which

$$\left[N^{o}_{ij}\right] = \left[\pi^{ij}_{o1}\right]\left[r'_{ij}\right]^{-1}\left[\left[I_{23}\right]\left[\pi^{ij}_{1o}\right]\left[M^{o}_{j} - M^{o}_{i}\right] - \left[SW'^{ij}_{pj}\right]\right] , \tag{5-35}$$

$$\left[N^{o}_{ji}\right] = \left[\pi^{ij}_{o1}\right]\left[r'_{ij}\right]^{-1}\left[\left[I_{23}\right]\left[\pi^{ij}_{1o}\right]\left[M^{o}_{j} - M^{o}_{i}\right] + \left[SW'^{ij}_{pi}\right]\right] . \tag{5-36}$$

If the end forces are related to the conventional sign of reactions,

$$\left[N^{o}_{ij}\right] = \left[\pi^{ij}_{o1}\right]\left[r'_{ij}\right]^{-1}\left[\left[I_{23}\right]\left[\pi^{ij}_{1o}\right]\left[M^{o}_{i} - M^{o}_{j}\right] + \left[SW'^{ij}_{pj}\right]\right] . \tag{5-37}$$

TABLE 5-4 END FORCES MEMBER ij

$$\begin{bmatrix} N_{y'} \end{bmatrix} = \begin{bmatrix} N_{ijx'} \\ N_{ijy'} \\ N_{ijz'} \end{bmatrix}, \quad \begin{bmatrix} N_{ji'} \end{bmatrix} = \begin{bmatrix} N_{jix'} \\ N_{jiy'} \\ N_{jiz'} \end{bmatrix}, \quad \begin{bmatrix} M_{ji'} \end{bmatrix} = \begin{bmatrix} M_{jix'} \\ M_{jiy'} \\ M_{jiz'} \end{bmatrix}, \quad \begin{bmatrix} M_{ji'} \end{bmatrix} = \begin{bmatrix} M_{jix'} \\ M_{jiy'} \\ M_{jiz'} \end{bmatrix}$$

$$\begin{bmatrix} r_{ij'} \end{bmatrix} = \begin{bmatrix} -\psi_{ij} & 0 & 0 \\ 0 & 0 & d_{ij} \\ 0 & -d_{ij} & 0 \end{bmatrix} \qquad \begin{bmatrix} r_{ip'} \end{bmatrix} = \begin{bmatrix} -\psi'_{pi} & 0 & 0 \\ 0 & 0 & x_{ip'} \\ 0 & -x_{ip'} & 0 \end{bmatrix} \qquad \begin{bmatrix} r_{pj'} \end{bmatrix} = \begin{bmatrix} -\psi'_{pj} & 0 & 0 \\ 0 & 0 & x_{pj'} \\ 0 & -x_{pj'} & 0 \end{bmatrix}$$

$$\begin{bmatrix} SW^{ij}_{pi'} \end{bmatrix} = \begin{bmatrix} I_{23} \end{bmatrix}\begin{bmatrix} Q'_i \end{bmatrix} - \begin{bmatrix} r_{ip'} \end{bmatrix}\begin{bmatrix} P_{p'} \end{bmatrix}$$

$$\begin{bmatrix} SW^{ij}_{pj'} \end{bmatrix} = \begin{bmatrix} I_{23} \end{bmatrix}\begin{bmatrix} Q'_i \end{bmatrix} - \begin{bmatrix} r_{pj'} \end{bmatrix}\begin{bmatrix} P_{p'} \end{bmatrix}$$

$$\begin{bmatrix} I_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} r_{ij'} \end{bmatrix}^{-1} = \begin{bmatrix} -1/\psi_{ij} & 0 & 0 \\ 0 & 0 & -1/d_{ij} \\ 0 & 1/d_{ij} & 0 \end{bmatrix}$$

By cyclosymmetry, the reference components of end forces at j, k of bar jk and related to reactions' sign convention are

$$\left[N^{o}_{jk}\right] = \left[\pi^{jk}_{o1}\right]\left[r''_{jk}\right]^{-1}\left[\left[I_{23}\right]\left[\pi^{jk}_{1o}\right]\left[M^{o}_{j} - M^{o}_{k}\right] + \left[SW''^{jk}_{pk}\right]\right] , \tag{5-38}$$

$$\left[N^{o}_{kj}\right] = \left[\pi^{jk}_{o1}\right]\left[r''_{jk}\right]^{-1}\left[\left[I_{23}\right]\left[\pi^{jk}_{1o}\right]\left[M^{o}_{k} - M^{o}_{j}\right] + \left[SW''^{jk}_{pj}\right]\right] . \tag{5-39}$$

Reactions at joint j along the reference system are

$$\left[R^{o}_{j}\right] = \left[N^{o}_{ji}\right] + \left[N^{o}_{jk}\right] . \tag{5-40}$$

Substitution of Equations 5-36, 38 into Equation 5-40 gives

$$\left[R^{o}_{j}\right] = \left[\pi^{ij}_{o1}\right]\left[r'_{ij}\right]^{-1}\left[\left[I_{23}\right]\left[\pi^{ij}_{1o}\right]\left[M^{o}_{j} - M^{o}_{i}\right] + \left[SW'^{ij}_{pi}\right]\right] + \left[\pi^{jk}_{o1}\right]\left[r''_{jk}\right]^{-1}\left[\left[I_{23}\right]\left[\pi^{jk}_{1o}\right]\left[M^{o}_{j} - M^{o}_{k}\right] + \left[SW''^{jk}_{pk}\right]\right] . \tag{5-41}$$

Denoting

$$\left[H_{ij}\right] = \left[\pi^{ij}_{o1}\right]\left[r'_{ij}\right]^{-1}\left[I_{23}\right]\left[\pi^{ij}_{1o}\right] \tag{5-42}$$

$$\left[H_{kj}\right] = \left[\pi^{jk}_{o1}\right]\left[r''_{jk}\right]^{-1}\left[I_{23}\right]\left[\pi^{jk}_{1o}\right] \tag{5-43}$$

$$\left[N^{L}_{ji}\right] = \left[\pi^{ij}_{o1}\right]\left[r'_{ij}\right]^{-1}\left[SW'^{ij}_{pi}\right] \tag{5-44}$$

$$\left[N^{L}_{jk}\right] = \left[\pi^{jk}_{o1}\right]\left[r''_{jk}\right]^{-1}\left[SW''^{jk}_{pk}\right] \tag{5-45}$$

$$\left[\Sigma N^{L}_{j}\right] = \left[N^{L}_{ji}\right] + \left[N^{L}_{jk}\right] , \tag{5-46}$$

Equation 5-41 becomes

$$\left[R_j^o\right] = -\left[H_{ij}\right]\left[M_i^o\right] + \left[H_{ij} + H_{kj}\right]\left[M_j^o\right] - \left[H_{kj}\right]\left[M_k^o\right] + \left[\Sigma N_j^{Lo}\right] \quad . \tag{5-47}$$

Equation 5-47 is a nine moment equation for reactions at j. Similar matric equations are available for other joints, and the reference components of joint reactions of the continuous beam 1 h i j k l 2 are

$$
\begin{bmatrix} [R_1^o] \\ [R_h^o] \\ [R_i^o] \\ [R_j^o] \\ [R_k^o] \\ [R_l^o] \\ [R_2^o] \end{bmatrix}
=
\begin{bmatrix}
[-H_{h1}] & & & & \\
[H_{ih}] & [-H_{ih}] & & & \\
[-H_{hi}] & [H_{hi}+H_{ji}] & [-H_{ji}] & & \\
 & [-H_{ij}] & [H_{ij}+H_{kj}] & [-H_{kj}] & \\
 & & [-H_{jk}] & [H_{jk}+H_{lk}] & [-H_{lk}] \\
 & & & [-H_{kl}] & [H_{kl}] \\
 & & & & [-H_{l2}]
\end{bmatrix}
\begin{bmatrix} [M_h^o] \\ [M_i^o] \\ [M_j^o] \\ [M_k^o] \\ [M_l^o] \end{bmatrix}
+
\begin{bmatrix} [\Sigma N_1^{oL}] \\ [\Sigma N_h^{oL}] \\ [\Sigma N_i^{oL}] \\ [\Sigma N_j^{oL}] \\ [\Sigma N_k^{oL}] \\ [\Sigma N_l^{oL}] \\ [\Sigma N_2^{oL}] \end{bmatrix} . \qquad (5\text{-}48)
$$

CHAPTER VI

STATIONARY AND MOVING LOADS

The beam functions for stationary and moving loading are investigated in this chapter. Moment equations in terms of angular load functions along respective initial systems are developed for stationary loads. Equations obtained are used for the derivation of moment equations due to moving loads. Influence values for different beam functions are outlined.

6-1. Stationary Loading

From the previous discussion (Art. 5-2), the angular load functions for the continuous beam in the reference system are

$$\left[\tau^{Lo}\right] = \left[\Pi\right]\left[\tau'^{L}\right] , \qquad (6\text{-}1)$$

and nomenclature of Equation 6-1 is explained in Table 6-1a.

If no twisting moments are applied to the continuous beam, the moment Equation 5-28 after substitution from Equation 6-1 becomes

$$\left[M^{R}\right] = -\left[A^{L}\right]^{-1}\left[\rho\right]\left[\Pi\right]\left[\tau'^{L}\right] . \qquad (6\text{-}2)$$

In this case, the angular load functions in the initial system ij at the ends i, j are

TABLE 6-1a

STATIONARY LOADING-MOMENT EQUATIONS

$$
\left[\tau'^{L}\right] = \begin{bmatrix} \left[\tau'^{L}_{h1}\right] \\ \left[\tau'^{L}_{hi}\right] \\ \left[\tau'^{L}_{ih}\right] \\ \left[\tau'^{L}_{ij}\right] \\ \left[\tau'^{L}_{ji}\right] \\ \left[\tau'^{L}_{jk}\right] \\ \left[\tau'^{L}_{kj}\right] \\ \left[\tau'^{L}_{kl}\right] \\ \left[\tau'^{L}_{lk}\right] \\ \left[\tau'^{L}_{l2}\right] \end{bmatrix}
$$

$$
\left[\Pi\right] = \begin{bmatrix}
\left[\pi^{1h}_{o1}\right] & \left[\pi^{hi}_{o1}\right] & & & & & & & & \\
& & \left[\pi^{hi}_{o1}\right] & \left[\pi^{ij}_{o1}\right] & & & & & & \\
& & & & \left[\pi^{ij}_{o1}\right] & \left[\pi^{jk}_{o1}\right] & & & & \\
& & & & & & \left[\pi^{jk}_{o1}\right] & \left[\pi^{kl}_{o1}\right] & & \\
& & & & & & & & \left[\pi^{kl}_{o1}\right] & \left[\pi^{12}_{o1}\right]
\end{bmatrix}
$$

$$
\left[\pi^{ij}_{o1yz}\right] = \begin{bmatrix} \pi^{ij}_{o1y} & \pi^{ij}_{o1z} \end{bmatrix}
$$

$$
\left[\tau'^{L}_{h1}\right]_{yz} = \begin{bmatrix} \tau'^{L}_{h1y} \\ \tau'^{L}_{h1z} \end{bmatrix}
$$

$$
\left[\tau'^{L}_{yz}\right] = \begin{bmatrix} \left[\tau'^{L}_{h1}\right]_{yz} \\ \left[\tau'^{L}_{yz}\right]_{hi} \\ \left[\tau'^{L}_{yz}\right]_{ij} \\ \left[\tau'^{L}_{yz}\right]_{jk} \\ \left[\tau'^{L}_{yz}\right]_{kl} \\ \left[\tau'^{L}_{12}\right]_{yz} \end{bmatrix}
$$

$$
\left[\Pi_{yz}\right] = \begin{bmatrix}
\left[\pi^{1h}_{o1yz}\right] & \left[\pi^{hi}_{o1yz}\right] & & & & & & & & \\
& & \left[\pi^{hi}_{o1yz}\right] & \left[\pi^{ij}_{o1yz}\right] & & & & & & \\
& & & & \left[\pi^{ij}_{o1yz}\right] & \left[\pi^{jk}_{o1yz}\right] & & & & \\
& & & & & & \left[\pi^{jk}_{o1yz}\right] & \left[\pi^{kl}_{o1yz}\right] & & \\
& & & & & & & & \left[\pi^{kl}_{o1yz}\right] & \left[\pi^{12}_{o1yz}\right]
\end{bmatrix}
$$

Equation 6-2 in terms of nonzero initial components of angular load functions, and nomenclature introduced in Table 6-1a is

$$\left[M^R\right] = -\left[A^L\right]^{-1}\left[\rho\right]\left[\Pi_{yz}\right]\left[\tau'^L_{yz}\right] . \qquad (6\text{-}8)$$

6-2. Moving Loads

A bar ij of the continuous beam is acted upon by a unit load in the reference direction Z ; respective initial components of P_{pz} are

$$\left[P'_p\right] = \begin{bmatrix} \gamma^{ij}_x \\ \gamma^{ij}_y \\ \gamma^{ij}_z \end{bmatrix} . \qquad (6\text{-}9)$$

The angular load functions at end i are

$$\left[\tau'^L_{ij}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \gamma^{ij}_z & 0 \\ 0 & 0 & -\gamma^{ij}_y \end{bmatrix} \begin{bmatrix} 0 \\ C'_y \\ C'_z \end{bmatrix} , \qquad (6\text{-}10)$$

where

C'_y = initial angular load function τ'^L_{ijy} due to $P'_{pz} = 1$,

C'_z = initial angular load function τ'^L_{ijz} due to $P'_{py} = 1$.

The non-zero angular load functions at i are

$$\begin{bmatrix} \tau'^{L}_{ijy} \\ \tau'^{L}_{ijz} \end{bmatrix} = \begin{bmatrix} \gamma^{ij}_{z} & 0 \\ 0 & -\gamma^{ij}_{y} \end{bmatrix} \begin{bmatrix} C'_{y} \\ C'_{z} \end{bmatrix} , \qquad (6\text{-}11)$$

and at the end j

$$\begin{bmatrix} \tau'^{L}_{jiy} \\ \tau'^{L}_{jiz} \end{bmatrix} = \begin{bmatrix} \gamma^{ij}_{z} & 0 \\ 0 & -\gamma^{ij}_{y} \end{bmatrix} \begin{bmatrix} C'_{1y} \\ C'_{1z} \end{bmatrix} , \qquad (6\text{-}12)$$

where

C'_{1y} = initial angular load function τ'^{L}_{jiy} due to $P'_{z} = 1$,

C'_{1z} = initial angular load function τ'^{L}_{jiz} due to $P'_{y} = 1$.

If the unit load P_z is applied to the member jk ,

$$\begin{bmatrix} \tau''^{L}_{jky} \\ \tau''^{L}_{jkz} \end{bmatrix} = \begin{bmatrix} \gamma^{jk}_{z} & 0 \\ 0 & -\gamma^{jk}_{y} \end{bmatrix} \begin{bmatrix} C''_{y} \\ C''_{z} \end{bmatrix} . \qquad (6\text{-}13)$$

For the continuous beam 1 h i j k l 2 acted upon by loads as shown (Fig. 6-1) ,

$$\begin{bmatrix} \tau'^{L}_{yz} \end{bmatrix} = \begin{bmatrix} \gamma \end{bmatrix} \begin{bmatrix} C \end{bmatrix} , \qquad (6\text{-}14)$$

and nomenclature appearing in Equation 6-14 is explained in Table 6-1b.

In case the unit load P_z is moving on the continuous beam, one value of each of C_y, C_z, C_{1y}, C_{1z} exists at a time. These are the respective influence values for the nonzero initial components of angular

TABLE 6-1b

MOVING LOADS - MOMENT EQUATIONS

$$
[\gamma] = \begin{bmatrix}
[\gamma^{1h}] & & & & & & & & & \\
& [\gamma^{hi}] & & & & & & & & \\
& & [\gamma^{hi}] & & & & & & & \\
& & & [\gamma^{ij}] & & & & & & \\
& & & & [\gamma^{ij}] & & & & & \\
& & & & & [\gamma^{jk}] & & & & \\
& & & & & & [\gamma^{jk}] & & & \\
& & & & & & & [\gamma^{kl}] & & \\
& & & & & & & & [\gamma^{kl}] & \\
& & & & & & & & & [\gamma^{12}]
\end{bmatrix}
, \quad
[C] = \begin{bmatrix}
C_1^{1h} \\ C^{hi} \\ C_1^{hi} \\ C^{ij} \\ C_1^{ij} \\ C^{jk} \\ C_1^{jk} \\ C^{kl} \\ C_1^{kl} \\ C^{12}
\end{bmatrix}
, \quad
[C]_{pz} = \begin{bmatrix}
[C_1^{1h}] & & & & & \\
& \begin{bmatrix} C^{hi} \\ C_1^{hi} \end{bmatrix} & & & & \\
& & \begin{bmatrix} C^{ij} \\ C_1^{ij} \end{bmatrix} & & & \\
& & & \begin{bmatrix} C^{jk} \\ C_1^{jk} \end{bmatrix} & & \\
& & & & \begin{bmatrix} C^{kl} \\ C_1^{kl} \end{bmatrix} & \\
& & & & & [C^{12}]
\end{bmatrix}_{Pz}
$$

$$
[\gamma^{ij}] = \begin{bmatrix} \gamma_z^{ij} & 0 \\ 0 & -\gamma_y^{ij} \end{bmatrix}
, \quad
[C^{ij}] = \begin{bmatrix} C'_y \\ C'_z \end{bmatrix}
, \quad
[C_1^{ij}] = \begin{bmatrix} C'_{1y} \\ C'_{1z} \end{bmatrix}
, \quad
\begin{bmatrix} C^{ij} \\ C_1^{ij} \end{bmatrix}_{Pz} =
\begin{bmatrix}
C'_{y1} & C'_{y2} & C'_{y3} & C'_{y4} \cdots\cdots \\
C'_{z1} & C'_{z2} & C'_{z3} & C'_{z4} \cdots\cdots \\
\hline
C'_{1y1} & C'_{1y2} & C'_{1y3} & C'_{1y4} \cdots\cdots \\
C'_{1z1} & C'_{1z2} & C'_{1z3} & C'_{1z4} \cdots\cdots
\end{bmatrix}
$$

load functions at the ends of the loaded panel.

Thus, the matrix $\begin{bmatrix} C_{pz} \end{bmatrix}$ for a moving unit load P_z is a diagonal matrix with maximum of four rows running simultaneously. Each row of the matrix contains the respective influence values along the initial directions specified (Table 6-1b).

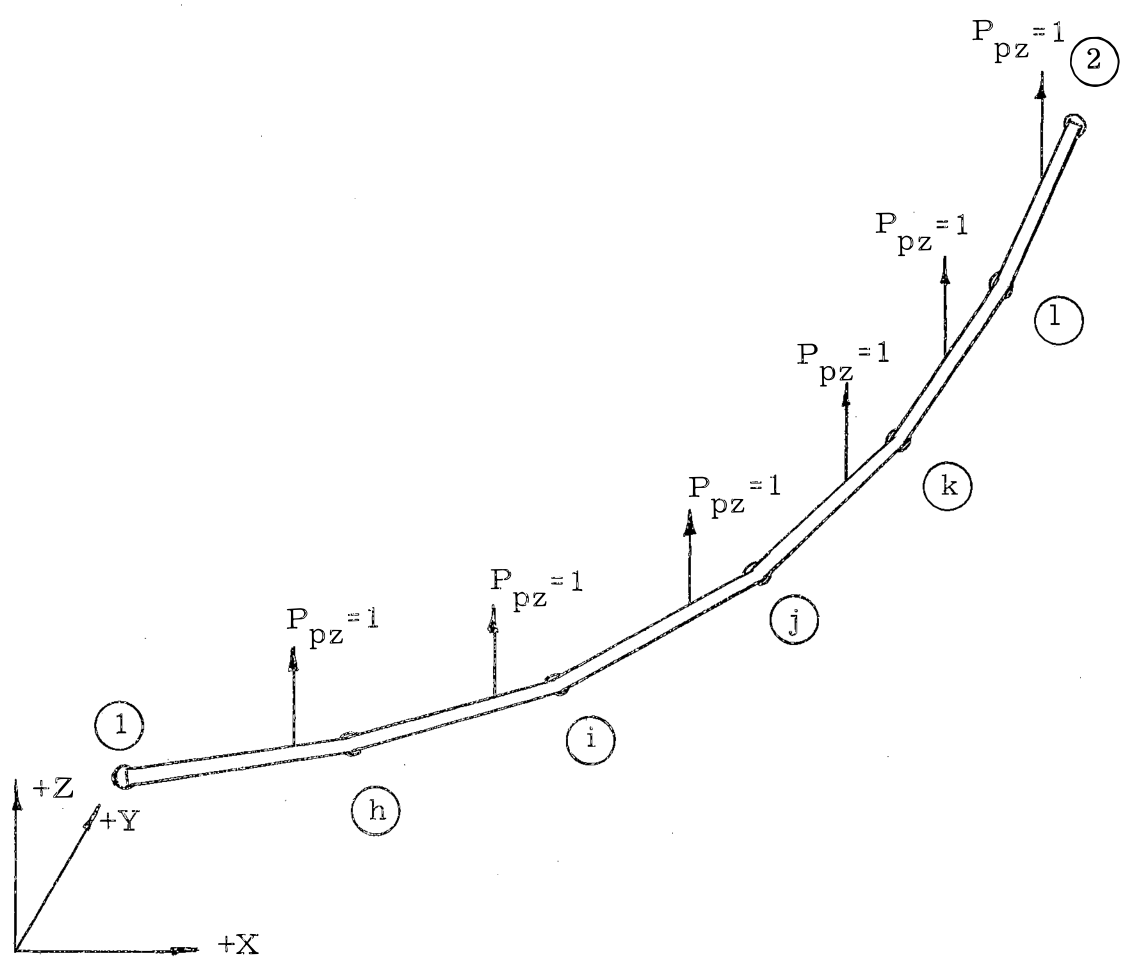


Figure 6-1
Continuous Beam in Space
Loaded Vertically

Using the moment Equation 6-8 derived in the first part of this chapter, and by substitution of Equation 6-14, the moment equation for a unit moving load P_z is

$$\left[M^R\right]_{P_z} = -\left[A^L\right]^{-1}\left[\rho\right]\left[\Pi_{yz}\right]\left[\gamma\right]\left[C\right]_{P_z} \quad . \qquad (6\text{-}15)$$

Denoting

$$\left[\tau^L\right]_{P_z} = \left[\rho\right]\left[\Pi_{yz}\right]\left[\gamma\right]\left[C\right]_{P_z} \quad ,$$

Equation 6-15 becomes

$$\left[M^R\right]_{P_z} = -\left[A^L\right]^{-1}\left[\tau^L\right]_{P_z} \quad . \qquad (6\text{-}16)$$

6-3. Influence Values

From the Equations 5-3, 8 outlined in previous discussion and Equation 6-14, the influence values of angular load functions included in respective elasto-static equations are as shown in Table 6-2.

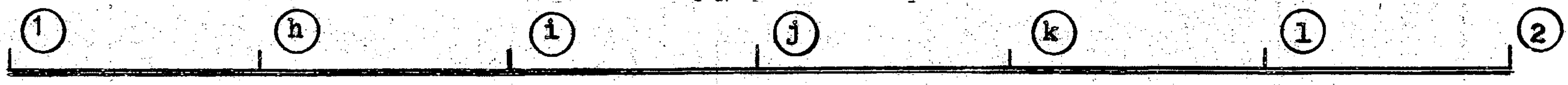
The influence values of redundant moments are determined by Equation 6-16; respective joint moments in the reference system are

$$\left[M^O\right]_{P_z} = \left[\rho\right]'\left[M^R\right]_{P_z} \quad . \qquad (6\text{-}17)$$

Substituting Equation 6-17 into Equation 5-47, the influence values for reactions are

$$\left[R^O\right]_{P_z} = \left[H\right]\left[M^O\right]_{P_z} + \left[N^L\right]_{P_z} \quad . \qquad (6\text{-}18)$$

TABLE 6-2 INFLUENCE VALUES OF ANGULAR LOAD FUNCTIONS CONTINUOUS BEAM



$$\left[\rho_{ho\beta}\right]\left[\bar{P}_h^o\right] = 0$$

$$\left[\rho_{ho\gamma}\right]\left[\bar{P}_h^o\right] + \left[\rho_{io\alpha}\right]\left[\bar{P}_i^o\right] = 0$$

$$\left[\rho_{io\beta}\right]\left[\bar{P}_i^o\right] = 0$$

$$\left[\rho_{io\gamma}\right]\left[\bar{P}_i^o\right] + \left[\rho_{jo\alpha}\right]\left[\bar{P}_j^o\right] = 0$$

$$\left[\rho_{jo\beta}\right]\left[\bar{P}_j^o\right] = 0$$

$$\left[\rho_{jo\gamma}\right]\left[\bar{P}_j^o\right] + \left[\rho_{ko\alpha}\right]\left[\bar{P}_k^o\right] = 0$$

$$\left[\rho_{ko\beta}\right]\left[\bar{P}_k^o\right] = 0$$

$$\left[\rho_{ko\gamma}\right]\left[\bar{P}_k^o\right] + \left[\rho_{lo\alpha}\right]\left[\bar{P}_l^o\right] = 0$$

$$\left[\rho_{lo\beta}\right]\left[\bar{P}_l^o\right] = 0$$

Nomenclature introduced in Equations 6-17, 18 is explained in Table 6-3.

TABLE 6-3 INFLUENCE VALUES MOMENTS AND REACTIONS

$$\left[\overset{o}{M}\right]_{Pz} = \begin{bmatrix} \left[\overset{o}{M}_h\right]_{Pz} \\ \left[\overset{o}{M}_i\right]_{Pz} \\ \left[\overset{o}{M}_j\right]_{Pz} \\ \left[\overset{o}{M}_k\right]_{Pz} \\ \left[\overset{o}{M}_l\right]_{Pz} \end{bmatrix} \qquad \left[\overset{o}{R}\right]_{Pz} = \begin{bmatrix} \left[\overset{o}{R}_h\right]_{Pz} \\ \left[\overset{o}{R}_i\right]_{Pz} \\ \left[\overset{o}{R}_j\right]_{Pz} \\ \left[\overset{o}{R}_k\right]_{Pz} \\ \left[\overset{o}{R}_l\right]_{Pz} \end{bmatrix}$$

$$\left[H\right] = \begin{bmatrix} [\Sigma H_h] & -[H_{ih}] & & & \\ -[H_{hi}] & [\Sigma H_i] & -[H_{ji}] & & \\ & -[H_{ij}] & [\Sigma H_j] & -[H_{kj}] & \\ & & -[H_{jk}] & [\Sigma H_k] & -[H_{lk}] \\ & & & -[H_{kl}] & [\Sigma H_l] \end{bmatrix}$$

$$\left[\overset{L}{N}\right]_{Pz} = \begin{bmatrix} \left[\overset{Lo}{\Sigma N_h}\right]_{Pz} \\ \left[\overset{Lo}{\Sigma N_i}\right]_{Pz} \\ \left[\overset{Lo}{\Sigma N_j}\right]_{Pz} \\ \left[\overset{Lo}{\Sigma N_k}\right]_{Pz} \\ \left[\overset{Lo}{\Sigma N_l}\right]_{Pz} \end{bmatrix}$$

$$\left[\overset{o}{M}_j\right]_{Pz} = \begin{bmatrix} M_{jx1} & M_{jx2} & M_{jx3} & M_{jx4} \cdots\cdots \\ M_{jy1} & M_{jy2} & M_{jy3} & M_{jy4} \cdots\cdots \\ M_{jz1} & M_{jz2} & M_{jz3} & M_{jz4} \cdots\cdots \end{bmatrix}$$

$$\left[\overset{Lo}{\Sigma N_j}\right]_{Pz} = \begin{bmatrix} \Sigma N_{jx1} & \Sigma N_{jx2} & \Sigma N_{jx3} & \Sigma N_{jx4} \cdots\cdots \\ \Sigma N_{jy1} & \Sigma N_{jy2} & \Sigma N_{jy3} & \Sigma N_{jy4} \cdots\cdots \\ \Sigma N_{jz1} & \Sigma N_{jz2} & \Sigma N_{jz3} & \Sigma N_{jz4} \cdots\cdots \end{bmatrix}$$

CHAPTER VII

PROCEDURE AND APPLICATION

7-1. Procedure of Analysis

The procedure of analysis for continuous beam in space investigated in this study is as follows:

(a) Select a reference system of axes X, Y, Z fixed in direction, and relate the continuous beam to it.

(b) Establish the transformation matrices relating the selected system to the initial systems of the continuous beam through respective direction parameters.

(c) Compute the direction parameters of the normal vectors at intermediate joints, and construct respective jointal and redundant transformation matrices.

(d) By the use of the transformation matrices computed in (b, c), construct the geometrical matrices $[\rho]$, $[\rho]'$, $[\Pi_{yz}]$, and $[\gamma]$.

(e) Compute the deformation constants along the reference system and construct the respective matrix for the beam $[FG^{O}]$.

(f) Compute the angular load functions for the members of the continuous beam; if moving loads are applied compute the influence matrix $[C]$.

(g) Compute the linear deformation constants ψ' for the members of the continuous beam.

(h) Substitute the geometrical and deformation matrices into the general moment equation, and solve for the redundant moments.

(i) Compute other beam functions using the respective matric equations.

7-2. Application

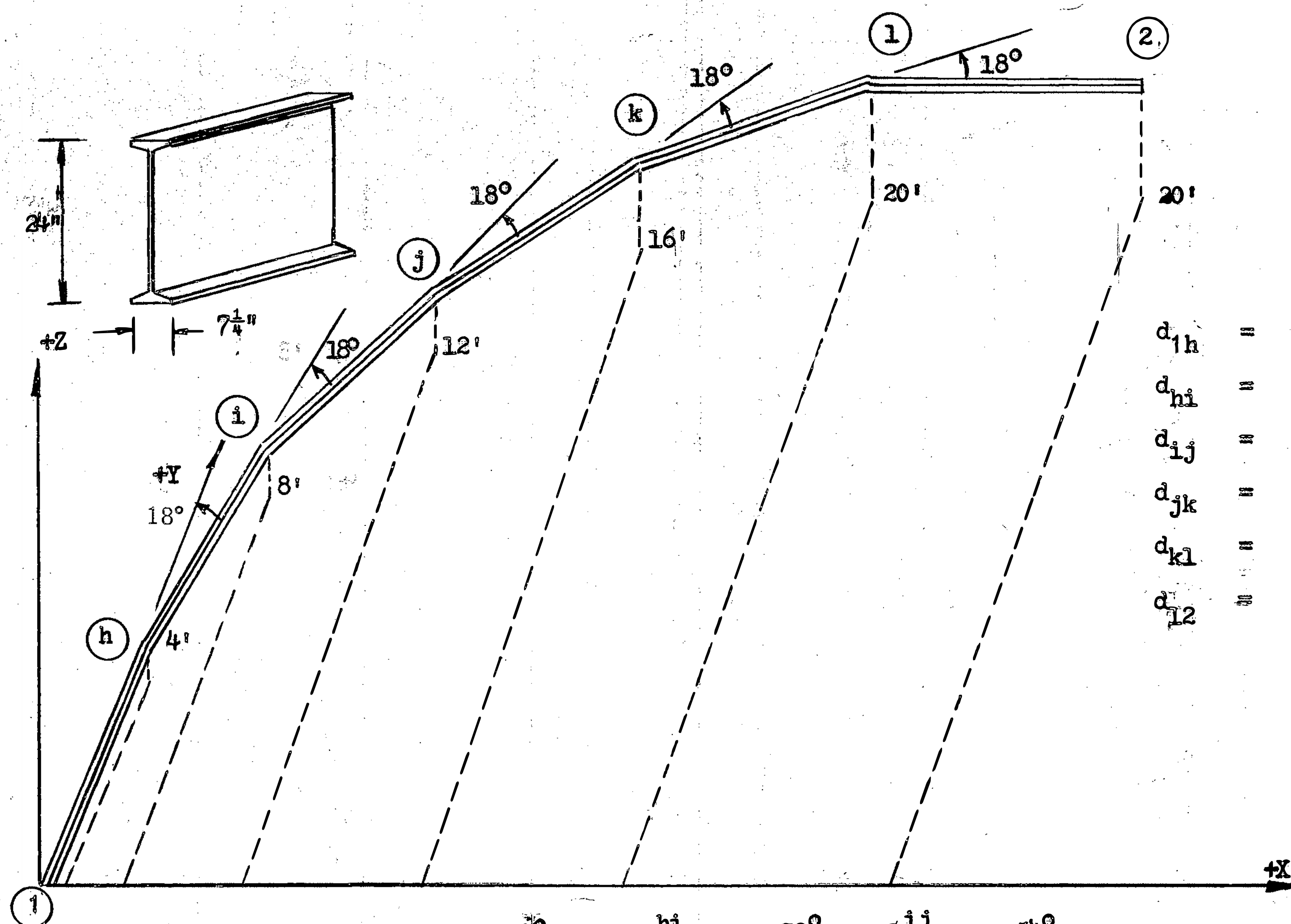
A space continuous steel girder 1 h i j k 1 2 of constant cross-section is considered. The geometry and dimensions are shown in Figure 7-1. The space continuous beam is loaded by a unit moving load in the Z direction and rests on spherical hinged supports at the respective joints. The angle of transformation ϕ_3 is zero for all the basic structures. The coordinates of the joints with respect to a selected reference system of axes X, Y, Z (Fig. 7-1) are indicated, and the transformation angles ϕ_1, ϕ_1' for the members are shown. The transformation angles ϕ_2 are the same for all the basic structures, and they are

$$\phi_2 = \sin^{-1}(-.1000).$$

The transformation matrices corresponding to the initial systems of the girder are recorded in Tables 7-1, 2. Jointal and redundant transformation matrices for this solution are obtained from Tables 7-3, 4. The geometry of redundants is indicated in Table 7-4.

The initial deformation constants for this solution are the same for all the basic structures (Fig. 7-2).

The inversion of the moment equation was carried out by an electronic computer, and the influence values for redundant moments are recorded in Tables 7-5, 6, 7, respectively.



$d_{1h} = 40'$

$d_{hi} = 40'$

$d_{ij} = 40'$

$d_{jk} = 40'$

$d_{kl} = 40'$

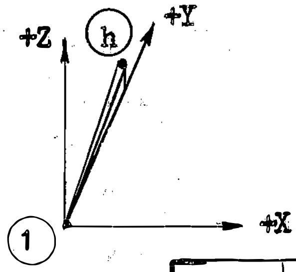
$d_{l2} = 40'$

$\phi_1^{1h} = 90°\ ; \quad \phi_1^{hi} = 72°\ , \quad \phi_1^{ij} = 54°$

$\phi_1^{jk} = 36°\ ; \quad \phi_1^{kl} = 18°\ , \quad \phi_1^{l2} = 0$

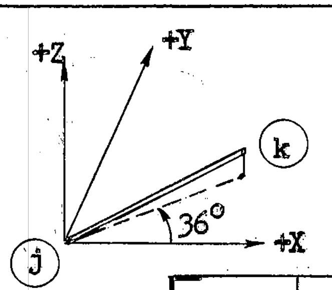
Figure 7-1
Space Continuous Beam

TABLE 7-1 TRANSFORMATION MATRICES



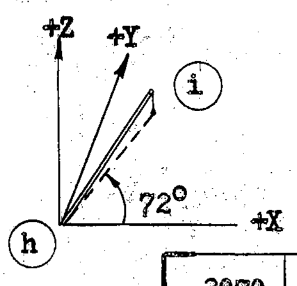
$\left[\frac{1h}{||o1}\right] =$

0000	-1.0000	0000
.9950	0000	-.1000
.1000	0000	.9950



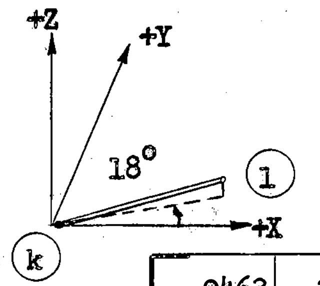
$\left[\frac{jk}{||o1}\right] =$

.8050	-.5878	-.0809
.5849	.8090	-.0588
.1000	0000	.9950



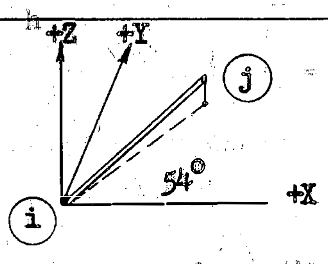
$\left[\frac{hi}{||o1}\right] =$

.3070	-.9511	-.0309
.9463	.3090	-.0951
.1000	0000	.9950



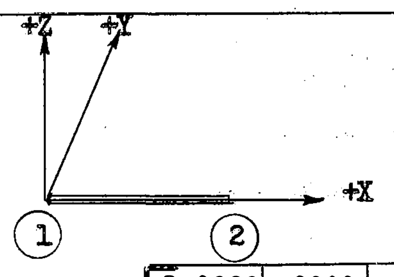
$\left[\frac{kl}{||o1}\right] =$

.9463	-.3090	-.0951
.3070	.9511	-.0309
.1000	0000	.9950



$\left[\frac{ij}{||o1}\right] =$

.5849	-.8090	-.0588
.8050	.5878	-.0809
.1000	0000	.9950



$\left[\frac{l2}{||o1}\right] =$

1.0000	0000	0000
0000	1.0000	0000
0000	0000	1.0000

TABLE 7-2 TRANSFORMATION MATRICES

$$\left[\Pi_{1o}^{1h}\right] = \begin{bmatrix} 0000 & .9950 & .1000 \\ -1.0000 & 0000 & 0000 \\ 0000 & -.1000 & .9950 \end{bmatrix},$$

$$\left[\Pi_{1o}^{jk}\right] = \begin{bmatrix} .8050 & .5849 & .1000 \\ -.5878 & .8090 & 0000 \\ -.0809 & -.0588 & .9950 \end{bmatrix},$$

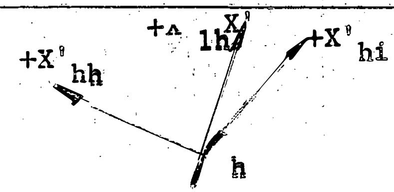
$$\left[\Pi_{1o}^{hi}\right] = \begin{bmatrix} .3070 & .9463 & .1000 \\ -.9511 & .3090 & 0000 \\ -.0309 & -.0951 & .9950 \end{bmatrix}$$

$$\left[\Pi_{1o}^{kl}\right] = \begin{bmatrix} .9463 & .3070 & .1000 \\ -.3090 & .9511 & 0000 \\ -.0951 & -.0309 & .9950 \end{bmatrix}$$

$$\left[\Pi_{1o}^{ij}\right] = \begin{bmatrix} .5849 & .8050 & .1000 \\ -.8090 & .5878 & 0000 \\ -.0588 & -.0809 & .9950 \end{bmatrix},$$

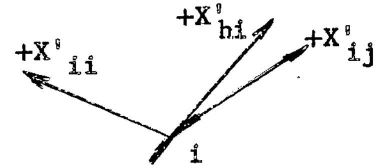
$$\left[\Pi_{1o}^{12}\right] = \begin{bmatrix} 1.0000 & 0000 & 0000 \\ 0000 & 1.0000 & 0000 \\ 0000 & 0000 & 1.0000 \end{bmatrix}$$

TABLE 7-3 JOINTAL TRANSFORMATION MATRICES



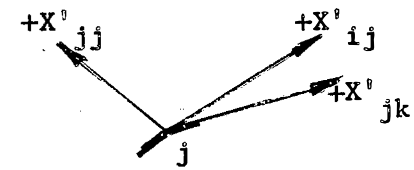
$[\rho_{oh}]$ =

0000	.0158	.3070
.9950	.1000	.9463
.1000	-.9949	.1000



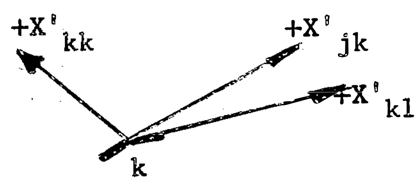
$[\rho_{oi}]$ =

.3070	.0459	.5849
.9463	.0902	.8050
.1000	-.9949	.1000



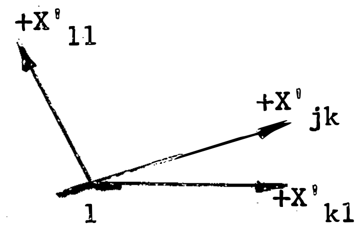
$[\rho_{oj}]$ =

.5849	.0716	.8050
.8050	.0716	.5849
.1000	-.9949	.1000



$[\rho_{ok}]$ =

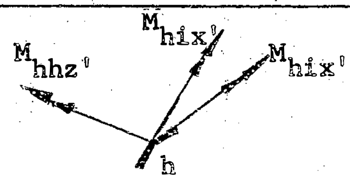
.8050	.0902	.9463
.5849	.0459	.3070
.1000	-.9949	.1000



$[\rho_{ol}]$ =

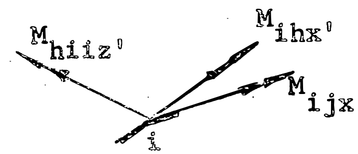
.9463	0000	1.0000
.3070	.3097	0000
.1000	-.9508	0000

TABLE 7-4 REDUNDANT TRANSFORMATION MATRICES



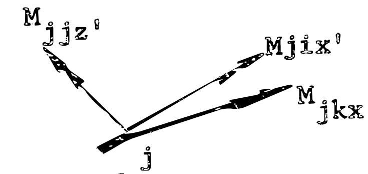
$[\Gamma_{ho}]$

0000	.9950	.1000
.0158	.0902	-.9949
.3070	.9463	.1000



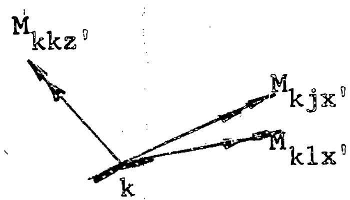
$[\Gamma_{io}]$

.3070	.9463	.1000
.0459	.0902	-.9949
.5849	.8050	.1000



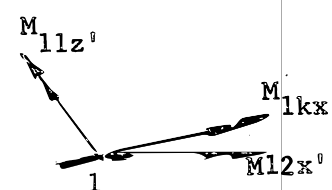
$[\Gamma_{jo}]$

.5849	.8050	.1000
.0716	.0716	-.9949
.8050	.5849	.1000



$[\Gamma_{ko}]$

.8050	.5849	.1000
.0902	.0459	-.9949
.9463	.3070	.1000



$[\Gamma_{lo}]$

.9463	.3070	.1000
0000	.3097	-.9508
1.0000	0000	0000

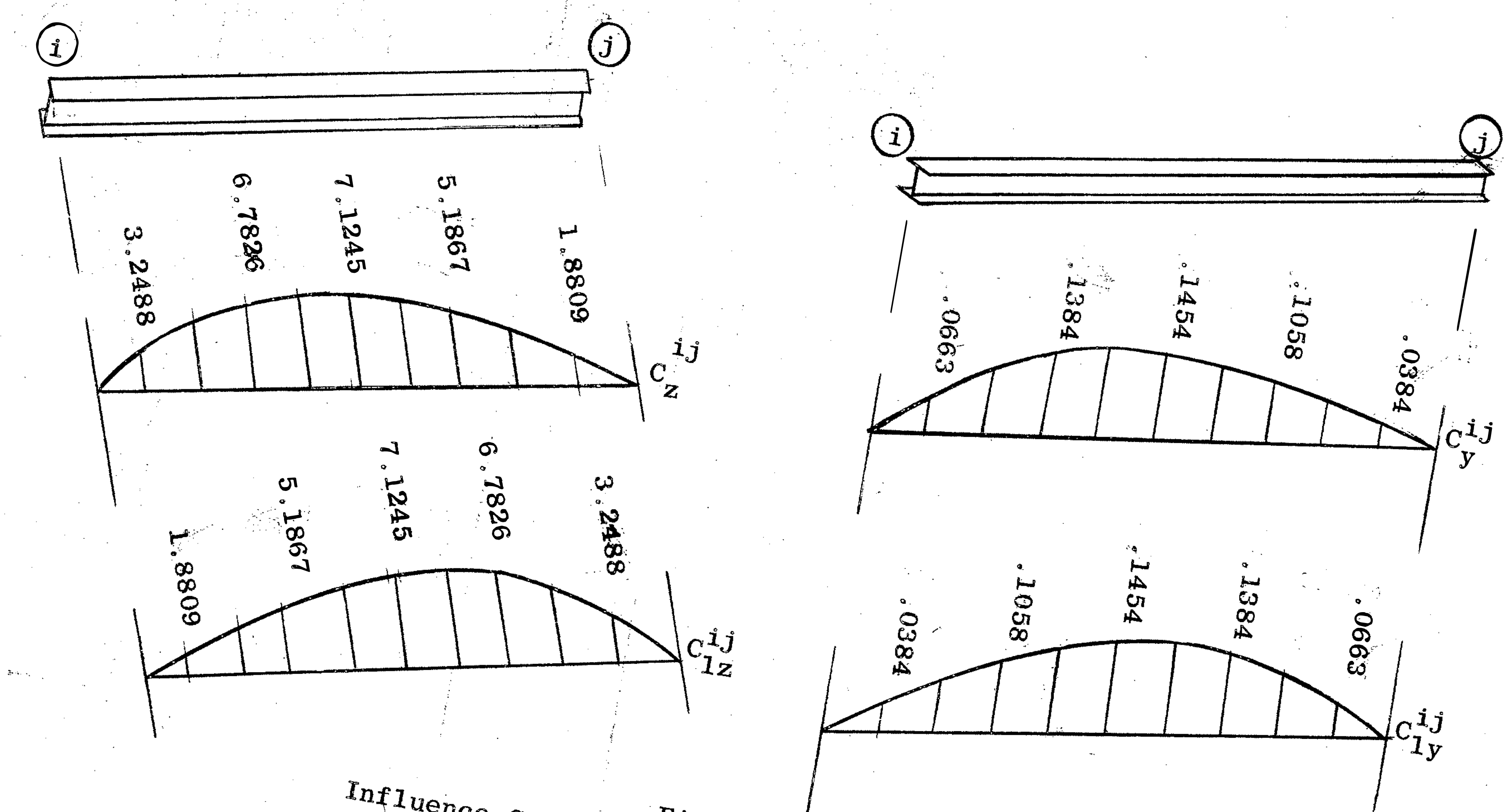


Figure 7-2
Influence Coefficients for Angular Load Functions

TABLE 7-5 INFLUENCE VALUES FOR REDUNDANT MOMENTS

M^R \ M	Redundant Moments For Member 1h x 10^2									
	-.1	-.2	-.3	-.4	-.5	-.6	-.7	-.8	-.9	1.0
$M_{hhz'}$	-.0322	-.0625	-.0889	-.1094	-.1222	-.1251	-.1163	-.0938	-.0557	0
$M_{hix'}$	-.2259	-.4380	-.6228	-.7665	-.8556	-.8760	-.8144	-.6570	-.3900	0
$M_{iiz'}$	+.0121	+.0235	+.0334	+.0411	+.0459	+.0470	+.0437	+.0352	+.0209	0
$M_{ijx'}$	-.0008	-.0016	-.0023	-.0029	-.0032	-.0032	-.0030	-.0025	-.0015	0
$M_{jjz'}$	-.0064	-.0124	-.0176	-.0217	-.0242	-.0248	-.0230	-.0186	-.0110	0
$M_{jkx'}$	-.0004	-.0008	-.0011	-.0014	-.0015	-.0016	-.0015	-.0011	-.0007	0
$M_{kkz'}$	+.0017	+.0033	+.0047	+.0058	+.0064	+.0066	+.0061	+.0049	+.0029	0
$M_{llz'}$	-.0005	-.0009	-.0013	-.0015	-.0017	-.0018	-.0016	-.0013	-.0008	0

TABLE 7-6 INFLUENCE VALUES FOR REDUNDANT MOMENTS

M^R \ M	Redundant Moments for Member hi $\times (10)^2$									
	-1	-2	-3	-4	-5	-6	-7	-8	-9	1.0
$M_{hzz'}$	+.0658	+.1118	+.1400	+.1522	+.1505	+.1367	+.1128	+.0807	+.0425	0
M_{hix}	-.1577	-.2112	-.1864	-.1092	-.0056	+.0985	+.1770	+.2040	+.1537	0
$M_{ijz'}$	-.0475	-.0894	-.1237	-.1488	-.1627	-.1635	-.1495	-.1188	-.0696	0
$M_{ijx'}$	-.2257	-.4374	-.6215	-.7645	-.8529	-.8729	-.8112	-.6542	-.3883	0
$M_{jjz'}$	+.0143	+.0282	+.0408	+.0510	+.0576	+.0596	+.0559	+.0455	+.0272	0
$M_{jkx'}$	-.0011	-.0021	-.0027	-.0031	-.0033	-.0031	-.0028	-.0021	-.0012	0
$M_{kkz'}$	-.0069	-.0136	-.0195	-.0242	-.0272	-.0280	-.0262	-.0212	-.0126	0
$M_{klx'}$	-.0004	-.0008	-.0011	-.0014	-.0015	-.0016	-.0015	-.0012	-.0007	0
$M_{llz'}$	+.0014	+.0028	+.0040	+.0050	+.0057	+.0059	+.0055	+.0044	+.0026	0

TABLE 7-7 INFLUENCE VALUES FOR REDUNDANT MOMENTS

M^R \ M	Redundant Moments For Member jk × $(10)^2$									
	-1	-2	-3	-4	-5	-6	-7	-8	-9	1.0
$M_{hzz'}$	+.0002	+.0012	+.0027	+.0044	+.0059	+.0069	+.0072	+.0063	+.0040	0
$M_{hix'}$	+.0007	+.0012	+.0015	+.0017	+.0016	+.0015	+.0012	+.0009	+.0004	0
$M_{ijz'}$	-.0007	-.0048	-.0107	-.0174	-.0234	-.0276	-.0287	-.0253	-.0161	0
$M_{ijx'}$	+.0011	+.0019	+.0025	+.0028	+.0029	+.0027	+.0024	+.0018	+.0010	0
$M_{jjz'}$	+.0230	+.0519	+.0825	+.1106	+.1323	+.1434	+.1399	+.1175	+.0722	0
$M_{jkx'}$	+.3885	+.6546	+.8119	+.8737	+.8538	+.7654	+.6223	+.4381	+.2261	0
$M_{kkz'}$	-.1182	-.2451	-.3676	-.4726	-.5470	-.5775	-.5512	-.4549	-.2755	0
$M_{klx'}$	-.1554	-.2075	-.1818	-.1043	-.0009	+.1027	+.1802	+.2063	+.1548	0
$M_{llz'}$	+.5404	+1.1385	+1.7278	+2.2415	+2.6131	+2.7751	+2.6617	+2.2057	+1.3406	0

CHAPTER VIII

SUMMARY AND CONCLUSIONS

8-1. Summary

The application of the method of elastic weights and the jointal and redundant systems to the analysis of continuous beams in space is presented in this study. The results of this investigation are general and can be applied to any space continuous beam loaded by any system of loads, stationary or moving.

The "Seven and Nine Moment Equations" are derived for the elasto-static compatibility. The determination of the independent redundants reduces the number of unknowns involved and makes the solution by matric inversion possible.

The procedure outlined for stationary loading is extended to moving loads, and simple matric equations are obtained for the influence values of the beam functions.

8-2. Conclusions

A method of analysis for continuous bent members in space has been developed. The method is based on angular flexibility coefficients and on moment redundants which are selected as the torsions in members and the bendings normal to the planes of adjacent members. This selection makes the redundant moments independent and offers a simple algebraic solution. The application of the normal and axial vectors to the elasto-statics simplifies the analysis of the deformations. The arrangement of the respective equations is attained by the use of the jointal and redundant transformation matrices. The moment matrix which is derived from this investigation is a "Seven and Nine Moment Matrix" similar to the "Three Moment Equation Matrix." This matrix offers an easy application to influence values and leads to a rapid evaluation of the influence areas. This method involves a small number of unknowns and requires less amount of numerical work than any other known method for the analysis of this problem.

The illustration of the theory is demonstrated by a numerical example.

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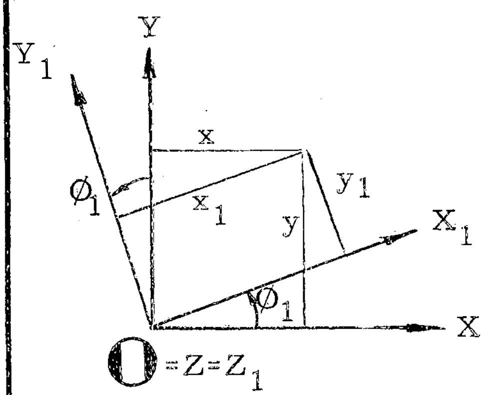
APPENDIX

Tables showing the formation of the transformation matrix are presented. They have been used in the calculations of the respective transformation matrices of the numerical example.

TABLE A-1 TRANSFORMATION OF COORDINATES



Rotation ϕ_1

$$x = x_1 \cos \phi_1 - y_1 \sin \phi_1$$

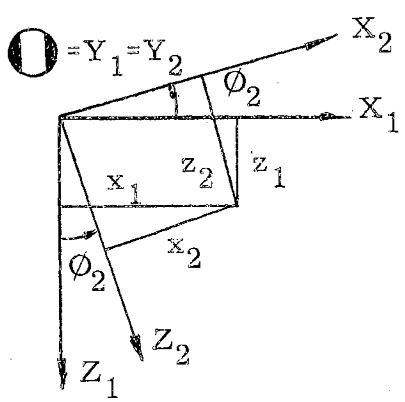
$$y = x_1 \sin \phi_1 + y_1 \cos \phi_1$$

$$z = z_1$$

$$x_1 = x \cos \phi_1 + y \sin \phi_1$$

$$y_1 = -x \sin \phi_1 + y \cos \phi_1$$

$$z_1 = z$$



Rotation ϕ_2

$$x_1 = x_2 \cos \phi_2 + z_2 \sin \phi_2$$

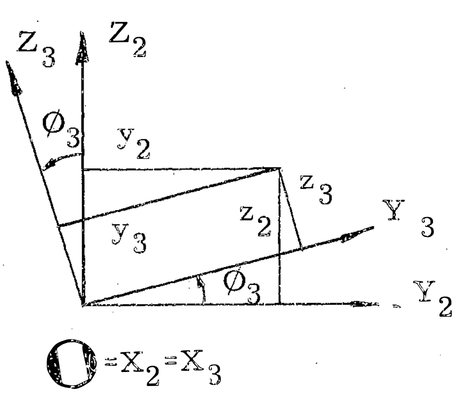
$$y_1 = y_2$$

$$z_1 = x_2 \sin \phi_2 + z_2 \cos \phi_2$$

$$x_2 = x_1 \cos \phi_2 - z_1 \sin \phi_2$$

$$y_2 = y_1$$

$$z_2 = x_1 \sin \phi_2 + z_1 \cos \phi_2$$



Rotation ϕ_3

$$x_2 = x_3$$

$$y_2 = y_3 \cos \phi_3 - z_3 \sin \phi_3$$

$$z_2 = y_3 \cos \phi_3 + z_2 \cos \phi_3$$

$$x_3 = x_2$$

$$y_3 = y_2 \cos \phi_3 + z_2 \sin \phi_3$$

$$z_3 = -y_2 \sin \phi_3 + z_2 \cos \phi_3$$

TABLE A-2 TRANSFORMATION MATRICES

	x'	y'	z'
x	α_x	α_y	α_z
y	β_x	β_y	β_z
z	γ_x	γ_y	γ_z

Transformation Matrix

$$\beta_x = \sin \phi_1 \cos \phi_2$$

$$\beta_y = \cos \phi_1 \cos \phi_3 + \sin \phi_1 \sin \phi_2 \sin \phi_3$$

$$\beta_z = - \cos \phi_1 \sin \phi_3 + \sin \phi_1 \sin \phi_2 \cos \phi_3$$

$$x = x' \alpha_x + y' \alpha_y + z' \alpha_z$$

$$y = x' \beta_x + y' \beta_y + z' \beta_z$$

$$z = x' \gamma_x + y' \gamma_y + z' \gamma_z$$

$$\alpha_x = \cos \phi_1 \cos \phi_2$$

$$\alpha_y = - \sin \phi_1 \cos \phi_3 + \cos \phi_1 \sin \phi_2 \sin \phi_3$$

$$\alpha_z = \sin \phi_1 \sin \phi_3 + \cos \phi_1 \sin \phi_2 \cos \phi_3$$

$$x' = x \alpha_x + y \beta_x + z \gamma_x$$

$$y' = x \alpha_y + y \beta_y + z \gamma_y$$

$$z' = x \alpha_z + y \beta_z + z \gamma_z$$

$$\gamma_x = - \sin \phi_2$$

$$\gamma_y = \cos \phi_2 \sin \phi_3$$

$$\gamma_z = \cos \phi_2 \cos \phi_3$$

VITA

Mohammad Esmet Kamel

Candidate for the Degree of

Doctor of Philosophy

Thesis: INFLUENCE AREAS FOR CONTINUOUS BEAMS IN SPACE

Major Field: Engineering

Biographical:

Personal Data: Born August 10, 1930, Asyut, Egypt, the son of the late and Mrs. Kamel Abdul Aaty.

Education: Graduated from Sohag High School, Egypt, in May, 1946. Received the degree of Bachelor of Science in Civil Engineering from Cairo University in July, 1952. Received the Master of Science degree from Cairo University with a major in Structural Engineering in 1958. Completed requirements for the degree of Doctor of Philosophy in August, 1962. Elected to associate membership in the Society of Sigma Xi, and to the membership of the honorary fraternity, Chi Epsilon.

Professional Experience: Bridge Engineer for the Egyptian State Railways, 1952-1960. Member of the U.A.R. Society of Civil Engineers, Vice-President of the E.S.R. Engineers. Supervisor, Department of Bridge Engineering, Cairo, Egypt, U.A.R.