CONVEX CONES OF SUPER-(L) FUNCTIONS

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PREFACE

A generalization of the concave function concept is the super-(L) function concept introduced by F. F. Bonsall [1]. (The number in the bracket refers to the bibliography.) Certain properties of super-(L) functions are investigated in this paper. The basic definitions and some properties of super-(L) functions are given in Chapter I, with the fundamental result being the fact that the set of non-negative super-(L) functions on $[0,1]$ forms a convex cone C. Chapters II through VI are devoted to consequences of this result, with continuity of the functions at the end points assumed in Chapters II through V.

In Chapter II, the extremal structure of the convex cone C is characterized, and a type of integral representation for the elements of C in terms of the extremal elements of C is developed in Chapter III. The structure of the linear space C-C is partially determined in Chapter IV. The relationship between the extremal elements of C and Green's function is discussed in Chapter V. In Chapter VI, the extremal structure of the convex cone of discontinuous super-(L) functions on $[0,1]$ is characterized. In Chapter VII, a partial solution is obtained to the problem of extending the preceding results to super-(L) functions on a convex compact domain in E^2, and the paper is ended with an indication of some unsolved problems.

Indebtedness is acknowledged to the members of my advisory committee; to Dr. L. Wayne Johnson, Head of the Department of Mathematics, for my graduate assistantship, for his friendly counsel, and for his thoughtfulness in the arrangement of my teaching assignment; and especially to Professor E. K. McLachlan for the inspiration and encouragement he provided before and during the writing of this paper.

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CHAPTER I

INTRODUCTION

One of the more interesting generalizations of the notion of a real-valued concave function of one real variable is that due to F. F. Bonsall [1]. It is this generalization that is investigated in this thesis.

Definition 1. Let y and z be arbitrary real numbers, and let u and v be real numbers such that $0 \leqq u < v \leqq 1$. Let $L(y) \equiv d^2y/dx^2 + p(x) \cdot dy/dx + q(x)y = 0$ be such that there exists a unique solution F on $[0,1]$ (where the appropriate one-sided derivatives are used at the end-points 0 and 1) for which $F(u) = y$ and $F(v) = z$. Then a real-valued function f is <u>super-(L)</u> on $[0,1]$ if $f(x) \geqq F(f,u,v;x)$ for all x, u, and v such that $0 \leqq u < v \leqq 1$ and $u \leqq x \leqq v$, where $F(f,u,v;x)$ is the solution of $L(y) = 0$ such that $F(f,u,v;u) = f(u)$ and $F(f,u,v;v) = f(v)$.

Definition 2. A function f is <u>concave</u> on $[a,b]$, where $a<b$, if $f(tx+(1-t)y) \geqq tf(x)+(1-t)f(y)$ for all x and y in $[a,b]$ and all t such that $0 \leqq t \leqq 1$.

Observe that the ordinary concave function definition, Definition 2, is the special case of Definition 1 obtained by using $L(y) \equiv d^2y/dx^2$.

Definition 3. A function f is <u>sub-(L)</u> on $[0,1]$ if $-f$ is super-(L) on $[0,1]$.

Definition 4. A function f is <u>convex</u> if $-f$ is concave.

Bonsall studied properties of sub-(L) functions on an arbitrary (bounded) open interval; however, it was found to be more fruitful in this thesis to use a closed interval. For convenience, the closed interval $[0,1]$ was chosen. All the results obtained are also valid in any closed interval. The definitions and results due to others have been rephrased, when desirable, to fit the particular setting used.

Generalizations of the convex function definition which contain the sub-(L) function definition as a special case have been considered by Valiron [2] and Beckenbach [3].

Bonsall characterized sub-(L) functions by proving that if f is sub-(L) in (a,b) then f has a second derivative almost everywhere in (a,b) and $L(f) \geqq 0$ at each point where the second derivative exists, and that if f has a continuous second derivative and $L(f) \geqq 0$ in (a,b), then f is sub-(L) in (a,b). He pointed out that this characterization makes it possible to use sub-(L) functions as an analytical tool in a manner similar to the use of convex functions.

Some basic properties of super-(L) functions are given in the next two theorems due to Bonsall.

Theorem 1 [1, p. 101]. If f is super-(L) on $[0,1]$, then $f(x) \leqq F(f,u,v;x)$ for all x in $[0,u]$ and $[v,1]$, where $u<v$.

Theorem 2 [1, p. 102]. If f is super-(L) on $[0,1]$, then f is continuous in $(0,1)$.

By Theorem 2, a super-(L) function is continuous in $(0,1)$. Unless stated to the contrary, it will be assumed in the remainder of this thesis that the super-(L) functions considered are also continuous at 0 and at 1.

The basic result which forms the starting point of the investigations made in this thesis is the fact that the set of non-negative super-(L) functions on $[0,1]$ is a convex cone.

Definition 5. Let A be a set in a real linear space. Then A is a <u>convex cone</u> if 1) for every f and g in A and every non-negative real number k, f+g and kf belong to A, and 2) f in A and -f in A imply f=0, the origin of the real linear space.

Lemma 1. If f and g are super-(L) functions on $[0,1]$ and k is a non-negative real number, then kf and f+g are super-(L) functions on $[0,1]$.

Proof. Let u and v be such that $0 \leqq u < v \leqq 1$. Since $L(y) = 0$ is a linear homogeneous differential equation, $kF(f,u,v;x)$ and $F(f,u,v;x) + F(g,u,v;x)$ are solutions. Then $F(kf,u,v;x) = kF(f,u,v;x)$ for all x in $[0,1]$ is the unique solution of $L(y) = 0$ such that $F(kf,u,v;u) = kF(f,u,v;u) = kf(u) = (kf)(u)$, $F(kf,u,v;v) = kF(f,u,v;v) = kf(v) = (kf)(v)$, and $(kf)(x) = kf(x) \geqq kF(f,u,v;x) = F(kf,u,v;x)$ for all x in $[u,v]$. Thus kf is a super-(L) function on $[0,1]$.

Next, $F(f+g,u,v;x) = F(f,u,v;x)+F(g,u,v;x)$ for all x in $[0,1]$ is the unique solution of $L(y) = 0$ such that $F(f+g,u,v;u) = F(f,u,v;u) + F(g,u,v;u) = f(u)+g(u) = (f+g)(u)$, $F(f+g,u,v;v) = F(f,u,v;v)+F(g,u,v;v) = f(v)+g(v) = (f+g)(v)$, and $(f+g)(x) = f(x)+g(x) \geqq F(f,u,v;x) + F(g,u,v;x) = F(f+g,u,v;x)$ for all x in $[u,v]$. Thus f+g is a super-(L) function on $[0,1]$.

Theorem 3. The set of non-negative super-(L) functions on $[0,1]$ forms a convex cone C.

Proof. Let k be a non-negative real number. Let f and g belong to C. By Lemma 1, kf and f+g are super-(L) functions on $[0,1]$. Since $k \geqq 0$, $f \geqq 0$, and $g \geqq 0$, it follows that $kf \geqq 0$ and $f+g \geqq 0$. Thus kf and f+g belong to C. Let h be any element of C such that -h is also an element of C. Then $h \geqq 0$ and $-h \geqq 0$ imply $h = 0$, the non-negative super-(L) function which is identically 0 on $[0,1]$. Thus C is a convex cone.

CHAPTER II

EXTREMAL STRUCTURE OF C

In this chapter, those elements which are extremal elements of the convex cone C of non-negative super-(L) functions on $[0,1]$ will be characterized.

Definition 6. Let A be a convex cone. An element f of A is called an <u>extremal element</u> of A if for every pair of elements g and h of A such that f = g+h there exists a real number k such that g = kf.

McLachlan [4] has completely characterized the extremal structure of the convex cone of non-negative concave functions on $[0,1]$. It will be shown in this chapter that the extremal structure of C is analogous to that obtained by McLachlan.

Definition 7. A real-valued function f on $[0,1]$ is said to be <u>(L)-linear on $[u,v]$</u> if $f(x) = F(f,u,v;x)$ for all x in $[u,v]$, where $0 \leqq u < v \leqq 1$.

Theorem 4. If f, g, and h are super-(L) functions on $[0,1]$ such that $f(x) = g(x)+h(x)$ for all x in $[u,v]$, where $0 \leqq u < v \leqq 1$, and f is (L)-linear on $[u,v]$, then g and h are (L)-linear on $[u,v]$.

Proof. Since f is (L)-linear on $[u,v]$, $f(x) = F(f,u,v;x)$ for all x in $[u,v]$ by definition. Then $g(x)+h(x) = (g+h)(x) = f(x) = F(f,u,v;x) = F(g+h,u,v;x) = F(g,u,v;x)+F(h,u,v;x)$ as in the proof of

Lemma 1. Since g and h are super-(L) on $[0,1]$, $g(x) \geqq F(g,u,v;x)$ and $h(x) \geqq F(h,u,v;x)$ for all x in $[u,v]$. If $g(w) > F(g,u,v;w)$ or $h(w) > F(h,u,v;w)$ for some w in $[u,v]$, then $g(w)+h(w) > F(g,u,v;w) + F(h,u,v;w)$, which contradicts $g(x)+h(x) = F(g,u,v;x)+F(h,u,v;x)$ for all x in $[u,v]$. Thus $g(x) = F(g,u,v;x)$ and $h(x) = F(h,u,v;x)$ for all x in $[u,v]$, and hence g and h are (L)-linear on $[u,v]$.

Theorem 5. If f, g, and h are elements of the convex cone C such that there exists a u in $[0,1]$ for which $f(u) = g(u)+h(u)$ and $f(u) = 0$, then $g(u) = 0$ and $h(u) = 0$.

Proof. Since g and h are elements of C, $g(u) \geqq 0$ and $h(u) \geqq 0$. Thus $g(u)+h(u) = 0$ implies $g(u) = 0$ and $h(u) = 0$.

Definition 8. A real-valued function f on $[0,1]$ is an <u>(L)-conical</u> function with its vertex over w in $[0,1]$ if 1) $f(w) > 0$, 2) $f(0) = f(1) = 0$ if $w \neq 0,1$; $f(0) = 0$ if $w = 1$; or $f(1) = 0$ if $w = 0$; and 3) f is (L)-linear on $[0,w]$ and on $[w,1]$.

Theorem 6. If f is an element of the convex cone C such that $f(w) = 0$ for some w in $(0,1)$, then $f = 0$.

Proof. Suppose there exists a u in $[0,1]$ such that $f(u) > 0$. Assume $u < w$. (A similar proof holds for the case $u > w$). Suppose there is a v in $(w,1]$ such that $f(v) = 0$. Since f is super-(L), $F(f,u,v;w) \leqq f(w) = 0$ and $F(f,u,v;u) = f(u) > 0$, so there exists a z in $(u,w]$ such that $F(f,u,v;z) = 0$ since $F(f,u,v;x)$ is a continuous function. (See Figure 1.) Then $F(f,u,v;x) \equiv 0$, the unique solution of $L(y) = 0$

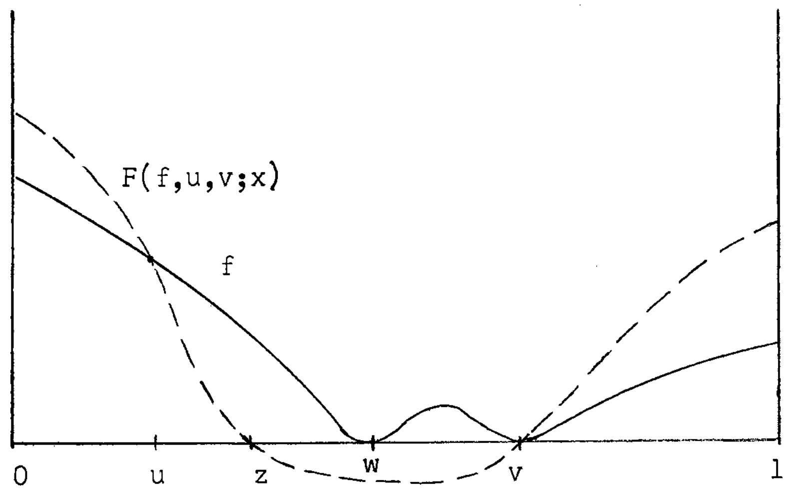

Figure 1. Theorem 6 proof. The assumption that $f(v) = 0$.

which has zero function value at two distinct points, since $F(f,u,v;v) = f(v) = 0$. This contradicts $F(f,u,v;u) = f(u) > 0$, so that $f(x) > 0$ for all x in $(w,1]$. Since f is super-(L), $F(f,0,1;w) \leqq f(w) = 0$. Suppose $F(f,0,1;w) < 0$. Now $F(f,0,1;1) = f(1) > 0$, so that there exists a z in $(w,1)$ such that $F(f,0,1;z) = 0$, since $F(f,0,1;x)$ is a continuous function. Also, since $F(f,0,1;0) = f(0) \geqq 0$ there exists a t in $[0,w)$ such that $F(f,0,1;t) = 0$. Thus $F(f,0,1;x) \equiv 0$, the unique solution of $L(y) = 0$ which has zero function value at two distinct points which contradicts $F(f,0,1;1) = f(1) > 0$. Therefore $F(f,0,1;w) = 0 = f(w)$. Since f is super-(L), $f(x) \geqq F(f,0,w;x)$ for all x in $[0,w]$ and $f(x) \geqq F(f,w,1;x)$ for all x in $[w,1]$. (See Figure 2.) By Theorem 1, $f(x) \leqq F(f,0,w;x)$ for all x in $[w,1]$ and $f(x) \leqq F(f,w,1;x)$ for all x in $[0,w]$. Then $F(f,0,w;0) = f(0) = F(f,0,1;0)$, $F(f,w,1;1) = f(1) = F(f,0,1;1)$, and $F(f,0,w;w) = F(f,w,1;w) = F(f,0,1;w) = f(w)$, so by uniqueness of solution to $L(y) = 0$, $F(f,0,w;x) \equiv F(f,w,1;x) \equiv F(f,0,1;x)$.

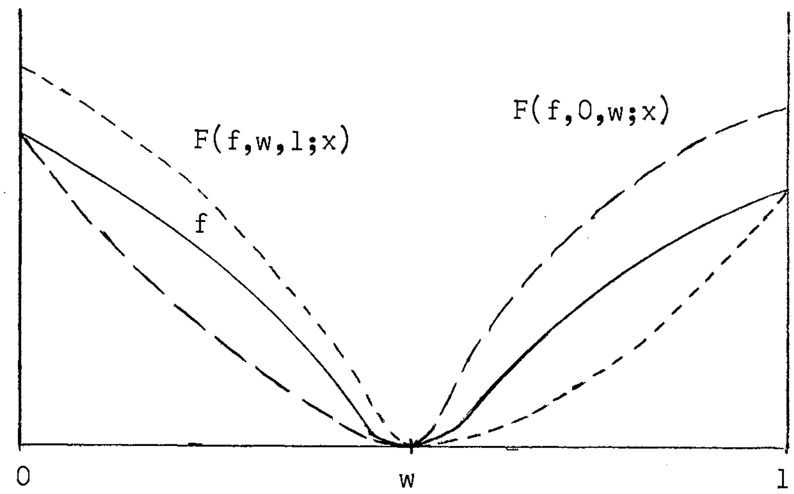

Figure 2. Theorem 6 proof. The proof that f is (L)-linear.

Thus $f(x) = F(f,0,1;x)$ for all x in $[0,1]$.

Let u and v be such that $0 \leqq u < w < v \leqq 1$. Let r and s be arbitrary positive real numbers. There exist non-negative real numbers m and n such that $r = mF(f,0,1;u)$ and $s = nF(f,0,1;v)$, since $F(f,0,1;x) > 0$ for $x \neq w$ in $[0,1]$. Since $L(y) = 0$ is a linear homogeneous differential equation, $mF(f,0,1;x)$ and $nF(f,0,1;x)$ are solutions. Let G be the solution of $L(y) = 0$ such that $G(u) = r$ and $G(v) = s$. If $m \geqq n$ then $mF(f,0,1;v) \geqq nF(f,0,1;v) = s = G(v)$. (See Figure 3.) Suppose $G(w) < 0$. Then since G is continuous and $G(v) > 0$, there exists a z in (w,v) such that $G(z) = 0$, and since $G(u) > 0$ there exists a t in (u,w) such that $G(t) = 0$. This implies $G(x) \equiv 0$, the unique solution of $L(y) = 0$ having 0 function value at two distinct points, which contradicts $G(u) > 0$. Therefore $G(w) \geqq 0 = F(f,0,1;w)$. Then since $G(v) \leqq mF(f,0,1;v)$ and $mF(f,0,1;x)$ is continuous, there exists a z' in $[w,v]$ such that $G(z') = mF(f,0,1;z')$. Then $G(x) \equiv mF(f,0,1;x)$ by uniqueness

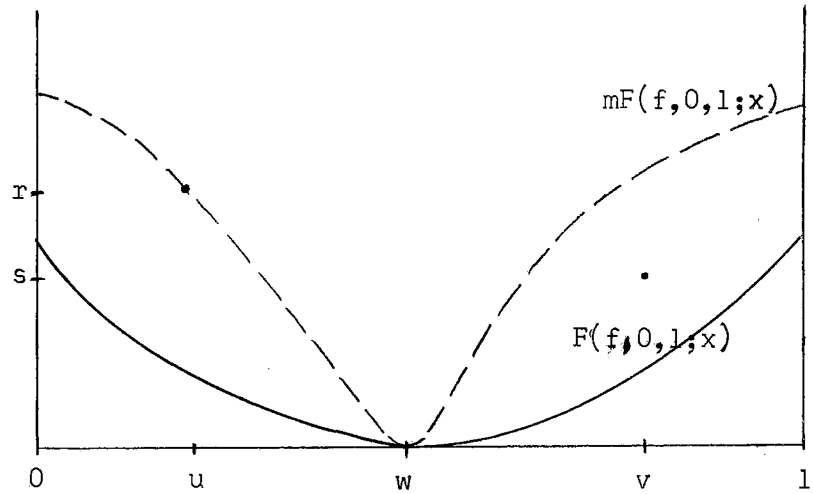

Figure 3. Theorem 6 proof. Contradiction of f being positive.

of solution to $L(y) = 0$, since $G(u) = r = mF(f,0,1;u)$. Similarly, if $n \geqq m$ then $G(x) \equiv nF(f,0,1;x)$, so that in either case $G(w) = 0$. Thus any solution of $L(y) = 0$ taking a positive function value in $[0,w)$ and a positive function value in $(w,1]$ is 0 at w, which contradicts the existence of a solution taking arbitrary positive function values at w and at $u \neq w$, since such a solution would be positive in an open interval containing w. Therefore there does not exist a u in $[0,1]$ for which $f(u) > 0$, and the theorem is proved.

It is now possible to characterize concisely the extremal structure of C; however, the proof of the characterization is long, so it will be developed in a series of lemmas.

Lemma 2. If f is an (L)-conical function with its vertex over 0 or 1, then f is an extremal element of C.

Proof. Let g and h be elements of C such that $f = g+h$. By

Definition 8, f is (L)-linear on $[0,1]$, and thus is super-(L) on $[0,1]$. Since f is zero at one end-point and positive at the other end-point, it is non-negative by uniqueness of solution to $L(y) = 0$. Hence f belongs to C. Assume f has its vertex over 0 since the proof for the other case is similar. By Theorem 4, g and h are (L)-linear on $[0,1]$. By Theorem 5, $g(1) = 0$ and $h(1) = 0$ since $f(1) = 0$. Since $g \neq 0$, $g(x) > 0$ for all x in $[0,1)$. Let $k = g(0)/f(0)$. Then g and kf intersect in two distinct points since $g(0) = kf(0)$ and $(kf)(1) = 0 = g(1)$, and so by uniqueness of solution to $L(y) = 0$, $g = kf$. Thus f is an extremal element of C.

If f is an (L)-conical function which has its vertex over w in $(0,1)$, then by Definition 8, $f(x) = F(f,0,w;x)$ for all x in $[0,w]$ and $f(x) = F(f,w,1;x)$ for all x in $[w,1]$. The relationship between these two (L)-linear functions is given in the following lemma.

Lemma 3. If f is an (L)-conical function which has its vertex over w in $(0,1)$, then $F(f,0,w;x) > F(f,w,1;x)$ for all x in $(w,1]$ and $F(f,0,w;x) < F(f,w,1;x)$ for all x in $[0,w)$.

Proof. If $F(f,0,w;x') = F(f,w,1;x')$ for some x' in $(w,1]$, then $F(f,0,w;x) = F(f,w,1;x)$ for all x in $[0,1]$ by uniqueness of solution to $L(y) = 0$, since $F(f,0,w;w) = F(f,w,1;w)$ and $x' \neq w$. Then $F(f,0,w;0) = 0$ and $F(f,0,w;1) = F(f,w,1;1) = 0$ imply $F(f,0,w;x) \equiv 0$, the unique solution of $L(y) = 0$ which is 0 at two distinct points. This contradicts $F(f,0,w;w) = f(w) > 0$. If $F(f,0,w;x') < F(f,w,1;x')$ for some x' in $(w,1)$, suppose that $F(f,0,w;1) \geqq F(f,w,1;1)$. Then there exists an x" in $(x',1]$ such that $F(f,0,w;x") = F(f,w,1;x")$ since $F(f,0,w;x)$ and $F(f,w,1;x)$ are continuous functions of x. Then by uniqueness of solution to $L(y) = 0$, $F(f,0,w;x) = F(f,w,1;x)$ for all x in $[0,1]$ since $F(f,0,w;w) = F(f,w,1;w)$, which, as before, leads to the contradiction that $F(f,0,w;x) \equiv 0$. Thus $F(f,0,w;1) < F(f,w,1;1) = 0$. Since $F(f,0,w;w) = f(w) > 0$ and $F(f,0,w;x)$ is continuous, there exists a t in $(w,1)$ such that $F(f,0,w;t) = 0$. Then $F(f,0,w;x) = 0$ for all x in $[0,1]$ by uniqueness of solution to $L(y) = 0$ since $F(f,0,w;0) = f(0) = 0$. This contradicts $F(f,0,w;w) = f(w) > 0$. Therefore $F(f,0,w;x) > F(f,w,1;x)$ for all x in $(w,1]$.

The proof for $F(f,0,w;x) < F(f,w,1;x)$ for all x in $[0,w)$ is similar to the above proof.

Lemma 4. If f is an (L)-conical function which has its vertex over w in $(0,1)$, then f is an extremal element of C.

Proof. The proof will be given in two parts: (i) it will be shown that f belongs to C, and (ii) it will be proved that f is an extremal element of C by showing that every (non-zero) decomposition of f is proportional to f.

(i). In order for f to be an element of C, it must be both non-negative and super-(L).

By uniqueness of solution to $L(y) = 0$, an (L)-linear function which is not identically zero can have zero function value at only one point. Thus f is non-negative on $[0,w]$ and $[w,1]$ and hence on $[0,1]$.

It will now be shown that f is super-(L). Let u and v be such that $0 \leqq u < v \leqq 1$. If u and v are both in $[0,w]$ or both in $[w,1]$, the result follows immediately since f is (L)-linear on $[0,w]$ and on $[w,1]$; hence it will be assumed that $u < w < v$. The proof will be by contraposition. Suppose there exists a z such that $u < z < v$ and

$f(z) < F(f,u,v;z)$. Suppose there is also a z' such that $u < z' < v$ and $F(f,u,v;z') \leqq f(z')$. Then there exists an x' in (u,v) such that $f(x') = F(f,u,v;x')$ since $f(x)$ and $F(f,u,v;x)$ are continuous functions of x. If x' is in $(u,w]$, $F(f,u,v;x) = F(f,0,w;x)$ for all x in $[0,1]$, since $F(f,u,v;u) = F(f,0,w;u) = f(u)$. Then $F(f,0,w;x) = F(f,w,1;x)$ for all x in $[0,1]$ by uniqueness of solution to $L(y) = 0$ since $F(f,0,w;w) = F(f,w,1;w)$, $F(f,0,w;v) = F(f,u,v;v) = F(f,w,1;v)$, and $w \neq v$. This implies $F(f,0,w;x) \equiv 0$ since $F(f,0,w;0) = f(0) = 0$ and $F(f,0,w;1) = F(f,w,1;1) = f(1) = 0$, which contradicts $F(f,0,w;w) = f(w) > 0$. Similarly, taking x' in $[w,v)$ leads to a contradiction. Thus $F(f,u,v;x) > f(x)$ for all x in (u,v).

In particular, $F(f,u,v;w) > f(w) = F(f,0,w;w)$. Then there exists an x' in (w,v) such that $F(f,0,w;x') = F(f,u,v;x')$, since $F(f,0,w;v)$ $F(f,w,1;v) = F(f,u,v;v)$ by Lemma 3, and since $F(f,0,w;x)$ and $F(f,u,v;x)$ are continuous. (See Figure 4.) Then $F(f,0,w;x) = F(f,u,v;x)$ for all

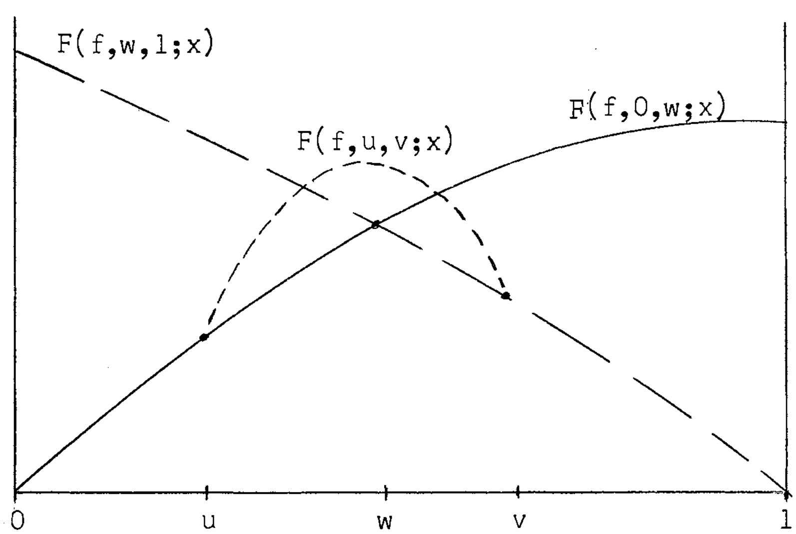

Figure 4. Lemma 4 proof.

x in $[0,1]$ by uniqueness of solution to $L(y) = 0$ since $F(f,0,w;u) = F(f,u,v;u)$ and $u \neq x'$. Then $F(f,0,w;x) = F(f,w,1;x)$ for all x in $[0,1]$ since $F(f,0,w;w) = F(f,w,1;w)$ and $F(f,0,w;v) = F(f,u,v;v) = F(f,w,1;v)$. Now $F(f,0,w;1) = F(f,w,1;1) = 0$ implies $F(f,0,w;x) \equiv 0$, which contradicts $F(f,0,w;w) > 0$. Therefore $F(f,u,v;x) \leqq f(x)$ for all x in $[u,v]$, and hence f is super-(L).

(ii). In this part it will be shown that every non-zero decomposition of f is proportional to f. Let g and h be any elements of C such that $f = g+h$. By Theorem 4, g and h are (L)-linear on $[0,w]$ and on $[w,1]$. Let $k = g(w)/f(w)$. Note that $g(w) \neq 0$ since $g(0) = 0$ and g is not identically zero. Then $kF(f,0,w;x)$ is the unique solution of $L(y) = 0$ such that $kF(f,0,w;0) = 0 = g(0)$ and $kF(f,0,w;w) = g(w)$. So $g(x) = F(g,0,w;x) = kF(f,0,w;x) = kf(x)$ for all x in $[0,w]$. Similarly, $kF(f,w,1;x)$ is the unique solution of $L(y) = 0$ such that $kF(f,w,1;w) = g(w)$ and $kF(f,w,1;1) = 0 = g(1)$. So $g(x) = F(g,w,1;x) = kF(f,w,1;x) = kf(x)$ for all x in $[w,1]$. Thus $g(x) = kf(x)$ for all x in $[0,1]$, and the decomposition is proportional. Therefore any (L)-conical function for which w is in $(0,1)$ is an extremal element of C.

Lemma 5. If f is an element of C which is not (L)-linear on $[0,1]$ and is such that either $f(0) > 0$ or $f(1) > 0$, then f is not an extremal element of C.

Proof. It will be assumed that $f(0) > 0$ since the proof for the case where $f(0) = 0$ and $f(1) > 0$ is similar.

Let $g(x) \equiv F(f,0,1;x)$ and $h = f-g$. (See Figure 5.) Suppose $F(f,0,1;u) < 0$ for some u in $[0,1]$. Then since $F(f,0,1;x)$ is continuous, and since $F(f,0,1;0) = f(0) > 0$ and $F(f,0,1;1) = f(1) \geqq 0$,

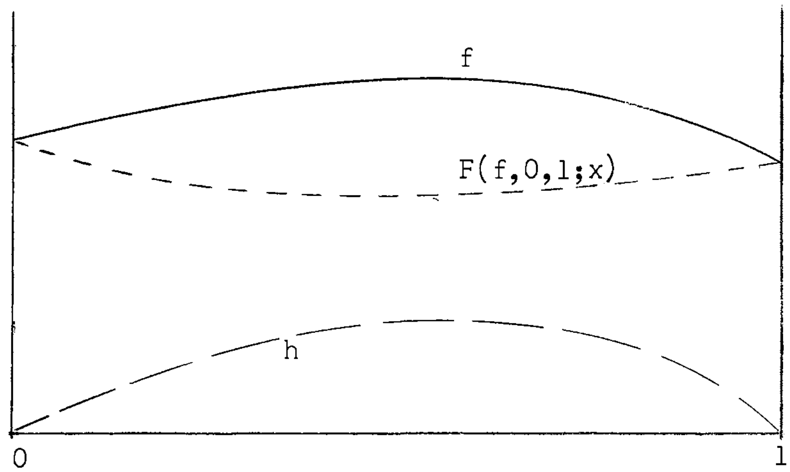

Figure 5. Lemma 5 proof.

there exist an x' in $(0,u)$ and an x'' in $(u,1]$ such that $F(f,0,1;x') = 0$ and $F(f,0,1;x'') = 0$. Thus $F(f,0,1;x) \equiv 0$ by uniqueness of solution to $L(y) = 0$, which contradicts $F(f,0,1;0) = f(0) > 0$. Therefore g is non-negative. Since $F(f,0,1;x)$ is (L)-linear, g is super-(L). Thus g belongs to C.

Since f is super-(L), $f(x) \geqq F(f,0,1;x)$ for all x in $[0,1]$. Thus $h(x) \equiv f(x)-F(f,0,1;x)$ is non-negative. By Lemma 1, h is a super-(L) function since $h(x) = f(x) + (-F(f,0,1;x))$ and $-F(f,0,1;x)$ is an (L)-linear function. Thus h belongs to C.

The decomposition is non-proportional since g is an (L)-linear function, but kf is not (L)-linear for any non-zero real number k. Therefore f is not an extremal element of C.

Lemma 6. If f is an element of C which is (L)-linear on $[0,1]$ and such that $f(0) > 0$ and $f(1) > 0$, then f is not an extremal element of C.

Proof. Let h be the (L)-linear function such that $h(0) = 0$ and $h(1) = f(1)$. Let g be the (L)-linear function such that $g(0) = f(0)$ and $g(1) = 0$. (See Figure 6.) By uniqueness of solution to $L(y) = 0$, a not-identically-zero (L)-linear function can be 0 at only one point, so that g and h are non-negative. Since g and h are (L)-linear they are super-(L). Thus g and h belong to C. Now g+h is the solution of $L(y) = 0$ such that $(g+h)(0) = g(0)+h(0) = f(0)$ and $(g+h)(1) = g(1) + h(1) = f(1)$, so that $f = g+h$ by uniqueness of solution to $L(y) = 0$.

Let k be any positive real number. Then $kf(0) \neq 0 = h(0)$, so $h \neq kf$. Thus g and h form a non-proportional decomposition of f, and hence f is not an extremal element of C.

To complete the characterization of the extremal structure of C, it will be shown in Lemma 8 that the non-(L)-conical functions which are 0 at both end-points are not extremal elements of C. For such a function, the non-proportional decomposition exhibited in the proof of Lemma 8 will be based on the following lemma.

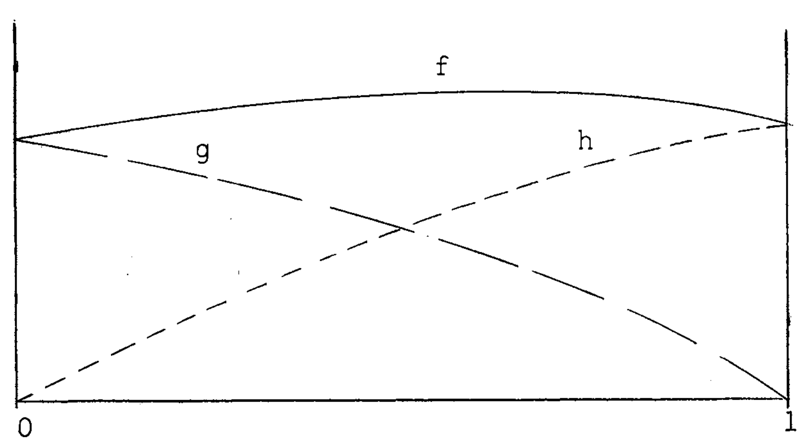

Figure 6. Lemma 6 proof.

Lemma 7. If g is a super-(L) function which is positive on $(w,1)$ for some w in $(0,1)$ and is such that $g(w) = g(1) = 0$, then there exists an (L)-linear function F which intersects g at least once in $(w,1)$ and is such that $F(x) \geqq g(x)$ for all x in $[w,1]$ and $F(0) = 0$.

Proof. Let $X = \{F: F$ is (L)-linear, $F(0) = 0$, $F \neq 0$, and $F(u) = g(u)$ for at least one u in $[w,1]\}$. By Definition 1, X is not empty. If F is in X, then $F(x) > 0$ for all x in $(0,1]$ by uniqueness of solution to $L(y) = 0$. Let $P(Fg) = \{x$: x is in $[0,1]$ and $F(x) = g(x)\}$. The structure of $P(Fg)$ will be considered in three cases:

Case A: There exists an F in X for which there exists an x' such that $P(Fg) = \{x'\}$.

Case B: There exists an F in X for which $P(Fg)$ consists of more than two points.

Case C: For every F in X, $P(Fg)$ consists of exactly two points.

If a function of the type described in Case A exists, then it will be used as the function F in the statement of the lemma. If no such function exists, then it will be proved that a function of the type described in Case B must exist and that it may be used as the function F in the statement of the lemma.

It will first be shown that for the F in Case B, $p(Fg)$ is a closed interval. Let $u < z < v$ be three points in $P(Fg)$. Then since F is the solution of $L(y) = 0$ such that $F(u) = g(u)$ and $F(z) = g(z)$, and since g is super-(L), it follows that $g(x) \geqq F(x)$ for all x in $[u,z]$. Also, since F is the solution of $L(y) = 0$ such that $F(z) = g(z)$ and $F(v) = g(v)$, it follows that $g(x) \leqq F(x)$ for all x in $[u,z]$ by Theorem 1. Thus $g(x) = F(x)$ for all x in $[u,z]$. Similarly $g(x) = F(x)$ for all x in

$[z,v]$, so P(Fg) is a convex set in $[0,1]$ and hence is an interval. Let $x' < x''$ be the end-points of the interval. Since F and g are continuous, $F(x') = g(x')$ and $F(x'') = g(x'')$. Therefore P(Fg) is a closed interval.

Next, it will be shown that Case C is impossible. Let A be an index set such that $X = \{F_\alpha : \alpha \text{ is in } A\}$. For each F_α in X, let x_α be the smaller point in $P(F_\alpha g)$, and let y_α be the larger.

If $F_\alpha \neq F_\beta$, then x_α , y_α , x_β , and y_β are all distinct since F_α and F_β can have no more than one point, (0,0), in common by uniqueness of solution to $L(y) = 0$. (e.g., if $x_\alpha = x_\beta$, then $F_\alpha(x_\alpha) = g(x_\alpha) = g(x_\beta) = F_\beta(x_\beta)$.)

Let F_α be in X. Suppose there exists an F_β in X such that $y_\beta \leqq x_\alpha$ and $F_\beta \neq F_\alpha$. Then $F_\beta \neq F_\alpha$ implies $y_\beta \neq x_\alpha$ by the above remark, so $y_\beta < x_\alpha$. Since g is super-(L), $F_\beta(x) \geqq g(x)$ for all x in $(y_\beta, 1]$ by Theorem 1. Since $P(F_\beta g) = \{x_\beta, y_\beta\}$ and $x_\beta < y_\beta$, $F_\beta(x) > g(x)$ for all x in $(y_\beta, 1]$. In particular, $F_\beta(x_\alpha) > g(x_\alpha)$. Similarly, $F_\alpha(x) > g(x)$ for all x in $[w, x_\alpha)$, and in particular, $F_\alpha(y_\beta) > g(y_\beta)$. (See Figure 7.)

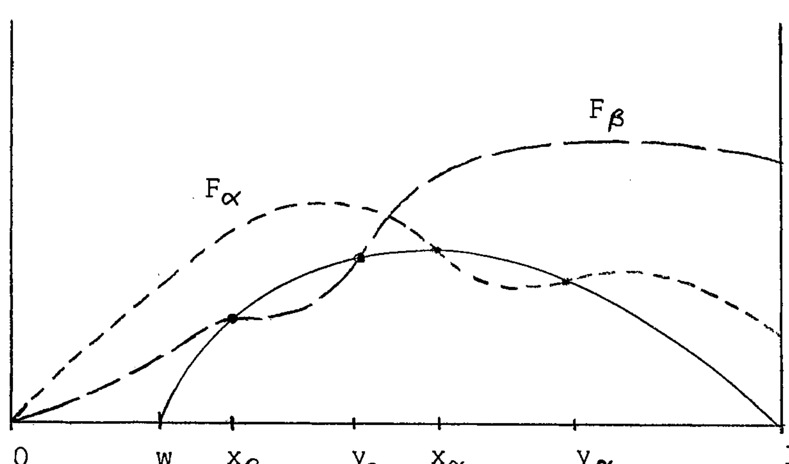

Figure 7. Lemma 7 proof. Assumption that $y_\beta \leqq x_\alpha$.

Then $F_\alpha(y_\beta)-F_\beta(y_\beta) > g(y_\beta)-F_\beta(y_\beta) = g(y_\beta)-g(y_\beta) = 0$, and $F_\alpha(x_\alpha)-F_\beta(x_\alpha) < F_\alpha(x_\alpha)-g(x_\alpha) = g(x_\alpha)-g(x_\alpha) = 0$. Thus there exists an x' in (y_β, x_α) such that $F_\alpha(x') = F_\beta(x')$ since F_α and F_β are continuous. Since $F_\alpha(0) = 0 = F_\beta(0)$, $F_\alpha = F_\beta$ by uniqueness of solution to $L(y) = 0$, which contradicts the assumption that $F_\alpha \neq F_\beta$. Therefore for every F_β in X, $x_\alpha < y_\beta$.

Then $\{x_\alpha : \alpha \text{ is in } A\}$ has an upper bound (any y_β), so by the completeness property of real numbers there exists a least upper bound z. Similarly, $\{y_\alpha : \alpha \text{ is in } A\}$ has a lower bound (any x_β), so that it has a greatest lower bound z'. Since $x_\alpha < y_\beta$ for all α, β in A, $z \leqq z'$. Suppose $z < z'$. Then there is an x' in (z, z'), and the points $(0,0)$ and $(x', g(x'))$ determine a solution of $L(y) = 0$ which is in X. This is a contradiction since then x' would have to be in $\{x_\alpha : \alpha \text{ is in } A\}$ or $\{y_\alpha : \alpha \text{ is in } A\}$, but be less than the greatest lower bound of $\{y_\alpha : \alpha \text{ is in } A\}$ and greater than the least upper bound of $\{x_\alpha : \alpha \text{ is in } A\}$. Thus $z = z'$.

Let F be the solution of $L(y) = 0$ determined by the two points $(0,0)$ and $(z, g(z))$. Then F belongs to X, so F intersects g in a second point, say $(z', g(z'))$, where $z' \neq z$ and z' is in $(w,1)$. There are two possible cases:

Case 1: Suppose $z' < z$. Let u be such that $z' < u < z$. Let G be the solution of $L(y) = 0$ determined by the two points $(0,0)$ and $(u, g(u))$. Then G is in X, so that G intersects g at a second point, say $(v, g(v))$. Suppose $v \geqq z$. (See Figure 8.) Since g is super-(L), $G(u) = g(u) \geqq F(u)$ and $F(z) = g(z) \geqq G(z)$. Then there exists a u' in $[u,v]$ such that $F(u') = G(u')$ since F and G are continuous. Thus $F = G$ by uniqueness of

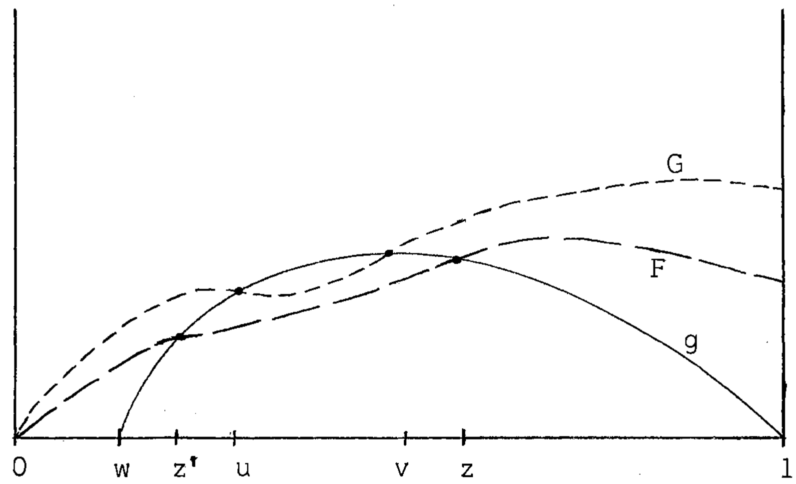

Figure 8. Lemma 7 proof. Assumption that $z' < z$.

solution to $L(y) = 0$. This contradicts $z' < u < z$ since F intersects g in only two distinct points. Similarly, assuming $v \leqq z'$ leads to a contradiction, so that $z' < v < z$. Then the larger of u and v is in $\{y_\alpha : \alpha \text{ is in } A\}$ and less than z, the greatest lower bound of $\{y_\alpha : \alpha \text{ is in } A\}$, a contradiction.

Case 2: Suppose $z' > z$. Let u be such that $z < u < z'$. Let G be the solution of $L(y) = 0$ determined by the two points (0,0) and $(u,g(u))$. Then as in Case 1, G intersects g at a second point v such that $z < v < z'$. The smaller of u and v is in $\{x_\alpha : \alpha \text{ is in } A\}$ and greater than z, the least upper bound of $\{x_\alpha : \alpha \text{ is in } A\}$, a contradiction.

Thus F intersects g only at $(z,g(z))$, and Case C is impossible.

Suppose for the F in Case A that $F(u) < g(u)$ for some u in $[w,1]$. Then $F(x) < g(x)$ for all x in $[w,1]$ such that $x \neq x'$, since $P(Fg) = \{x'\}$. This contradicts F being non-negative. In Case B, let

$x' = \sup P(Fg)$ and $x'' = \inf P(Fg)$. Then $F(x) \geqq g(x)$ for all x in $[w,x'']$ and $[x',1]$ by Theorem 1. Thus in either case, $F(x) \geqq g(x)$ for all x in $[w,1]$, and the proof of Lemma 7 is completed.

Lemma 8. If $f \neq 0$ is an element of C which is not (L)-conical and is such that $f(0) = f(1) = 0$, then f is not an extremal element of C.

Proof. Let w be in $(0,1)$. Since f is not (L)-conical, it is either non-(L)-linear on $[0,w]$ or on $[w,1]$. It will be assumed that f is non-(L)-linear on $[w,1]$, since the proof for the other case is similar.

Let $g(x) = f(x)-F(f,w,1;x)$ for all x in $[0,1]$. (See Figure 9.) It will be shown that g satisfies the hypotheses of Lemma 7. The (L)-linear function of the conclusion of Lemma 7 will then be used to exhibit a non-proportional decomposition of f. By Lemma 1, g is a super-(L) function since $-F(f,w,1;x)$ and f are super-(L) functions. Next, $g(w) = f(w)-F(f,w,1;w) = f(w)-f(w) = 0$ and $g(1) = f(1)-F(f,w,1;1) = 0$. Finally, it is necessary to show that g is positive on $(w,1)$.

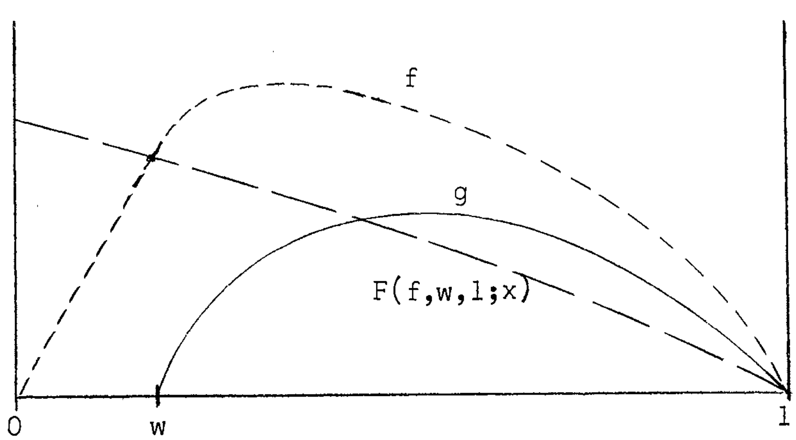

Figure 9. Lemma 8 proof. The function g.

Since f is super-(L), $f(x) \geqq F(f,w,1;x)$ for all x in (w,1), so that $g(x) \geqq 0$ for all x in (w,1). Suppose there exists a u in (w,1) for which $g(u) = 0$. Then $F(f,w,1;u) = f(u)$ and $F(f,w,1;w) = f(w)$ imply by uniqueness of solution to $L(y) = 0$ that $F(f,w,1;x) \equiv F(f,w,u;x)$. Similarly, $F(f,w,1;u) = f(u)$ and $F(f,w,1;1) = f(1)$ imply by uniqueness of solution to $L(y) = 0$ that $F(f,w,1;x) \equiv F(f,u,1;x)$. By Theorem 1, $f(x) \leqq F(f,u,1;x) = F(f,w,1;x)$ for all x in $[w,u]$ and $f(x) \leqq F(f,w,u;x) = F(f,w,1;x)$ for all x in $[u,1]$. Since f is super-(L), $f(x) \geqq F(f,w,1;x)$ for all x in $[w,1]$. Thus $f(x) = F(f,w,1;x)$ for all x in $[w,1]$, which contradicts the assumption that f is non-(L)-linear on $[w,1]$. Therefore $g(x) > 0$ for all x in (w,1).

By Lemma 7 there exists an (L)-linear function F which intersects g at least once in (w,1) and which is such that $F(0) = 0$ and $F(x) \geqq g(x)$ for all x in $[w,1]$. Let $z = \sup P(Fg)$. Suppose $z = 1$. Then $F(1) = g(1) = f(1)-F(f,w,1;1) = f(1)-f(1) = 0$, and since $F(0) = 0$, $F = 0$ by uniqueness of solution to $L(y) = 0$. This contradicts $F(x) \geqq g(x)$ for all x in (w,1). Thus $z < 1$. Now, $f(z)-F(z) = f(z)-g(z) = F(f,w,1;z) > 0$ and $f(1) = 0 < F(1)$, so that there exists an x' in (z,1) such that $f(x') = F(x')$. That is, $F(x) \equiv F(f,0,x';x)$.

Let $G(x) = F(f,0,x';x)$ for all x in $[0,z]$ and $f(x)-F(f,w,1;x)$ for all x in $[z,1]$. Let $H = f-G$.

To complete the proof that f is not an extremal element of C, it will be shown that G and H belong to C and that f is not proportional to G. The proofs that G and H belong to C are each given in two parts: (i) each function is shown to be non-negative, and (ii) each is shown to be super-(L).

(i). Suppose there exists a u in $(0,1]$ such that $F(f,0,x';u) \leqq 0$. Then $F(f,0,x';1) > g(1) = 0$ and continuity of $F(f,0,x';x)$ imply that there exists a v in $[u,1]$ such that $F(f,0,x';v) = 0$. So $F(f,0,x';x) \equiv 0$ by uniqueness of solution to $L(y) = 0$, since $F(f,0,x';0) = 0$. This contradicts $F(f,0,x';1) > 0$. Therefore $F(f,0,x';x) \geqq 0$ for all x in $[0,z]$. Since f is super-(L), $f(x) \geqq F(f,w,1;x)$ for all x in $[w,1]$, so $f(x)-F(f,w,1;x) \geqq 0$ for all x in $[w,1]$. Thus G is non-negative.

By its definition, $H(x) = f(x)-F(f,0,x';x)$ for all x in $[0,z]$ and $F(f,w,1;x)$ for all x in $[z,1]$.

Since f is super-(L), $f(x) \geqq F(f,0,x';x)$ for all x in $[0,x']$, so $H(x) = f(x)-F(f,0,x';x) \geqq 0$ for all x in $[0,z]$ since $z < x'$. Since $F(f,w,1;z) > 0$ and $F(f,w,1;1) = 0$, $F(f,w,1;x) \geqq 0$ for all x in $[z,1]$ because otherwise $F(f,w,1;x)$ would be zero at some $u < 1$ and by uniqueness of solution to $L(y) = 0$ $F(f,w,1;x)$ would be identically zero, a contradiction. Thus H is non-negative on $[0,1]$.

(ii). Since $F(f,0,x';x)$ is (L)-linear, G is super-(L) on $[0,z]$. By Lemma 1, G is super-(L) on $[z,1]$. Let u be in $[0,z)$ and v in $(z,1]$. Suppose $F(G,u,v;u') > G(u')$ for some u' in (u,v). (See Figure 10.) Then $F(G,u,v;z) > G(z)$, since if $F(G,u,v;z) \leqq G(z)$, $F(G,u,v;u') > G(u')$ and continuity of $F(G,u,v;x)$ and G imply the existence of a v' between z and u' such that $F(G,u,v;v') = G(v')$. If v' is in $(u,z]$, $F(G,u,v;x) \equiv F(f,0,x';x)$ by uniqueness of solution to $L(y) = 0$, since $F(G,u,v;u) = F(f,0,x';u)$ and $u \neq v'$. This contradicts $F(f,0,x';v) > G(v) = F(G,u,v;v)$. If v' is in (z,v), then $u < z \leqq v' < u' < v$ and $G(u') < F(G,u,v;u') = F(G,v',v;u')$ contradict G being super-(L) on $[z,1]$. Then $F(G,u,v;z) > G(z) = F(f,0,x';z)$, $F(G,u,v;v) = G(v) < F(f,0,x';v)$, and continuity of $F(G,u,v;x)$ and $F(f,0,x';x)$ imply the existence of a z' in (z,v)

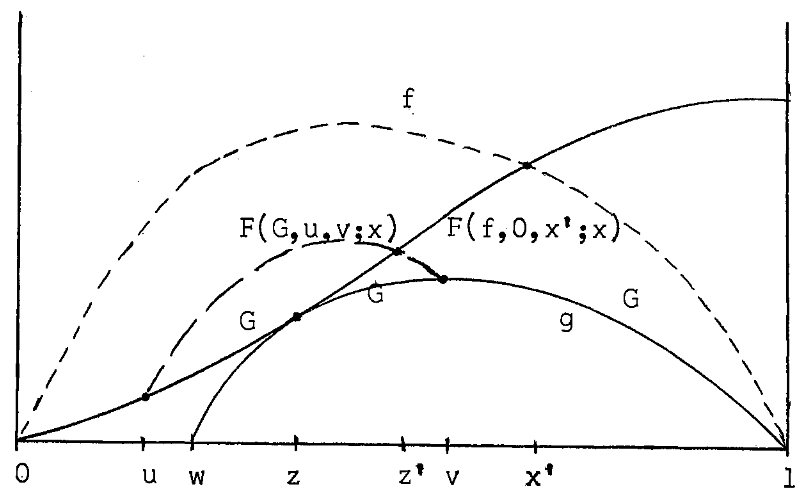

Figure 10. Lemma 8 proof. The function G.

such that $F(G,u,v;z') = F(f,0,x';z')$. By uniqueness of solution to $L(y) = 0$, $F(G,u,v;x) \equiv F(f,0,x';x)$ since $F(G,u,v;u) = G(u) = F(f,0,x';u)$ and $u \neq z'$. This contradicts $F(f,0,x';v) > F(G,u,v;v)$. Therefore $G(x) \geqq F(G,u,v;x)$ for all x in $[u,v]$, and G is super-(L) on $[0,1]$. Therefore G belongs to C.

In order to prove that H is super-(L) on $[0,1]$, it is first necessary to show that $H(x) \leqq F(f,w,1;x)$ for all x in $[0,z]$. Suppose there exists a u in $[0,z)$ such that $F(f,w,1;u) < H(u) = f(u)-F(f,w,1;u)$. Then $F(f,0,x';u) < f(u)-F(f,w,1;u)$ for some u in $[0,z)$. As shown previously, $F(f,0,x';x) > 0$ for all x in $(0,1]$, so $F(f,0,x';w) > f(w)-F(f,w,1;w) = 0$. By continuity of $f(x)-F(f,w,1;x)$ and $F(f,0,x';x)$, there exists a v between w and u such that $F(f,0,x';v) = f(v)-F(f,w,1;v)$. Suppose $u < v < w$. Then since f is super-(L), $f(v) \leqq F(f,w,1;v)$ by Theorem 1. Then $F(f,0,x';v) = f(v)-F(f,w,1;v) \leqq 0$, which contradicts $F(f,0,x';v) > 0$. If $w < v < u$ then P(Fg) contains v and z but not u, which contradicts $P(Fg)$ being a single point or a closed interval. (Recall that $F(x) \equiv F(f,0,x';x)$ and $g(x) \equiv f(x)-F(f,w,1;x)$.) Thus $F(f,0,x';x) \geqq f(x)-F(f,w,1;x)$ for all x in $[0,z)$, or in other words, $F(f,w,1;x) \geqq H(x)$ for all x in $[0,z]$.

Since $F(f,w,1;x)$ is (L)-linear, H is super-(L) on $[z,1]$. By Lemma 1, H is super-(L) on $[0,z]$. Let u be in $[0,z)$ and v in $(z,1]$. Suppose $F(H,u,v;u') > H(u')$ for some u' in (u,v). (See Figure 11.) If $F(H,u,v;z) \leqq H(z)$ then there exists a v' between u' and z such that $H(v') = F(H,u,v;v')$ since H and $F(H,u,v;x)$ are continuous. If v' is in (u,z), then $u < u' < v' < z < v$ and $H(u') < F(H,u,v;u') = F(H,u,v';u')$ contradict H being super-(L) on $[0,z]$. If v' is in $[z,v)$, then $F(H,u,v;x) \equiv F(f,w,1;x)$ by uniqueness of solution to $L(y) = 0$ since $F(f,w,1;v) = F(H,u,v;v)$ and $v' \neq v$. This contradicts $F(f,w,1;u) > H(u) = F(H,u,v;u)$. Thus $F(H,u,v;z) > H(z)$. Then $F(H,u,v;z) > H(z) = F(f,w,1;z)$ and $F(f,w,1;u) \geqq H(u) = F(H,u,v;u)$ imply the existence of a z' in $[u,z)$ such that $F(H,u,v;z') = F(f,w,1;z')$ since $F(H,u,v;x)$ and $F(f,w,1;x)$ are continuous. By uniqueness of solution to $L(y) = 0$, $F(H,u,v;x) \equiv F(f,w,1;x)$ since $F(H,u,v;v) = H(v) = F(f,w,1;v)$ and $z' \neq v$. This contradicts $F(H,u,v;z) > H(z) = F(f,w,1;z)$. Therefore $H(x) \geqq F(H,u,v;x)$ for all x in $[u,v]$, and H is super-(L) on $[0,1]$. Thus H belongs to C.

Suppose there exists a real number k such that $G = kf$. Since $G \neq 0$, $k \neq 0$. Then f is (L)-linear on $[0,z]$ since G is. Since $f = G+H$ and $H \neq 0$, it follows that $k \neq 1$. Then $H = f-G = f-kf = (1-k)f$, so that f is (L)-linear on $[z,1]$ since H is. Thus f is (L)-conical since $f(z) > 0$, which contradicts the original assumption that f is not (L)-conical. Therefore G and H form a non-proportional decomposition of f, and hence f is not an extremal element of C.

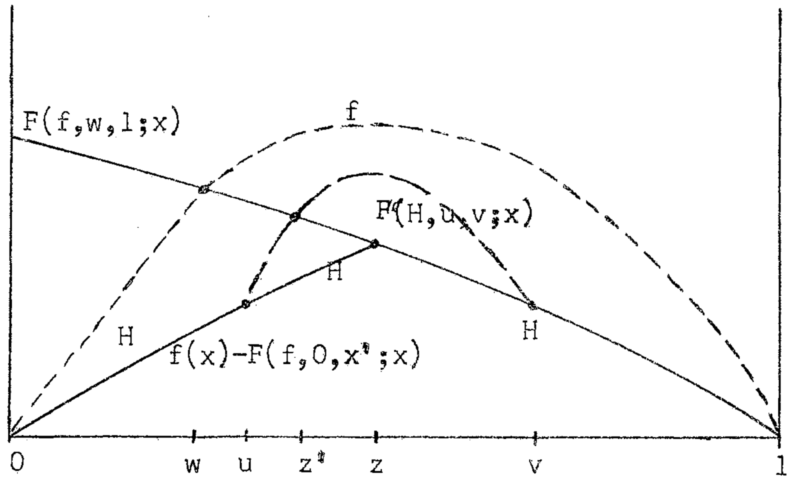

Figure 11. Lemma 8 proof. The function H.

<u>Note</u>: The function 0 is trivially an extremal element of C; hence, the not-identically-zero extremal elements will be called "non-trivial."

Theorem 7. A function f is a non-trivial extremal element of C if and only if f is an (L)-conical function.

Proof. The proof follows directly from Lemmas 2, 4, 5, 6, and 8.

CHAPTER III

INTEGRAL REPRESENTATION

This chapter is concerned with the existence of a type of integral representation (Radon measure) for the elements of the convex cone C based on the following theorem due to Choquet.

Theorem 8 [5, p. 237]. If the linear space L is a locally convex Hausdorff space, and if A is a convex compact subset of L, then for every x in A there exists a non-negative Radon measure [6] on the closure of the set of extreme points of A whose center of gravity is x.

Definition 9. Let A and B be subsets of a real linear space L. Then $A+B = \{x+y: x \text{ is in } A, \ y \text{ is in } B\}$, $-A = \{x: -x \text{ is in } A\}$, and $A-B = A+(-B)$.

The theorem will be applied in the following way. First, it is known that C-C is a real linear space such that the vertex of C is the origin of C-C [7, p. 47]. It is also known that when C-C is topologized with the topology of simple convergence, it is a locally convex Hausdorff space [5, p. 236]. (The topology of simple convergence is the induced product topology of $R^{[0,1]}$. A neighborhood basis at 0 for C-C consists of the sets $\{f: |f(x_i)| < \varepsilon \text{ for } i = 1,\ldots,n\}$, where ε is a positive real number.) Then it will be shown that $B = \{f: f \text{ is in } C, f(w) = 1\}$, where w is a fixed real number in (0,1), is a convex compact subset of C–C which meets each ray of C once and only once and does not contain 0. It will also be shown that the set of extreme points of B is closed in C–C for the topology of simple convergence. So by Theorem 8, there will exist an integral representation of each f in B in terms of extreme points of B. It will then follow that there is an integral representation of each g in C in terms of extremal elements of C, since B meeting each ray of C once and only once implies that there exists a real number k such that kg belongs to B and that the set of extremal elements of C is the same as the set $\{mf$: f is an extreme point of B, m is a non-negative real number$\}$.

The proof that B is closed will be based on the following theorem and lemmas.

Theorem 9 [8, p. 218]. In order that a family F of functions on a set X to a topological space Y be compact relative to the topology of simple convergence it is sufficient that

(a) F be pointwise closed in Y^X, and

(b) for each point x of X the set $F[x] = \{f(x)$: f is in $F\}$ has a compact closure.

Lemma 9. Let $B = \{f$: f is in C, $f(w) = 1\}$, where w is a fixed real number in $(0,1)$. Let G be the solution of $L(y) = 0$ determined by the points $(w,1)$ and $(1,0)$, and let H be the solution determined by the points $(0,0)$ and $(w,1)$. Then for each x in $[0,w]$, $\{f(x)$: f is in $B\} = [H(x),G(x)]$, and for each x in $[w,1]$, $\{f(x)$: f is in $B\} = [G(x),H(x)]$.

Proof. Suppose there is an x' in $[0,1)$ for which $G(x') \leqq 0$. Then $G(w) = 1$ and continuity of G imply that there exists an x'' between w and x' such that $G(x'') = 0$, and hence $G = 0$ by uniqueness of solution to

$L(y) = 0$. This contradicts $G(w) = 1$. Thus $G(x) > 0$ for all x in $[0,1)$. Similarly, $H(x) > 0$ for all x in $(0,1]$, so that G and H belong to B.

Suppose there exists an x' in $[0,w)$ such that $G(x') \leqq H(x')$. Then $G(0) > 0 = H(0)$ and continuity of G and H imply that there exists an x'' in $(0,w)$ such that $G(x'') = H(x'')$. Then $G(w) = 1 = H(w)$ implies $G = H$ by uniqueness of solution to $L(y) = 0$. Therefore $G(x) > H(x)$ for all x in $[0,w)$. Similarly, $H(x) > G(x)$ for all x in $(w,1]$.

Note that since G and H are continuous on $[0,1]$, each assumes its maximum on $[0,1]$.

Let f be any element of B. It will be shown that f is bounded between G and H. Suppose there exists a u in $[0,w)$ such that $f(u) > G(u)$. Then suppose there exists a v in $[0,w)$ such that $f(v) \leqq G(v)$. Now $f(u) > G(u)$, $f(v) \leqq G(v)$, and continuity of f and G imply that there exists an x' between u and v for which $f(x') = G(x')$. If $v \leqq x' < u$ then $G(x) \equiv F(f,x',w;x)$. (See Figure 12.) By Theorem 1, $f(x) \leqq G(x)$ for all x in $[w,1]$. Then $f(1) \leqq G(1) = 0$ implies $f(1) = 0$ since f

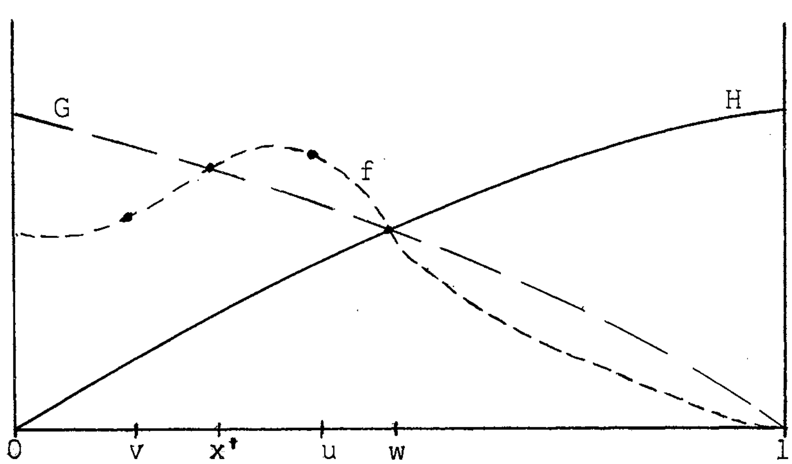

Figure 12. Lemma 9 proof. The case $v \leqq x' < u$.

is non-negative. So $G(x) \equiv F(f,w,1;x)$, and by Theorem 1 $f(x) \leqq G(x)$ for all x in $[0,w]$, which contradicts $f(u) > G(u)$. Thus $v \leqq x' < u$ is impossible. Suppose then that $u < x' \leqq v$. (See Figure 13.) Now $G(x) \equiv F(f,x',w;x)$ and f super-(L) imply that $f(x) \leqq F(f,x',w;x)$ for all x in $[0,x']$ by Theorem 1, which contradicts $f(u) > G(u)$. Thus $f(u) > 0$ implies $f(x) > G(x)$ for all x in $[0,w)$. Suppose there exists a z in $(w,1]$ for which $f(z) \leqq G(z)$. (See Figure 14.) If $f(1) \geqq G(1) = 0$, then $f(z) \leqq G(z)$ and continuity of f and G imply the existence of a z' in $[z,1]$ such that $f(z') = G(z')$. Then since $G(x) \equiv F(f,w,z';x)$ and f is super-(L), $f(x) \leqq G(x)$ for all x in $[0,w)$ by Theorem 1, which contradicts $f(u) > G(u)$. Thus $f(1) < G(1) = 0$, which contradicts f being non-negative. Thus $f(x) > G(x)$ for all x in $(w,1]$. Since $-G$ is a super-(L) function, $f-G$ is a super-(L) function by Lemma 1. Now $(f-G)(w) = f(w)-G(w) = 0$ and w is in $(0,1)$, so $f-G = 0$ by Theorem 6 since $f-G$ is non-negative. This contradicts $f(u) > G(u)$. Therefore $f(x) \leqq G(x)$ for all x in $[0,w]$.

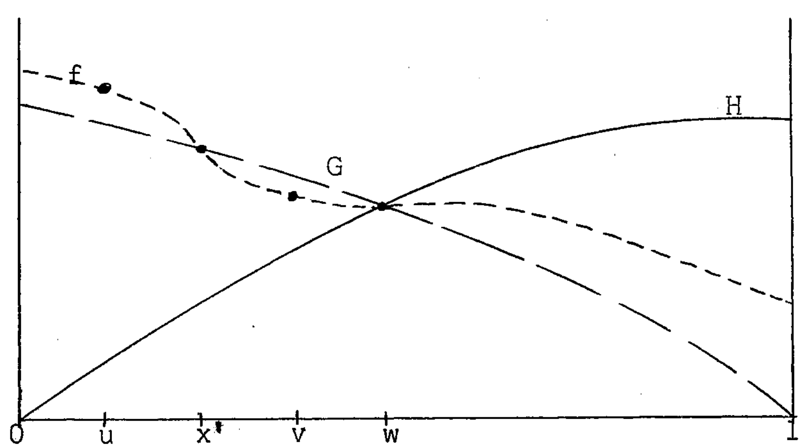

Figure 13. Lemma 9 proof. The case $u < x' \leqq v$.

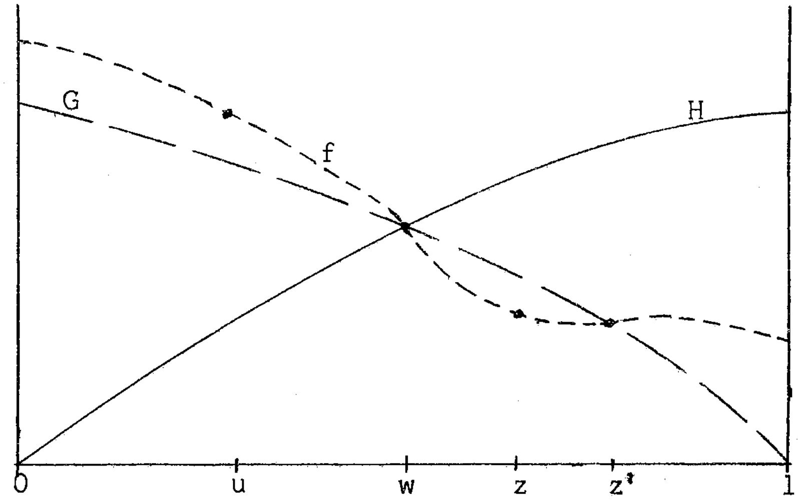

Figure 14. Lemma 9 proof. Assumption that $f(z) \leqq G(z)$.

Similarly, $f(x) \leqq H(x)$ for all x in $[w,1]$.

Suppose there exists a u in $[0,w)$ such that $f(u) < H(u)$. (See Figure 15.) If $f(0) \geqq H(0)$ then continuity of f and H implies that there exists an x' such that $0 \leqq x' < u$ and $f(x') = H(x')$. Now $H(x) \equiv F(f,x',w;x)$, $f(u) < F(f,x',w;u)$ and $x' < u < w$ contradict f being super-(L). Then $f(0) < H(0)$ contradicts f being non-negative. Thus $f(x) \geqq H(x)$ for all x in $[0,w]$. Similarly, $f(x) \geqq G(x)$ for all x in $[w,1]$.

Therefore for each x in $[0,w]$, $\{f(x)\colon f \text{ is in } B\} \subset [H(x),G(x)]$, and for each x in $[w,1]$, $\{f(x)\colon f \text{ is in } B\} \subset [G(x),H(x)]$. Let x and y be such that $0 \leqq x < w$ and $H(x) \leqq y \leqq G(x)$. By Definition 1, there exists a unique (L)-linear function F such that $F(w) = 1$ and $F(x) = y$. Since F is in C, it follows that F is in B. Thus for each x in $[0,w)$, $[H(x),G(x)] = \{f(x)\colon f \text{ is in } B\}$. For $x = w$, $\{f(w)\colon f \text{ is in } B\} = \{1\} = [H(w),G(w)]$. Similarly, for each x in $[w,1]$ $\{f(x)\colon f \text{ is in } B\} = [G(x),H(x)]$, and the lemma is proved.

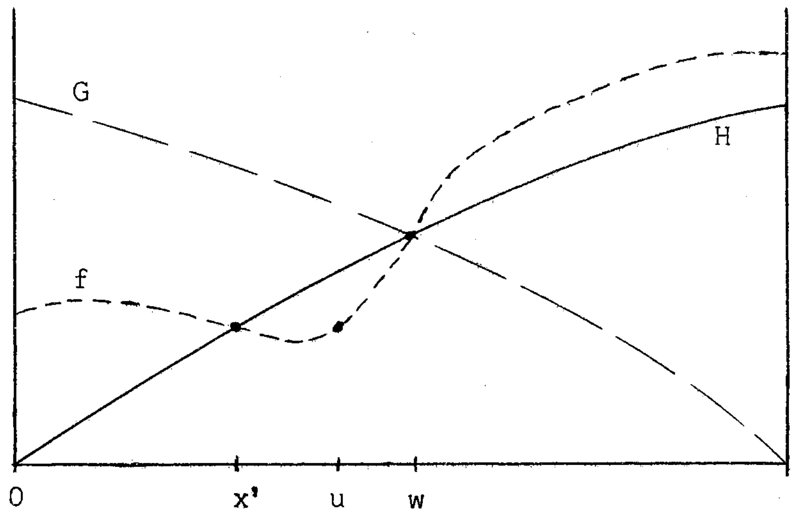

Figure 15. Lemma 9 proof. The case $f(u) < H(u)$.

Lemma 10. The convex cone C is closed in $R^{[0,1]}$ for the topology of simple convergence.

Proof. Let f belong to the complement of C. It will be shown that the complement of C is open by exhibiting a neighborhood of f contained entirely in the complement. The proof will be given in two parts based on the fact that if f is not in C, then either f is non-negative and not super-(L) or f is not non-negative.

Case I. Suppose f is not non-negative. Then there exists an x' in $[0,1]$ such that $f(x') < 0$. Let $\varepsilon = -(f(x')/2)$. Let g be in $U(f;x';\varepsilon)$, an ε-neighborhood of f in the topology of simple convergence. Since $|g(x')-f(x')| < \varepsilon$, $g(x')-f(x') < -f(x')/2$, and so $g(x') < f(x')/2 < 0$. Thus g is not non-negative and hence not in C. Therefore $U(f;x';\varepsilon)$ is contained in the complement of C.

Case II. Suppose f is non-negative and not super-(L). Then there exist u,z, and v such that $u < z < v$ and $f(z) < F(f,u,v;z)$. For convenience, denote $F(f,u,v;x)$ by $F(x)$.

If $F(u) = f(u) > 0$ and $F(v) = f(v) > 0$, let

$\varepsilon = [1/2]\min\{F(u)[F(z)-f(z)]/[F(u)+F(z)],\ F(v)[F(z)-f(z)]/[F(v)+F(z)]\}$.

Observe that $\varepsilon > 0$ since $F(u) > 0$, $F(v) > 0$, and $F(z) > f(z) \geqq 0$.

Then:

$$\varepsilon < F(u)[F(z)-f(z)]/[F(u)+F(z)]$$
$$[F(u)+F(z)]\varepsilon < F(u)[F(z)-f(z)]$$
$$F(u)\varepsilon+F(z)\varepsilon < F(u)F(z)-F(u)f(z)$$
$$F(u)f(z)+F(u)\varepsilon < F(u)F(z)-F(z)\varepsilon$$
$$F(u)[f(z)+\varepsilon] < F(z)[F(u)-\varepsilon]$$
$$F(u)[f(z)+\varepsilon]/F(z) < F(u)-\varepsilon .$$

Similarly, $F(v)[f(z)+\varepsilon]/F(z) < F(v)-\varepsilon$. Let $k = [f(z)+\varepsilon]/F(z)$. By Lemma 1, kF is an (L)-linear function. Note that $kF(z) = f(z)+\varepsilon$. Suppose there exists a g in $C \cap U(f;u,v,z;\varepsilon)$. (See Figure 16.) Then $g(u) > f(u)-\varepsilon = F(u)-\varepsilon > kF(u)$, $g(v) > f(v)-\varepsilon = F(v)-\varepsilon > kF(v)$, and $g(z) < f(z)+\varepsilon = kF(z)$. Since g and kF are continuous, there exists an x' in (u,z) such that $g(x') = kF(x')$, and there exists an x'' in (z,v)

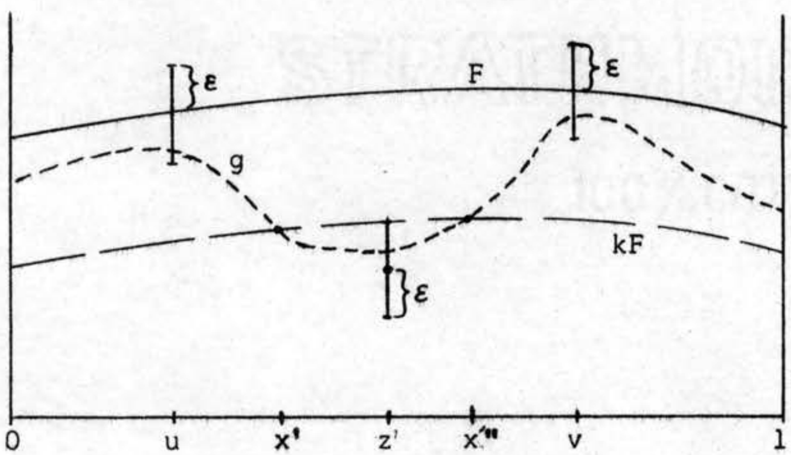

Figure 16. Lemma 10 proof.

such that $g(x'') = kF(x'')$. Then $kF(x) \equiv F(g,x',x'';x)$ and $g(z) < F(g,x',x'';z)$ contradict g being super-(L). Thus $U(f;u,v,z;\varepsilon)$ is in the complement of C.

If $F(u) = 0$, then $F(v) = 0$ implies $F = 0$ by uniqueness of solution to $L(y) = 0$, and $F(z) = 0$ contradicts $F(z) > f(z) \geqq 0$. Thus $F(v) > 0$. Let $\varepsilon = [1/2]F(v)\,[F(z)-f(z)]/[F(v)+F(z)]$. As shown in the preceding paragraph, $\varepsilon > 0$ and $kF(v) < F(v)-\varepsilon$. If there exists a g in $C \cap U(f;u,v,z;\varepsilon)$, then $g(v) > f(v)-\varepsilon = F(v)-\varepsilon > kF(v)$. Since g and kF are continuous, there exists an x' such that $g(x') = kF(x')$ and $u \leqq x' < z$, and there exists an x'' such that $g(x'') = kF(x'')$ and $z < x'' < v$. Then $kF(x) \equiv F(g,x',x'';x)$ and $g(z) < F(f,x',x'';z)$ contradict g being super-(L), so that $U(f;u,v,z;\varepsilon)$ is in the complement of C.

Similarly, if $F(v) = 0$ then $F(u) > 0$ and $U(f;u,v,z;\varepsilon)$ is in the complement of C, where $\varepsilon = [1/2]F(u)\,[F(z)-f(z)]\,/[F(u)+F(v)]$.

Thus the complement of C is open in $R^{[0,1]}$ for the topology of simple convergence, and the lemma is proved.

Theorem 10. The set $B = \{f\colon f \text{ is in } C,\ f(w) = 1\}$, where w is a fixed real number in $(0,1)$, is a convex compact subset of C-C which meets each ray of C once and only once and which does not contain 0.

Proof. First, it will be shown that B is a convex set. Let f and g belong to B, and let k be any real number such that $0 < k < 1$. Then by Lemma 1, $kf+(1-k)g$ is in C. Since $(kf+(1-k)g)(w) = kf(w)+(1-k)g(w) = k+1-k = 1$, $kf+(1-k)g$ is in B. Thus B is convex.

Next, compactness of B will be proved by applying Theorem 9. To prove that B is closed, it will be shown that the complement of B relative to C is open. Then since C is closed relative to $R^{[0,1]}$ by

Lemma 10, B will be closed relative to $R^{[0,1]}$. Let f be any element in the complement of B relative to C. Then since f is super-(L), $f(w) \neq 1$. Let $\varepsilon = |f(w)-1|$. The neighborhood $U(f;w;\varepsilon)\cap C$ of f is in the complement of B relative to C, since if g is any element in $U(f;w;\varepsilon)\cap C$, $|g(w)-f(w)|<\varepsilon$ implies $|g(w)-f(w)| < |f(w)-1|$. Then $|g(w)-1| = |g(w)-f(w)+f(w)-1| \geqq |f(w)-1| - |g(w)-f(w)| > 0$, so that $g(w) \neq 1$, and hence g is not in B. Thus f is an interior point of the complement of B relative to C. Therefore B is closed relative to $R^{[0,1]}$. Then by Lemma 9 and Theorem 9, B is compact.

Clearly $f = 0$ is not in B since $f(w) = 0 \neq 1$.

To complete the proof of the theorem, it will be shown that B intersects each ray of C in one and only one point. Let H be any ray of C. Then there exists an $f \neq 0$ in C such that $H = \{kf;\ k$ is a non-negative real number$\}$. Since f is continuous on $[0,1]$ f(w) is finite. By Theorem 6, $f(w) \neq 0$ since w is in $(0,1)$. Then $k = 1/f(w)$ is the unique real number such that $kf(w) = 1$. Thus the intersection of B with H exists and is unique.

Lemma 11. If f is an element of $B\setminus e(B)$, the complement of e(B) relative to B (where e(B) is the set of extreme points of B), then either there exists a u in $[0,w)$ such that $H(u) < f(u) < G(u)$ or there exists a u in $(w,1]$ such that $G(u) < f(u) < H(u)$, where G and H are the functions of Lemma 9.

Proof. Suppose the conclusion of the lemma is false. By Lemma 9, $H(x) \leqq f(x) \leqq G(x)$ for all x in $[0,w]$ and $G(x) \leqq f(x) \leqq H(x)$ for all x in $[w,1]$. Since f is continuous and $G(x) \neq H(x)$ for any x in $[0,w)$ or $(w,1]$, there are four possible cases: 1) $f(x) = G(x)$ for all x in

$[0,w]$ and $f(x) = H(x)$ for all x in $[w,1]$, 2) $f(x) = H(x)$ for all x in $[0,w]$ and $f(x) = G(x)$ for all x in $[w,1]$, 3) $f(x) = G(x)$ for all x in $[0,1]$, and 4) $f(x) = H(x)$ for all x in $[0,1]$. In cases 2, 3, and 4, f is an (L)-conical function and thus in e(B), which contradicts f being in $B \setminus e(B)$. In case 1, it will be shown that f cannot be super-(L). Let u be in $[0,w)$, v in $(w,1]$. Then $F(f,u,v;w) \leqq f(w) = G(w)$, $F(f,u,v;v) = H(v) > G(v)$, and continuity of $F(f,u,v;x)$ and $G(x)$ imply the existence of an x' in $[w,v)$ such that $F(f,u,v;x') = G(x')$. Then $F(f,u,v;x) \equiv G(x)$ by uniqueness of solution to $L(y) = 0$. This contradicts $G(v) < H(v) = F(f,u,v;v)$, and the lemma is proved.

Theorem 11. The set e(B) of extremal elements of B is closed in C-C for the topology of simple convergence.

Proof. By Theorem 10, B is closed relative to C-C, so to show e(B) is closed relative to C-C it is only necessary to prove e(B) is closed relative to B. It will be shown that $B \setminus e(B)$ is open relative to B. Let f belong to $B \setminus e(B)$. Using the conclusion of Lemma 11 and the fact that f is continuous, it will be assumed that there exists a u in $[0,w)$ such that $H(u) < f(u) < G(u)$. The proof for u in $(w,1]$ such that $G(u) < f(u) < H(u)$ is similar, and will be omitted.

Let P be the (L)-conical function in e(B) determined by the points (0,0) and (u,f(u)). Note that any (L)-conical function in e(B) is either 1) identical with H on $[0,r]$ and identical with the (L)-linear function determined by the points (r,H(r)) and (1,0) on $[r,1]$, where r is in $[w,1]$, or 2) identical with G on $[r,1]$ and identical with the (L)-linear function determined by the points (0,0) and (r,G(r)) on $[0,r]$ where r is in $[0,w]$. Let z be the x-coordinate of the vertex of P. Note that $z < w$, since if P intersects H at any point in $[0,w]$, then uniqueness of solution to $L(y) = 0$ implies $P(x) = H(x)$ for all x in $[0,w]$, which contradicts $P(u) = f(u) > H(u)$. Suppose $f(x) = P(x)$ for all x in $[0,z]$. Then f super-(L) and $G(x) \equiv F(f,z,w;x)$ imply $f(x) \geqq G(x)$ for all x in $[z,w]$ and $f(x) \leqq G(x)$ for all x in $[w,1]$, by Theorem 1. By Lemma 9, $f(x) \leqq G(x)$ for all x in $[0,w]$ and $f(x) \geqq G(x)$ for all x in $[w,1]$. Thus $f(x) = G(x)$ for all x in $[z,1]$, and hence $f = P$, which contradicts f not being in e(B). Therefore there exists a v in $[0,z]$ such that $f(v) \neq P(v)$.

It will now be determined how ε may be chosen so that $U(f;u,v;\varepsilon)$ is a neighborhood of f contained entirely in $B \setminus e(B)$. (See Figure 17.) Let Q be the (L)-conical function in e(B) determined by the points (0,0) and $(v,[1/2][f(v)+P(v)])$. Now $|Q(v)-P(v)| = |[1/2]f(v)+[1/2]P(v)-P(v)| = [1/2]|f(v)-P(v)| > 0$. Thus $Q(v) \neq P(v)$, which implies by uniqueness of solution to $L(y) = 0$ that $Q(u) \neq P(u) = f(u)$, since $Q(0) = P(0)$. Let $\varepsilon = [1/2]\min\{|f(v)-P(v)|, |f(u)-Q(u)|\}$.

Suppose there exists a g in $U(f;u,v;\varepsilon) \cap e(B)$. Then $|f(u)-g(u)| < \varepsilon$

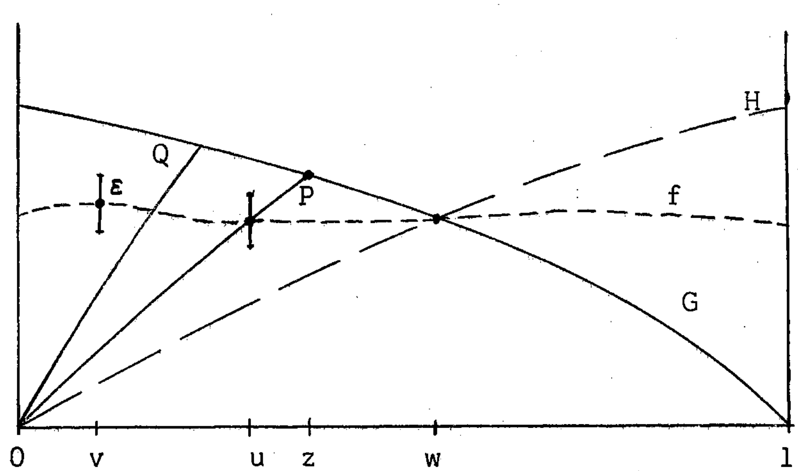

Figure 17. Theorem 11 proof.

and $|f(v)-g(v)|<\varepsilon$. Suppose first that $P(v)>Q(v)$. Then $f(u) = P(u)>Q(u)$ by uniqueness of solution to $L(y) = 0$, and $Q(v) = [1/2]f(v)+[1/2]P(v)> [1/2]f(v)+[1/2]Q(v)$, so that $Q(v)>f(v)$. Thus $f(u)-g(u)<\varepsilon<f(u)-Q(u)$ implies $g(u)-Q(u)>0$, and $Q(v)-f(v) = [1/2]f(v)+[1/2]P(v)-f(v) = [1/2][P(v)-f(v)] \geqq \varepsilon > g(v)-f(v)$ implies $g(v)-Q(v)<0$. Also, $G(v) \geqq P(v)>Q(v) = [1/2][f(v)+P(v)] = f(v)+[1/2][P(v)-f(v)] \geqq f(v)+\varepsilon$, so that G is not in $U(f;u,v;\varepsilon)$. Suppose next that $P(v)<Q(v)$. Then $f(u) = P(u)<Q(u)$ by uniqueness of solution to $L(y) = 0$, and $Q(v) = [1/2]f(v)+[1/2]P(v)<[1/2]f(v)+[1/2]Q(v)$, so that $Q(v)<f(v)$. Thus $g(u)-f(u)< \varepsilon<Q(u)-f(u)$ implies $g(u)-Q(u)<0$, and $f(v)-Q(v) = f(v)-[1/2]f(v)-[1/2]P(v) = [1/2][f(v)-P(v)] \geqq \varepsilon > f(v)-g(v)$ implies $g(v)-Q(v)>0$. Also, $G(u) \geqq Q(u) = Q(u)-f(u)+f(u) \geqq f(u)+\varepsilon$, so that G is not in $U(f;u,v;\varepsilon)$. Thus in both cases, g-Q changes sign between u and v, and G is not in $U(f;u,v;\varepsilon)$.

Since g-Q is continuous, there exists an x' strictly between u and v such that $g(x') = Q(x')$. Then since $g \neq G$, $g(0) = Q(0)$ and uniqueness of solution to $L(y) = 0$ imply $g = Q$. This is a contradiction since $|f(u)-Q(u)|>\varepsilon$ implies Q is not in $U(f;u,v;\varepsilon)$. Thus f is an interior point of $B\setminus e(B)$, and $e(B)$ is closed relative to C-C.

It now follows from Theorems 8, 10, and 11 that there exists an integral representation (Radon measure) for each f in C in terms of extremal elements of C.

Note. If $e(B)$ is dense in B, then the integral representation of Theorem 8 is of little value. By the above theorem, $e(B)$ is closed, so that if $e(B)$ is dense in B then $e(B) = B$. To see that this is not the case, observe that $[1/2](G+H)$, where G and H are the functions of Lemma 9, belongs to B but not to $e(B)$ since $[1/2](G+H)(0) \neq 0$ and $[1/2](G+H)(1) \neq 0$.

CHAPTER IV

THE STRUCTURE OF C-C

Let K be the convex cone of non-negative concave functions on $[0,1]$. The relationship between the real linear spaces C-C and K-K will be investigated by considering some results obtained by Hartman [9] and Bonsall [1].

Definition 10. A real-valued function f defined on a convex domain D is a <u>d.c. function</u> on D if there exist continuous convex functions g and h on D such that $f = g-h$.

Definition 11. A real-valued function f is <u>d.c. at a point</u> u in D if there exists a convex neighborhood U of u such that f is d.c. on $U \cap D$. When f is d.c. at every point in D, it is called <u>locally d.c. on D</u>.

Theorem 12 [9, p. 707]. If f is locally d.c. on an open interval I, then f is d.c. on I.

Theorem 13 [1, p. 105]. If f is super-(L) on $[0,1]$, then, given a and b in $(0,1)$, f is d.c. on (a,b).

Theorem 14. If f is super-(L) on $[0,1]$, then f is d.c. on $(0,1)$.

Proof. Let u be in $(0,1)$. Then there exist a and b such that $0 < a < u < b < 1$. By Theorem 13, f is d.c. on (a,b). Thus f is locally d.c. on $(0,1)$, and by Theorem 12 f is d.c. on $(0,1)$.

Theorem 15. If $L(y)$ is such that all non-negative (L)-linear functions are convex, then K-K is a subspace of C-C.

Proof. Let f belong to K-K. Then there exist g and h in K such that f=g-h. Let $u<v$ be in $[0,1]$. Since g is concave, the line segment joining the points $(u,g(u))$ and $(v,g(v))$ lies below $g(x)$ for all x in $[u,v]$. Since $F(g,u,v;x)$ is convex, it lies below the line segment, and thus below $g(x)$ for all x in $[u,v]$. Thus g is super-(L). Similarly, h is super-(L), and hence f belongs to C-C. Therefore $K-K\subset C-C$, and hence K-K is a subspace of C-C.

To see that not all $L(y)$ satisfy the hypothesis of Theorem 15, observe that $L(y)\equiv d^2y/dx^2-[1/(x-2)]\,dy/dx=0$ satisfies Definition 1, but the solution $y=-(x-2)^2+4$ is non-negative and not convex on $[0,1]$.

CHAPTER V

REPRESENTATION IN TERMS OF GREEN'S FUNCTIONS

This chapter is concerned with the demonstration of the existence of a type of integral representation in terms of Green's functions related to $L(y) = 0$. This representation will be for simple modifications of elements of the convex cone C.

Let $C' = \{f: f \text{ is in } C, f(0) = f(1) = 0\}$. Clearly C' is a subcone of C. Let $B' = B \cap C'$.

Lemma 12. A function f is an extremal element of C' if and only if f is an (L)-conical function with its vertex over w in (0,1).

Proof. If f is not an (L)-conical function, then f is not an extremal element of C' by Lemma 8 since the functions used for the non-proportional decomposition in the proof of Lemma 8 are elements of C'. If f is an (L)-conical function with its vertex over 0 or 1, then f is not an element of C' and hence not an extremal element of C'.

If f is an (L)-conical function with its vertex over w in (0,1), then f is an extremal element of C' by Lemma 4 since the proof of Lemma 4 uses only elements of C'.

Definition 12. A real-valued function $K(x,t)$ is a <u>Green's function</u> of $L(y) = 0$ with boundary conditions $y(0) = y(1) = 0$ if 1) for each t, $K(x,t)$ is a continuous function of x and satisfies the boundary conditions, 2) for $x \neq t$ the first and second derivatives of $K(x,t)$ with respect to x are continuous functions of x in $[0,1]$, and at $x = t$ the first derivative has the jump discontinuity $\lim_{x \to t^+} dK/dx - \lim_{x \to t^-} dK/dx = -1$, and 3) $K(x,t)$ as a function of x satisfies $L(y) = 0$ throughout $[0,1]$ except at $x = t$.

Lemma 13. For each x in $(0,w]$, $\{f(x): f \text{ is in } B'\} = [H(x),G(x)]$, and for each x in $[w,1)$, $\{f(x): f \text{ is in } B'\} = [G(x),H(x)]$. Also, $\{f(0): f \text{ is in } B'\} = \{0\}$ and $\{f(1): f \text{ is in } B'\} = \{0\}$.

Proof. Let f be an element of B'. Since B' is a subset of B, $H(x) \leqq f(x) \leqq G(x)$ for all x in $(0,w]$ and $G(x) \leqq f(x) \leqq H(x)$ for all x in $[w,0)$. Let x and y be such that $0 \leqq x < w$ and $H(x) \leqq y \leqq G(x)$. By Definition 1, there exists a unique (L)-linear function F such that $F(0) = 0$ and $F(x) = y$. Then there exists an x' in $(0,w]$ such that $F(x') = G(x')$. The (L)-conical function defined as $F(x)$ on $[0,x']$ and $G(x)$ on $[x',1]$ belongs to B'. Hence for each x in $(0,w)$, $\{f(x): f \text{ is in } B'\} = [H(x),G(x)]$. Similarly, $\{f(x): f \text{ is in } B'\} = [G(x),H(x)]$ for each x in $(w,1)$. For $x = w$ $\{f(w): f \text{ is in } B'\} = \{1\}$. For $x = 0$ $\{f(0): f \text{ is in } B'\} = \{0\}$, and for $x = 1$ $\{f(1): f \text{ is in } B'\} = \{0\}$.

Lemma 14. The convex cone C' is closed in $R^{[0,1]}$ for the topology of simple convergence.

Proof. Let f be an element in the complement of C' relative to $R^{[0,1]}$. If f is in the complement of C, Lemma 10 applies directly to show that f is an interior point of the complement of C'. Let f be in $C \setminus C'$. Then f is a non-negative super-(L) function which is non-zero at 0 or at 1. Let $\varepsilon = [1/2]\max\{f(0),f(1)\}$. Then $U(f;0,1;\varepsilon)$ contains f but does not intersect C'. Thus the complement of C' relative to $R^{[0,1]}$ is open, and hence C' is closed in $R^{[0,1]}$.

Theorem 16. The set B' = {f: f is in C', f(w) = 1}, where w is some point in (0,1), is a convex compact subset of C'-C' which meets each ray of C' once and only once and which does not contain 0.

Proof. The proof is the same as the proof of Theorem 10 with B replaced by B', C by C', Lemma 9 by Lemma 13, and Lemma 10 by Lemma 14.

Lemma 15. If f is an element of $B'\setminus e(B')$, then either there exists a u in (0,w) such that $H(u) < f(u) < G(u)$ or there exists a u in (w,1) such that $G(u) < f(u) < H(u)$, where G and H are the functions of Lemma 9.

Proof. The supposition that the conclusion is false implies $f(x) = H(x)$ for all x in [0,w] and $f(x) = G(x)$ for all x in [w,1]. This f is an (L)-conical function, which contradicts f being in $B'\setminus e(B')$.

Theorem 17. The set e(B') of extremal elements of B' is closed in C'-C' for the topology of simple convergence.

Proof. The proof is the same as the proof of Theorem 11 with B replaced by B', C by C', Lemma 9 by Lemma 13, Lemma 11 by Lemma 15, and Theorem 10 by Theorem 16.

Theorem 18. If f is an element of C, then there exists an integral representation (Radon measure) for f(x)-F(f,0,1;x) in terms of Green's functions of L(y) = 0 with boundary conditions y(0) = y(1) = 0.

Proof. By Lemma 1, f(x)-F(f,0,1;x) belongs to C'. By Theorem 8, there exists an integral representation for each function in B' in terms of extreme points of B'. It then follows that there is an integral representation for each function in C' in terms of extremal elements of C', since B' meets each ray of C' once and only once. Since the extremal elements of C' are (L)-conical functions with vertices over points in (0,1), and since for each such function g there exists a positive real number k such that kg is a Green's function of $L(y) = 0$ with boundary conditions $y(0) = y(1) = 0$, it follows that the integral representation for $f(x)-F(f,0,1;x)$ is in terms of Green's functions of $L(y) = 0$ with boundary conditions $y(0) = y(1) = 0$.

Note. Martin [10] has obtained a result similar to that of Theorem 18 for the special convex cone K of non-negative concave functions on $[0,1]$.

CHAPTER VI

DISCONTINUOUS SUPER-(L) FUNCTIONS

The larger cone C" of non-negative super-(L) functions which may be discontinuous at 0 or at 1 will be considered in this chapter. For f in C", $F(f,0,u;x) \leqq f(x)$ for all x in $[0,u]$ and continuity of $F(f,0,u;x)$ imply $f(0) = \lim_{x \to 0^+} F(f,0,u;x) \leqq \lim_{x \to 0^+} f(x)$. Similarly, $f(1) \leqq \lim_{x \to 1^-} f(x)$. From this and from Theorem 4 it is easy to see that any (L)-conical function is also an extremal element of C". Also, since C is a subcone of C", any element of C which is not an extremal element of C is not an extremal element of C". Therefore to determine the extremal structure of C" it is only necessary to consider those elements which are discontinuous at 0 or at 1.

Lemma 16. If f is an element of C" such that $0 < f(0) < \lim_{x \to 0^+} f(x)$ or $0 < f(1) < \lim_{x \to 1^-} f(x)$, then f is not an extremal element of C".

Proof. Let $g(x) = F(f,0,1;x)$ for all x in $[0,1]$, and let $h = f-g$. Then g is in C" since $F(f,0,1;x)$ is a non-negative (L)-linear function. Since $f(x) \geqq F(f,0,1;x)$ for all x in $[0,1]$, h is non-negative. By Lemma 1, h is super-(L) and hence is in C". Suppose there exists a positive real number k such that $g = kf$. Then $f(0) = g(0) \neq 0$ or $f(1) = g(1) \neq 0$ implies $k = 1$, which contradicts $\lim_{x \to 0^+} f(x) > f(0)$ or

$\lim_{x \to 1^-} f(x) > f(1)$ since $\lim_{x \to 0^+} g(x) = g(0)$ and $\lim_{x \to 1^-} g(x) = g(1)$. Thus g and h form a non-proportional decomposition of f, and hence f is not an extremal element of C".

Lemma 17. If f is an element of C" such that $f(0) = 0 < \lim_{x \to 0^+} f(x)$ or $f(1) = 0 < \lim_{x \to 1^-} f(x)$ and f is not (L)-linear on (0,1), then f is not an extremal element of C".

Proof. Let g on (0,1) be the (L)-linear function determined by the two points $(0, \lim_{x \to 0^+} f(x))$ and $(1, \lim_{x \to 1^-} f(x))$. Define $g(0) = f(0)$ and $g(1) = f(1)$. Let $h = f-g$. (See Figure 18.) Since g is (L)-linear on (0,1), $g(0) \leqq \lim_{x \to 0^+} g(x)$ and $g(1) \leqq \lim_{x \to 1^-} g(x)$, g is a non-negative super-(L) function. Define $G(x) = f(x)$ for all x in (0,1), $G(0) = \lim_{x \to 0^+} f(x)$ and

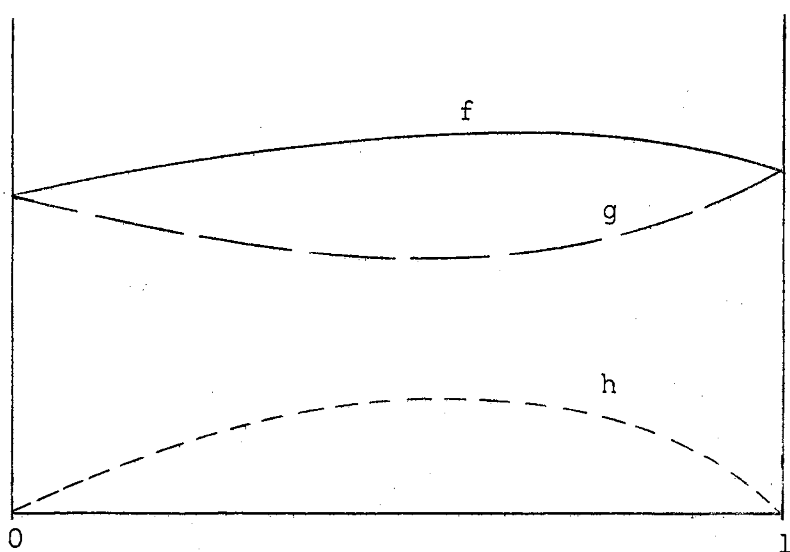

Figure 18. Lemma 17 proof.

$G(1) = \lim_{x \to 1^-} f(x)$. Then $g(x) = F(G,0,1;x)$ for all x in (0,1), so that $h(x) = f(x)-g(x) = G(x)-F(G,0,1;x) \geqq 0$ for all x in (0,1) since G is super-(L). Since $h(0) = h(1) = 0$, h is non-negative. By Lemma 1, h is a super-(L) function. Since g is (L)-linear on (0,1) and f is not, $g \neq kf$ for any real number k. Thus g and h form a non-proportional decomposition of f, and hence f is not an extremal element of C".

Lemma 18. If f is an element of C" such that f is (L)-linear on (0,1), $f(0) = 0 < \lim_{x \to 0^+} f(x)$ and $0 < \lim_{x \to 1^-} f(x)$, or $f(1) = 0 < \lim_{x \to 1^-} f(x)$ and $0 < \lim_{x \to 0^+} f(x)$, then f is not an extremal element of C".

Proof. Only the first case will be proved, since the proof for the other case is similar. Let g on [0,1) be the (L)-linear function determined by the two points (0,0) and $(1, \lim_{x \to 1^-} f(x))$, and define $g(1) = f(1)$. Let h on (0,1] be the (L)-linear function determined by the two points $(0, \lim_{x \to 0^+} f(x))$ and (1,0), and define $h(0) = 0$. (See Figure 19.) Then $(g+h)(0) = g(0)+h(0) = \lim_{x \to 0^+} f(x)$ and $(g+h)(1) = g(1)+h(1) = \lim_{x \to 1^-} f(x)$, so g+h is the unique solution of $L(y) = 0$ which is $\lim_{x \to 0^+} f(x)$ at 0 and $\lim_{x \to 1^-} f(x)$ at 1. Thus $f(x) = g(x)+h(x)$ for all x in (0,1). Now $f(0) = 0 = g(0)+h(0) = (g+h)(0)$ and $f(1) = g(1) = g(1)+h(1) = (g+h)(1)$, so that $f = g+h$. Suppose there exists a positive real number k such that $g = kf$. Then $\lim_{x \to 0^+} kf(x) = \lim_{x \to 0^+} g(x) = 0$ which contradicts $\lim_{x \to 0} f(x) > 0$. Thus g and h form a non-proportional

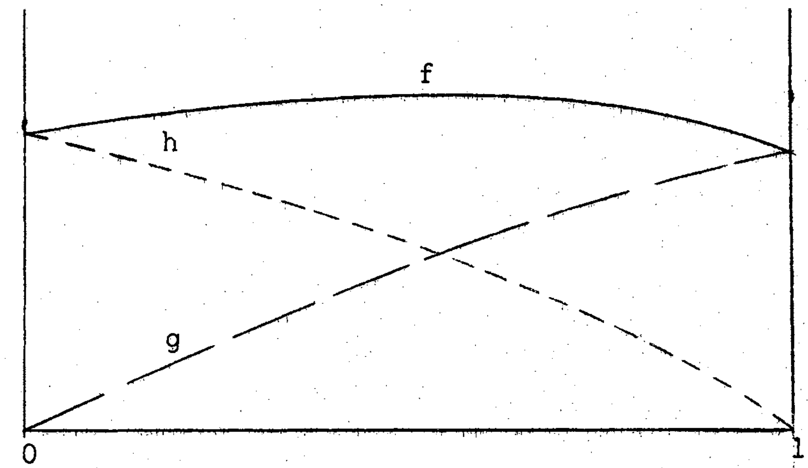

Figure 19. Lemma 18 proof.

decomposition of f, and hence f is not an extremal element of C".

Lemma 19. If f is an element of C" which is either (L)-linear on $(0,1]$ and such that $f(0) = 0 < \lim_{x\to 0^+} f(x)$ and $f(1) = 0$, or (L)-linear on $[0,1)$ and such that $f(1) = 0 < \lim_{x\to 1^-} f(x)$ and $f(0) = 0$, then f is an extremal element of C".

Proof. Let g and h be any two elements of C" such that f = g+h. Only the proof for the first case will be given, since the proof for the other case is similar. By Theorem 4 g and h are (L)-linear on $(0,1]$. Let $k = \lim_{x\to 0^+} g(x)/\lim_{x\to 0^+} f(x)$. Then $kf(x) = g(x)$ for all x in $(0,1]$ by uniqueness of solution to $L(y) = 0$ since $g(1) = 0 = kf(1)$ and $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} kf(x)$. Also, $kf(0) = 0 = g(0)$, so that g = kf and the decomposition is proportional. Thus f is an extremal element of C".

Theorem 19. A function f is an extremal element of C" if and only if it is an extremal element of C or a function of the type described in the statement of Lemma 19.

Proof. The proof follows directly from Lemmas 16, 17, 18 and 19.

CHAPTER VII

A CONE OF CONCAVE FUNCTIONS

One extension of the concave function definition (Definition 2) to domains of dimension higher than one is the following.

Definition 13. Let D be a convex compact set in a real linear space L. Then a real-valued function f is a <u>concave function on D</u> if f is a concave function when restricted to any line interval contained in D.

It is known that the set K(n) of real-valued non-negative concave functions on a convex compact domain in E^n is a convex cone. McLachlan [4] has obtained some results concerning the extremal structure of this cone. In this chapter a certain subcone of this cone will be investigated and its extremal structure completely characterized.

Unless stated to the contrary, the domain of the real-valued functions considered in this chapter will be a compact convex subset D of E^2, and the functions will be assumed to be continuous on D. For each such function f, define $V(f) = \{(r,\theta,z): (r,\theta) \text{ is in } D,\ 0 \leqq z \leqq f(r,\theta)\}$, where r, θ, and z are cylindrical coordinates, with $0 \leqq \theta < 2\pi$ and $r \geqq 0$. It will be assumed that the point $(0,\theta)$ is an interior point of D and that $r = h(\theta)$ is the equation of the boundary of D.

Let $Q = \{f$: f is a non-negative concave function on D which is continuous on the boundary of D, $f(h(\theta),\theta) = 0$ for all θ in $[0,2\pi)$, and for each $\bar{z}$ such that $0 \leqq \bar{z} \leqq \sup\{f(r,\theta): (r,\theta)$ is in $D\}$ the boundary of the intersection of $z = \bar{z}$ with $V(f)$ has equation $r = kh(\theta)$ for some constant k in $[0,1]\}$.

Theorem 20. The set Q is a convex cone.

Proof. Let f and g belong to Q, and let k be a non-negative real number. Since K(2) is a convex cone, kf and f+g are non-negative concave functions on D. For all θ in $[0,2\pi)$, $(kf)(h(\theta),\theta) = kf(h(\theta),\theta) = 0$ and $(f+g)(h(\theta),\theta) = f(h(\theta),\theta)+g(h(\theta),\theta) = 0$.

To complete the proof it will be shown that kf and f+g satisfy the last condition of the definition of Q. Let $\bar{z}$ be such that $z = \bar{z}$ intersects $V(kf)$. Assume $k > 0$, since the function which is identically zero clearly belongs to Q. Then the boundary of the intersection of $z = \bar{z}/k$ with $V(f)$ has equation $r = mh(\theta)$ for some m in $[0,1]$ since f is in Q, so that the boundary of the intersection of $z = \bar{z}$ with $V(kf)$ has equation $r = mh(\theta)$, since the boundary of $\{(r,\theta): f(r,\theta) = \bar{z}/k\}$ is the boundary of $\{(r,\theta): kf(r,\theta) = \bar{z}\}$. Let z' be such that $z = z'$ intersects $V(f+g)$. Let (r',θ',z') be a point in the boundary of this intersection. Let $u = f(r',\theta')$ and $v = g(r',\theta')$. Then the boundaries of the intersections of $z = u$ with $V(f)$ and $z = v$ with $V(g)$ have equations $r = mh(\theta)$ and $r = nh(\theta)$ respectively, for some constants m and n. Note that $r' = \sup\{r: (f+g)(r,\theta') = z'\}$, and that $(f+g)(r,\theta') < z'$ for all $r > r'$ since f+g is concave. Similarly, $mh(\theta') = \sup\{r: f(r,\theta') = u\}$ and $nh(\theta') = \sup\{r: g(r,\theta') = v\}$, $f(r,\theta') < u$ for all $r > mh(\theta')$, and $g(r,\theta') < v$ for all $r > nh(\theta')$. If $mh(\theta') < r'$ or $nh(\theta') < r'$, then $f(r',\theta') < u$ or $g(r',\theta') < v$, which contradicts the definitions of u and v. Thus $mh(\theta') \geqq r'$ and $nh(\theta') \geqq r'$. If $mh(\theta') > r'$ then $f(r,\theta') = u$

for all r in $[r', mh(\theta')]$ since $f(r', \theta') = u$, $f(mh(\theta'), \theta') = u$ and f is non-increasing. Similarly, if $nh(\theta') > r'$ then $g(r, \theta') = v$ for all r in $[r', nh(\theta')]$. Let $\bar{r} = \min\{mh(\theta'), nh(\theta')\}$. Then $\bar{r} > r'$ and $(f+g)(\bar{r}, \theta') = u+v = z'$, which contradicts $r' = \sup\{r: (f+g)(r, \theta') = z'\}$. Therefore either $r' = mh(\theta')$ or $r' = nh(\theta')$. Since m and n are independent of r' and θ', the equation of the boundary of the intersection of $z = z'$ with $V(f+g)$ is $r = mh(\theta)$ or $r = nh(\theta)$. Therefore kf and f+g are in Q, and hence Q is a convex cone.

Lemma 20. Let F be a concave function on $[0,a]$, where $a > 0$, for which $F_+'(0) \leqq 0$ and $F(a) = 0$. Then $f(r,\theta) = F(ar/h(\theta))$ belongs to Q.

Proof. Since $(0,\theta)$ is an interior point of D, $h(\theta) > 0$ for all θ in $[0, 2\pi)$, and since D is a convex set each θ-coordinate line intersects the boundary of D in exactly one point; hence, $f(r,\theta)$ is well defined for all (r,θ) in D. Let (r_1,θ_1) and (r_2,θ_2) be any two distinct points in D, where $0 \leqq \theta_2 \leqq \theta_1 < 2\pi$. Let $R(r_1,\theta_1,f(r_1,\theta_1))$ and $S(r_2,\theta_2,f(r_2,\theta_2))$ be the corresponding points on the graph of f. Let $\overline{RS}$ be the line segment joining R and S. If $\theta_1 = \theta_2$ then $\overline{RS}$ lies below the graph of f since F is concave. If $\theta_1 = \theta_2 + \pi$ then $\overline{RS}$ lies below the graph of f since F is concave and $F_+'(0) \leqq 0$.

The intersection of $z = z_1$ with $V(f)$ will now be shown to be a convex set. Let $r_1 = \sup\{r: F(r) = z_1\}$. Note that F can only be constant on $[0,u]$ for some u in $(0,a)$ since F_+' is monotone non-increasing and $F_+'(0) \leqq 0$. If $r = r_1 h(\theta)/a$ then $f(r,\theta) = f(r_1 h(\theta)/a, \theta) = F(r_1) = z_1$. If $r > r_1 h(\theta)/a$ then $f(r,\theta) = F(ar/h(\theta)) < z_1$ since r_1 is the maximum r for which $F(r) = z_1$. Thus $r = r_1 h(\theta)/a$ is the equation of the boundary of the intersection of $z = z_1$ with $V(f)$, and hence that intersection is convex since D is convex and a positive homothety of a convex set is convex. Thus if R and S lie in a plane $z = z_1$, $\overline{RS}$ lies in $V(f)$.

Assume now that $\theta_1 \neq \theta_2$, $\theta_1 \neq \theta_2 + \pi$, and R and S do not lie in any plane $z = z_1$. Let H_i be the plane $z = f(r_i, \theta_i)$, and let H_{i+2} be the half-plane $\theta = \theta_i$, for $i = 1,2$. Denote by $\widehat{ABCD}$ the section of the graph of f bounded by H_1, H_2, H_3, and H_4 which is nearest $\overline{RS}$, where A, B, C, and D are the points of intersection. Take A = R, D = S, B in $z = f(r_1, \theta_1)$, and C in $z = f(r_2, \theta_2)$. (See Figure 20.) Denote projection onto the (r,θ)-plane by P. Choose $r_1' = \sup \{r\colon f(r,\theta_1) = f(r_2,\theta_2)\}$ and $r_2' = \sup \{r\colon f(r,\theta_2) = f(r_1,\theta_1)\}$. (See Figure 21.) Then $F(ar_1/h(\theta_1)) = f(r_1,\theta_1) = f(r_2',\theta_2) = F(ar_2'/h(\theta_2))$ and $F(ar_2/h(\theta_2)) = f(r_2,\theta_2) = f(r_1',\theta_1) = F(ar_1'/h(\theta_1))$ imply $ar_1/h(\theta_1) = ar_2'/h(\theta_2)$ and $ar_2/h(\theta_2) = ar_1'/h(\theta_1)$ since F is strictly monotone decreasing on $[0,a]$ or constant on $[0,u]$ and strictly monotone decreasing on $[u,a]$ for some u in $(0,a)$. Then $r_1/r_1' = r_2'/r_2$, and hence $\overline{P(A)P(B)}$ is parallel to

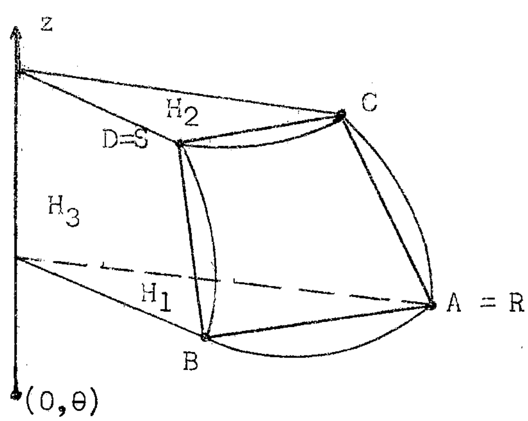

Figure 20. The section $\widehat{ABCD}$.

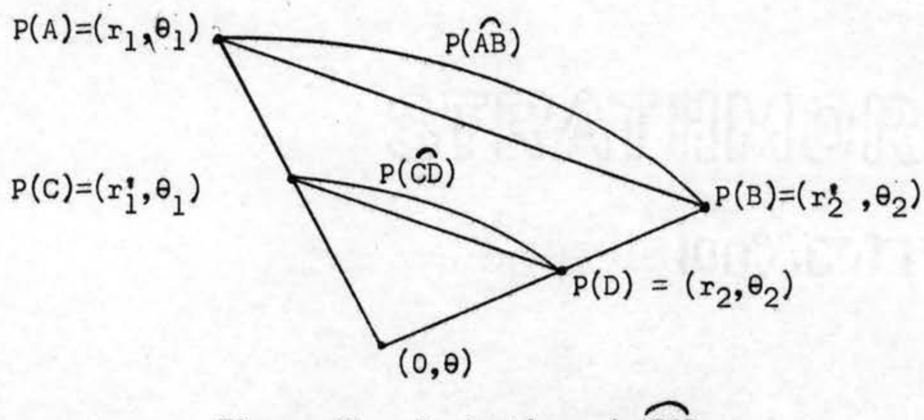

Figure 21. Projection of $\widehat{ABCD}$.

$\overline{P(C)P(D)}$. Since H_1 and H_2 are parallel to the (r,θ)-plane, $\overline{AB}$ is parallel to $\overline{CD}$. Thus conv$\{A,B,C,D\}$, the convex hull of the points A, B, C, and D, is a plane surface. As shown previously, the intersection of $z = f(r_i,\theta_i)$ with $V(f)$ is a convex set, for $i = 1,2$. Hence $\overline{AB}$ and $\overline{CD}$ lie inside the arcs $\widehat{AB}$ and $\widehat{CD}$ of $\widehat{ABCD}$. The intersection of $\widehat{ABCD}$ with $\theta = \overline{\theta}$ is a concave function graph for all θ in $[\theta_2,\theta_1]$ since F is a concave function, so the intersection of conv$\{A,B,C,D\}$ with $\theta = \overline{\theta}$ is inside that graph since $\overline{AB}$ and $\overline{CD}$ lie inside the arcs $\widehat{AB}$ and $\widehat{CD}$ of $\widehat{ABCD}$. Thus conv$\{A,B,C,D\}$, and hence $\overline{RS}$, lies inside $V(f)$. Therefore f is concave on D.

Clearly f is non-negative, and $f(h(\theta),\theta) = F(a) = 0$ for all θ in $[0,2\pi)$. Hence f belongs to Q.

Definition 14. If f is an element of Q such that for each θ in $[0,2\pi)$ f is constant ($\neq 0$) on $[(0,\theta),(kh(\theta),\theta)]$ and linear on $[(kh(\theta),\theta),(h(\theta),\theta)]$ for some k in $[0,1)$, then f is a <u>truncated conical function</u>. If $k = 0$

f will be called a <u>conical function</u>.

The following result due to McLachlan will be used in the characterization of the extremal structure of Q.

Theorem 21 [11]. Let f be a real-valued convex function on $[a,b]$ such that $f'_+(a)$ and $f'_-(b)$ are finite. Suppose f is not piecewise linear on three or fewer non-overlapping segments whose union is $[a,b]$. Then there exist real-valued convex functions g and h on $[a,b]$ that differ from f on $[a,b]$, but have the same values and derivatives as f at the end-points and for some k, $0 < k < 1$, $kg(x)+(1-k)h(x) = f(x)$ for all x in $[a,b]$.

Theorem 22. A function f is an extremal element of Q if and only if f is a truncated conical function or a conical function.

Proof. First, it will be shown that if f is a truncated conical function or a conical function, then it is an extremal element of Q. By Definition 14, f belongs to Q. Let G and H be any two elements of Q such that f = G+H. By Theorem 4, G is linear on $I(\theta) = [(0,\theta),(kh(\theta),\theta)]$ and on $[kh(\theta),\theta),(h(\theta),\theta)]$ for each θ in $[0,2\pi)$. Note that $\lim_{r\to 0^+} \partial G/\partial r \leqq 0$ and $\lim_{r\to 0^+} \partial H/\partial r \leqq 0$ for each θ in $[0,2\pi)$, since G and H are concave. Then $\lim_{r\to 0^+} \partial f/\partial r = 0$ and f = G+H imply $\lim_{r\to 0^+} \partial G/\partial r = \lim_{r\to 0^+} \partial H/\partial r = 0$, and hence G and H are constant on $I(\theta)$ for each θ. Let f = b and G = mb on $I(\theta)$ for θ. Then for each θ, the graph of mf is the unique line segment determined by $(h(\theta),\theta,G(h(\theta),\theta))$ and $(kh(\theta),\theta,G(kh(\theta),\theta))$ since $mf(h(\theta),\theta) = 0 = G(h(\theta),\theta)$ and $mf(kh(\theta),\theta) = mb = G(kh(\theta),\theta)$. Therefore g = mf on $[(0,\theta),(h(\theta),\theta)]$ for each θ, and hence G and H form a proportional decomposition of f. Thus f is an extremal element of Q.

Next, let f be any element of Q which is not a truncated conical function or a conical function. Let F(r) be the restriction of f to M = $[(0,0),(h(0),0)]$. There are three cases to consider:

Case A. Let f be such that F is not piecewise linear on three or fewer non-overlapping segments whose union is M. Clearly there exists a u in (0,h(0)) for which $F'_-(u)$ is finite. By Theorem 21 there exist concave functions G and H on J = $[(0,0),(u,0)]$ that differ from F on J but have the same values and derivatives as F at r = 0 and r = u, and such that for some k in (0,1) $kG(x)+(1-k)H(x) = F(x)$ for all x in J. Define $G(x) = H(x) = F(x)$ for all x in K = $[(u,0),(h(0),0)]$. Since $G'_-(u) = H'_-(u) = F'_-(u)$ and $G(u) = H(u) = F(u)$, G and H are concave on M. By Lemma 20 the functions $g(r,\theta) = G(rh(0)/h(\theta))$ and $h(r,\theta) = H(rh(0)/h(\theta))$ belong to Q since $G'_+(0) = H'_+(0) = F'_+(0) \leqq 0$. Now $f(0,\theta) = g(0,\theta)$, but $f \neq g$ since F differs from G on J, so kg and (1-k)h form a non-proportional decomposition of f. Thus f is not an extremal element of Q.

Case B. Let f be such that F is piecewise linear on three segments in M, say J = $[(0,0),(u,0)]$, K = $[(u,0),(v,0)]$, and L = $[(v,0),(h(0),0)]$. Let $k = 1-(F'_-(v)/F'_+(v))$. Since $F'_+(v) < F'_-(v) < 0$, $0 < F'_-(v)/F'_+(v) < 1$ and so $0 < k < 1$. Let $G(x) = kF(v)$ for all x in $[0,v]$ and $G(x) = kF(x)$ for all x in $[v,h(0)]$. By Lemma 20 $g(r,\theta) = G(rh(0)/h(\theta))$ belongs to Q. Let $H(x) = F(x)-G(x)$ for all x in M. Since $H'_-(v) = F'_-(v)-G'_-(v) = F'_-(v) = (1-k)F'_+(v) = F'_+(v)-G'_+(v) = H'_+(v)$ and $H'_-(u) = F'_-(u)-G'_-(u) = F'_-(u) > F'_+(u) = F'_+(u)-G'_+(u) = H'_+(u)$, H is concave on M. By Lemma 20 $h(r,\theta) = H(rh(0)/h(\theta))$ belongs to Q. Since G is linear on $J \cup K$ and F is not, g and h form a non-proportional decomposition of f, and hence f is not an extremal element.

Case C. Let f be such that F is piecewise linear on two segments in M, say $J = [(0,0),(v,0)]$ and $K = [(v,0),(h(0),0),0)]$, and $F'_+(0) < 0$. Let $k = 1-(F'_-(v)/F'_+(v))$ as in Case B. Define g and h as in Case B. Since H is linear on M and F is not, g and h form a non-proportional decomposition of f. Hence f is not an extremal element of Q, and the theorem is proved.

This thesis will be concluded with some remarks concerning unsolved problems. It would be interesting to know if the integral representation developed in Chapter III is unique. The characterization of C-C has been only partially determined in Chapter IV. A converse to Theorem 18 would be of some value in the application of the results of Chapter V. The bulk of the unsolved problems occur in the present chapter. The ultimate aim is the extension of the results of the previous chapters to the cone of non-negative super-(L) functions on a compact convex subset of E^n, where the super-(L) function is defined analogously to the concave function of Definition 13. Two sub-problems leading in this direction are : 1) the extension of the results of this chapter to the cone of non-negative concave functions on D, and 2) the generalization of the results of this chapter to a set of super-(L) functions defined analogously to the set Q. An important step in this generalization would be the proof of an analogue to Theorem 21.

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APPENDIX

NUMBERED RESULTS

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