THE DISTORTION OF THE CROSS SECTION OF THIN-WALLED CYLINDERS DUE TO LATERAL VIBRATIONS

by

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PREFACE

In 1958 Douglas Aircraft Corporation, under contract to the U. S. Government, fitted a second stage, designated the Delta, to the Thor booster rocket. A 12 inch pipe was weighted and machined so as to simulate this two stage rocket dynamically. Shake tests were made on this pipe, and it was found that there was no agreement between these tests and zero air speed modes as calculated by an IBM 701 vibration analysis program. The effort to find this discrepancy resulted in the study of nonlinear vibrations due to cross sectional distortion as described in this thesis.

I am grateful to my supervisor, Mr. B. M. Hall, Chief of the Dynamics Group of Douglas Aircraft Corp. in Tulsa, for the privilege of using some of the results of my work there in the preparation of this thesis. I am also grateful to Mr. Hall's leadership in the conduct of this study. I am indebted to Mr. Joe Butler, who shared with me his knowledge during our work together on previous related studies and his counsel during this study. Finally I wish to thank Oklahoma State University's Professor L. J. Fila for checking and reviewing this thesis and for his guidance and counsel during its preparation.

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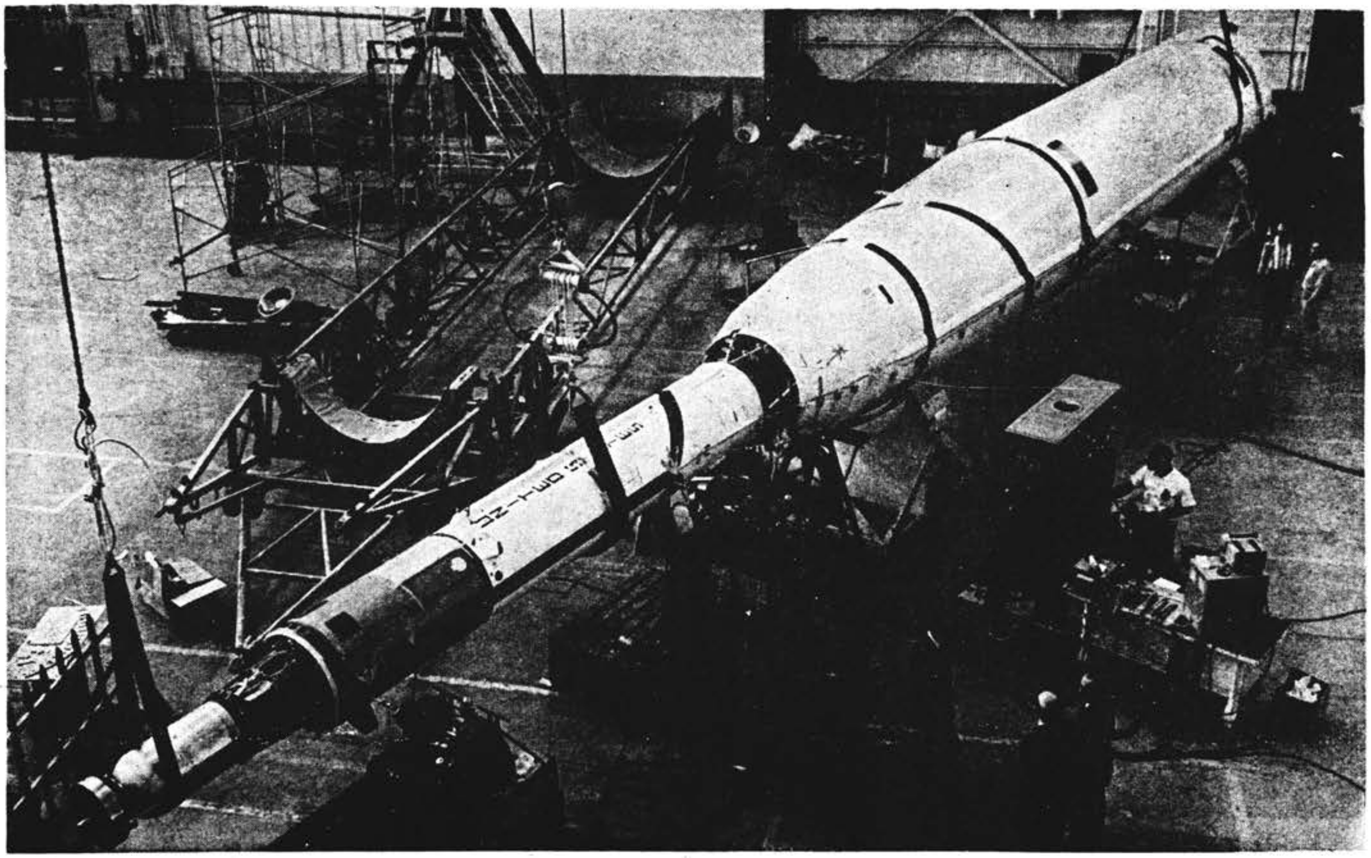
LIST OF SYMBOLS

D	A determinant
E	Modulus of elasticity
EI	Modulus of rigidity
f	Frequency
G	Angular modulus of elasticity
h	Lateral deflection of a beam
I	Area moment of inertia about a transverse axis
J	Polar moment of elasticity
K	A correction to the area moment of inertia about a transverse axis
k	Spring constant
L,x	Axial distance along a beam
M	Moment of force
[M]	A square matrix containing the mass and inertia terms
S	Vertical shear
T	Torsional moment, kinetic energy
V	Potential energy
Θ	The angle of lateral bending of a beam
ϕ	Angle of torsion of a beam
ω	Circular frequency
GL	Spring Matrix, contains elastic moduli and distances

LIST OF SUBSCRIPTS

b	Due to bending
d	Applies to the dynamic state
ex	Due to external loads
i	The number of the mass reference point
s	Due to shear
s	Applies to the static state
θ	In the θ direction
ϕ	In the ϕ direction



Vibration Test Performed on Delta Vehicle

Fully-assembled Douglas Delta space vehicle undergoes a vibration test at Douglas Missile and Space Systems Division, Santa Monica, Calif. Test determines control stability and body bending modes of the 80-ft.-long, 8,428-lb. vehicle.

CHAPTER I

INTRODUCTION

When a thin-walled circular cylinder is bent by external moments, the cross section of that cylinder evolves into an ellipse. It follows that the area moment of inertia of an ellipse about the neutral axis of a bent beam is less than that of the circle from which it evolved. Furthermore, the more an initially circular cylindrical beam is bent, the less will be the area moment of inertia of its cross section.

In the differential equation for undamped free vibration,

$$m\ddot{h} + kh = 0,$$

the k, commonly referred to as the spring constant, is not a constant for thin-walled circular cylinders under lateral vibrations; instead k is a function of the displacement, h. This is because the eccentricity of the elliptical cross section will vary constantly during a vibration cycle. Therefore, k is a function of the varying moment of inertia of this cross section. It may be noted here that a circle is an ellipse with an eccentricity that approaches zero; and, therefore, the previous definition of varying eccenticity is valid. Since the displacement, h, is a function of the time, t, k may be expressed as a function of time. Hence, the equation of undamped free vibration of a thin-walled cylinder may be written

$$m\ddot{h} + k(t)h = 0.$$

Characteristics of this nonlinear spring constant, together with an application to rocket vibrations, are discussed in this thesis.

In order to find the effect of cross sectional distortion on thin-walled cylinders undergoing lateral vibration, a search of the literature was made. Since no satisfactory literature was found on this subject, a search of the literature on cross sectional distortion of thin-walled circular cylinders under static load was made. The best work that was found on the subject was written by Theodore von Karman in 1911.[1] This work consisted of a derivation of a correction to the elementary beam equations under static load conditions. S. Timoshenko translated and revised a part of von Karman's work.[2] R. A. Clark and E. Reissner reviewed von Karman's work and demonstrated that von Karman's assumption in using the semiaxes of an ellipse in the place of the mean geometrical radius of a circle was a reasonably valid mathematical relation.[3]

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[1]Theodore von Karman, "Über die Formänderung dünnwandiger Rohre, insbesondere federnder Ausgleichsrohre", <u>The Collected Works of Theodore von Karman</u>, I (Toronto, 1956), p. 304.

[2]S. Timoshenko, <u>Strength of Materials</u>, I (New York, 1955), p. 455.

[3]R. A. Clark and E. Reissner, "Bending of Curved Tubes", <u>Advances in Applied Mechanics</u>, II ed. R. von Mises and T. von Karman (New York, 1951), p. 120.

Von Karman's derivation, in effect, describes cross sectional distortion in terms of work and energy relationships. Because this approach is used in the derivation of matrices for complex vibration problems, the entire article was translated, and reproduced in Appendix A of this thesis.

CHAPTER II

MYKLESTAD'S LINEAR ANALYSIS

Introduction

In any beam, regardless of the loading configuration, the following relation exists:[1]

$$EI\frac{d^2h}{dL^2} = M. \tag{1}$$

The moment, M, may be broken into two parts. One of these parts will be called the external moment. This will consist of all moments except those caused by vertical shears which hitherto had not been included in the external moment. The sum of the vertical shears, S, about any point may, by vector addition, be considered to be acting at a single point. The product of the sum of the vertical shears, S, times their distance, L, to the point considered may be called the shear moment, M_S. Equation (1) then becomes at a distance from $x = 0$ in Fig. 1:

$$EI\frac{d^2h}{dx^2} = M_{ex} + S(L-x). \tag{2}$$

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[1]Timoshenko, Strength of Materials, I (New York, 1955), p. 246.

The integral of (2) is

$$EI\frac{dh}{dx} = M_{ex}x + SLx - S\frac{x^2}{2} + C_1. \quad (3)$$

If (3) is integrated, the result is

$$EIh = M_{ex}\frac{x^2}{2} + SL\frac{x^2}{2} - S\frac{x^3}{6} + C_1x + C_2. \quad (4)$$

In the cantilever of Fig. 1 at x = 0, dh/dx and h are both zero. If these values are substituted into equation (3), then C_1 is zero. Likewise, if these values are substituted into equation (4), C_2 is zero; and when x is L, equation (4) for the cantilever becomes

$$EIh = \frac{M_{ex}L^2}{2} + S\frac{L^3}{3}. \quad (5)$$

dh/dx is the tangent to the bending curve of the beam. For small angles of bending, where θ is the angle of bending, dh/dx is very nearly equal to θ. If the values, dh/dx is θ, x is L, and C_1 is 0 are substituted into equation (3) then the result is

$$EI\theta = M_{ex}L + S\frac{L^2}{2}. \quad (6)$$

The equation for angular distortion from twisting of a beam under a torsional moment, T, is given by the formula,[2]

[2]Timoshenko and MacCullough, Elements of Strength of Materials, (New York, 1952) p. 258.

$$TL = GJ\phi, \qquad (7)$$

where G is the angular modulus of elasticity, J is the polar moment of inertia, and ϕ is the angle of twist.

Equations (5), (6), and (7) are dependent and represent a system of simultaneous equations which may be arrayed as follows:

$$\begin{aligned} h &= \frac{SL^3}{3EI} + \frac{ML^2}{2EI} + 0 \\ \theta &= \frac{SL^2}{2EI} + \frac{ML}{EI} + 0 \\ \phi &= 0 + 0 + \frac{TL}{GJ}. \end{aligned} \qquad (8)$$

This array may be expressed in matrix form:[3]

$$\begin{bmatrix} h \\ \theta \\ \phi \end{bmatrix} = \begin{bmatrix} \frac{L^3}{3EI} & \frac{L^2}{2EI} & 0 \\ \frac{L^2}{2EI} & \frac{L}{EI} & 0 \\ 0 & 0 & L/GJ \end{bmatrix} \begin{bmatrix} S \\ M \\ T \end{bmatrix}. \qquad (9)$$

The General Myklestad Matrix

Since components and structures are different throughout the length and breadth of a rocket, analysis cannot be made on the basis of any functional continuity. However, an analysis

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[3]Compare with R. H. Scanlon and R. Rosenbaum, <u>Aircraft Vibration and Flutter</u>, (New York, 1951), p. 21.

can be made by an approximation. Consider a rocket divided into sections of finite length as shown in Figure 2. The masses of the chosen sections are considered to be concentrated at the selected reference points. The structural rigidity is assumed to be a constant between adjacent mass reference points; and the length, ${}_{i-1}L_i$, between adjacent mass reference points is considered to constitute the length of a cantilever beam with the concentrated mass of the higher numbered section at its free end. With the nomenclature of Fig. 3, equation (9) is, for the ith section,

$$\begin{bmatrix} \Delta h_i \\ \Delta \theta_i \\ \Delta \phi_i \end{bmatrix} = \begin{bmatrix} \frac{{}_{i-1}L_i^3}{3EI} & \frac{{}_{i-1}L_i^2}{2EI} & 0 \\ \frac{{}_{i-1}L^2}{2EI} & \frac{{}_{i-1}L_i}{EI} & 0 \\ 0 & 0 & \frac{{}_{i-1}L_i}{GJ} \end{bmatrix} \begin{bmatrix} S_i \\ M_i \\ T_i \end{bmatrix}. \tag{10}$$

Likewise, these relations may be seen from Fig. 3:

$$h_{i-1} = h_i - \Delta h_i - {}_{i-1}L_i\,(\theta_i - \Delta\theta_i), \tag{11a}$$

$$\theta_{i-1} = \theta_i - \Delta\theta_i, \tag{11b}$$

and

$$\phi_{i-1} = \phi_i - \Delta\phi_i. \tag{11c}$$

If Δh_i, as given in line 1 of matrix (10) is substituted into equation (11a), then there follows the expression:

$$h_{i-1} = h_i - \frac{S_i\,{}_{i-1}L_i^3}{3EI} - \frac{M_i\,{}_{i-1}L_i^2}{2EI} - \theta_i\,{}_{i-1}L_i + \Delta\theta_i\,{}_{i-1}L_i. \tag{12a}$$

If $\Delta\theta_i$ from line 2 of matrix (10) is substituted into (12a), then the result is

$$h_{i-1} = h_i - \frac{S_{i\ i-1}L_i^3}{3EI} - \frac{M_{i\ i-1}L_i^2}{2EI} - \theta_{i\ i-1}L_i + \frac{S_{i\ i-1}L_i^3}{2EI} + \frac{M_{i\ i-1}L_i^2}{EI}. \qquad (12b)$$

This is equivalent to

$$h_{i-1} = h_i + \frac{S_{i\ i-1}L_i^3}{6EI} + \frac{M_{i\ i-1}L_i^2}{2EI} - \theta_{i\ i-1}L_i. \qquad (12c)$$

Now, $\Delta\theta_i$ from line 2 of matrix (10) is substituted into equation (11b), and the result is

$$\theta_{i-1} = \theta_i - \frac{S_{i\ i-1}L_i^2}{2EI} - \frac{M_{i\ i-1}L_i}{EI}. \qquad (12d)$$

Finally $\Delta\phi_i$ of line 3 of matrix (10) is substituted into (11c), and the following expression is obtained:

$$\phi_{i-1} = \phi_i - \frac{T_{i\ i-1}L_i}{GJ_i}. \qquad (12e)$$

(12c), (12d), and (12e) are dependent simultaneous equations and may be written in matrix form:

$$\begin{bmatrix} h_{i-1} \\ \theta_{i-1} \\ \phi_{i-1} \end{bmatrix} = \begin{bmatrix} \frac{{}_{i-1}L_i^3}{6EI} & \frac{{}_{i-1}L_i^2}{2EI} & 0 & 1 & -{}_{i-1}L_i & 0 \\ \frac{-{}_{i-1}L_i^2}{2EI} & \frac{-{}_{i-1}L_i}{EI} & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-{}_{i-1}L_i}{GJ_i} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_i \\ M_i \\ T_i \\ h_i \\ \theta_i \\ \phi_i \end{bmatrix}. \qquad (13)$$

This matrix may be abbreviated to

$$\begin{bmatrix} h_{i-1} \\ \\ \theta_{i-1} \\ \\ \phi_{i-1} \end{bmatrix} = \begin{bmatrix} GL \end{bmatrix} \begin{bmatrix} S_i \\ M_i \\ T_i \\ h_i \\ \theta_i \\ \phi_i \end{bmatrix}. \qquad (13a)$$

The kinetic energy just to the right of m_{i-1}, due to the concentrated mass, m_i, is

$$T_i = \frac{m_i}{2}({}_{i-1}L_i\,\dot{\theta}_{i-1} + L_{\phi i}\dot{\phi}_{i-1} + \dot{h}_{i-1})^2;$$

where T is the kinetic energy of the ith section, and L_ϕ is the distance from the reference point to the point of torsional distortion.

The forces in the q directions may be found by the Lagrange operator,[4]

$$\frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{q}}\right) = \text{the inertia force, } Q_I.$$

For the h, θ, and ∅ directions, equations are found by performing the Lagrange operations respectively as indicated:

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[4]Timoshenko & Young, Advanced Dynamics, (New York, 1948) p. 214.

$$\frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{h}_i}\right) = m_i(\ddot{h}_{i-1} + {}_{i-1}L_i\,\ddot{\theta}_{i-1} + L_{\phi i}\,\ddot{\phi}_{i-1}), \tag{14a}$$

$$\frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{\theta}_i}\right) = m_i\,{}_{i-1}L_i(\ddot{h}_{i-1} + {}_{i-1}L_i\,\ddot{\theta}_{i-1} + L_{\phi i}\,\ddot{\phi}_{i-1}), \tag{14b}$$

$$\frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{\phi}_i}\right) = m_i L_{\phi i}(\ddot{h}_{i-1} + {}_{i-1}L_i\,\ddot{\theta}_{i-1} + L_{\phi i}\,\ddot{\phi}_{i-1}). \tag{14c}$$

Where S_ϕ is the static unbalance in roll, $m_i\ L_{\phi i}$;

S_θ is the static unbalance in pitch, $m_i\ {}_{i-1}L_i$;

I_ϕ is the moment of inertia in roll, $m_i\ L_{\phi i}^2$;

I_θ is the moment of inertia in pitch, $m_i\ {}_{i-1}L_i^2$;

$I_{\theta\phi}$ is the product of inertia in roll pitch, $m_i\ {}_{i-1}L_i\ L_{\phi i}$;

then with the preceding notation, equations (14a), (14b), and (14c) become

$$\frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{h}_i}\right) = m_i\ddot{h}_{i-1} + S_{\theta i}\,\ddot{\theta}_{i-1} + S_{\phi i}\,\ddot{\phi}_{i-1}, \tag{15a}$$

$$\frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{\theta}_i}\right) = S_{\theta i}\ddot{h}_{i-1} + I_{\theta i}\ddot{\theta}_{i-1} + I_{\theta\phi i}\,\phi_{i-1}, \tag{15b}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}_i}\right) = S_{\phi i}\,\ddot{h}_{i-1} + I_{\theta\phi i}\,\ddot{\theta}_{i-1} + I_{\theta i}\ddot{\phi}_{i-1}. \tag{15c}$$

The right hand terms of this system of equations, (15a, 15b, 15c), may be written in matrix form:

$$\begin{bmatrix} m_i & S_{\theta i} & S_{\phi i} \\ S_{\theta i} & I_{\theta i} & I_{\theta\phi i} \\ S_{\phi i} & I_{\theta\phi i} & I_{\phi i} \end{bmatrix} \begin{bmatrix} \ddot{h}_{i-1} \\ \ddot{\theta}_{i-1} \\ \ddot{\phi}_{i-1} \end{bmatrix}. \tag{16}$$

For small deflections, since $\ddot{q}$ is equivalent to $q\omega^2$, matrix (16) may be written[5]

$$\begin{bmatrix} m_i & S_{\theta i} & S_{\phi i} \\ S_{\theta i} & I_{\theta i} & I_{\theta\phi i} \\ S_{\phi i} & I_{\theta\phi i} & I_{\theta i} \end{bmatrix} \begin{bmatrix} h_{i-1} \\ \theta_{i-1} \\ \phi_{i-1} \end{bmatrix} \omega^2 . \tag{17}$$

Matrix (13a) substituted into matrix (17) yields

$$\begin{bmatrix} m_i & S_{\theta i} & S_{\phi i} \\ S_{\theta i} & I_{\theta i} & I_{\theta\phi i} \\ S_{\phi i} & I_{\theta\phi i} & I_{\phi i} \end{bmatrix} \begin{bmatrix} GL_i \end{bmatrix} \begin{bmatrix} S_i \\ M_i \\ T_i \\ h_i \\ \theta_i \\ \phi_i \end{bmatrix} \omega^2 , \tag{18}$$

This may be abbreviated to

$$\begin{bmatrix} M_i \end{bmatrix} \begin{bmatrix} GL_j \end{bmatrix} \begin{bmatrix} S_i \\ M_i \\ T_i \\ h_i \\ \theta_i \\ \phi_i \end{bmatrix} \omega^2 . \tag{18a}$$

The potential energy resulting from the generalized forces in the h_{i-1}, θ_{i-1}, and ϕ_{i-1} directions at the cantilevered mass,

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[5]See Scanlon and Rosenbaum, p. 66.

m_i, is[6]

$$V_i = S_i h_i + \frac{EI(\Delta\theta_i)^2}{2\,{}_{i-1}L_i} + \frac{GJ(\Delta\phi_i)^2}{2\,{}_{i-1}L_i}.$$

From the preceding equation the inertia forces are found to be

$$\frac{\partial V_i}{\partial h_i} = S_i, \tag{19a}$$

$$\frac{\partial V_i}{\partial \theta_1} = \frac{EI\Delta\theta_i}{{}_{i-1}L_i} = \left(\frac{EI}{{}_{i-1}L_1}\right)\left(\frac{M_T\,{}_{i-1}L_i}{EI}\right) = M_T = S_i\,{}_{i-1}L_i + M_i, \tag{19b}$$

$$\frac{\partial V_i}{\partial \phi_i} = \frac{GJ\Delta\phi_i}{{}_{i-1}L_i} = \left(\frac{GJ}{{}_{i-1}L_i}\right)\left(\frac{T_i\,{}_{i-1}L_i}{GJ}\right) = T_i. \tag{19c}$$

Equations (19) may be written in matrix form:

$$\begin{bmatrix} 1 & 0 & 0 \\ {}_{i-1}L_i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_i \\ M_i \\ T_i \end{bmatrix}. \tag{20}$$

The sum of matrix (18a) and matrix (20) is the sum of the inertia and elastic forces of the ith section. The sum of (18a) and (20) may be equated to the generalized forces of the (i-1)th section:

[6]Timoshenko & Young, p. 214.

$$\begin{bmatrix} S_{i-1} \\ M_{i-1} \\ T_{i-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ {}_{i-1}L_i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_i \\ M_i \\ T_i \end{bmatrix} + \omega^2 [M_i][GL_i] \begin{bmatrix} S_i \\ M_i \\ T_i \\ h_i \\ \theta_i \\ \phi_i \end{bmatrix} \tag{21}$$

When (21) is augmented to the equations of restraint, (13a), the following system for a cantilever is obtained:

$$\begin{bmatrix} S_{i-1} \\ M_{i-1} \\ T_{i-1} \\ h_{i-1} \\ \theta_{i-1} \\ \phi_{i-1} \end{bmatrix} = \left\{ \left[\begin{array}{c|c} H_i & \phi_1 \\ \hline \multicolumn{2}{c}{GL_i} \end{array} \right] + \left[\begin{array}{c} [M_i][GL_i]\omega^2 \\ \hline \phi_2 \end{array} \right] \right\} \begin{bmatrix} S_i \\ M_i \\ T_i \\ h_i \\ \theta_i \\ \phi_i \end{bmatrix} \tag{22}$$

where GL_i and M_i are previously defined and

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ {}_{i-1}L_i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \phi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The second term right of the equal sign of equation (22) is, in fact,

$$\left[\begin{array}{cccccc} & [M_i] & & [GL] & & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} S_i \\ M_i \\ T_i \\ h_i \\ \theta_i \\ \phi_i \end{bmatrix} \omega^2 .$$

Hence by adding the two terms to the right of the equal sign of equation (22), a square matrix is formed. This matrix leads to the equation

$$\begin{bmatrix} S_{i-1} \\ M_{i-1} \\ T_{i-1} \\ h_{i-1} \\ \theta_{i-1} \\ \phi_{i-1} \end{bmatrix} = \begin{bmatrix} D_1 \end{bmatrix} \begin{bmatrix} S_i \\ M_i \\ T_i \\ h_i \\ \theta_i \\ \phi_i \end{bmatrix} .$$

It follows, for example, that for a cantilever analysis of three sections, there will be a realtionship for each of the three sections:

$$\begin{bmatrix} S_{i-1} \\ M_{i-1} \\ T_{i-1} \\ h_{i-1} \\ \theta_{i-1} \\ \phi_{i-1} \end{bmatrix} = \begin{bmatrix} D_1 \end{bmatrix} \begin{bmatrix} S_i \\ M_i \\ T_i \\ h_i \\ \theta_i \\ \phi_i \end{bmatrix} , \quad \begin{bmatrix} S_{i-2} \\ M_{i-2} \\ T_{i-2} \\ h_{i-2} \\ \theta_{i-2} \\ \phi_{i-2} \end{bmatrix} = \begin{bmatrix} D_2 \end{bmatrix} \begin{bmatrix} S_{i-1} \\ M_{i-1} \\ T_{i-1} \\ h_{i-1} \\ \theta_{i-1} \\ \phi_{i-1} \end{bmatrix} , \quad \begin{bmatrix} S_o \\ M_o \\ T_o \\ h_o \\ \theta_o \\ \phi_o \end{bmatrix} = \begin{bmatrix} D_3 \end{bmatrix} \begin{bmatrix} S_{i-2} \\ M_{i-2} \\ T_{i-2} \\ h_{i-2} \\ \theta_{i-2} \\ \phi_{i-2} \end{bmatrix} . \qquad (23)$$

The continued substitution of equations (23) reduces to the single equation:

$$\begin{bmatrix} S_o \\ M_o \\ T_o \\ h_o \\ \theta_i \\ \phi_i \end{bmatrix} = \begin{bmatrix} D_1 \end{bmatrix} \begin{bmatrix} D_2 \end{bmatrix} \begin{bmatrix} D_3 \end{bmatrix} \begin{bmatrix} S_i \\ M_i \\ T_i \\ h_i \\ \theta_i \\ \phi_i \end{bmatrix} . \qquad (24)$$

Now the free end conditions, $M = S = T = 0$, and initial conditions, $h_o = \theta_o = \phi_o = 0$, may be utilized to produce:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} D_1 \end{bmatrix} \begin{bmatrix} D_2 \end{bmatrix} \begin{bmatrix} D_3 \end{bmatrix} \begin{bmatrix} S_i \\ M_i \\ T_i \\ h_i \\ \theta_i \\ \phi_i \end{bmatrix} . \qquad (25)$$

Solutions of equation (25) exist only for values of ω^2 that cause the determinant $\begin{bmatrix} D_1 \end{bmatrix}\begin{bmatrix} D_2 \end{bmatrix}\begin{bmatrix} D_3 \end{bmatrix}$ to vanish.[7] Expansion of the determinant of the equation will yield a determinantal equation in ω^2. The expansion of this determinant will not be attempted here. After expansion, the determinental equation may be solved

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[7]Frazer, Duncan, Collar, Elementary Matrices and Some Applications to Dynamics and Differential Equations, (New York, 1946) p. 57, p. 61, p. 288.

for ω^2 in a variety of ways on a digital computer.[8] For any chosen value of h_i, θ_i and ϕ_i can be found. The unknown deflections, moments and forces of equations (23) can be found in order from right to left; the values of ω^2 are substituted into each determinant.

[8]See for example, Chapter XV, Muir and Metzler, Theory of Determinants, (New York, 1933) p. 603.

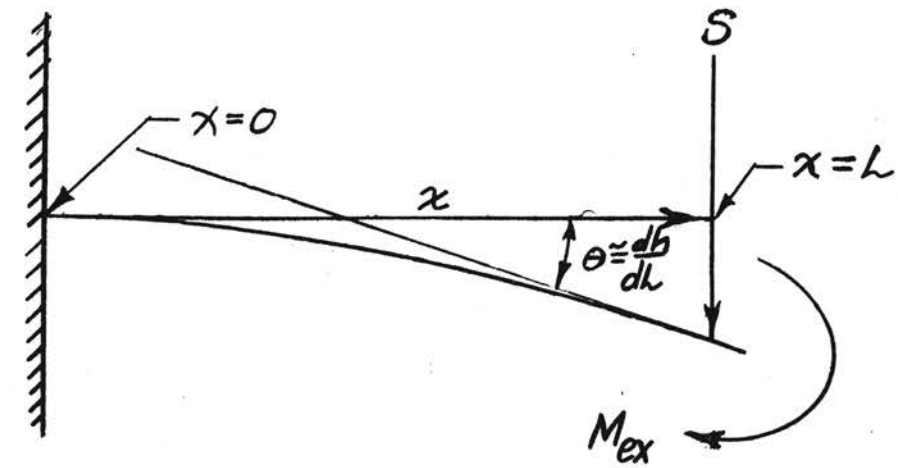

Fig. 1 The Bending of a Cantilever

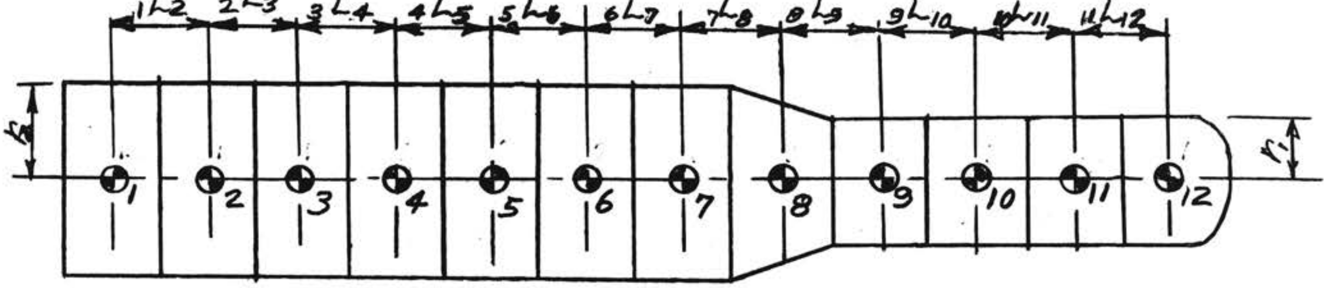

Fig. 2 Typical Sectional Breakdown

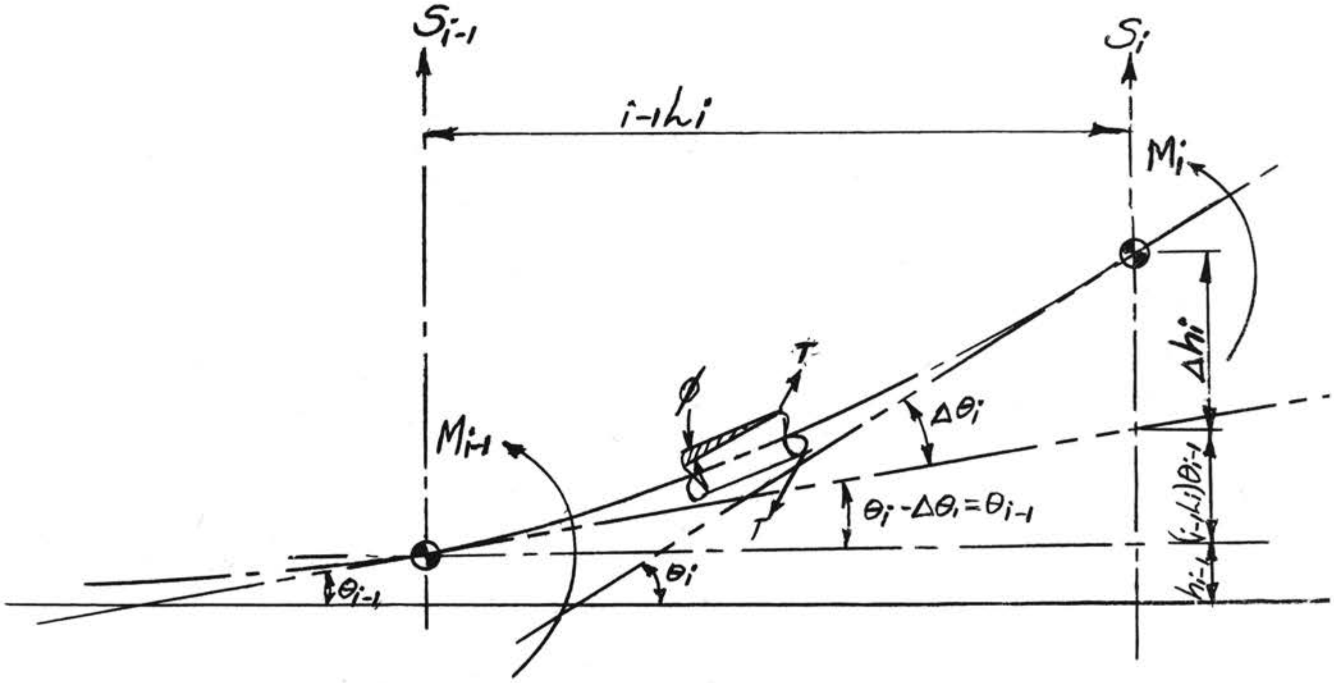

Fig. 3 Nomenclature for Bending of Adjacent Sections

CHAPTER III

THE DISTORTION OF THE CROSS-SECTION

Consider a thin-walled tube with a circular cross-section that is bent by a moment, M_b. See Figure 4. The angle of curvature of this bent circular tube is designated as $(\theta_1 - \theta_{i-1})$. A moment of magnitude M_b, however, would distort the cross section of an initially straight, unbent circular tube into an ellipse. The elliptical cross section has a smaller section modulus, EI, about its major axis than the circular cross section from which it evolved as a result of the application of the moment, M_b. Therefore, the actual angle of curvature of the more limber elliptical tube is greater than the angle of curvature of a circular tube by an angle, $\Delta(\theta_i - \theta_{i-1})$.

Because of the external moment, M_b, there are tensile forces, T, at the convex side of the tube and compressive forces, C, at the concave side of the tube which form a couple. The compressive and tensile forces of this couple have resultants, R and -R, in the direction of the neutral axis. These resultants cause the outer fiber $\overline{ab}$ to be displaced a distance, W, toward the neutral axis and to a position $\overline{a_1b_1}$.

If $\Delta(\theta_1 - \theta_{i-1})$ and W of Fig. 4 are small, then $\overline{e_1b_1} \cong \overline{eb}$. Then, from Fig. 4 the following relation is observed:

$$(\overline{a_1b_1} - \overline{2a_1e_1}) - (\overline{ab} - \overline{2a_1e_1}) \cong \overline{a_1b_1} - \overline{ab}. \quad (1)$$

Likewise, the following relation may be observed:

$$(\overline{a_1b_1} - \overline{2a_1e_1}) = (r - W)\ \Delta(\theta_i - \theta_{i-1}). \quad (2)$$

If the ratio r/R is small then $(r - W) \cong r$ and equation (2) may be written as

$$(\overline{a_1b_1} - \overline{2a_1e_1}) \cong r\ \Delta(\theta_1 - \theta_{i-1}). \quad (3)$$

From Fig. 4 it is evident that the following relation exists:

$$(\overline{ab} - \overline{a_1b_1}) = W(\theta_1 - \theta_{i-1}). \quad (4)$$

When equations (3) and (4) are substituted in equation (1), the result is

$$r\ \Delta(\theta_1 - \theta_{i-1}) - W(\theta_1 - \theta_{i-1}) = (\overline{ab} - \overline{2a_1e_1}). \quad (5)$$

Now, $(\overline{ab} - \overline{2a_1e_1})$ is the total elongation of the fiber $\overline{ab}$, which is defined as

$$\overline{ab} = (R + r)(\theta_i - \theta_{i-1}).$$

The unit elongation or strain may be found by dividing the elongation of the fiber $\overline{ab}$ by its length:

$$\xi = \frac{r\Delta(\theta_i - \theta_{i-1})}{(R+r)(\theta_i - \theta_{i-1})} - \frac{W(\theta_i - \theta_{i-1})}{(R+r)(\theta_i - \theta_{i-1})}. \quad (6)$$

The first term of equation (6) is the strain in the outer fiber from the rotation of the cross section $\overline{bd}$ with respect to the cross section $\overline{ac}$. The second term of equation (6) is the result of the flattening of the cross section. The maximum stress (which is in the outer fiber) is, from equation (6) and Hooke's law,

$$\sigma = \xi E = \frac{Er[\Delta(\theta_i - \theta_{i-1})]}{(R+r)(\theta_i - \theta_{i-1})} - \frac{EW(\theta_i - \theta_{i-1})}{(R+r)(\theta_i - \theta_{i-1})}. \qquad (7)$$

The effect of the second term may be of considerable importance. For example, if $(R + r)$ is 60 inches and W is .02 inches, then $W/(R + r)$ is 1/3,000; and the corresponding stress in steel is 10,000 psi. This value is nearly equal to the fatigue working stress for mild steel, (15,000 psi).

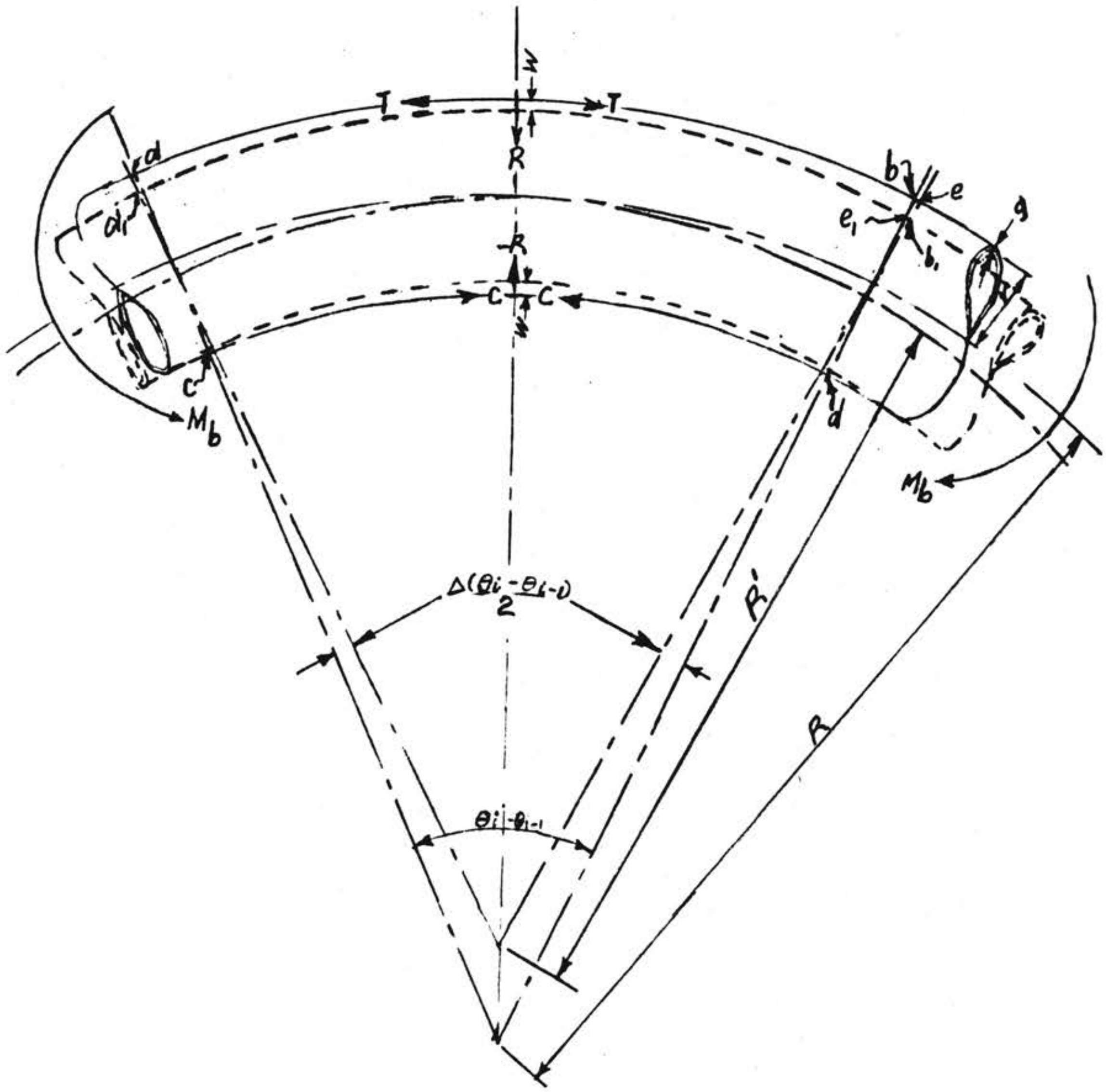

Fig. 4. The Bending of Thin-Walled Circular Cylinders.

CHAPTER IV

NONLINEAR MODES

Introduction

In Chapter II, a method of linear modal analysis, suitable for digital computers, was developed. In Chapter III, a method of calculating the effect of cross sectional distortion on the stresses and strains of thin-walled cylinders under static loading conditions was developed. It is the purpose of this chapter to show how the distortion of the cross section as discussed in Chapter III, can be combined with the linear modal analysis of Chapter II to evaluate the nonlinear lateral vibration modes of thin-walled cylinders.

Von Karman's Correction During A Vibration Cycle

In Appendix B a correction factor is given to be multiplied by the area moment of inertia of thin-walled cylinders. This correction factor is needed when the cross section of a thin-walled cylinder is distorted. Therefore, this factor is needed when the cross section of a thin-walled cylinder is distorted during lateral vibrations. This correction at any time, t, during a vibration cycle is

$$K_s = 1 - \frac{9}{10 - 12\left(\frac{gL}{r^2 \Delta\theta}\right)^2}.$$

Now, in order to simplify the presentation, Q shall be a constant equal to $-12(gL/r^2)^2$. Using this constant, Q, the preceding equation may be written

$$K_s = 1 - \frac{9}{10 + \frac{Q}{(\Delta\theta)^2}}.$$

This is equivalent to

$$K_s = 1 - \frac{9(\Delta\theta)^2}{10(\Delta\theta)^2 + Q}.$$

This may be further simplified:

$$K_s = \frac{10(\Delta\theta)^2 + Q - 9(\Delta\theta)^2}{10(\Delta\theta)^2 + Q}.$$

If the terms of the numerator are collected, the expression becomes

$$K_s = \frac{Q + (\Delta\theta)^2}{Q + 10(\Delta\theta)^2}.$$

The numerator and denominator are multiplied by $Q - 10(\Delta\theta)^2$, which yields

$$K_s = \frac{Q^2 + Q(\Delta\theta)^2 - 10Q(\Delta\theta)^2 - 10(\Delta\theta)^4}{Q^2 - 100(\Delta\theta)^4}.$$

In the case of rockets of cylindrical shape, $\Delta\theta$ is small. For, example, even as small a value of $\Delta\theta$ as .01 is too large to be encountered under all designed for conditions. Therefore, all powers of $\Delta\theta$ to the fourth power may be dropped without appreciable error. It follows that, with $(\Delta\theta)^4$ dropped, the preceding equation is approximately

$$K_s \cong \frac{Q^2 - 9Q(\Delta\theta)^2}{Q^2}.$$

The equation may now be written in a form like that of the von Karman correction:

$$K_s = 1 - \frac{9(\Delta Q)^2}{Q}. \qquad (1)$$

Since the maximum angular deflection is given at each mass reference point in the usual linear modal analysis, this factor, $\Delta\theta_{max} = \theta_i - \theta_{i-1}$ shall be used in a dynamic correction factor to the linear modal analysis. Over a complete cycle, according to linear vibration theory, the angular difference, $\Delta\theta$, varies in a sinusoidal manner. This may be expressed, at any time in a cycle, as

$$\Delta\theta = \Delta\theta_{max} \sin 2\pi ft; \qquad (2)$$

where f is the frequency in cycles per unit time, and t is the time from the beginning of a vibration cycle. Then combining (1) and (2), the value of the correction factor, K, is

$$K_s = 1 - \frac{9(\Delta\theta_{max} \sin 2\pi ft)^2}{Q}. \qquad (3)$$

By the substitution of the double angle equivalent of $\sin^2 2\pi ft$, equation (3) becomes

$$K_s = 1 - \frac{g(\Delta\theta_{max})^2\left(\frac{1}{2} - \frac{1}{2}\cos 2\pi f t\right)}{Q}. \quad (4)$$

In order to find the effective value of equation (4) over a vibration cycle, the second term of that equation may be intregated over the time interval of one cycle while it is divided by the time interval, 1/f, of one cycle as follows:

$$K_d = 1 - \frac{g(\Delta\theta_{max})^2 f \int_0^{1/f}\left(\frac{1}{2} - \frac{1}{2}\cos 2\pi f t\right)dt}{Q}$$

The integration of this dynamic correction factor yields.

$$K_d = 1 - \frac{g(\Delta\theta_{max})^2 f}{Q}\left|\frac{t}{2} - \frac{\sin 2\pi f t}{4\pi f}\right|_0^{1/f}.$$

When the limits are taken the equation becomes

$$K_d = 1 - \frac{g(\Delta\theta_{max})^2 f}{Q}\left(\frac{1}{2f}\right)$$

which reduces to

$$K_d = 1 - \frac{g(\Delta\theta_{max})^2 r^4}{24(gL)^2}. \quad (5)$$

Thus the second term of the dynamic correction factor is ½ of the value of the 2nd term of the static correction factor. Since equations (1) and (5) are in the form of the von Karman equation it follows that von Karman's factor may be used in place of equation (5). If the von Karman correction factor, K_s, is taken from Fig. 1 of Appendix B, then the dynamic correction factor, k_d, is

$$K_d = \frac{1 + K_s}{2}$$

Hoop Modes

When the cross section of a thin-walled cylinder is distorted into an ellipse, the mass in the shell has inertia toward and away from the longitudinal axis of the cylinder. This inertia is resisted by opposing stresses in the pipe walls. This balance of kinetic and strain energy can affect the lateral modes, with or without the distortion correction factor. For this reason, it is believed that hoop mode equations should be a part of the modal linear analysis.[1] Then when the dynamic cross sectional correction factor, K_d, is applied to each mass reference point's area moment of inertia, possible coupling of lateral bending and hoop modes will be combined with the effect of cross sectional distortion to expose any dangerous condition. In addition to hoop modes, there exists the possibility of exciting stringer modes. This is true especially if the stringers are spaced somewhat widely apart. In this mode the stringers combined with the skin to which they are welded may be thought of as fixed ended beams that are rigidly attached to the stiffer sections of a rocket. These modes could

[1]This mode is discussed as the vibration of a circular ring: S. Timoshenko, Vibration Problems in Engineering, (New York, 1955) p. 425.

conceivable occur in addition to the modes associated with the bending of a thin shell.

The Effect of Cross Sectional Distortion
On the Modes of the Thor-Delta Rocket

In order to find the effect of cross sectional distortion on the Thor booster Delta second stage combination, a linear Myklestad analysis was made.[2] The two stage rocket was divided into a total of 20 mathematical mass reference sections; and masses, moments, moments of inertia, and area moments of inertia associated with each section and its mass reference point were calculated. An initial deflection was assumed and modal frequencies and maximum deflections at each mass reference point were obtained.

It may here be observed that equation (5) may be written

$$K_d = 1 - \frac{9(.707\,\Delta\theta_{max})^2 r^4}{12(gL)^2} .$$

This shows that if $.707\,\Delta\theta_{max}$ is used in place of $\Delta\theta_{max}$, that the static von Karman distortion correction factor, K_s, as shown in Fig. 1 of Appendix B may be used. The hoop and stringer modes were not included in this analysis. In general, the neglect of degrees of freedom results in a low frequency which is on the high side. The neglect of modes is

[2]This analysis was run under the direction of Burt Hall, Chief of the Dynamics Group, Douglas Aircraft Corporation in Tulsa, Oklahoma in 1958.

equivalent to an increase of stiffness which should result in smaller distortions. Consequently, it was decided to increase the term .707 $\Delta\theta_{max}$ to simply $\Delta\theta_{max}$. Thus, in effect, the static correction factor, K_s, was used in place of the dynamic correction factor, K_d. The relation $K_d = \frac{1 + K_s}{2}$, should be kept in mind.

$\Delta\theta_{max}$ is the difference between the maximum longitudinal angular deflections, θ_{max}, of adjacent mass reference points. The values of $\Delta\theta_{max}$ found in this manner were used in the von Karman correction factor, K_s, as taken from Fig. 1 of Appendix B. This correction factor multiplied by the area moment of inertia at each mass reference point gave a new, or corrected, area moment of inertia. The linear analysis was then repeated using these new or corrected area moments of inertia at each mass reference point.

The results of this linear simulation of the nonlinear case discussed showed that the first bending mode was quite close to the uncorrected bending mode. The second bending mode, however, changed from a frequency of 11.45 cps to a frequency of 9.43 cps. Fig. 5 shows a shifting of the node point after consideration of cross sectional distortion. This node point moved from the booster stage, through the conical transition region and to the base of the delta second stage after the linear analysis was corrected for cross sectional distortion.

Closing Comments

It should be observed that each lateral bending mode studied requires a separate correction for cross sectional distortion. Thus a corrected modal analysis is valid only for the deflections of the mode being studied. This is because the deflections of other modes would be quite different causing quite different area moments of inertia to be used in the analysis.

Bending modes may be obtained independently by Myklestad's method and lower modes need not be computed.[3] However, these modes should not be left out of the mass or spring matrices. Particularly adjacent modes should not be left out. These adjacent modes may be torsional, stringer, hoop, fore and aft or any other conceivable mode. By adjacent modes it is meant modes that have the next higher and next lower frequency to that of the mode being studied. Studies on this subject by Joe Butler and W. J. Slagle at Douglas Aircraft Co. of Tulsa, Oklahoma reveal that the leaving out of adjacent modes can cause errors up to 10% in frequency. These errors in frequency could cause even greater errors in deflections, particularly in the case of self-excited vibrations. Present linear

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[3] R. L. Bisplinghoff, H. Ashley, and R. L. Halfman, Aeroelasticity, (Cambridge, 1955), p. 163.

programs for the analysis of vibrations in rockets could be modified to feed in increments of initial end deflections. For each end deflection a linear analysis would be run. This linear analysis could be corrected from tables in core storage of the computer for each mode desired. Present day high speed computers make such an analysis feasible.

The thickness, g, of the skin was calculated as the equivalent thickness that the ribbed cylinder actually used would have if its cross sectional area moment of inertia were the same as an equivalent thin-walled cylinder. However, for mass and inertia terms the actual masses were used. Periodic longitudinal stiffeners should have little effect on the bending for the first two bending modes.[4]

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[4]This statement is due to Mr. William Pierpont, Chief Dynamics Group, Beech Aircraft Corp, Wichita, Kansas. Mr. Pierpont has been a vibration engineer since 1945.

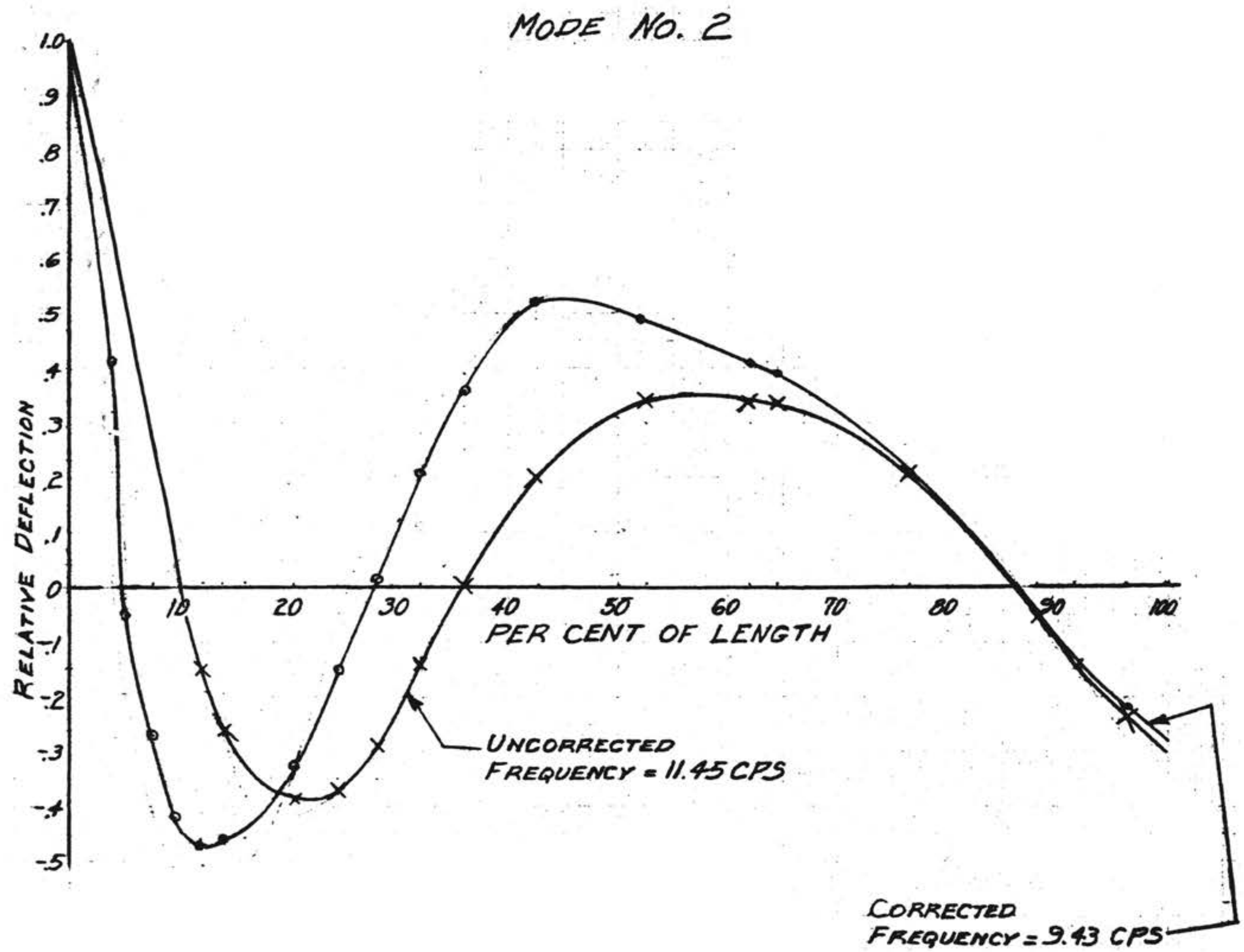

Fig. 5. The Effect of the von Karman Correction.

CHAPTER V

SUMMARY AND CONCLUSIONS

The purpose of this thesis is to show a method of making vibration modal analyses of thin-walled cylinders that have nonlinear characteristics which are due to dynamic cross sectional distortions. It is possible that this method of analysis may utilize existing computer programs designed to perform linear vibration modal analyses.

The method consists of first dividing the cylinder into several mathematical sections and performing the linear modal analysis for a given end deflection. The deflections at each mass reference point, as found by the linear analysis for a given mode, are then used to calculate new area moments of inertia about a transverse axis. These new area moments of inertia represent the effective value of the area moment of inertia over a vibration cycle. These new moments of inertia may be found by

$$I_n = I\left(1 - \frac{9}{2[10 + 12(gL/r^2\Delta\theta)^2}\right) \cong I\left(1 - \frac{9(.707\Delta\theta_{max})^2 r^4}{12(gL)^2}\right)$$

where g is the thickness of the cylinder wall, r is the radius of the cylinder, L is the distance between mass reference points and $\Delta\theta$ is the difference between two successive angles, at their reference points, with the horizontal.

In the case considered, a loading configuration of the Thor-Delta rocket, there is a marked change in the deflection curve and the frequency as a result of considering the cross sectional distortion.

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APPENDIX A

ÜBER DIE FORMÄNDERUNG DÜNNWANDIGER ROHRE, INSBESONDERE FEDERNDER AUSGLEICHSROHRE

by

THEODORE VON KARMAN

TRANSLATED

by

WILLIAM JOSEPH SLAGLE

THE DEFORMATION

OF THIN-WALLED PIPES

ESPECIALLY SPRING COMPENSATOR PIPES*

The reason for writing the following article is found in a research paper by Professor A. Bantlin on the deformation of spring compensator pipes. This paper was published in the second issue, 1910, of this periodical (Zeitschrift für des Vereines deutcher Ingenieure) and in the ninety-second issue of "Mitteilungen über Forschungsarbeiten." This investigation, which is worthy of the utmost consideration, led to the remarkable result that the measured deflection was four to five times greater than that which had been ascertained by calculation. In the following pages I would like to show that this significant difference between theory and measurement arises solely because the customary bending theory, which remains mostly silent on the assumed hypothesis that the cross section of a rod remains unchanged, can lead to very great errors if applied directly to the bending of thin-walled pipes having an initial curvature. Now consider that if an initially curved

*Spring compensator pipes are pipe bends in which the spring or elasticity of the bend is used to compensate for thermal, pressure, shock, or vibration strains of connected unbent pipes or attached plant equipment.

rod is bent, the individual fibers will be distorted to conform to the expected new curvature provided that these fibers are not displaced toward the neutral axis; and that the actual displacement will be insignificantly larger than such displacement as will be indicated by the usual bending theory.

The following case applies particularly to pipes with wall thicknesses that are small in comparison to their internal diameter.[1] It is easy to see that a proportionally small flattening of the cross section is enough to alter considerably the entire stress distribution. Thus a pipe with a radius of curvature of 800 millimeters, flattened one millimeter in diameter, already has a diminishing of the stress of about

$$(2{,}000{,}000 \times .5)/800 = 1250 \text{ kg./cm.}^2$$

(by the definition of the modulus of elasticity). That is to say, that the outer fibers, which carry the greatest tensil and compressive stresses according to the conventional bending theory, will be greatly relieved under these conditions. It is easy to understand beforehand from Figures 1 and 2 that there must be a tendency of the cross section to flatten. It may be seen that as long as the resultant of the tension as well as the resultant of the compression in the outer fibers run perpen-

[1] Very commonly the bending theory can lead to very great errors if one dimension of a cross section is small compared to its other dimension. It is a pleasant duty for me to mention that this point, and particularly the case of thin-walled cylinders, had already been brought to my attendion several years ago by Professor Prandtl.

dicular to the pipe wall, that a bending moment so applied as to increase the curvature will press together the inner and outer pipe walls toward the neutral axis, and that a moment in the opposite sense will tend to pull them apart. It may be seen in either case that there will be a decrease in strain. It follows that a pipe, distorted by bending, having a known measured circumferentially distributed stress and a small bending moment will behave as indicated in the usual bending theory: a given moment will not significantly increase the cross sectional distortion. In the case to be shown, the following typical calculation, together with formula II which follows, will develop a completely logical measurement for the difference between theory and practice by consideration of the flattening of the cross section.[2] The then still remaining difference of about 20% is probably due to the difference in the thickness of the pipe and other geometrical assumptions not actually true in the physically tested body.

The Theorem of the Minimum Work of Distortion

It would present no fundamental difficulty to develop again the formulas for calculating the distortion by the so-named theory of thin shells as they would be developed by the mathematical theory of elasticity. It should be kept in mind

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[2] It follows that due to this flattening, the moment of inertia will decrease. This effect is, however, of very little importance.

that this method will succeed but has little practical value because of the long tedious calculations involved. Fortunately for the art, the extremely simple, exact, and complete theorem of the minimum work of distortion has been adopted. It was namely W. Ritz[3] who developed a different and very valuable method. This consists of approximate relaxations by the method of variations. I believe that the approximate solution of Göttinger, who died too early in life, should merit the special attention of all engineers as it creates a somewhat different approach to the theories of elastic stability and hydrodynamics; and since there are many other directly applicable cases in ideal theory in which this fundamental formula will achieve with simplicity the complete exactness required by technology.[4]

Before I present the applications of these methods, I would like to present completely the simplest form of the problem. A pipe with a circular cross section and circular center line, Fig. 1, will be uniformly bent so that the angle between both end cross sections that previously had a value of θ will have a value of $\theta + \Delta\theta$.

(a) How will the circular form of the cross section change?

(b) How large is the change in the angle, $\Delta\theta$, in relation to the bending moment and in relation to

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[4]Recently, the Ritz approximation was modified by Mr. Timoshenko of Kiev so as to give a linear solution.

the work of distortion which is the half product of the change in the angle and the bending moment?

By not considering the cross section it may be seen, by the common bending theory, that the approximate value of the moment is[5]

$$M_b = EI\frac{\Delta\theta}{R\theta}; \qquad (1)$$

where I is the moment of inertia of the cross section;

E is the modulus of elasticity of the material; and

R is the radius of curvature of the centerline. The work of distortion is found by the common formulas,

$$W = \tfrac{1}{2}\, M_b\,\Delta\theta$$

and

$$W = \frac{EI}{2}\frac{(\Delta\theta)^2}{R\theta}. \qquad (2)$$

Otherwise, the theorem of the least work of distortion gives the following reply to our questions: the cross section will be distorted in such a manner that the work of distortion, using equal values of $\Delta\theta$, becomes the minimum value determined by the work actually required by the necessary moment, M_b, needed

[5]The formulas do not indicate the influence of the finite curvature on the stress distribution; it is merely indicated, in these first formulas, that only a large initial curvature can cause any appreciable effect. See Z. 1910 s. 1675.

for the bending of the pipe.

The work of distortion, in the case where the distortion of the cross section occurs, falls into two parts: that work done in straining the longitudinal fibers, and that work done in distorting the cross section. It is now clear that, in a curved rod, the longitudinal strain of the fibers will steadily decrease as the elements of the cross section are displaced toward the neutral axis. Now with solid rods or with thick-walled pipes having no increase in transverse strain, this is impossible; so that if the first part of the work of distortion decreases, the second part will increase approximately the same amount. In fact, the smallest significant value for the work of distortion that can be easily calculated for especially simple forms of solid rods and thick-walled pipes differs very little from the value yielded by the usual bending theory, so that both computation procedures agree closely. However, the relations are entirely different concerning thin-walled pipes (or more generally when a measurement of the solid part of the cross section is small compared to that which is left over). In this case the resistance to the distortion of the cross section is relatively small, so that it may be expected that the resultant work of distortion obtained by distorting the cross section can be considerably less. One need not consider, therefore, distortions in which the centerline of the pipe wall is not changed. Of course, it is easy to comprehend that the transverse strain of the pipe wall that is caused by the work of distortion will likewise increase approximately in the same magnitude as is taken away by the decrease of the longitudinal strain, so that thereby no considerable decrease of the work of distortion can occur.

With these reservations, the entire work of distortion is divided into two parts, which are:

(a) the longitudinal strain of the fibers, and

(b) the transverse bending of the pipe wall.

In the following pages we would like to calculate the sum of both of these work factors.

The Expression for the Work of Distortion

In Figure 3, R is the radius of curvature of the center-line,

r is the mean radius of the pipe cross section, and

g is the thickness of the pipe wall.

We next place a coordinate system, xy, through the center, O, of the pipe cross section, so that the y axis is coincident with the center of curvature. The angle, B, is shown between the radius, OP, and the x axis. See Figure 4. We represent the displacement of the point, P, first as the components w_x, w_y in the directions of the x and y axes and then in the directions of the tangent and the radius of curvature. See Fig. 5.

We next calculate the strain of a fiber that is perpendicular to the plane of the figure and that goes through P. The calculation consists of two parts. For the first part, we may set by the conventional bending theory:

$$e_1' = \frac{y}{y+R}\frac{\Delta\theta}{\theta}. \qquad (3a)$$

The second part results from the displacement of the point in the y direction and obviously amounts to

$$e_1'' = \frac{w_y}{y+R}. \qquad (3b)$$

We would like, in order to simplify the summation, to neglect y next to R, and thus write approximately[6] for the resultant longitudinal strain:

$$e_1 = \frac{y\Delta\theta}{R\theta} + \frac{w_y}{R}, \qquad (3c)$$

or by introduction of the equation, $w_y = w_t \cos B + w_r \sin B$:

$$e_1 = \frac{y\Delta\theta}{R\theta} + \frac{w_t \cos B + w_r \sin B}{R}. \qquad (3d)$$

We must restrict ourselves to the strainless movement of the pipe wall by setting the transverse strain equal to zero and thereby succeed in finding a relation between both of the components of the displacement. If the points are displaced

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[6]This approximation gives results of exactly the same accuracy as the approximate values found by formula (1). One may easily be convinced since if, by the examination of the stress distribution, an error of the order of (r/R) is committed; that the error of the resulting distortion is only of the order $(r/R)^2$.

to w_t and w_r in the directions of the tangents and the radius, the transverse strain of a circularly formed ring is given by the expression,

$$e_z = \frac{\frac{dw_t}{dB} + w_r}{r}. \tag{4}$$

Therefore, if we set

$$\frac{dw_t}{dB} + w_r = 0,$$

then there results

$$w_r = -\frac{dw_t}{dB}. \tag{5}$$

Therefore, the result is that the distortion of the pipe cross section will be fixed by one component, w_t. Finally, in order to find the transverse bending of the pipe, we must calculate the change of curvature of the pipe centerline. This is likewise given by the theory of the bending of curved rods and particularly for circular cylinders by the formula,[7]

$$\frac{1}{r'} - \frac{1}{r} = \frac{1}{-r^2}\left(\frac{d^2 w_r}{dB^2} + w_r\right); \tag{6}$$

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[7]Compare, for example, B. Föppl, "Vorlesungen über technische Mechanik," Bd. III, Absatz #36. Moreover, the formula can easily be derived by noticing that the first term expresses the rate, $-d^2w/ds^2$; that a straight rod becomes curved; and that the second term, however, represents the change in curvature caused by the distortion, w_r, from its original finite curvature.

where r' means the radius of curvature of the pipe wall after distortion. Putting in the value of w_r from formula (5) we finally obtain the expressions: for the longitudinal strain,

$$e_1 = \frac{1}{R}\left(\frac{\Delta\theta}{\theta} r \sin B + w_r \cos B - \frac{dw_r}{dB} \sin B\right)$$

and for the transverse bending,

$$\frac{1}{r'} - \frac{1}{r} = \frac{1}{r^2}\left(\frac{d^3 w_r}{dB^3} + \frac{dw_r}{dB}\right) \qquad (7)$$

With the use of these formulas, we are now ready to write the expressions for the work of distortion, which are obvious enough if we consider only cylinders of length, 1.

The first expression, the magnitude of which is dependent upon the longitudinal strain for a unit length cylinder is given for a unit arc length by $\frac{1}{2}Ee_1^{\,2}$, so for an entire cylinder of unit length:

$$W_1 = \frac{rgE}{2}\int_0^{2\pi} e_1^2\, dB = \frac{rgE}{2R^2}\int_0^{2\pi}\left(\frac{r\Delta\theta}{\theta} + w_r \cos B + w_r \sin B\right)^2 dB.$$

The second expression, the magnitude of which is dependent upon the transverse bending component, is, for an element of the ring of Fig. 4 equal to

$$\frac{1}{2}\,\frac{g^3 E}{12}\left(\frac{1}{r'} - \frac{1}{r}\right)^2 r\, dB;$$

where $g^3/12$ is the moment of inertia of a cylindrical section of unit length and of unit arc length. Hence for the entire (unit length) cylinder,

$$W_2 = \frac{g^3 rE}{24}\int_0^{2\pi}\left(\frac{1}{r'} - \frac{1}{r}\right)^2 dB = \frac{g^3 E}{24r^3}\int_0^{2\pi}\left(\frac{d^3 W_t}{dB^3} + \frac{dW_t}{dB}\right)^2 dB.$$

So, with this we have the expression of all the work:

$$W = W_1 + W_2 = \frac{rgE}{2R^2}\int_0^{2\pi}\left(\frac{r\Delta\theta}{\theta} + W_t \cos B - \frac{dW_t}{dB}\sin B\right)^2 dB + \frac{g^3 E}{24r^3}\int_0^{2\pi}\left(\frac{d^3 W_t}{dB^3} + \frac{dW_t}{dB}\right) dB.$$

or

$$W = \frac{rgE}{2R^2}\left\{\int_0^{2\pi}\left(\frac{r\Delta\theta}{\theta} + W_t \cos B - \frac{dW_t}{dB}\sin B\right)^2 dB + \frac{g^2 R}{12r^4}\int_0^{2\pi}\left(\frac{d^3 W_t}{dB^3} + \frac{dW_t}{dB}\right)^2 dB\right. \quad \ldots\ldots\ldots (I)$$

In order to determine how (I) might be made a minimum, we have obtained the displacement, w_t, as a function of B. The minimum value of (I) would then give the correct value of the work of distortion; and the function, w_t, which makes the integral a minimum will result in a valid flattening of the cross section.

We would like to solve this problem by an approximation. The approximation should be correct if we substitute into the integral expression for the work of distortion, (I), the approximate expression with undetermined coefficients,

$$w_t = c_1 \sin 2B + c_2 \sin 4B + \ldots\ldots c_n \sin 2nB, \quad (8)$$

and calculate the unkown coefficients, $c_1, c_2, \ldots\ldots c_n$, so that expression (I) will be a minimum. We, therefore, have not used all possible functions of w_t that would yield a compatible solution, but have confined ourselves to such functions as would give us the trigometric series of equation (8). So, we do not actually get the exact value of the displacement, w_t, but, so to say, the best approximation by means of the trigometric functions. In general, it is not necessary to use only trigometric functions; but we may choose the functions that give the easiest solution. The better that one chooses, the faster will be the reduction to a good approximation. Theoretically, it is necessary only to choose functions that develop a solution and satisfy the boundary and symmetric conditions. Practically, however, it is necessary to choose those functions that lead to the easiest calculations. In this case, the above mentioned series expansion is especially useful. Next, it is clear from the statement of the initial conditions that the distortion of the cross section is of the same magnitude in both the x and y directions. Because of this, equation (I) may be calculated easily, and the first approximation from a single term gives a value of sufficient accuracy for practical purposes.

We set as the first approximation

$$w_t = c \sin 2B \tag{9}$$

and place this in equation (I) of the work of distortion and determine the constant c in such a manner that the result,

$$W = \frac{rgE}{2R^2}\left\{\int_0^{2\pi}\left(\frac{r\Delta\theta}{\theta} + c \,\mathrm{Sin}\, 2B \,\mathrm{Cos}\, B - 2c \,\mathrm{Cos}\, 2B \,\mathrm{Sin}\, B\right)^2 dB + \frac{g^2R^2}{12r^4}\, 36c^2 \int_0^{2\pi} \mathrm{Cos}^2 2B \, dB\right\}, \tag{I'}$$

will decrease in value.

We set

$$\sin 2B \cos B = \tfrac{1}{2}(\sin 3B + \sin B),$$

$$\cos 2B \sin B = \tfrac{1}{2}(\sin 3B - \sin B), \text{ and}$$

$$\lambda = \frac{gB}{r^2}$$

and remembering that

$$\int_0^{2\pi} \sin^2 B \, dB = \int_0^{2\pi} \sin^2 2B \, dB = \int_0^{2\pi} \sin^2 3B \, dB = \pi,$$

hence

$$\int_0^{2\pi} \sin B \sin 3B \, dB = 0;$$

so the work becomes

$$W = \frac{\pi r g E}{2R^2}\left\{2r^2\left(\frac{\Delta\theta}{\theta}\right)^2 - 3r\left(\frac{\Delta\theta}{\theta}\right)c + \left(\frac{5}{2} + 3\lambda^2\right)c^2\right\}.$$

W contains, as a single variable, the unknown constant, c, and we get the minimum of W if we set dW/dc = 0. Performing this operation gives

$$-3r\frac{\Delta\theta}{\theta} + (5 + 6\lambda^2)c = 0,$$

from which it follows:

$$c = \frac{3}{5 + 6\lambda^2} r \frac{\Delta\theta}{\theta}.$$

The value for the minimum of W then becomes

$$W_{min} = \frac{E\pi r^3 g}{2R^2}\left(\frac{\Delta\theta}{\theta}\right)^2\left(1 - \frac{9}{10+12\lambda^2}\right).$$

Or if we replace $\pi r^3 g$ for the moment of inertia, I, we have

$$W_{min} = \frac{EI}{2R^2}\left(\frac{\Delta\theta}{\theta}\right)^2\left(1 - \frac{9}{10+12\lambda^2}\right).$$

Otherwise, in most cases, the work of distortion has the value

$$W_{min} = \frac{1}{2} M_b \frac{\Delta\theta}{R\theta}.$$

from which the bending moment may be seen to be

$$M_b = \frac{2 W_{min} R\theta}{\Delta\theta}$$

Therefore, we get as a first approximation, the relation between the bending moment, M_b, and the change in the angle, $\Delta\theta/\theta$,

$$M_b = EI\frac{\Delta\theta}{R\theta}\left(1 - \frac{9}{10+12\lambda^2}\right), \qquad (II)$$

instead of, by the usual bending theory,

$$M_b = EI\frac{\Delta\theta}{R\theta}.$$

The two equations differ from each other by the correction factor,

$$K_s = \left(1 - \frac{9}{10 + 12\lambda^2}\right).$$

This depends, as we can see, only upon the factor,

$$\lambda = \frac{gR}{r^2},$$

that is to say, upon the ratio of the wall thickness times the radius of curvature to the geometric average radius of the cross section.

We get the second approximation from the relation,

$$w_t = c_1 \sin 2B + c_2 \sin 4B.$$

As in the first approximation of relation (I) for the work of distortion, we determine the value of the constants c_1 and c_2 so that W will become a minimum. The minimum value of W is found to be

$$W_{min} = \frac{EI}{2R^2}\left(\frac{\Delta\theta}{\theta}\right)^2\left(1 + \frac{102 + 3600\lambda^2}{105 + 4136\lambda^2 + 4800\lambda^4}\right),$$

so that the bending moment for the expression becomes

$$M_b = \frac{EI(\Delta\theta)}{R\theta}\left(1 - \frac{102 + 3600\lambda^2}{105 + 4136\lambda^2 + 4800\lambda^4}\right).$$

These equations differ from the equation of the usual bending theory again by a correction factor:

$$K_s = \left(1 - \frac{102 + 3600\lambda^2}{105 + 4136\lambda^2 + 4800\lambda^4}\right)$$
$$= \left(\frac{3 + 536\lambda^2 + 4800\lambda^4}{105 + 4136\lambda^2 + 4800\lambda^2}\right).$$

If we replace the expression for the correction by its symbol, K_s, the form of the equation remains

$$M_b = K_s \frac{EI\Delta\theta}{R\theta};$$

but we now have a somewhat more accurate value of the correction factor, K_s. However, the following table and the corresponding Figure 6 show that the first approximation has already obtained sufficient accuracy for most practical cases of importance. The table contains values of the factor, K_s, for the first three approximations. A noticeable difference between the first and second approximations occurs only for small values of the factor, $\lambda = gR/r^2$; approximately when λ is less than 0.3. For $\lambda > 0.3$ the difference is less than .01. The second and third approximations differ mostly in the fourth or fifth places, and only when λ is less than .02 in the third place. Thus, the approximation coverges well even for very small values of λ. The convergence is somewhat slower for $\lambda = 0$. The first three approximations run about 0.1, 0.029 and 0.012, while the real value is $K_s = 0$. Meanwhile, we find for practical spring compensator pipe cases, and for other thin-walled pipe cases, that λ is greater than 0.3; so that we can, for the most part, obtain a satisfactory correction from the first approximation.

We can, therefore, sum up the results of the preceding investigation by stating that for originally bent pipes of the usual bending equation,

$$M_b = \frac{EI\Delta\theta}{R\theta},$$

the moment of inertia, I, is replaced by a corrected value, $K_s I$. K_s may be approximated by

$$K_s = 1 - \frac{9}{10 + 12\left(\frac{gR}{r^2}\right)^2}.$$

Instead of the change of the angle, $\Delta\theta/\theta$, the change in the curvature may be used so that the equation is

$$M_b = EI\left(\frac{1}{R'} - \frac{1}{R}\right)\left(1 - \frac{9}{10 + 12\left(\frac{gR}{r^2}\right)^2}\right), \qquad (III)$$

instead of

$$M_b = EI\left(\frac{1}{R'} - \frac{1}{R}\right)$$

as found by the usual bending theory.

The predicted change due to the flattening of the cross section is very accurate if $\lambda = gR/r^2$ is very small, that is, when the wall thickness compared to the internal diameter is very small and at the same time, however, the radius of curvature compared to the internal diameter is not too large. We shall see how this case behaves as a wrought iron spring compensator pipe.

Consideration of a Spring Compensator Pipe

By considering the research of Mr. Bantlin on compensator pipes, we must keep in mind that the pipe has two parts with distinguishably different radii of curvature. We should pause for a moment to reconnoiter and use the value of the correction factor that corresponds to the larger radius of curvature (so that a correction that is still too small occurs). Doing this gives

$$R = 831.0 \text{ cm},\ r = 10.43 \text{ cm},\ g = 0.665 \text{ cm},$$
$$\lambda = 0.507, \text{ and } K_s = 0.312.$$

For the given force our calculations already give, for the first approximation, 1/0.312 = <u>3.2 times the distortion of the usual bending theory</u>. That is to say, this has already accounted for a considerable part of the difference between measurement and theory.

For a closer agreement with the measurement, we should use, in our calculations, different correction factors for both parts of the compensator pipe. We obtain:

$$\text{For } R = 831.0 \text{ cm},\quad \lambda = 0.507 \text{ and } K_s = 0.312;$$
$$\text{For } R = 558.5 \text{ cm},\quad \lambda = 0.340 \text{ and } K_s = 0.212.$$

We have, consequently, divided the translation, Δx, of the point F, Fig. 7, into two parts and inserted, for the part corresponding to the bending of arc, EG, 0.312 I, and for the part corresponding to the bending of arc, GJ, 2.10 I, instead of I, or multiplying the values of displacement as found by the usual bending theory by 1/0.213 = 3.2 and 1/2.10 = 4.75, respectively and then adding the results. We shall now turn from these quite involved formulas to the graphic method of von Bauman.[8] For a load of 300 kg one may obtain

for the first part . . . . $\Delta x_1 = 0.397$ cm,

for the second part . . . . $\Delta x_2 = 0.033$ cm.

The final displacement will therefore be:

by the bending theory . . . . . . $\Delta x_1 + \Delta x_2 = 0.43$ cm,

by our formula . . . . $3.2\Delta x_1 + 4.75\Delta x_2 = 1.43$ cm.

resulting from observation . . . . $= 1.72$ cm.

The ratio, $\frac{\text{Measurement}}{\text{Calculation}}$, is found from this to be

for the usual bending theory 4.00,

for our formula . . . . 1.20.

It may be seen that a discrepancy of about 300 per cent for the distortion of the cross section has become only 20 per cent.

It is similar with a compensator pipe which is 125 mm in inside diameter. Its dimensions are given as

$r = 6.55$ cm, and $g = 0.419$ cm;

for $R_1 = 802.4$ cm,

---

[8]See for example, 1910 issue of this periodical page, 1675. A simple example of the Bauman method is obtained: if the work is taken first as the product of the force P and the distance Δx, and second as the true value of the work of distortion. This gives a relation between the two as $\frac{1}{2}P\Delta x = \frac{1}{2}\int(P^2b^2/EI)ds$ and from this $\Delta x = P\int(b^2/EI)ds$. If we replace I with K_sI, corresponding to the actual radius of curvature, our result can be generalized for any variable radius of curvature.

$$\lambda_1 = 0.811 \text{ and } K_{s1} = 0.497,$$

for $R_2 = 0.440$ and $K_{s2} = 0.269$.

The resulting displacement of the point of application of the force will be for a load of 100 kg:

by the bending theory	0.691 cm,
by our formula	1.44 cm,
obtained from observation	1.75 cm.

The ratio, $\frac{\text{Measurement}}{\text{Calculation}}$, from this is,

for the bending theory	2.54,
for our formula	1.21.

As we can see, the measured value of the distortion is still about 20 per cent greater than that which was calculated by the correction theory. One cannot expect a very close agreement because the unmachined pipe that was purchased had an irregular form and an unequal wall thickness. Furthermore, no doubt, springing on the internal wall causes a roll of discernable waves and depressions, which was already known by Mr. Bantlin. Of course, I cannot altogether agree with Mr. Bantlin that the large difference between measurement and theory is due to the heretofore mentioned springing. From the above divided calculations, it appears to me that, certainly, the results come from the known characteristics of the flattening of the cross section. One can determine an approximate estimate of the influence of these waves, if one chooses the seeming coefficient of elasticity for calculations on a pipe with a corregated wall. If the half amplitude of the wave is indicated by d, the wall thickness again by g, and the coefficient of elasticity by E; we have for the apparent coefficient of elasticity, the formula

$$E' = \frac{E}{1 - 6(d/g)^2}.$$

We shall use as our example a spring compensator pipe with an inside diameter of 200 mm and a thickness, g = 6.65 mm and by hindsight note that the deepest corregation, d = 1 mm. From this we work out that the decrease of the coefficient of elasticity due to the waves is about 13 per cent. The influence of the depressions is, therefore, not large enough to account for the difference of 300 per cent between the usual bending theory and the measurement.

The Stress Distribution

I will likewise remark that one can only hope for a first approximation for the bending stresses in a bent pipe, since we have permitted the displacement of the neutral axis of the cross section toward the center of curvature. This, like the case of solid rods, is a very small influence in the final deformation, but on the other hand, it may cause a considerable error in the assertainment of the stress distribution.

We have, for the strain of a longitudinal fiber, according to equation (7)

$$e_1 = \frac{1}{R}\left(\frac{\Delta\theta}{\theta} r \sin B + w_t \cos B - \frac{dw_t}{dB} \sin B\right),$$

and for w_t, in the first approximation, we have placed

$$w_t = c \sin 2B.$$

We found by the method of the least value of the work of distortion that for the constant c,

$$c = \frac{3}{5+6\lambda^2} r \frac{\Delta\theta}{\theta}.$$

If we place this value of c in e_1, we obtain

$$e_1 = \frac{\Delta\theta}{\theta} \frac{r}{R}\left(\sin B - \frac{6}{5+6\lambda^2} \sin^3 B\right).$$

By using the relation of the corrdinate $y = r \sin B$ we obtain from the previous equation

$$e_1 = \frac{\Delta\theta}{\theta} \frac{y}{R}\left(1 + \frac{6}{5+6\lambda^2}\left(\frac{y}{r}\right)^2\right)$$

instead of

$$e_1 = \frac{\Delta\theta}{\theta} \frac{y}{R}$$

by the usual bending theory.

In Fig. 8 the stress distribution for three pipe cross sections is shown for λ = 0.2, 0.5, 1.0 and for one full circular cross section $\lambda = \infty$; and also shown is the corresponding change in the radius of curvature.[9] It may be seen, for example, that for $\lambda = 0.5$ (for the spring compensator pipe of 200 mm inside diameter), the stress in the outer fiber is only about 1/13 of the stress as calculated on the basis of the usual bending theory. Furthermore, the highest stress, that is calculated from the bending moment, is higher than the real stress occurring in the outer fiber. The highest stress does not usually occur in the highest fiber, Fig. 9; and for a small value of λ (occurring with very thin pipe walls) the stress can be considerably larger than that of the outer fiber.

Discussion of the Theory of the Borden-Manometer

Professor H. Lorenz in the "Zeitschrift des Vereines Ingenieure" in 1910 on page 1865 and the following pages has shown a relation between the pressure and the distortion by means of the well known elastic properties of pipes as used in the Borden Manometer. He uses for the usual relation

$$M_b = EI\left(\frac{1}{R'} - \frac{1}{R}\right);$$

where M_b is the bending moment,

[9]The second approximation should also be kept in mind for the curve $\lambda = 0.2$.

I is the moment of inertia of the cross section,

R, R', is the radius of curvature before and after bending.

Since he used here a pipe with an elliptical cross section, we cannot turn directly to our formula. We raise, however, the first assumption that instead of the radius of the cross section of the circle, we may use the mean of the large and small half axes in the formula:

$$M_b = EI\left(\frac{1}{R'} - \frac{1}{R}\right)\left(1 - \frac{9}{10 + 12(gR/r^2)^2}\right).$$

For the example given by Professor Lorenz we will get for

$R = 6.0$ cm, $g = 0.02$ cm, $a = 1.0$ cm, $b = 0.3$ cm;

$r_m = (a + b)/2 = 0.65$ cm, $\lambda = (Rg/r_m^2) = 0.285$ and

$K_s = 0.18$.

We could substitute 0.18 I for the moment of inertia, I. It is now easy to see that the resistance against the transverse bending is somewhat smaller with the flattened ellipse than with the circle, so that the difference between the published bending theory and that obtained by a flattened ellipse might be larger than that obtain by a corresponding circular cross section. Instead of the value of the distortion as calculated by Professor Lorenz we obtained a value about five times as much, although, naturally, the relation, worked out by Professor Lorenz between the pressure and the bending moment and ex-pecially the proof of the proportionality between pressure and distortion, remains unchanged.

Summary

(1) In general, the bending theory gives much too small a value for the distortion, when thin-walled pipes with originally curved centerlines are considered; since in following the flattening of the cross section the actual strain is significantly smaller than the value obtained by the unusual bending theory. The real relation is obtained between the bending moment and the distortion, if in the equation

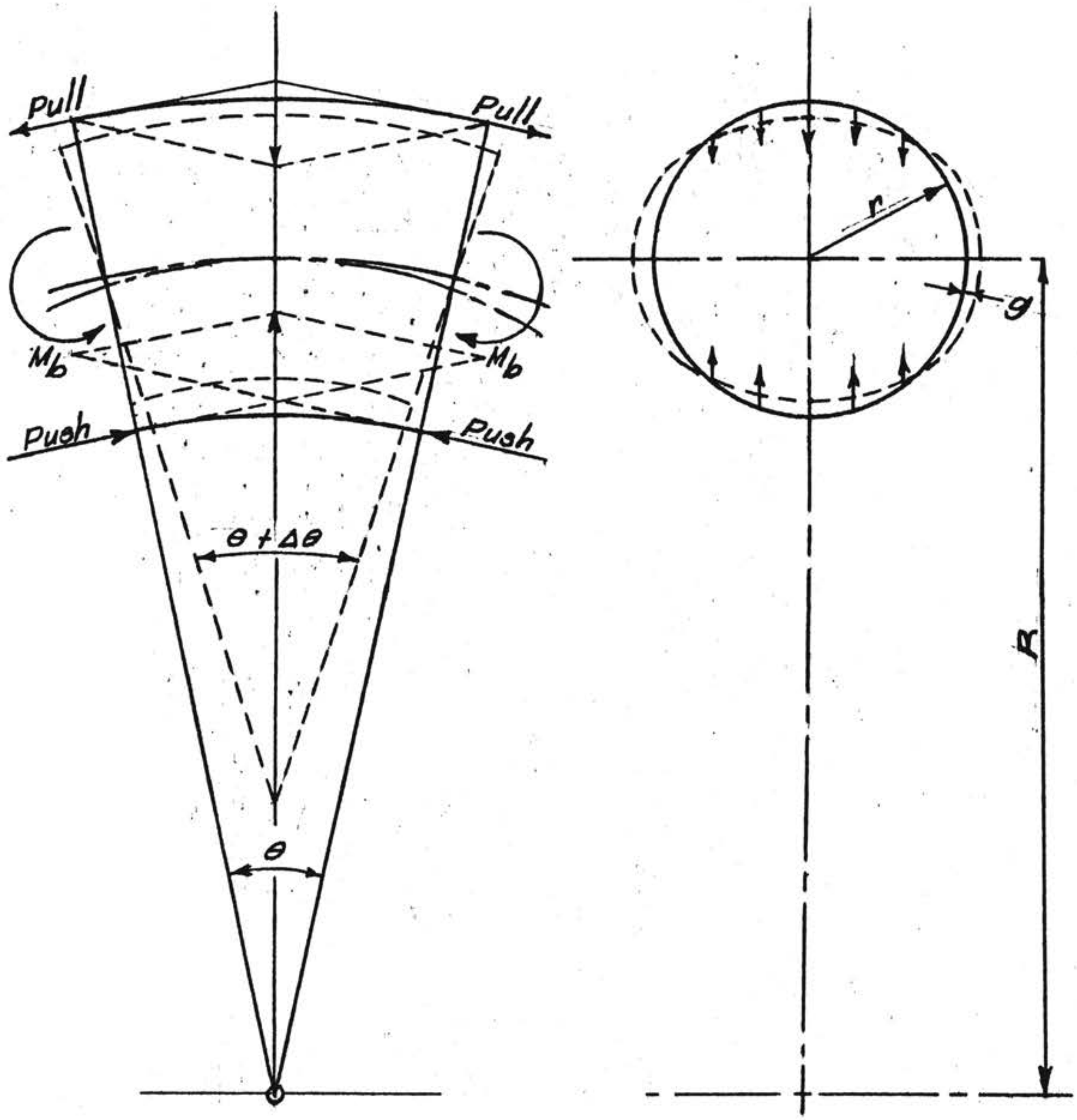
$$M_b = EI\left(\frac{1}{R'} - \frac{1}{R}\right),$$

the moment of inertia, I is replaced by the corrected value, $K_s I$; since K_s can give a good approximation (if gR/r^2 is greater than 0.3):

$$K_s = 1 - \frac{9}{10 + 12(gR/r^2)}.$$

(2) The difference between calculation and measurement is decreased from about 300 per cent to about 20 per cent by consideration of the relations concerning spring compensator pipes made of wrought iron.

(3) Also, by the relations of pipe springs as developed for the Borden-Manometer, the difference between the bending formulas is used to prove that the neglected effect of the flattening of the cross section, as practiced in the usual bending theory, can lead to large errors.



Figures 1 and 2. The Bending of Thin-Walled Pipes.

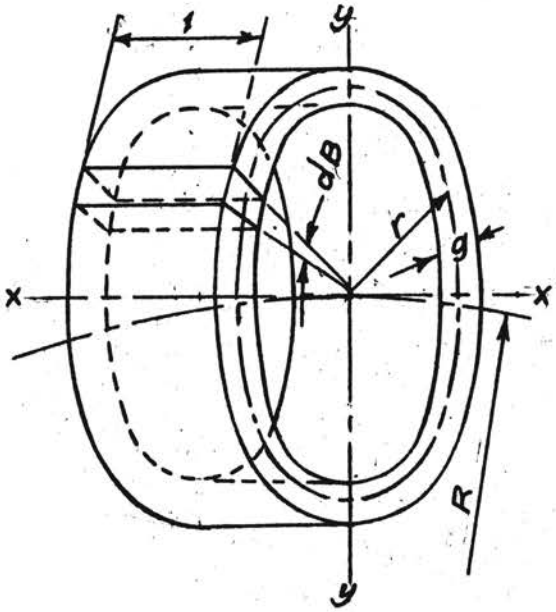


Fig. 3. The Pipe Cross Section.

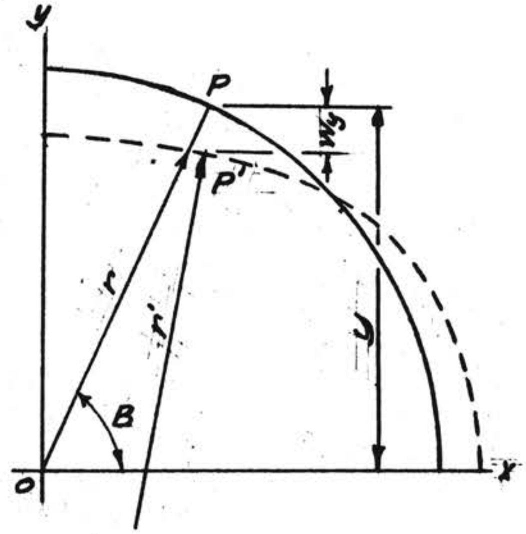


Fig. 4. The Distortion of the Cross Section

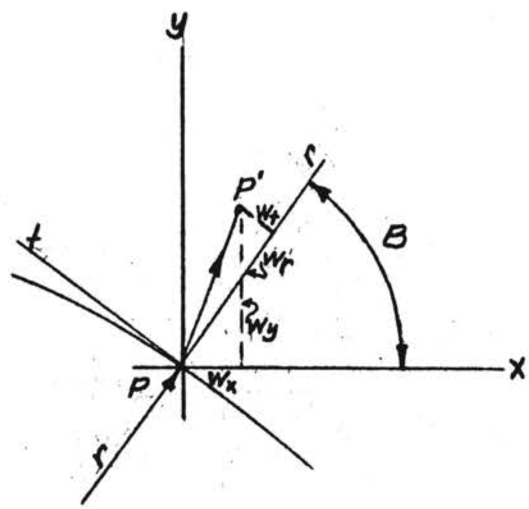


Fig. 5. The Coordinates of the Displacement.

TABLE I

A TABLE OF VALUES FOR THE CORRECTION FACTOR n

$\lambda = \frac{gR}{r^2}$	0	0.1	0.2	0.3	0.4	0.5	1.0	1.5	2.0	3.0
First Approx. n_1	.100	.111	.141	.188	.245	.308	.5910	.757	.845	.926
Second Approx. n_2	.029	.060	.115	.175	.238	.304	.5905	.757	.845	.926
Third Approx. n_3	.012	.058	.115	.175	.238	.304	.5905	.757	.845	.926

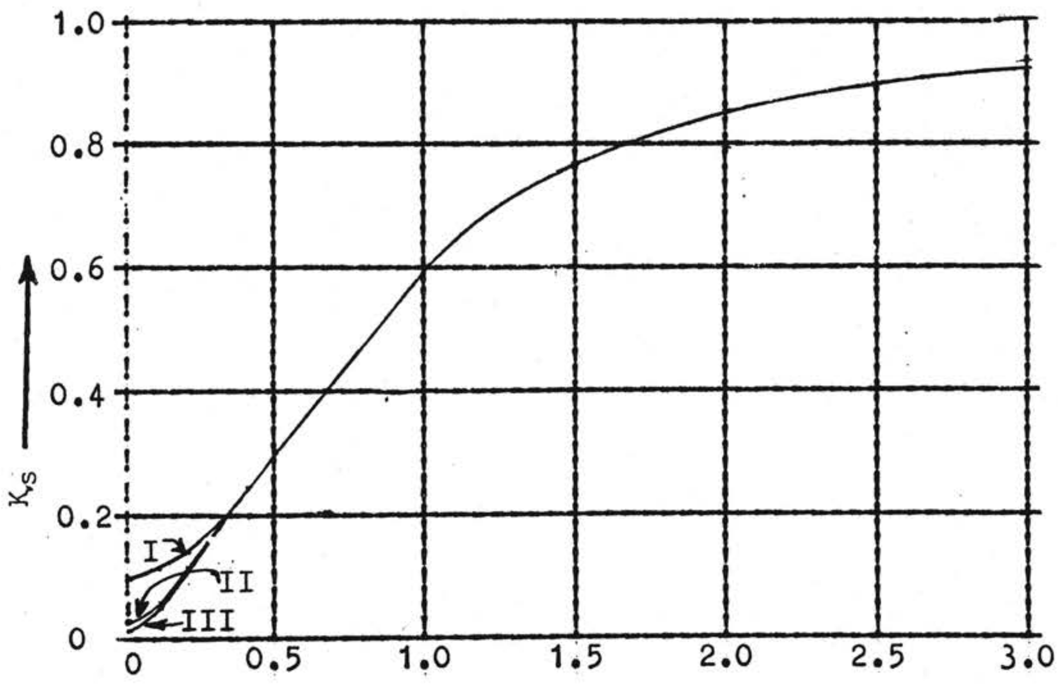


Fig. 6. The Correction Factor, K_s, as a function of the Proportional Number, $\lambda = gR/r^2$. I, II, and III are the First, Second and Third Approximations respectively.

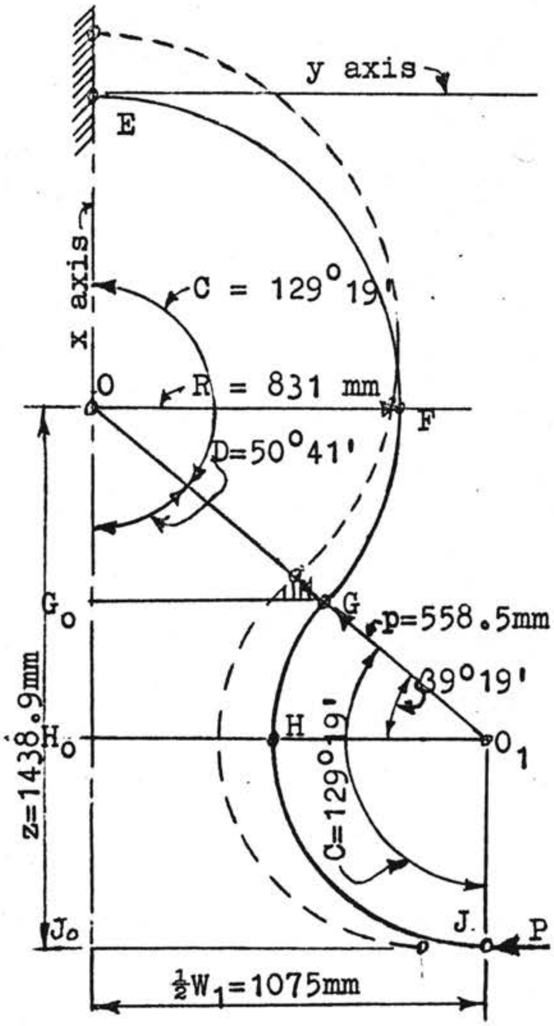


Fig. 7. The distortion of a pipe used as an elastic compensator [or an elastic coupling].

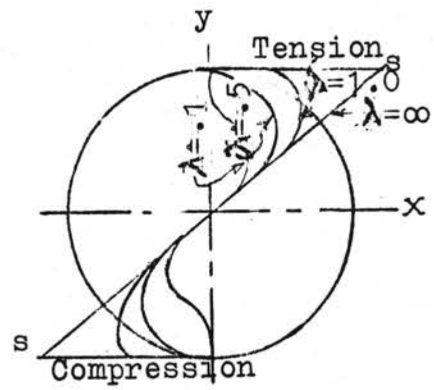


Fig. 8. Stress Distribution in the cross section of a pipe for different values of the ratio $\lambda = gR/r^2$.

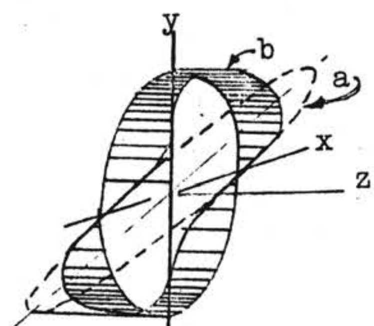


Fig. 9. Stress Distribution in the cross section of a thin-walled pipe ($gR/r^2 = 0.5$)

(a) by the usual bending theory.

(b) with consideration of the flattening.

APPENDIX B

VON KARMAN'S CORRECTION

The following statement quoted from Timoshenko[1] indicates that a correction might be applied to the area moment of inertia of thin-walled cylinders:

> The fibers of the tube fartherest from the neutral axis do not take the share of the stresses which the ordinary theory of bending indicates. This effects the bending in the same way as a decrease in its moment of inertia.

As stated in the introduction, work on the bending of thin-walled cylinders subsequent to von Karman's work have either verified that work or approximated it. The potential energy of strain as given by von Karman is[2]

$$V = \frac{EI}{2R^2}\left(\frac{\Delta\theta}{\theta}\right)^2\left(1 - \frac{9}{10+12\lambda^2}\right). \qquad (1)$$

V is defined as the potential energy of strain; I is defined as the moment of inertia of the cross section of the circular cylinder (i.e. a pipe bend radius of curvature); θ is the central angle of curvature of the bend as found by the usual

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[1]Timoshenko, Strength of Materials, I (New York, 1955), p. 455.

[2] von Karman, p. 312.

linear binding theory; $\Delta\theta$ is the increase of angle of curvature due to the distortion of the cross section into an ellipse; and λ^2 is given by

$$\lambda^2 = \left(\frac{gR}{r^2}\right)^2. \qquad (2)$$

g is the thickness of the thin-walled cylinder, and r is the radius of the thin-walled cylinder prior to bending. If L is the length of an arc on the center line of the thin-walled cylinder, then it follows that

$$L = R\theta. \qquad (3)$$

The value, L, is conveniently used as the distance between mass points in a typical linear analysis. $\Delta\theta$ is easily found as an output on most linear analysis programs. To obtain the potential energy of strain in terms of $\Delta\theta$ and L, (2) and (3) may be substituted into (1) to obtain

$$V = \frac{EI}{2L^2}(\Delta\theta)^2\left[1 - \frac{9}{10 + 12\left(\frac{gL}{\theta r^2}\right)^2}\right] \qquad (4)$$

Von Karman then gives the bending moment acting on a thin-walled cylinder as,[3]

$$M_b = EI\left(\frac{\Delta\theta}{R\theta}\right)\left(1 - \frac{9}{10 + 12\lambda^2}\right). \qquad (5)$$

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[3] Ibid. p. 312.

Then if (2) and (3) are substituted into (5) the moment is

$$M_b = EI\left(\frac{\Delta\theta}{L}\right)\left[1 - \frac{9}{10 + 12\left(\frac{gL}{\theta r^2}\right)^2}\right]. \qquad (6)$$

In the case of a rocket there is no initial curvature. Therefore, $\theta_o = 0$. At any time θ is the same as $\Delta\theta$ in the preceding equations. Therefore, equation (4) for the potential energy of strain of a rocket, at any time, t, and for any length, L, is

$$V = \frac{EI}{2L}(\Delta\theta)^2\left[1 - \frac{9}{10+12\left(\frac{gL}{\Delta\theta r^2}\right)^2}\right]. \qquad (7)$$

Likewise, the moment causing a given angular distortion for a length, L, along a rocket is, at any time, t,

$$M_b = EI\left(\frac{\Delta\theta}{L}\right)\left[1 - \frac{9}{10 + 12\left(\frac{gL}{\Delta\theta r^2}\right)^2}\right]. \qquad (8)$$

The preceding equations were derived for the convenience of the reader. It seems possible that other more direct calculations might, eventually, use these equations instead of the correction factor approach. The term on the right,

$$K_s = 1 - \frac{9}{10 + 12\left(\frac{gL}{r^2\Delta\theta}\right)^2},$$

is termed the von Karman correction factor. The von Karman factor, K_s, was calculated for two cases indicated, by table I. It was found that by plotting these values, as shown on Fig. 1, that the graph could be used instead of further calculations. Thus, the reader may use this graph for values of von Karman's factor instead of calculating them himself. Since W is a known property in the correction factor of equations (7) and (8), it follows that this factor may be found if $(\theta_i - \theta_{i-1}) = \Delta\theta$ is known for a given value of L. See Fig. 3 of Chapter III.

TABLE I

TABULATION OF THE CORRECTION FACTOR

$$K_s = 1 - 9/\left[10 + 12(gL/r^2\Delta\theta)^2\right]$$

		Mode 1, Case 19			Mode 2, Case 24		
Point No.	tL/r^2	$\Delta\theta$ $(\theta_i - \theta_{i-1})$	$tL/r^2\Delta\theta$	K	$\Delta\theta$	$tL/r^2\Delta\theta$	K
1	.021230	+.00003896	547.00000	.9999	+.00064249	33.10000	.9993
2	.018260	+.00022633	80.60000	.9999	+.00271901	6.72000	.9837
3	.018260	+.00013888	12.13000	.9949	+.00187394	.90030	.5439
4	.000957	+.00016808	5.70000	.9774	+.00200892	.47680	.2926
5	.003260	+.00027035	12.08000	.9948	+.00291056	1.11870	.6403
6	.003920	+.00028943	13.54000	.9957	+.00278806	1.40500	.7331
7	.005810	+.00041172	14.11000	.9963	+.00240203	2.41868	.8878
8	.005810	+.00060290	9.65000	.9920	+.00166887	3.48200	.9421
9	.008370	+.00073160	11.42000	.9938	+.00040458	20.70000	.9983
10	.002930	+.00014929	15.10000	.9967	-.00030155	9.72400	.9921
11	.001318	+.00028826	4.56500	.9654	-.00083359	1.58000	.7750
12	.001630	+.00067285	2.42500	.8886	-.00268748	.60700	.3759
13	.004520	+.00035143	12.86000	.9955	-.000127735	3.53800	.9438
14	.003120	+.00025139	12.43000	.9952	-.00102322	3.05000	.9260
15	.000578	+.00007361	7.85000	.9880	-.00032836	1.76100	.8094
16	.003692	+.00019462	18.96000	.9979	-.00099167	3.72000	.9490
17	.003520	+.00000613	53.20000	.9997	-.00049920	7.06000	.9852
18	.000946	+.00000223	424.00000	.9999	-.00005629	16.82000	.9974
19	.001756	+.00000044	3994.00000	.9999	-.00001243	141.30000	.9999

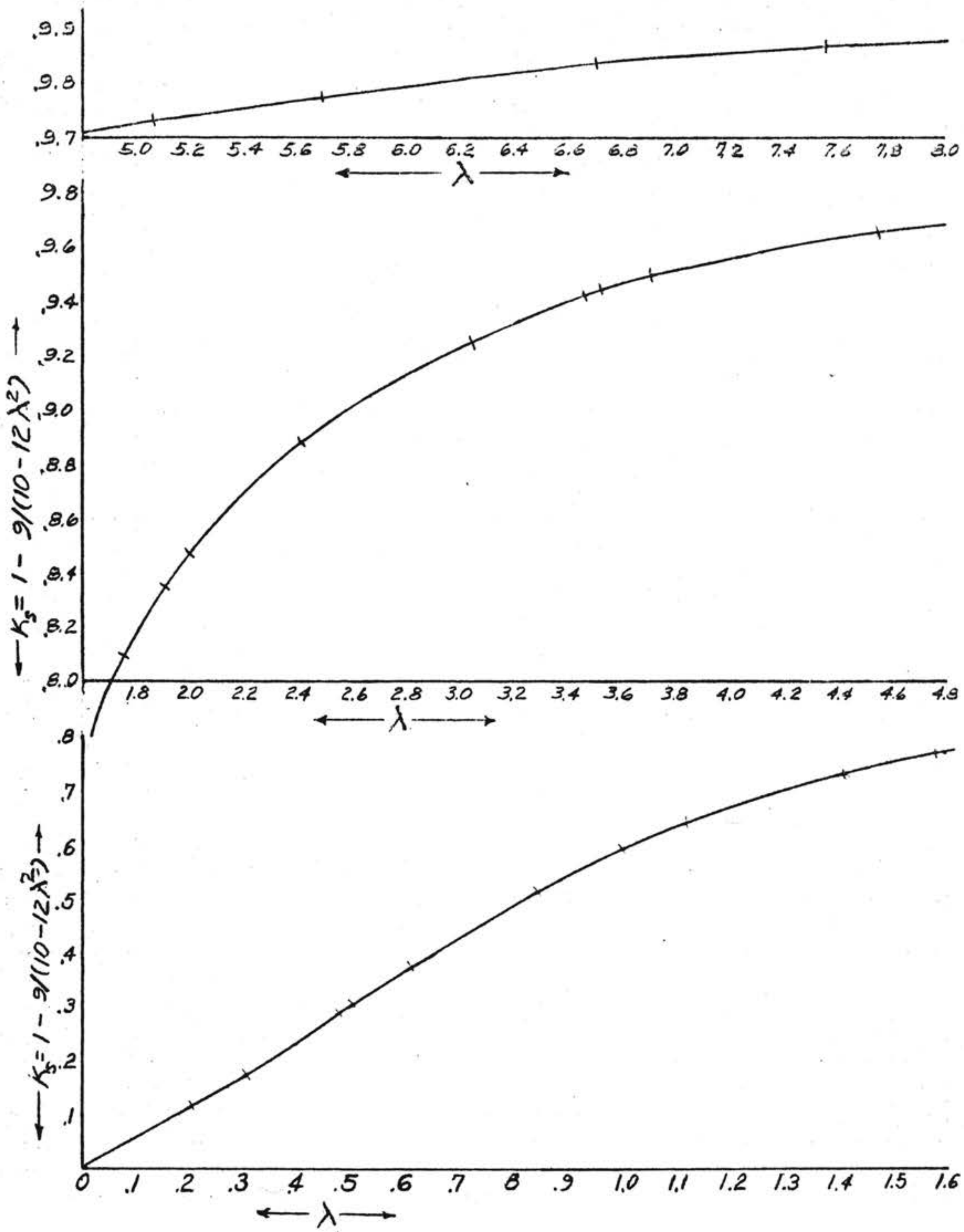

Fig. 1 - Graph of the Correction, K_S

VITA

WILLIAM JOSEPH SLAGLE

Candidate for the Degree of

Master of Science

Thesis: THE DISTORTION OF THE CROSS SECTION OF THIN-WALLED CYLINDERS DUE TO LATERAL VIBRATIONS.

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Undergraduate Study: Electronics Technician, U. S. Navy 1943 and 1944; received the Bachelor of Science degree in Mechanical Engineering from the University of Missouri in August, 1951.

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