USE OF THE DIGITAL COMPUTER WITH THE

PHASE-PLANE DELTA METHOD

By

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PREFACE

The phase-plane delta method is a graphical procedure useful in the solution of nonlinear and time varying linear second-order differential equations. This paper demonstrates how implementation of a digital computer can in many cases simplify or improve the solutions of problems obtained by the phase-plane delta method.

Appreciation is expressed to Dr. R. L. Lowery for the suggestion and supervision of this study, and to Associate Professor Edwin J. Waller and Robert K. Reed for their assistance.

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CHAPTER I

INTRODUCTION

The phase-plane delta method formulated by L. S. Jacobsen is a graphical procedure useful in the solution of nonlinear second-order differential equations.

Robert Buland (1) gives a summary of the usefulness of the method.

>   1. The method offers a powerful means of solving nonlinear second order differential equations and time varying linear differential equations of second order.

>   2. Most servo systems can be expressed as a series of second order expressions. The method can then be used to solve for the transient response since any arbitrary forcing function is acceptable for solution by this method. Therefore the output of one phase-plane plot can be used as the input to the next phase-plane plot until the loop is closed.

>   3. Nonlinearities and forcing functions are introduced graphically, in many cases, directly from experimental data, thus alleviating the necessity for finding an analytical expression for them.

>   4. The method may be mechanized to solve equations automatically on an analog computer. This might prove advantageous in solving complicated nonlinear problems.

Jacobsen (3) states: "The delta method is extremely simple to apply, even to complicated equations. A subprofessional assistant can be instructed quickly in how to use the method, and ordinarily will develop enough speed to become useful in a short time.

Buland's suggestion of using an analog computer in solving complicated problems points out a weakness in the phase-plane delta method.

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The weakness is that the method can become very tedious or difficult if the delta function becomes too involved. The solution does not become more complicated theoretically, but the graphical difficulties are multiplied. This indicates that if the graphical procedures can be performed numerically, a solution using a digital computer may be useful.

Jacobsen's suggestion of using a subprofessional assistant also indicates that a digital computer can be useful since the digital computer is ideally suited to perform tedious repetive processes such as those called for in the construction of phase-plane plots.

A personal search disclosed no successful usage of the digital computer with the phase-plane delta method. One previous attempt was evidently unsuccessful because of difficulties encountered in crossing axes; this difficulty probably arose from some limitation of the particular computer used.

This paper will present a successful implementation of the digital computer with the phase-plane delta method. The difficulty of crossing axes was overcome easily using a modern computer. Modification of the phase-plane delta method for use with a digital computer consists of converting the method from a graphical procedure into a numerical one. In cases in which solution by hand is particularly difficult, the computer solution can improve the accuracy of the solution.

CHAPTER II

BASIC THEORY OF THE PHASE-PLANE DELTA METHOD

The phase-plane delta method is a graphical procedure useful in the solution of the standard second-order differential equation:

$$\ddot{x} + f(x, \dot{x}, t) = 0. \tag{1}$$

The following development is taken from Jacobsen's paper (1) and his book (3). The first step in the procedure consists of rewriting equation (1) in the standard delta form.

$$\ddot{x} + p^2 (x + \delta) = 0 \tag{2}$$

This is accomplished by adding and subtracting $p^2 x$ from equation (1).

$$\ddot{x} + [f(x, \dot{x}, t) - p^2 x] + p^2 x = 0 \tag{3}$$

then

$$\delta = \frac{1}{p^2} [f(x, \dot{x}, t) - p^2 x] \tag{4}$$

In this paper the variables x, $\dot{x}$, and t will represent displacement, velocity, and time.

The phase-plane coordinates are defined as:

$$x = x \quad \text{and} \quad \dot{x}/p = v. \tag{5}$$

The following relationships can be developed from the v coordinate.

$$\ddot{x} = p \, dv/dt = p \, (dv/dx) \, dx/dt = p^2 v \, dv/dx \tag{6}$$

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Substituting for $\ddot{x}$ in equation (2) yields a first order differential e-quation.

$$v \, dv/dx + x + \delta = 0 \tag{7}$$

This equation leads to the geometrical relation

$$dx/dv = \frac{-v}{x + \delta} \tag{8}$$

If δ is held constant over a finite time interval, Δt, equation (8) can be integrated; the result being

$$(x + \delta)^2 + v^2 = c^2 . \tag{9}$$

This is the equation of a circle, and the instantaneous phase trajectory is a circular arc passing through the point (x, v) with its center at -δ on the x axis as shown in Figure 1.
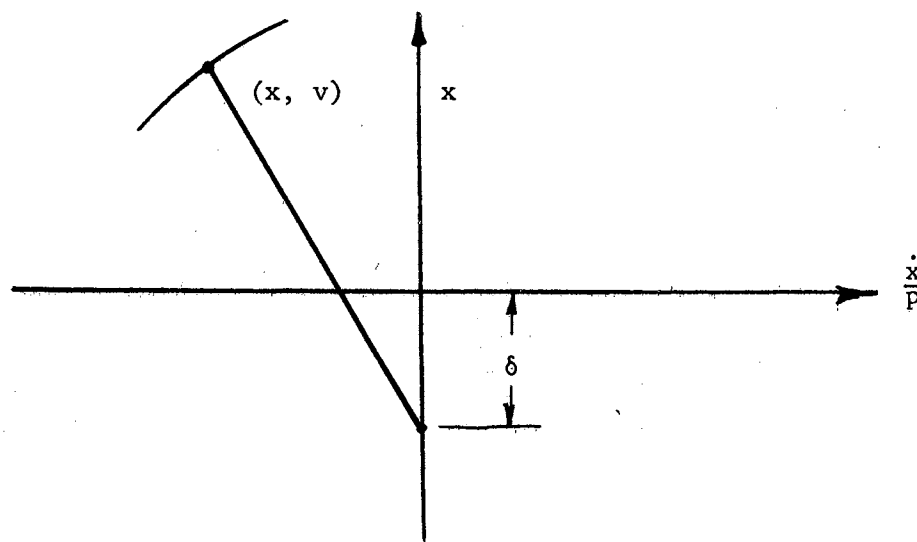

Figure 1. Construction of Circular Arcs in the Phase-Plane.

An explicit expression for time may be obtained from equation (6).

$$dt = dx/pv \tag{10}$$

Referring again to Figure 1, it can be seen that arc length is given by

$$ds = \sqrt{(dx)^2 + (dv)^2}, \tag{11}$$

and that the radius of curvature, R, is given by

$$R = \sqrt{(x + \delta)^2 + v^2}, \tag{12}$$

so that the incremental angle of swing is

$$d\theta = \frac{ds}{R} = \frac{\sqrt{(dx)^2 + (dv)^2}}{\sqrt{(x + \delta)^2 + v^2}} = \frac{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}}{\sqrt{1 + \left(\frac{x + \delta}{v}\right)^2}} \frac{dx}{v}. \tag{13}$$

Since

$$\left(\frac{dx}{dv}\right)^2 = \left(\frac{v}{x + \delta}\right)^2, \tag{14}$$

equation (13) reduces to

$$d\theta = \frac{ds}{R} = \frac{dx}{V}. \tag{15}$$

Substituting $\frac{dx}{v} = d\theta$ into equation (10) yields

$$dt = \frac{1}{p} d\theta, \tag{16}$$

or for finite increments

$$\Delta t = \frac{1}{p} \Delta \theta. \tag{17}$$

Thus angular arcs on the phase-plane are proportional to time.

Proceeding with the graphical integration, it is necessary to plot the δ function or functions. This plot or plots may be superimposed on the phase-plane diagram. However, if the δ function is time dependent it may be necessary to make an auxiliary plot of δ versus time.

Then, given initial conditions, the phase-plane plot can be constructed (see Figure 2). For example, suppose that at time zero the displacement and velocity are represented by point P_1 on Figure 2. In this example δ had been taken as a function of x only.
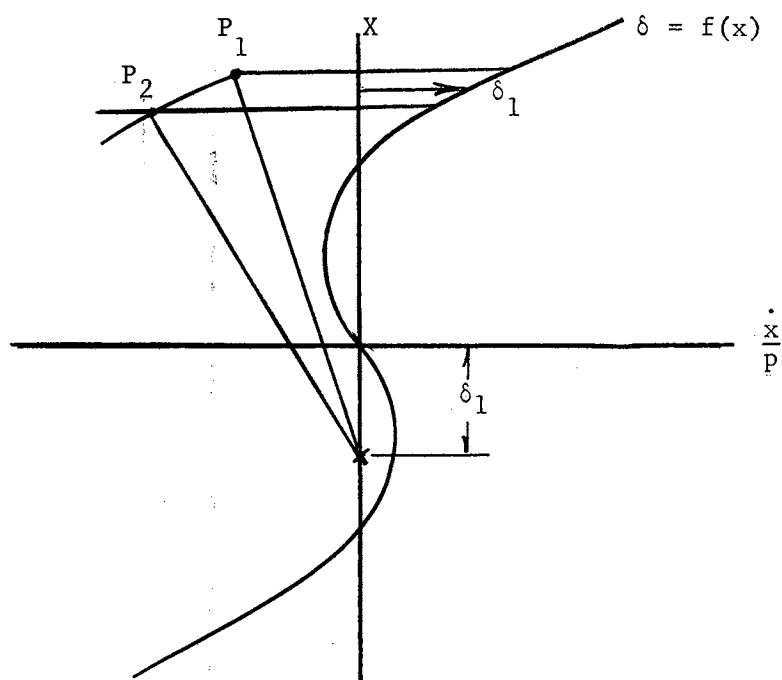
Figure 2. Construction of the Phase-Plane Trajectory.

An average value of δ for a small arc is easily approximated here so that the center on the x axis can be located and the arc constructed. This establishes a new point P_2 which determines new values for x and $\dot{x}/p$. By successive construction of these circular arcs the phase-plane trajectory can be traced out. For a more thorough discussion of this procedure the reader is referred to Jacobsen (2) and (3) or Buland (1).

CHAPTER III

MODIFICATION OF THE DELTA METHOD FOR USE

WITH THE DIGITAL COMPUTER

As explained in Chapter II, the phase-plane delta method is a graphical procedure useful in solving the differential equation

$$\ddot{x} + f(\dot{x}, x, t) = 0. \tag{18}$$

To set up the problem for programming on a digital computer the first step is to determine the δ function just as in the graphical procedure. In general δ will be a function of x, $\dot{x}$, and t. Now consider a general point on the phase-plane diagram. It can be seen that, in general, this point will be a function of x, $\dot{x}/p$, and t (see Figure 3).
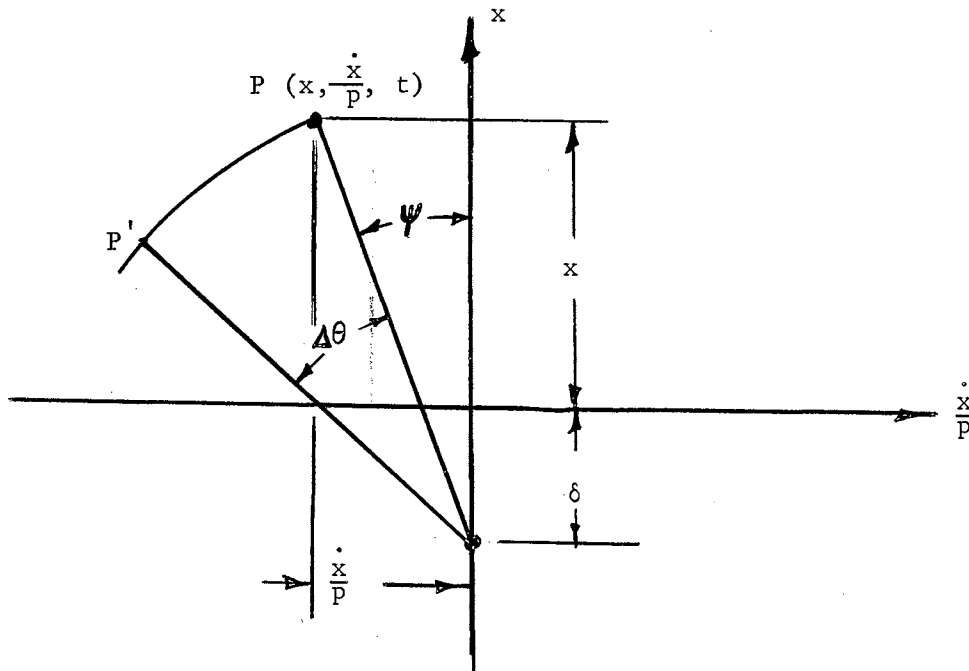

Figure 3. Phase-Plane Construction Used with the Digital Computer.

The δ function can be programmed into the computer so that for any point P, a value of δ may be calculated. As explained in Chapter II the instantaneous phase trajectory is a circular arc passing through the point P, with its center at $-\delta$ on the x axis. When δ has been calculated the center of the arc can be located.

It can be seen from the figure that the radius of the arc may be calculated from the equation

$$R = \sqrt{(x + \delta)^2 + (\dot{x}/p)^2}. \tag{19}$$

Now, if an arc through the point P with the center at $-\delta$ on the x axis is swung through a small angle $\boldsymbol{\Delta}\theta$ a new point P$'$ can be established. P$'$ establishes a new value for each of the variables x, $\dot{x}/p$, and t. These new values are given by the following equations:

$$(x)' = R \cos (\boldsymbol{\psi} + \boldsymbol{\Delta}\theta) - \delta$$

$$\left(\frac{x}{p}\right)' = -R \sin (\boldsymbol{\psi} + \boldsymbol{\Delta}\theta) \tag{20}$$

$$t' = t + \frac{\boldsymbol{\Delta}\theta}{p}$$

where

$$\boldsymbol{\psi} = \text{Arctan} \left(\frac{-v}{x + \delta}\right).$$

If the above equations are programmed into the computer a new set of variables can be obtained. The process can then be repeated until a full set of points for a phase-plane plot has been obtained.

In the graphical solution the angle $\boldsymbol{\Delta}\theta$ must be reasonably large (on the order of five to ten degrees or more) in order to keep the figure from becoming too confused and the labor involved prohibitive. Also, Jacobsen

(3) notes that cumulative graphical errors increase as the number of steps becomes large. At the same time the angle must not be made too large as this invalidates the assumption that δ remains constant for the increment. Jacobsen (3) discusses at length various means of obtaining an average value of δ so that larger angular increments can be taken.

In the digital computer solution it is possible to take much smaller angular increments since there is no plot to become too confused or cluttered with construction lines and no cumulative graphical errors to build up. If very great accuracy is desired in the solution it might be advisable to introduce some sort of iterative procedure for averaging δ over the angular increment $\Delta\theta$. However, in most cases the desired accuracy can be obtained by simply taking smaller angular increments.

In almost any case where phase-plane plots can be made by the graphical procedure, these same plots could be made by the digital computer although in many cases it would not be advantageous to use the computer. One must use a little common sense to determine when the use of the computer is justified.

There are several types of problems for which the advantage of a computer solution can be seen immediately.

Consider equations of the form

$$\ddot{x} + f(x, \dot{x}) = 0. \tag{21}$$

In this case δ will not be a function of time. It is possible to program a general δ function, consisting, for example, of polynomial terms, trigonometric terms and exponential terms for both x, and $\dot{x}/p$. Then by choosing coefficients this general program could be used for solving a great number of problems. This sort of program is illustrated by the first example in

Chapter IV. There are several advantages this type of program might have

over a solution by hand. First, a great number of problems can be solved

with one computer program by simply changing coefficients. If δ is a

function of both x and $\frac{\dot{x}}{p}$, a suitable averaging of δ which is necessary for

the graphical solution may prove to be very difficult or tedious. With

a computer solution no such difficult averaging procedure is necessary.

The problem is no more difficult than one for which δ is a simple func-

tion of x alone. This constitutes a very significant advantage of com-

puter procedures over the graphical one. Also variation of parameters

for one particular problem can be accomplished easily. In some problems it

may be that the parameters are such that the graphical solution becomes

very difficult or awkward whereas the computer solution presents no problem

at all. This last advantage arises from the ability to take very small

angular increments easily when using the computer.

It may be that in solving a particular problem of the form

$$\ddot{x} + f(x, \dot{x}) = 0 \tag{21}$$

the δ function which is needed cannot be obtained from any general program

which is available or there may be no such program available. Still it

may appear that a computer solution will be advantageous. For example,

it might be that it is desired to study the variation of parameters or to

obtain phase-plane plots for a large number of different starting condi-

tions. In this case a computer program can very easily be written by tak-

ing the general program mentioned above and inserting a specialized δ

function in the place of the general one. This is actually not a great

alteration since most of the computing and incrementing sections of the

program will not be changed. The advantages that apply in the previous

case also apply here. This sort of problem is illustrated by example two in Chapter IV.

Consider now the more general case

$$\ddot{x} + f(x, \dot{x}, t) = 0. \tag{22}$$

In this case δ is in general a function of all three variables also. In this type of problem, determination of a suitable average value of δ for an arc may be even more difficult or tedious than the case where δ is as function of x and $\dot{x}$. Again in most cases this presents no problems using the computer since no averaging of δ is necessary. A computer program for this type of problem can be developed in exactly the same manner as in the previous example. That is, it is only necessary to substitute a new δ function into the same general program. Example three in Chapter IV illustrates this type of problem.

Another type of problem which might be handled advantageously by computer procedures is the forced vibration problem. The differential equation to be solved in this case is

$$\ddot{x} + f(x, \dot{x}, t) = \frac{1}{m} F(t). \tag{23}$$

As indicated by Jacobsen (3), this type of problem can be handled quite easily by rearranging the equation.

$$\ddot{x} + p^2 (x + \delta - \frac{1}{k} F(t)) = 0 \tag{24}$$

then

$$\delta_{eff} = \delta - \frac{1}{k} F(t) \tag{25}$$

A problem of this type is solved in example four in Chapter IV.

The output from a computer program will usually be in digital form. In the programs which have been written the output may be either typed,

or punched on cards, or both. The important variables included in the output are x, v, and total angular displacement θ. In most cases angular displacement is related to time by the equation

$$\Delta t = \frac{1}{p} \Delta \theta. \tag{26}$$

The output data from the computer program is numerical and can be analyzed in the numerical form or it can be plotted for easier study or better presentation. If x is plotted against v, a conventional phase-plane plot results. Velocity or displacement versus time plots as well as an example of the digital output are included in Chapter IV. The variable or coordinate which is labeled velocity in the figures presented is actually the phase-plane coordinate, $v = \dot{x}/p$.

Plotting can be done by hand, or some sort of mechanical plotter may be used. Most of the plots included in this paper were made with the auto-plotter system which is an off-line plotter used with the IBM 1620 computer system. This system plots points with a resolution of one hundreth of an inch and also prints labels and scales.

CHAPTER IV

EXAMPLE PROBLEMS

Example Number One.

Consider the solution of the equation

$$\ddot{x} + f(x, \dot{x}/p) = 0, \tag{27}$$

such that the delta function turns out to be the sum of two functions, $f_1(x)$ and $f_2(\dot{x}/p)$. In order to facilitate the solution of a large number of problems of this sort with one computer program it is possible to set up general functions f_1 and f_2 consisting of a sum of polynomial, trigonometric, and exponential terms. A block diagram, outlining in general terms the operation of a computer program for the phase-plane delta method, is given in Figure 4 and an example computer program, written in Fortran is given in Table I.

As an example of the use of this program, consider a system with linear restoration and quadratic velocity damping. The equation of motion for such a system can be written as

$$\ddot{x} + q|\dot{x}|\dot{x} + p^2 x = 0 \tag{28}$$

so that the delta function is

$$\delta = q\left|\frac{\dot{x}}{p}\right|\frac{\dot{x}}{p}. \tag{29}$$

13
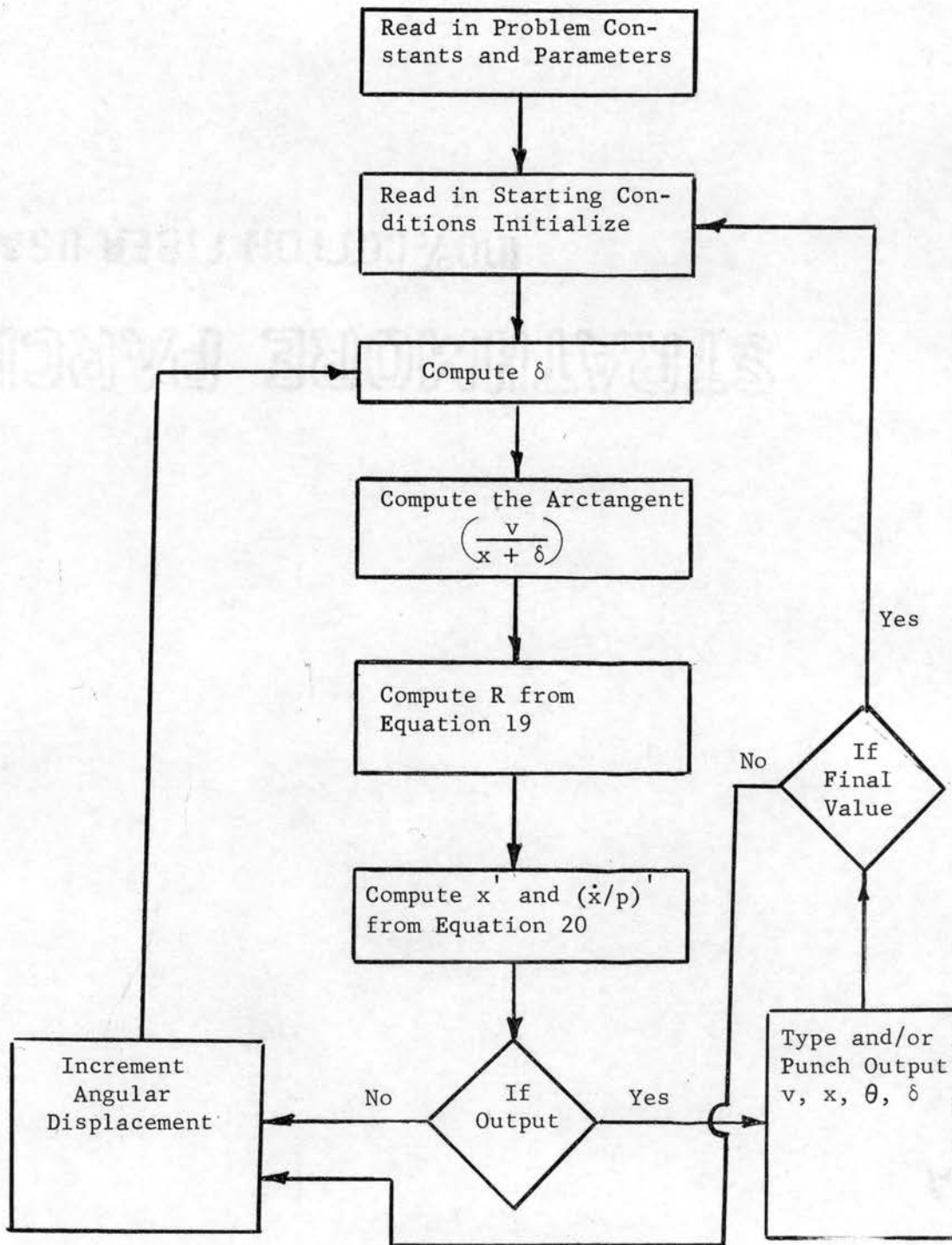
Figure 4. Block Diagram for Phase-Plane Delta Computer Program.

TABLE I

FORTRAN PROGRAM, δ = f (x, ẋ)

```
C    PHASE PLANE DELTA, SIMPSON
     DIMENSION A(61)
3    DO 4 1 = 1,61
4    A(1) = 0.0
5    READ, JJ, ADAM
     IF (JJ -62) 6,7,6
6    I = JJ
     A (1) = ADAM
     GO TO 5
7    CONTINUE
8    READ, XSTAR, VSTAR, DBETA, XSTP, BESTP, BTYPE
     IF (XSTAR - 44444.4) 9, 3, 9
9    BETA = 0.0
     DTHET = DBETA * 0.017453292
     X = XSTAR
     V = VSTAR
     DEL = 0.0
     DL = 0.0
     GO TO 75
10   DELX = 0.0
     DELV = 0.0
     IF (A(53)) 11,12,11
11   DELX = DELX + A(1) + A(2)*X + A(3)*(X**2) + A(4)*(X**3)
     DELX = DELX + A(5)*(X**4) + A(6)*(X**5)
     IF (X) 87,88,89
87   XX = -X
     DELX = DELX - A(7)*(XX**A(8))
     GO TO 12
88   DELX = DELX + 0.0
     GO TO 12
89   DELX = DELX + A(7)*(X((A(8))
12   IF (A(54)) 13, 18, 13
13   IF (X) 14, 16, 15
14   DELX = DELX - A(9)*(X**2) -A(10)*(X**4)
     GO TO 18
15   DELX = DELX + A(9)*(X**2) + A(10)*(X**4)
     GO TO 18
16   DELX = DELX + 0.0
18   IF (A(55)) 19, 20, 19
19   DELX = DELX + A(11)*SIN(A(12)*X + A(13))
     DELX = DELX + A(14)*SIN(A(15)*X + A(16))
     DELX = DELX + A(17)*X*SIN(A(18)*X + A(19))
20   IF (A(56)) 91,22,91
91   IF (X) 92,94,21
92   XX = -X
     DELX = DELX - A(20)*EXP(A(21)*XX) - A(22)*XX*EXP(A(23)*XX)
     DELX = DELX - A(24)*EXP(A(25)*(XX**A(26))) + A(20) + A(24)
     GO TO 22
94   DELX = DELX + 0.0
     GO TO 22
21   DELX = DELX + A(20)*EXP(A(21)*X) + A(22)*X*EXP(A(23)*X)
     DELX = DELX + A(24)*EXP(A(25)*(X**A(26))) - A(20) - A(24)
22   IF (A(57)) 23,24,23
23   DELV = DELV + A(27) + A(28)*V + A(29)*(V**2) + A(30)*(V**3)
     DELV = DELV + A(31)*(V**4) + A(32)*(V**5)
     IF (V) 101,102,103
101  VV = -V
     DELV = DELV - A(51)*(VV**A(52))
     GO TO 24
102  DELV = DELV + 0.0
     GO TO 24
103  DELV = DELV + a(51)*(V**A(52))
24   IF(A(58)) 25,30,25
25   IF (V) 26,28,27
26   DELV = DELV - A(33)*(V**2) - A(34)*(V**4)
     GO TO 30
27   DELV = DELV + A(33)*(V**2) + A(34)*(V**4)
     GO TO 30
```

TABLE I (Continued)

```
28    DELV = DELV + 0.0
30    IF (A(59)) 31,32,31
31    DELV = DELV + A(35)*XIN(A(36)*V + A(37))
      DELV = DELV + A(38)*SIN(A(39)*V + A(40))
      DELV = DELV + A(41)*V*SIN(A(42)*V + A(43))
32    IF(A(60)) 95,35,95
95    IF (V) 96,97,33
96    VV = - V
      DELV = DELV - A(44)*EXP(A(45)*VV) - A(46)*VV*EXP(A(47)*VV)
      DELV = DELV - A(48)*EXP(A(49)*(VV**A(50))) + A(44) + A(48)
      GO TO 35
97    DELV = DELV + 0.0
      GO TO 35
33    DELV = DELV + A(44)*EXP(A(45)*V) + A(46)**V*EXP(A(47)*V)
      DELV = DELV + A(48)*EXP(A(49)*(V**A(50)))  - A(44) - (48)
35    DL = DELV + DELX
      IF (X+DL) 54,40,50
40    IF (V) 42,46,44
42    ALPHA = 0.5 * 3.14159265359
      GO TO 70
44    ALPHA = 1.5 * 3.14159265359
      GO TO 70
46    ZZ = 99999.9
      TYPE, ZZ
      GO TO 8
50    IF (V) 60,52,57
52    ALPHA = 0.0
      GO TO 70
54    IF (V) 59,56,58
56    ALPHA = 3.14159265359
      GO TO 70
57    ALPHA = ATN((X + DL) /V) + 1.5 * 3.14159265359
      GO TO 70
58    ALPHA = ATN(V/( -1.0 *(X + DL))) + 3.14159265359
      GO TO 70
59    ALPHA = ATN((X = DL)/V) + 0.5 * 3.14159265359
      GO TO 70
60    ALPHA = ATN((-V)/(X+DL))
70    RAD = SQR ((X+DL)**2 + V**2)
      V = RAD * SIN (ALPHA + DTHET) *(-1.0)
      X = RAD * COS (ALPHA + DTHET) - DL
      BETA = BETA + DBETA
      IF (BETA - DEL*BTYPE) 1,75,75
75    IF (A(61)) 72,74,76
72    TYPE, V, X, BETA, DL
      GO TO 78
74    PUNCH, V, X, BETA, DL
      GO TO 78
76    TYPE, V, X, BETA, DL
      PUNCH, V, X, BETA, DL
78    IF (XSTP - X) 80, 80, 82
80    ZZ = 88888.8
      TYPE, ZZ
      GO TO 8
82    IF (BESTP - BETA) 84,83,83
83    DEL = DEL + 1.0
      GO TO 10
84    ZZ = 77777.7
      TYPE, ZZ
      GO TO 8
      END
```

An exact solution exists for this problem (5) and Jacobsen (2) uses the problem as an example illustrating the accuracy of the phase-plane delta method. He varies the number of steps used and compares the results with the exact solution. The same type of procedure will serve to evaluate the accuracy of the digital computer solution.

For the computer solution the parameters and initial conditions were taken as in Jacobsen (3):

$q = 0.3$

$x_o = 5.0$

$\dot{x}_o = 0.0,$

Parameters fed into the computer program were:

A(58) = 1.0

A(33) = 0.3

A(61) = 1.0 .

The exact solution (5) indicates that the phase-plane trajectory should cross the x axis at the following points:

$x = -1.54$ $\theta = 205^\circ$

$x = 0.942$ $\theta = 389^\circ.$

The data obtained by taking angular increments of thirty degrees is presented in Table II. This is the form in which the results come from the computer. It can be seen that the velocity becomes zero for a value of x of about -1.2 to -1.3. The results then are not very accurate. This is a consequence of the relatively large angular increments which were taken, which demonstrates that small angular increments must be used in

the computer solution. Since no method of averaging of δ is employed in the computer solution, the angle of swing must be small enough so that there is very little variation of δ.

Figure 5 is a plot of x and $\dot{x}/p$ versus angular rotation θ, for data obtained by using 5 degree computation increments. The velocity becomes zero at approximately

$$x = -1.45 \qquad \theta = 200^{\circ}$$

and

$$x = 1.0 \qquad \theta = 395^{\circ}.$$

Figure 6 is a similar plot of data obtained by using 1 degree increments with printout of results every 10 degrees. The cross over points in this case are approximately

$$x = -1.52 \qquad \theta = 205^{\circ}$$
$$x = .95 \qquad \theta = 190^{\circ}.$$

Figure 7 is a conventional phase-plane plot of the same data.

Figure 8 is a plot of x and $\dot{x}/p$ versus θ for data obtained by using 0.5 degree increments with printout every 5 degrees. This figure is hardly distinguishable from Figure 6. It is extremely difficult to read these plots to more than two significant figures; consequently, the accuracy of the solutions can be inferred much more readily from the numerical data. The numerical data for 0.5 degree computation increments and 5 degree printout increments yields cross over points which compare very favorably with exact solution.

$$x = -1.53 \qquad \theta = 205^{\circ} \qquad\qquad (-1.54,\ 205^{\circ})$$
$$x = .94 \qquad \theta = 388^{\circ} \qquad\qquad (-.942,\ 389^{\circ})$$

TABLE II

OUTPUT DATA AS IT COMES FROM THE COMPUTER

FOR 30 DEGREE COMPUTATION INCREMENTS

| $\dfrac{\dot{x}}{p}$	x	θ	δ
.00000000	5.0000000	.00000000	.00000000
-2.4999998	4.3301269	30.000000	.00000000
-3.3926267	2.7512025	60.000000	-1.8749997
-2.5872150	1.1489093	90.000000	-3.4529748
-1.8109966	-2.9587400E-02	120.00000	-2.0081045
-1.0616192	-.79930210	150.00000	-.98391261
-.35068309	-1.1777271	180.00000	-.33811059
.30360959	-1.1903403	210.00000	-3.6893589E-02
.84427673	-.88276517	240.00000	2.7653635E-02
1.0656271	-.37100813	270.00000	.21384096
.93803009	.16587005	300.00000	.34066833
.59743791	.57729721	330.00000	.26397014
.17520825	.78432707	360.00000	.10707962
-.24503330	.76561753	390.00000	9.2093793E-03
-.58600764	.54294081	420.00000	-1.8012395E-02
-.72745718	.19099901	450.00000	-.10302149
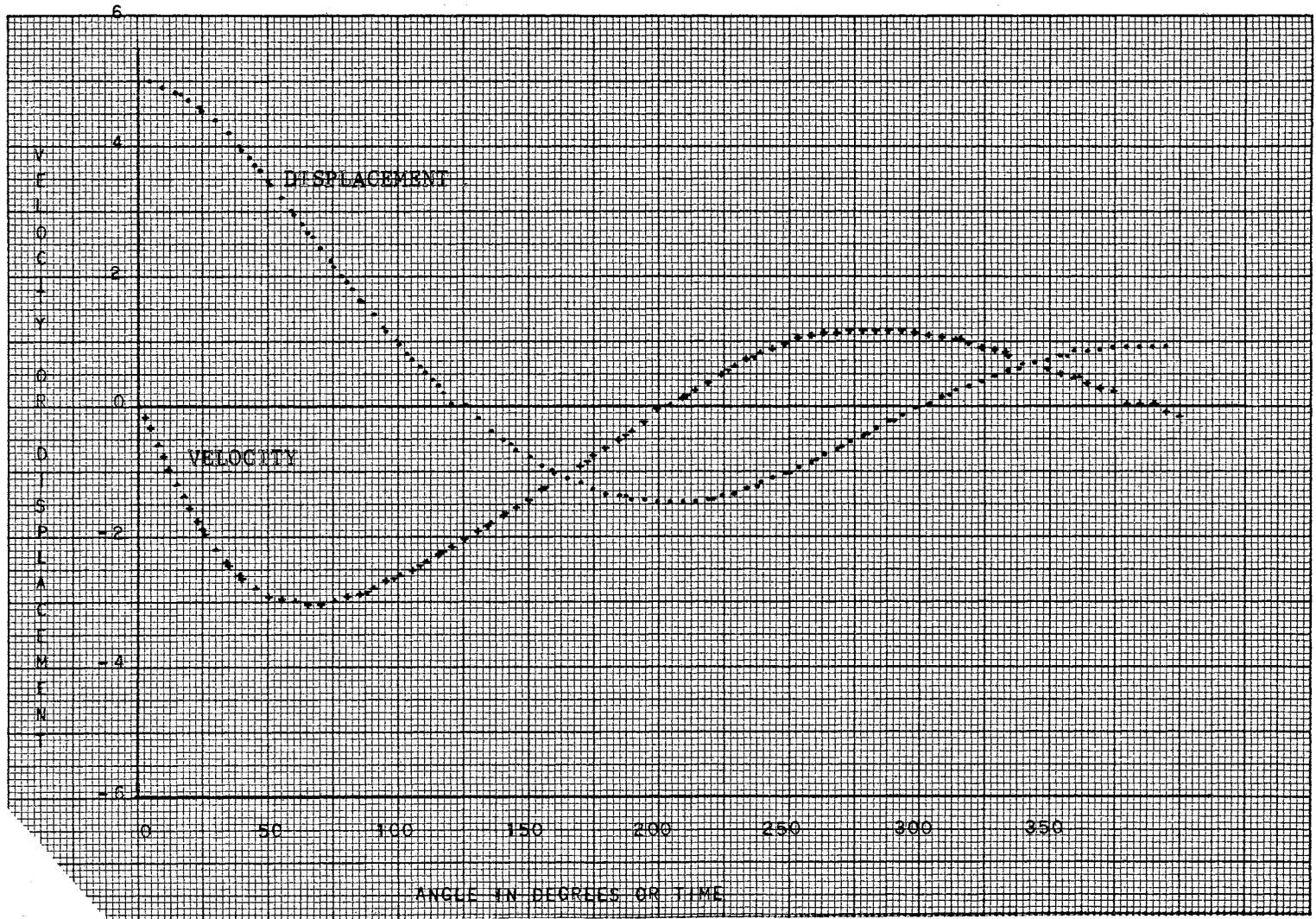
Figure 5. Displacement and Velocity ($\overset{.}{x}$/p) Versus Angular Rotation for 5 Degree Computation
Increments.

Figure 6. Displacement and Velocity (ẋ/p) Versus Angular Rotation for 1 Degree Computation Increments and 10 Degree Printout Increments.

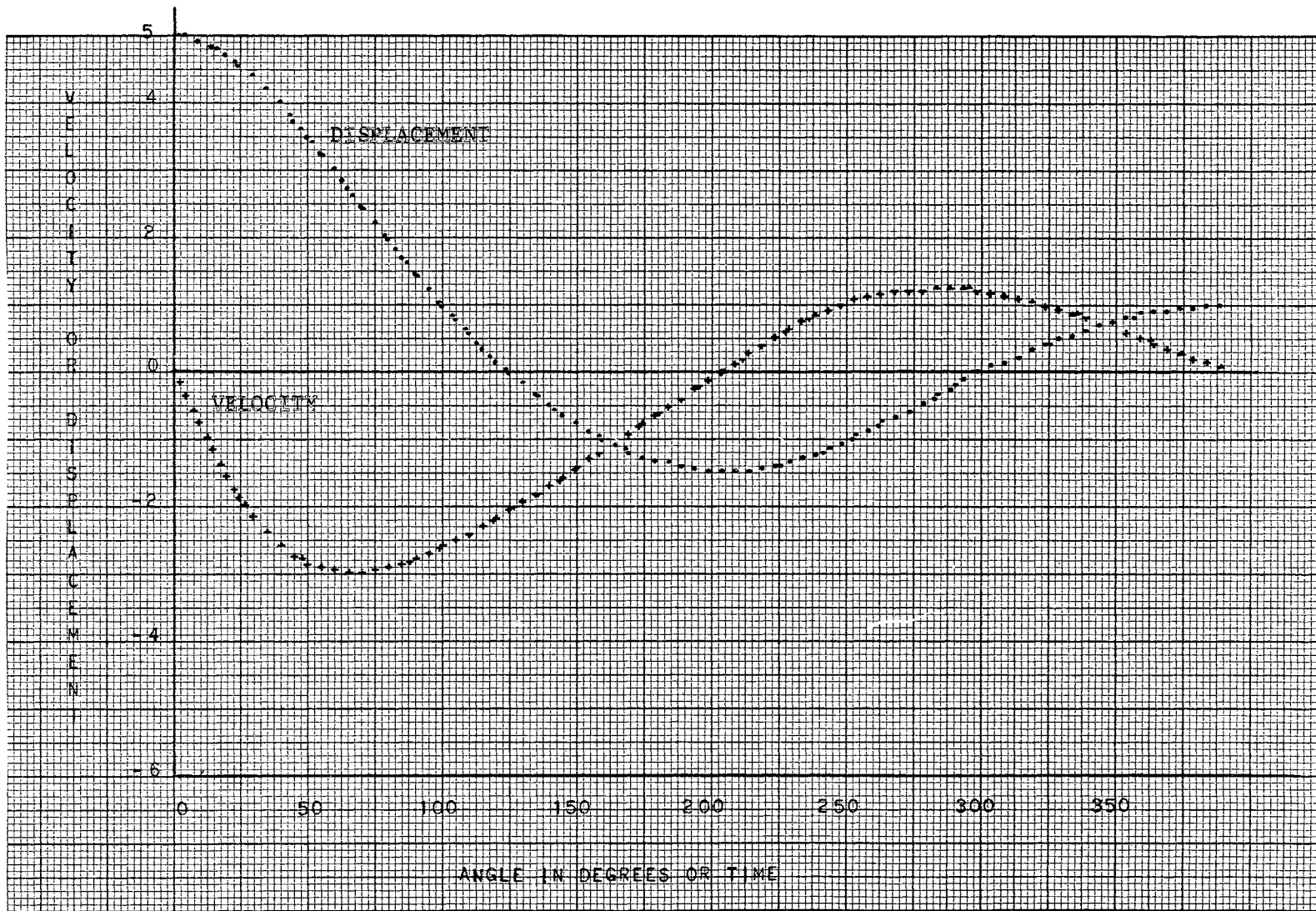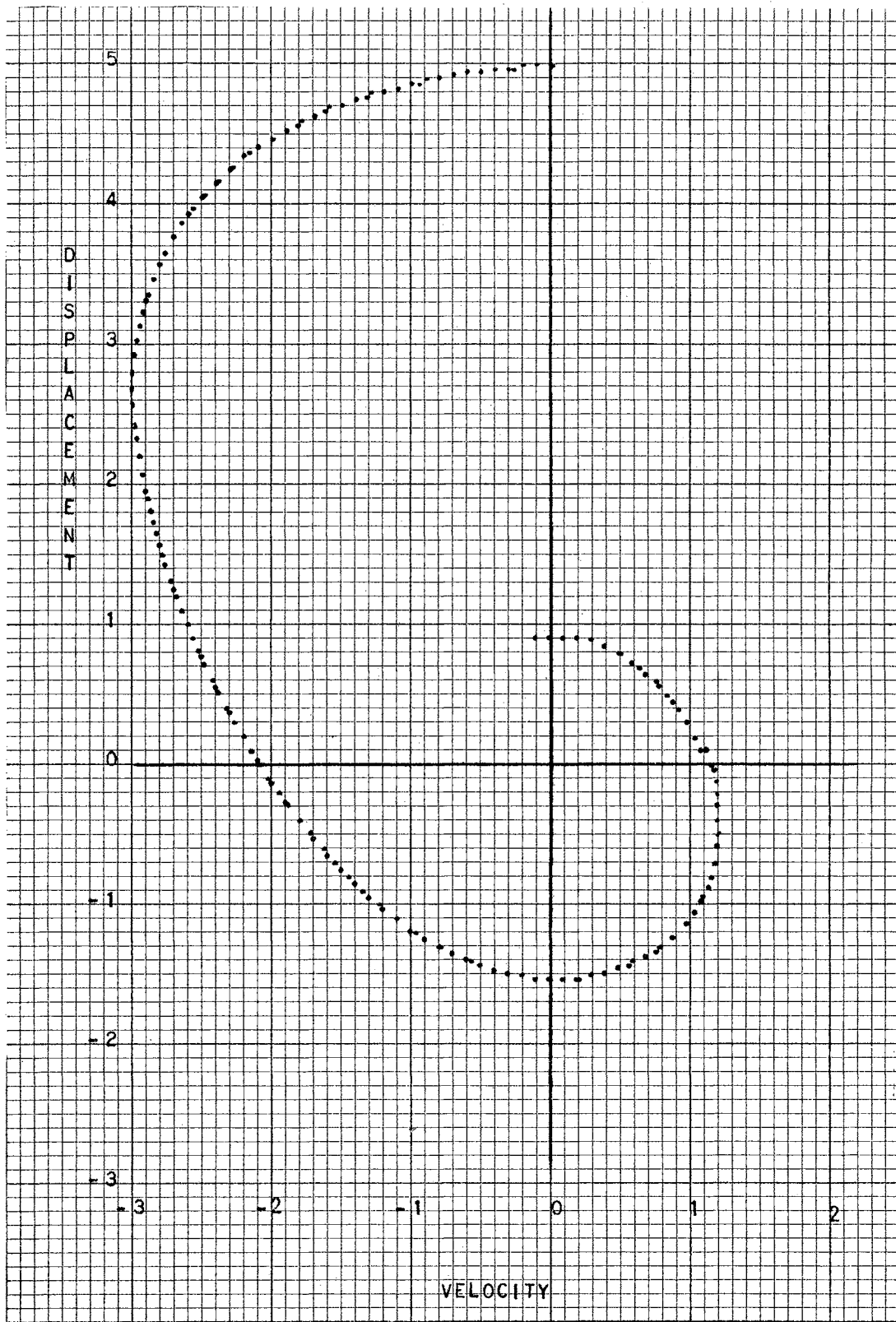Figure 7. Conventional Phase-Plane Plot of the Same Data Presented
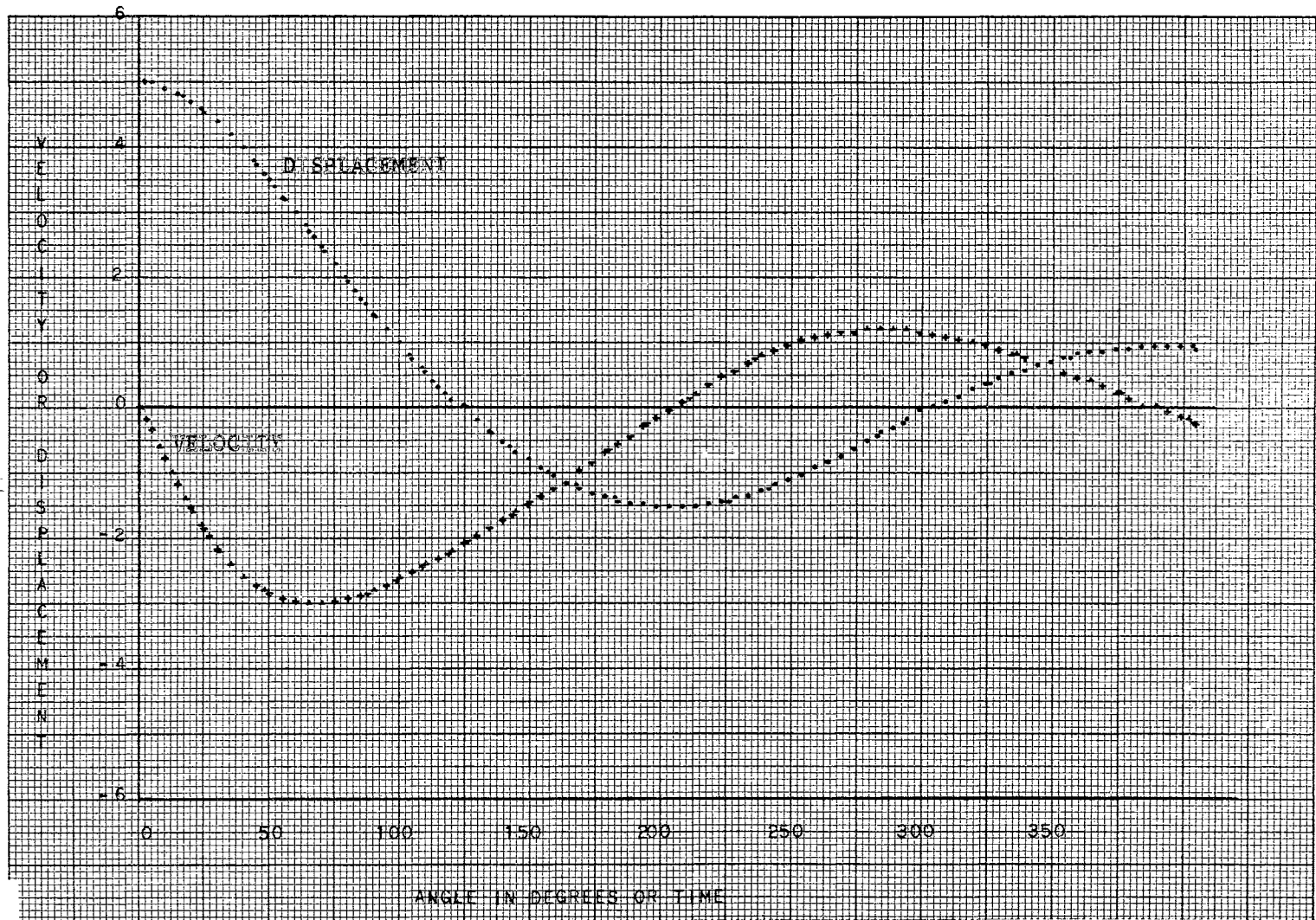in Figure 6.

Figure 8. Displacement and Velocity ($\dot{x}/p$) Versus Angular Rotation for 0.5 Degree Computation Increments and 5 Degree Printout Increments.

Exact evaluation of the accuracy of the method in cases where exact
solutions do not exist is not possible. One must simply accept that very
small increments yield good answers for problems of short time duration.
Jacobsen (1) sums up the accuracy evaluation problem very well:

> Accuracy of the graphical method depends on the judicious
> choice of the size of steps. Test cases involving elliptic inte-
> grals, Bessel and Neumann functions, show good accuracy for practi-
> cal size of steps. In a general case the degree of approximation
> can be inferred empirically only.

> Solutions involving long durations, as for instance in the
> Mathieu types of equations, will suffer in accuracy by the many
> steps necessary. Consequently, the method is not recommended
> for long-duration phenomena.

Use of the digital computer reduces considerably the restriction on
the number of steps which can be used, and seems to give the best answers
if large numbers of very small increments are used.

Example Number Two.

Consider Van Der Pol's equation

$$m\ddot{x} + b(a^2 - x^2)\ \dot{x} + kx = 0. \tag{30}$$

The delta function in this case is

$$\delta = -\frac{b}{mp}\ (a^2 - x^2)\ \frac{\dot{x}}{p} \tag{31}$$

where

$$p^2 = k/m. \tag{32}$$

This problem cannot be handled by the general program which was used
in example number one; hence, it will be necessary to write a program spe-
cifically for the solution of this problem. However, this does not mean
that it is necessary to start completely anew. The needed program can be
developed quite easily by modifying the program used for example number

TABLE III

FORTRAN PROGRAM VAN DER POL'S

EQUATION

```
C     PHASE PLANE DELTA, SIMPSON
C     SOLUTION TO VAN DER POLS EQUATION
3     READ, B, A2, AMASS,AP
      TYPE, B, A2, AMASS,AP
      PUNCH, B, A2, AMASS,AP
8     READ, XSTAR, VSTAR, DBETA, XSTP, BESTP, BTYPE, CAT
      IF (XSTAR - 44444.4) 9,3,9
9     BETA = 0.0
      DTHET = DBETA * 0.017453292
      X = XSTAR
      V = VSTAR
      DEL = 0.0
      DL = 0.0
      GO TO 75
10    DL = -1.0*(B/AMASS*AP)*(A2 - X**2)*V
      IF (X+DL) 54,40,50
40    IF (V) 42,46,44
42    ALPHA = 0.5 * 3.14159265359
      GO TO 70
44    ALPHA = 1.5 * 3.14159265359
      GO TO 70
46    ZZ = 99999.9
      TYPE, ZZ
      GO TO 8
50    IF (V) 60,52,57
52    ALPHA = 0.0
      GO TO 70
54    IF (V) 59,56,58
56    ALPHA = 3.14159265359
      GO TO 70
57    ALPHA = ATN((X + DL) /V) + 1.5 * 3.14159265359
      GO TO 70
58    ALPHA = ATN(V/( -1.0 *(X + DL))) + 3.14159265359
      GO TO 70
59    ALPHA = ATN((X + DL)/V)+ 0.5 * 3.14159265359
      GO TO 70
60    ALPHA = ATN((-V)/(X+DL))
70    RAD = SQR ((X+DL)**2 + V**2)
      V = RAD * SIN (ALPHA + DTHET) *(-1.0)
      X = RAD * COS (ALPHA + DTHET) - DL
      BETA = BETA + DBETA
      IF(BETA - DEL*BTYPE) 10,75,75
      IF(CAT) 72,74,76
75    IF(CAT) 72,74,76
72    TYPE, V, X, BETA, DL
      GO TO 78
74    PUNCH, V, X, BETA, DL
      GO TO 78
76    TYPE, V, X, BETA, DL
      PUNCH, V, X, BETA, DL
78    IF (XSTP - X) 80,80,82
80    ZZ = 88888.8
      TYPE, ZZ
      GO TO 8
82    IF (BESTP - BETA) 84,83,83
83    DEL = DEL + 1.0
      GO TO 10
84    ZZ = 77777.7
      TYPE, ZZ
      GO TO 8
      END
```
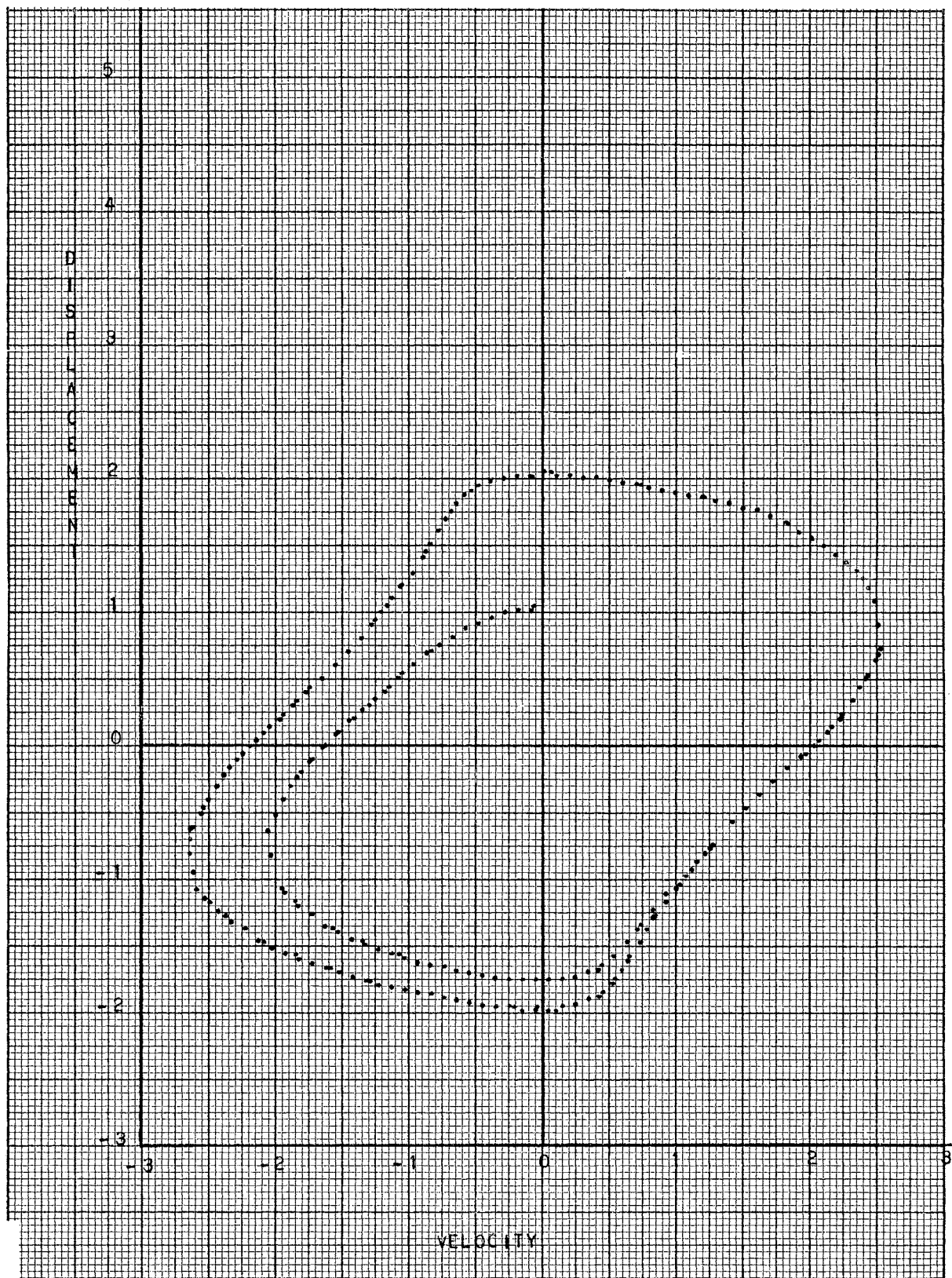
Figure 9. Phase-Plane Trajectory of Van Der Pol's Equation for a Pure
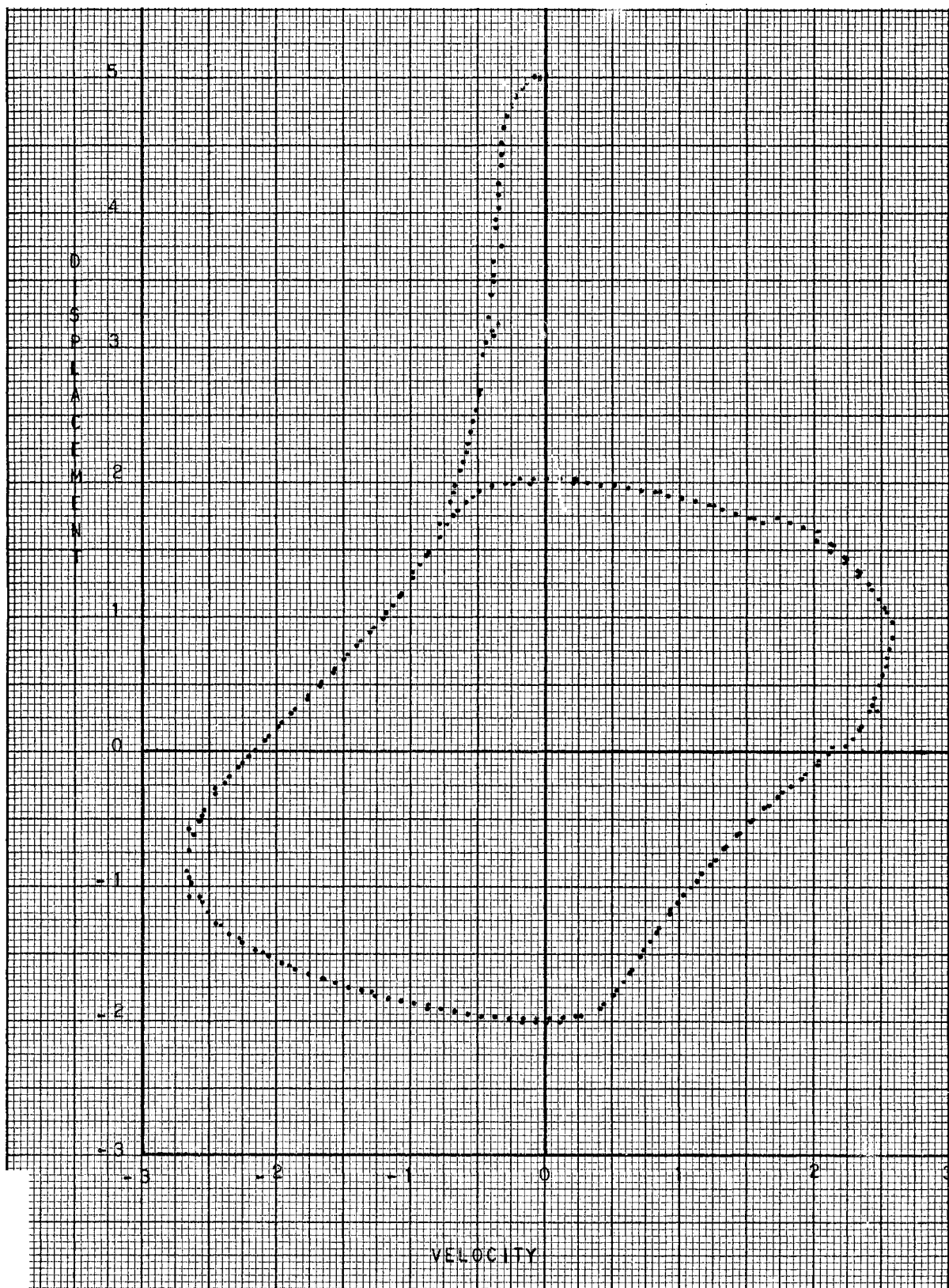Displacement Start Inside the Limit Cycle.

Figure 10. Phase-Plane Trajectory for a Pure Displacement State Outside the Limit Cycle.

one. The incrementing and computing parts of the program remain the same
so that it is only necessary to define a new delta function. A Fortran
program for the solution of this problem is given in Table III. For a nu-
merical example let the constants be

$b = 1.0$

$m = 1.0$

$a^2 = 1.0$

$k = 1.0.$

For any starting condition, the phase plane trajectory very quickly
converges to a limit cycle. This is illustrated in Figure 9 for a start
inside the limit cycle and in Figure 10 for a start outside the limit
cycle. The data for these plots was obtained using one degree computa-
tion increments and ten degree printout increments. The plots were made
using the autoplotter system. The system uses a polynomial fill which
causes some difficulty when data points are too far apart. Also the system
will not plot points of a second curve which is very nearly superimposed
on a first curve. This makes the trajectory appear to converge on the
limit cycle faster than it actually does; however, these plots do define
the limit cycle very well. In this case slightly better plots could be
done by hand. For further discussion of the Van Der Pol equation as solved
by the phase-plane method see Jacobsen (1) or Ku (4).

Example Number Three.

As an example with a time dependent delta function consider Weber's
equation and its associate.

$$x + (n + \tfrac{1}{2} \mp \tfrac{1}{4} a^2 t^2) \, x = 0 \tag{33}$$

A computer program can be developed as in the case of Van Der Pol's equation by substituting a new value of delta. A Fortran program for the problem is given in Table IV.

The delta function for this case is

$$\delta = \frac{1}{p^2}(n + \tfrac{1}{2} \mp \tfrac{1}{4} a^2 t^2 - p^2)\, x. \tag{34}$$

Setting $\quad t = \theta^2/p^2$

and letting $\quad n = 0$ and $p^2 = \tfrac{1}{2}$,

the delta function becomes

$$\delta = \mp a^2\theta^2 x. \tag{35}$$

The negative sign goes with Weber's equation while the positive sign goes with its associate.

Phase-plane plots of these two equations are given in Figures 11 and 12 for a pure displacement starting condition.

Example Number Four.

As an example of a forced vibration problem consider the linear case

$$m\ddot{x} + c\dot{x} + kx = P_o SIN\omega t. \tag{36}$$

Taking $t = \theta/p$ the equation for the delta function becomes

$$\delta = \frac{c}{k}\dot{x} - \frac{P_o}{k} SIN\ \omega\frac{\theta}{p} \tag{37}$$

where

$$p^2 = \frac{k}{m}.$$

The Fortran program is set up again by simply substituting the new delta function into the same program which has been used before. This

TABLE IV

FORTRAN PROGRAM WEBER'S

EQUATION

```
C      PHASE PLANE DELTA, SIMPSON
C      SOLUTION TO WEBER, S EQUATION
3      READ, AN, A2, AP2, C
       TYPE, AN, A2, AP2, C
       PUNCH, AN, A2, AP2, C
8      READ, XSTAR, VSTAR, DBETA, XSTP, BESTP, BTYPE, CAT
       IF (XSTAR - 44444.4) 9,3,9
9      BETA = 0.0
       DTHET = DBETA * 0.017453292
       X = XSTAR
       V = VSTAR
       DEL = 0.0
       DL = 0.0
       GO TO 75
10     DOG = (1.0/AP2)*((BETA*0.017453292)**2)
       DL = (1.0/AP2)*(AN + 0.5 + 0.25*C*A2*DOG - AP2)*X
       IF (X+DL) 54,40,50
40     IF (V) 42,46,44
42     ALPHA = 0.5 * 3.14159265359
       GO TO 70
44     ALPHA = 1.5 * 3.14159265359
       GO TO 70
46     ZZ = 99999.9
       TYPE, ZZ
       GO TO 8
50     IF (V) 60,52,57
52     ALPHA = 0.0
       GO TO 70
54     IF (V) 59,56,58
56     ALPHA = 3.14159265359
       GO TO 70
57     ALPHA = ATN((X + DL) /V) + 1.5 * 3.14159265359
       GO TO 70
58     ALPHA = ATN(V/( -1.0 *(X + DL))) + 3.14159265359
       GO TO 70
59     ALPHA = ATN((X + DL)/V) + 0.5 * 3.14159265359
       GO TO 70
60     ALOHA = ATN((-V)/(X + DL))
70     RAD = SQR ((X + DL)**2 + V**2)
       V = RAD * SIN (ALPHA + DTHET) * (-1.0)
       X = RAD * COS (ALPHA + DTHET) - DL
       BETA = BETA + DBETA
       IF (BETA - DEL*BTYPE) 10,75,75
75     IF (CAT) 72,74,76
72     RYPW, V, X, BETA, DL
       GO TO 78
74     PUNCH, V, X, BETA, DL
       GO TO 78
76     TYPE, V, X, BETA, DL
       PUNCH, V, X, BETA, DL
78     IF (XSTP - X) 80,80,82
80     ZZ = 88888.8
       TYPE, ZZ
       GO TO 8
82     IF (BESTP - BETA) 84,83,83
83     DEL = DEL + 1.0
       GO TO 10
84     ZZ = 77777.7
       TYPE, ZZ
       GO TO 8
       END
```
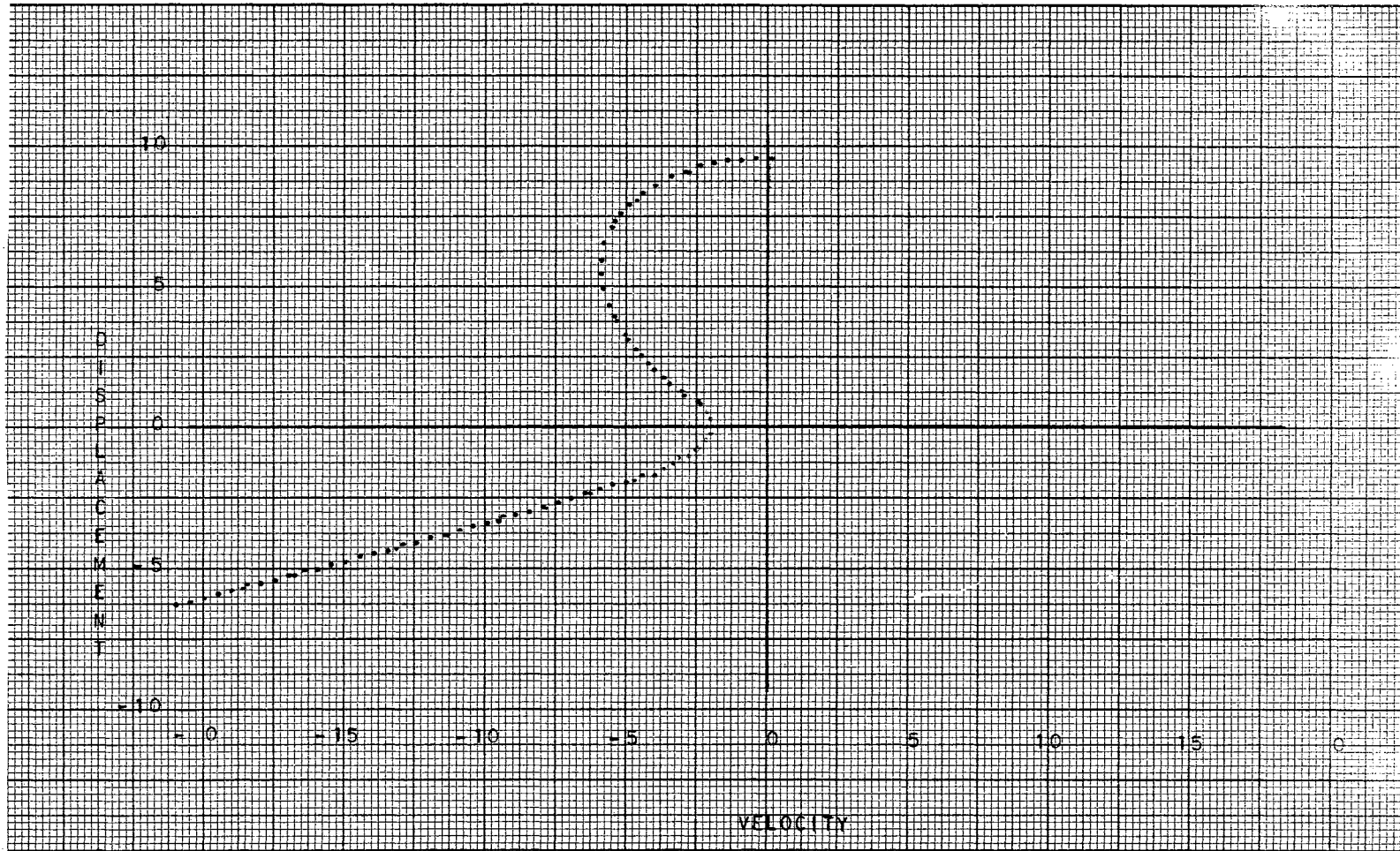
Figure 11. Phase-Plane Trajectory of Weber's Equation for a Pure Displacement Start.
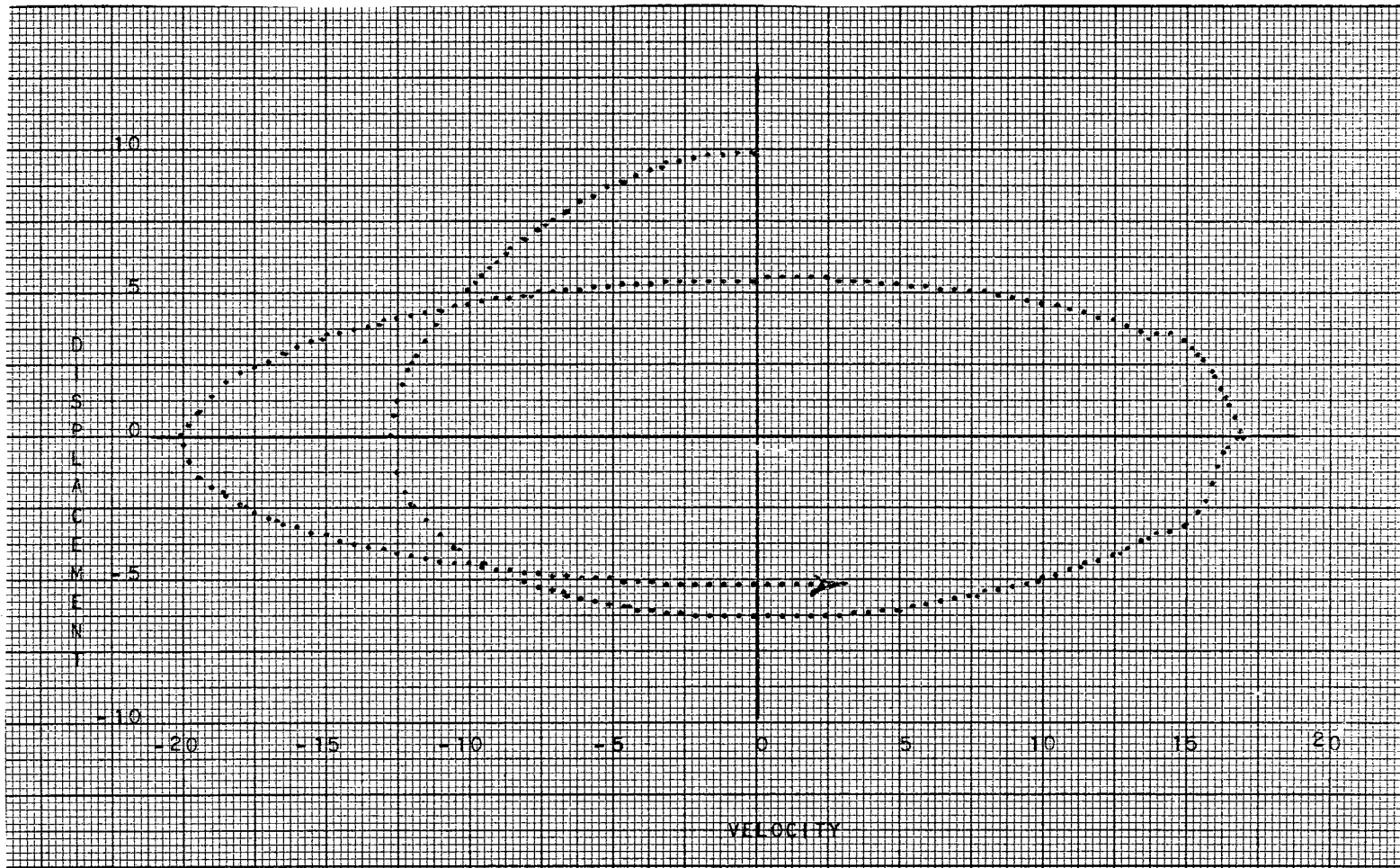
Figure 12. Phase-Plane Trajectory of Weber's Associate Equation for a Pure Displacement Start.

program is given in Table V. Figure 13 is a plot of x and $\dot{x}/p$ versus θ or time, for the nondamped case with initial velocity and displacement equal to zero. The exact solution for this case is

$$x = \frac{-x_{st} \, {}^{\omega}/_{p}}{1-(\omega/p)^2} \quad SINpt + \frac{x_{st} \quad \omega}{1-(\omega/p)^2} \, SIN\omega t \tag{38}$$

where

$$x_{st} = \frac{P_o}{k}.$$

The parameter values used to obtain the data plotted in Figure 12 were

$$P_o = 16.0$$

$$K = 16.0$$

$$m = 1.0$$

$$c = 0.0$$

$$\omega = \sqrt{8} \, .$$

The plot agrees very well with the exact solution.

TABLE V

FORTRAN PROGRAM

$$m\ddot{x} + c\dot{x} + kx = P_o \sin\omega t$$

```
C       FORCED VIBRATIONS WITH VISCOUS DAMPING
C       LINEAR SYSTEM ONE DEGREE OF FREEDOM
C       PHASE PLANE DELTA, SIMPSON
3       READ, AMASS, C, AK, APO, W
        TYPE, AMASS, C, AK, APO, W
        PUNCH, AMASS, C, AK, APO, W
8       READ, XSTAR, VSTAR, DBETA, XSTP, BESTP, BTYPE, CAT
        IF (XSTAR - 44444.4) 9,3,9
9       BETA = 0.0
        DTHET = DBETA * 0.017453292
        X = XSTAR
        V = VSTAR
        DEL = 0.0
        DL = 0.0
        A1 = C/AK
        A2 = APO/AK
        AP = SQR (AK/AMASS)
        W2 = (W/AP_ * 0.017453292
        GO TO 75
10      DL = A1*V - A2*SIN(W2*BETA)
        IF (X+DL) 54,40,50
40      IF (V) 42,46,44
42      ALPHA = 0.5 * 3.14159265359
        GO TO 70
46      ZZ = 99999.9
        TYPE, ZZ
        GO TO 88
50      IF (V) 60,52,57
52      ALPHA = 0.0
        GO TO 70
54      IF (V) 59,56,58
56      ALPHA = 3.14159265359
        GO TO 70
57      ALPHA = ATN((X + DL) /V) + 1.5 * 3.14159265359
        GO TO 70
58      ALPHA = ATN(V/( -1.0 *(X + DL))) + 3.14159265359
        GO TO 70
59      ALPHA = ATN((X + DL)/V) + 0.5 * 3.14159265359
        GO TO 70
60      ALPHA = ATN((-V)/(X+DL))
70      RAD = SQR ((X+DL)**2 + V**2)
        V = RAD * SIN (ALPHA + DTHET) * (-1.0)
        X = RAD * COS ( ALPHA + DTHET) - DL
88      BETA = BETA + DBETA
        IF (BETA - DEL * BTYPE) 10,75,75
75      IF (CAT) 72,74,76
72      TYPE, V, X, BETA, DL
        GO TO 78
74      PUNCH, V, X, BETA, DL
        GO TO 78
76      TYPE, V, X, BETA, DL
        PUNCH, V, X, BETA, DL
78      IF (XSTP - X) 80,80,82
80      ZZ = 88888.8
        TYPE, ZZ
        GO TO 8
82      IF (BESTP - BETA) 84,83,83
83      DEL = DEL + 1.0
        GO TO 10
84      ZZ = 77777.7
        TYPE, ZZ
        GO TO 8
        END
```
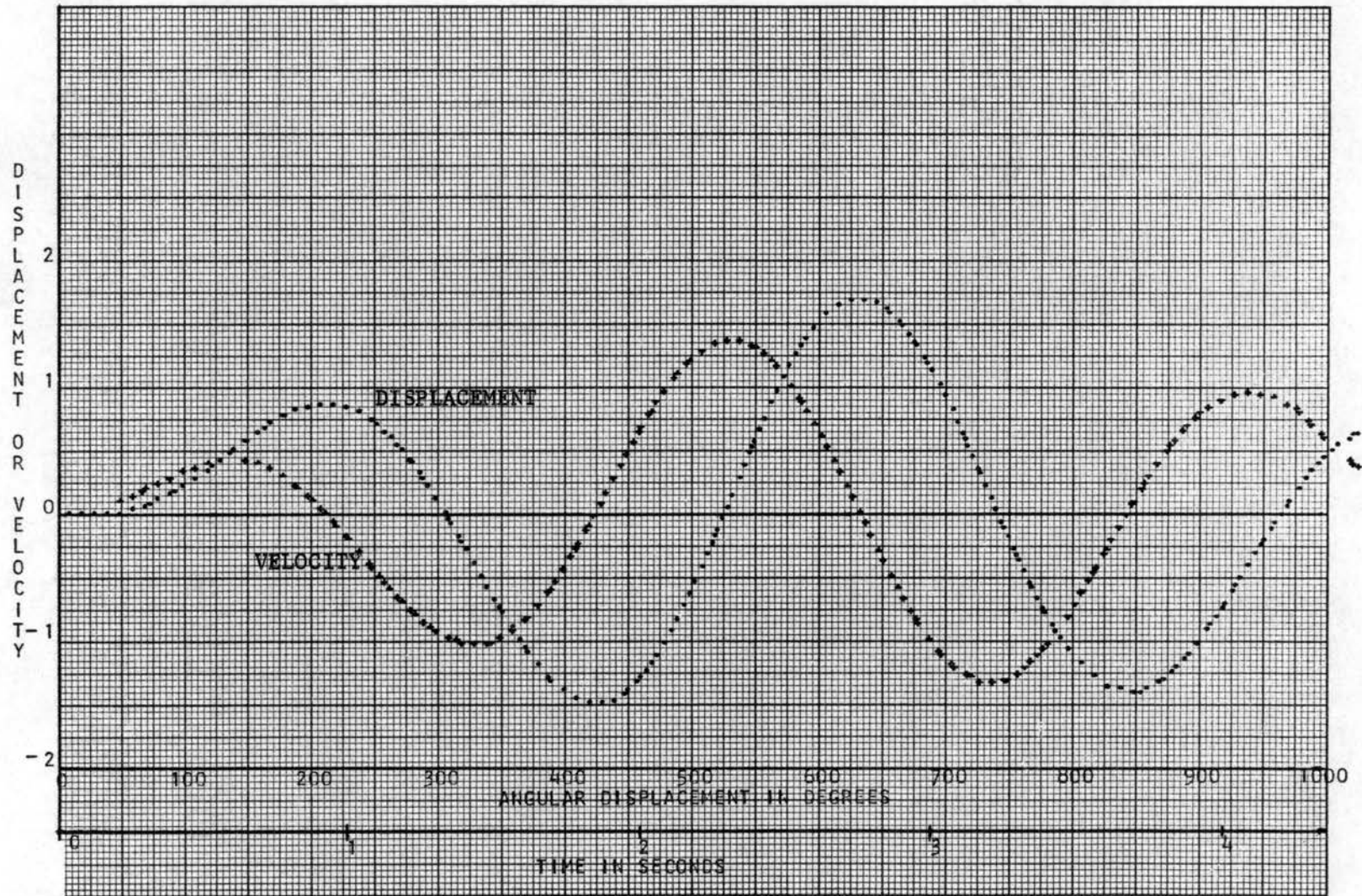
Figure 13. Linear Forced Vibrations, Displacement and Velocity Versus Angular Displacement or Time.

CHAPTER V

CONCLUSIONS

In almost any case where a differential equation can be solved successfully with the graphical phase-plane delta method, it can also be solved by the phase-plane method as adopted to the digital computer. However, it should be kept in mind that use of a computer will not always result in an easier or faster solution. In some cases use of the computer will represent a definite improvement of the method in terms of speed and ease of solution while in other cases the computer solution may be more difficult.

Several possible advantages of the digital computer solution can be enumerated. A general program such as the one given in example number one of Chapter IV allows the solution of a great number of different problems very quickly by merely changing coefficients. Any program which allows for variation of parameters and of starting conditions can be used quite effectively to study the results of variations of parameters and starting conditions. In cases where it is difficult to obtain a suitable average value of δ for a reasonable size arc with the graphical method, the computer may greatly reduce the difficulties of solution and may very well produce greater accuracy.

The restriction of the graphical procedure to short time phenomena due to the build up of graphical errors resulting from a large number of steps can be relaxed a little with the computer solution at least in the

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respect that a much greater number of steps may be employed. However ac-
cumulative numerical errors can be expected to build up with the number
of steps taken just as in the graphical procedure except at a much slower
rate with respect to the number of steps.

One of the advantages of the graphical method is that a δ function or
a forcing function need not be expressed in mathematical form. This allows
experimental data to be introduced into the solution in a graphical form
making it unnecessary to find an analytical expression for such data. This
advantage does not apply to the computer solution. In general if the δ
function can be handled more easily graphically and less easily expressed
as a mathematical function, the problem can probably be solved more easily
with graphical procedures. In cases where the δ function or the averaging
procedure become difficult to manage graphically, and yet is easily ex-
pressed mathematically the numerical computer methods will probably be
superior.

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