EFFECTS OF INITIAL CONDITIONS ON PERFORMANCE OF CONTROL SYSTEMS

By

NARENDRAKUMAR M. SHAH

Bachelor of Engineering

University of Bombay

Bombay, India

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Thesis Approved:

E. C. Fitch Jr.

Thesis Adviser

M. K. Jovanovic

Robert MacVicar

Dean of the Graduate School

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NOMENCLATURE

The nomenclature used is patterned after the standard nomenclature and symbols of the American Standards Association[1]. Capital letters are used to represent the Laplace transforms of the time functions.

a, A, b, B, c, d	Arbitrary constant and/or coefficient for differential equation
$c(t)$	Controlled variable (function of time)
c_p	Peak value of $c(t)$
$e(t)$	Actuating signal (function of time)
i	i^{th} term in a series, used as subscript
j	Complex number $\sqrt{-1}$
k	Gain constant
m	Used as a subscript to denote m^{th} term in series and/or m^{th} power
M_m	Maximum value of $\left\|\frac{C}{R}(j\omega)\right\|$
n	Used as a subscript to denote n^{th} term in series and/or n^{th} power
N	Number of oscillations up to settling time
$p, p^2, \ldots$	Differential operator, $p = \frac{d}{dt}$, $p^2 = \frac{d^2}{dt^2}, \ldots$
$p^n c_o$	Initial value of the n^{tn} derivative of $c(t)$
p_n	n^{th} pole
$r(t)$	Reference input variable (function of time)

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[1]Letter symbols for Feedback Control Systems, ASA Y 10-13-1955, American Standards Association, New York, July, 1955.

NOMENCLATURE (Cont.)

s	Laplace Transform Operator $= \sigma \pm j\omega$
t	Time seconds
t_r	Rise time, seconds
t_d	Delay time, seconds
t_p	Time to first peak of transient
t_s	Settling time, seconds
T	Time constant, seconds
u(t)	Disturbance variable (function of time)
x(t), y(t)	Variables used when standard terminology for feedback system is not applicable
z_n	n^{th} zero
ξ	Relative damping, damping factor
Π	Product sign
$\sum$	Summation sign
ω	Angular frequency radians/second
ω_m	Angular frequency at which M_m occurs, radians/second
ω_n	Natural angular frequency radians/second
$\therefore$	Therefore

CHAPTER I

INTRODUCTION

1.1. Subject

It is known that the initial conditions do not affect the stability of a linear feedback control system as a whole but they do affect its transient performance. The present trend in the design of linear feedback control systems is to design a system with the assumption that the initial conditions are absent. But this is an idealized assumption and the control system that meets the performance specifications under idealized conditions will not necessarily be within allowable tolerances when the initial conditions are present. Thus the designs based solely on neglecting the initial conditions may at times prove to be disastrous in actual working as either the system itself may be damaged and lose its accuracy or the departure from the designed characteristics of systems such as emergency control systems like fire control systems may result in heavy losses. For underdamped systems such as used for regulators, meters and position servomechanisms, apart from stability, peak transient overshoot is one of the most important design criteria. It can be shown that the presence of the initial conditions considerably changes the magnitude of the peak overshoot. For systems which require rapid synchronization like fire control systems settling time is an important point to be kept in mind while designing. For overdamped systems such as used in process controls, rise time and dead time are important criteria in judging the performance. An attempt, therefore, is made in this thesis to explore the effects of initial conditions on these parameters grouped under criteria for judging transient performance in Section 1.2. In the discussion here onward both the qualitative and the quantitative aspects of the effects of initial conditions are pointed out and it is also shown how the components of the system govern the effects of the initial conditions.

Methods have been developed which show how these effects can be taken into account while designing a control system and a large number of curves for response of a control system with different initial conditions have been plotted to serve as a general guide. A method has also been shown by which a transfer function can be approximated for a control system with the initial conditions present when output and input characteristics are available from experimental results.

1.2. Criteria for Judging Transient Performance. (1)[1]

1. Peak transient overshoot c_p.

2. Time t_p to reach peak overshoot.

3. Delay time t_d required for the response to reach fifty percent of its final value.

4. Rise time t_r required for the response to rise from ten percent to ninety percent of its final value.

5. Settling time t_d. The limit used in this thesis is ± 2 percent of final value.

6. Number of oscillations N up to settling time.

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[1]Note: ( ) refers to Selected Bibliography.

1.3. Plan of Development

Before attempting to explore the nature of the effects of the initial conditions on the performance of a control system, a detailed analysis of linear feedback control systems is performed in Chapter III and a general transform equation is derived which includes the initial conditions. An analogy is shown between the effects of initial conditions and those due to impulse disturbances.

Both the qualitative and the quantitative aspects of the effects of initial conditions are discussed in Chapter IV using pole-zero configuration analysis. Various methods such as analytical, graphical and numerical have been discussed to show how these effects can be determined. Dependence of the effects of the initial conditions on various control system components is also pointed out.

Chapter V shows how the conventional frequency response methods (2) can be extended to evaluate the transient response of systems when initial conditions are present. A method has been shown by which a higher order feedback control system with non-zero initial conditions can be approximated by an equivalent second-order or first-order system. The accuracy and the ease with which a problem can be tackled is illustrated by an example. A mathematical explanation of the nature of the effects of initial conditions is given which explains the results obtained in Chapter IV.

A method has been developed in Chapter VI, by which given the required response c(t) and excitation r(t), both with the initial conditions present, in graphical form, a linear differential equation representing the system can be determined. Although the method developed allows approximation by a second-order system, it may be extended to determine a higher order approximation; but the accuracy of the method may be deteriorated.

The results and conclusions are contained in Chapter VII.

1.4. Limitations

The effects of initial conditions on linear feedback control systems only are considered. Also the effects on system optimization are not considered. Several graphical methods which are usually associated with non-linear systems but which can be also applied to linear systems, such as the phase-plane method, are not discussed here as they have been treated fully elsewhere (3,4).

CHAPTER II

PREVIOUS INVESTIGATIONS

The effects of initial conditions on performance of feedback control systems have been discussed in a wide variety of literature on control systems but almost in all the cases the treatment is superficial.

Francis Raven (5) , J. C. Truxal (3) and many others have shown that the initial conditions do not affect the stability of a linear feedback control system as a whole, where a stable system may be defined as one, the real parts of the roots of whose characteristic equation are either negative or zero and whose response practically attain a steady state value after some time interval has elapsed. They have also indicated that the initial conditions affect only the transient performance and that these effects are of the same nature irrespective of the type of the input signal. Further, they mention that the response due to each initial condition can be found out separately, just in the same manner as that due to the external excitation. Evans (6), without going into further details, has suggested that the effects of initial conditions can be looked upon as similar to those due to impulse disturbances introduced at suitable points in the control loop.

CHAPTER III

GENERAL THEORY

3.1. Control System

A feedback control system can in general be represented by the block diagram shown in Figure 3.1. in which the control elements have been cascaded to produce the over-all forward and feedback transfer functions $G_1(p)$, $G_2(p)$, and $H(p)$.

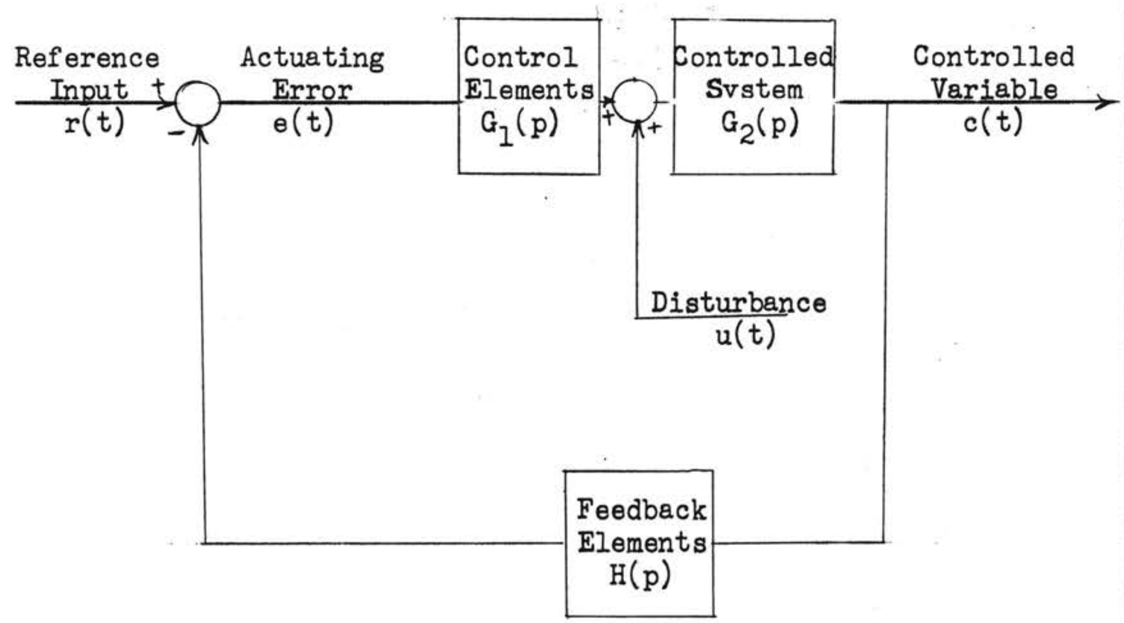


Figure 3.1. Block Diagram of a Feedback Control System.

3.2. Mathematical Representation of Control Systems

The equations describing the control system shown in Figure 3.1. may be expressed as:

$$c(t) = \left[e(t)\ G_1(p) + u(t)\right]\ G_2(p) \tag{3.1}$$

$$e(t) = r(t) - c(t)H(p) \tag{3.2}$$

The combination of Equations (3.1) and (3.2) gives controlled variable

$$c(t) = \frac{G_1(p)G_2(p)}{1 + G_1(p)G_2(p)H(p)}\ r(t) + \frac{G_2(p)}{1 + G_1(p)G_2(p)H(p)}\ u(t) \tag{3.3}$$

It follows from Equation (3.3) that the effects due to reference input r(t) and disturbance u(t) may be superimposed to have combined output c(t). Again, since the characteristic equation is the same in both the cases, the response due to u(t) can be calculated in the same manner as that due to r(t). Hence for further discussions disturbance u(t) is neglected.

The Equation (3.3) can now be written as:

$$c(t) = \frac{G_1(p)G_2(p)}{1 + G_1(p)G_2(p)H(p)}\ r(t) = \frac{G(p)}{1 + G(p)H(p)}\ r(t) \tag{3.4}$$

where $G(p) = G_1(p)G_2(p)$

and is known as combined forward transfer function.

The function G(p) and H(p) can be expressed as:

$$G(p) = \frac{K_1 \prod_{W=1}^{W} (p + a_W)}{\prod_{Z=1}^{Z} (p + b_Z)}$$

$$H(p) = \frac{K\prod_{U=1}^{U}(p + c_U)}{\prod_{V=1}^{V}(p + d_V)}$$

where K_1 = open loop gain

K_1K_2 = static loop sensitivity

a's, b's, c's and d's may be real, complex or zero.

Substitution for G(p) and H(p) into Equation (3.4) and simplifying, c(t) can be expressed as

$$c(t) = \frac{A_m p^m + A_{m-1}p^{m-1} + \text{-----} + A_1 p + A_o}{B_n p^n + B_{n-1}p^{n-1} + \text{-------} + B_1 p + B_o}\ r(t) \qquad (3.5)$$

where A's and B's are real constants and m and n are positive integers, and because of the nature of the functions G(p) and H(p), in general $n \geq m$.

The Equation (3.5) can also be written as:

$$c(t) = \frac{\sum_{i=0}^{m} A_i p^i}{\sum_{i=0}^{n} B_i p^i}\ r(t) \qquad (3.5)$$

Since $Lp^n y(t) = S^n y(S) - S^{n-1} y_o - S^{n-2} p y_o \text{-----} p^n y_o$.

The Laplace transform of Equation (3.5) can on simplification be written as:

$$C(S) = \frac{\sum_{i=0}^{m} A_i S^i}{L(S)} R(S) + \left[\frac{\sum_{i=1}^{n} B_i S^{i-1}}{L(S)} c_o + \frac{\sum_{i=2}^{n} B_i S^{i-2}}{L(S)} pc_o + --- + \frac{B_n}{L(S)} p^{n-1} c_o \right]$$

$$- \left[\frac{\sum_{i=1}^{m} A_i S^{i-1}}{L(S)} r_o + \frac{\sum_{i=2}^{m} A_i S^{i-2}}{L(S)} pr_o + --- + \frac{A_m}{L(S)} p^{m-1} r_o \right] \quad (3.6)$$

Where $L(S) = \sum_{i=0}^{n} B_i S^i$ and is called the characteristic equation.

The first term on the right hand side of Equation (3.6) represents response due to reference input r(t) only, terms in the first [ ] represent response due to c_o, pc_o,---$p^{n-1}c_o$ respectively and the negative of the terms in the second [ ] represent response due to p_o, pr_o,---$p^{m-1}r_o$ respectively.

Case I: All the initial conditions are zero. In this case, the Equation (3.6) reduces to

$$C(S) = \frac{\sum_{i=0}^{m} A_i S^i}{L(S)} R(S) \quad (3.7)$$

Since $\mathcal{E}^n Y(t) = S^n Y(S)$, with all the initial conditions zero.

$$c_A(t) = \int_0^t w(t)dt \quad (3.8)$$

$$c_B(t) = \int_0^t c_A(t)dt \quad (3.9)$$

$$c_C(t) = \int_0^t \int_0^{t_1} c_A(t)dt_1 dt \quad (3.10)$$

Where $w(t)$ = response due to unit impulse

$c_A(t)$ = response due to unit step input

$c_B(t)$ = response due to unit ramp input

and $c_C(t)$ = response due to unit parabolic input

Case II: All the initial conditions except c_o and r_o are zero. The Equation (3.6) can now be written as

$$C(S) = \frac{\sum_{i=0}^{m} A_i S^i}{L(S)} R(S) + \frac{1}{S} c_o - \frac{\sum_{i=1}^{m} A_i S^i}{L(S)} \frac{r_o}{S}$$

$$C(S) = \frac{\sum_{i=0}^{m} A_i S^i}{L(S)} R(S_{+o}) + \frac{c_o}{S} \tag{3.11}$$

where $R(S_{+o}) = Lr(t_{+o})$

Also $L^{-1} \frac{c_o}{S} = c_o$

Hence response $c(t)$ can easily be obtained by finding response due to $r(t_{+o})$ and adding c_o to it.

Case III: Non-zero initial conditions. In this case the system is represented by Equation (3.6). A study of Equation (3.6) indicates that:

1. Effects of initial conditions are the same irrespective of the type of the input signal.
2. By superposition theorem, response due to each initial condition can be found out separately and added together to that due to input $r(t)$ to have the resultant output $c(t)$.
3. Effects of initial conditions are the same as those due to impulse disturbances. Hence initial conditions can be

included in the block diagram by applying impulse disturbances at suitable points in the loop; e.g.,

$$c(t) = \frac{\omega_n^2}{p^2 + 2\xi\omega_n p + \omega_n^2}\, r(t) \qquad c_o \neq o,\ pc_o \neq 0$$

can be represented by the block diagram of Figure 3.2. where u_1 indicates unit impulse.

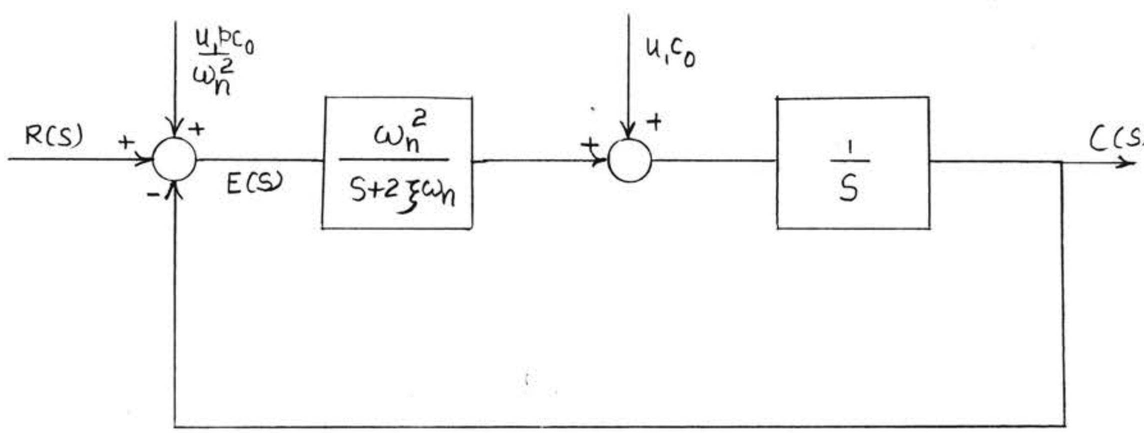


Figure 3.2. Block Diagram Representation of Initial Conditions.

$$c(t) = \frac{\omega_n^2}{p^2 + 2\xi\omega_n p + \omega_n^2}\, r(t),\ c_o \neq o,\ pc_o \neq o$$

$$C(S) = \frac{\omega_n^2}{S^2 + 2\xi\omega_n S + \omega_n^2} R(S) + \frac{1}{S^2 + 2\xi\omega_n S + \omega_n^2} pc_o + \frac{S + 2\xi\omega_n}{S^2 + 2\xi\omega_n S + \omega_n^2} c_o$$

CHAPTER IV

POLE-ZERO CONFIGURATION ANAYSIS

4.1. Introduction

A qualitative, as well as quantitative measure of the effects of initial conditions on performance of a control system can be easily obtained by pole-zero configuration anaylsis.

The Equation (3.6) indicates that r_o, pr_o,---,$p^{m-1}r_o$ affect the system in the same manner as c_o , pc_o,---,$p^{n-1}c_o$. Hence for further analysis only effects of c_o, pc_o,---, $p^{n-1}c_o$ are considered; ro, pr_o,---,$p^{m-1}r_o$ being assumed zero.

4.2. Qualitative Anaysis

Qualitative anaysis can be explained best by considering a specific problem.

Example: Consider a closed loop control system with

$$G(p) = \frac{K(p+z_1)}{S(p+p_1)(p+p_2)} \tag{4.1}$$

$$H(p) = 1 \tag{4.2}$$

Substituting in Equation (3.4), output c(t) can be expressed as

$$c(t) = \frac{K(p+z_1)}{p(p+p_1)(p+p_2)+K(p+z_1)} \tag{4.3}$$

which on simplification gives

$$c(t) = \frac{K(p+z_1)}{p^3+(p_1+p_2)\ p^2+(p_1p_2+K)\ p+Kz_1}\ r(t) \tag{4.4}$$

Taking Lapace transform of both the sides and rearranging the terms as in Equation (3.6).

$$C(S) = \frac{K(S+z_1)}{L(S)}\ R(S) + \frac{S^2(p_1+p_2)S\ +(p_1p_2+K)}{L(S)}\ c_o$$

$$+\frac{S+(p_1+p_2)}{L(S)}\ pc_o\ +\frac{1}{L(S)}\ p^2c_o \tag{4.5}$$

where $L(S) = S^3+ (p_1+p_2)S^2+ (p_1p_2+ K)S + Kz_1$

The Equation(4.5) can be further simplified to

$$C(S) = \frac{G(S)}{1+ G(S)}\ R(S)+\frac{(S+q_1)(S+q_2)}{K(S+z_1)}\ X\ \frac{G(S)}{1+(G(S)}\ c_o$$

$$+\frac{S\ (p_1+p_2)}{K(S+z_1)}\ X\ \frac{G(S)}{1+G(S)}\ pc_c+\frac{1}{K(S+z_1)}\ X\ \frac{G(S)}{1+G(S)}\ p^2c_o$$

(4.6)

where q_1 and q_2 are roots of the equation

$$S^2 + (p_1+p_2)S + (p_1p_2 + K) = 0 \tag{4.7}$$

and are given by

$$q_1,q_2 = \frac{-(p_1+p_2)\ \pm \sqrt{(p_1-p_2)^2 - 4K}}{2} \tag{4.8}$$

Equation (4.6) provides the base for the discussion of qualitative effects of c_o, pc_o and p^2c_o, for the given control system.

1. Effect of c_o. If the ratio $\frac{(S+q_1)(S+q_2)}{(S+z_1)}$ describes a lead network, i.e., if q_1 and q_2 are much smaller than z_1, the effect of c_o will be more pronounced. Thus the effect of c_o is governed by relative values of poles and zeros of GH. Presence of zeros of the control ratio $\frac{c(t)}{r(t)}$ make the effect of c_o less pronounced.

It is evident from Equations (4.6) and (4.7) that increasing the gain K reduces the effect of c_o. This is true irrespective of the type of the system.

2. Effect of pc_o. Here again effect of pc_o depends upon the ratio $\frac{S+(p_1+p_2)}{S+z_1}$, the effect being more pronounced if the ratio represents a lead network. Also increasing the value of gain K reduces the effect of pc_o.

3. Effect of p^2c_o. Remarks for effect of c_o apply here too. A higher value of the zero of the control ratio makes the effect of p^2c_o less pronounced, while decreasing the gain increases this effect.

General Conclusions

It follows from the discussion in Section 4.2. that

1. Effect of initial conditions depend upon the relative values of poles and zeros of GH.
2. Zeros of control ratio $\frac{c(t)}{r(t)}$ make the effect of the initial conditions less pronounced. Higher the values of the zeros the less pronounced this effect will be.
3. Increasing the gain K reduces the effects due to initial conditions.

4. Effect of the initial condition $p^n c_o$ decreases as n increases.

4.3. Quantitative Analysis

Quantitative analysis consists of determining the roots of the characteristic equation and evaluating the coefficient of each exponential and power series term in the inverse Laplace transform. If the open-loop transfer function is expressed as

$$G(p)\,H(p) = \frac{K \prod_{m=1}^{m} (p + z_m)}{\prod_{n=1}^{n} (p + p_n)} \qquad (4.9)$$

where the a's and b's may be real, complex or zero and K is defined as static loop sensitivity or simply gain, the characteristic equation may be written as

$$\prod_{n=1}^{n} (S + p_n) + K \prod_{m=1}^{m} (S + z_m) = 0 \qquad (4.10)$$

Various methods are available to determine the roots of the Equation (4.10) as gain K is varied from zero to infinity or some large value. This process is known as root-locus finding. All these methods are based on the fact that the roots of the characteristic Equation (4.10), are related to the poles and zeros of the open-loop transfer function GH. More precisely, this fact may be expressed as

$$G(S)H(S) = 1 \qquad (4.11)$$

$$G(S)H(S) = (1 + 2i)\ 180^o \qquad (4.12)$$

where $i = 0, \pm 1, \pm 2$, ---

4.3.1 Methods for Plotting Root-Locus

1. Graphical Method. This method has been discussed in much detail by Evans (6). Specific points on root-locus are determined using geometrical short-cuts. These points are then plotted and connected by the use of a French curve or other means to form the root-locus. Use of Spirule (6) facilitates this procedure. However, the accuracy is limited by the skill and experience of the person performing the calculations and manipulating the Spirule.

2. Automatic Computation of Root-Locus Using Digital Computer. Numerical methods (7) employed for computation involve iterative processes in which an initial approximation z_0 to a real root $S = x$ is estimated by graphical methods or other means, and a recurrence relation is used to generate a sequence of successive approximations x_1, x_2,---, x_n, which converges to the limit x. Unfortunately, in all these methods convergence is either uncertain, slow and excessively laborious or critical concerning the initial estimate of the root.

The problem of uncertain convergence may be overcome by using two different methods simultaneously. A combination of the Bairstow and Newton-Raphson methods results in a powerful iterative process for solving polynomial equations because the class of polynomials for which the Bairstow Method fails is generally not the same class as that for which the Newton-Raphson Method fails. The equation should be scaled and coefficients should be reversed, when necessary, to prevent loss of significance in the synthetic division.

A method has been developed by Stuart Brown Herndon (8) in Fortran language which simultaneously employs Bairstow and Newton-Raphson Methods. The Bairstow iteration is performed to find a quadratic factor of the type $S^2 + PS + Q$ of the polynomial and a consecutive Newton-Raphson iteration is performed to find the linear factors before attempting another Bairstow iteration with corrected values of P and Q. Figure 4.1. shows a general flow chart of the composite program. The gain K cannot be varied from zero to infinity because the largest permissible number in Fortran is 10^{50}, however, gain K can be varied to a sufficiently large value to obtain the useful part of the root-locus.

3. Automatic Computation of Root-Locus Using Analog Computer (9). The points which lie on the root-locus must satisfy Equation (4.12). However, for values of **S** which are not on the root-locus the relationship may be expressed as

$$\angle G(\mathbf{S})H(\mathbf{S}) - (1 + 2i)\ 180^0 = E \qquad (4.19)$$

where angle E represents the error in satisfying the angle condition. The vector **S** can also be represented as

$$\mathbf{S} = S_o + \Delta\mathbf{S} \qquad (4.20)$$

where $\mathbf{S}_o$ is measured from the origin to one of the open-loop poles and ΔS is an increment measured from that open-loop pole.

The Equations (4.19) and (4.20) are used to determine the computer setup. A functional block diagram of the computer setup is shown in Figure 4.2. In order to make E go to zero, the computer setup must operate as a Type I system. By means of a potentiometer, the rate of increase $|\Delta S|$, and therefore, the speed of plotting the root-locus can be controlled. The real and the imaginary parts of the open-loop poles and zeros are inserted by means of the potentiometers. The sign of these terms is determined by the polarity of the 100 V source.

This method has been described in detail by Liethen (9). The system requires one more resolver than the total number of poles and zeros of the open-loop transfer function. The large number of resolvers required may be considered the major objection to this method.

Example: Consider a type 1 position control servomechanism whose open-loop transfer function is given by

$$G(p)H(p) = \frac{K_1}{p(1+T_m p)(1+T_f p)} \qquad (4.13)$$

where

$$H(p) = 1 \qquad (4.14)$$

$$T_m = 1 \qquad (4.15)$$

$$T_f = 0.2 \qquad (4.16)$$

The Equation (4.13) may be written as

$$G(p)H(p) = \frac{K}{p(p+1)(p+5)} \qquad (4.17)$$

where gain K is equal to $5K_1$.

The characteristic equation is obtained with the help of Equation (3.10) as

$$L(S) = S(S+1)(S-5) + K = 0 \qquad (4.18)$$

The root-locus plot is shown in Figure 4.3.

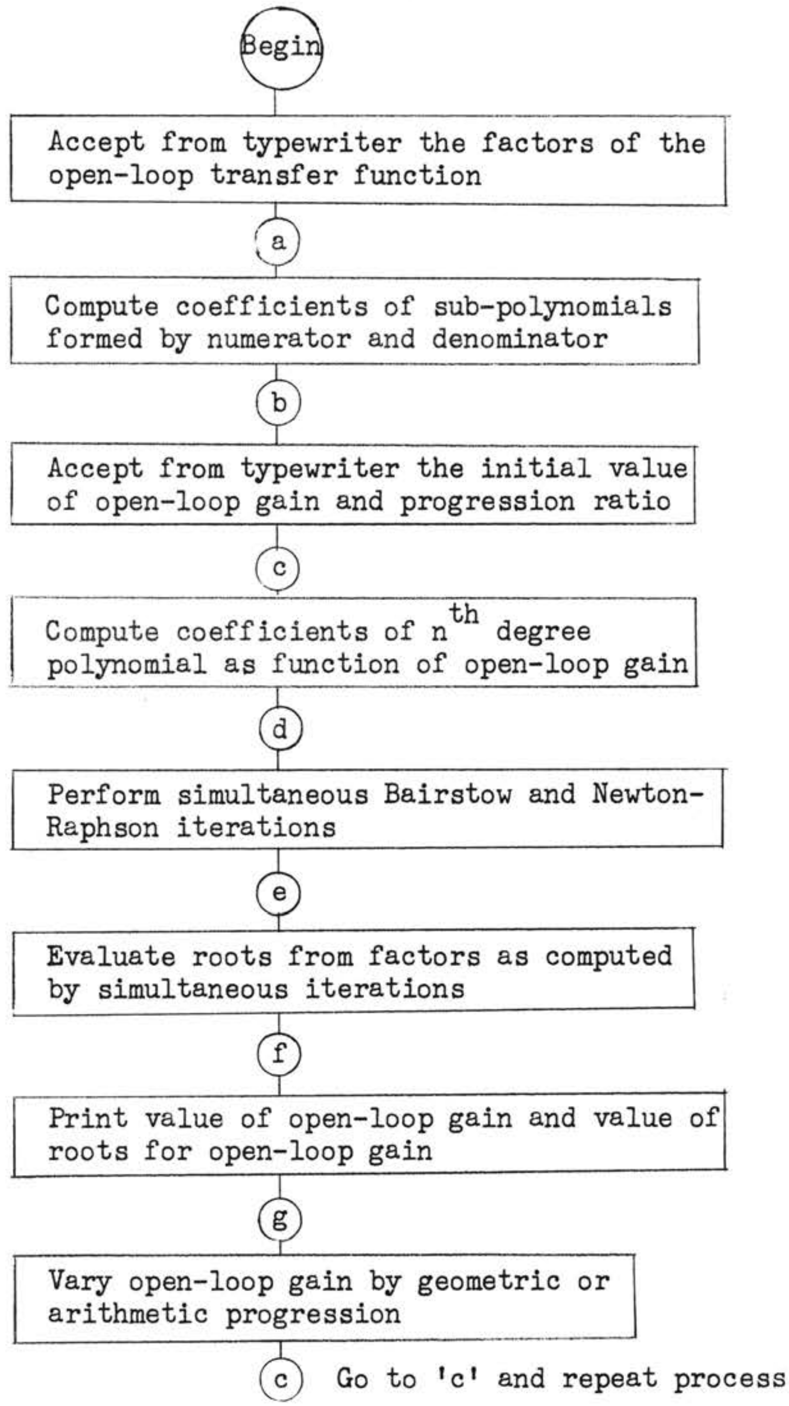


Figure 4.1. General Flow Chart of Composite Program.

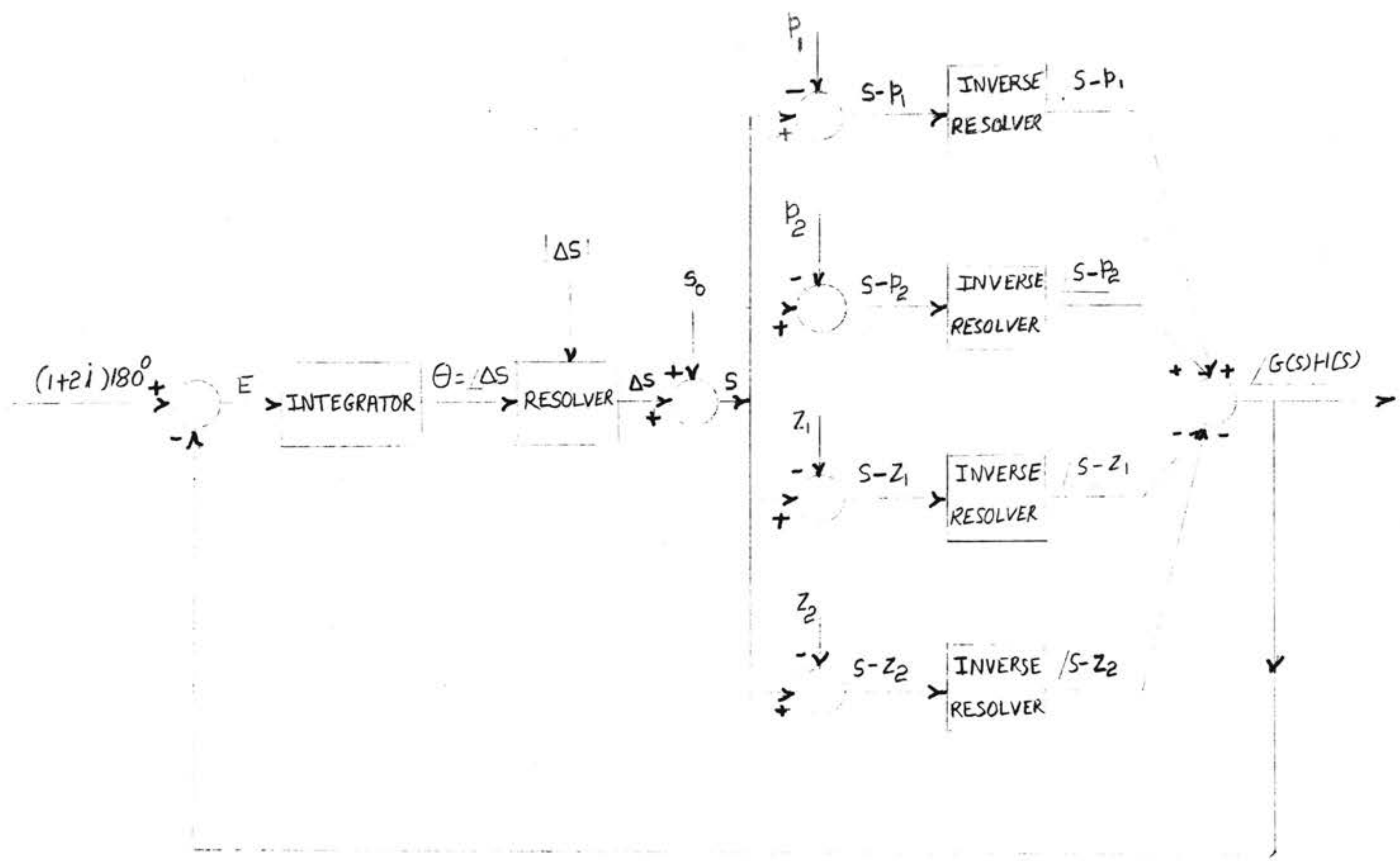


Figure 4.2. Functional Block Diagram of Computer Setup

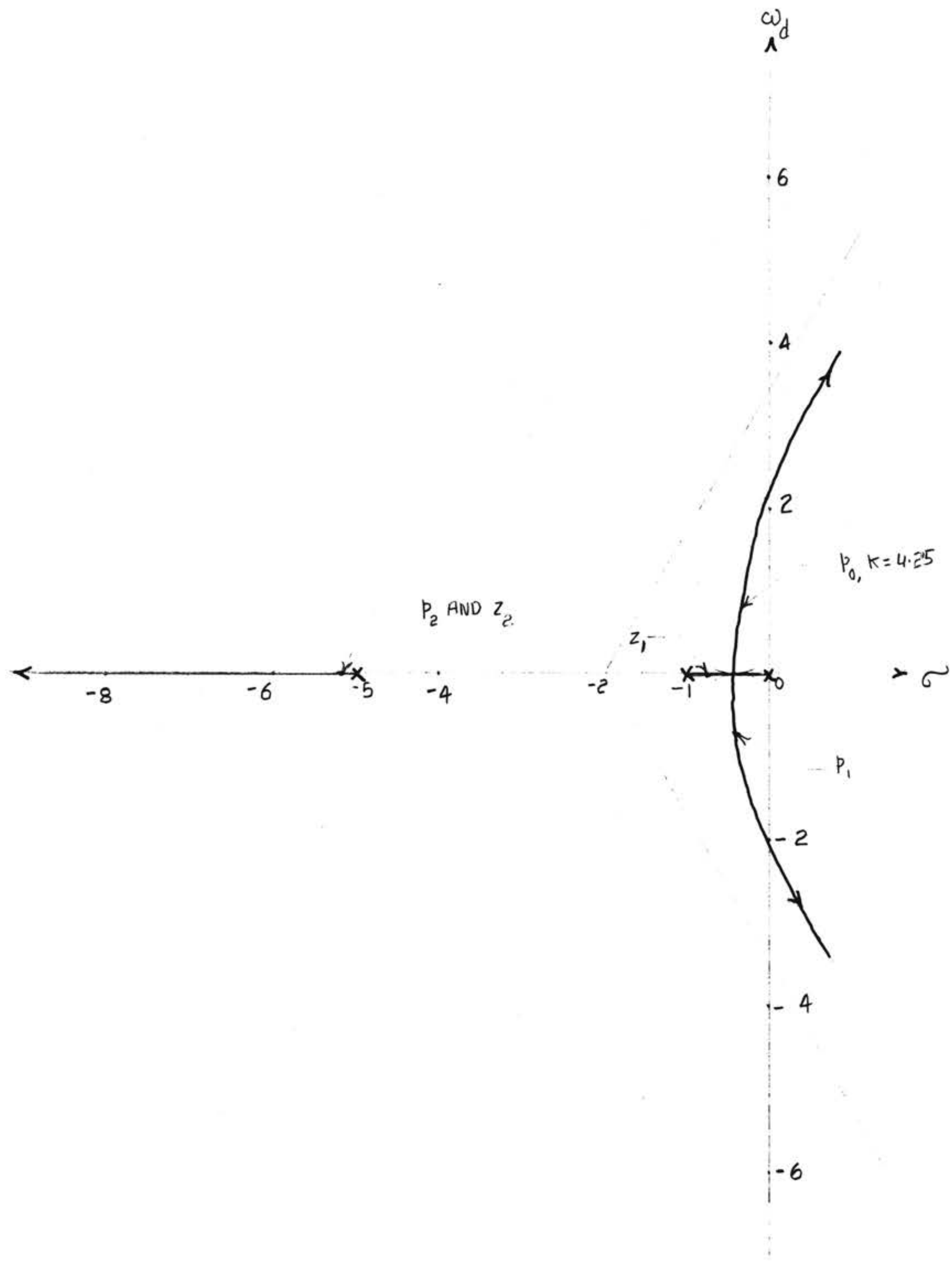


Figure 4.3. Root-Locus for Illustrative Problem.

4.3.2 Transient Response for Non-Zero Initial Conditions

Various methods are available to determine the transient response of a control system with non-zero initial conditions.

1. Analytical Method. Having determined the value of the gain K and the roots of the characteristic equation the transient response can be evaluated by finding out the coefficients of the expotential and power series term of $L^{-1}C(S)$ by the method of partial fraction expansion (3). The following example illustrates the method.

Illustrative example: The example of Section 4.3.1 is considered here to find out the transient response due to unit step input. The root-locus plot for this system is given in Figure 4.3. A value of gain K equal to 4.25 will be a fair choice. The roots of the characteristic equation can be obtained from the root-locus plot as

$$S_1 = -0.40 + j\,0.80 \tag{4.21}$$

$$S_2 = -0.40 - j\,0.80 \tag{4.22}$$

$$S_3 = -5.20 \tag{4.23}$$

The Equations (3.17) and (2.4) give the control ratio

$$\frac{c(t)}{r(t)} = \frac{K}{p(p+1)(p+5)+K}$$

which can be simplified as

$$\frac{c(t)}{r(5)} = \frac{4.25}{p^3+6p^2+5p+4.25} \tag{4.24}$$

The Laplace transform of Equation (4.24) is given by Equation (3.6) as

$$C(S) = \frac{4.25}{L(S)} R(S) + \frac{S^2+6S+5}{L(S)} c_o + \frac{S+6}{L(S)} pc_o + \frac{1}{L(S)} p^2 c_o \qquad (4.25)$$

where

$$\begin{aligned} L(S) &= S^3 + 6S^2 + 6S + 4.25 \\ &= (S-S_1)(S-S_2)(S-S_3) \\ &= (S+0.40 - j0.80)(S+0.40 + j0.80)(S+5.20) \qquad (4.26) \end{aligned}$$

As $R(S) = \frac{1}{S}$ for unit step input, substituting for R(S) and rearranging the terms Equation (4.25) reduces to

$$C(S) = \frac{c_o S^3 + (6\, c_o + pc_o)S^2 + (5\, c_o + 6pc_o + p^2 c_o)\, S + 4.25}{S(S + 0.40 - j0.80)(S + 0.40 + j0.80)(S + 5.20)} \qquad (4.27)$$

Using the partial fraction expansion method, Equation (4.27) can be written as

$$C(S) = \frac{K_1}{S} + \frac{K_2}{S + 0.40 - j0.80} + \frac{K_3}{S + 0.40 + j0.80} + \frac{K_4}{S + 5.20} \qquad (4.28)$$

where K_1, K_2, K_3 and K_4 are determined as

$$K_1 = \lim_{S \to 0} \left[S\, C(S) \right] = 1$$

$$K_2 = \lim_{S \to (-0.40+j0.80)} \left[(S + 0.40 - j0.80) C(S) \right] = \frac{1}{1.6j} K(-0.40+j0.80)$$

where $K(0.40+j0.80) = \lim_{S \to (-0.40+j0,80)} \left[(S+0.40-j0.80) C(S) \right]$

$K_3 = \frac{1}{1.6j} K(-0.40-j0.40)$ and is complex conjugate of K_2

$$K_4 = \lim_{S \to 5.20} \left[(S + 5.20)C(S)\right]$$

The response c(t) can be written as

$$c(t) = 1 + \frac{1}{1.6j} \left|K(-0.40 + j0.80)\right| e^{-0.4t} \mathrm{Sin}\,(0.8t + a) + K_4 e^{-5.2t} \qquad (4.29)$$

where $a = \tan^{-1}\left(\frac{0.80}{-0.40}\right)$

The method involves laborious numerical calculations. Also, the quantities of practical importance to us, namely peak overshoot, time at peak overshoot, delay time, rise time, settling time and number of oscillations up to settling time cannot be obtained directly unless a curve of c(t) versus t is plotted for the transient period.

2. Graphical Method. When the transient response due to a step input is to be evaluated, a graphical method based on the pole-zero configuration, usually employed (2) for systems with zero initial conditions, can be extended to determine peak overshoot c_p, time t_p to reach peak overshoot, settling time t_s and number of oscillations N up to settling time. If the system has a dominant complex pole $p_o = \sigma + j\omega_d$, then t_p, c_p, t_s and N are given by

$$t_p \simeq \frac{1}{\omega_d}\left[\frac{\pi}{2} - \begin{pmatrix}\text{Sum of angles from} \\ \text{zeros of } \frac{C(S)}{R(S)} \text{ to} \\ \text{dominant pole } p_o\end{pmatrix} + \begin{pmatrix}\text{Sum of angles of all other} \\ \text{poles of } \frac{C(S)}{R(S)} \text{ to dominant} \\ \text{pole } p_o\text{, including conjugate} \\ \text{pole}\end{pmatrix}\right] \qquad (4.30)$$

$$c_p = \frac{C(S)}{R(S)}\Big|_{S=0} + \frac{\begin{bmatrix}\text{Product of distances from all poles of } \frac{C(S)}{R(S)} \\ \text{to origin excluding distances of two domi-} \\ \text{nant poles from origin}\end{bmatrix}}{\begin{bmatrix}\text{Product of distances from all other poles} \\ \text{of } \frac{C(S)}{R(S)} \text{ to dominant pole } p_o\text{, excluding} \\ \text{distance between dominant poles}\end{bmatrix}} \times \frac{\begin{bmatrix}\text{Product of distances from all zeros of } \frac{C(S)}{R(S)} \\ \text{to dominant pole } p_o\end{bmatrix}}{\begin{bmatrix}\text{Product of distances from all zeros of} \\ \frac{C(S)}{R(S)} \text{ to origin}\end{bmatrix}} \times e^{\sigma t_p} \tag{4.31}$$

$$t_s \simeq \frac{4}{|\sigma|} \tag{4.32}$$

$$N = \frac{2\omega_d}{\pi|\sigma|} \tag{4.33}$$

Illustrative Example: The illustrative problem treated by analytical method is used to explain this method. Using Equation (4.27) the ratio $\frac{C(S)}{R(S)}$ can be expressed as

$$\frac{C(S)}{R(S)} = \frac{c_oS^3 + (6c_o+pc_o)S^2 + (5c_o+6pc_o+p^2c_o)S + 4.25}{(S + 0.40-j0.80)(S + 0.40+j0.80)(5 +5.20)} \tag{4.34}$$

For the sake of simplicity, c_o and p^2c_o are assumed zero and pc_o is assumed equal to unity.

The Equation (4.34) can be written as

$$\frac{C(S)}{R(S)} = \frac{pc_o\left[S(S + 6) + \frac{4.25}{pc_o}\right]}{(S+ 0.40-j0.80)(S+ 0.40+j0.80)(S + 5.20)}$$

Substituting for pc_o and simplifying,

$$\frac{C(S)}{R(S)} = \frac{(S + 5.18)(S + 0.82)}{(S + 0.40-j0.80)(S + 0.40+j0.80)(S + 5.20)} \tag{4.35}$$

Where $p_o = -0.40 + j0.80$ is a dominant complex pole.

The values of poles and zeros of $\frac{C(S)}{R(S)}$ are plotted in Figure 4.3. With the help of this figure and using Equations (4.30), (4.31), (4.32) and (4.33), the values of t_p, c_p, t_s and N may be determined as:

$$t_p = \frac{1}{0.80}\left[\frac{\pi}{2} - (9^{\circ}+65^{\circ}) + (90^{\circ}+9^{\circ})\right]\frac{\pi}{180} = 2.51 \text{ secs}$$

$$c_p = 1 + \frac{5.20 \times 4.85 \times 0.9}{4.86 \times 5.18 \times 0.82} e^{-0.4 \times 2.51} = 1.39$$

$$t_s = \frac{4}{0.40} = 10 \text{ secs}$$

$$N = \frac{2 \times 0.80}{\pi \times 0.40} = 1.275$$

These values are in agreement with those obtained by using an analog computer.

3. Using Analog or Digital Computer. The use of computers has supplanted much of the paper design study. This approach allows rapid and complete evaluation of the expected system performance. Also, if gain K is determined from other considerations, one can avoid drawing of a root-locus plot to determine the roots of the characteristic equation.

Various methods (7) for the numerical solutions of ordinary differential equations can be used to evaluate the transient response of a feedback control system with non-zero initial conditions using a digital computer (10). Methods generally used, namely the Euler's Method and the Runge-Kutta Method are self starting and require the knowledge of initial conditions only in order to start the solution. Other methods like Euler's Modified Method, Adam's Method, etc. are non-self-starting and depend on knowledge of the value of the function at two or more points.

These values can be determined using Taylor series expansion or by graphical means.

A feedback control system can be simulated and its transient response can be automatically plotted using an electronic analog computer (11). The relative ease with which systems can be simulated allows analysis and design calculations which would be otherwise practically impossible. The initial conditions can be varied by varying the initial voltage input to an integrator. Similarly, the gain K can be varied by using a potentiometer. Thus a family of curves can be easily plotted for various initial conditions and gain K.

Illustrative Example: Problem of Section 4.3.1 is considered here to show the relative ease with which this problem can be solved using an analog computer.

From Equations (4.14) and (4.17)

$$G(p) = \frac{c(t)}{e(t)} = \frac{K}{p(p+1)(p+5)}$$

and

$$H(p) = 1$$

$$\left[p^3+6p^2+5p \right] \; c(t) = K \; e(t) \tag{4.36}$$

and $$e(t) = r(t) - c(t) \tag{4.37}$$

Equations (4.36) and (4.37) are used to determined the computer setup.

Figure 4.4 shows computer setup for a step input r(t) = 100 units. Amplitude and time scaling is employed to prevent any of the amplifiers from being overloaded. Figure 4.5 shows the computer setups for obtaining ramp, parabolic and sinusoidal input signals. Amplitude scaling of the computer setup shown in Figure 4.4 may be modified for a ramp, parabolic or sinusoidal input. Results obtained for value of gain K equal to 4.25 and different initial conditions for a step, ramp, parabolic and sinsusoidal input are shown in Figures 4.8 through 4.17. Plots for c(t) when system is excited only by the initial conditions are given in Figures 4.6 and 4.7.

Tables 4.1 through 4.3 give values of peak overshoot c_p, peak time t_p, delay time t_d, rise time t_r, settling time t_s and number of oscillations N for different values of c_o, pc_o and p^2c_o respectively for the case of step input signal.

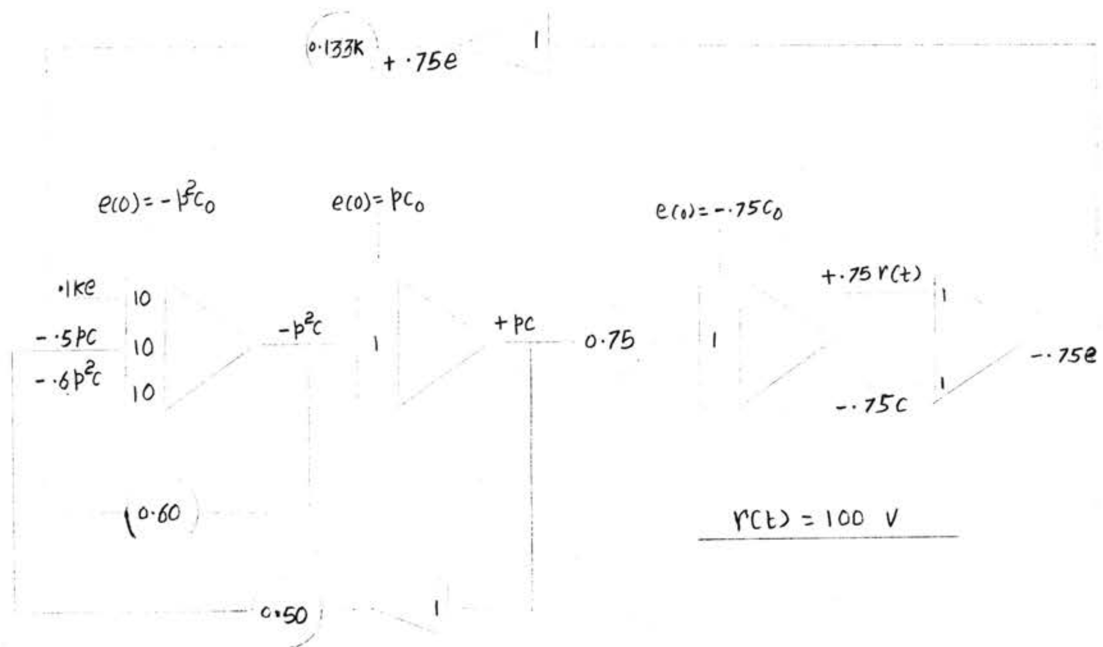


Figure 4.4. Computer Setup for Equations (3.36) And (3.37).

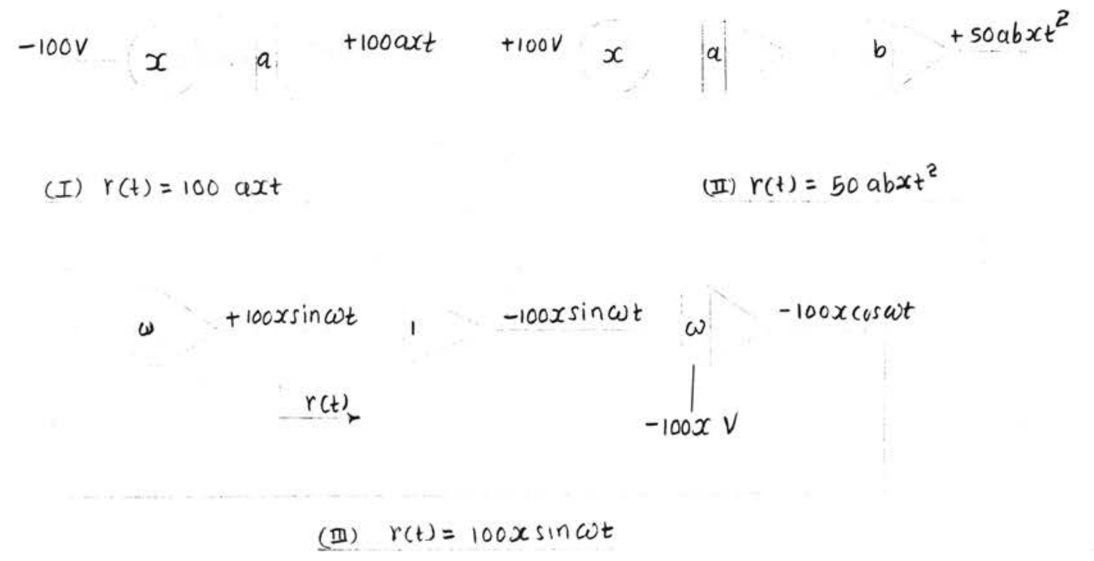


Figure 4.5. Computer Setups for Ramp, Parabolic, and Sinusoidal Signals.

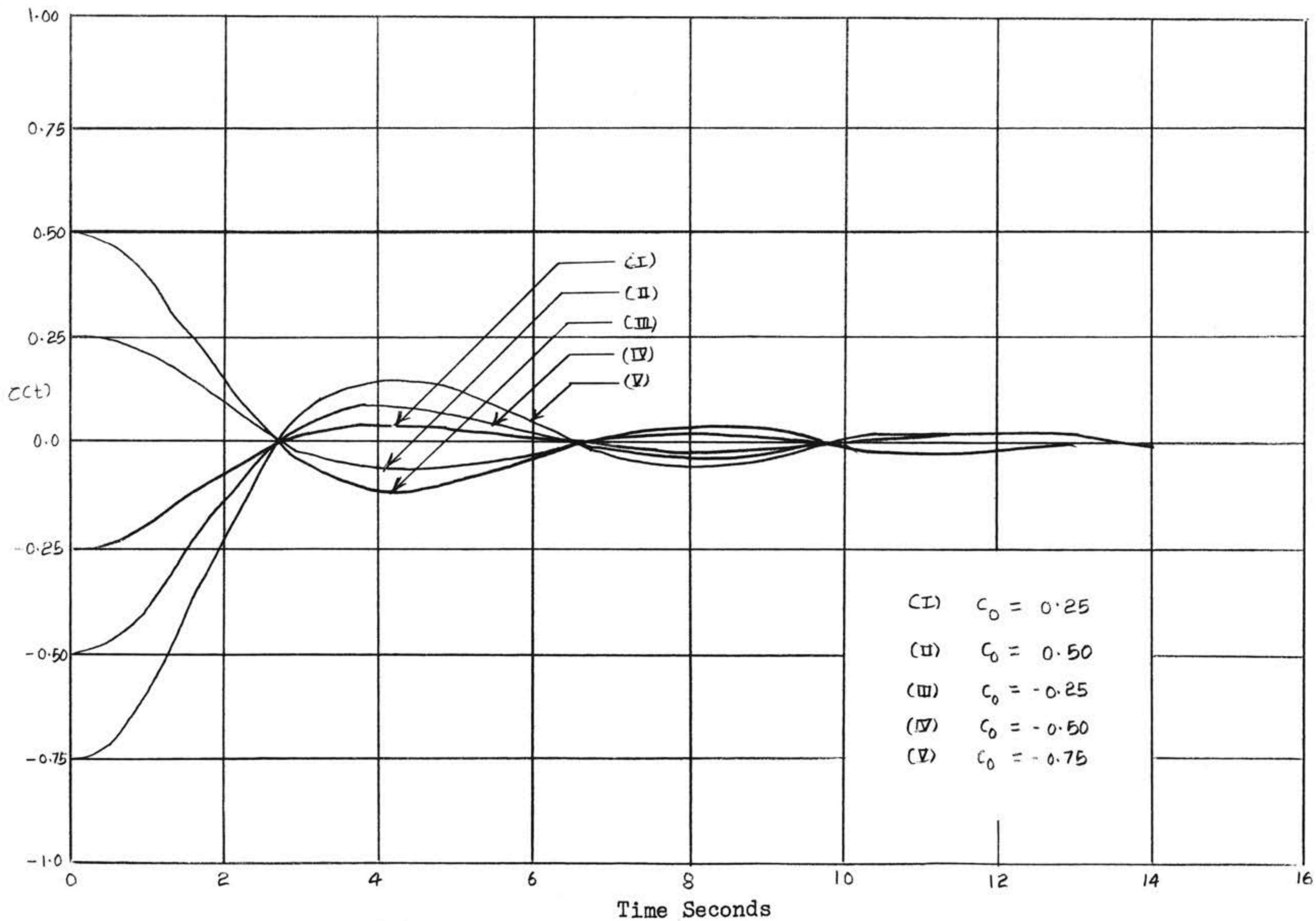


Figure 4.6. Response Due to c_o Alone $\left[r(t) = 0\right]$

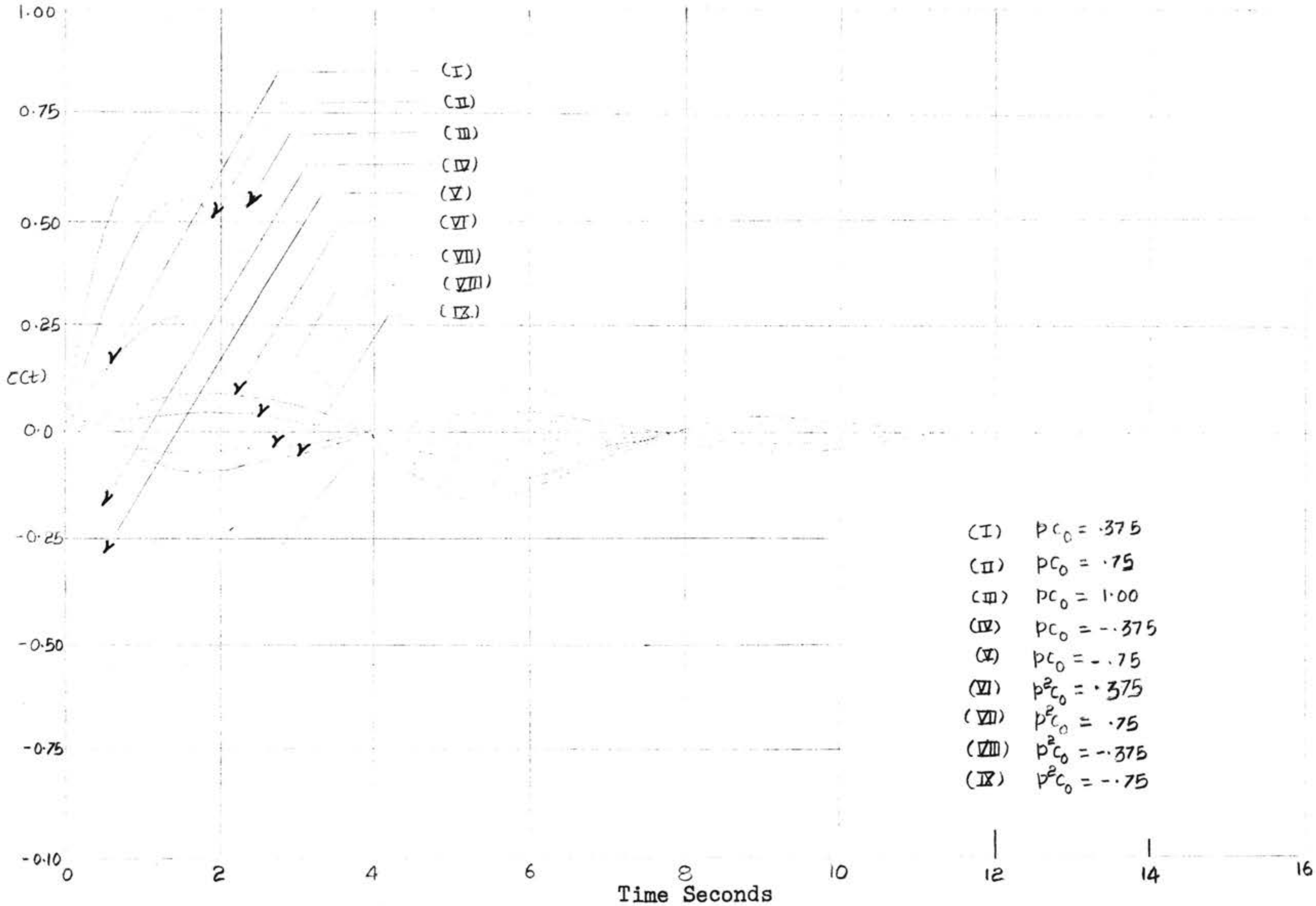


Figure 4.7. Responses Due to pc_o And p^2c_o $\left[r(t) = 0.0\right]$

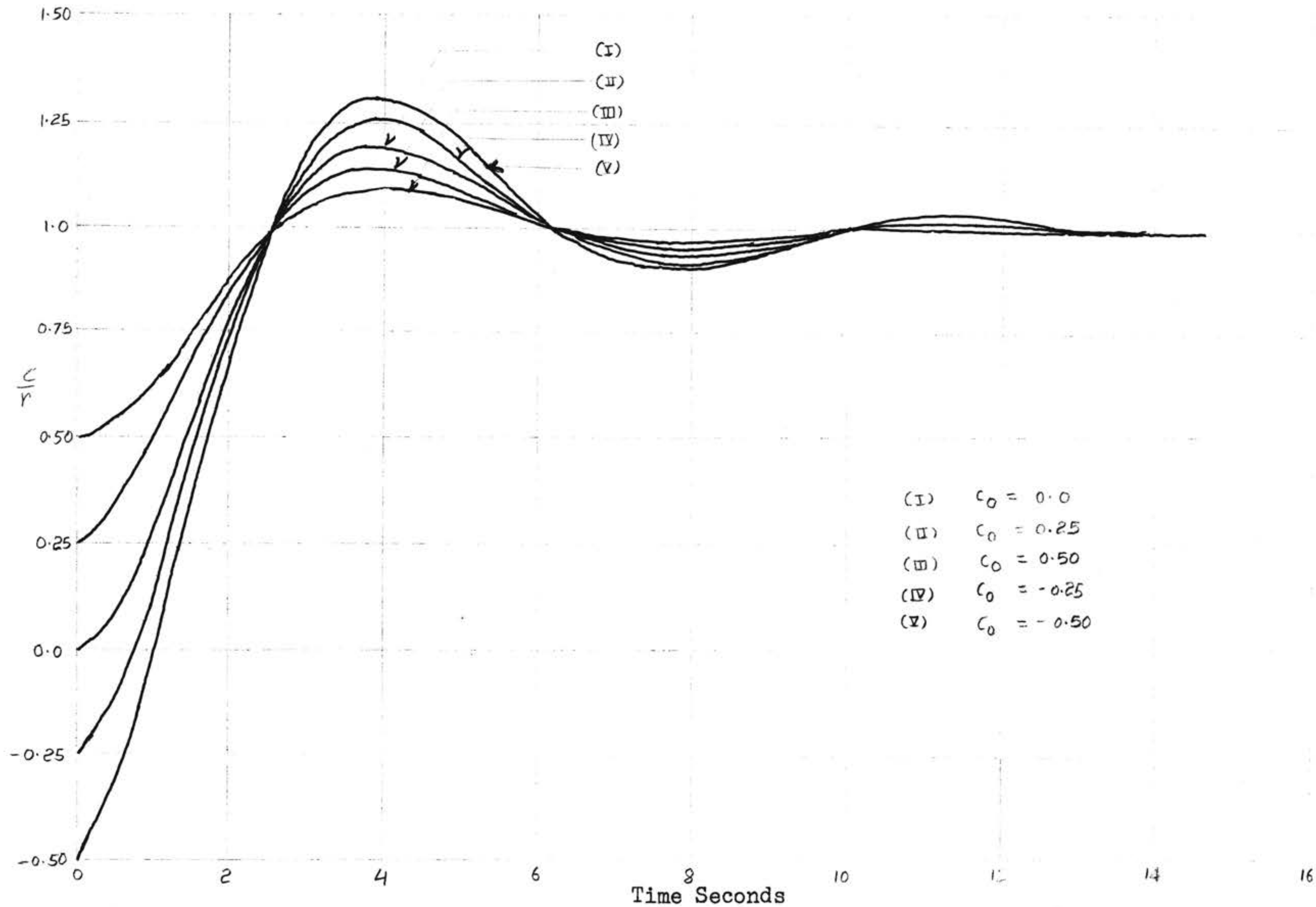


Figure 4.8. Effect of c_o -(Unit Step Input)

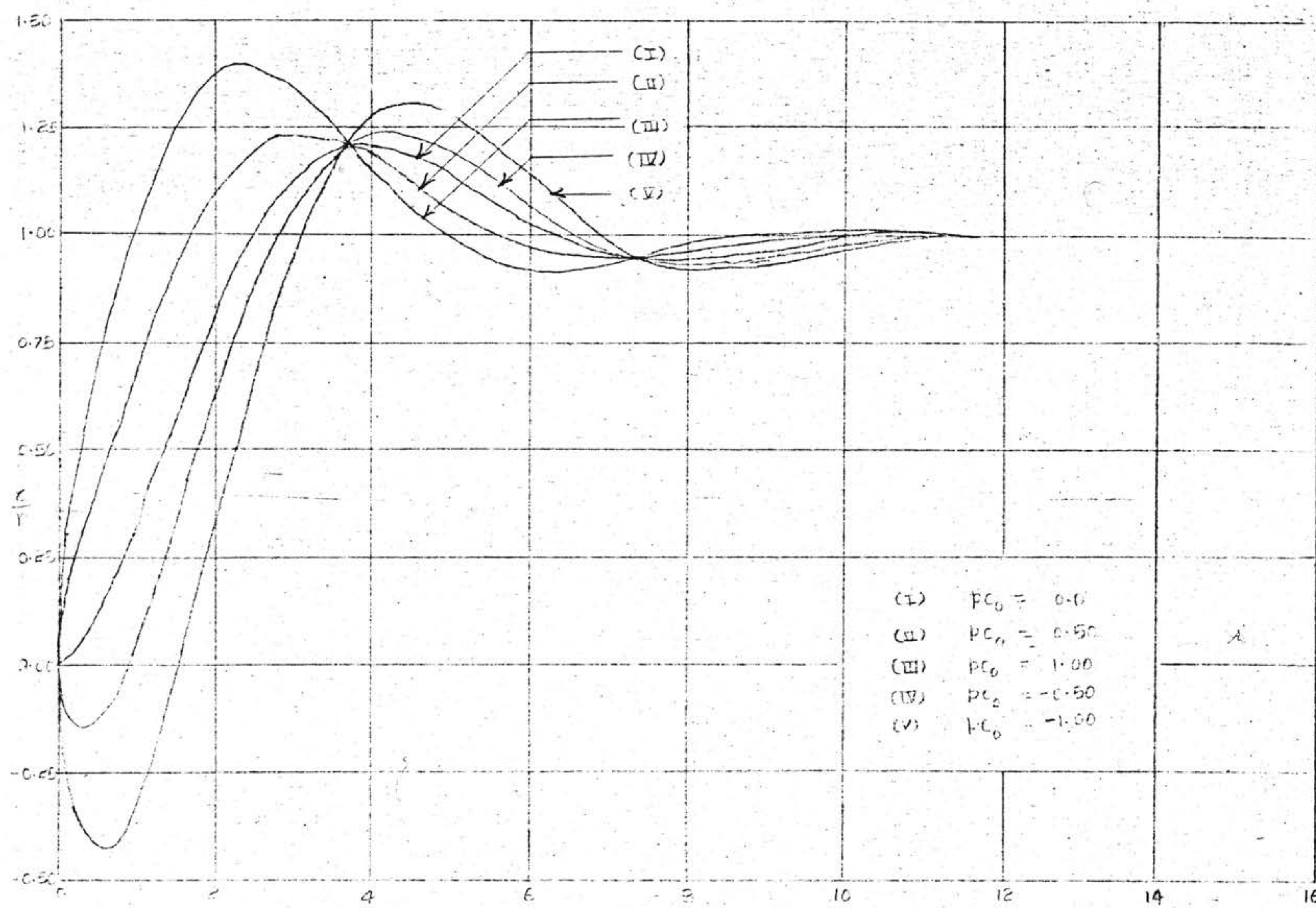


Figure 4.9. Effect of pc_o (Unit Step Input)

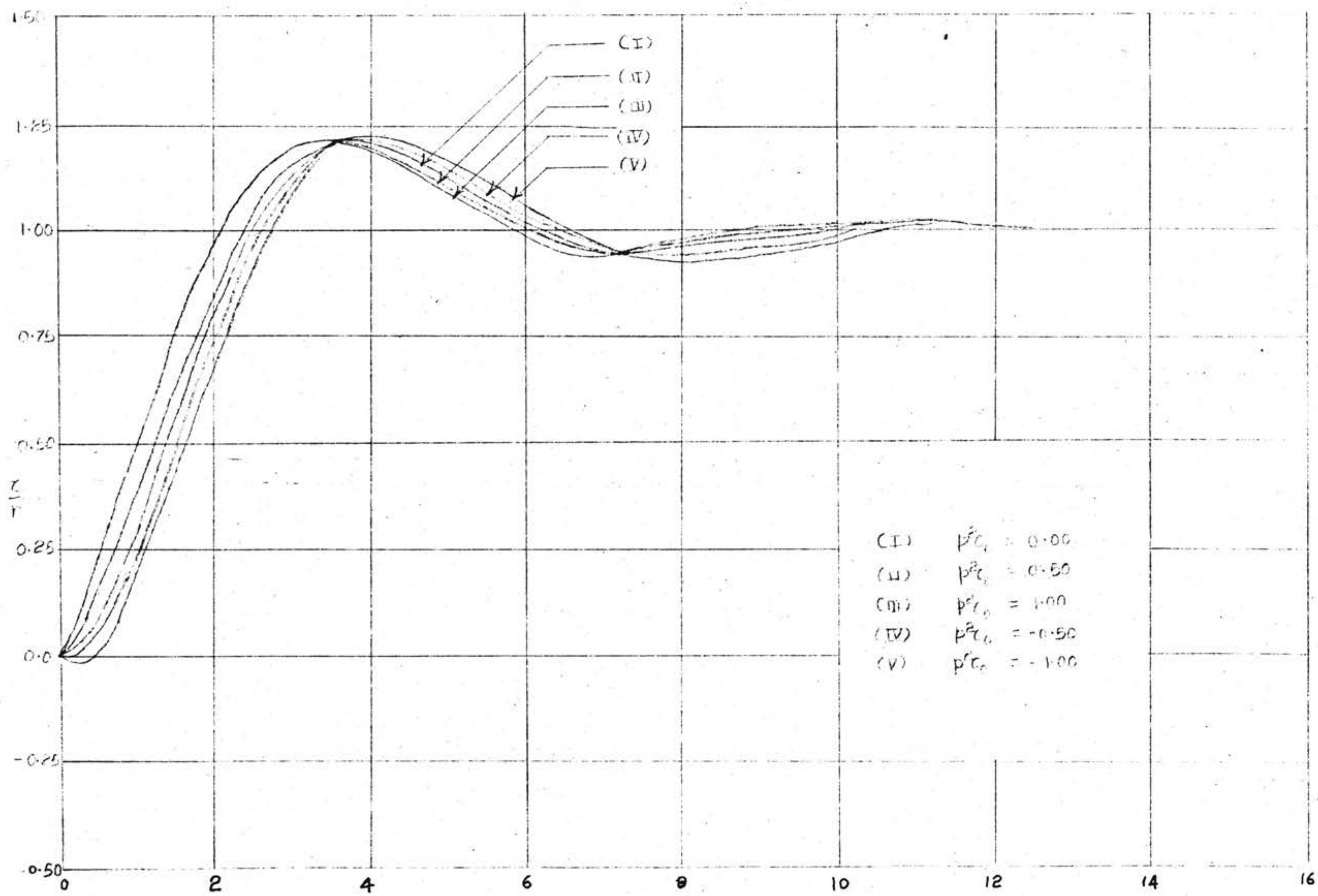


Figure 4.10. Effect of p^2c_0 (Unit Step Input)

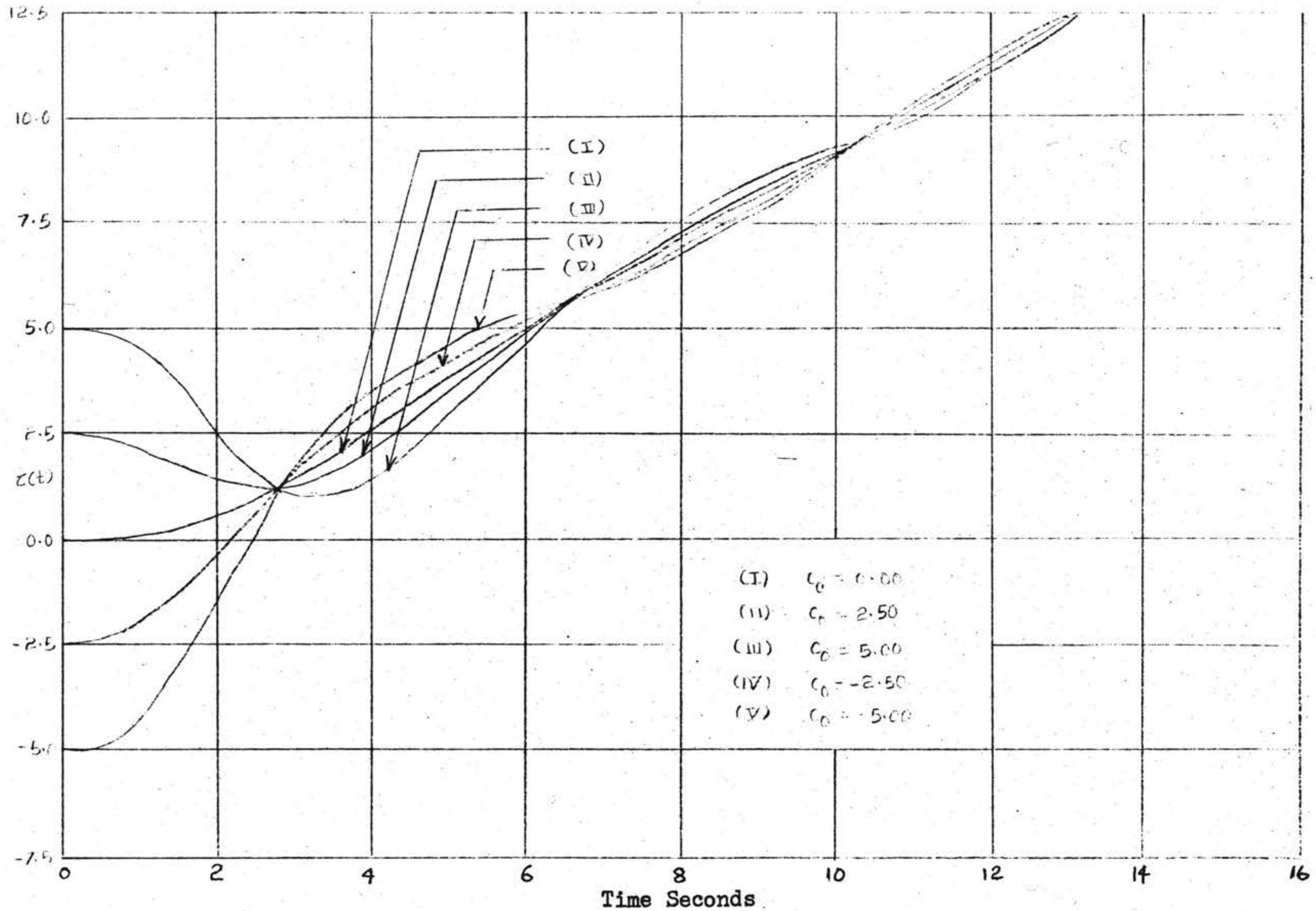


Figure 4.11. Effect of c_o (Unit Ramp Input)

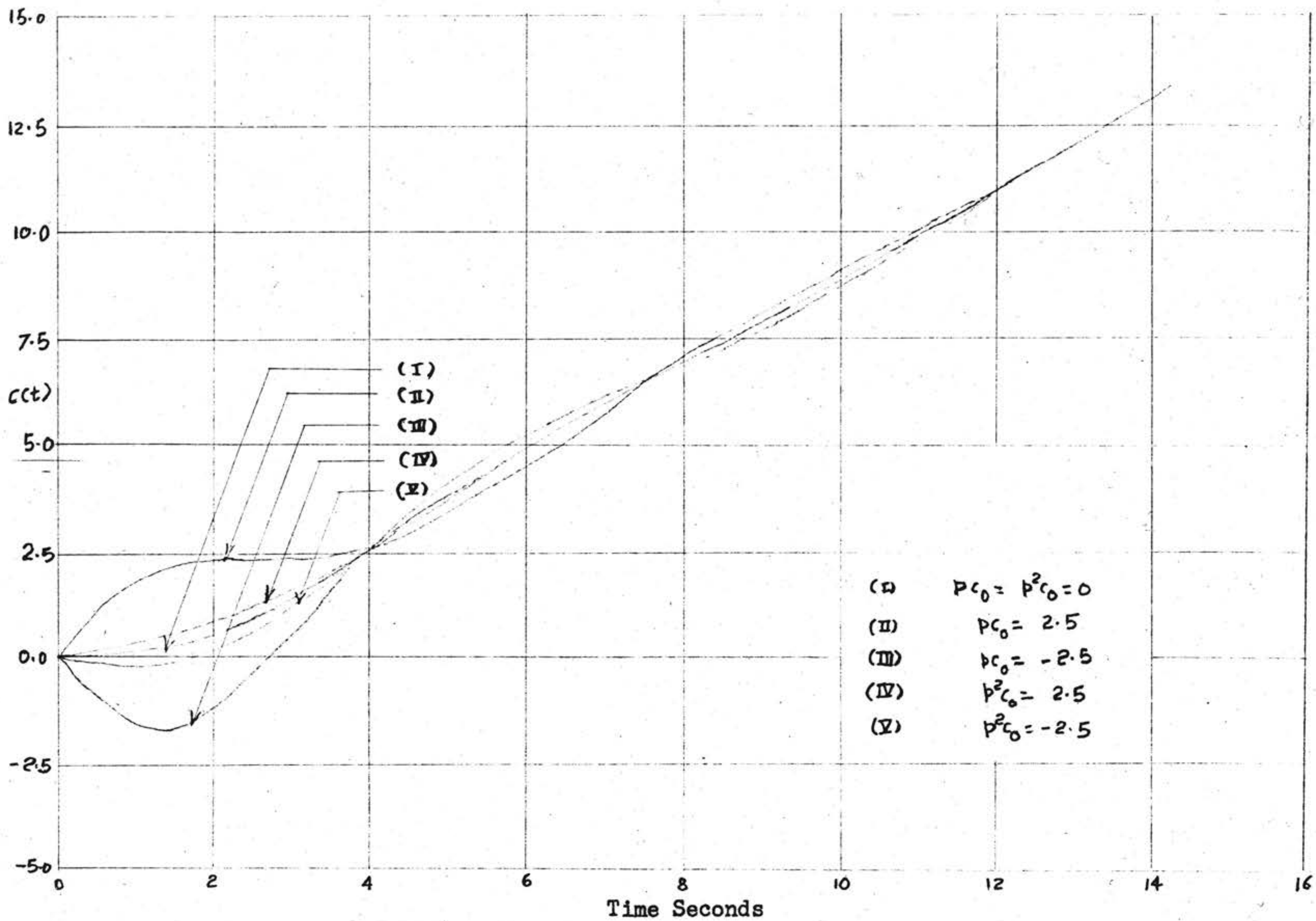


Figure 4.12. Effects of pc_o and p^2c_o (Unit Ramp Input)

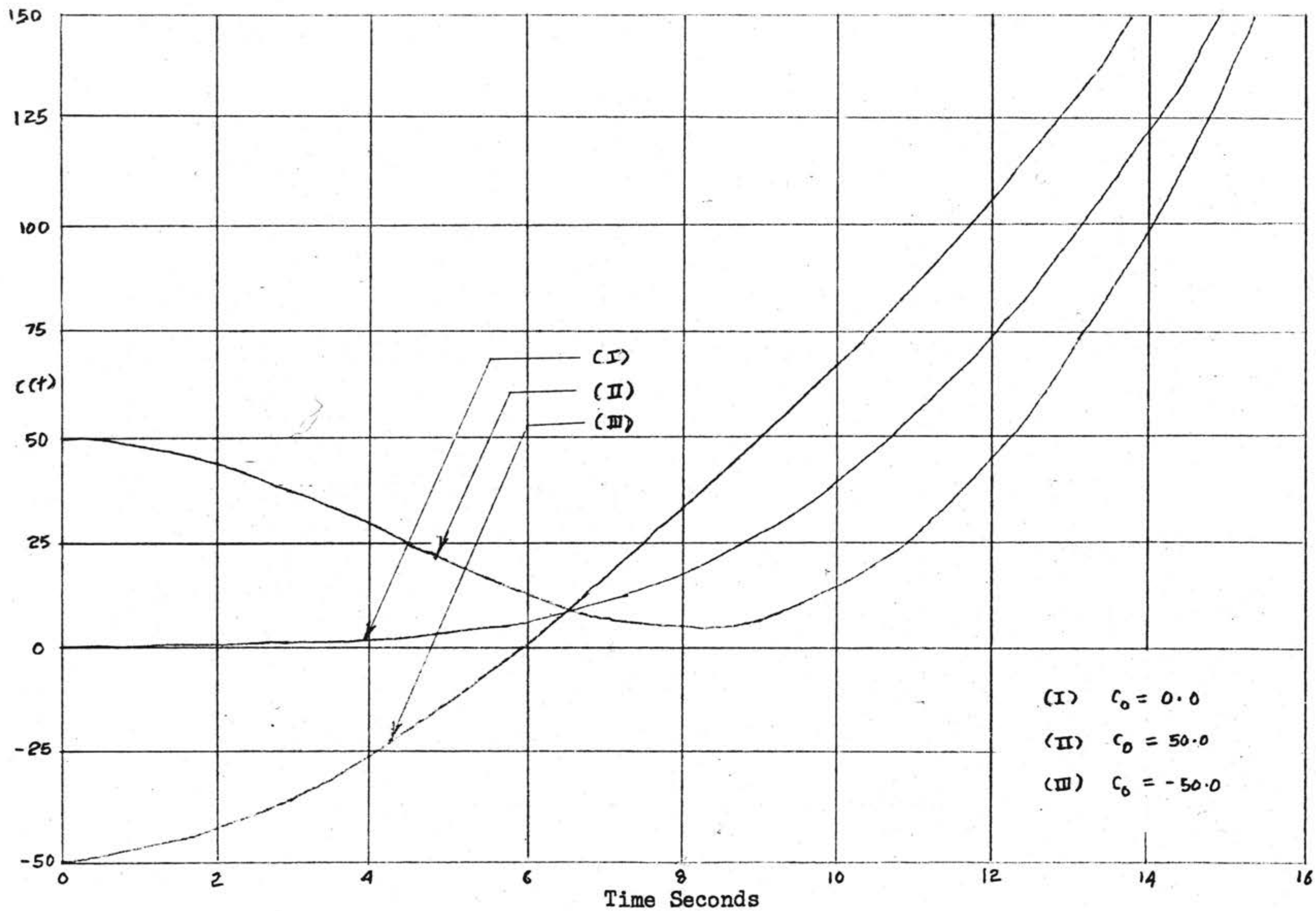


Figure 4.13. Effect of c_o (Unit Parabolic Input)

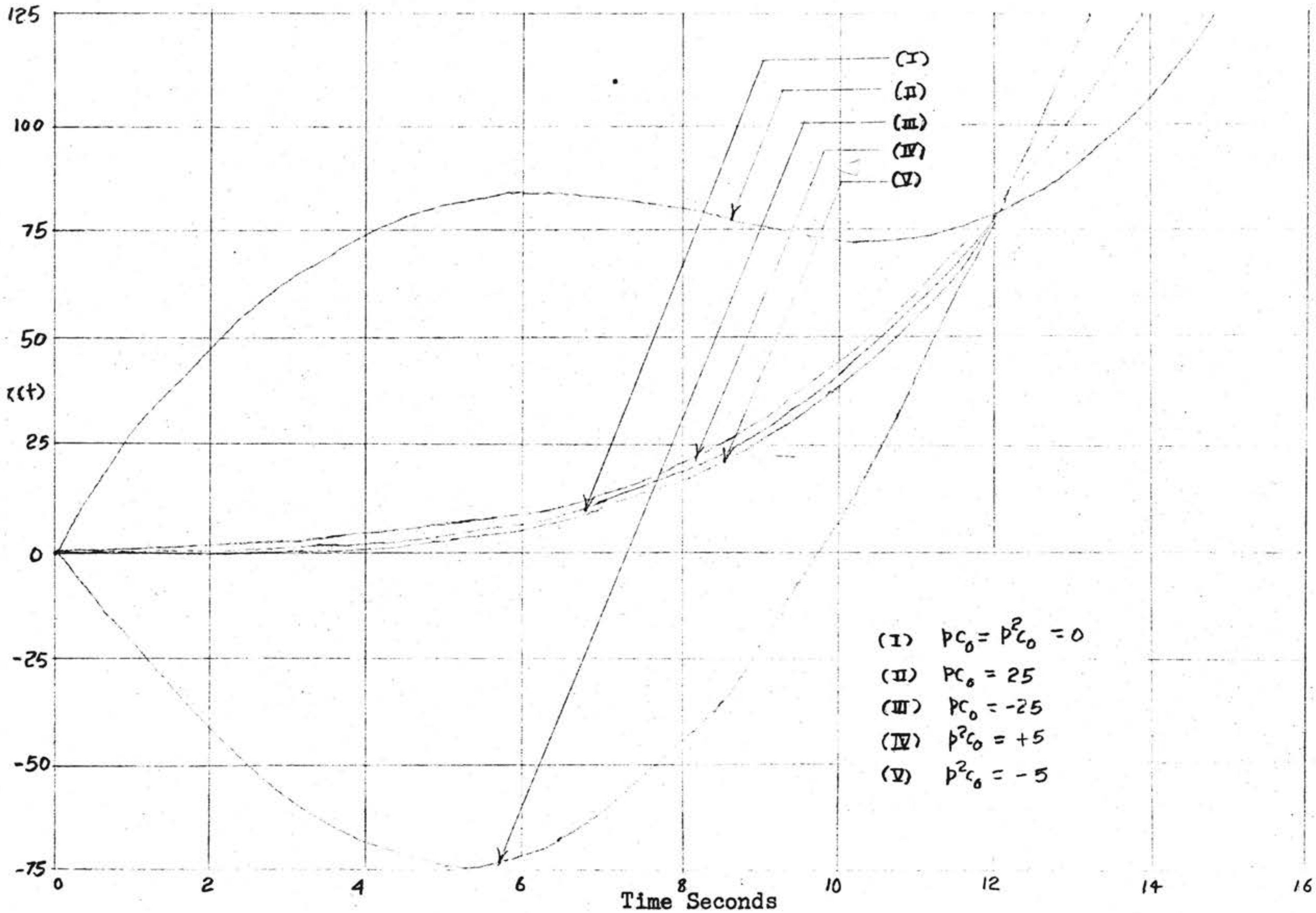


Figure 4.14. Effects of pc_o and p^2c_o (Unit Parabolic Input)

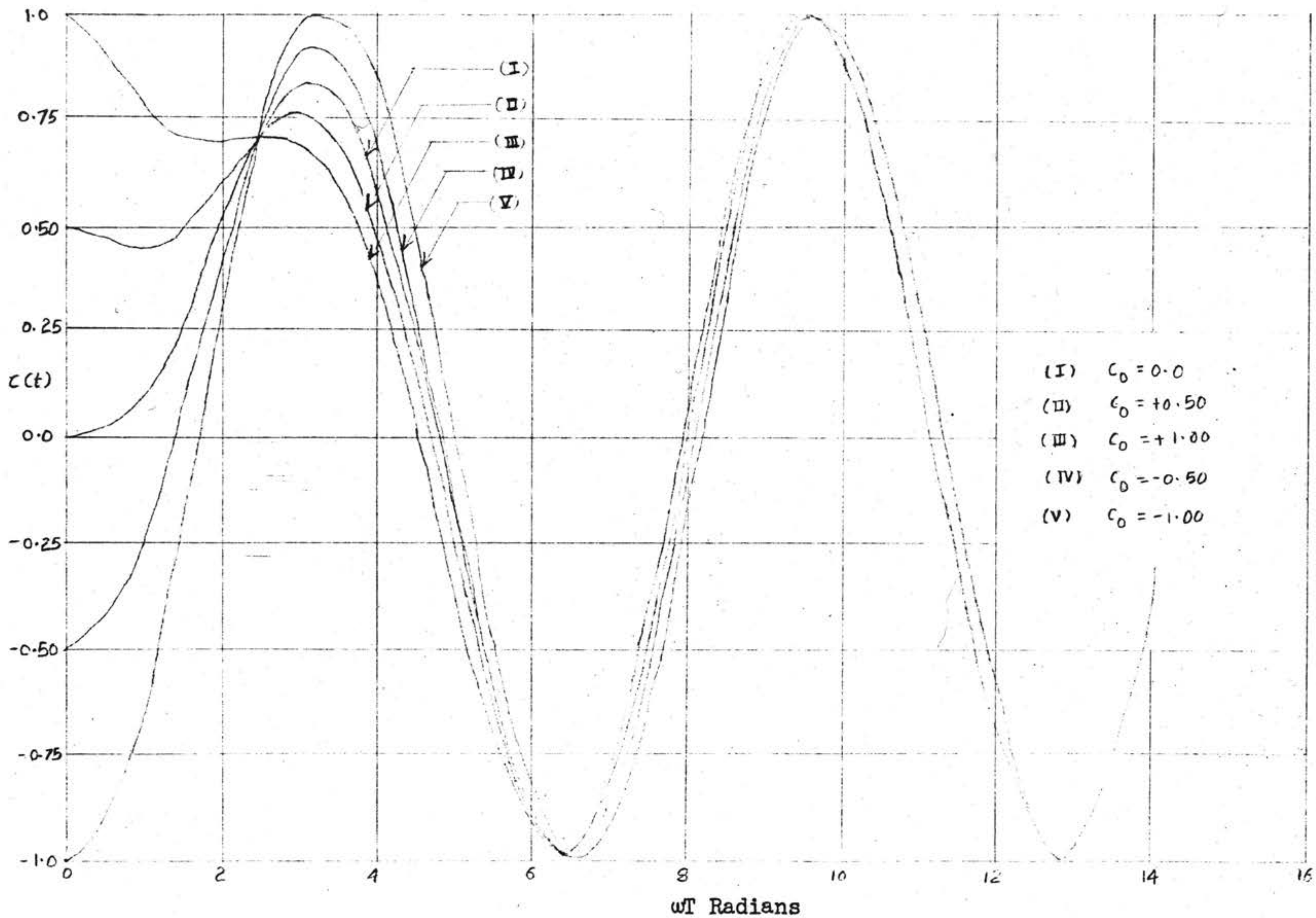
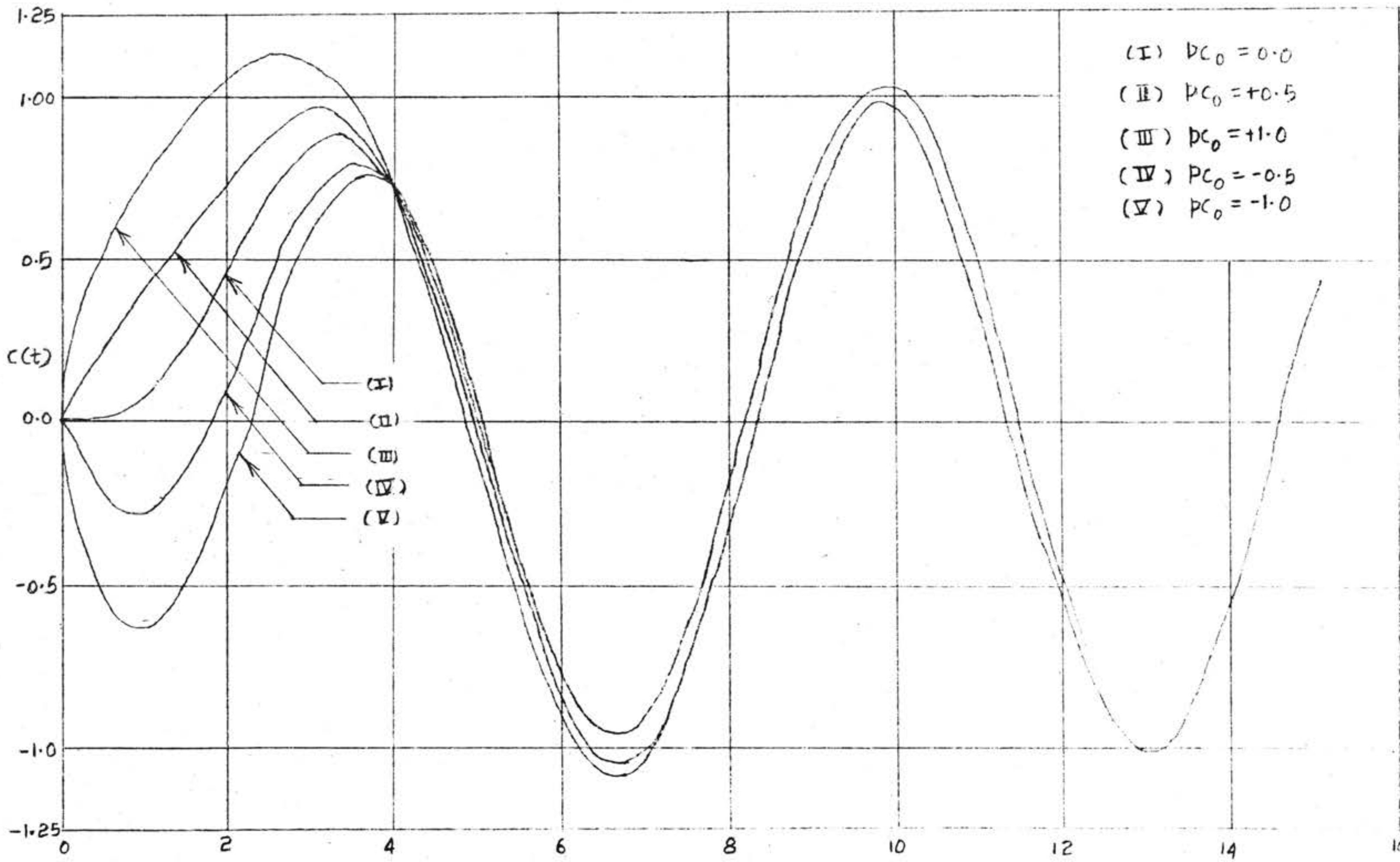


Figure 4.15 Effect of c_o (Unit Sinusoidal Input)



ωT Radians

Figure 4.16. Effect of pc_o (Unit Sinusoidal Input)

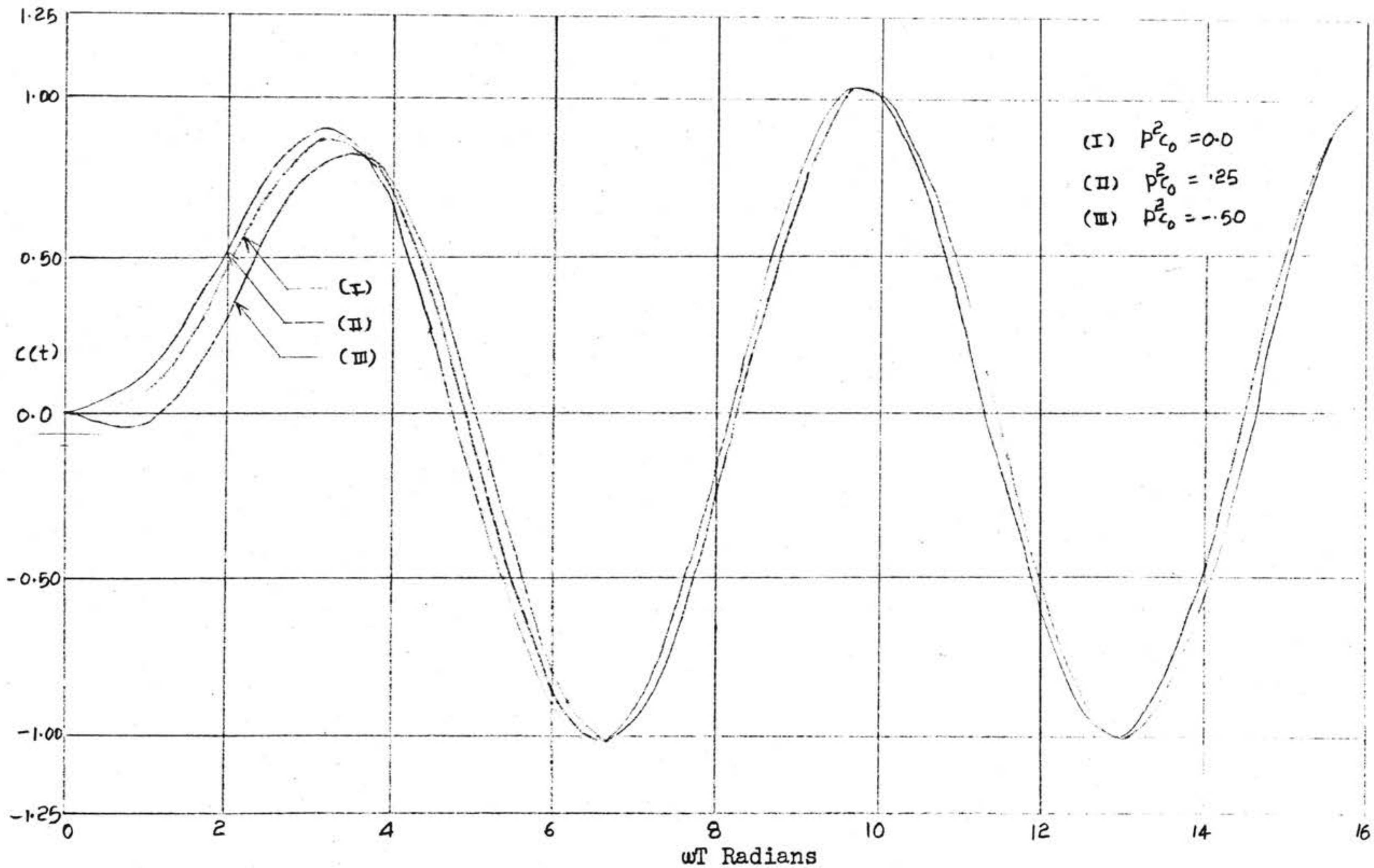


Figure 4.17. Effect of p^2c_o (Unit Sinusoidal Input)

TABLE 4.1

COMPARISON OF TRANSIENT RESPONSE (UNIT STEP INPUT) WHEN THE c_o VARIES ($pc_o = p^2c_o = 0$)

S.No	c_o	c_p	% overshoot	t_p secs.	t_d secs.	t_r secs.	t_s secs.	N
1	0	1.20	20.0	3.80	1.42	1.60	9.80	1.295
2	±.25	1.14	14.3	3.80	0.95	-	9.50	1.255
3	±.50	1.10	10.0	3.80	0.0	-	9.20	1.215
4	-.25	1.25	25.0	3.80	1.80	1.26	9.80	1.295
5	-.50	1.31	31.0	3.80	1.95	1.10	9.80	1.295

TABLE 4.2

COMPARISON OF TRANSIENT RESPONSE (UNIT STEP INPUT) WHEN THE pc_o VARIES ($c_o = p^2c_o = 0$)

S.No	c_o	c_p	% overshoot	t_p secs.	t_d secs.	t_r secs.	t_s secs.	N
1	0	1.20	20.0	3.80	1.42	1.60	9.80	1.295
2	0.5	1.24	24.0	3.00	0.85	1.26	8.80	1.260
3	1.0	1.38	38.0	2.40	0.31	0.80	8.20	1.185
4	-0.5	1.22	22.0	4.18	1.90	1.50	9.90	1.305
5	-1.0	1.30	30.0	4.58	2.35	1.20	10.20	1.345

TABLE 4.3

COMPARISON OF TRANSIENT RESPONSE (UNIT STEP INPUT)
WHEN THE p^2c_o VARIES ($c_o = pc_o = 0$)

S.No	p^2c_o	c_p	% overshoot	t_p secs.	t_d secs.	t_r secs.	t_s secs.	N
1	0	1.20	20	3.80	1.42	1.60	9.80	1.295
2	0.50	1.20	20	3.75	1.26	1.65	9.60	1.270
3	1.00	1.21	21	3.75	1.05	1.27	9.20	1.215
4	-0.50	1.20	20	3.95	1.55	1.70	9.90	1.305
5	-1.00	1.21	21	4.21	1.65	1.70	10.00	1.32

CHAPTER V

FREQUENCY RESPONSE ANALYSIS

5.1. Introduction

The conventional frequency response methods (2) used to analyze the feedback control system with zero initial conditions can be extended to analyze the control systems with non-zero initial conditions. The method is based on the assumption that the transient behavior of a higher order system for which the maximum closed-loop frequency response $\frac{c(j\omega)}{r(j\omega)}$ is greater than one, may be approximated by a second-order system.

$$\frac{c(t)}{r(t)} = \frac{\omega_n^2}{p^2+2\xi\omega_n p + \omega_n^2} \tag{5.1}$$

where damping ratio ξ and natural frequency ω_n are given by the equations

$$M_m = \frac{c(j\omega)}{r(j\omega)} = \frac{1}{2\xi\sqrt{1-\xi^2}}, \qquad M_m \geqq 1 \tag{5.2}$$

$$\omega_m = \omega_n\sqrt{1-2\xi^2}, \qquad M_m \geqq 1 \tag{5.3}$$

ω_m is the frequency at which the maximum value M_m occurs. This assumption is practical since the roots of the characteristic equation which are located nearest the imaginary axis have a predominant effect upon the transient behavior. The systems for which M_m is less than one, the transient behavior can be approximated by a first order system

$$\frac{c(t)}{r(t)} = \frac{1}{1 + Tp} \tag{5.4}$$

where time constant $T = \frac{1}{\omega}$, ω being the frequency at which the magnitude of the forward transfer function $G(j\omega)$ of the system is unity.

5.2. From Frequency Response to Transient Response

It has been already shown that the effects of initial conditions 'die out' once the steady state is reached. Hence by replacing p for $j\omega$ and applying the conventional methods (2) the values of the maximum closed-loop frequency response M_m and the frequency ω_m at which it occurs can be determined. If M_m is found to be greater than or equal to unity, the system will be approximated by the second-order system represented by Equation (5.1). If M_m is less than unity, it will be approximated by the first-order system of Equation (5.4).

Case I. $M_m \geq 1$. The values of ξ and ω_n can be determined by using Equations (5.2) and (5.3). ξ can more easily be determined if a plot of M_m against ξ is available (2). It is evident from the Equation (5.2) that the value of ξ for this case must lie in the range of 0 to 0.707.

For the purpose of analysis it will be assumed that the initial conditions of the approximated Equation (5.1) are identical to those of the system under consideration. It will be possible only to explore the effects of c_o and pc_o, but it has been already indicated in Section 4.2 that the effect of $p^n c_o$ decreases as n increases. The results of Section 4.3.2 also show that the effect of $p^2 c_o$ is negligible.

The block diagram representation of Equation (5.1) which includes the initial conditions is given in Figure 3.2. From this block diagram or by taking Laplace transform of Equation (5.1), C(S) can be expressed as

$$C(S) = \frac{\omega_n^2}{L(S)} R(S) + \frac{S + 2\xi\omega_n}{L(S)} c_o + \frac{1}{L(S)} pc_o \qquad (5.5)$$

where
$$L(S) = S^2 + 2\xi\omega_n S + \omega_n^2 \qquad (5.6)$$

For unit step input r(t),

$$R(S) = \frac{1}{S}$$

Therefore, substituting in Equation (5.5)

$$C(S) = \frac{(S^2 + 2\xi\omega_n S)\, c_o + S\, pc_o + \omega_n^2}{S\, L(S)} \qquad (5.7)$$

Taking Laplace inverse transform (12), the response c(t) is given by

$$c(t) = 1 + Me^{-\omega_n t} \quad \text{Sin}\, (\omega_n \sqrt{1-\xi^2}\, t + \varphi) \qquad (5.8)$$

where

$$M = \left[\frac{(1-a\xi\omega_n - b\omega_n^2 + 2b\xi^2\omega_n^2)^2 + \omega_n^2(1-\xi^2)(a-2b\xi\omega_n)^2}{1 - \xi^2}\right]^{\frac{1}{2}} \qquad (5.9)$$

$$a = \frac{2\xi\omega_n c_o + pc_o}{\omega_n^2} \qquad (5.10)$$

$$b = \frac{c_o}{\omega_n^2} \qquad (5.11)$$

$$\varphi = \tan^{-1}\left[\frac{\omega_n\sqrt{1-\xi^2}\ (a-2b\xi\omega_n)}{b\ \omega_n(2\xi^2-1) + 1-a\xi\omega_n}\right] - \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{-\xi}\right] \tag{5.12}$$

Equations (5.8) through (5.12) give the transient response of the system under consideration.

Case II. $M_m < 1$. The value of t in Equation (5.4) can be determined from the Bode plot representation of $G(j\omega)$ by taking the frequency ω at which the magnitude of $G(j\omega)$ is unity.

The Laplace transform of Equation (5.4) gives

$$C(S) = \frac{1}{1 + TS}\quad R(S) + \frac{c_oT}{1 + TS} \tag{5.13}$$

For unit step input r(t),

$$C(S) = \frac{1 + c_oTS}{S(1 + TS)} \tag{5.14}$$

Inverse transform of Equation (5.14) gives

$$c(t) = 1 - (1-c_o)e^{-t/T} \tag{5.15}$$

In this case it will be possible only to explore the effects of c_o. But it can be shown that the effect of p^nc_o where $n > 0$ will be negligible for this type of systems.

5.3. Mathematical Analysis of the Effects of Initial Conditions

The results obtained in Section 4.3.2 can now be explained mathematically.

Case I. $M_m \geq 1$. The maximum and minimum values of c(t) and the time at which they occur can be obtained by equating $\frac{d\,c(t)}{dt}$ equal to zero and substituting the value of t thus obtained in Equation (5.8).

Equating $\frac{d\,c(t)}{dt}$ to zero gives

$$-\xi\omega_n \operatorname{Sin}(\omega_n\sqrt{1-\xi^2}t + \varphi) + \omega_n\sqrt{1-\xi^2}\cos(\omega_n\sqrt{1-\xi^2}t + \varphi) = 0$$

$$\therefore \tan(\omega_n\sqrt{1-\xi^2}t + \varphi) = \frac{\sqrt{1-\xi^2}}{\xi}$$

Substituing for φ from Equation (5.12) and simplifying

$$t_m = \frac{n\pi}{\omega} - \frac{1}{\omega}\tan^{-1}\left[\frac{\omega_n\sqrt{1-\xi^2}\,(a-2b\xi\omega_n)}{b\,\omega_n\,(2\xi^2-1) + 1-a\xi\omega_n}\right] \tag{5.16}$$

where t_m indicates time for maxima or minima,

$$\omega = \omega_n\sqrt{1-\xi^2} \quad \text{and } n = 0, 1, 2, \ldots.$$

Substituting $t = t_m$ in Equation (5.8)

$$c_m = 1 + Me^{(-\xi\omega_n t_m)} \operatorname{Sin}(\omega_n\sqrt{1-\xi^2}t_m + \varphi) \tag{5.17}$$

where it may be noted that the peak overshoot c_p corresponds to value of t_m when $n = 1$.

Effect of c_o. $(p^n c_o = 0$ where $n > 0)$

Substituting for a and b from Equations (5.10) and (5.11), the Equations (5.9) and (5.16) may be simplified to

$$M = \frac{(1-c_o)}{\sqrt{1-\xi^2}} \tag{5.18}$$

$$t_m = \frac{n\pi}{\omega} \tag{5.19}$$

Substituting for M and t_m in Equations (5.8) and (5.17)

$$c(t) = 1 + \frac{(1-c_o)}{\sqrt{1-\xi^2}} e^{-\omega_n t} \operatorname{Sin}(\omega_n\sqrt{1-\xi^2}t + \varphi) \tag{5.20}$$

$$c_m = 1 - (1-c_o)(-1)^n \; e^{\left(-\frac{n\pi\xi}{\sqrt{1-\xi^2}}\right)} \tag{5.21}$$

A study of Equations (5.19) and (5.20) indicate that c_o does not affect the times at which the response c(t) reaches its maximum or minimum values or the times at which c(t) is unity. Thus for any value of c_o, time t_p at peak overshoot is the same and effect of c_o is zero at times when c(t) is unity. Equation (5.21) shows that the peak overshoot (c_p-1) is proportional to $(1-c_o)$. These remarks explain the nature of the curves shown in Figures 4.6, 4.8, 4.11, 4.13, and 4.15.

Effect of pc_o. $(c_o = 0,\ p^n c_o = 0$ where $n > 1)$

Equations (5.9), (5.12) and (5.16) can be simplified to

$$M = \left[\frac{1 - \frac{2pc_o\xi}{\omega_n} + \frac{(pc_o)^2}{\omega_n^2}}{1 - \xi^2}\right]^{\frac{1}{2}} \tag{5.22}$$

$$\varphi = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}\,pc_o}{\omega_n-\xi pc_o}\right] - \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{-\xi}\right] \tag{5.23}$$

$$t_m = \frac{n\pi}{\omega} - \frac{1}{\omega} \tan^{-1}\left[\frac{\sqrt{1-\xi^2}\,pc_o}{\omega_n-\xi pc_o}\right] \tag{5.24}$$

Substituting these values and n =1 in Equation (5.17)

$$c_p = 1 + \left[1 - \frac{2pc_o\xi}{\omega_n} + \frac{(pc_o)^2}{\omega_n^2}\right]^{\frac{1}{2}} e^{-\xi\omega_n t_m} \tag{5.25}$$

When pc_o is equal to zero, Equation (5.24) gives

$$t = t_m = \frac{n\pi}{\omega} \tag{5.26}$$

Substituting for M, φ and t from Equations (5.22), (5.23) and (5.26) into Equation (5.8), and simplifying

$$c(t) = 1 + \left[\frac{\dfrac{1-2pc_o\xi}{\omega_n} + \dfrac{(pc_o)^2}{\omega_n^{\,2}}}{1-\xi^2}\right]^{\frac{1}{2}} e^{\left(\frac{-n\pi\xi}{\sqrt{1-\xi^2}}\right)} \text{Sin}\left[\pi n + \tan^{-1}\frac{N+\frac{\sqrt{1-\xi^2}}{\xi}}{1-\frac{N\sqrt{1-\xi^2}}{\xi}}\right] \qquad (5.27)$$

where

$$N = \frac{\sqrt{1-\xi^2}\,pc_o}{\omega_n-\xi pc_o}$$

On further simplification, Equation (5.27) reduces to

$$\overline{c}(t) = 1 - (-1)^n\, e^{-\left[\frac{n\pi\xi}{1-\xi^2}\right]} \qquad (5.28)$$

But, the left hand side of Equation (5.28) is equal to the maxima or minima c_m when all the initial conditions are zero.

$$c(t) = c_m \text{ when } p^n c_o = 0 \qquad (5.29)$$

Equations (5.24) and (5.29) explain the nature of the curves shown in Figures 4.7, 4.9, 4.12, 4.14, and 4.16 which indicate that the time to reach maxima or minima is offset by the amount $\frac{1}{\omega}\tan^{-1}\left[\frac{\sqrt{1-\xi^2}\,pc_o}{\omega_n-\xi pc_o}\right]$ and that the effect of pc_o is zero at the times when response reaches its maximum or minimum values if all the initial conditions are zero

Case II. $M_m < 1$. A study of Equation (5.15) indicates that there will not be any overshoot irrespective of the value of c_o, except for c_o greater than unity which is not practical. Also the dump $\left[1-c(t)\right]$ is proportional to $(1-c_o)$.

5.4. Illustrative Example

The illustrative example of Section 4.3.2 is solved here to show the accuracy of this method and the relative ease with which the problem can be solved.

The control ratio is given by (4.24) as

$$\frac{c(t)}{r(t)} = \frac{4.25}{p^3 + 6p^2 + 5p + 4.25} \tag{5.30}$$

A polar plot for Equation (5.30) is given in Figure 5.1. From the polar plot

$$M_m = 1.25$$

$$\omega_m = 0.72$$

Using Equations (5.2), (5.3) and above values

$$\xi = 0.45 \tag{5.31}$$

$$\omega_n = 0.923 \tag{5.32}$$

$$\omega = \omega_n \sqrt{1-\xi^2} = 0.82 \tag{5.33}$$

Effect of c_o. $(pc_o = p^2 c_o = 0)$

Substituting for ξ and ω in Equations (5.19) and (5.21) and noting that n is equal to unity for first maxima or peak overshoot.

$$t_p = \frac{\pi}{0.82} = 3.83 \tag{5.34}$$

$$c_p = 1 + (1-c_o) e^{\left(-\frac{0.45\pi}{0.89}\right)} \tag{5.35}$$

Table 5.1 gives values of c_p and t_p for different values of c_o. A comparison of this table with Table 4.1 indicates that the results obtained by two methods are more or less identical.

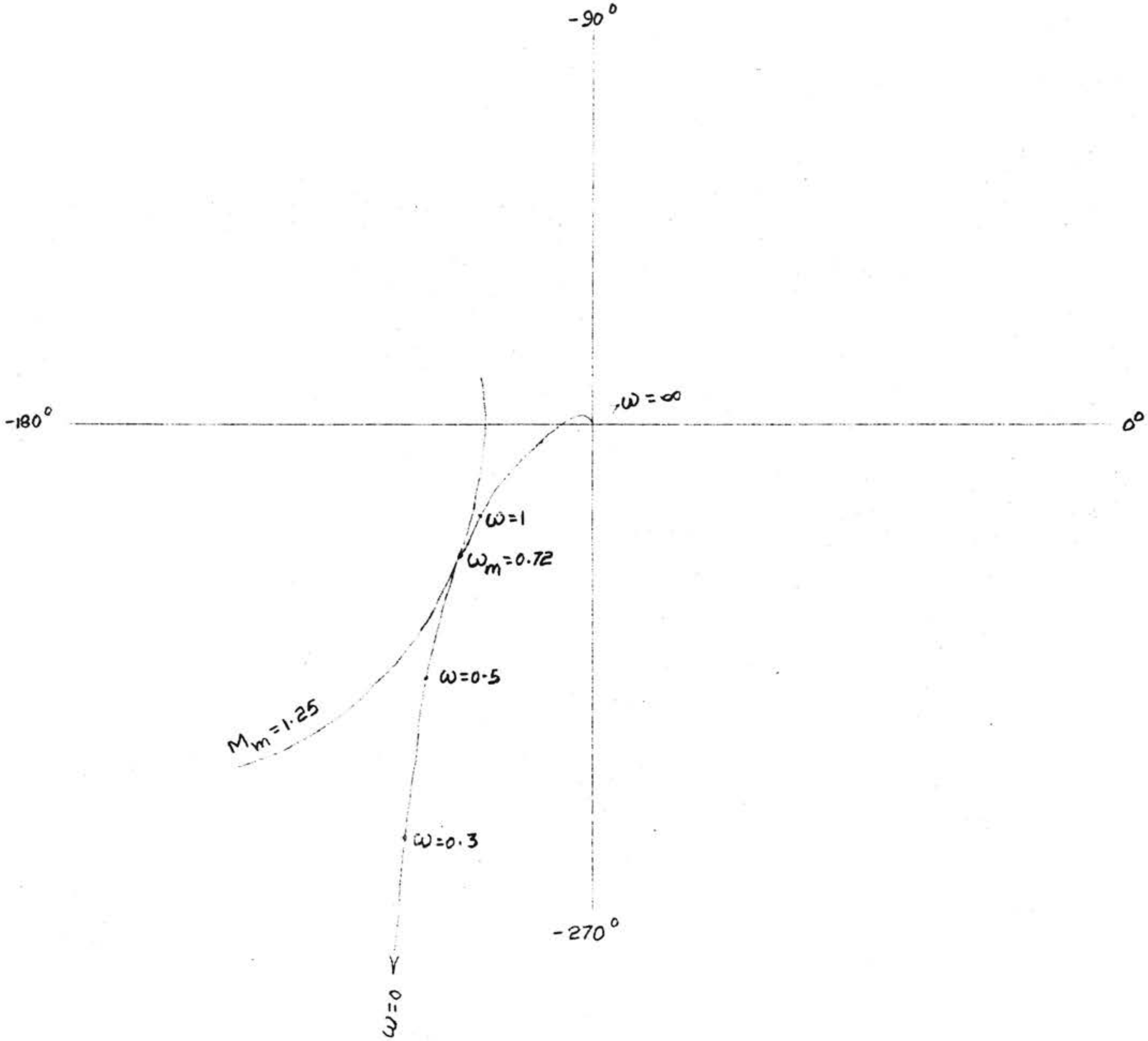


Figure 5.1. Polar Plot for Illustrative Problem

TABLE 5.1

COMPARISON OF TRANSIENT RESPONSE (UNIT STEP INPUT) WHEN THE c_o VARIES ($pc_o = p^2c_o = 0$)

S. No	c_o	c_p	% overshoot	t_p secs.
1	0	1.204	20.4	3.83
2	+.25	1.153	15.3	3.83
3	+.50	1.102	10.2	3.83
4	-.25	1.255	25.5	3.83
5	-.50	1.306	30.6	3.83

Effect of pc_o. Substituting for ξ, ω_n and ω in Equations (5.24) and (5.25) and taking n equal to unity for peak overshoot

$$t_p = 3.83 - 1.22 \tan^{-1} \left[\frac{0.89\ pc_o}{0.923 - 0.45\ pc_o}\right] \qquad (5.36)$$

$$c_p = 1 + \left[1 - 0.975\ pc_o + 1.175\ (pc_o)^2\right]^{\frac{1}{2}} e^{(-0.416\ t_p)} \qquad (5.37)$$

Values of c_p and t_p for different values of pc_o are given in Table 5.2. These values show good agreement with those given in Table 4.2.

TABLE 5.2

COMPARISON OF TRANSIENT RESPONSE (UNIT STEP INPUT)
WHEN THE pc_o VARIES ($c_o = 0, p^2c_o = 0$)

S. No	pc_o	c_p	% overshoot	t_p secs.
1	0	1.204	20.4	3.830
2	0.50	1.242	24.2	3.143
3	1.00	1.385	38.5	2.510
4	-0.50	1.222	22.2	4.280
5	-1.00	1.28	28.2	4.530

CHAPTER VI

FROM GRAPHICAL TO MATHEMATICAL REPRESENTATION

(Non-Zero Initial Conditions)

6.1. Introduction

A problem of practical importance with which a control system designer is often faced with is; given the desired response curve c(t) and the input r(t), how to determine the equation representing the system. The problem becomes more difficult when the initial conditions are present so that the conventional methods (13) used for systems with zero initial conditions cannot be utilized.

It has been already shown that a higher order system can be represented by an equivalent second-order system. Thus a method may be developed by which a second-order equation of the form

$$\frac{c(t)}{r(t)} = \frac{A_2p^2 + A_1p + 1}{B_2p^2 + B_1p + B_o} \qquad (6.1)$$

can be determined to represent the given response curve c(t) and the excitation curve r(t).

6.2. Formulation

Integrating the Equation (6.1) for the interval t_1 to t_2:

$$B_2 p \left[c(t_2) - c(t_1)\right] + B_1 \left[c(t_2) - c(t_1)\right] + B_o \int_{t_1}^{t_2} c(t)dt =$$

$$A_2 p \left[r(t_2) - r(t_1)\right] + A_1 \left[r(t_2) - r(t_1)\right] + \int_{t_1}^{t_2} r(t)dt \qquad (6.2)$$

Integrating the Equation (6.1) for the interval t to t_2:

$$B_2 p \left[c(t_2) - c(t)\right] + B_1 \left[c(t_2) - c(t)\right] + B_o \int_{t}^{t_2} c(t)dt =$$

$$A_2 p \left[r(t_2) - r(t)\right] + A_1 \left[r(t_2) - r(t)\right] + \int_{t}^{t_2} r(t)dt \qquad (6.3)$$

Integrating the Equation (6.3) for the range t_1 to t_2:

$$B_2 \left[pc(t_2)(t_2-t_1) - c(t_2) + c(t_1)\right] + B_1 \left[c(t_2)(t_2-t_1) - \int_{t_1}^{t_2} c(t)dt\right]$$

$$+ B_o \int_{t_1}^{t_2} \int_{t}^{t_2} c(t)dt^2 =$$

$$A_2 \left[p\, r(t_2)(t_2-t_1) - r(t_2) + r(t_1)\right] + A_1 \left[r(t_2)(t_2-t_1) - \int_{t_1}^{t_2} r(t)dt\right]$$

$$+ \int_{t_1}^{t_2} \int_{t}^{t_2} r(t)dt^2 \qquad (6.4)$$

Similarly, triple integrating the Equation (6.1), first for the interval t to t_2, then again for the interval t to t_2 and the finally for the interval t_1 to t_2

$$B_2\left[pc(t_2)\ \frac{(t_2-t_1)^2}{2} - c(t_2)(t_2-t_1) + \int_{t_1}^{t_2} c(t)dt\right]$$

$$+ B_1\left[c(t_2)\ \frac{(t_2-t_1)^2}{2} - \int_{t_1}^{t_2}\int_{t}^{t_2} c(t)dt^2\right] + B_o\int_{t_1}^{t_2}\int_{t}^{t_2}\int_{t}^{t_2} c(t)dt^3 =$$

$$A_2\left[pr(t_2)\ \frac{(t_2-t_1)^2}{2} - r(t_2)(t_2-t_1) + \int_{t_1}^{t_2} r(t)dt\right] +$$

$$+ A_1\left[r(t_2)\ \frac{(t_2-t_1)^2}{2} - \int_{t_1}^{t_2}\int_{t}^{t_2} r(t)dt^2\right] + \int_{t_1}^{t_2}\int_{t}^{t_2}\int_{t}^{t_2} r(t)dt^3 \qquad (6.5)$$

Similarly a series of equations may be written by integrating the Equation (6.1) once more each time for the range t to t_2, the last integration in each case being from t_1 to t_2.

Equations (6.2), (6.4), (6.5), ... may be solved simultaneously to find the coefficients B_2, B_1, B_o, A_2 and A_1. To determine these five coefficients five equations will be necessary. The problem will be further simplified if input r(t) is of the form such that both A_1 and A_2 are zero in which case only three equations will be required to determine B_2, B_1, and B_o. Also, since the coefficient B_o is associated with the steady state value of response c(t), it may be determined from other considerations. Now the problem consists only of solving two simultaneous linear differential Equations (6.2) and (6.4) to find B_2 and B_1.

The time instants t_1 and t_2 are so selected that $pc(t_2)$, $pc(t_1)$, $pr(t_2)$ and $pr(t_1)$ can be accurately determined from the curves. Usually, the starting point and a point of maxima or minima will be the best choice.

6.3. Illustrative Example

Consider a control system, the response $c(t)$ for which is given by the curve (III) of Figure 4.9, the excitation $r(t)$ being a unit step input. It is given that the steady state value of response $c(t)$ is unity.

The second-order equation approximating the system will be of the form

$$\frac{c(t)}{r(t)} = \frac{1}{B_2p^2 + B_1p + 1} \tag{6.6}$$

the B_o being unity as it is known that the steady state value of response $c(t)$ is unity. As only two unknowns B_2 and B_1 are to be determined, only two Equations, (6.2) and (6.4), need to be solved.

A study of the response curve $c(t)$ indicates that the values of $pc(t)$ can be easily determined at the starting point and at the point where $c(t)$ is maximum. Hence the points t_1 and t_2 are chosen as

$$t_1 = 0 \text{ secs.} \tag{6.7}$$

$$t_2 = 2.4 \text{ secs.} \tag{6.8}$$

From the curve

$$c(t_1) = 0, \quad pc(t_1) = 1 \quad p\,r(t_1) = 0$$

$$c(t_2) = 1.38, \quad pc(t_2) = 0 \quad p\,r(t_2) = 0 \tag{6.9}$$

The $\int_{t_1}^{t_2} r(t)dt$ and $\int_{t_1}^{t_2}\int_{t}^{t_2} r(t)dt^2$ of Equations (6.2) and (6.4) can be evaluated analytically.

$$\int_{t_1}^{t_2} r(t)dt = \int_0^{2.4} dt = 2.4 \tag{6.10}$$

$$\int_{t_1}^{t_2}\int_{t}^{t_2} r(t)dt^2 = \int_0^{2.4} (2.4 - t)dt = 2.88 \tag{6.11}$$

The $\int_{t_1}^{t_2} c(t)dt$ and $\int_{t_1}^{t_2}\int_{t}^{t_2} c(t)dt^2$ can be evaluated by using trapezoidal rule (4), as shown in Table 6.1.

$$\int_0^{2.4} c(t)dt = \Delta t\left[\frac{c_o}{2} + \sum_{n=1}^{11} c_n + \frac{c_{12}}{2}\right] \tag{6.12}$$

Δt is choosen as 0.2 secs. $\int_{t}^{t_2} c(t)dt$ suggests that the integration should be started backward. Making use of Equation (6.12).

$$10\int_{2.2}^{2.4} c(t)dt = 1.38 + 1.37 = 2.75$$

$$10\int_{2.0}^{2.4} c(t)dt = 1.38 + 1.37 + 1.37 + 1.36 = 5.48$$

$$10\int_{0}^{2.4} c(t)dt = 23.80 + 0.38 + 0.0 = 24.18 \tag{6.13}$$

$$100\int_0^{2.4}\int_t^{2.4} c(t)dt = 10\left[\int_{2.2}^{2.4} c(t)dt + 2\left\{\int_{2.0}^{2.4} c(t)dt + \int_{1.8}^{2.4} c(t)dt\right\} + \int_0^{2.4} c(t)dt\right]$$

$$100\int_0^{2.4}\int_t^{2.4} c(t)dt = 351.77 \tag{6.14}$$

TABLE 6.1

CALCULATIONS OF $\int_0^{2.4} c(t)\,dt$ AND $\int_0^{2.4}\int_t^{2.4} c(t)dt^2$

S. No n	t_n	$c_n(t)$	$10\int_{t_n}^{2.4} c(t)dt$	$100\int_0^{2.4}\int_{t_n}^{t_2} c(t)dt^2$
0	0	0.0	24.18	351.77
1	0.2	0.38	23.80	
2	0.4	0.56	22.86	
3	0.6	0.76	21.54	
4	0.8	0.89	19.89	
5	1.0	1.03	17.97	
6	1.2	1.17	15.77	
7	1.4	1.24	13.36	
8	1.6	1.30	10.82	
9	1.8	1.34	8.18	
10	2.0	1.36	5.48	
11	2.2	1.37	2.75	
12	2.4	1.38		

Substituting the values of the coefficients of B_2, B_1 and B_o and noting that B_o is equal to unity and A_2 and A_1 are each equal to zero, the Equations (6.2) and (6.4) may be simplified as

$$-B_2 + 1.38\, B_1 = -.018 \qquad (6.15)$$

$$-1.38\, B_2 + .89\, B_1 = -.64 \qquad (6.16)$$

Solving Equations (6.15) and (6.16) simultaneously, B_1 and B_2 can be obtained as

$$B_2 = 0.853$$

$$B_1 = 0.605$$

Substituting for B_2 and B_1 in Equation (6.6) and simplifying

$$\frac{c(t)}{r(t)} = \frac{1.16}{p^2 + .72\,p + 1.16} \qquad (6.17)$$

The Equation (6.17) approximates the system under consideration. The values of the damping ratio ξ and natural frequency ω_n may be noted as 0.34 and 1.07 respectively. A second-order approximation for the same system was obtained in Section 5.4 and the values of ξ and ω_n were obtained as 0.45 and 0.923 respectively. It may be noted that more accuracy could have been obtained by evaluating the integrals more accurately.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

7.1. Conclusions

The purpose of this study was (1) to show how the inital conditions affect the performance of linear feedback control systems with constant coefficients and (2) to develop analytical, graphical and numerical methods by which these effects can be easily taken into account while designing such control systems.

The writer's conclusions are summarized in the following list:

1. The effects of initial conditions are governed by the relative values of poles and zeros of the open-loop transfer function G(p)H(p).
2. Increasing the static loop sensitivity K reduces the effects of initial conditions.
3. The effects of initial condition $p^n c_o$ where n = 0,1,2,... decrease as n increases. Mostly the effects of c_o and pc_o are more pronounced.

The effects of c_o and pc_o, which are more pronounced and are of more practical importance, are as follows:

The Effects of c_o.

1. A positive value of c_o "smoothens out" the response while a negative value causes the

response to be "rougher". As the value of c_o decreases, the response becomes more and more "rough."

2. The presence of c_o does not affect the time at which the response attains its maximum or minimum values. If the control ratio $\frac{c(t)}{r(t)}$ contains any zeros, the criteria may not be absolutely true, but for all practical purposes the deviations will be negligible.
3. The peak overshoot (c_p-1) is approximately proportional to $(1-c_o)$ if all other conditions remain unchanged.
4. The effects of c_o vanish at times when $c(t)$ is unity.

The Effects of pc_o.

1. The presence of pc_o causes an increase in overshoot, whether it be positive or negative.
2. As the value of pc_o increases, the response becomes faster and the values of t_p, t_d, t_r, t_s, and N progressively decrease. Thus a positive value of pc_o causes the response to be faster while a negative value of pc_o slows down the response.
3. The effects of pc_o vanish at times when response $c(t)$ would reach its maximum or minimum values if all the initial conditions are zero.

The writer is of the opinion that the methods using digital or analog computers as described in Section 4.3.2 particularly the latter one, are the most complete and handy methods by which the effects of initial conditions can be taken into account while designing a control system. But, failing to have such costly equipment, the combined graphical and analytical method developed in Chapter V and the purely graphical method developed in Section 4.3.2 are more useful and accurate for all practical purposes. Even if the computers are available it is necessary to understand these methods because at the present state of the art, it is not possible to obtain a complete design from the computer without interpretation at various steps by the design engineer. Also, when an analog computer facility is available, these techniques will be required for prelinimary, order-of-magnitude estimates and for verifying computer solutions.

7.2. Recommendations for Future Study

1. As the initial conditions affect the transient performance, they also affect the criteria for optimization of feedback control systems. A study of these effects may be of considerable practical importance.

2. As discussed before, the initial conditions may be considered identical to impulse disturbances introduced at appropriate points in the control loop. Hence, a further investigation may be made by applying the theory of multivariable input signals.

3. The effects of initial conditions on linear systems with time varying coefficients and also on non-linear systems should be studied in detail as they represent the more practical systems.

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VITA

Narendrakumar M. Shah

Candidate for the Degree of

Master of Science

Thesis: EFFECTS OF INITIAL CONDITIONS ON PERFORMANCE OF CONTROL SYSTEMS.

Major Field: Mechanical Engineering

Biographical:

Personal Data: Born June 1, 1939, in Gujarat, India, the son of Vijyalaxmi and Manilal B. Shah.

Education: Graduated from Sheth N. L. High School, Bombay, India, in 1956; received the degree of Bachelor of Engineering, Hons., in Mechanical Engineering at University of Bombay, June, 1961; completed the requirements for the Master of Science degree in May, 1963.

Experience: Employed as Maintenance Engineer at Indian Plastics Ltd., Bombay, India, in 1961-1962. Empolyed at Ex-Cell-O India, Ltd., Thana, India, for summer, 1961.

Professional Organizations: American Society of Mechanical Engineers.