INFLUENCE LINES FOR CONTINUOUS CURVED MEMBERS WITH LATERAL LOADS

By

MUPPIDI NARSIMHA REDDY

Bachelor of Civil Engineering

Osmania University

Hyderabad - (Dn) - India

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Thesis Approved:

[illegible]

Thesis Adviser

Roger L. Flanders

[illegible]

Dean of the Graduate School

PREFACE

The work presented in this thesis is an outgrowth of the ideas expressed by Professor Jan J. Tuma in the summer of 1961.

Upon completing the requirements for the degree of Master of Science, the writer wishes to express his sincere appreciation and gratitude to the following persons:

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M. N. R.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Figure		Page

NOMENCLATURE

1, ···, i, j, k, ··· , n, n+1	Notation for Supports of n-Span Curved Girder.
x_i, y_i, z_i	Cartesian Axes of i th Span.
o_i	Center of Curvature in i th Span.
o'_i	Center of Circular Arc ij.
q, r, t	Notation for Cross-Section.
R_i	Radius of Curvature in i th Span.
L_i	Length of Major Chord of i th Span.
x_{ri}, y_{ri}	Cartesian Coordinates of r in i th Span.
x'_{ti}	$= (L_i - x_{ti})$.
E	Modulus of Elasticity.
G	Modulus of Rigidity.
I	Moment of Inertia.
J	Equivalent Polar Moment of Inertia.
P	Real Concentrated Load.
M	Real Concentrated Moment.
Z_{ij}, Z_{ji}	End Reactive Forces of Span ij .
X_{ij}, Y_{ij}, X_{ji}, Y_{ji}	End Reactive Moments of Span ij .
bZ_{ij}, bZ_{ji}, bX_{ij}	End Reactions of Basic Span ij Due to Applied Loads.

D_{zi}, D_{qzi} . . . . .Reaction and Shear Influence Coefficients of Basic Span ij Due to $Y_{ij} = 1$.

A_{qti}, B_{qyi}, C_{qti} . . . . .Cross-Sectional Twisting Moment Influence Coefficients at q in Basic Span ij Due to Unit Values of X_{ij}, Y_{ij}, Y_{ji}, Respectively.

A_{qbi}, B_{qbi}, C_{qbi} . . . .Cross-Sectional Bending Moment Influence Coefficients at q in Basic Span ij Due to Unit Values of X_{ij}, Y_{ij}, Y_{ji}, Respectively.

V_{gzi}, M_{qti}, M_{qbi} . . . .Cross-Sectional Shear, Twisting, and Bending Moments, Respectively, at q in Span ij.

bV_{qzi}, bM_{qti}, bM_{qbi} . . .Cross-Sectional Shear, Twisting, and Bending Moments, Respectively, at q in Basic Span ij Due to Loads.

ds . . . . .Differential Length.

$d\lambda_t$, $d\lambda_b$ . . . . .Differential Angle Changes of Twisting and Bending Moments, Respectively.

U_s . . . . .Strain Energy.

U_R . . . . .Work of Reactions.

U . . . . .Total Potential Energy.

f_{ijxx}, f_{ijxy} . . . . .Angular Flexibility and Near Carry-Over Value of Basic Span ij.

g_{ijxy}, g_{ijyy} . . . . .Angular Far Carry-Over Values of Basic Span ij.

$t_{ijx}^{(L)}$, $t_{ijy}^{(L)}$, $t_{jiy}^{(L)}$ . . . . .Angular Load Functions of Basic Span ij.

$t_{ij}^{(\Delta)}$. . . . . .Angular Displacement Function of Span ij.

Z_i . . . . .Reactive Force at ith Support of n-Span Structure.

BZ_i, BY_{ij}, BY_{ji} . . . . . .Reactive, Force at ith Support and Moments at the Ends of Span ij of n-Span Basic Structure Due to Loads.

$XZ_i^{(ij)}$, $BZ_i^{(ij)}$ . . . . . . .Influence Coefficients of Reaction at ith Support of n-Span Basic Structure Due to X_{ij} = Unity and bX_{ij} = Unity, Respectively.

$Xv_{qzi}^{(ij)}$, $Xm_{qti}^{(ij)}$, $Xm_{qbi}^{(ij)}$ . . Influence Coefficients of Cross-Sectional Shear, Twisting and Bending Moments, Respectively, at q in Span ij of n-Span Basic Structure Due to X_{ij} = Unity.

BV_{qzi}, BM_{qti}, BM_{qbi} . . Cross-Sectional Shear, Twisting and Bending Moments, Respectively at q in Span ij of n-Span Basic Structure Due to Loads.

$Bv_{qzi}^{(ij)}$, $Bm_{qti}^{(ij)}$, $Bm_{qbi}^{(ij)}$, . Influence Coefficients of Cross-Sectional Shear, Twisting, and Bending Moments, Respectively, at q in Span ij of n-Span Basic Structure Due to bX_{ij} = Unity.

F_i, G_{hi}, G_{gi} . . . . . . Angular Flexibility, Near and Far Carry-Over Values of n-Span Basic Structure.

$\tau_i^{(L)}$, $\tau_i^{(\Delta)}$ . . . . Angular Load and Displacement Functions of n-Span Basic Structure.

r_{ij} . . . . Carry-Over Moment Factor.

$m_i^{(L)}$, $m_i^{(\Delta)}$ . . . . . . Load and Displacement Starting Moments.

m_{ij}, a_{ij} . . . . Redundant Moment Influence Coefficients.

$\overline{p}_t$, $\overline{p}_b$ . . . . Distributed Conjugated Loads.

$\overline{P}_{ijx}$, $\overline{P}_{ijy}$ . . . . Total Conjugate Loads in Span ij.

$\overline{X}_{ij}$, $\overline{Y}_{ij}$, $\overline{X}_{ji}$, $\overline{Y}_{ji}$ . . . . End Reactive Forces of Conjugate Span ij.

$\overline{M}_{rzi}$ . . . . Conjugate Bending Moment at r in Span ij.

$\overline{RM}_{rzi}$, $\overline{BM}_{rzi}$ . . . . . Conjugate Bending Moment at r of Span ij Due to Conjugate Reactions and Loads, Respectively.

T_{ijx}, T_{ijy} . . . . End Slopes of n-Span Continuous Beam at Simple End i.

α, β, γ . . . . Angular Coordinates of q, t, r, Respectively

$2\omega_i$ . . . . Angular Length of Span ij.

Ω_i . . . . Geometric Angle Change.

ψ_i, η_i	Cosec Ω_i and cot Ω_i, Respectively.
θ	Angular Rotations.
Δ	Linear Displacement.
$\sum$	Summation
*	Superscript Denoting "Modified."

CHAPTER I

INTRODUCTION

1.1 General

The purpose of this presentation is the development of a general five moment equation and its application to the evaluation of influence values for any function in a laterally loaded continuous curved beam. This thesis is the outgrowth of lectures on space structures delivered by Tuma[1], in the summer of 1961. The writer's contribution is the derivation of the five-moment equation and other analytical expressions related to it, and the presentation of influence lines for reactive, cross-sectional elements and deformations.

The curved girder with its axis in the x-y plane, and loaded out of this plane, finds a variety of uses in buildings, theater balconies, elevated tanks, bridges, etc. The analysis of such structures differs from that of a continuous straight beam in that the torsional moments are induced in the former in addition to bending and shear.

1.2 Review of Literature

The analysis of a single span circular beam with fixed ends and loaded normal to its plane has been studied by a number of investigators. Grashoff[2], the first to analyze such a structure, examined a circular ring cut at a section with two equal and opposite loads applied at the cut ends. Thereby he neglected the effect of the bending and twisting moments at the cut points. Gibson and Ritchie[3], and Oesterblom[4] obtained curves of bending and twisting moments for various values of span and cross-sectional variation. Pippard and Barrow[5], Moorman[6], Schulz and Chedraui[11] approached the problem in a similar way. Moorman[7] presented the influence lines for bending and twisting moments in a one span curved beam with fixed ends. Volterra[8, 9] analyzed circular rings by integrating the differential equations of the elastic curve. Michalos[10] gave curves of the torsional and bending moments in one span arch ribs for lateral wind loads. It may be noted that the above literature survey shows the study restricted to single span structures only.

Veltuni[12] appears to be the first to have analyzed a continuous curved beam. In his analysis, all the spans were equal and had the same center of curvature. He applied the moment distribution method, considering the radial (bending) and tangential (twisting) moments. Michalos[13, 14] generalized the method for space structures. Pippard[15] solved for the redundant reactions and moments, by the method of forces, making use of the deformation functions of a circular cantilever arc. He also analyzed circular rings on many simple supports by the principle of leastwork. Fickel[16], assuming torsional restraints at all the supports and selecting an indeterminate basic structure, developed a three moment equation. Thereby he imposed a condition, which can not be realised in an actual structure. He also presented influence lines for the bending and twisting moments and shear force in single span and continuous girders. Tuma[1] applied the principle of elastic weights and analyzed complex space structures with bent and curved members. Patel[17] analyzed a four span curved girder with fixed ends by making use of the angular functions of a one span basic structure.

The carryover moment method developed by Tuma[18, 19, 20, 28] and applied to continuous beams, is extended in this presentation to the analysis of laterally loaded structures. This method was also applied to the analysis of flat plates by Tuma, Havner, and French[21], and to the continuous prestressed concrete beams by Munshi[29]. The five warping moment equation developed by Gillespie[22] was applied in carryover form to space trusses. Oden[23] employed the flexibility approach for the analysis of the complex plate-beam structures.

Little can be found in the literature on influence lines for space structures. Chamecki[25] developed influence lines by the methods of forces and displacements. Kamel[30] used the concept of elastic weights to the evaluation of influence areas for continuous bent beams in space. In this thesis, flexibility method is adopted to develop the influence lines of reactive moments. Maxwell-Mohr theorems[14, 26] are found efficient for influence lines of deformations.

In addition to the general review, basic principles such as coordinate systems, formation of basic structures, assumptions and the sign convention are also discussed in this chapter. Chapters II and III are devoted to the stereostatics and the deformations of a one span basic structure. Chapter IV deals with the stereostatics of an n-span basic structure, and in the fifth chapter the five-moment equations is developed along with its carryover form. Chapters VI and VII cover a discussion on influence lines and derivation of analytical expressions for the angular and other elastic functions. Finally in Chapter eight, the theory, developed in this presentation, is demonstrated by numerical problems.

Before going into details, it is necessary to establish reference systems to be followed throughout this thesis. This is done as follows.

Fig. 1.1 shows the cartesian set of axes, where axes x, y are located in the plane of the paper, and the third axis is normal to the x-y plane.

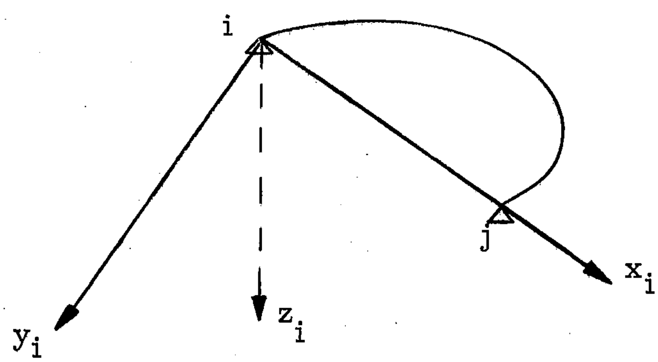


Fig. 1.1

Cartesian Set of Axes

The orientation of the x, y axes is varied from span to span for convenience. The subscript stands for the span. This is illustrated in the above figure. The x-axis of the span ij is directed along its major chord, and the y-axis perpendicular to the chord as shown.

The calculation of the angular functions of a basic span involves integrals, as will be seen later. The evaluation of these integrals becomes easier with polar coordinates. Such a system is illustrated in Fig. 1.2.

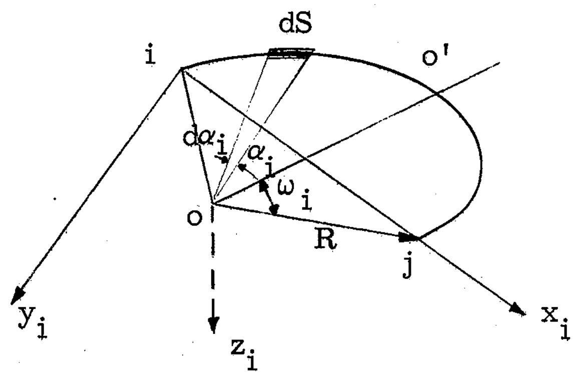


Fig. 1.2

Cylindrical Coordinates

In the above figure, o is the center of curvature of the circular arc ij lying in the x-y plane. oo', the bisector of the arc, is taken as a reference line, with respect to which the angular coordinate of any element is measured. In this presentation positive angular coordinate is measured counterclockwise. In the Fig. 1.2, the element at q is located by the coordinates (R_i, α_i, o), where R_i is its radius of curvature.

1.3 Statement of the Problem

A continuous curved beam with its centroidal axis in the x-y plane and loaded by out of plane forces and moments is considered

(Fig. 1.3).

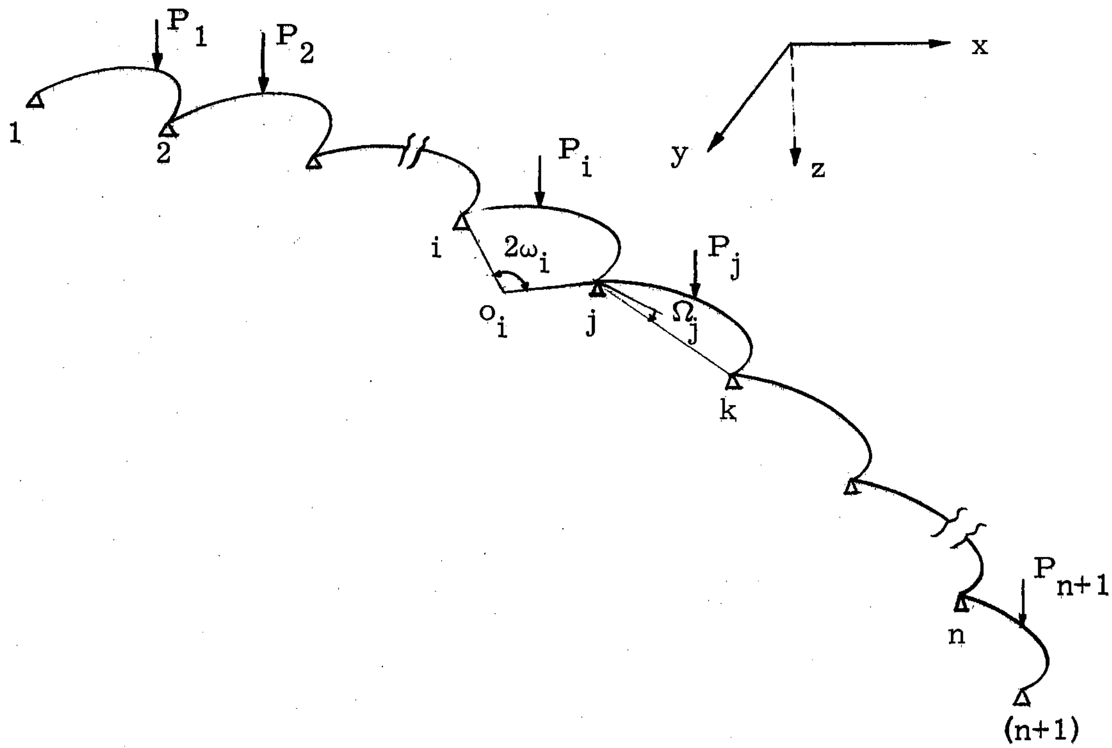


Fig. 1.3

Continuous Curved Beam in Space

The supports are denoted by 1, 2, · · ·, i, j, k, · · ·, n, n+1. The spans are numbered after their left supports. R_1, R_2, · · ·, R_1, R_j, R_k, · · ·, R_n, are the radii of curvature of the spans 1, 2, · · ·, i, j, k, · · ·, n, respectively. $2\omega_1$, $2\omega_2$, · · ·, $2\omega_i$, $2\omega_j$, $2\omega_k$, · · ·, $2\omega_n$, are the angles subtended by the spans at their centers of curvature. Ω_2, · · ·, Ω_i, Ω_j, Ω_k, · · ·, Ω_n are the positive geometric angle changes between major chords of the adjacent spans at the interior supports 2, · · ·, i, j, k, · · ·, n, respectively. The interior supports are spherical hinges. The structure considered may have any end conditions.

It can be easily seen, that the continuous structure (Fig. 1. 3), with both ends fixed against deformations, involves a total of (n+5) unknowns and therefore is statically indeterminate to the (n+2) degree. The general method for the analysis of such a structure consists of its replacement by a primary or basic structure whose behavior under any type of loading is known, and on which the redundant unknowns are applied as external loading. For this purpose two types of basic structures are found necessary and their details are given below.

a) One-Span Basic Structure.

A one-span curved beam (Fig. 1. 4) rests on simple supports at i and j, and is restrained at the right support j against rotation about its major chord ij. R_i is its radius of curvature and $2\omega_i$, the angle subtended by the span at its center of curvature o_i. The structure, under the action of lateral loads and moments, develops that reactive forces Z_{ij}, Z_{ji} and moment X_{ji}, which are statically determinate and sufficient for its stability.

b) n-Span Basic Structure.

Consider the n-span structure (Fig. 1. 3) with simple ends. This end condition is chosen here only for simplicity. For any other situation, the modifications necessary to tackle the problem will be shown later. The total number of redundants in this case is equal to (n-2). The structure is rendered determinate by providing (n-2) hinges one at the left end of each of the (n-2) interior spans (Fig. 1. 5). This device allows free rotation of each span at the hinge about its major card, while it maintains perfect continuity in the perpendicular y-direction of each of the (n-2) interior spans. Also, the stability of the structure is preserved.

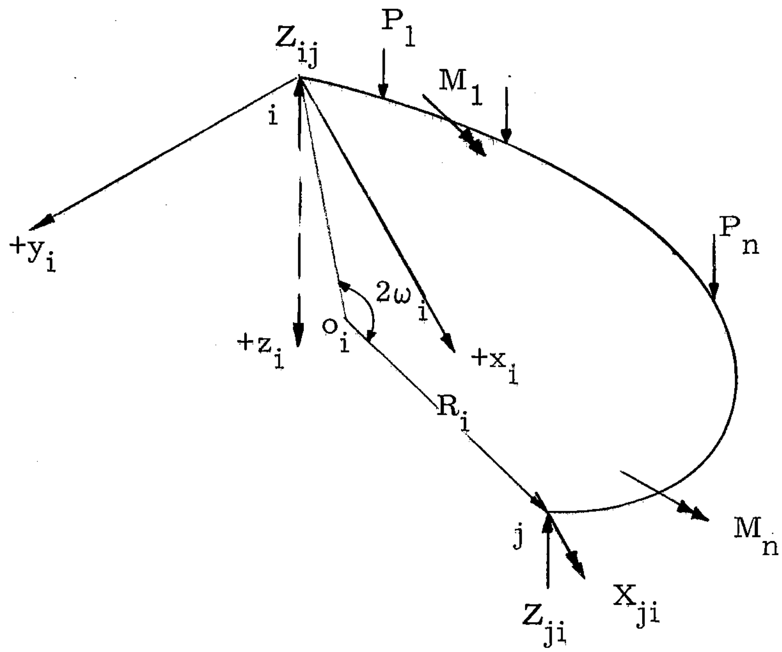


Fig. 1.4

One-Span Basic Structure

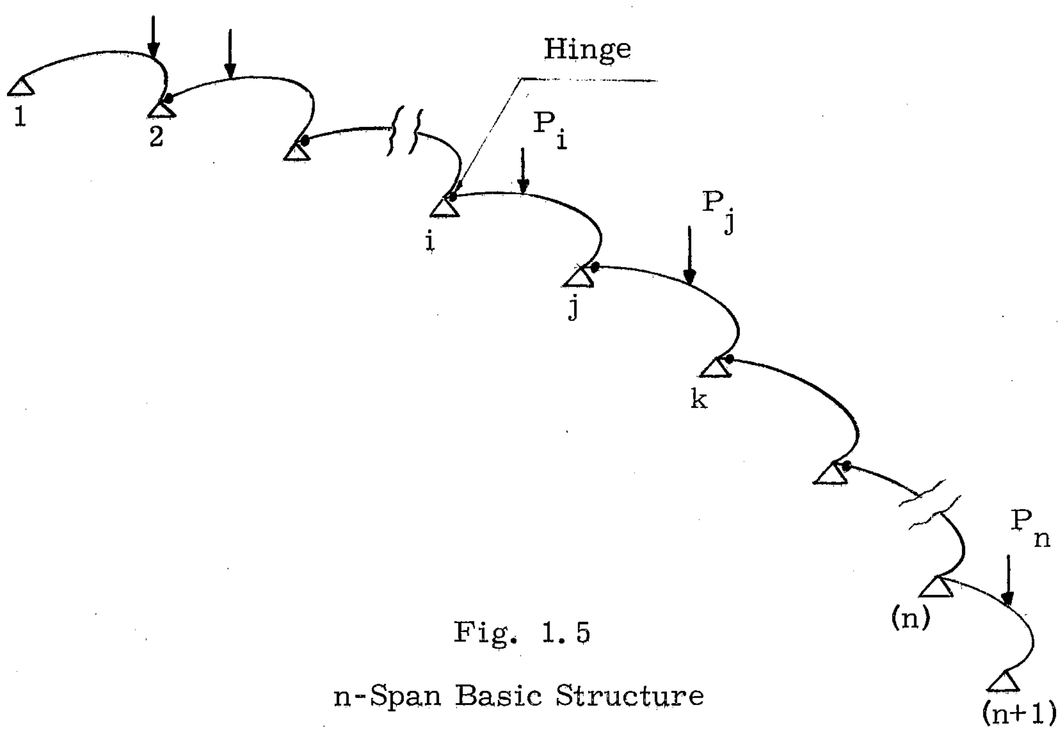


Fig. 1.5

n-Span Basic Structure

1.4 Assumptions

In this presentation, the following assumptions are made.

a) The material is homogeneous and isotropic.

b) Plane sections before deformation remain plane after deformation.

c) Deformations are small, and elastic.

d) Shear deformations are small and can be neglected.

e) The support displacements, if there are any, are rigid.

1.5 Sign Convention and Notation

This is shown in Table 1.1 only for those forces and deformations, which occur in the structures under the type of loading considered in this thesis.

TABLE 1.1 SIGN CONVENTION

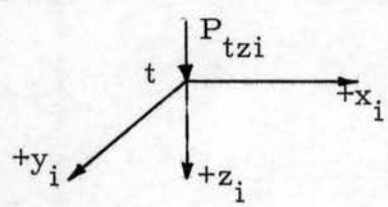
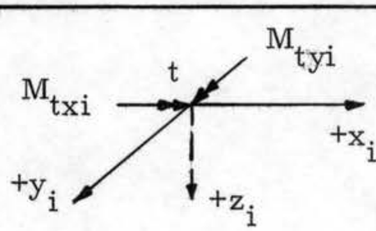
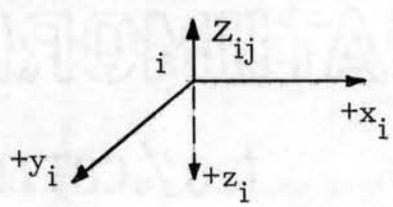
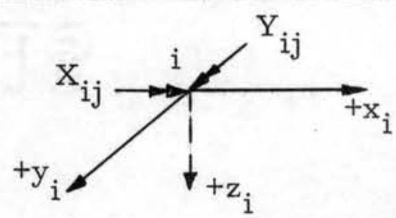
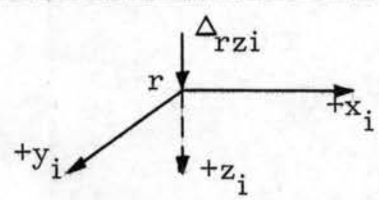
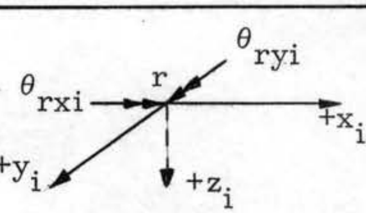
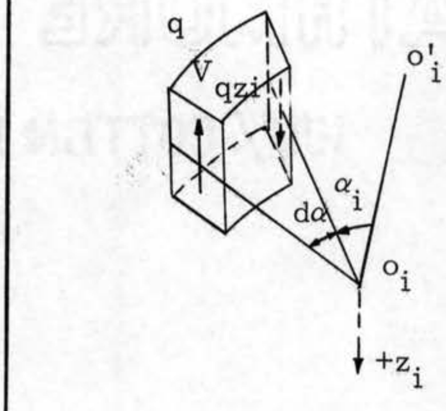
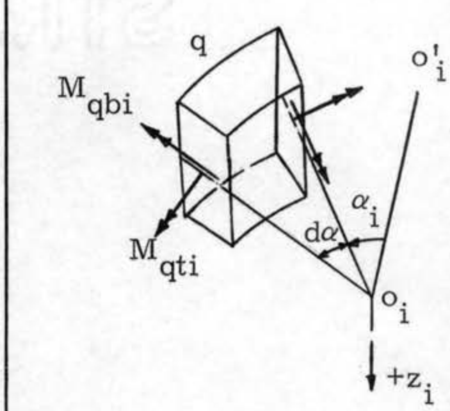
Positive Applied Loads	 Forces	 Moments
Positive Reactions	 Forces	 Moments
Positive Deformations	 Deflections	 Rotations

TABLE 1.1 (continued)

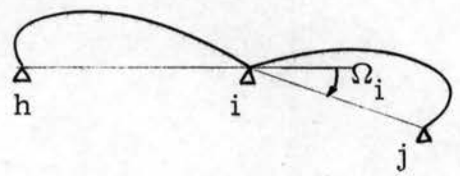
Positive Cross-Sectional Elements



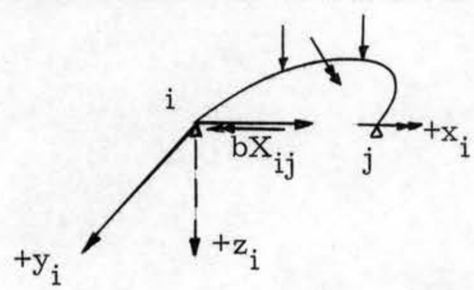
Forces



Moments

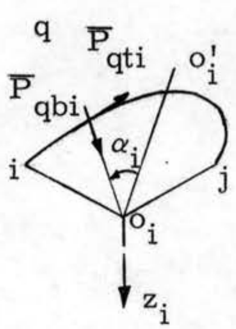


Positive Geometric Angle Change

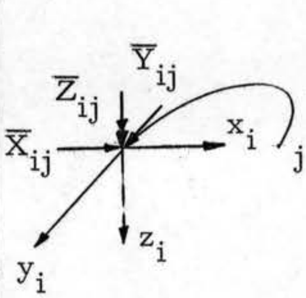


Positive Static Moment

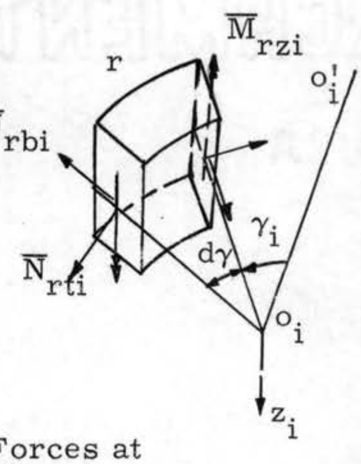
Positive Conjugate Loads and Moments



Loads



End Reactions



Forces at Section

CHAPTER II

STEREO-STATICS OF A ONE-SPAN CURVED BAR

2.1 Statement

Span ij of the continuous curved beam (Fig. 1.3) is isolated and shown below (Fig. 2.1) as a freebody in its plan view. Positive z-axis is oriented normal into the paper. The forces are represented by the symbols ●, ⊙ . The first stands for a force acting in the positive z-direction and the second for one in the opposite direction. Vector notation is used for moments, and their direction is determined by the familiar right hand rule. The moments acting at both ends are shown in the same directions, since for a symmetrical span, this results in identical values of angular functions.

In the Fig. 2.1, $P_1, \cdots, P_n$ and $M_1, \cdots M_n$ are the applied forces and moments, respectively. Z_{ij}, Z_{ji} and X_{ij}, Y_{ij}, X_{ji}, Y_{ji}, are the unknown positive reactive forces and moments, respectively. An inspection of the span ij shows that it is indeterminate to third degree. Therefore X_{ij}, Y_{ij}, and Y_{ji} are chosen as the redundants and are treated as applied arbitrary moments. Z_{ij}, Z_{ji}, and X_{ji} are expressed in terms of the redundants and applied loads by using the three equations of static equilibrium of span ij as shown in Section 2.2.

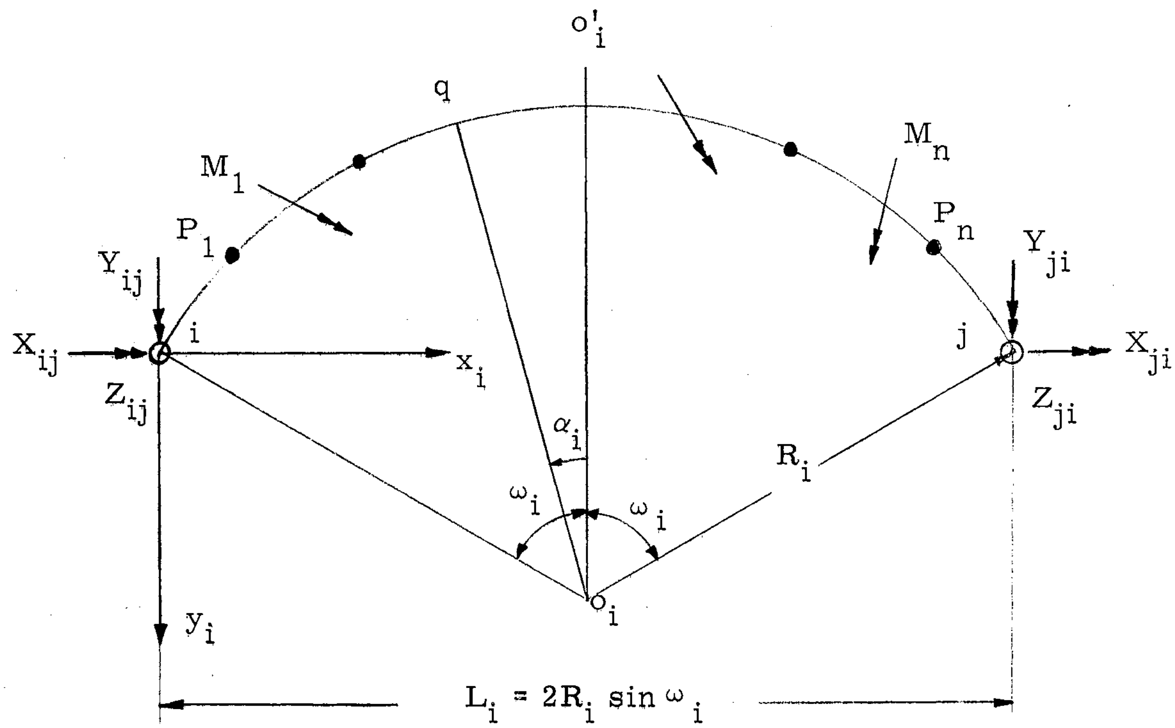


Fig. 2.1

Freebody of Span ij

2.2 Stereo-Statics of the Curved Bar ij

By summing moments about chord ij

$$\sum_{i}^{j} M_{jx} = 0$$

or

$$X_{ji} = - X_{ij} + bX_{ij} \tag{2.1}$$

where bX_{ij} is the positive static moment of all applied loads about chord ij.

By summing moments in the y-direction at j

$$\sum_{i}^{j} M_{jy} = 0$$

or

$$Z_{ij} = \frac{1}{2R_i \sin \omega_i} (Y_{ij} + Y_{ji}) + bZ_{ij} \tag{2.2}$$

Finally by summing forces in the z-direction;

$$\sum_{i}^{j} F_z = 0$$

or

$$Z_{ji} = -\frac{1}{2R_i \sin \omega_i} (Y_{ij} + Y_{ji}) + bZ_{ji} \tag{2.3}$$

where bZ_{ij}, bZ_{ji} are the positive end reactive forces due to the applied loads only acting on the basic span.

2.3 Cross-sectional Elements

q is a section in the span ij (Fig. 2.1), located by the polar coordinates (R_i, α_i). Fig. 2.2 shows the freebody of isolated part iq.

By employing the three equations of static equilibrium of iq (Fig. 2.2), the cross-sectional shear V_{qzi} and moments M_{qti}, M_{qbi} at section q are calculated as follows;

$$V_{qzi} = \frac{1}{2R_i \sin \omega_i} (Y_{ij} + Y_{ji}) + bV_{qzi} \tag{2.4a}$$

$$M_{qti} = X_{ij}(-\cos \alpha_i) + Y_{ij} \frac{1-\cos(\omega_i+\alpha_i)}{2 \sin \omega_i} + Y_{ji} \frac{1-\cos(\omega_i-\alpha_i)}{2 \sin \omega_i} + bM_{qti} \tag{2.4b}$$

$$M_{qbi} = X_{ij}(-\sin\alpha_i) + Y_{ij}\left(-\frac{\sin(\omega_i+\alpha_i)}{2\sin\omega_i}\right) +$$

$$+ Y_{ji}\frac{\sin(\omega_i-\alpha_i)}{2\sin\omega_i} + bM_{qbi}$$

(2.4c)

where bV_{qzi} and bM_{qti}, bM_{qbi} are the positive shear and moments at q respectively due only to the applied loads acting on the basic span ij.

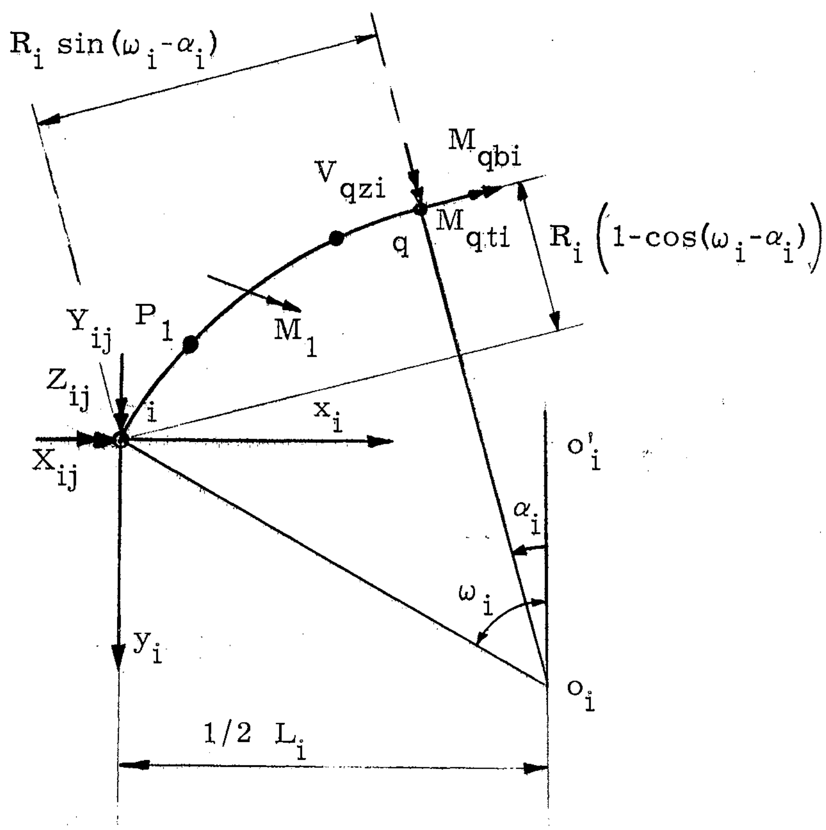


Fig. 2.2

Cross-Sectional Elements at q

Introducing the following notation,

$$\left.\begin{array}{llll} (-\cos \alpha_i) = A_{qti} & ; & (-\sin \alpha_i) = A_{qbi} & \\ \dfrac{1-\cos(\omega_i+\alpha_i)}{2 \sin \omega_i} = B_{qti} & ; & -\dfrac{\sin(\omega_i+\alpha_i)}{2 \sin \omega_i} = B_{qbi} & \\ \dfrac{1-\cos(\omega_i-\alpha_i)}{2 \sin \omega_i} = C_{qti} & ; & \dfrac{\sin(\omega_i-\alpha_i)}{2 \sin \omega_i} = C_{qbi} & \\ \text{and} & & \dfrac{1}{2R_i \sin \omega_i} = D_{zi} = D_{qzi} & \end{array}\right] \quad (2.5)$$

The reactive and cross-sectional elements in span ij from Eqs. (2.1 to 2.4) are expressed in matrix language as shown below.

a) Static Equilibrium Matrix 2.1

$$\begin{bmatrix} X_{ji} \\ Z_{ij} \\ Z_{ji} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & D_{zi} & D_{zi} \\ 0 & -D_{zi} & -D_{zi} \end{bmatrix} \begin{bmatrix} X_{ij} \\ Y_{ij} \\ Y_{ji} \end{bmatrix} + \begin{bmatrix} bX_{ij} \\ bZ_{ij} \\ bZ_{ji} \end{bmatrix}$$

where the use of subscripts for reactive forces and moments is straight forward. The first subscript gives the location and the second, the other end of the span, on which they act.

b) Cross-Sectional Force Matrix 2.2

$$\begin{bmatrix} V_{qzi} \\ M_{qti} \\ M_{qbi} \end{bmatrix} = \begin{bmatrix} 0 & D_{qzi} & D_{qzi} \\ A_{qti} & B_{qti} & C_{qti} \\ A_{qbi} & B_{qbi} & C_{qbi} \end{bmatrix} \begin{bmatrix} X_{ij} \\ Y_{ij} \\ Y_{ji} \end{bmatrix} + \begin{bmatrix} bV_{qzi} \\ bM_{qti} \\ bM_{qbi} \end{bmatrix}$$

where the subscripts used for the cross-sectional elements have certain significance. The last subscript denotes the span ij. The first subscript stands for the location of the cross-sectional element, and the second, for the nature of the force or moment.

By changing the subscripts, the above matrices (2.1) and (2.2) may be applied to any other span. For example, for span KL, i and j are replaced by K and L respectively.

CHAPTER III

DEFORMATIONS OF A ONE-SPAN BASIC STRUCTURE

In this chapter algebraic expressions are developed for the end angular functions of a basic span ij (Fig. 2.1) using the energy principles.

3.1 Strain Energy

In the light of the fourth assumption made in Chapter I, the strain energy stored in the circular bar ij (Fig. 2.1) due to the applied loads and the arbitrary end moments becomes

$$U_{si} = \frac{1}{2} \int_j^i (M_{qti})^2 \, d\lambda_t + \frac{1}{2} \int_j^i (M_{qbi})^2 \, d\lambda_b \qquad (3.1)$$

where $d\lambda_t$ and $d\lambda_b$ are the differential angle changes of an element of length dS_i due to unit causes in torsion and bending respectively, and they are:

$$d\lambda_t = \frac{dS_i}{GJ_{qi}} = \frac{R_i \, d\alpha_i}{GJ_{qi}}$$

and

$$d\lambda_b = \frac{dS_i}{EI_{qi}} = \frac{R_i \, d\alpha_i}{EI_{qi}}$$

where GJ_{qi} and EI_{qi} are the torsional and flexural rigidities of the bar at q.

3.2 Deformation Equations

By employing castigliano's first theorem, the total angular deformations at the ends of the bar ij (Fig. 2.1) are obtained as

$$\theta_{ijx} = \frac{\partial U_{si}}{\partial X_{ij}} = \int_j^i M_{qti} \frac{\partial M_{qti}}{\partial X_{ij}} \, d\lambda_t + \int_j^i M_{qbi} \frac{\partial M_{qbi}}{\partial X_{ij}} \, d\lambda_b \qquad (3.2a)$$

$$\theta_{ijy} = \frac{\partial U_{si}}{\partial Y_{ij}} = \int_j^i M_{qti} \frac{\partial M_{qti}}{\partial Y_{ij}} \, d\lambda_t + \int_j^i M_{qbi} \frac{\partial M_{qbi}}{\partial Y_{ij}} \, d\lambda_b \qquad (3.2b)$$

$$\theta_{jiy} = \frac{\partial U_{si}}{\partial Y_{ji}} = \int_j^i M_{qti} \frac{\partial M_{qti}}{\partial Y_{ji}} \, d\lambda_t + \int_j^i M_{qbi} \frac{\partial M_{qbi}}{\partial Y_{ji}} \, d\lambda_b \qquad (3.2c)$$

where θ_{ijx}, θ_{ijy}, and θ_{jiy} are the angular rotations at the ends i and j in the direction of the arbitrary applied moments X_{ij}, Y_{ij}, and Y_{ji}, respectively.

The first partial derivatives appearing in Eqs. (3.2) are given in the following table.

TABLE 3.1 PARTIAL DERIVATIVES

First Partial Derivative			
of Dependent Variable	with Respect to Independent Variable		
	X_{ij}	Y_{ij}	Y_{ji}
M_{qti}	A_{qti}	B_{qti}	C_{qti}
M_{qbi}	A_{qbi}	B_{qbi}	C_{qbi}

TABLE 3.2 ALGEBRAIC EXPRESSIONS FOR ANGULAR FUNCTIONS

Deformation / Caused By	At i	At j
$X_{ij} = 1$	$f_{ijxx} = \int_{-\omega_i}^{+\omega_i} (A_{qti})^2 d\lambda_t + \int_{-\omega_i}^{+\omega_i} (A_{qbi})^2 d\lambda_b$ $f_{ijyx} = \int_{-\omega_i}^{+\omega_i} (A_{qti})(B_{qti}) d\lambda_t + \int_{-\omega_i}^{+\omega_i} (A_{cbi})(B_{qbi}) d\lambda_b$	$g_{jiyx} = \int_{-\omega_i}^{+\omega_i} (A_{qti})(C_{qti}) d\lambda_t + \int_{-\omega_i}^{+\omega_i} (A_{qbi})(C_{qbi}) d\lambda_b$
$Y_{ij} = 1$	$f_{ijyy} = \int_{-\omega_i}^{+\omega_i} (B_{qti})^2 d\lambda_t + \int_{-\omega_i}^{+\omega_i} (B_{qbi})^2 d\lambda_b$ $f_{ijxy} = \int_{-\omega_i}^{+\omega_i} (A_{qti})(B_{qti}) d\lambda_t + \int_{-\omega_i}^{+\omega_i} (A_{qbi})(B_{qbi}) d\lambda_b$	$g_{jiyy} = \int_{-\omega_i}^{+\omega_i} (B_{qti})(C_{qti}) d\lambda_t + \int_{-\omega_i}^{+\omega_i} (B_{qbi})(C_{qbi}) d\lambda_b$
$Y_{ji} = 1$	$g_{ijyy} = \int_{-\omega_i}^{+\omega_i} (B_{qti})(C_{qti}) d\lambda_t + \int_{-\omega_i}^{+\omega_i} (B_{qbi})(C_{qbi}) d\lambda_b$ $g_{ijxy} = \int_{-\omega_i}^{+\omega_i} (A_{qti})(C_{qti}) d\lambda_t + \int_{-\omega_i}^{+\omega_i} (A_{qbi})(C_{qbi}) d\lambda_b$	$f_{jiyy} = \int_{-\omega_i}^{+\omega_i} (C_{qti})^2 d\lambda_t + \int_{-\omega_i}^{+\omega_i} (C_{qbi})^2 d\lambda_b$
Loads	$t_{ijx} = \int_{-\omega_i}^{+\omega_i} (bM_{qti})(A_{qti}) d\lambda_t + \int_{-\omega_i}^{+\omega_i} (bM_{qbi})(A_{qbi}) d\lambda_b$ $t_{ijy} = \int_{-\omega_i}^{+\omega_i} (bM_{qti})(B_{qti}) d\lambda_t + \int_{-\omega_i}^{+\omega_i} (bM_{qbi})(B_{qbi}) d\lambda_b$	$t_{jiy} = \int_{-\omega_i}^{+\omega_i} (bM_{qti})(C_{qti}) d\lambda_t + \int_{-\omega_i}^{+\omega_i} (bM_{qbi})(C_{qbi}) d\lambda_b$

Substituting for the moments from the matrix (2. 2),for the partial derivatives from the Table (3.1), the deformation Eqs. (3. 2) reduce to

$$\theta_{ijx} = X_{ij}(f_{ijxx}) + Y_{ij}(f_{ijxy}) + Y_{ji}(g_{ijxy}) + (t_{ijx}) \qquad (3.\ 3a)$$

$$\theta_{ijy} = X_{ij}(f_{ijyx}) + Y_{ij}(f_{ijyy}) + Y_{ji}(g_{ijyy}) + (t_{ijy}) \qquad (3.\ 3b)$$

$$\theta_{jiy} = X_{ij}(g_{jiyx}) + Y_{ij}(g_{jiyy}) + Y_{ji}(f_{jiyy}) + (t_{jiy}) \qquad (3.\ 3c)$$

where the coefficients in the parenthesis are the angular functions defined algebraically in Table (3. 2), and interpreted physically in the following Article 3. 3.

3. 3 Physical Interpretation of the One-Span Angular Functions

Fig. 3 illustrates the angular deformations of a basic span ij under the action of either a unit end moment or the loads applied across its span. The cause is shown on the left side of the page, and the corresponding angular functions on the right side. For example, the basic span ij, under the action of a unit end moment X_{ij} (Fig. 3. 1a), undergoes the end rotations f_{ijxx}, f_{ijyx}, g_{jiyx}, as shown in Fig. 3. 1b. The rotations are represented by double headed vectors and their direction is determined by the familiar right hand rule. For convenience let us call the left and right ends of a basic span as near and far ends, respectively.

f_{ijxx}, defined as the angular flexibility, is the rotation at the near end of the basic bar ij about its x-axis.

f_{ijyx}, defined as the angular near carry-over value, is the rotation at the near end of the bar ij about its y-axis.

g_{jiyx}, defined as the angular far carry-over value, is the rotation at the far end of the bar ij about its y-axis.

Similar explanation follows for the angular functions due to the unit moments, Y_{ij} and Y_{ji}, shown in the Figs. 3.2 and 3.3, respectively.

It may be noted here that, by the Maxwell's reciprocal theorem, the angular carry-over values f_{ijxy}, g_{ijyy}, and g_{jiyx} are equal to f_{ijyx}, g_{jiyy}, and g_{ijxy}, respectively.

The physical interpretation of the angular load functions t_{ijx}, t_{ijy}, and t_{jiy} is given in Fig. 3.4. Any general system of out-of-plane loads are shown in Fig. 3.4a, and the corresponding end deformations, defined as the angular load functions, are shown in Fig. 3.4b. A comparison with the Fig. 3.1b, reveals that t_{ijx}, t_{ijy}, and t_{jiy} are similar to the angular rotations f_{ijxx}, f_{ijyx}, and g_{jiyx}, respectively.

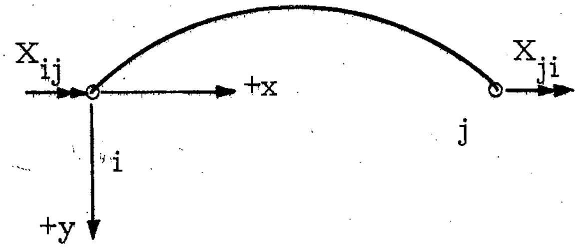


Fig. 3.1a

Applied Moment $X_{ij}=1$

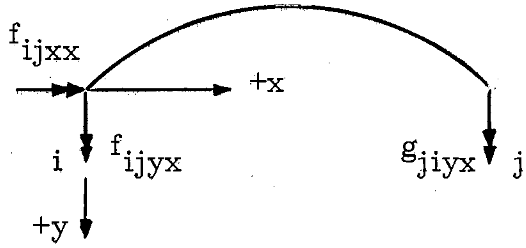


Fig. 3.1b

End Slopes

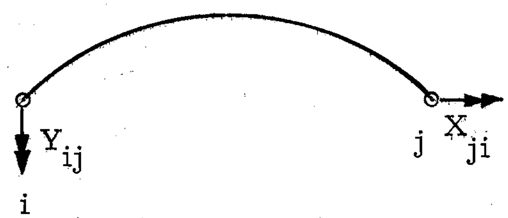


Fig. 3.2a

Applied Moment $Y_{ij}=1$

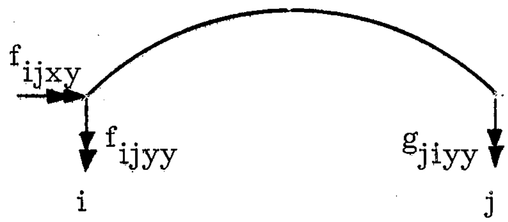


Fig. 3.2b

End Slopes

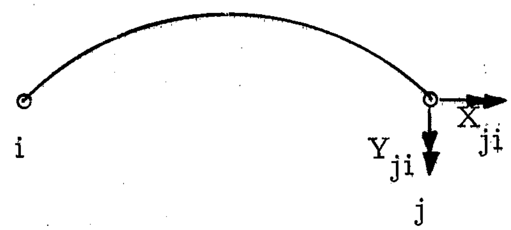


Fig. 3.3a

Applied Moment $Y_{ji}=1$

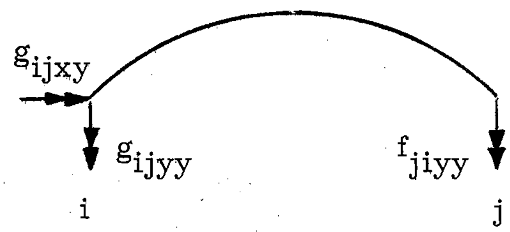


Fig. 3.3b

End Slopes

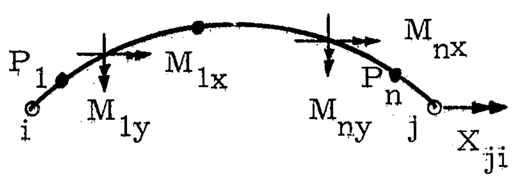


Fig. 3.4a

Applied Loads

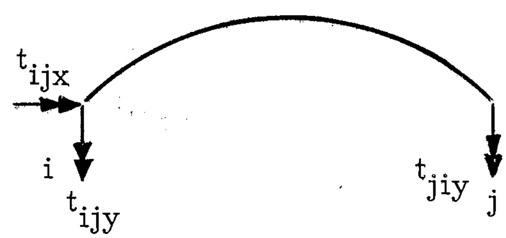


Fig. 3.4b

End Slopes

Fig. 3

Loads and Corresponding End Slopes of One-Span Basic Structure

CHAPTER IV

STEREO-STATICS OF AN n-SPAN BASIC STRUCTURE

An n-span continuous curved beam with a general system of normal loads is shown in Fig. 1. 3. The corresponding n-span determinant structure (Fig. 1. 5) is obtained as explained in Article 1. 3. In this continuous basic structure, the angular deformation caused by the applied loads at the near end of each span about its major chord is not continuous. The chord moments necessary to bring about the continuity of the said deformations are the redundant unknowns and they are shown as external loads in addition to the applied ones in Fig. 4. 1.

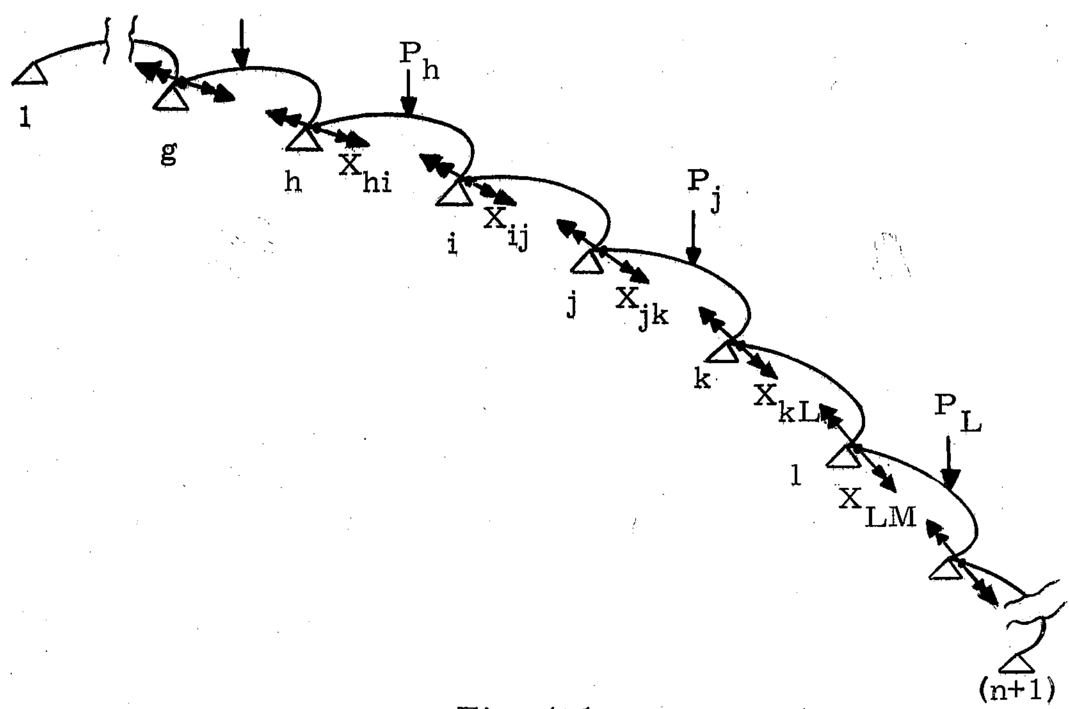


Fig. 4. 1

n-Span Basic Structure

4.1 Stereo-Statics and Joint Equilibrium

Span ij, isolated from Fig. 4.1, is shown as a freebody in Fig. 2.1. The matrix (2.1) defines the static equilibrium of the span ij. An orderly change of subscripts provides such equations for any other span, as explained at the end of Article 2.3.

To express the unknown moments Y_{ij} and Y_{ji}, consider the equilibrium of the forces and moments at joint i (Fig. 4.2).

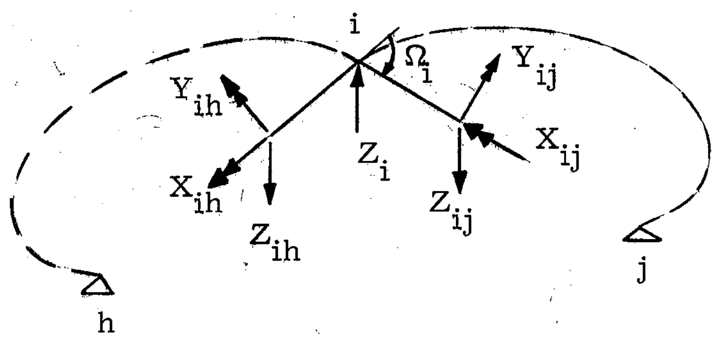


Fig. 4.2

Freebody of Joint i

In the above freebody sketch, Z_{ih}, X_{ih}, Y_{ih} are the positive reactions acting at the far end of the span hi, and Z_{ij}, X_{ij}, Y_{ij} are those at the near end of the span ij. Z_i is the positive reactive force provided by the simple support at i.

The unknown force Z_i and moments Y_{ih}, Y_{ij} are expressed in terms of the redundant moments X_{hi}, X_{ij} and the applied loads, by employing the equilibrium condition of the joint i.

By summing forces in the z-direction

$$Z_i = Z_{ih} + Z_{ij} \tag{4.1a}$$

By summing moments about chord ij

$$Y_{ih} = - X_{ih} \cot \Omega_i - X_{ij} \operatorname{cosec} \Omega_i \tag{4.1b}$$

By summing moments about chord hi

$$Y_{ij} = X_{ih} \operatorname{cosec} \Omega_i + X_{ij} \cot \Omega_i \tag{4.1c}$$

Denote for the trigonometric functions

$$\operatorname{cosec} \Omega_i = \psi_i$$

$$\cot \Omega_i = n_i$$

and for the reaction influence coefficients

$$XZ_i^{(gh)} = D_{zh} \psi_h$$

$$XZ_i^{(hi)} = - \left(D_{zh}(n_h + n_i) + D_{zi} \psi_i\right)$$

$$XZ_i^{(ij)} = \left(D_{zh} \psi_i + D_{zi}(n_i + n_j)\right)$$

$$XZ_i^{(jk)} = - D_{zi} \psi_j$$

$$BZ_i^{(gh)} = - D_{zh} \psi_h$$

$$BZ_i^{(hi)} = (D_{zh} n_i + D_{zi} \psi_i)$$

$$BZ_i^{(ij)} = - D_{zi} n_i \tag{4.2}$$

Substituting from the static equilibrium matrix (2. 1), and using the notation defined vide Eqs. (4. 2), the joint equilibrium Eqs. (4. 1) reduce to the following:

Joint Equilibrium Matrix 4. 1

$$\begin{bmatrix} Y_{ih} \\ Y_{ij} \\ Z_i \end{bmatrix} = \begin{bmatrix} - & n_i & -\psi_i & - \\ - & -\psi_i & n_i & - \\ XZ_i^{(gh)} & XZ_i^{(hi)} & XZ_i^{(ij)} & XZ_i^{(jk)} \end{bmatrix} \begin{bmatrix} X_{gh} \\ X_{hi} \\ X_{ij} \\ X_{jk} \end{bmatrix} + \begin{bmatrix} BY_{ih} \\ BY_{ij} \\ BZ_i \end{bmatrix}$$

where BY_{ih}, BY_{ij}, and BZ_i are the corresponding reactions at i due to the applied loads only. They are

Joint Equilibrium Matrix 4. 2

$$\begin{bmatrix} BY_{ih} \\ BY_{ij} \\ BZ_i \end{bmatrix} = \begin{bmatrix} - & -n_i & - \\ - & \psi_i & - \\ BZ_i^{(gh)} & BZ_i^{(hi)} & BZ_i^{(ij)} \end{bmatrix} \begin{bmatrix} bX_{gh} \\ bX_{hi} \\ bX_{ij} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ bZ_i \end{bmatrix}$$

where bZ_i is the sum of bZ_{ih} and bZ_{ij} which are the positive reactions at end i of the one span basic structures hi and ij, respectively.

The use of sub and superscripts for the reaction influence coefficients needs explanation. The main letters X and B preceding Z denote the influence of the redundant moment and applied loads, respectively. The subscript stands for the support and the superscript for the influencing redundant or applied chord moment.

Again, by proper change of the script-letters, the matrices (4.2) and (4.3) give the necessary equilibrium equations of any other joint.

4.2 Cross-Sectional Elements.

The cross-sectional elements, at any section q of span ij (Figs. 2.1, 2.2), are available from the matrix (2.2) in terms of the moments X_{ij}, Y_{ij}, Y_{ji}, and the applied loads.

Denote for the cross-sectional influence coefficients:

$$-D_{qzi}\psi_i = Xv_{qzi}^{(hi)} \quad ; \quad D_{qzi}(n_i + n_j) = Xv_{qzi}^{(ij)}$$

$$-D_{qzi}\psi_j = Xv_{qzi}^{(jk)} \quad ; \quad -B_{qti}\psi_i = Xm_{qti}^{(hi)}$$

$$(A_{qti} + n_i B_{qti} + n_j C_{qti}) = Xm_{qti}^{(ij)} \quad ; \quad -C_{qti}\psi_j = Xm_{qti}^{(jk)}$$

$$-B_{qbi}\psi_i = Xm_{qbi}^{(hi)} \quad ; \quad (A_{qbi} + n_i B_{qbi} + n_j C_{qbi}) = Xm_{qbi}^{(ij)}$$

$$-C_{qbi}\psi_j = Xm_{qbi}^{(jk)} \quad ; \quad D_{qzi}\psi_i = Bv_{qzi}^{(hi)}$$

$$-D_{qzi}n_j = Bv_{qzi}^{(ij)} \quad ; \quad B_{qti}\psi_i = Bm_{qti}^{(hi)}$$

$$-C_{qti}n_j = Bm_{qti}^{(ij)} \quad ; \quad B_{qbi}\psi_i = Bm_{qbi}^{(hi)}$$

and

$$-C_{qbi}n_j = Bm_{qbi}^{(ij)} \qquad (4.3)$$

Now substituting for Y-moments from the matrices (4.1) and (4.2), using the above notation, the cross-sectional force matrix (2.2) for span ij becomes:

Cross-Sectional Force Matrix 4.3

$$\begin{bmatrix} V_{qzi} \\ M_{qti} \\ M_{qbi} \end{bmatrix} = \begin{bmatrix} Xv_{qzi}^{(hi)} & Xv_{qzi}^{(ij)} & Xv_{qzi}^{(jk)} \\ Xm_{qti}^{(hi)} & Xm_{qti}^{(ij)} & Xm_{qti}^{(jk)} \\ Xm_{qbi}^{(hi)} & Xm_{qbi}^{(ij)} & Xm_{qbi}^{(jk)} \end{bmatrix} \begin{bmatrix} X_{hi} \\ X_{ij} \\ X_{jk} \end{bmatrix} + \begin{bmatrix} BV_{qzi} \\ BM_{qti} \\ BM_{qbi} \end{bmatrix}$$

where the main letter X and the superscripts denote the redundant moment influence. The load terms are:

Cross-Sectional Force Matrix 4.4

$$\begin{bmatrix} BV_{qzi} \\ BM_{qti} \\ BM_{qbi} \end{bmatrix} = \begin{bmatrix} Bv_{qzi}^{(hi)} & Bv_{qzi}^{(ij)} \\ Bm_{qti}^{(hi)} & Bm_{qti}^{(hi)} \\ Bm_{qbi}^{(hi)} & Bm_{qbi}^{(ij)} \end{bmatrix} \begin{bmatrix} bX_{hi} \\ bX_{ij} \\ \end{bmatrix} + \begin{bmatrix} bV_{qzi} \\ bM_{qti} \\ bM_{qbi} \end{bmatrix}$$

where the main letter B and the superscripts denote the influence of the applied load chord moment. The shear bV_{qzi} and the moments bM_{qti}, bM_{qbi} are due to the applied loads acting on the basic span ij, as defined in Chapter II.

The matrices (4.3) and (4.4) are general and therefore may be applied to any other span. This is done simply by switching from letters h, i, j, · · ·, to k, l, m, · · ·, respectively.

CHAPTER V

DERIVATION OF FIVE-MOMENT EQUATION

In this Chapter, the principle of least work is employed to derive the five-moment equation. Its application for various end conditions of a continuous structure is discussed. Also, the carry-over form of this equation is developed in Article 5.4.

Consider the n-span continuous curved beam (Fig. 1.3). The corresponding basic structure with the redundant moments and applied loads is shown in Fig. 4.1. The supports 1, 2, · · ·, i, j, k, · · ·, n, n+1, are assumed to undergo positive displacements Δ_{1z}, Δ_{2z}, · · ·, Δ_{iz}, Δ_{jz}, Δ_{kz}, · · ·, Δ_{nz}, $\Delta_{n+1,z}$, respectively.

5.1 Total Potential Energy

The total energy consists of the strain energy of the curved bar minus the work of reactions.

Neglecting the shear contribution, the total strain energy is given as

$$U_s = \sum_{i=1}^{n} U_{si}$$

or

$$U_s = \sum_{i=1}^{n} \frac{1}{2} \int_{-w_i}^{+w_i} \left((M_{qti})^2 \, d\lambda_t + (M_{qbi})^2 \, d\lambda_b \right) \qquad (5.1)$$

The work done by the reactions is

$$U_R = \sum_{i=1}^{n+1} U_{Ri} \tag{5.2a}$$

or

$$U_R = \sum_{i=1}^{n+1} (-Z_i \Delta_{iz}) \tag{5.2b}$$

where the cross-sectional moments M_{qti}, M_{qbi}, and support reactions Z_i are available from matrices (4.3, 4.4) and (4.1, 4.2), respectively.

Thus the total potential energy of the n-span structure becomes

$$U = (U_S - U_R)$$

or

$$U = \sum_{i=1}^{n} \frac{1}{2} \int_{-w_i}^{+w_i} \left((M_{qti})^2 \, d\lambda_t + (M_{qbi})^2 d\lambda_b \right) + \sum_{i=1}^{n+1} (Z_i \Delta_{iz}) \tag{5.3}$$

5.2 Compatibility Conditions

The condition of consistent deformation across the continuous structure would provide compatibility equations. The number of available compatibility equations is equal to the total number of redundant moments. It is sufficient to consider one equation, since all the compatibility equations are similar in form, before any modifications for the end conditions are applied.

Mathematically, such a condition for the continuous angular deformation across the span ij of the n-span continuous curved beam (Fig. 1.3) is stated, by the least work theorem, as

$$\frac{\partial U}{\partial X_{ij}} = 0$$

where X_{ij} is the redundant moment in span ij.

Using the Eq. (5. 3) for the total energy, the above condition becomes

$$\sum_{i=1}^{n} \int_{-\omega_i}^{\omega_i} \left(M_{qti} \frac{\partial M_{qti}}{\partial X_{ij}} d\lambda_t + M_{qbi} \frac{\partial M_{qbi}}{\partial X_{ij}} d\lambda_b \right) + \sum_{i=1}^{n+1} \frac{\partial Z_i}{\partial X_{ij}} \Delta_{iz} = 0$$

A close examination of the matrices (4. 1) and (4. 3) reveals that the redundant moment X_{ij} can influence, at the most, the reactions provided by the supports at h, i, j, k, and the cross-sectional elements in spans hi, ij, jk. Thus the above equation reduces to

$$\sum_{i=h}^{j} \int_{-\omega_i}^{\omega_i} \left(M_{qti} \frac{\partial M_{qti}}{\partial X_{ij}} d\lambda_t + M_{qbi} \frac{\partial M_{qbi}}{\partial X_{ij}} d\lambda_b \right) + \sum_{i=h}^{k} \frac{\partial Z_i}{\partial X_{ij}} \Delta_{iz} = 0 \qquad (5.4)$$

The first partial derivatives appearing in Eq. (5. 4) are tabulated below.

TABLE 5. 1 FIRST PARTIAL DERIVATIVES

Support	Span ↓ / Partial Derivative →	$\frac{\partial M_{qt}}{\partial X_{ij}}$	$\frac{\partial M_{qb}}{\partial X_{ij}}$	$\frac{\partial Z}{\partial X_{ij}}$
h				$XZ_h^{(ij)}$
	hi	$Xm_{qth}^{(ij)}$	$Xm_{qbh}^{(ij)}$	
i				$XZ_i^{(ij)}$
	ij	$Xm_{qti}^{(ij)}$	$Xm_{qbi}^{(ij)}$	
j				$XZ_j^{(ij)}$
	jk	$Xm_{qtj}^{(ij)}$	$Xm_{qbj}^{(ij)}$	
k				$XZ_k^{(ij)}$

5.3 Five-Moment Equation

Substituting for, the cross-sectional moments from the matrix (4.3), the partial derivatives from the Table (5.1), and introducing the notation of Table (5.2), the Eq. (5.4) reduces to

$$\begin{bmatrix} X_{gh}\ (G_{gi}) & & \\ X_{hi}\ (G_{hi}) & & \\ X_{ij}\ \ (F_i) \quad + & (\tau_i^{(L)}) & (\tau_i^{(\Delta)}) \\ X_{jk}\ (G_{ji}) & & \\ X_{kl}\ (G_{ki}) & & \end{bmatrix} = 0 \qquad (5.5)$$

which is the desired five-moment equation. It may be noted, that each of the terms in the above Eq. (5.5) represents an angular rotation of the n-span basic structure about chord ij due to the redundant moments, or the applied loads or the vertical displacements of supports.

In an n-span continuous beam 1, 2, $\cdots$, g, h, i, j, k, l, $\cdots$, n, n+1, the spans hi, jk, and gh, kl, are denoted as the near and far spans, respectively, with respect to the span ij.

The quantities in the parenthesis of Eq. (5.5) are defined below, and their physical interpretation is given in the next article.

G_{gi} and G_{ki} are defined as the angular far carry-over values from the far spans gh and kl, respectively, to the span ij of the n-span basic structure.

G_{hi} and G_{ji} are defined as the angular near carry-over values from the near spans hi and jk, respectively, to the span ij of the n-span basic structure.

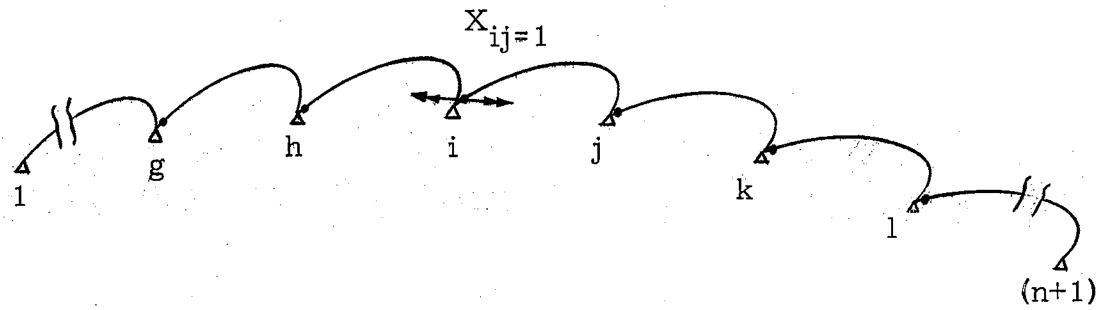


Fig. 5.1a Applied Unit Moment x_{ij}

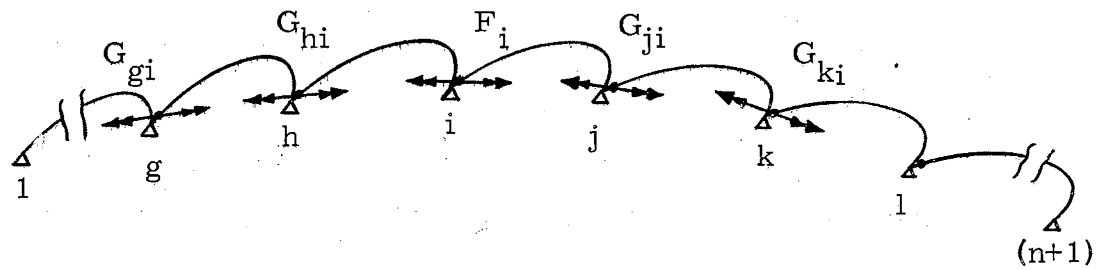


Fig. 5.1b Angular Flexibilities and Carry-Over Values

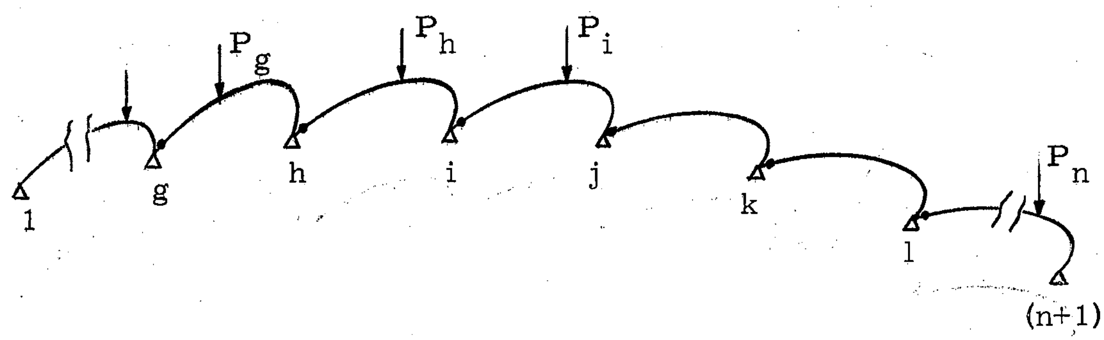


Fig. 5.1c Applied Loads

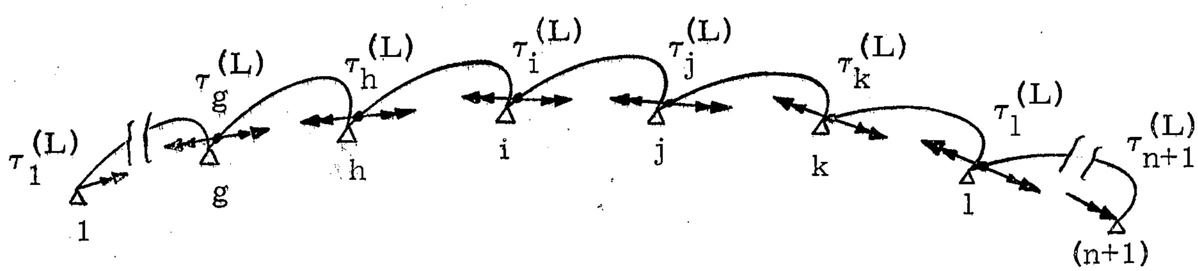


Fig. 5.1d Angular Load Functions

Fig. 5.1

n-Span Basic Structure - Physical Interpretation of Angular Functions

F_i is the angular flexibility of the span ij of the n-span basic structure.

$\tau_i^{(L)}$ and $\tau_i^{(\Delta)}$ are the angular load and displacement functions of the span ij of the n-span basic structure due to the applied loads and the support displacements, respectively.

5.4 Physical Interpretation of the n-Span Angular Functions

Fig. 5.1 illustrates the n-span basic angular functions. A unit couple X_{ij} is applied as shown in Fig. 5.1a. The corresponding angular deformations are G_{gi}, G_{hi}, F_i, G_{ji}, G_{ki}, shown in Fig. 5.1b. These are actually relative rotations at the corresponding hinges. Therefore, they are represented by two double headed vectors, as shown.

The loads are applied across the n-span basic structure (Fig. 5.1c). The corresponding relative rotations are $\tau_1^{(L)}, \cdots, \tau_g^{(L)}, \tau_h^{(L)}, \tau_i^{(L)}, \cdots$, at the hinges at 1, $\cdots$, g, h, i, $\cdots$ (Fig. 5.1d).

In Fig. 5.1c, if the applied loads are zero, and instead, the supports undergo vertical displacements, the corresponding relative rotations in Fig. 5.1d will be $\tau_1^{(\Delta)}, \cdots, \tau_g^{(\Delta)}, \tau_h^{(\Delta)}, \tau_i^{(\Delta)}, \cdots$, etc.

5.5 Modification of the Five-Moment Equation for Various End Conditions

The application of the five-moment Equation (5.5) to the spans near the ends needs few modifications. This is explained below for all possible cases.

a) Both Ends on Simple Supports

This can best be explained through an illustration. For this consider a six-span continuous beam ghijklm on simple supports (Fig. 5.2) loaded by a general system of loads.

TABLE 5.2 ANGULAR FUNCTIONS OF n-SPAN BASIC STRUCTURE

Angular Deformation About / Caused By	Chord ij of n-Span Basic Structure
$X_{gh} = 1$	$G_{gi} = \int_i^h (Xm_{qth}^{(gh)})(Xm_{qth}^{(ij)})\,d\lambda_t + \int_i^h (Xm_{qbh}^{(gh)})(Xm_{qbh}^{(ij)})\,d\lambda_b$
$X_{hi} = 1$	$G_{hi} = \int_i^h (Xm_{qth}^{(hi)})(Xm_{qth}^{(ij)})\,d\lambda_t + \int_i^h (Xm_{qbh}^{(hi)})(Xm_{qbh}^{(ij)})\,d\lambda_b$ $\int_j^i (Xm_{qti}^{(hi)})(Xm_{qti}^{(ij)})\,d\lambda_t + \int_j^i (Xm_{qbi}^{(hi)})(Xm_{qbi}^{(ij)})\,d\lambda_b$
$X_{ij} = 1$	$\int_i^h (Xm_{qth}^{(ij)})^2\,d\lambda_t + \int_i^h (Xm_{qbh})^2\,d\lambda_b$ $F_i = \int_j^i (Xm_{qti}^{(ij)})^2\,d\lambda_t + \int_j^i (Xm_{qbi}^{(ij)})^2\,d\lambda_b$ $\int_k^j (Xm_{qtj}^{(ij)})^2\,d\lambda_t + \int_k^j (Xm_{qbj}^{(ij)})^2\,d\lambda_b$
$X_{jk} = 1$	$G_{ji} = \int_j^i (Xm_{qti}^{(jk)})(Xm_{qti}^{(ij)})\,d\lambda_t + \int_j^i (Xm_{qbi}^{(jk)})(Xm_{qbi}^{(ij)})\,d\lambda_b$ $\int_k^j (Xm_{qtj}^{(jk)})(Xm_{qtj}^{(ij)})\,d\lambda_t + \int_k^j (Xm_{qbj}^{(jk)})(Xm_{qbj}^{(ij)})\,d\lambda_b$
$X_{kl} = 1$	$G_{ki} = \int_k^j (Xm_{qtj}^{(kl)})(Xm_{qtj}^{(ij)})\,d\lambda_t + \int_k^j (Xm_{qbj}^{(kl)})(Xm_{qbj}^{(ij)})\,d\lambda_b$
Loads	$\int_i^h (BM_{qth})(Xm_{qth}^{(ij)})\,d\lambda_t + \int_i^h (BM_{qbh})(Xm_{qbh}^{(ij)})\,d\lambda_b$ $\tau_i^{(L)} = \int_j^i (BM_{qti})(Xm_{qti}^{(ij)})\,d\lambda_t + \int_j^i (BM_{qbi})(Xm_{qbi}^{(ij)})\,d\lambda_b$ $\int_k^j (BM_{qtj})(Xm_{qtj}^{(ij)})\,d\lambda_t + \int_k^j (BM_{qbj})(Xm_{qbj}^{(ij)})\,d\lambda_b$
Displacements	$\tau_i^{(\Delta)} = (XZ_h^{(ij)})\,\Delta_{hz} + (XZ_i^{(ij)})\,\Delta_{iz} + (XZ_j^{(ij)})\,\Delta_{jz} + (XZ_k^{(ij)})\,\Delta_{kz}$

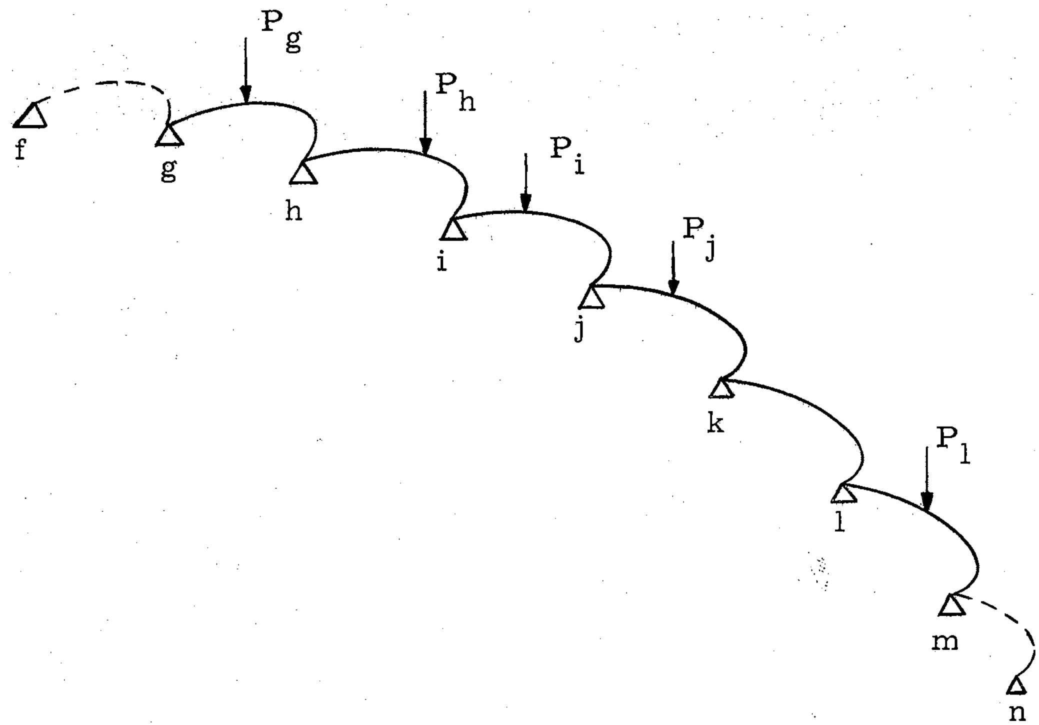


Fig. 5.2

Six-Span Continuous Girder with Simple Ends

It may be noted, that the five-moment equation is developed in terms of the redundant chord moments. In this case there are six chords, with which six chord moments X_{gh}, X_{hi}, X_{ij}, X_{jk}, X_{kl}, and X_{lm} may possibly be associated. The moments X_{gh} and X_{lm} are statically determinate, and they are obtained from the known end conditions and the statics of the end spans as

$$X_{gh} = 0 \tag{5.6a}$$

$$X_{lm} = bX_{lm} \tag{5.6b}$$

The structure is statically indeterminate to the 4th degree. X_{hi}, X_{ij}, X_{jk}, and X_{kl} are the four redundants. The structure is extended at both ends as shown by broken lines only for convenience in the application of the five-moment equation to the end redundants X_{hi} and X_{kl}. The chord moments and all the load and angular functions associated with these imaginary spans fg and mn are zero. With the above information, the application of the five-moment equation to the four redundants becomes straight forward.

b) Both Ends Fixed Along x and y Axes of the End Spans

A six-span continuous structure ghijklm with both ends fixed (Fig. 5.3) is indeterminate to the 8th degree. The symbol "•——•" stands for an end fixed about an axis perpendicular to it.

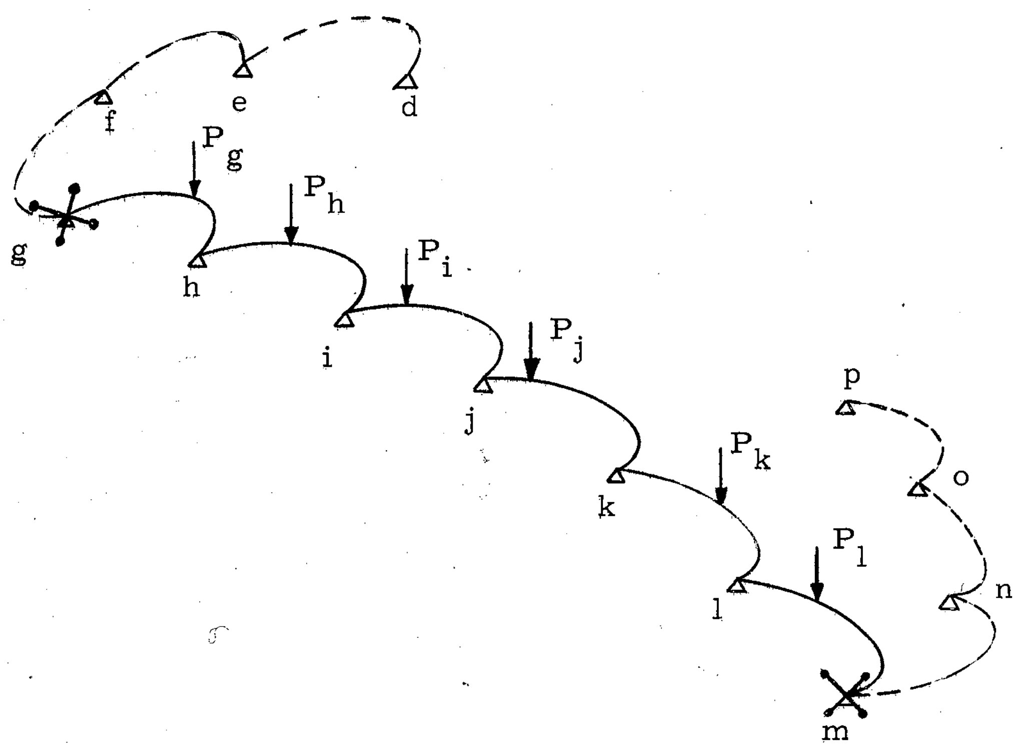


Fig. 5.3

Six-Span Continuous Girder with Fixed Ends

Y_{gh}, X_{gh}, X_{hi}, X_{ij}, X_{jk}, X_{kl}, X_{lm}, Y_{ml} are the redundant moments. The five-moment equation can not take care of the Y-moment redundants directly. Therefore, it is necessary to transform them into chord moments by some means. The Y-moments at the ends may be easily visualized as chord moments associated with zero spans. In other words, this is equivalent to saying that they are associated with the imaginary spans fg and mn, whose load and angular functions are zero. These imaginary spans are oriented such that the redundants Y_{gh} and Y_{ml} are respectively equal to the chord moments X_{fg} and X_{mn}. This can be easily verified by examining the so-called freebodies of the joints g and m (Fig. 5.4). Thus

$$Y_{gh} = X_{fg} \quad , \qquad \Omega_g = -\frac{\pi}{2} \tag{5.7a}$$

and

$$Y_{ml} = X_{mn} \quad , \qquad \Omega_m = -\frac{\pi}{2} \tag{5.7b}$$

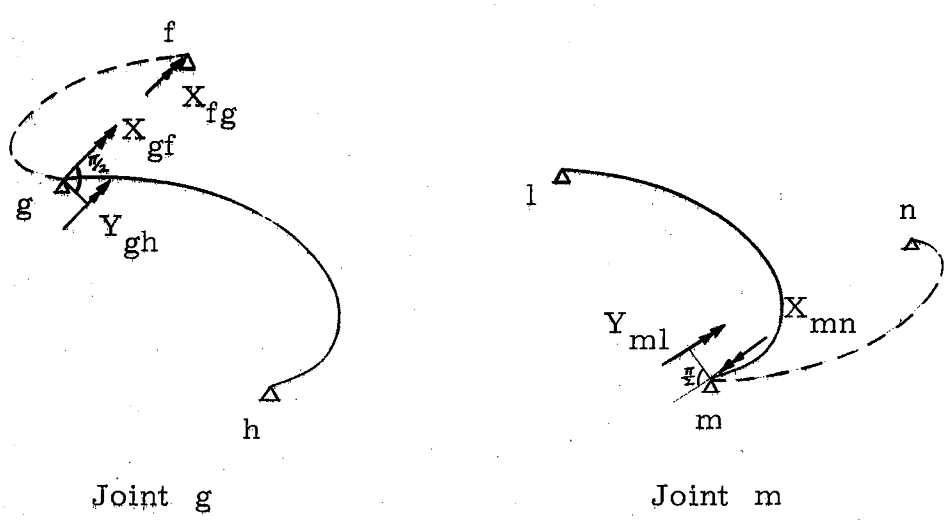


Fig. 5.4

Joint Free-Bodies

The new set of redundants are X_{fg}, X_{gh}, X_{hi}, X_{ij}, X_{jk}, X_{kl}, X_{lm}, and X_{mn}. The imaginary spans de, ef, no, op (Fig. 5. 3) are added only to facilitate the application of the five-moment equation to the redundants X_{fg}, X_{gh}, X_{lm}, and X_{mn}. The moments and the angular functions associated with these spans are zero. Thus with this modification the application of the five-moment equation to any redundant becomes simple.

Any other combination of the end conditions is simply a special case of (a) and (b) explained above. Therefore, with the tools developed in (a) and (b) any end conditions can be easily tackled.

5. 6 Carry-Over Five-Moment Equation

Solving the five-moment Equation (5. 5) for X_{ij}, we get

$$X_{ij} = \left[\begin{array}{l} X_{gh} \left(-\frac{G_{gi}}{F_i}\right) + X_{hi} \left(-\frac{G_{hi}}{F_i}\right) \\ \\ X_{kl} \left(-\frac{G_{ki}}{F_i}\right) + X_{jk} \left(-\frac{G_{ji}}{F_i}\right) \end{array} + \left(-\frac{\tau_i^{(L)}}{F_i}\right) + \left(-\frac{\tau_i^{(\Delta)}}{F_i}\right) \right] \tag{5. 8}$$

Define the carry-over moment factors

$$\left(-\frac{G_{gi}}{F_i}\right) = r_{gi} \quad ; \quad \left(-\frac{G_{hi}}{F_i}\right) = r_{hi} \tag{5. 9a}$$

$$\left(-\frac{G_{ki}}{F_i}\right) = r_{ki} \quad ; \quad \left(-\frac{G_{ji}}{F_i}\right) = r_{ji} \tag{5. 9b}$$

The subscripts denote the influence of one redundant upon the other. For example r_{gi} is the carry-over from redundant X_{gh} to the redundant X_{ij}.

The load and displacement functions on the extreme right side of Eq. (5.8) are defined as the corresponding starting moments associated with the redundant moment X_{ij}, and they are denoted respectively as

$$\left(-\frac{\tau_i^{(L)}}{F_i}\right) = m_i^{(L)} \qquad ; \qquad \left(-\frac{\tau_i^{(\Delta)}}{F_i}\right) = m_i^{(\Delta)} \qquad (5.10)$$

Using the above notations in Eq. (5.8), we get the desired carry-over five-moment equation as

$$X_{ij} = \left[\begin{matrix} X_{gh}\,(r_{gi}) + X_{hi}\,(r_{hi}) \\ \\ X_{kl}\,(r_{ki}) + X_{jk}\,(r_{ji}) \end{matrix} \quad + \quad m_i^{(L)} \quad + \quad m_i^{(\Delta)}\right] \qquad (5.11)$$

The physical interpretation of the moment functions appearing in Eq. (5.11) is given in Article 5.5.

5.5 Physical Interpretation of the Moment Functions

a) Carry-Over Moment Factors

An n-span continuous structure indeterminate to degree one is considered (Fig. 5.5). This differs from the n-span basic structure (Fig. 1.5) in that there is no hinge placed in its span ij. No loads are applied across its span. It is assumed that the supports do not undergo any displacements.

A unit couple X_{gh} is applied as shown. The redundant moment X_{ij} can be solved by employing Eq. (5.11), in which the chord moments X_{hi}, X_{jk}, X_{kl}, and the starting moments $m_i^{(L)}$, $m_i^{(\Delta)}$ are zero. Thus, the carry-over moment factor r_{gi} is the chord moment X_{ij} developed at the near end of the span ij.

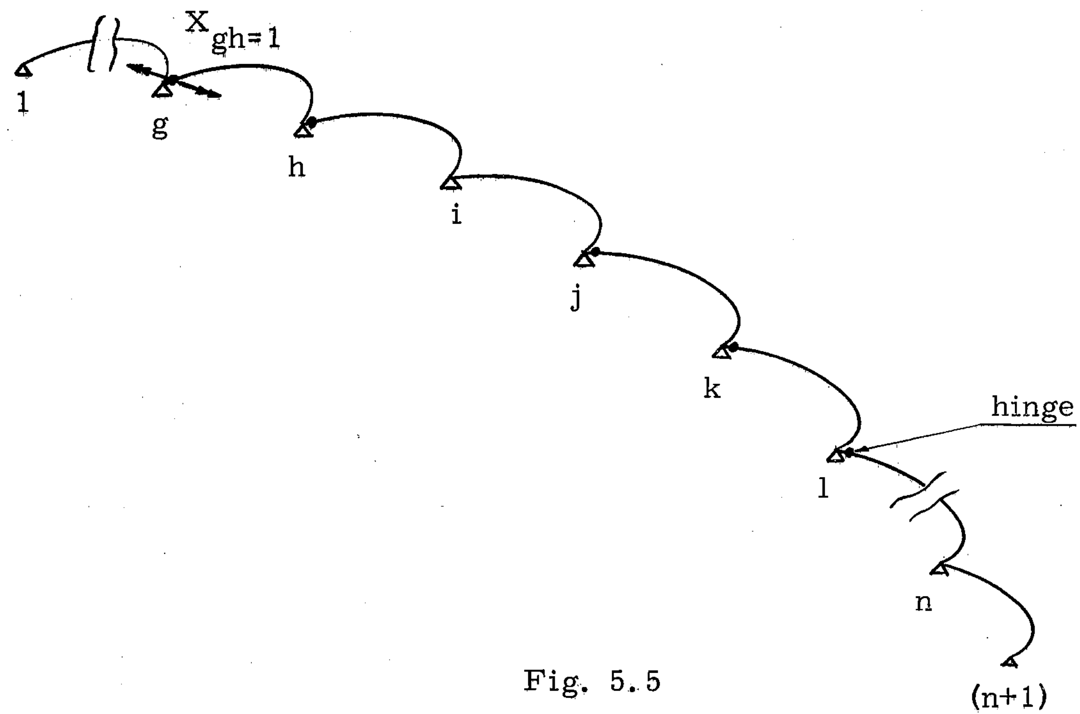


Fig. 5.5

n-Span Continuous Curved Beam

Similarly the carry-over factors r_{hi}, r_{ji}, or r_{ki} may be defined as the chord moment X_{ij} due to unit couples X_{hi}, X_{ji}, or X_{ki}, respectively. In view of the relative location of the spans gh, hi, jk, kl, with respect to the span ij, r_{hi}, r_{ji}, and r_{gi}, $_{ki}$ are called the near and far carry-over factors, respectively.

b) Starting Moment Functions

Consider the n-span continuous beam (Fig. 5.6). A general system of loads is assumed to act across the structure. It is assumed that there are no support displacements.

In Eq. (5.11), on the right side, all the terms except $m_i^{(L)}$ are zero. Thus, the redundant X_{ij} equals $m_i^{(L)}$. This defines the load starting moment $m_i^{(L)}$, as the chord moment X_{ij} developed by the applied loads only.

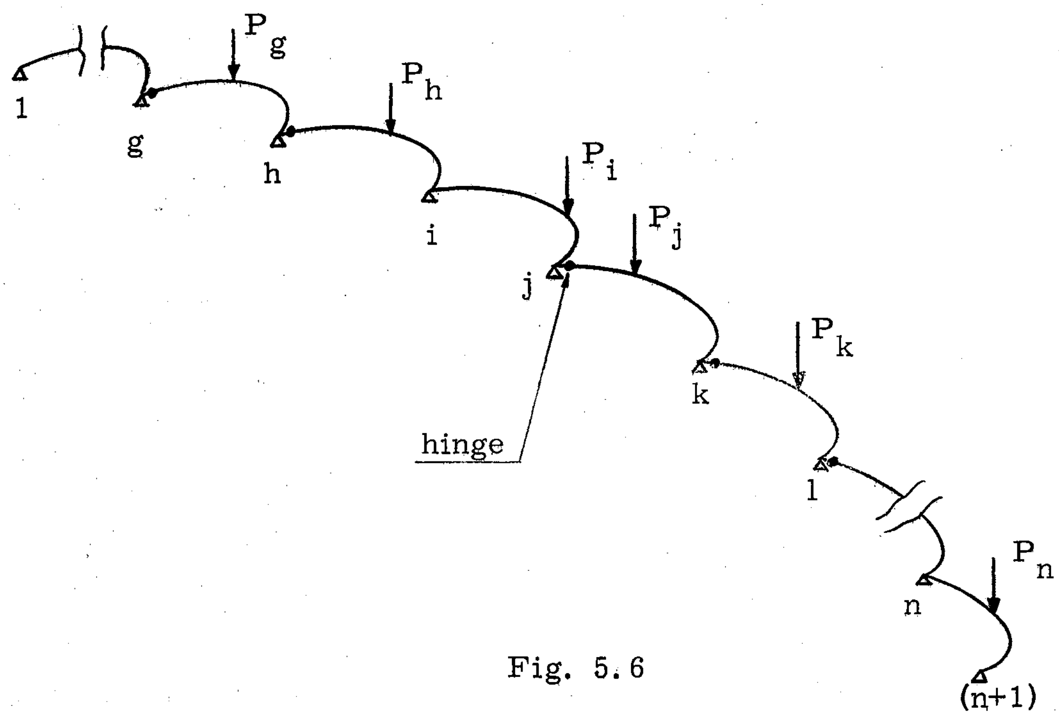


Fig. 5.6

n-Span Continuous Curved Beam

Similarly in Fig. 5.6, if all the applied loads are zero and instead, the supports undergo displacements, the redundant moment X_{ij} becomes equal to the displacement starting moment $m_i^{(\Delta)}$.

CHAPTER VI

INFLUENCE LINES

An influence line is a diagram which shows for a particular section the variation of any function - force, moment, or deformation - due to a unit load moving across the structure. Its analysis in the case of a statically determinate structure is simply a matter of statics. In this chapter, the procedure and necessary analytical expressions for obtaining such diagrams are developed for the case of indeterminate structures. It is assumed, throughout this chapter, that the supports do not undergo any displacements.

6.1 Influence Lines for Forces and Moments

The first and the most important step consists of the determination of the redundant moment influence equations. The carry-over procedure renders the analysis practical, provided that the number of the redundants does not exceed three. In case of more redundants, the compatibility matrix is solved for the desired equations by means of a digital computer. Both the methods are explained below separately.

a) Algebraic Carry-Over Moment Method

Consider a structure ghijkl with simple ends (Fig. 6.1), in which X_{hi}, X_{ij}, and X_{jk} are the three redundant chord moments.

The carry-over moment factors defined vide Eq. (5.9) are noted in Table (6.1). Let the load-starting moments, corresponding to the redundants, for any position of a moving unit load be $m_h^{(L)}$, $m_i^{(L)}$, and

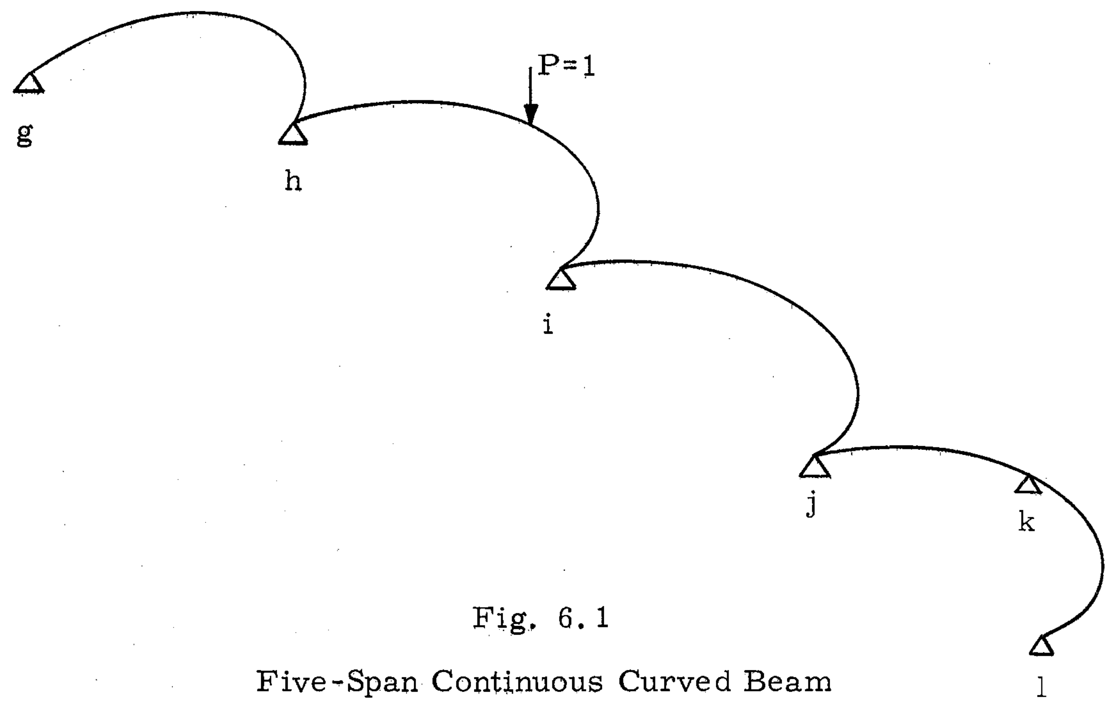


Fig. 6.1

Five-Span Continuous Curved Beam

$m_j^{(L)}$, respectively.

The carry-over procedure, explained else where[18, 19, 20], is an iteration process to solve simultaneous linear equations. As the principle of superposition holds, the redundant moments are solved for individual effects of the starting moments. Table (6.1) illustrates the carry-over process for a unit starting moment $m_h^{(L)}$.

In the last row of Table (6.1) are shown the moment influence coefficients m_{hh}, m_{hi}, and m_{hj} obtained by summing the solumns. These are due to a unit value of $m_h^{(L)}$.

Similar procedure for unit values of $m_i^{(L)}$ and $m_j^{(L)}$ will provide the corresponding moment influence coefficient m_{ih}, m_{ii}, m_{ij} and m_{jh}, m_{ji}, m_{jj}, respectively.

Superposing the results, the redundant moments are expressed as:

TABLE 6.1 MOMENT INFLUENCE COEFFICIENTS

FOR $m_h^{(L)} =$ Unity

Redundants	X_{hi}	X_{ij}	X_{jk}
Carry-Over Factors	r_{hi} →	→	
	r_{hj} →	→	→
		r_{ij} →	→
	←	← r_{ih}	
	←	←	← r_{jh}
		←	← r_{ji}
Starting Moments	1	0	0
Carry-Over Process			
C. O. No: 1		r_{hi}	r_{hj}
C. O. No: 2	$(r_{ih}r_{hi})$		$(r_{ij}r_{hi})$
C. O. No: 3	$r_{jh}(r_{hj}+r_{ij}r_{hi})$	$r_{ji}(r_{hj}+r_{ij}r_{hi})$	
C. O. No: 4			
and so on till the process converges			
$\sum$	m_{hh}	m_{hi}	m_{hj}

Redundant Moment Influence Matrix 6.1

$$\begin{bmatrix} X_{hi} \\ X_{ij} \\ X_{jk} \end{bmatrix} = \begin{bmatrix} m_{hh} & m_{ih} & m_{jh} \\ m_{hi} & m_{ji} & m_{ji} \\ m_{hj} & m_{ii} & m_{jj} \end{bmatrix} \begin{bmatrix} m_h \\ m_i \\ m_j \end{bmatrix}$$

Substituting for the starting moments vide notation of Eq. (5.10), and incorporating the angular flexibility terms with the moment influence coefficients, the above matrix becomes

Redundant Moment Influence Matrix 6.2

$$\begin{bmatrix} X_{hi} \\ X_{ij} \\ X_{jk} \end{bmatrix} = \begin{bmatrix} a_{hh} & a_{ih} & a_{jh} \\ a_{hi} & a_{ii} & a_{ji} \\ a_{hj} & a_{ij} & a_{jj} \end{bmatrix} \begin{bmatrix} -\tau_h \\ -\tau_i \\ -\tau_j \end{bmatrix}$$

where the "a" terms, for example, are

$$a_{ih} = \frac{m_{ih}}{F_i} \quad , \quad a_{ji} = \frac{m_{ji}}{F_j} \quad , \quad a_{hj} = \frac{m_{hj}}{F_h}$$

and so on.

Thus, with the angular load functions known for any position of the moving unit load, the redundant moments are determined by matrix (6.2).

b) Matrix Solution of Redundants

As mentioned earlier, this method finds an efficient tool when the number of redundants in a structure exceeds three. Consider, for illustration, an eight-span continuous beam with simple end supports (Fig. 6.2).

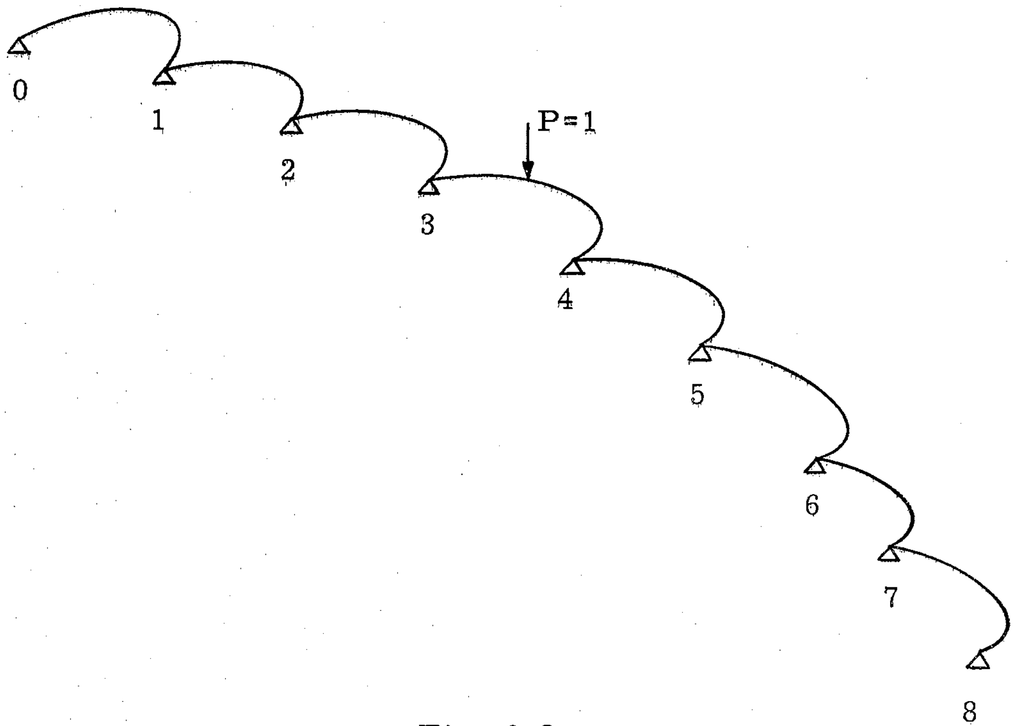


Fig. 6.2

Eight-Span Continuous Beam

An inspection of this structure reveals that the chord moments X_{12}, X_{23}, X_{34}, X_{45}, X_{56}, and X_{67} are the six redundants. The statically determinate quantities are:

$$X_{01} = 0$$

$$X_{78} = bX_{78} \qquad (6.1)$$

Also, the angular displacement functions $\tau^{(\Delta)}$ vanish, since there are no support displacements. After substituting for the statically determinate chord moments X_{01} and X_{78}, the five-moment Equations (5.5) for the six redundants are arranged in the matrix form as

Compatibility Matrix 6.3

$$\begin{bmatrix} F_1 & G_{21} & G_{31} & - & - & - \\ G_{12} & F_2 & G_{32} & G_{42} & - & - \\ G_{13} & G_{23} & F_3 & G_{43} & G_{53} & - \\ - & G_{24} & G_{34} & F_4 & G_{54} & G_{64} \\ - & - & G_{35} & G_{45} & F_5 & G_{65} \\ - & - & - & G_{46} & G_{56} & F_6 \end{bmatrix} \begin{bmatrix} X_{12} \\ X_{23} \\ X_{34} \\ X_{45} \\ X_{56} \\ X_{67} \end{bmatrix} = (-1) \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5^* \\ \tau_6^* \end{bmatrix}$$

where

$$\tau_5^* = (\tau_5 + bX_{78}\, G_{75})$$

and (6.2)

$$\tau_6^* = (\tau_6 + bX_{78}\, G_{76})$$

In the abreviated form, the above matrix may also be written as

$$[F]\ [X] = (-1)\ [\tau]$$

Solving for the redundant moments

$$[X] = [F]^{-1}[-\tau]$$

This is expressed in its complete form as

Redundant Influence Matrix 6.4

$$\begin{bmatrix} X_{12} \\ X_{23} \\ X_{34} \\ X_{45} \\ X_{56} \\ X_{67} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} & a_{51} & a_{61} \\ a_{12} & a_{22} & a_{32} & a_{42} & a_{52} & a_{62} \\ a_{13} & a_{23} & a_{33} & a_{43} & a_{53} & a_{63} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{54} & a_{64} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{65} \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} \end{bmatrix} \begin{bmatrix} -\tau_1 \\ -\tau_2 \\ -\tau_3 \\ -\tau_4 \\ -\tau_5^* \\ -\tau_6^* \end{bmatrix}$$

where

$$[a] = [F]^{-1}$$

Comparison of matrices (6.2) and (6.4) shows their similarity. The flexibility matrix [F] is inverted in the first method by the carry-over process, whereas in the second method this is done by the classical matrix approach or by means of digital computer.

Having known the redundant moments for all positions of a unit load moving across the structure, the determination of the other reactive elements and the forces at any cross-section is a matter of statics, and are found by the equations developed in Chapters II and V. This is illustrated by a numerical example in Chapter VIII.

6.2 Influence Lines for Deformations

For this purpose, the Maxwell and Mohr theorems are employed. The former states that the influence line for a linear displacement or an angular rotation at a section of a structure is the elastic curve, itself, developed by the application of a unit force or a unit couple at that section acting in the direction of the deformation. The Mohr's theorem states that the elastic curve of a structure is equal to the bending moment diagram of the corresponding conjugate structure acted upon by the elastic loads.

In this presentation the influence lines are developed only for linear displacements in the Z-direction. By a similar procedure, those for the angular deformations could be obtained. Fig. 6.3 shows an n-span continuous structure, 1 2 3, · · ·, i j k, · · ·, (n+1). In span ij, t is a section for which the influence line for positive deflection is desired. Therefore, in accordance with the Maxwell's theorem a unit positive force is applied at that section.

This indeterminate structure, under the action of the unit concentrated force, is analyzed for the redundant chord moments by either the carry-over method or the matrix method as explained earlier in this chapter. By using the matrices (4.1, 2) and (2.1, 2), the reactive elements other than the redundants and the cross-sectional elements in any span are determined.

The next step is the determination of the corresponding elastic curve. This is done by applying the Mohr's principle to the structure under consideration.

Fig. 6.4 shows an n-span conjugate structure corresponding to the real one of Fig. 6.3. The intermediate joints "2, · · ·, i, j, k, · · ·, n," are hinges, which can transmit only elastic forces, in the plane of the structure, between the adjacent spans. The ends "1 and (n+1)" are simple supports and develop conjugate reactive forces only, under the action of the elastic loads $\overline{p}_t$ and $\overline{p}_b$. For example $\overline{X}_{12}$ and $\overline{Y}_{12}$ (Fig. 6.4) are such reactions, which are the corresponding end

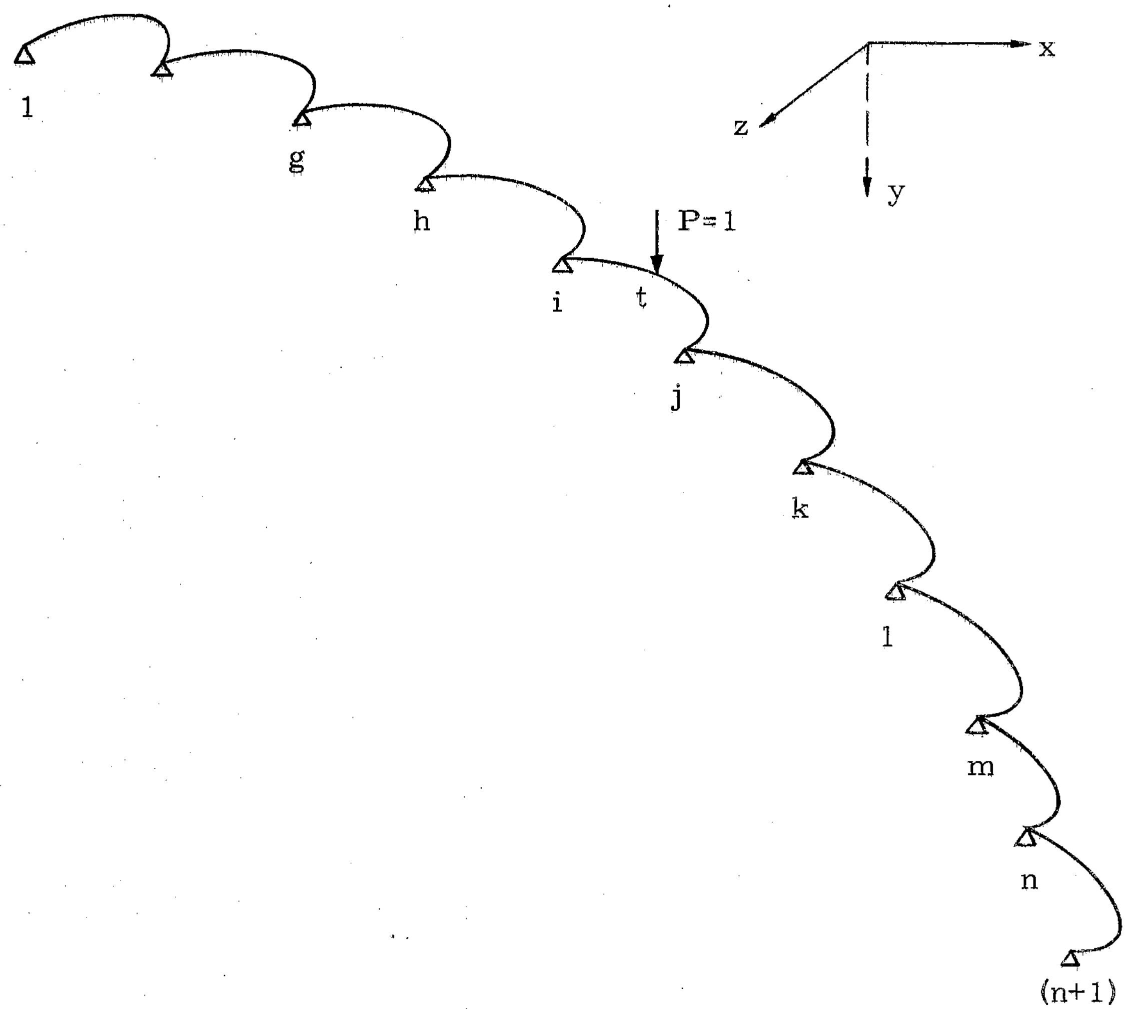


Fig. 6.3

n-Span Continuous Structure

slopes of the n-span continuous structure (Fig. 6.3).

If the ends of the real structure are fixed, the corresponding conjugate structure will have free ends. The intensity of elastic loads $\overline{p}_t$ and $\overline{p}_b$, defined in Chapter VII, act in the plane of the structure. Thus, the conjugate structure behaves like a planar structure (curved arch).

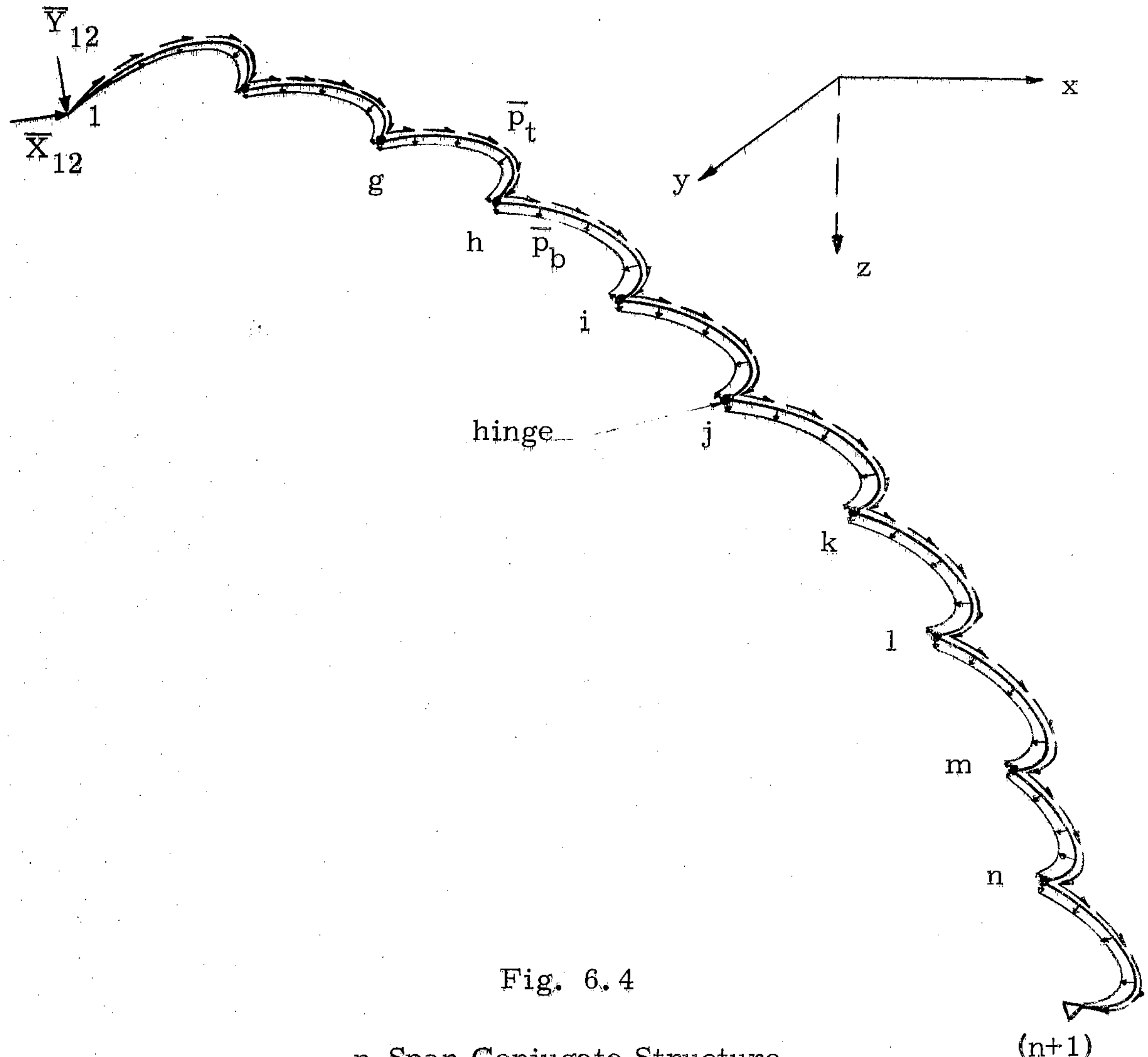


Fig. 6.4

n-Span Conjugate Structure

a) Elastostatic Equilibrium

Consider the end span 12 isolated as a freebody from the n-span conjugate structure. This is shown in Fig. 6.5.

The distributed elastic forces $\overline{p}_t$ and $\overline{p}_b$ are replaced by their statically equivalent concentrated forces $\overline{P}_{12x}$ and $\overline{P}_{12y}$ defined by Eqs. (7.32). The conjugate reactions $\overline{X}_{12}$ and $\overline{Y}_{12}$ are known by virtue of the Eqs. (7.33, 34). The unknown reactions $\overline{X}_{21}$ and $\overline{Y}_{21}$ provided by the hinge, are determined by the two conditions of the force equilibrium (namely $\Sigma \overline{F}_x = 0$ and $\Sigma \overline{F}_y = 0$) only.

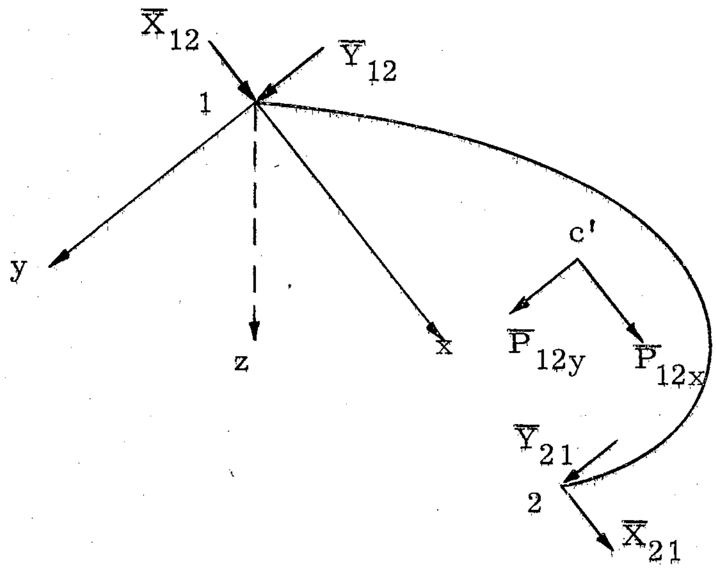


Fig. 6.5

Freebody of Conjugate Span 12

The location C' of $\overline{P}_{12x}$ and $\overline{P}_{12y}$ is immaterial for our problem, therefore, the moment equilibrium condition is not used. By carrying out the said force equilibrium operations, we get the following:

$$\overline{X}_{21} = -(\overline{X}_{12} + \overline{P}_{12x}) \tag{6.3a}$$

$$\overline{Y}_{21} = -(\overline{Y}_{12} + \overline{P}_{12y}) \tag{6.3b}$$

Consideration of the force equilibrium of joint 2 provides the left end reactions, $\overline{X}_{23}$ and $\overline{Y}_{23}$ of the conjugate span 23 (Fig. 6.6). They are given by:

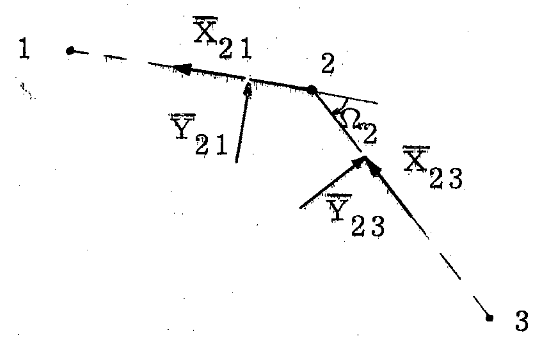


Fig. 6.6

Freebody of Joint 2

Conjugate Reaction Matrix 6.5

$$\begin{bmatrix} \overline{X}_{23} \\ \overline{Y}_{23} \end{bmatrix} = \frac{1}{\psi_2} \begin{bmatrix} -n_2 & -1 \\ 1 & -n_2 \end{bmatrix} \begin{bmatrix} \overline{X}_{21} \\ \overline{Y}_{21} \end{bmatrix}$$

where

$$\psi_2 = \text{cosec}\ \Omega_2$$

$$n_2 = \cot \Omega_2$$

as defined in Chapter IV.

Similarly proceeding from span to span the end reactions of all the conjugate spans are found by employing Eq. (6.3) and matrix (6.5). With this information at hand, the elastic curve in all the spans is determined as explained in the following section.

b) Elastic Curve

The general expressions for the bending moment at any section r of the span ij of the n-span conjugate structure (Fig. 6.4) are derived below. This obviously applies to any other span. Fig. 6.7 shows the isolated conjugate span ij with the known end reactions $\overline{X}_{ij}$, $\overline{Y}_{ij}$ and $\overline{X}_{ji}$, $\overline{Y}_{ji}$ at i and j , respectively.

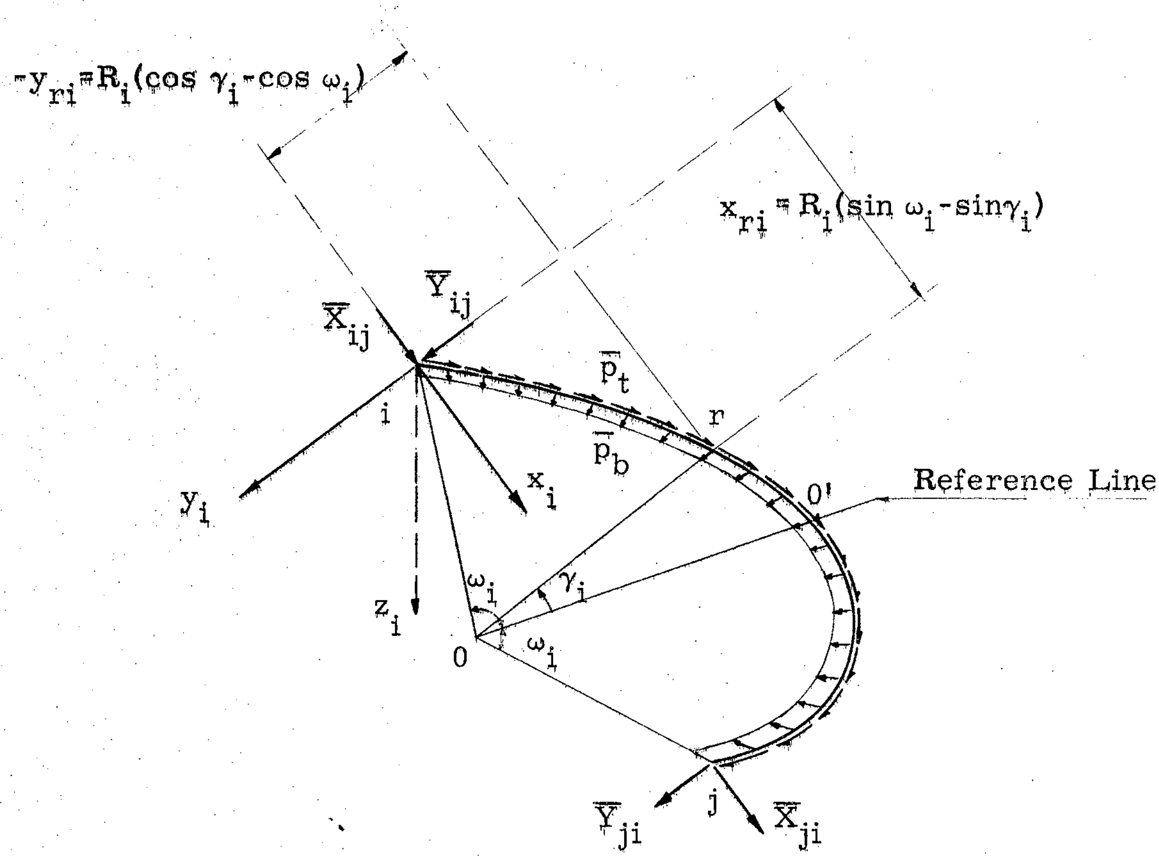


Fig. 6.7

Freebody of Conjugate Span ij

The bending moment, $\overline{M}_{rzi}$, at the section r is represented as

$$\overline{M}_{rzi} = \overline{RM}_{rzi} + \overline{BM}_{rzi} \tag{6.4}$$

where $\overline{RM}_{rzi}$ is the bending moment at r due to the conjugate reactions only. And from Fig. 6.7 this is as follows:

$$\overline{RM}_{rzi} = \overline{X}_{ij}\,(y_{ri}) - \overline{Y}_{ij}\,(x_{ri}) \tag{6.5}$$

and $\overline{BM}_{rzi}$ is that at r due to the distributed elastic loads $\overline{p}_t$ and $\overline{p}_b$ only. This follows from the Eqs. (7.8, 9, 11, 12, 13) as given below.

$$\overline{BM}_{rzi} = \begin{bmatrix} X_{ij}\,\overline{BM}_{rzi}^{(xi)} + Y_{ij}\,\overline{BM}_{rzi}^{(yi)} + Y_{ji}\,\overline{BM}_{rzi}^{(yj)} \\ + P\left(\overline{BM}_{rzi}^{(PtI)} + \overline{BM}_{rzi}^{(PtII)}\right) \end{bmatrix} \tag{6.6}$$

where $\overline{BM}_{rzi}^{(PtII)}$ is valid only when the section r is located to the right of the point of application, t, of the concentrated load P.

As mentioned earlier, the bending moment is the deflection at that section, or

$$\Delta_{rzi} = \overline{M}_{rzi} \tag{6.7}$$

Thus, when the applied load is unity and $\overline{X}_{ij}$, $\overline{Y}_{ij}$, X_{ij}, Y_{ij}, and Y_{ji} are the corresponding end reactions, the Eqs. (6.4-7) define the influence line for the deflection at t (Fig. 6.3).

CHAPTER VII

SPECIAL DERIVATIONS

In the preceding chapters of this presentation, the basic theory is developed in terms of the angular and elasto static functions. The application of the theory is possible, only if the said functions are defined mathematically in detail. Therefore, in this chapter analytical expressions are derived for these functions.

7.1 One-Span Angular Load Functions

Fig. 7.1 shows a basic span ij. A unit positive load moving across the span is located at r by an angular distance γ_i measured counterclockwise from the center of the span. Maxwell and Mohr theorems are employed to develop influence values for the corresponding end rotations t_{ijx}, t_{ijy}, and t_{jiy} .

a) Influence Values of t_{ijx}

The basic span ij loaded by a unit positive moment X_{ij} is now shown in Fig. 7.2. The resulting elastic curve is the desired influence line for t_{ijx}.

From the influence matrix (2.1), the end reactions are

$$Z_{ij} = 0 \tag{7.1a}$$

$$Z_{ji} = 0 \tag{7.1b}$$

$$X_{ji} = -1 \tag{7.1c}$$

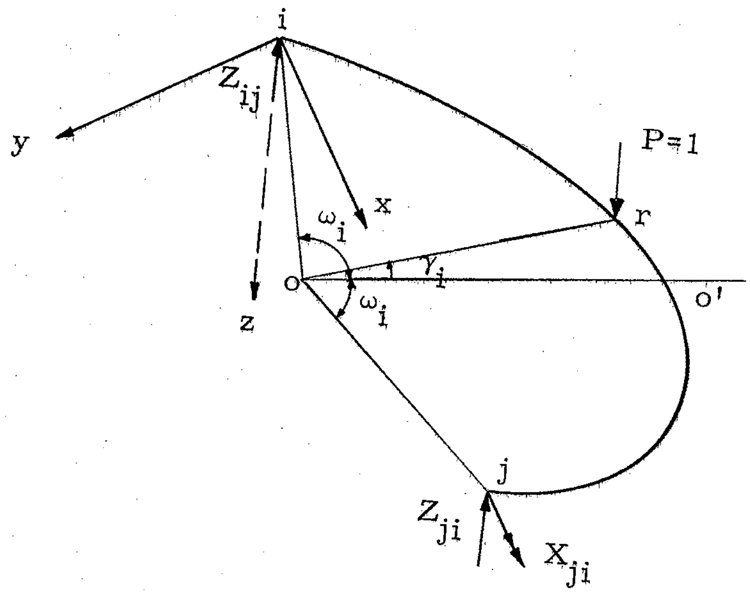


Fig. 7.1

Basic Span ij

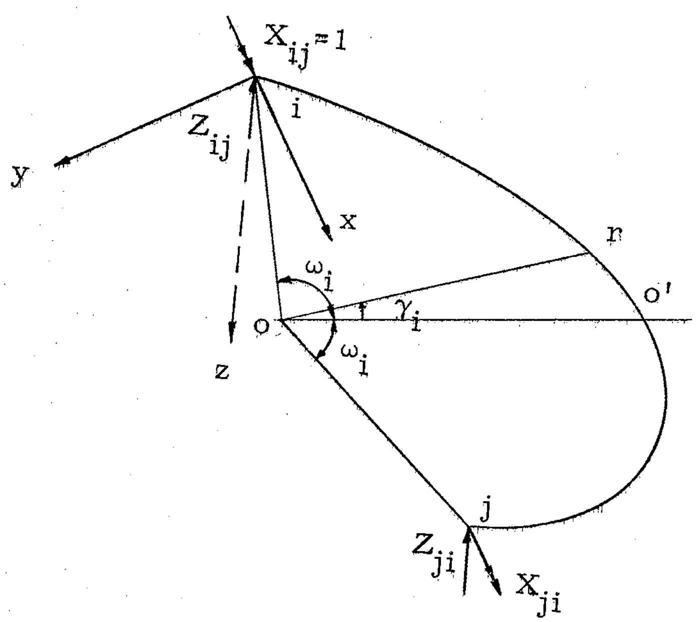


Fig. 7.2

Basic Span ij

The freebody of ir is shown in Fig. 7.3, in which the moment X_{ij} is resolved into components along and perpendicular to the chord ir. They are

$$X_{ir} = \cos \frac{\omega_i + \gamma_i}{2} \tag{7.2a}$$

$$Y_{ir} = \sin \frac{\omega_i + \gamma_i}{2} \tag{7.2b}$$

The forces and moments at r, referred to the chord ir, are determined from its static equilibrium as

$$Z_{ri} = 0 \tag{7.3a}$$

$$X_{ri} = -\cos \frac{\omega_i + \gamma_i}{2} \tag{7.3b}$$

$$Y_{ri} = -\sin \frac{\omega_i + \gamma_i}{2} \tag{7.3c}$$

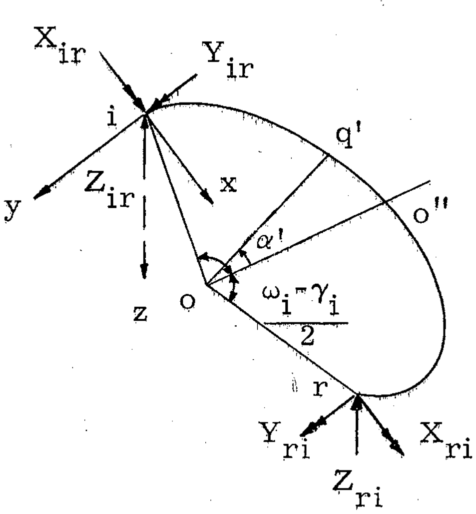


Fig. 7.3

Freebody Sketch of ir

In this figure q' is any section located at an angular distance α' measured counter clockwise from the center of the arc ir. Using the influence matrix (2.2), the positive cross-sectional moments at q' are written as

$$M_{q'ti} = \cos\frac{\omega_i+\gamma_i}{2} A_{q'ti} + \sin\frac{\omega_i+\gamma_i}{2} B_{q'ti} - \sin\frac{\omega_i+\gamma_i}{2} C_{q'ti}$$

$$M_{q'bi} = \cos\frac{\omega_i+\gamma_i}{2} A_{q'bi} + \sin\frac{\omega_i+\gamma_i}{2} B_{q'bi} - \sin\frac{\omega_i+\gamma_i}{2} C_{q'bi}$$

(7.4)

where

$$A_{q'ti} = -\cos\alpha' \qquad A_{q'bi} = -\sin\alpha'$$

$$B_{q'ti} = \frac{1-\cos\left(\frac{\omega_i-\gamma_i}{2}+\alpha'\right)}{2\sin\frac{\omega_i-\gamma_i}{2}} \qquad B_{q'bi} = -\frac{\sin\left(\frac{\omega_i-\gamma_i}{2}+\alpha'\right)}{2\sin\frac{\omega_i-\gamma_i}{2}}$$

$$C_{q'ti} = \frac{1-\cos\left(\frac{\omega_i-\gamma_i}{2}-\alpha'\right)}{2\sin\frac{\omega_i-\gamma_i}{2}} \qquad C_{q'bi} = \frac{\sin\left(\frac{\omega_i-\gamma_i}{2}-\alpha'\right)}{2\sin\frac{\omega_i-\gamma_i}{2}}$$

(7.5)

Conjugate Span ij

Fig. 7.4 shows the conjugate span corresponding to the real one of the Fig. 7.2.

$\overline{p}_t$ and $\overline{p}_b$ are the variable intensities of the distributed elastic loads acting in tangential and radial directions, respectively. At any section q', they are related to the corresponding cross-sectional moments by the expressions:

$$\bar{p}_{q'ti} = \frac{1}{GJ_{q'}} \left(M_{q'ti}\right) \tag{7.6a}$$

$$\bar{p}_{q'bi} = \frac{1}{EI_{q'}} \left(M_{q'bi}\right) \tag{7.6b}$$

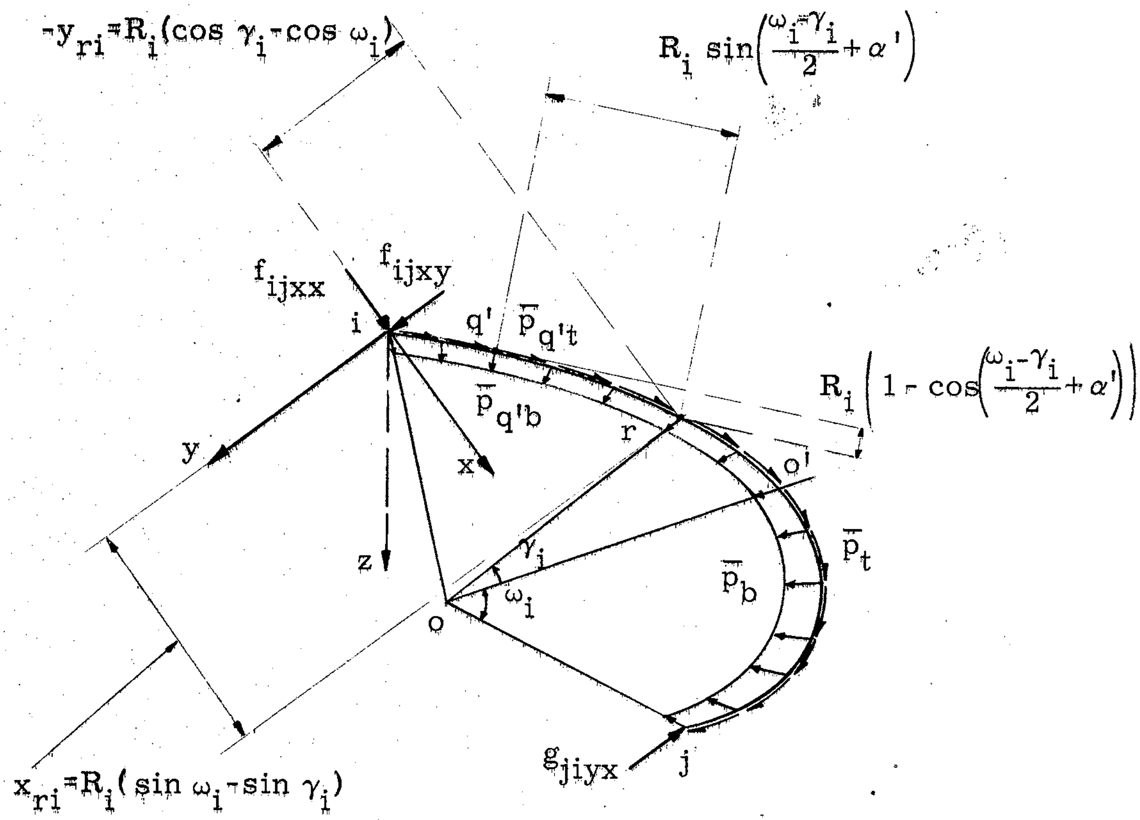


Fig. 7.4

Conjugate Span ij

Referring to the real structure (Fig. 7.2), the only possible deformations, consistent with the constraints, are the end rotations f_{ijxx}, f_{ijyx} at the left end and g_{jiyx} at the right end, as shown in Fig. 3.1. Therefore, the corresponding conjugate reactions, as shown in the Fig. 7.4, are

$$\overline{X}_{ij} = f_{ijxx} \qquad (7.7a)$$

$$\overline{Y}_{ij} = f_{ijyx} \qquad (7.7b)$$

$$\overline{Y}_{ji} = - g_{jiyx} \qquad (7.7c)$$

Elastic Curve

Following the Mohr's theorem, the positive deflection at section r is equal to the conjugate moment at that section. Thus

$$\Delta_{rzi} = \overline{M}^{(xi)}_{rzi} = t_{ijx} \qquad (7.8)$$

where the subscripts r, z, i indicate the point of investigation, the direction of moment or deflection, and the span, respectively. The superscripts xi stand for the cause. For example, $\overline{M}^{(xi)}_{rzi}$ is the conjugate bending moment acting in the Z-direction at section r caused by the application of a unit moment X_{ij} at the near end of basic span ij.

The bending moment at any section can be expressed as the sum of $\overline{RM}_{rzi}$ due to conjugate reactions and $\overline{BM}_{rzi}$ due to the elastic loads $\overline{p}_t$ and $\overline{p}_b$. Thus

$$\overline{M}^{(xi)}_{rzi} = \overline{RM}^{(xi)}_{rzi} + \overline{BM}^{(xi)}_{rzi} \qquad (7.9)$$

Consider the freebody ir (Fig. 7.5) isolated from the conjugate span ij (Fig. 7.4).

It follows from the above sketch that

$$\overline{RM}^{(xi)}_{rzi} = f_{ijxx}(y_{ri}) - f_{ijyx}(x_{ri}) \qquad (7.10)$$

and

$$\overline{BM}_{rzi}^{(xi)} = R_i^2 \int_{-\frac{\omega_i-\gamma_i}{2}}^{+\frac{\omega_i-\gamma_i}{2}} \left(\overline{p}_{q't} \left[1 - \cos\left(\frac{\omega_i-\gamma_i}{2} + \alpha'\right)\right] - \overline{p}_{q'b} \sin\left(\frac{\omega_i-\gamma_i}{2} + \alpha'\right)\right) d\alpha'$$

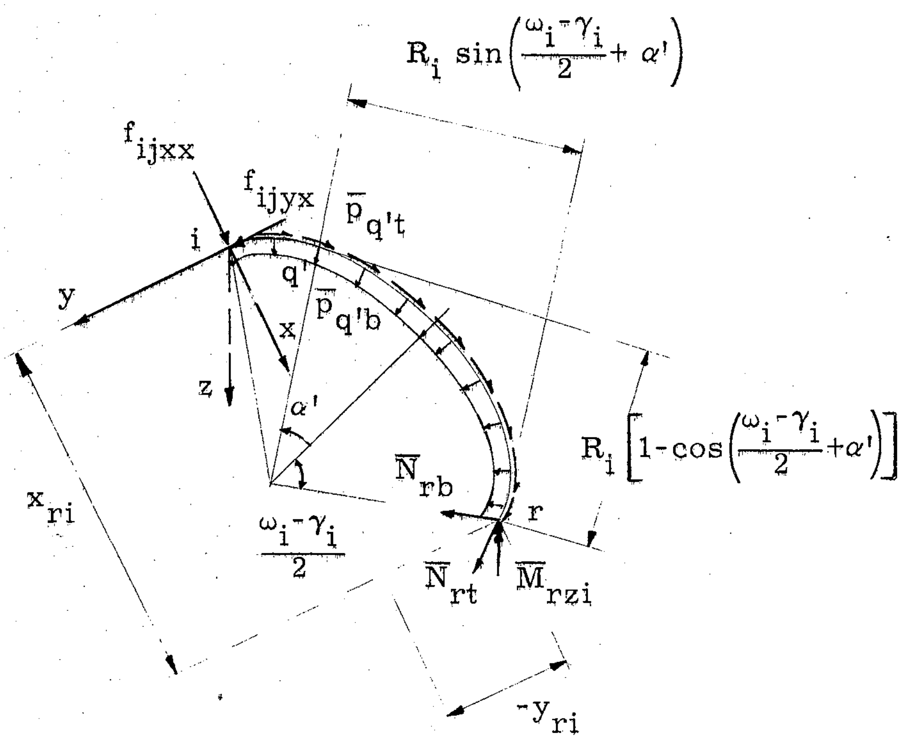


Fig. 7.5

Conjugate Freebody ir

Substituting for the elastic loads from the Eqs. (7.4, 6), and using the notation vide Eq. (7.5) and Table (3.2), the above equation reduces to

$$\overline{BM}_{rzi}^{(xi)} = (g_{iryy} - f_{iryy}) \quad (y_{ri}) + f_{iryx} (x_{ri}) \tag{7.11}$$

The Eq. (7.9), together with the expressions (7.10, 11), defines the influence values of the angular load function t_{ijx} of a basic span ij.

The procedure for developing analytical expressions for the influence values of the angular load functions t_{ijy} and t_{jiy} is exactly similar to that for t_{ijx} explained in detail. Therefore, only the final expressions are given below.

b) Influence Values of t_{ijy}

$$\Delta_{rzi} = \overline{M}_{rzi}^{(yi)} = t_{ijy} \tag{7.12a}$$

$$\overline{M}_{rzi}^{(yi)} = \overline{RM}_{rzi}^{(yi)} + \overline{BM}_{rzi}^{(yi)} \tag{7.12b}$$

$$\overline{RM}_{rzi}^{(yi)} = f_{ijxy} (y_{ri}) - f_{ijyy} (x_{ri}) \tag{7.12c}$$

$$\overline{BM}_{rzi}^{(yi)} = (f_{iryx} + \cot \omega_i \, g_{iryy})(y_{ri}) + f_{iryy}(x_{ri}) \tag{7.12d}$$

c) Influence Values of t_{jiy}

$$\Delta_{rzi} = \overline{M}_{rzi}^{(yi)} = t_{jiy} \tag{7.13a}$$

$$\overline{M}_{rzi}^{(yj)} = \overline{RM}_{rzi}^{(yj)} + \overline{BM}_{rzi}^{(yj)} \tag{7.13b}$$

$$\overline{RM}_{rzi}^{(yj)} = g_{ijxy} (y_{ri}) - g_{ijyy} (x_{ri}) \tag{7.13c}$$

$$\overline{BM}_{rzi}^{(yj)} = \Big((x_{ri}) + \cot \omega_i \, (y_{ri})\Big) \, g_{iryy} \tag{7.13d}$$

7.2 Angular Functions of the n-Span Basic Structure

Analytical expressions for the angular functions defined in Chapter V are now derived. These are quite general and therefore apply to continuous structures with variable cross-section.

Angular Flexibility and Carry-Over Values

Substituting the notation vide Eq. (4.3) and using the Table (3.2), the first five expressions of the Table (5.2) become, respectively,

$$G_{gi} = \psi_h \psi_i g_{hiyy} \tag{7.14}$$

$$G_{hi} = G_{hi}^{(L)} + G_{hi}^{(R)} \tag{7.15a}$$

where

$$G_{hi}^{(L)} = -\psi_i (g_{ihyx} + n_h g_{ihyy}) \tag{7.15b}$$

and

$$G_{hi}^{(R)} = -\psi_i \left(f_{ijyx} + n_i \Sigma (f_{iyy}) + n_j g_{ijyy} \right) . \tag{7.15c}$$

$$F_i = F_i^{(L)} + F_i^{(C)} + F_i^{(R)} \tag{7.16a}$$

where

$$F_i^{(L)} = (\psi_i^2 f_{ihyy} + n_i^2 f_{ijyy} + n_i n_j g_{ijyy} + n_i f_{ijyx}) \tag{7.16b}$$

$$F_i^{(C)} = (+ n_i f_{ijxy} + f_{ijxx} + n_j g_{ijxy}) \tag{7.16c}$$

$$F_i^{(R)} = (\psi_j^2 f_{jkyy} + n_j^2 f_{jiyy} + n_j n_i g_{jiyy} + n_j g_{jiyx}) . \tag{7.16d}$$

$$G_{ji} = G_{ji}^{(L)} + G_{ji}^{(R)} \tag{7.17a}$$

where

$$G_{ji}^{(L)} = -\psi_j (g_{jiyx} + n_i g_{jiyy}) \tag{7.17b}$$

$$G_{ji}^{(R)} = -\psi_j (f_{jkyx} + n_j \Sigma(f_{jyy}) + n_k g_{jkyy}) \tag{7.17c}$$

$$G_{ki} = \psi_k \psi_j g_{kjyy} \tag{7.18}$$

As defined in Chapter V, G_{gi} and G_{ki} are the angular far carry-over values. G_{hi} and G_{ji} are the angular near carry-over values. F_i is the angular flexibility of span ij of the n-span basic structure. The Eqs. (7.15 - 17) show the subdivision of these functions. It has no special significance, except that it becomes easy to deal with them.

Angular Load Function

Upon substitution of the notations vide Eq. (4.3), Table (3.2) and Eqs. (7.15, 16), the sixth expression of the Table (5.2) reduces to

$$\tau_i^{(L)} = \begin{bmatrix} -\psi_i t_{ihy}^{(L)} + n_i t_{ijy}^{(L)} + t_{ijx}^{(L)} + n_j t_{jiy}^{(L)} - \psi_j t_{jky} \\ -bX_{gh} G_{gi} - bX_{hi} G_{hi}^{(R)} - bX_{ij} F_i^{(R)} + bX_{jk} \left(\frac{n}{\psi}\right)_k G_{ki} \end{bmatrix} \tag{7.19}$$

Angular Displacement Function

From the 7th expression of the Table (5.2)

$$\tau_i^{(\Delta)} = -\psi_i t_{hi}^{(\Delta)} + (n_i + n_j) t_{ij}^{(\Delta)} - \psi_j t_{jk}^{(\Delta)} \tag{7.20}$$

where

$$t_{hi} = \frac{\Delta_h - \Delta_i}{L_h}$$

and so on.

7.3 Angular Functions of a Basic Span for Constant Cross Section

The inertial properties of the section of the basic span ij (Fig. 2.1) are assumed constant across its length.

Substituting into the expressions of Table (3.2), for notation, Eq. (2.7), and performing the integration between the indicated limits, the angular flexibilities and carry-over values reduce to

$$f_{ijxx} = \frac{R}{2\ EI}\left(\mu\ (2\omega_i \quad \sin 2\omega_i) + (2\omega_i - \sin 2\omega_i)\right) \tag{7.21a}$$

$$f_{ijyy} = f_{jiyy}$$

$$= \frac{R}{8\ EI \sin^2 \omega_i}\left(\mu\ (6\omega_i - 4 \sin 2\omega_i + \frac{1}{2} \sin 4\omega_i) + (2\omega_i - \frac{1}{2} \sin 4\omega_i)\right) \tag{7.21b}$$

$$f_{ijxy} = f_{ijyx} = g_{ijxy} = g_{jiyx} \tag{7.21c}$$

$$= \frac{R}{8\ EI \sin \omega_i}\left(\mu\ (\sin 3\omega_i + 4\omega_i \cos \omega_i - 7 \sin \omega_i) + (4\omega_i \cos \omega_i - \sin 3\omega_i - \sin \omega_i)\right)$$

and

$$g_{ijyy} = g_{jiyy}$$

$$= \frac{R}{8\ EI\ \sin^2 \omega_i} \ \mu\ (4\omega_i + 2\omega_i \cos 2\omega_i - 3\sin 2\omega_i)$$

$$+ \ (2\omega_i \cos 2\omega_i - \sin 2\omega_i) \qquad (7.21d)$$

where $\mu = \frac{EI}{GJ}$, the ratio of bending and torsional rigidities.

Thus, from the above it is obvious, that for a basic span with cross-sectional variation symmetrical with respect to its center, there are only four distinct angular functions.

7.4 Elastic Curve of a Basic Span ij Due to a Unit Concentrated Force

A basic span ij, with a unit concentrated load applied at Section t, is shown in Fig. 7.6. The point of application of the unit force is located by polar coordinates (R_i, β_i). The purpose of this article is to develop necessary analytical expressions for the deflection at any section r, located in the following figure by polar coodinates (R_i, γ_i).

By employing the three equations of static equilibrium of the basic span, the reactive forces bZ_{ij}, bZ_{ji} and moment bX_{ji} are expressed as

$$bZ_{ij} = \frac{\sin \omega_i + \sin \beta_i}{2 \sin \omega_i} = \frac{x'_{ti}}{L_i} \qquad (7.22a)$$

$$bZ_{ji} = \frac{\sin \omega_i - \sin \beta_i}{2 \sin \omega_i} = \frac{x_{ti}}{L_i} \qquad (7.22b)$$

$$bX_{ji} = R_i(\cos \beta_i - \cos \omega_i) = - y_{ti} \qquad (7.22c)$$

The cross-sectional forces at any section are visualized as a sum of two effects due to a) reaction bZ_{ij}, and b) applied force.

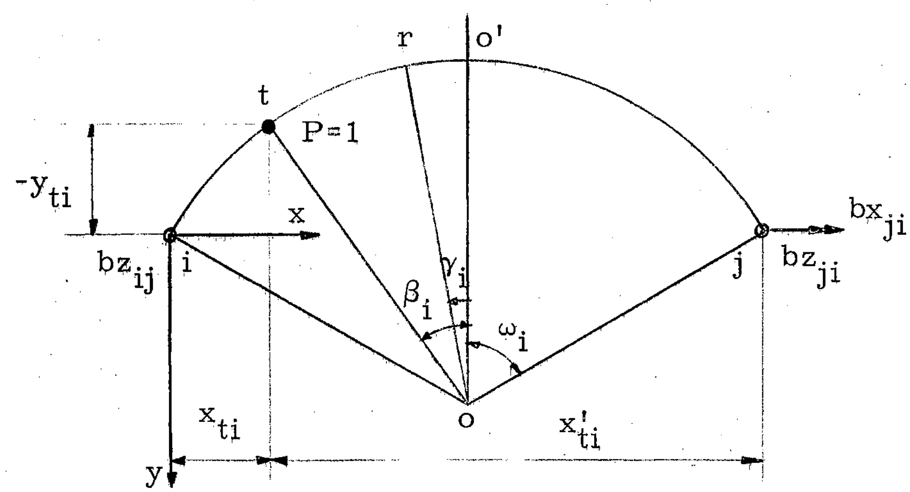


Fig. 7.6

Basic Span ij

The first effect is considered as follows. Consider the freebody in (Fig. 7.7) The reactive force bZ_{ij} is treated as an applied load. q' is any section located by polar coordinates (R_i, α'_i), as shown.

The reactive forces at r required for the static equilibrium of ir are

$$Z_{ri} = - bZ_{ij} \tag{7.23a}$$

$$X_{ri} = 0 \tag{7.23b}$$

$$Y_{ri} = bZ_{ij} \left(2R_i \sin \frac{\omega_i - \gamma_i}{2}\right) \tag{7.23c}$$

With the help of the influence matrix (2.2) the positive cross-sectional moments at q' are

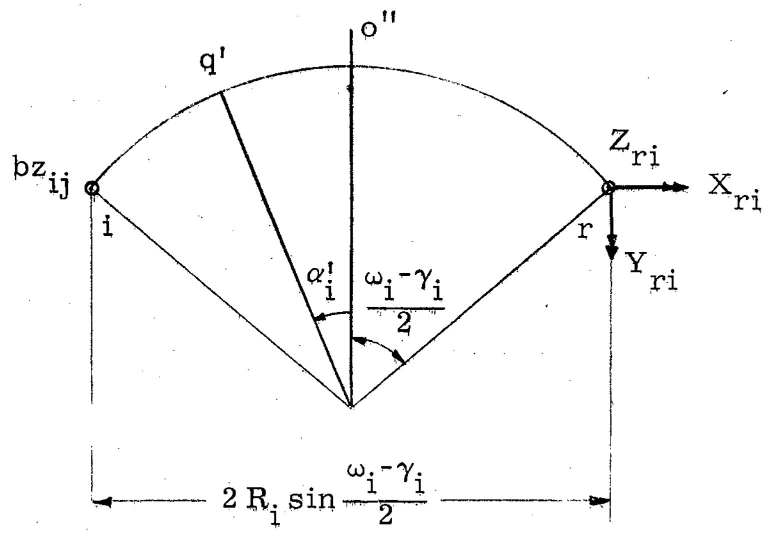


Fig. 7.7

Freebody of ir - First Effect

$$M_{q'ti} = bZ_{ij}\left(2R_i \sin\frac{\omega_i - \gamma_i}{2}\right) C_{q'ti} \tag{7.24a}$$

and

$$M_{q'bi} = bZ_{ij}\left(2R_i \sin\frac{\omega_i - \gamma_i}{2}\right) C_{q'bi} \tag{7.24b}$$

where

$$C_{q'ti} = \frac{1 - \cos\left(\frac{\omega_i - \gamma_i}{2} - \alpha'_i\right)}{2 \sin\frac{\omega_i - \gamma_i}{2}}$$

$$C_{q'bi} = \frac{\sin\left(\frac{\omega_i - \gamma_i}{2} - \alpha'_i\right)}{2 \sin\frac{\omega_i - \gamma_i}{2}}$$

The intensities of the corresponding elastic loads are given by the following equations

$$\overline{p}_{q'ti} = \frac{1}{GJ_{q'}} (M_{q'ti}) \tag{7.25a}$$

$$\overline{p}_{q'bi} = \frac{1}{EI_{q'}} (M_{q'bi}) \tag{7.25b}$$

The Eqs. (7.24, 25) are valid for any location of q' across the span ij.

To determine the contribution of the applied unit load, consider the free body of ir as shown in Fig. 7.8. The overhang "it" (Fig. 7.8) is not stressed, as there are no loads acting on that portion of span ij. q'' is any section located by (R_i, α_i'') as shown below.

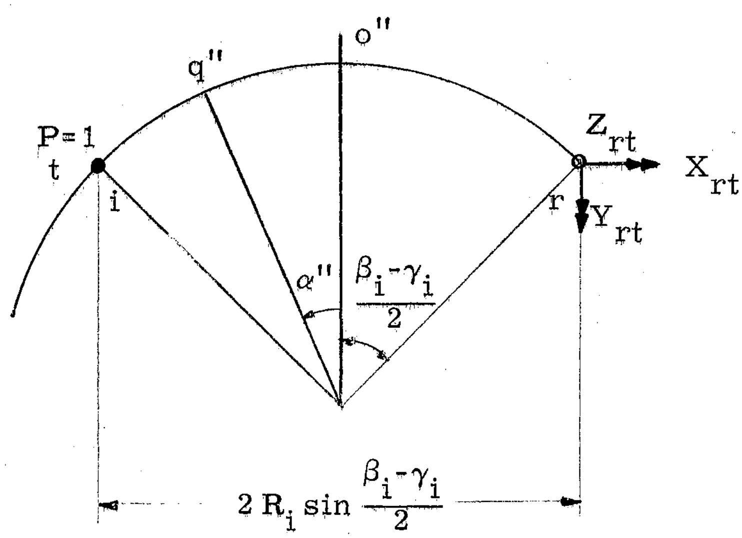


Fig. 7.8

Free Body of ir - Second Effect.

The reactions at r required for the equilibrium of ir are

$$Z_{rt} = 1 \tag{7.26a}$$

$$X_{rt} = 0 \tag{7.26b}$$

$$Y_{rt} = -2\,R_i \sin\frac{\beta_i - \gamma_i}{2} \tag{7.26c}$$

Using the influence matrix (2.2), the positive moments at the section q'' are expressed as

$$M_{q''ti} = -2R_i \sin\frac{\beta_i - \gamma_i}{2}\, C_{q''ti} \tag{7.27a}$$

$$M_{q''bi} = -2\,R_i \sin\frac{\beta_i - \gamma_i}{2}\, C_{q''bi} \tag{7.27b}$$

where

$$C_{q''ti} = \frac{1 - \cos\left(\frac{\beta_i - \gamma_i}{2} - \alpha_i''\right)}{2\sin\frac{\beta_i - \gamma_i}{2}}$$

$$C_{q''bi} = \frac{\sin\left(\frac{\beta_i - \gamma_i}{2} - \alpha_i''\right)}{2\sin\frac{\beta_i - \gamma_i}{2}}$$

The corresponding elastic weights at q'' by their intensities become

$$\overline{p}_{q''ti} = \frac{1}{GJ_{q''}}\,(M_{q''ti}) \tag{7.28a}$$

$$\overline{p}_{q''bi} = \frac{1}{EI_{q''}}\,(M_{q''bi}) \tag{7.28b}$$

The expressions given by the Eqs. (7.27, 28) are valid as long as β_i is greater than γ_i.

With this information available, the next step is to develop an expression for the deflection at r by employing the Mohr's principle. Fig. 7.9 shows the conjugate span ij corresponding to the real structure of the Fig. 7.6.

q' and q'' are shown as located by $(R_i, \frac{\omega_i - \gamma_i}{2} + \alpha'_i)$ and $(R_i, \frac{\beta_i - \gamma_i}{2} + \alpha_i'')$, respectively, relative to r.

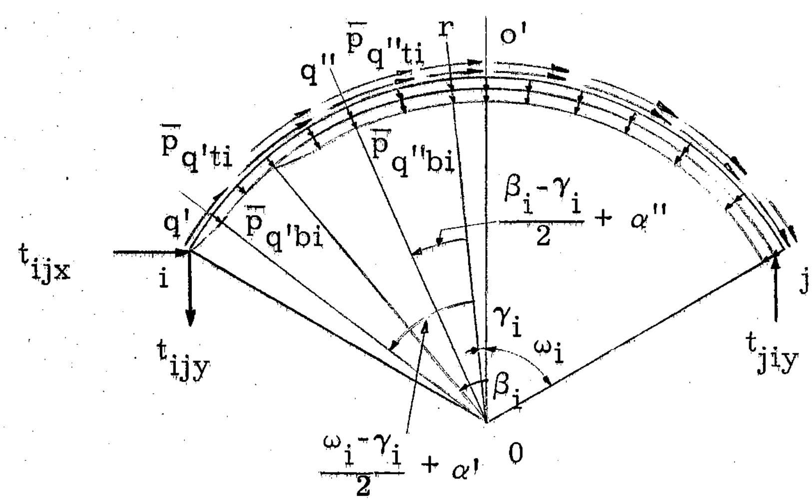


Fig. 7.9

Conjugate Span ij

t_{ijx}, t_{ijy}, and t_{jiy} are the angular load functions of the basic span ij (Fig. 7.6). Therefore, they are shown as the conjugate reactions. The elastic weights due to the reaction bX_{ij} and the unit load are shown separately with single and double primes, respectively. The elastic bending moment $\overline{M}_{rzi}^{(Pt)}$ at the section r, which is equal to the deflection at that section is given by

$$\overline{M}_{rzi}^{(Pt)} = \overline{RM}_{rzi} + \overline{BM}_{rzi}^{(PtI)} + \overline{BM}_{rzi}^{(PtII)} \qquad (7.29)$$

where $\overline{RM}_{rzi}$ is the elastic moment due to conjugate reactions only, and $\overline{BM}_{rzi}^{(PtI)}$, $\overline{BM}_{rzi}^{(PtII)}$ are due to the elastic weights corresponding to the first and second effects, respectively.

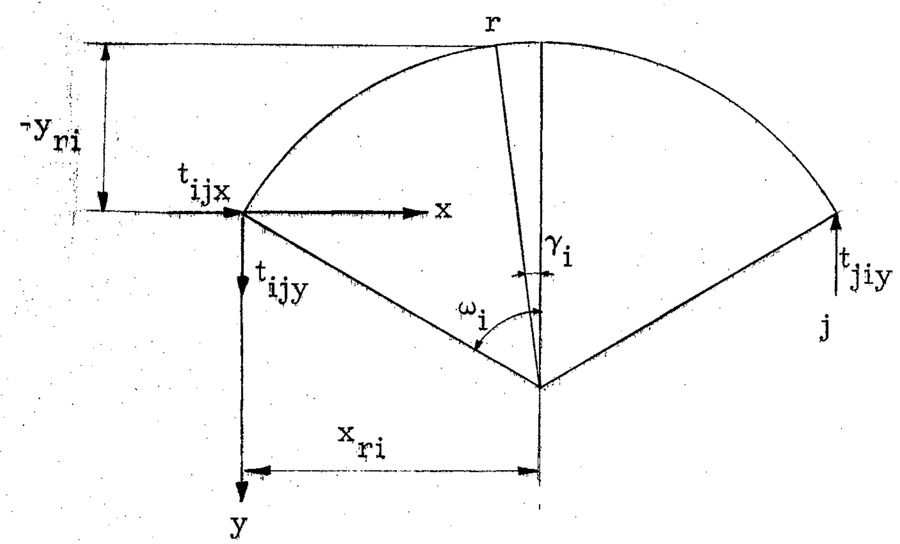
Explanation for the subscripts is given on page 62. The superscript letters Pt stand for the load applied at t. I and II denote the first and second effects, respectively. To avoid confusion, expressions for the functions $\overline{RM}$ and $\overline{BM}$ are derived separately in the following.

Fig. 7.10a shows the conjugate reactions only. From this figure it follows that

$$\overline{RM}_{rzi} = t_{ijx}\,(y_{ri}) - t_{ijy}\,(x_{ri}) \qquad (7.30)$$

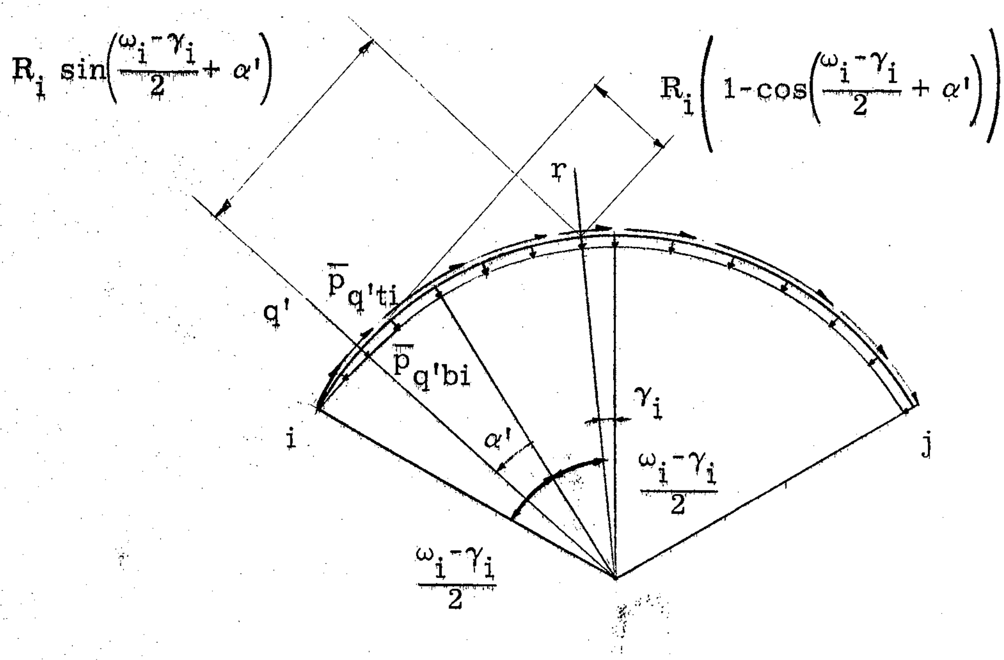
The elastic loads due to the first effect are shown in Fig. 7.10b. In this figure, the summation of moments at r in Z-direction gives

$$\overline{BM}_{rzi}^{(PtI)} = R_i^2 \int_{-\frac{\omega_i-\gamma_i}{2}}^{+\frac{\omega_i-\gamma_i}{2}} (\overline{p}_{q'ti}) \left(1 - \cos \frac{\omega_i-\gamma_i}{2} + \alpha'_i \right) d\alpha'_i \; -$$



(a)

Reactions Only



(b)

First Effect

Fig. 7.10 Conjugate Span ij

$$-R_i^2 \int_{-\frac{\omega_i-\gamma_i}{2}}^{+\frac{\omega_i-\gamma_i}{2}} (\overline{p}_{q'bi}) \left(\sin \frac{\omega_i-\gamma_i}{2} + \alpha_i'\right) \quad d\alpha_i'$$

Denoting:

$$\frac{1 - \cos\left(\frac{\omega_i - \gamma_i}{2} + \alpha_i'\right)}{2 \sin \frac{\omega_i - \gamma_i}{2}} = B_{q'ti}$$

$$-\frac{\sin\left(\frac{\omega_i - \gamma_i}{2} + \alpha'\right)}{2 \sin \frac{\omega_i - \gamma_i}{2}} = B_{q'bi}$$

and substituting for elastic loads from the Eqs. (7.24, 25), using the Table (3.2), the above expression simplifies to

$$\overline{BM}_{rzi}^{(PtI)} = x_{ti}' \ (x_{ri} + y_{ri} \cot \omega_i) \ g_{iryy} \qquad (7.31)$$

Similar procedure for the second effect results in the following equation.

$$\overline{BM}_{rzi}^{(PtII)} = -2 R_i^2 \left(1 - \cos(\beta_i - \gamma_i)\right) g_{tryy} \qquad (\text{for } \gamma_i < \beta_i \text{ only})$$

$$\overline{BM}_{rzi}^{(PtII)} = 0 \qquad (\text{for } \gamma_i > \beta_i)$$

(7.32)

7.5 Total Elastic Weights of Span ij

After analyzing the continuous beam (Fig. 6.3) for a unit concentrated force, the span ij is isolated and shown as a free body in Fig. 7.11.

Z_{ij}, Z_{ji}, and X_{ij}, X_{ji}, Y_{ij}, Y_{ji} are the reactive forces and moments, respectively.

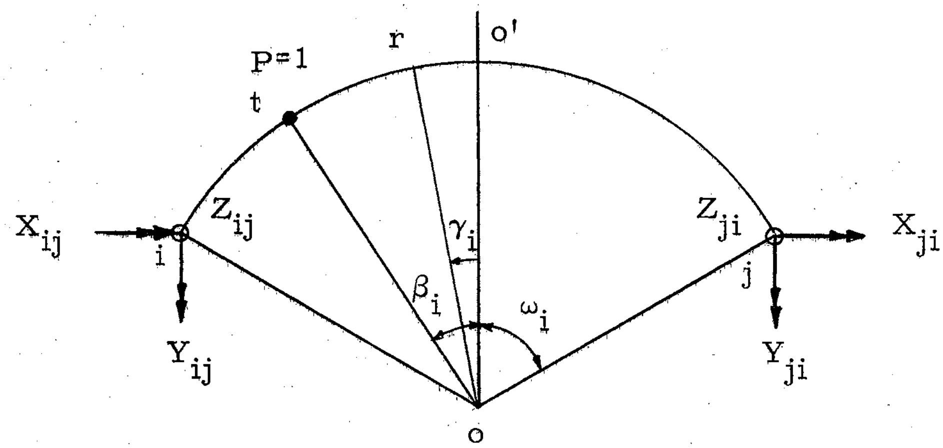


Fig. 7.11

Free Body of Span ij

X_{ij}, Y_{ij}, and Y_{ji} are treated as applied moments in addition to the unit concentrated load.

Referring back to Fig. 7.4, the resultant elastic weights, due to unit moment X_{ij}, follow from the elasto static equilibrium of the conjugate span ij. Thus

$$\overline{XP}_{ijx}^{(ij)} = -f_{ijxx}$$

$$\overline{XP}_{ijy}^{(ij)} = (g_{ijyx} - f_{ijyx})$$

Similarly for unit end moments Y_{ij}, Y_{ji}, and applied loads,

$$\overline{YP}_{ijx}^{(ij)} = - f_{ijxy}$$

$$\overline{YP}_{ijy}^{(ij)} = (g_{jiyy} - f_{ijyy})$$

$$\overline{YP}_{ijx}^{(ji)} = - g_{ijxy}$$

$$\overline{YP}_{ijy}^{(ji)} = (f_{jiyy} - g_{ijyy})$$

$$\overline{BP}_{ijx}^{(Pt)} = - t_{ijx}$$

$$\overline{BP}_{ijy}^{(Pt)} = (t_{jiy} - t_{ijy})$$

Thus, the total elastic weights on span ij (Fig. 7.11) become

$$\overline{F}_{ijx} = -(X_{ij} f_{ijxx} + Y_{ij} f_{ijxy} + Y_{ji} g_{ijxy} + t_{ijx}) \qquad (7.33a)$$

$$\overline{F}_{ijy} = \left[\begin{array}{l} X_{ij} (g_{jiyx} - f_{ijyx}) + Y_{ij} (g_{jiyy} - f_{ijyy}) \\ \\ Y_{ji} (f_{jiyy} - g_{ijyy}) + (t_{jiy} - t_{ijy}) \end{array} \right] \qquad (7.33b)$$

7.6 End Slopes of an n-Span Structure

Fig. 7.12 shows three left side spans of a continuous beam. i is the end support.

Y_{ij}, X_{ij}, X_{jk}, X_{kl}, X_{lm}, etc., are the redundant moments. hi is an imaginary extension to take care of the redundant Y_{ij}, as explained in Chapter V.

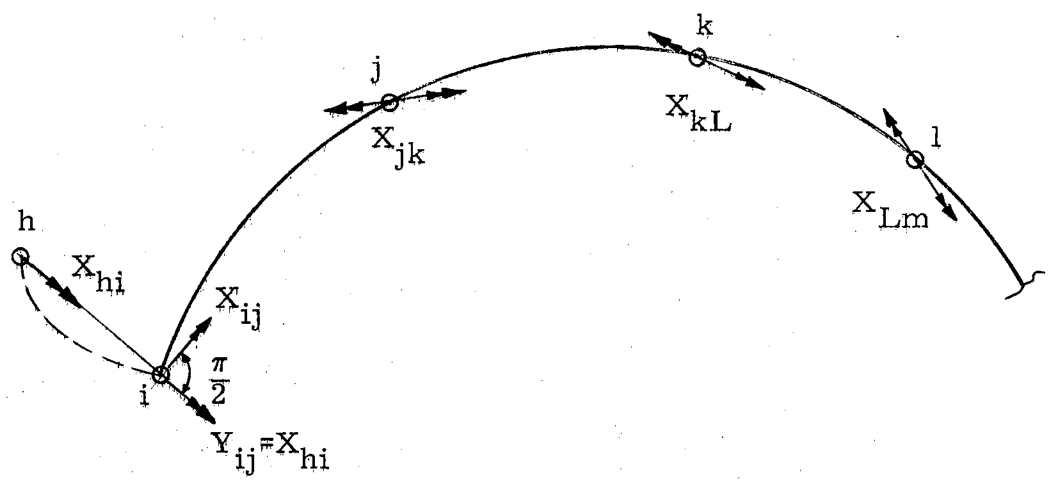


Fig. 7.12

Continuous Girder i j k l m · · ·

An examination of the five-moment equation, Eq. (5.5), written for redundant X_{ij}, reveals that each term is a part of the total relative rotation about ij contributed by different causes.

If the end condition is such that the structure is free to rotate about chord ij, then the total rotation about the chord ij is given by the following expression:

$$T_{ijx} = \left(X_{hi}\, G_{hi} + \tau_i^{(L)} + \tau_i^{(\Delta)} + X_{jk}\, G_{ji} + X_{kl}\, G_{ki}\right) \qquad (7.34)$$

where T_{ijx} is the total end rotation about end chord ij.

Similarly if the structure is free to rotate at its end about an axis perpendicular to the end chord, then the corresponding total slope T_{ijy} is expressed as:

$$T_{ijy} = \left(\tau_h^{(L)} + \tau_h^{(\Delta)} + X_{ij} G_{ih} + X_{jk} G_{jh}\right) \tag{7.35}$$

If the end is free to rotate about both the axes, then the corresponding total slopes are obtained by substituting zero for X_{hi} and X_{ij} in Eqs. (7.34) and (7.35), respectively.

CHAPTER VIII

APPLICATION

The numerical application of the theory developed in the preceding chapters is demonstrated in this chapter by three examples. The first presents the analysis of a two span beam with one end fixed by the carry-over moment method. The influence lines for forces, moments and deflections are investigated in the other examples by the matrix method.

A procedure of analysis by the carry-over moment method is outlined below.

1) Break the structure into single basic spans, and evaluate the corresponding angular functions (f, g, t) of each span.

2) Selecte the n-span basic structure and recognize the redundants.

3) Calculate the n-span angular functions (F, G, τ), bearing in mind their modifications, if necessary, for the existing end conditions.

4) Determine the carry-over moment functions (r, m).

5) Perform the carry-over process in a tabular form to get the redundants.

6) Determine the other forces and moments by statics.

The following numerical example illustrates the above sequence of steps.

a) Numerical Problem No. 1

A two-span continuous circular beam with left end fixed and loaded as shown (Fig. 8.1) is analyzed by the method of carry-over moments. The radius of curvature of the beam is 60 ft. Both the spans are equal and each subtends a 30° angle at its center of curvature. The cross-section is constant, and the ratio EI/GJ is assumed equal to 2.

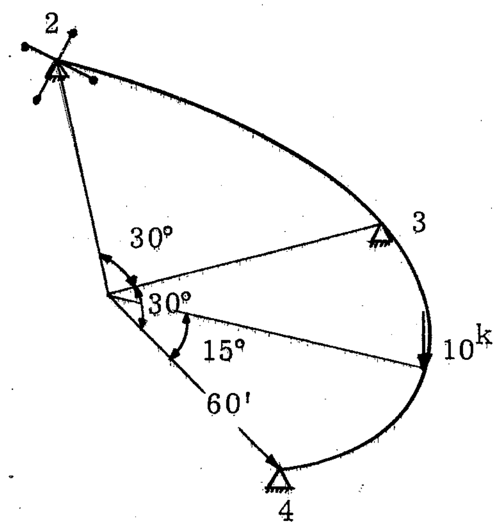


Fig. 8.1

Two-Span Continuous Curved Beam

1.) The two spans of the beam, considered, are identical, therefore, the corresponding basic spans are the same as shown in Fig. 8.2.

One-span angular functions are calculated by substituting for the geometric and elastic properties of the structure into the Eqs. (7.8 - 13, 21).

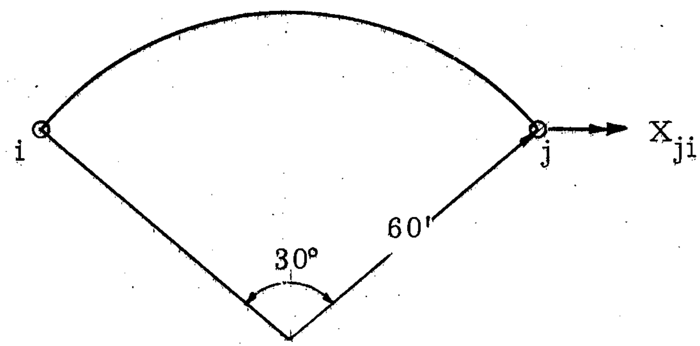


Fig. 8.2

One-Span Basic Structure

Angular Flexibilities and Carry-Over Values

$$f_{23xx} = f_{34xx} = (1.03540)\frac{R}{EI}$$

$$f_{23yy} = f_{32yy} = f_{34yy} = f_{43yy} = (0.18325)\frac{R}{EI}$$

$$g_{23yy} = g_{32yy} = g_{34yy} = g_{43yy} = (-0.08445)\frac{R}{EI}$$

$$f_{23xy} = f_{23yx} = g_{23xy} = g_{32yx} = (-0.06792)\frac{R}{EI}$$

$$f_{34xy} = f_{34yx} = g_{34xy} = g_{43yx} = (-0.06792)\frac{R}{EI}$$

Angular Load Functions

$$t_{23x} = t_{23y} = t_{32y} = 0$$

$$t_{34x} = (-10.58410)\frac{R}{EI}$$

$$t_{34y} = (-9.69851)\frac{R}{EI}$$

$$t_{43y} = (11.08004)\frac{R}{EI}$$

2.) The corresponding n-span basic structure is shown in Fig. 8.3, wherein 12 is an imaginary extension to take care of end moment Y_{23} ($=X_{12}$) as explained in Chapter V. The geometric angle changes at supports 2 and 3 are $-\frac{\pi}{2}$ and $\frac{\pi}{6}$ radians, respectively.

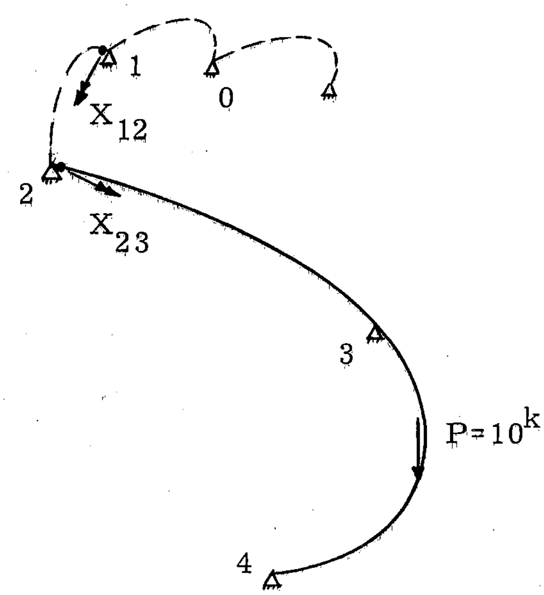


Fig. 8.3

n-Span Basic Structure - Redundants

X_{12}, X_{23} are the redundant chord moments and X_{34} is statically determinate and is equal to the static moment bX_{34}.

$$X_{34} = bX_{34} = 20.44452 \quad \text{k-ft.}$$

3.) The n-span angular functions are determined by Eqs. (7.14 - 7.19).

Flexibilities and Carry-Over Values

$$F_1^{(L)} = F_1^{(C)} = 0$$

$$F_1^{(R)} = (0.18325)\frac{R}{EI}$$

Therefore,

$$F_1 = (0.18325)\frac{R}{EI}$$

$$F_2^{(L)} = 0$$

$$F_2^{(C)} = (0.91776)\frac{R}{EI}$$

$$F_2^{(R)} = (1.16512)\frac{R}{EI}$$

Therefore,

$$F_2 = (2.08288)\frac{R}{EI}$$

$$G_{12}^{L} = 0$$

$$G_{12}^{R} = (-0.21419)\frac{R}{EI}$$

Therefore,

$$G_{12} = (-0.21419)\frac{R}{EI}$$

And by reciprocity

$$G_{21} = (-0.21419) \frac{R}{EI}$$

$$G_{31} = (0.16889) \frac{R}{EI}$$

$$G_{32} = (-0.99792) \frac{R}{EI}$$

Angular Load Functions

$$\tau_1 = 0.0$$

$$\tau_2 = (19.39702) \frac{R}{EI}$$

4.) The carry-over moment functions are calculated by employing Eqs. (5.9, 10).

Carry-Over Factors

$$r_{12} = 0.10283$$

$$r_{21} = 1.16882$$

$$r_{31} = -0.92165$$

$$r_{32} = 0.47911$$

Starting Moments

$$m_1 = 0$$

$$m_2 = -9.30894$$

$$m_3{}^* = X_{34} = bX_{34} = 20.44452 \text{ k-ft.}$$

5.) The carry-over process is performed in the following table.

TABLE 8.1 CARRY-OVER PROCESS

Chord Moments	12	23	34
Carry-Over Factors	+0.10283 →		
	←	+1.16882	
	←		-0.92165
		←	+0.47911
Starting Moments	0.0	-9.30894	20.44452
C. O. (1)	-18.84275	9.79513	
C. O. (2)		-1.93764	
		-1.45145	
C. O. (3)	- 1.69648		
C. O. (4)		-0.17445	
C. O. (5)	- 0.20390		
C. O. (6)		-0.02097	
C. O. (7)	- 0.02451		
C. O. (8)		-0.00252	
C. O. (9)	- 0.00295		
Final Chord Moments, Σ	-20.77059 k-ft	-1.64939 k-ft	20.44452 k-ft

The same problem has been worked out by the method of forces, selecting the reactive forces Z_3 and Z_4 as redundants. The following results are obtained for the corresponding chord moments.

Chord Moments -20.77444 k-ft -1.65249k-ft 20.44452 k-ft

It is interesting to note that inspite of some carry-over factors equal to and greater than unity, the convergence of the iteration process is so rapid that only four carry-over operations are sufficient in this case to bring the results within 1 1/2 % error.

6.) With the redundants known, the determination of the forces and moments across the structure is a matter of stereo statics.

Matrix method is employed to develop the influence lines for forces and moments in a four-span continuous circular beam. The following steps lead to the solution of the desired influence lines.

1) Break the continuous structure into single basic spans and evaluate their angular functions (f, g, t)

2) Select the n-span basic structure and recognize the redundant moments (X).

3) Calculate the angular functions (F, G) of the n-span basic structure.

4) Find the influence matrix of the angular load function (τ) of the n-span basic structure.

5) Write the matrix of the compatability (or five-moment) equations in terms of the redundants and the n-span angular functions from the steps 4 and 5. In abbreviated matrix notation, this is expressed as follows:

$$[F]_{nxn} [X]_{nxn'} = [-\tau]_{nxn'}$$

where n' represents the number of the points of the application of the unit load moving across the structure,

6) Solve for the redundant moment influence matrix as

$$[X]_{nxn'} = [F]^{-1}_{nxn} [-\tau]_{nxn'}$$

7) The other forces at any section is then a matter of stereo statics.

8) The final step is the graphical representation of the influence values.

The following numerical example demonstrates the above steps.

Numerical Problem No. 2

A four-span continuous circular beam 23456 with fixed ends is shown in Fig. 8.4. The radius of curvature is 60 ft. Each span subtends a 30° angle at the center of curvature. The cross-section is constant throughout the girder. The ratio (EI/GJ) is assumed equal to 2.

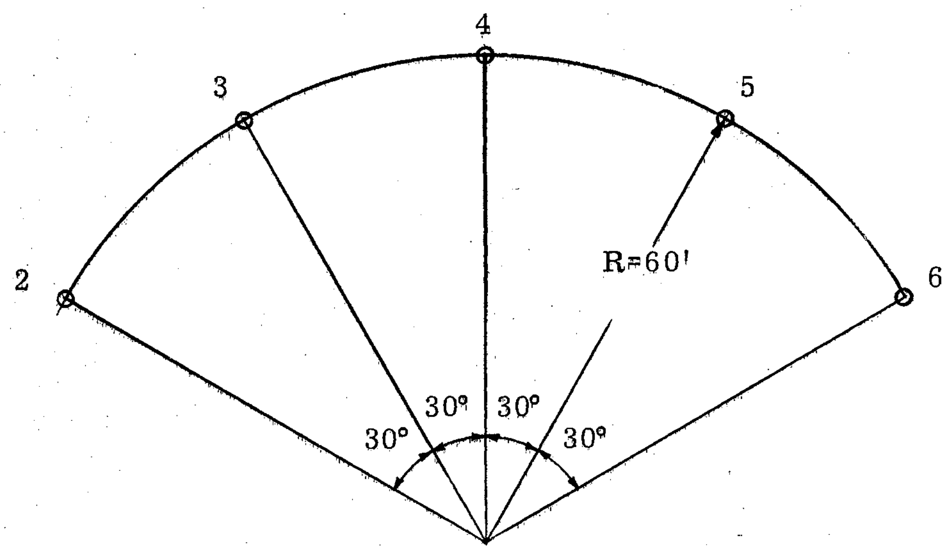


Fig. 8.4

Four-Span Continuous Girder With Fixed Ends

1.) All the four spans being identical in dimensions, Fig. 8.2 represents the corresponding basic span.

The positions of a moving unit load across any span are selected 5° apart. The angular functions (f, g) of various angular spans 5°, 10°, 15°, 20°, 25°, 30° are necessary for the determination of the angular load functions (t) for various positions of the unit load. They are obtained by employing Eqs. (7.21) and are recorded in Table (8.2). The basic span angular load functions (t) for each position of the unit load across the span are calculated vide Eqs. (7.8 - 13), and are noted in Table (8.3).

2.) Fig. 8.5 shows the corresponding n-span basic structure, where the broken line represents the imaginary extensions to take care of the end moments Y_{23} and Y_{65}. The geometric angle changes at 2, 3, 4, 5, 6 are -90°, 30°, 30°, 30°, -90°, respectively.

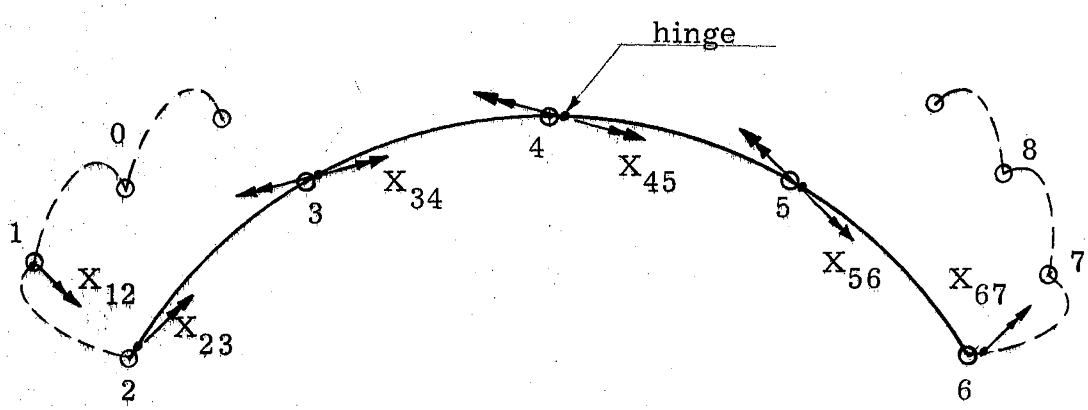


Fig. 8.5

n-Span Basic Structure

TABLE 8.2 BASIC SPAN ANGULAR FUNCTIONS

$2w_i$	f_{ijxx} $\frac{EI}{R}$	f_{ijyy}, f_{jiyy} $\frac{EI}{R}$	f_{ijxy}, f_{ijyx}, g_{ijxy}, g_{jiyx} $\frac{EI}{R}$	g_{ijyy}, g_{jiyy} $\frac{EI}{R}$
5°	0.17448	0.02913	-0.00190	-0.01453
10°	0.34862	0.05850	-0.00761	-0.02899
15°	0.52211	0.08836	-0.01710	-0.04328
20°	0.69461	0.11895	-0.03034	-0.05735
25°	0.86581	0.15050	-0.04730	-0.07110
30°	1.03540	0.18325	-0.06792	-0.08445

TABLE 8.3 BASIC SPAN ANGULAR LOAD FUNCTIONS

Unit Load at	t_{ijx} $\frac{EI}{R}$	t_{ijy} $\frac{EI}{R}$	t_{jiy} $\frac{EI}{R}$
i	0.00	0.00	0.00
1	-0.78619	-0.64712	0.49561
2	-1.10032	-0.95147	0.89085
3	-1.05841	-0.96985	1.10800
4	-0.78010	-0.76813	1.07420
5	-0.38683	-0.41905	0.72368
j	0.00	0.00	0.00

X_{12} (= Y_{23}), X_{23}, X_{34}, X_{45}, X_{56}, and X_{67} (= Y_{65}) are the redundant moments to be determined for all positions of a moving unit load.

3.) Eqs. (7.14 - 18) provide the following n-span angular functions:

<u>Angular Flexibilities</u>

$$F_1^{(L)} = F_1^{(C)} = 0$$

$$F_1^{(R)} = (0.18325)\frac{R}{EI}$$

Therefore,

$$F_1 = (0.18325)\frac{R}{EI}$$

$$F_2^{(L)} = 0$$

$$F_2^{(C)} = (0.91776)\frac{R}{EI}$$

$$F_2^{(R)} = (1.16511)\frac{R}{EI}$$

Therefore,

$$F_2 = (2.08287)\frac{R}{EI}$$

$$F_3^{(L)} = (0.91177)\frac{R}{EI}$$

$$F_3^{(C)} = (0.80012)\frac{R}{EI}$$

$$F_3^{(R)} = (0.91177)\frac{R}{EI}$$

Therefore,

$$F_3 = (2.62366)\frac{R}{EI}$$

$$F_4^{(L)} = (0.91177)\frac{R}{EI}$$

$$F_4^{(C)} = (0.80012)\frac{R}{EI}$$

$$F_4^{(R)} = (0.91177)\frac{R}{EI}$$

Therefore,

$$F_4 = (2.62366)\frac{R}{EI}$$

$$F_5^{(L)} = (1.16511)\frac{R}{EI}$$

$$F_5^{(C)} = (0.91776)\frac{R}{EI}$$

$$F_5^{(R)} = 0$$

Therefore,

$$F_5 = (2.08287)\frac{R}{EI}$$

$$F_6^{(L)} = (0.18325)\frac{R}{EI}$$

$$F_6^{(C)} = F_6^{(R)} = 0$$

Therefore,

$$F_6 = (0.18325)\frac{R}{EI}$$

Near Carry-Over Factors

$$G_{12}^{(L)} = 0$$

$$G_{12}^{(R)} = (-0.21419)\frac{R}{EI}$$

Therefore,

$$G_{21} = G_{12} = (-0.21419)\frac{R}{EI}$$

$$G_{23}^{(L)} = (0.13585)\frac{R}{EI}$$

$$G_{23}^{(R)} = (-0.84123)\frac{R}{EI}$$

Therefore,

$$G_{32} = G_{23} = (-0.70538)\frac{R}{EI}$$

$$G_{34}^{(L)} = (0.42838)\frac{R}{EI}$$

$$G_{34}^{(R)} = (-0.84123)\frac{R}{EI}$$

Therefore,

$$G_{43} = G_{34} = (-0.41285)\frac{R}{EI}$$

$$G_{45}^{(L)} = (0.42838)\frac{R}{EI}$$

$$G_{45}^{(R)} = (-1.13376)\frac{R}{EI}$$

Therefore,

$$G_{54} = G_{45} = (-0.70538)\frac{R}{EI}$$

$$G_{56}^{(L)} = (-0.21419)\frac{R}{EI}$$

$$G_{56}^{(R)} = 0$$

Therefore,

$$G_{65} = G_{56} = (-0.21419)\frac{R}{EI}$$

<u>Far Carry-Over Values</u>

$$G_{13} = G_{31} = (0.16890)\frac{R}{EI}$$

$$G_{24} = G_{42} = (-0.33779)\frac{R}{EI}$$

$$G_{35} = G_{53} = (-0.33779)\frac{R}{EI}$$

$$G_{46} = G_{64} = (0.16890)\frac{R}{EI}$$

4.) Eq. (7.19), after substituting for the known quantities, yields the following expressions for the n-span angular load functions.

$$\tau_1 = (t_{23y} + 0.1462677\ (\frac{R}{EI})\ bX_{23})$$

$$\tau_2 = \begin{pmatrix} t_{23x} + \sqrt{3}\ t_{32y} - 2\ t_{34y} \\ -1.16511\ (\frac{R}{EI})\ bX_{23} - 0.29254\ (\frac{R}{EI})\ bX_{34} \end{pmatrix}$$

$$\tau_3 = \begin{pmatrix} -2(t_{32y} + t_{45y}) + \sqrt{3}\ (t_{34y} + t_{43y}) + t_{34x} \\ +0.84123\ (\frac{R}{EI})\ bX_{23} - 0.91177\ (\frac{R}{EI})\ bX_{34} - 0.29254\ (\frac{R}{EI})bX_{45} \end{pmatrix}$$

$$\tau_4 = \begin{pmatrix} -2(t_{43y} + t_{56y}) + \sqrt{3}\,(t_{45y} + t_{54y}) + t_{45x} \\ +0.33779\,(\frac{R}{EI})\,bX_{23} + 0.84123\,(\frac{R}{EI})\,bX_{34} - 0.91177\,(\frac{R}{EI})\,bX_{45} \end{pmatrix}$$

$$\tau_5 = \begin{pmatrix} t_{56x} + \sqrt{3}\,t_{56y} - 2\,t_{54y} \\ +0.33779\,(\frac{R}{EI})\,bX_{34} + 1.13376\,(\frac{R}{EI})\,bX_{45} \end{pmatrix}$$

$$\tau_6 = (t_{65y} - 0.16890\,(\frac{R}{EI})\,bX_{45})$$

The compatability equations are as follows:

$$X_{12}\,F_1 + X_{23}\,G_{21} + X_{34}\,G_{31} + \tau_1 = 0$$

$$X_{12}\,G_{12} + X_{23}\,F_2 + X_{34}\,G_{32} + X_{45}\,G_{42} + \tau_2 = 0$$

$$X_{12}\,G_{13} + X_{23}G_{23} + X_{34}\,F_3 + X_{45}\,G_{43} + X_{56}\,G_{53} + \tau_3 = 0$$

$$X_{23}\,G_{24} + X_{34}\,G_{34} + X_{45}\,F_4 + X_{56}\,G_{54} + X_{67}\,G_{64} + \tau_4 = 0$$

$$X_{34}\,G_{35} + X_{45}\,G_{45} + X_{56}\,F_5 + X_{67}\,G_{65} + \tau_5 = 0$$

$$X_{45}\,G_{46} + X_{56}\,G_{56} + X_{67}\,F_6 + \tau_6 = 0$$

To save some labor, the following substitutions are made into the above equations.

$$X_{45} = -X_{54} + bX_{45}$$

$$X_{56} = -X_{65} + bX_{56}$$

Thus, the compatibility equations become:

$$X_{12} F_1 + X_{23} G_{21} + X_{34} G_{31} + \tau_1 = 0$$

$$X_{12} G_{12} + X_{23} F_2 + X_{34} G_{32} + X_{54} (-G_{42}) + \tau_2^* = 0$$

$$X_{12} G_{13} + X_{23} G_{23} + X_{34} F_3 + X_{54} (-G_{43}) + $$
$$+ X_{65} (-G_{53}) + \tau_3^* = 0$$

$$X_{23} G_{24} + X_{34} G_{34} + X_{54} (-F_4) + X_{65} (-G_{54}) +$$
$$+ X_{67} G_{64} + \tau_4^* = 0$$

$$X_{34} G_{35} + X_{54} (-G_{45}) + X_{65} (-F_5) + X_{67} G_{65} + \tau_5^* = 0$$

$$X_{54} (-G_{46}) + X_{65} (-G_{56}) + X_{67} F_6 + \tau_6^* = 0$$

where

$$\tau_2^* = \tau_2 + bX_{45} G_{42}$$

$$\tau_3^* = \tau_3 + bX_{45} G_{43} + bX_{56} G_{53}$$

and the angular load functions τ_6^*, τ_5^*, and τ_4^* are symmetrical with respect to τ_1, τ_2^*, and τ_3^*, respectively.

These angular load functions are recorded in Table 8.4 for all necessary positions of moving unit load.

5.) The compatibility and the redundant influence matrices are given as follow:

TABLE 8.4 INFLUENCE VALUES OF THE

ANGULAR LOAD FUNCTIONS

Unit Load at	$(\tau_1)\frac{EI}{R}$	$(\tau_2*)\frac{EI}{R}$	$(\tau_3*)\frac{EI}{R}$	$(\tau_4*)\frac{EI}{R}$	$(\tau_5*)\frac{EI}{R}$	$(\tau_6*)\frac{EI}{R}$
②	0.00	0.00	0.00	0.00	0.00	0.00
1	-0.48141	-1.24774	-0.03819	+0.38269	0.00	0.00
2	-0.68583	-1.67332	-0.25393	+0.61347	0.00	0.00
3	-0.67081	-1.52132	-0.49616	+0.69060	0.00	0.00
4	-0.50249	-1.03554	-0.62062	+0.61347	0.00	0.00
5	-0.25334	-0.45354	-0.49433	+0.38269	0.00	0.00
③	0.00	0.00	0.00	0.00	0.00	0.00
1	0.00	+0.96283	-2.08158	-0.03896	+0.38269	0.00
2	0.00	+1.37166	-2.86122	-0.25518	+0.61347	0.00
3	0.00	+1.34163	-2.68320	-0.49758	+0.69060	0.00
4	0.00	+1.00497	-1.90587	-0.62187	+0.61347	0.00
5	0.00	+0.50669	-0.89216	-0.49510	+0.38269	0.00
④	0.00	0.00	0.00	0.00	0.00	0.00
1	0.00	-0.38269	+0.49510	+0.89216	-0.50669	0.00
2	0.00	-0.61347	+0.62187	+1.90587	-1.00497	0.00
3	0.00	-0.69060	+0.49758	+2.68320	-1.34163	0.00
4	0.00	-0.61347	+0.25518	+2.86122	-1.37166	0.00
5	0.00	-0.38269	+0.03896	+2.08158	-0.96283	0.00
⑤	0.00	0.00	0.00	0.00	0.00	0.00
1	0.00	0.00	-0.38269	+0.49433	+0.45354	+0.25344
2	0.00	0.00	-0.61347	+0.62062	+1.03554	+0.50249
3	0.00	0.00	-0.69060	+0.49616	+1.52132	+0.67081
4	0.00	0.00	-0.61347	+0.25393	+1.67332	+0.68583
5	0.00	0.00	-0.38269	+0.03819	+1.24774	+0.48141
⑥	0.00	0.00	0.00	0.00	0.00	0.00

COMPATIBILITY MATRIX 8. 1

$$
\underbrace{\begin{bmatrix}
0.18325 & -0.21419 & 0.16890 & - & - & - \\
-0.21419 & 2.08287 & -0.70538 & -0.33779 & - & - \\
0.16890 & -0.70538 & 2.62366 & -0.41285 & -0.33779 & - \\
- & -0.33779 & -0.41285 & 2.62366 & -0.70538 & 0.16890 \\
- & - & -0.33779 & -0.70538 & 2.08287 & -0.21419 \\
- & - & - & 0.16890 & -0.21419 & 0.18325
\end{bmatrix}}_{\text{Flexibility Matrix}}
\underbrace{\begin{bmatrix}
X_{12} \\ X_{23} \\ X_{34} \\ -X_{54} \\ -X_{65} \\ X_{67}
\end{bmatrix}}_{\text{"X" Matrix}}
= (-)
\underbrace{\begin{bmatrix}
\tau_1 \\ \tau_2{}^* \\ \tau_3{}^* \\ \tau_4{}^* \\ \tau_5{}^* \\ \tau_6{}^*
\end{bmatrix}}_{\text{Load Matrix}}
\left(\frac{EI}{R}\right)
$$

REDUNDANT MOMENT INFLUENCE MATRIX 8.2

$$
\begin{bmatrix} X_{12} \\ X_{23} \\ X_{34} \\ -X_{54} \\ -X_{65} \\ X_{67} \end{bmatrix}
=
\begin{bmatrix}
6.361741 & 0.571662 & -0.256694 & 0.027110 & -0.039802 & 0.071507 \\
0.571662 & 0.611175 & 0.154822 & 0.122407 & 0.062470 & -0.039802 \\
0.256694 & 0.154822 & 0.474853 & 0.125819 & 0.122407 & 0.027110 \\
0.027110 & 0.122407 & 0.125819 & 0.474853 & 0.154822 & -0.256694 \\
-0.039802 & 0.062470 & 0.122407 & 0.154822 & 0.611175 & 0.571662 \\
-0.071507 & -0.039802 & 0.027110 & -0.256694 & 0.571662 & 6.361741
\end{bmatrix}
\begin{bmatrix} -\tau_1 \\ -\tau_2^* \\ -\tau_3^* \\ -\tau_4^* \\ -\tau_5^* \\ -\tau_6^* \end{bmatrix}
\left(\frac{EI}{R} \right)
$$

6.) The influence values of the redundant moments are obtained by using the angular functions of Table (8.4) in the matrix (8.2), and they are recorded in Table (8.5)

7.) The influence values of the reaction Z_3 and the cross-sectional elements at the section 2 of span 34 are developed by making use of the matrices (4.1-4). They are recorded in Table (8.6).

8.) The influence lines are drawn for the reactions X_{34} and Z_3 (Fig. 8.6) and the cross-sectional elements V_{2Z3}, M_{2t3}, and M_{2b3} (Fig. 8.7).

The influence values for deflection at the center of the span 34 (Fig. 8.4) are developed in the third numerical example which follows shortly.

The following sequence of procedure is found helpful in solving for the desired influence line.

1) Analyze the structure for a unit load applied at the section considered.

2) Recognize the corresponding conjugate structure.

3) Calculate the total elastic weights and the elasto-static moments for all the spans.

4) Employ the elasto-static equilibrium, and determine the conjugate end reactions and the elastic curve in all the spans.

5) Draw the influence line.

These steps are demonstrated in the following example.

TABLE 8.5 INFLUENCE VALUES OF REDUNDANT MOMENTS

Unit Load at	Y_{23}	X_{23}	X_{34}	X_{54}	X_{65}	Y_{65}
(2)	0.00	0.00	0.00	0.00	0.00	0.00
1	3.755739	0.996864	0.039585	0.011133	-0.004211	0.015182
2	5.237838	1.378977	0.126410	0.035941	-0.013338	0.048716
3	4.952971	1.305553	0.212053	0.061100	-0.022151	0.082204
4	3.612725	0.941142	0.248855	0.072844	-0.025679	0.097152
5	1.733710	0.451707	0.191770	0.057141	-0.019510	0.075468
(3)	0.00	0.00	0.00	0.00	0.00	0.00
1	-1.068452	-0.285320	0.797436	-0.103297	0.033205	-0.134016
2	-1.487249	-0.402434	1.103309	-0.218289	0.070886	-0.284040
3	-1.414744	-0.386787	1.044485	-0.302729	0.100408	-0.396372
4	-1.022455	-0.281345	0.752569	-0.317098	0.108148	-0.418662
5	-0.490011	-0.134851	0.360647	-0.226081	0.079683	-0.301505
(4)	0.00	0.00	0.00	0.00	0.00	0.00
1	0.301505	0.079683	-0.226081	0.360647	-0.134851	0.490011
2	0.418662	0.108148	-0.317098	0.752569	-0.281345	1.022455
3	0.396372	0.100408	-0.302729	1.044485	-0.386787	1.414744
4	0.284040	0.070886	-0.218289	1.103309	-0.402434	1.487249
5	0.134016	0.033205	-0.103297	0.797436	-0.285320	1.068452
(5)	0.00	0.00	0.00	0.00	0.00	0.00
1	-0.075468	-0.019510	0.057141	0.191770	0.451707	-1.733709
2	-0.097152	-0.025679	0.072844	0.248855	0.941142	-3.612725
3	-0.082204	-0.022151	0.061100	0.212053	1.305553	-4.952971
4	-0.048716	-0.013338	0.035941	0.126410	1.378977	-5.237838
5	-0.015182	-0.004211	0.011133	0.039585	0.996864	-3.755739
(6)	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 8.6 INFLUENCE VALUES OF SUPPORT REACTION AND THE CROSS-SECTIONAL ELEMENTS

Unit Load at	Z_3	V_{2Z3}	M_{2t3}	M_{2b3}
②	0.00	0.00	0.00	0.00
1	0.067641	0.013893	0.002920	-0.198070
2	0.240066	0.044564	0.009976	-0.635806
3	0.470556	0.075167	0.018081	-1.073383
4	0.705665	0.088792	0.023116	-1.269285
5	0.898915	0.068935	0.019489	-0.986610
③	1.00	0.00	0.00	0.00
1	0.974610	-0.127054	-0.056884	1.851873
2	0.843383	-0.297993 0.702007	-0.222505	4.512449
3	0.642277	0.507896	-0.352451	2.627830
4	0.409055	0.311982	-0.302675	1.325842
5	0.182229	0.135708	-0.157624	0.491531
④	0.00	0.00	0.00	0.00
1	-0.108788	-0.080078	0.104209	-0.265761
2	-0.150806	-0.110820	0.148155	-0.364373
3	-0.142480	-0.104624	0.143008	-0.341279
4	-0.101969	-0.074814	0.103954	-0.242573
5	-0.048081	-0.035263	0.049379	-0.114003
⑤	0.00	0.00	0.00	0.00
1	0.027176	0.019979	-0.026686	0.065709
2	0.035054	0.025803	-0.033574	0.085641
3	0.029713	0.021896	-0.027823	0.073252
4	0.017635	0.013008	-0.016197	0.043805
5	0.005502	0.004062	-0.004973	0.013752
⑥	0.00	0.00	0.00	0.00

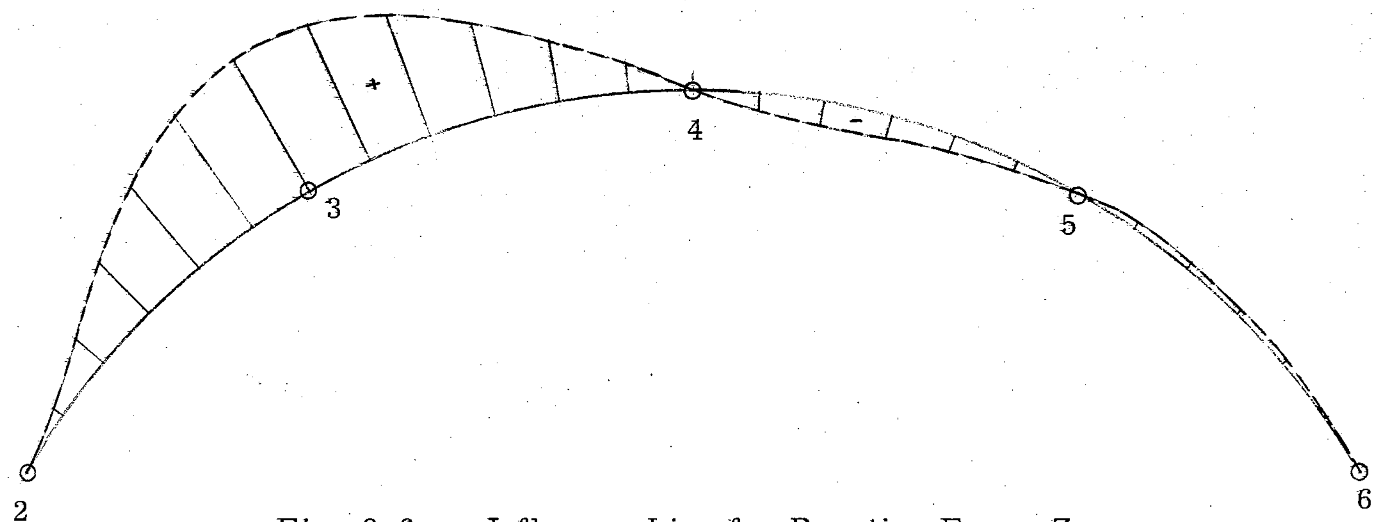


Fig. 8-6a. Influence Line for Reactive Force Z_3

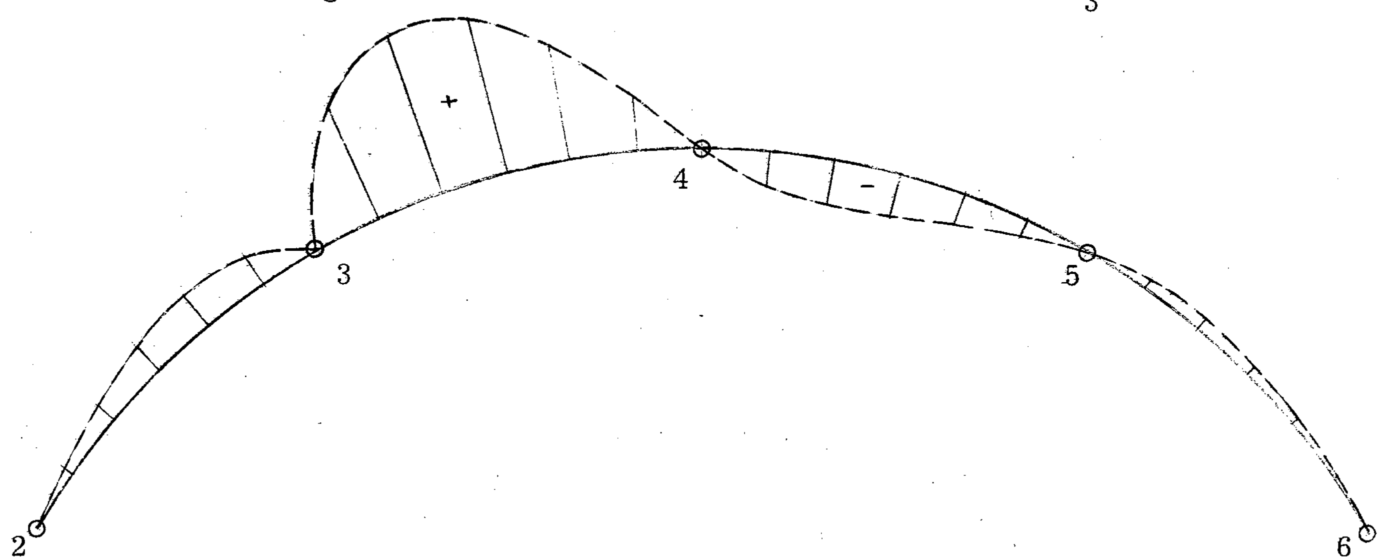


Fig. 8-6b. Influence Line for Chord Moment X_{34}

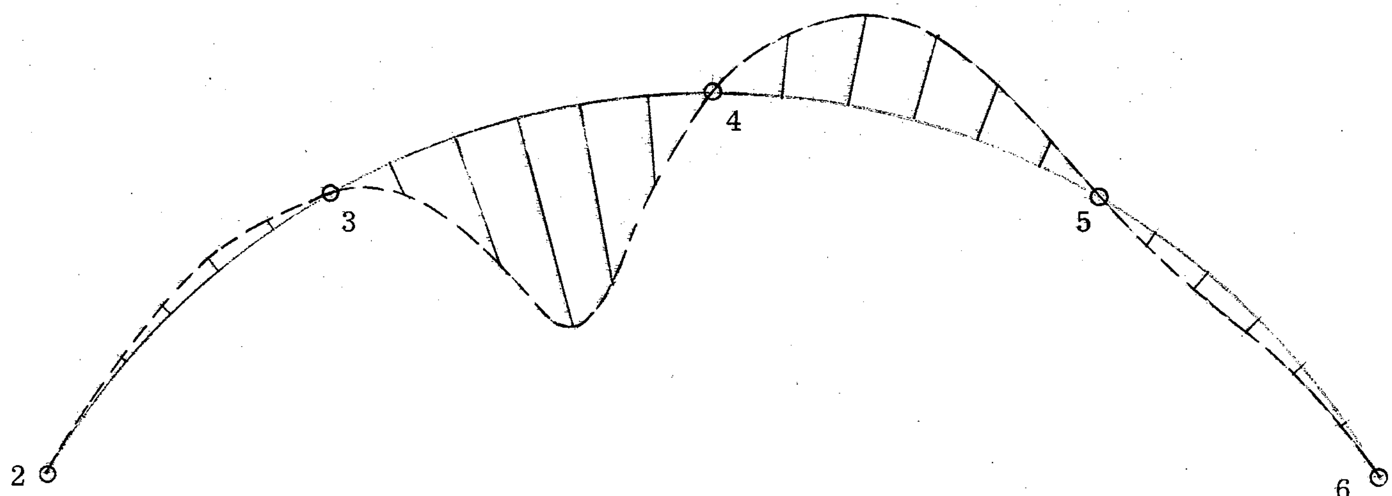


Fig. 8-7a. Influence Line for Twisting Moment M_{2t3}

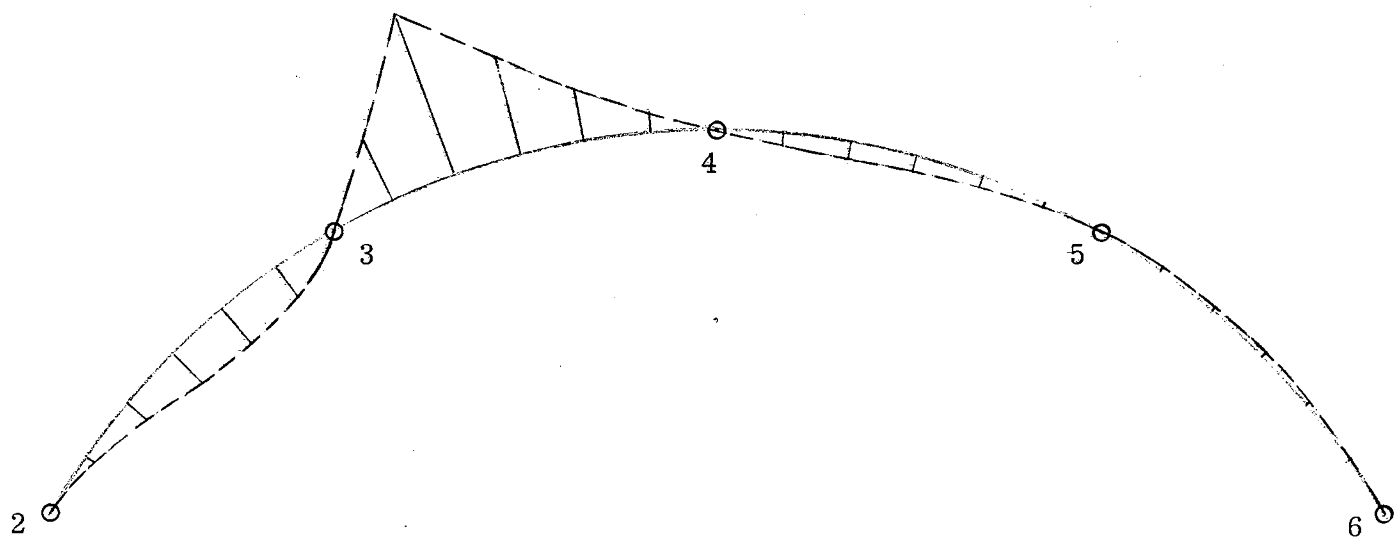


Fig. 8-7b. Influence Line for Bending Moment M_{2b3}

c) Numerical Example No. 3

The influence line for deflection at the section 2 in the span 34 (Fig. 8.4) is investigated. Therefore a unit load is applied at that section.

1.) The redundant moments due to the applied unit load are available from the preceding example. The other end moments of all the spans are determined by employing the stereo static matrices (2.1) and (4.1). They are recorded in the following table.

TABLE (8.7) END MOMENTS DUE TO UNIT LOAD

End Moment \ Span	23	34	45	56
X_{ij}	-0.40243	+1.10331	+0.218289	-0.07089
Y_{ij}	-1.48725	+2.71586	+1.803730	-0.55935
Y_{ji}	-2.90365	-1.67122	+0.519860	-0.28404

2.) The corresponding conjugate structure (Fig. 8.8) is free at the ends. The four spans are connected by hinges at the interior supports.

3.) The elastostatic moments and the total elastic weights given in Table (8.8) are calculated by employing Eqs. (7.11 - 13, 31, 32) and (7.33), respectively. The functions t_{34x}, t_{34y}, and t_{43y} are available from the second example.

4.) The elastic curve is determined by using the matrix (6.5) and Eq. (6.6). The influence values of the deflection are recorded in Table (8.9)

5) Finally the influence line for the deflection at section 2 in span 34 is drawn as shown in Fig. 8.9.

TABLE 8.8 ELASTO STATIC MOMENTS AND ELASTIC WEIGHTS

r	Elasto Static Moments for Unit Causes				Total Elastic Weights		
	$\overline{BM}_{rzi}^{(xi)}$ EI	$\overline{BM}_{rzi}^{(yi)}$ EI	$\overline{BM}_{rzi}^{(yj)}$ EI	$\overline{BM}_{rzi}^{(P2)}$ EI	ij	$\overline{P}_{ijx}$ EI	$\overline{P}_{ijy}$ EI
ⓘ	0.00	0.00	0.00	0.00	23	7.106752	-22.750223
1	2.384245	12.743392	-0.773091	-16.048191			
2	4.831874	48.757662	-6.137641	-127.408141	34	1.734414	40.074479
3	0.219570	104.221517	-20.535397	-402.395620			
4	-18.576745	174.752509	-48.129686	-808.867251	45	-4.091739	-20.621530
5	-58.573194	255.540772	-92.671001	-1286.575182			
ⓙ	-126.570074	341.488463	-157.368005	-1772.672547	56	0.966648	4.422083

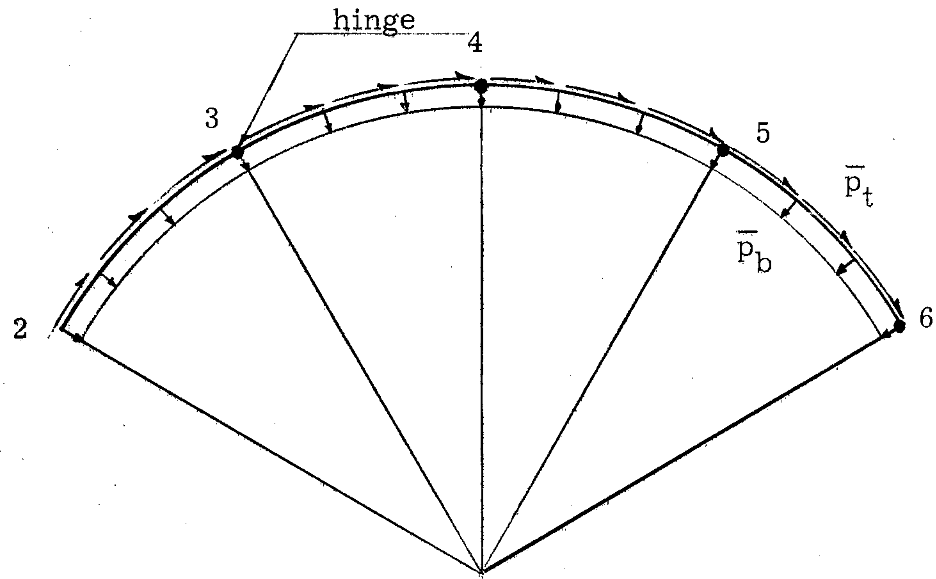


Fig. 8.8

Conjugate Structure

TABLE 8.9 INFLUENCE VALUES OF DEFLECTION Δ_{2Z3}

Unit Load at	$EI(\Delta_{2Z3})$	Unit Load at	$EI(\Delta_{2Z3})$
(2)	0.00	(4)	0.00
1	- 17.667309	1	-66.343528
2	- 56.377010	2	-91.954794
3	- 95.464006	3	-86.918261
4	-112.672602	4	-62.199720
5	- 87.396359	5	-29.316459
(3)	0.00	(5)	0.00
1	147.240166	1	16.497150
2	269.608582	2	21.160949
3	287.029684	3	17.716899
4	217.907936	4	10.127602
5	107.038778	5	2.473411
(4)	0.00	(6)	0.00

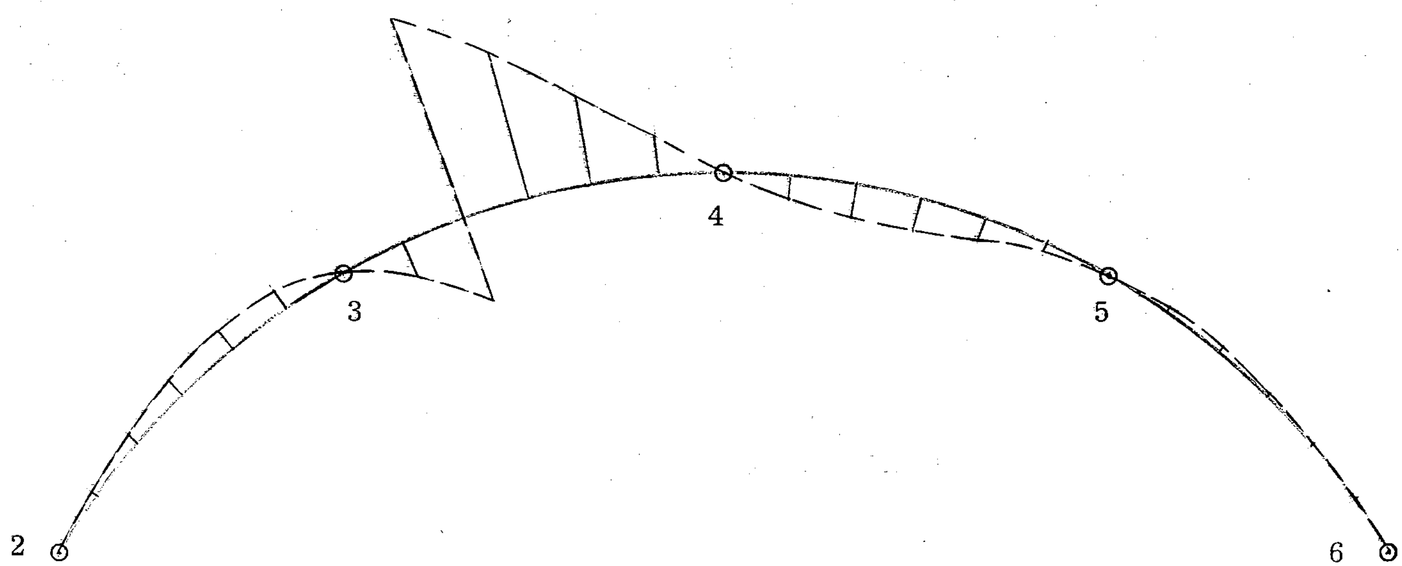


Fig. 8-7c. Influence Line for Shear V_{2Z3}

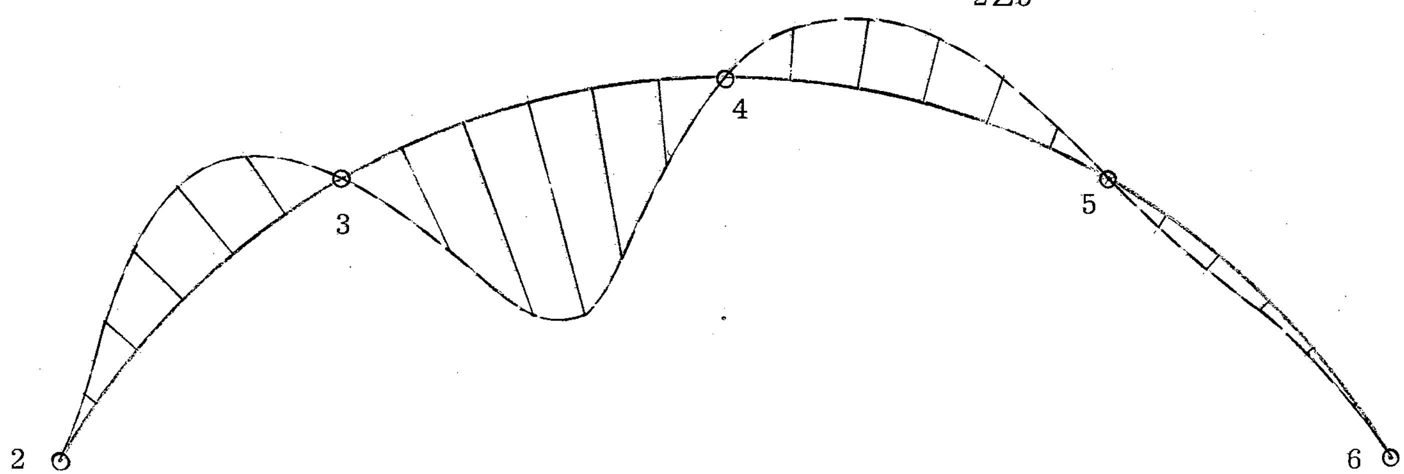


Fig. 8-9. Influence Line for Deflection Δ_{2Z3}

CHAPTER IX

SUMMARY AND CONCLUSION

9.1 Summary

The analysis of planar continuous curved beams, loaded laterally, by the flexibility method is presented in this study.

Chord moments are chosen as redundants. The other reactive forces and moments are expressed in terms of the redundants by stereo static and joint equilibrium conditions. Two basic structures, one-span and n-span, are selected. The n-span angular functions (F, G, τ) are derived in terms of one-span angular functions (f, g, t) and the static moments bX_{ij}. A five-moment equation is derived for the deformation compatibility. The influence values of the one-span angular functions (t) are expressed in terms of the angular flexibilities and carry-over values (f, g). The carry-over form of the five-moment equation is presented.

The theory presented in this thesis is illustrated by three numerical examples.

9.2 Conclusions

A five-moment equation has been derived in terms of angular flexibilities and redundant moments. This provides a systematic procedure for the analysis of planar continuous curved and bent members. Provision is made for the displacements of the rigid supports normal to the plane of the beam.

In problems involving a smaller number of redundants, the analysis by the carry-over moment method becomes faster in convergence - especially when the ends of the structure are free to rotate about y-axis of the end spans. If a structure with up to six redundants is symmetrical, the applied loads can be resolved into symmetrical and anti-symmetrical systems and the carry-over moment method can be very efficiently applied for its analysis.

In this thesis, the theory developed is applied to a continuous circular beam. However, this is applicable for any other curved members if the angular functions of a basic span are available.

9.3 Extension

The flexibility approach may be extended to make provision for the elastically supported planar continuous curved or bent structures.

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VITA

Muppidi Narsimha Reddy

Candidate for the Degree

of Master of Science

Thesis: INFLUENCE LINES FOR CONTINUOUS CURVED MEMBERS WITH LATERAL LOADS.

Major Field: Civil Engineering

Biographical:

Personal Data: Born February 11, 1933 in Hyderabad, India, the son of Gopal Reddy and Manikyamma.

Education: Graduated from Government High School, Karimnagar, India, June, 1951. Passed the Intermediate Examination, Arts and Science College, Warangal, India, May, 1953. Received the Degree of Bachelor of Science in Civil Engineering from Osmania University, Hyderabad, India, April, 1956. Member of Chi Epsilon and Associate Member of the Society of the Sigma Xi. Completed the requirements for the degree of Master of Science in Civil Engineering at the Oklahoma State University, Stillwater, Oklahoma, August, 1963.

Professional Experience: Worked as a trainee, with the Public Works Department, Hyderabad, India, from June, 1956 to October, 1956; as a Junior Engineer in Nagarjunasagar Left Bank Canal Project, Hyderabad, India, from November, 1956, to November, 1960.