QUEUING OF MULTIPLE POISSON INPUTS TO A SINGLE CHANNEL WITH ERLANG SERVICE

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PREFACE

The study of variable input to a service facility is a common problem to the Industrial Engineer. Such situations as men servicing machine breakdowns, telephone calls to a switchboard and others present a challenge to the industrial researcher. The conventional single channel model with a single Poisson input to an exponential service facility has been fully explored, but little work has been done on the general model of multiple inputs with a general distribution of service time.

The importance of any analytical development is its practical use in industrial applications. The problem of this thesis was first encountered in the study of the logistics of a maintenance and supply operation of a commercial airline. The purpose of this study is to present the theoretical model and compare with the actual situation.

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CHAPTER I

INTRODUCTION

A queuing process is composed of a service system and an input source which combines to establish an output (1). The study of operational systems with variable demands and service is a problem in congestion. Most of the previous work in queuing theory develops the single channel model with input from a homogeneous single Poisson process with service times of an exponential nature. The developments of interest are divided into the conventional model, the Erlang service case and the multiple input case.

Conventional Single Channel

The conventional single channel model designed for a single Poisson input (λ) and with exponential service (μ) is the most widely acknowledged of the queuing models. The complete development will not be shown, but its most interesting facets will be mentioned.

The single channel model is characterized by having a single facility servicing one kind of unit. The mean rate of arrival (λ) and the mean service time (μ) determines the model. The utilization factor $\rho = \frac{\lambda}{\mu}$ is used in most cases to simplify the results. The steady-state equations which describes the model are derived by the transition of units arriving and serviced through the facility. These transitions are products of P_n, the probability of (n) units in the system.

The pure birth and death process of Feller (2) describes the basic theory of queues. The system changes only through transitions from one state to another in single steps. The transitions are postulates of $E_n \longrightarrow E_{n+1}$ or $E_{n-1} \longrightarrow E_n$ where E_n is the state of (n) items in the system. The probability $P_n(t+\Delta t)$ of (n) items in the system at a finite period of time $(t+\Delta t)$ is composed of four probabilities during the interval of time $(t, t+\Delta t)$.

1. No change, $n \longrightarrow n$
2. One change from $n-1 \longrightarrow n$
3. One change from $n \longrightarrow n+1$
4. Two or more transitions which are of higher order and assumed zero.

The transition into and out of each state is the product of a rate and a probability describing the state. For the first case of $E_n \longrightarrow E_n$, where there is no change in the interval $(t, t+\Delta t)$ the transition is the product of no arrivals, no service and the probability of no change in state E_n.

$P_n(t)$ = (n) units in the system at time (t)

$1 - \lambda\Delta t$ = zero arrivals in the interval (Δt)

$1 - \mu\Delta t$ = zero service in the interval (Δt).

In the limit as $\Delta t \longrightarrow 0$ the transitional probabilities of state E_n becomes $-(\lambda+\mu)P_n$. The same reasoning is applied throughout the development of the equations of balance or steady-state solutions for $E_{n+1} \longrightarrow E_n$ and $E_{n-1} \longrightarrow E_n$. The complete transitional probabilities will be described in Chapter II. This limited discussion of the conventional Single Channel Model was intended to formulate the basic idea of the pure birth and death process.

Erlang Service

The Erlang Service distribution is a general type of distribution having (k) individual phases or stages linked together to simulate a continuous service facility of varying service time. Queuing models utilizing this distribution involve the simulation of phase type service. The earliest work of Erlang (3), further work of Jackson (4), and extensive display by Morse (5) utilizes the Erlang distribution to simulate general service times.

The Erlang density function is given by the expression:

$$s(t;k) = \frac{(k\mu)^k t^{k-1} e^{-k\mu t}}{(k-1)!}$$

The distribution function S(t) by definition is:

$$S(t) = \int_t^\infty s(x)dx = e^{-k\mu t} \sum_{n=0}^{k-1} \frac{(k\mu t)^n}{n!}$$

Then the mean rate of completion of service T_s is:

$$T_s = \int_0^\infty S(t)dt = \frac{1}{\mu}$$

For (k) phases each of service rate ($k\mu$) makes for a total service rate of $\frac{1}{\mu}\left(\frac{k}{k\mu} = \frac{1}{\mu}\right)$.

The importance of the Erlang service function is the fact that each phase has a service rate of ($k\mu$). As was shown in the Conventional Single Channel System, the transitional probabilities are the product of a rate and a finite probability describing the postulates of the states. Therefore the service rate of progression from phase $k \longrightarrow k-1$ or $s+1 \longrightarrow s$ becomes ($k\mu$) times the probability of the initial state. For example, the transition probability from state $E_{n,s+1} \longrightarrow E_{ns}$ become $k\mu P_{n,s+1}$ where $P_{n,s+1}$ is a finite number or probability of being in state $E_{n,s+1}$ for (n) units in the system and a unit in phase ($s+1$).

Multiple Inputs

The only work found resembling the model presented in this thesis was done by Ancker and Gafarian (6) and comparable to Pollocyek (7). The signification of this work was to develop the steady-state equations, the expression for (P_n), probability of (n) units in the queue, and the expected number of tasks in the queue. It was shown that the results derived were comparable with earlier results of Pollocyek.

The general equations of balance and the transitional probabilities are of particular interest. The probability notation ${}_jP_n$ is defined for (n) units in the queue and the (j^{th}) type in service. The initial equations of transition for zero units in the system:

$$0 = -\lambda \; {}_0P_0(t) + \sum_{i=1}^{i=m} \mu_i \; {}_iP_0(t) \qquad 1.1$$

Equation 1.1 is the sum of the independent probabilities of transition for the state $E_n \longrightarrow E_n$ from (n=0) units in the system with no arrivals or service terminations and $E_{n+1} \longrightarrow E_n$ for (n=1) units in the system with (i^{th}) type in service with one unit of any (i^{th}) type completing service in (Δt).

The second equation of balance for one unit in the queue and type (j) in service:

$$0 = -(\lambda+\mu_j) \; {}_jP_0(t) + \lambda_j \; {}_0P_0(t) + \frac{\lambda_j}{\lambda} \sum_{i=1}^{i=m} \mu_i \; {}_iP_1(t) \qquad 1.2$$

Equation 1.2 is the sum of three independent probabilities of transition for the state $E_n \longrightarrow E_n$, $E_{n-1} \longrightarrow E_n$ and $E_{n+1} \longrightarrow E_n$ with (n=0). The transition E_n is the product of no arrivals and no service of any of the (j) items. The transition E_{n-1} is the product of no service and one arrival of any (j) type. The transition E_{n+1}

is the product of the completion rate of any of (i) items and the chance the next unit to enter service being the (j^{th}) type. The final equation of balance is for (n) units in the system (n>1).

$$0 = -(\lambda+\mu_j)\ {}_jP_n(t) + \lambda\ {}_jP_{n-1}(t) + \frac{\lambda_j}{\lambda} \sum_{i=1}^{i=m} \mu_i\ {}_iP_{n+1}(t) \qquad 1.3$$

Equation 1.3 is developed in a similar manner as the other equations of balance with the fact that n>1. The transition E_n is for (n) units in the queue with no arrivals or completions of service. E_{n-1} is no service completions and no arrivals of the (j^{th}) type. E_{n+1} is the product of the chance the next unit in service is the (j^{th}) type times the probability of service completion of any (i^{th}) unit.

CHAPTER II

MULTIPLE INPUT MODEL

The queuing model is characterized by multiple inputs to an operational system of phase type service. Special notation is required to describe the probabilities of j^{th} arrival with i^{th} unit in the k^{th} phase.

The development of the analytical solution for the multiple input queue model is composed of simulating the transitions by a set of linear equations. The set of linear equations are solved for the basic probabilities P_n for $n=0,1,2, \ldots$ of (n) units in the system.

Input Description

The model to be presented in this thesis is one in which items arrive in Poisson fashion at a single channel servicing facility. Each input has an independent service time density function characterized by the Erlang function (3). The calling populations are mutually exclusive and independently distributed multiple Poisson variables. An unbounded queue is allowed for waiting. Arrivals are serviced on a first-come first-serviced basis.

The density function for the time between arrivals is the joint probability function of independently distributed exponential random variables. This joint probability function is called a Multiple Poisson Distribution by Feller (3) who gives the proof as:

$s(t)$ = density function

ω_i = inter-arrival times for the i^{th} item

$S(T)$ = distribution function

$$S(T) = 1 - \rho\left\{\omega_i > T,\ i=1,\ 2\ \ldots m\right\} = 1 - \prod_{i=1}^{m} \rho\left\{\omega_i > T\right\}$$

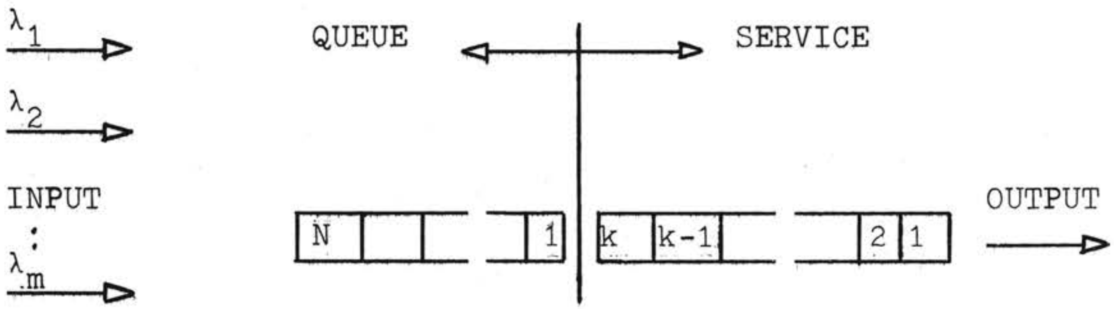
$$S(T) = 1 - \prod_{i=1}^{m} \int_{t}^{\infty} \lambda_i e^{-\lambda_i T} dT = 1 - \prod_{i=1}^{m} e^{-\lambda_i T} = 1 - e^{-\lambda T}$$

where $\lambda = \sum_{i=1}^{m} \lambda_i$ and λ_i = mean of the i^{th} arrival

thus $s(t) = \frac{dS(T)}{dt} = \lambda e^{-\lambda t}$

Operational System

The single channel facility allows but one input (customer/item) at a time to enter the service channel, with all other arrivals required to wait until the previous customer is discharged. With Erlang Service, the channel is a sequence of phases, each phase having its service time distributed as an exponential function. Figure 1 illustrates the system to be investigated in this thesis.



OPERATION SYSTEM

FIGURE 1

Each arrival (input) is from a mutually exclusive, independently distributed Poisson source. With an item in the k^{th} phase of the service channel, an arrival waits in the queue, in its order of arrival. The item in service progresses to the head of the service facility which is numbered in reverse, by the sequence k, k-1, ...s+1, s...2, 1. The service channel has a total of (k) phases, which must be traversed in turn before a new arrival may be admitted.

Notation

The following mathematical notation will be used throughout the analytical development.

$_jP_{ns}$ = Probability of (n) units in the queue with type (j) in service phase (s)

$_0P_{00}$ = Probability of zero units in the queue and in service

$P_n = \sum_{j=1}^{m} {}_jP_n$ = Probability of (n) units in the queue and one of any type in service

${}_jP_n = \sum_{s=1}^{k} {}_jP_{ns}$ = Probability of (n) units in the queue with type (j) in service

$\lambda = \sum_{j=1}^{m} \lambda_j$ = arrival rate of j^{th} unit

$k_1\mu_1, k_2\mu_2, \ldots k_i\mu_i$ = service rate of i^{th} phase

$$\alpha_n = \sum_i \sum_{s=0}^{k-1} \binom{s+n-1}{n-1} \frac{\lambda_i}{(\lambda+k_i\mu_i)^n} \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^s$$

Where α_n is a constant relating the transitional recurrence.

Steady-State Equations

The equations of balance are a set of homogeneous linear difference equations representing the transitions into and out of the queue states. Each equation is a composition of the probabilities for (n) units in the respective phases of k, (k=1, 2, ...s+1, s...). The transitions describing the probability are each a composition of the states of (j) unit in service phase k with (n) units in the queue. There are five independent probabilities that describe the steady-state equations. These are shown with the initial and final state probabilities as follows:

INITIAL STATE	FINAL STATE	PROBABILITY	DESCRIPTION
000	000	$-\lambda\ {}_0P_{00}$	No Arrival - No Service
j01	000	$\sum_{j=1}^{m} k_j\mu_j\ {}_jP_{01}$	No Arrival - Service Completed phase 1 by j^{th} item
j0k	j0k	$-(\lambda+k_j\mu_j)\ {}_jP_{0k}$	No Arrival - No Service from phase k
000	j0k	$\lambda_j\ {}_0P_{00}$	Arrival to phase k
j11	j0k	$\frac{\lambda_j}{\lambda}\sum_{i=1}^{m} k_i\mu_i\ {}_iP_{11}$	No Arrival-Service completed from phase 1 by (j^{th}) item times probability next item is j^{th} type
j0s	j0s	$-(\lambda+k_j\mu_j)\ {}_jP_{0s}$	No Arrival - No Service
j0,s+1	j0s	$k_j\mu_j\ {}_jP_{0,s+1}$	Service from phase s+1 $\longrightarrow$ s
jns	jns	$-(\lambda+k_j\mu_j)\ {}_jP_{ns}$	No Arrival - No Service
j,n-1,s	jns	$\lambda\ {}_jP_{n-1,s}$	Arrival - No Service
j,n,s+1	jns	$k_j\mu_j\ {}_jP_{n,s+1}$	No Arrival - Service phase s+1 $\longrightarrow$ s
jnk	jnk	$-(\lambda+k_j\mu_j)\ {}_jP_{nk}$	No Arrival - No Service

INITIAL STATE	FINAL STATE	PROBABILITY	DESCRIPTION
j,n-1,k	jnk	$\lambda_j P_{n-1,k}$	Arrival - No Service
j,n+1,1	jnk	$\frac{\lambda_j}{\lambda} \sum_{i=1}^{m} k_i\mu_i \; {}_iP_{n+1,1}$	No Arrival - Service completed from phase 1

The steady equations then become the sum of the independent probabilities for the transitions described above.

$$\left.\begin{aligned}
&\sum_{i=1}^{m} k_i\mu_i \; {}_iP_{01} = \lambda \; {}_0P_{00} && 2.1\\
&(\lambda+k_j\mu_j)\,{}_jP_{0k} = \lambda_j \; {}_0P_{00} + \frac{\lambda_j}{\lambda} \sum_{i=1}^{m} k_i\mu_i \; {}_iP_{11} && 2.2\\
&(\lambda+k_j\mu_j)\,{}_jP_{0s} = k_j\mu_j \; {}_jP_{0,s+1} && 2.3
\end{aligned}\right\} \begin{matrix} n=0 \\ 1\leq s<k \end{matrix}$$

$$(\lambda+k_j\mu_j)\,{}_jP_{ns} = \lambda_j P_{n-1,s} + k_j\mu_j \; {}_jP_{n,s+1} \quad \text{for } n>1 \quad 1\leq s<k \qquad 2.5$$

$$(\lambda+k_j\mu_j)\,{}_jP_{nk} = \lambda \; {}_jP_{n-1,k} + \frac{\lambda_j}{\lambda} \sum_{i=1}^{m} k_i\mu_i \; {}_iP_{n+1,1} \quad \text{for } n>1 \qquad 2.5$$

Solution For P_0

The characteristics of the system defining all conditions of the queuing model has as its beginning the development of P_0, the probability of zero units in the queue and any of the type (j) units in service. P_0 is likened to the first term of an infinite sequence. The sum of sequence is one (unity) in order that the sequence can be defined as a probability density function.

The first step in developing P_0 is to find a recursion relation connecting the initial equations. This recursion follows from the initial equation 2.1, 2.2 and 2.3. By induction the following is true:

$$\lambda \sum_j {}_jP_{n-1} = \Sigma k_i \mu_i \; {}_iP_{n1} \qquad 2.6$$

For n=1 in equation 2.6

$$\lambda \sum_j {}_jP_0 = \sum_i k_i \mu_i \; {}_iP_{11}$$

The proof for n=1 follows from summing the j's in equation 2.2, substituting equation 2.1 and adding the results to equation 2.3 after summing the j's and s's in the following manner,

$$\lambda \sum_j {}_jP_{0k} + \lambda \sum_{js}\sum {}_jP_{0s} + \lambda \; {}_0P_{00} + \sum_j k_j \mu_j \; {}_jP_{0k} = \lambda \; {}_0P_{00} + \sum_i k_i \mu_i \; {}_iP_{11}$$

$$+ \sum_j k_j \mu_j \; {}_jP_{0k}.$$

Next substitute equation 2.6 into equation 2.2, rearrange and combine. The expression for the initial probability ${}_0P_{00}$ is then

$${}_0P_{00} = \frac{\lambda + k_j \mu_j}{\lambda_j} \; {}_jP_{0k} - \sum_i {}_iP_0 \qquad 2.7$$

Equation 2.7 contains (m) equations in (m) unknowns that is solved by repeated substitution to give:

$$ {}_jP_{0k} = \frac{\lambda_j}{(\lambda+k_j\mu_j)(1-\alpha_1)}\ {}_0P_{00} \qquad 2.8a $$

Substituting 2.8a into equation 2.3 after developing a recursion relation from equation 2.3 as follows:

$$ {}_jP_{0s} = \left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^{k-s} {}_jP_{0k} = \left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^{k-s} \frac{\lambda_j}{\lambda+k_j\mu_j}\,\frac{{}_0P_{00}}{(1-\alpha_1)} \qquad 2.8b $$

By Definition: ${}_jP_0 = \Sigma_s\, {}_jP_{0s} + {}_jP_{0k}$

Substituting 2.8a and b in the above definition, changing the order of summation, and combine terms yields the equation:

$$ {}_jP_0 = \sum_{s=0}^{k-1} \left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^{s} \frac{\lambda_j}{\lambda+k_j\mu_j}\,\frac{{}_0P_{00}}{(1-\alpha_1)} \qquad 2.8c $$

Then by Definition: $P_0 = \Sigma_j\, {}_jP_0$

Substituting this definition into equation 2.8c yields the expression for P_0:

$$ P_0 = \sum_j \sum_{s=0}^{k-1} \frac{\lambda_j}{\lambda+k_j\mu_j}\left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^{s} \frac{{}_0P_{00}}{1-\alpha_1} $$

$$ P_0 = \frac{\alpha_1}{1-\alpha_1}\ {}_0P_{00} \qquad 2.8 $$

Solution for P_n

The probabilities defining the balance of the states of the queuing model for n>1 are derived from the steady state equations 2.4 and 2.5. These two equations are the queue equations that define the transitions for higher states (n>1).

The recursion equation 2.6 for the initial equations is likewise true for the queue equations. The proof follows in a similar manner by summing the s's in equation 2.4, the j's in equation 2.5, and combine.

$$\lambda \sum_j {}_jP_n + \sum_j k_j\mu_j \; {}_jP_n = \lambda \sum_j {}_jP_{n-1} + \sum_j \sum_{s=2}^{k} k_j\mu_j \; {}_jP_{ns} + \sum_i k_i\mu_i \; {}_iP_{n+1,1}$$

Substitute equation 2.6 for n=1

$$\lambda \sum_j {}_jP_{n-1} = \sum_i k_i\mu_i \; {}_iP_{n1} \quad \text{for } n>1 \qquad 2.6$$

Q.E.D.

Substituting equation 2.6 into the queue equation 2.5 gives the recursion relation for the probabilities of n>1. Combining and rearrange develops:

$$\frac{\lambda}{\lambda_j} {}_jP_{n-1,k} = \frac{\lambda + k_j\mu_j}{\lambda_j} {}_jP_{nk} - \sum_j {}_jP_n \quad \text{for } n>1 \qquad 2.9$$

Equation 2.9 contains (m) equations in (m) unknown that can be solved by repeated substitution to yield the expression for ${}_jP_{nk}$. Since $\sum_j {}_jP_n = \sum_j \sum_{s=1}^{k} {}_jP_{ns}$ then combining the queue equation 2.4 defines an expression in (m) equations with (m) unknowns for the transitions for all (k) phases and for n>1.

Solving equation 2.4 by repeated substitution yields an expression:

$$ {}_jP_{ns} = \sum_{r=0}^{n} \binom{s+r-1}{s-1} \left(\frac{\lambda}{\lambda+k_j\mu_j}\right)^r \left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^s {}_jP_{n-r,k} \qquad 2.10a $$

Solving the (m) equations of equation 2.9b simultaneously and substitute equation 2.10a, by induction yields the following:

$$ {}_jP_{nk} = \frac{\lambda_j}{(\lambda+k_j\mu_j)(1-\alpha_1)} \sum_i \left[\sum_{r=1}^{n} \sum_{s=0}^{k-1} \binom{s+r-1}{r} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^r \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^s {}_iP_{n-r,k} \right. $$

$$ \left. + \frac{\lambda}{\lambda+k_i\mu_i} \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^s {}_iP_{n-1,k} + \delta_{ij}(1-\alpha_1)\frac{\lambda}{\lambda_j}\, {}_jP_{n-1,k} \right] $$

$$ \text{where } \delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \qquad 2.10b $$

Substituting equation 2.10b into equation 2.9 yields an expression for P_n in terms of the probability of being in the (k^{th}) phase after making the transition from any other phase (s).

$$ P_n = \frac{1}{1-\alpha_1} \sum_i \sum_{r=1}^{n} \sum_{s=0}^{k-1} \left[\binom{s+r-1}{r} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^r \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^s {}_iP_{n-r,k} \right. $$

$$ \left. + \frac{\lambda}{\lambda+k_i\mu_i} \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^s {}_iP_{n-1,k} \right] \qquad 2.10c $$

The term ${}_iP_{n-r,k}$ in equation 2.10c then has to be solved in terms of the initial condition ${}_0P_{00}$. The probabilities P_m that appears describes the (m) transitional probabilities of prior states. Therefore expanding equation 2.9 for n=1,2,3,... and substituting equation 2.8a yields the expression:

$$ {}_jP_{nk} = \frac{\lambda^{n-1}\lambda_j}{(\lambda+k_j\mu_j)^n}\,\frac{{}_0P_{00}}{1-\alpha_1} + \sum_{m=1}^{n} \frac{\lambda^{n-m}\lambda_j}{(\lambda+k_j\mu_j)^{n+1-m}} P_m \qquad 2.10d $$

Equation 2.10c is simplified further by expansion; letting n=1,2,3,... and substituting equation 2.8a and 2.10d respectively to yield:

$$ P_n = \frac{\lambda^n \alpha_{n+1}\ {}_0P_{00}}{(1-\alpha_1)^2} + \sum_{r=1}^{n-1} \frac{\lambda^r \alpha_{r+1}}{1-\alpha_1} P_{n-r} \text{ for } n>1 \qquad 2.10 $$

Solution For ${}_0P_{00}$

The solution for ${}_0P_{00}$ is derived from the definition of a probability density function which states that a sum of all the probabilities must equal unity.

$$ 1 = {}_0P_{00} + P_0 + \sum_{n=1}^{\infty} P_n \qquad 2.11a $$

Rearrange equation 2.11

$$ \sum_{n=1}^{\infty} P_n = 1-{}_0P_{00}-P_0 \qquad 2.11b $$

Equation 2.8 and 2.10 contains all the probabilities necessary to evaluate the expression above. Rearranging equation 2.10 and summing like terms yields the following:

$$\sum_{n=1}^{\infty} P_n = \frac{{}_0P_{00}}{(1-\alpha_1)^2} \sum_{n=1}^{\infty} \lambda^n \alpha_{n+1} + \frac{\lambda\alpha_2}{1-\alpha_1} \sum_{n=1}^{\infty} P_n + \frac{\lambda^2\alpha_3}{1-\alpha_1} \sum_{n=1}^{\infty} P_n + \dots$$

$$\sum_{n=1}^{\infty} P_n = \frac{{}_0P_{00}}{(1-\alpha_1)^2} \sum_{n=1}^{\infty} \lambda^n \alpha_{n+1} + \frac{(1-{}_0P_{00}-P_0)}{1-\alpha_1} \sum_{n=1}^{\infty} \lambda^n \alpha_{n+1} \qquad 2.11c$$

Therefore substituting 2.11b, combining terms and substituting equation 2.8 for P_0 gives:

$${}_0P_{00} = 1-\alpha-\alpha_1 = 1-\sum_{i=1}^{m} \frac{\lambda_i}{\mu_i} \qquad 2.11$$

where $\alpha = \sum_{n=1}^{\infty} \lambda^n \alpha_{n+1}$ and the term $\alpha+\alpha_1 = \sum_{n=0}^{\infty} \lambda^n \alpha_{n+1}$

is evaluated in Appendix C.

CHAPTER III

SYSTEM CHARACTERISTICS

The measure of effectness of the system can be described by the mean number of tasks in the queue and the mean number of tasks in the total system, that is, in the queue plus in service. The Generating Function (3) is utilized in this chapter to facilitate the operational method in the development of the expected values of the system characteristics.

The Generating Function

The operational method utilizing the Generating Function is defined as F(z) where (z) is an arbitrary variable of convenience. The Generating Function is by definition:

$$F(z) = {}_0P_{00} + P_0 + \sum_{i=1}^{\infty} z^i P_i$$

Substituting equation 2.10 for the P_i's in the Generating Function yields:

$$F(z) = {}_0P_{00} + P_0 + \frac{\lambda\alpha_2}{(1-\alpha_1)^2} z\, {}_0P_{00} + \sum_{i=2}^{\infty} \frac{z^i}{1-\alpha_1}\left[\frac{\lambda\alpha_{i+1}}{1-\alpha_1}\, {}_0P_{00} + \sum_{r=1}^{i=1} \alpha_{i+1-r} P_r\right] \qquad 3.1a$$

Expanding F(z) for all i's (i=1,2,3,...) and summing like probabilities

$$F(z) = {}_0P_{00} + P_0 + \frac{{}_0P_{00}}{(1-\alpha_1)^2}\sum_{i=1}^{\infty}(\lambda z)^i\alpha_{i+1} + \frac{zP_1}{1-\alpha_1}\sum_{i=1}^{\infty}(\lambda z)^i$$

$$\alpha_{i+1} + \frac{z^2P_2}{1-\alpha_1}\sum_{i=1}^{\infty}(\lambda z)^i\alpha_{i+1} + \ldots \qquad 3.1b$$

Substituting $F(z) - {}_0P_{00} - P_0 = \sum_{i=1}^{\infty} z^i P_i$

$$F(z) = {}_0P_{00} + P_0 + \left[\frac{{}_0P_{00}}{(1-\alpha_1)^2} + \frac{F(z) - {}_0P_{00} - P_0}{1-\alpha_1}\right]\sum_{i=1}^{\infty}(\lambda z)^i\alpha_{i+1}$$

3.1c

Combining like terms and substitution equation 2.8 for P_0

$$F(z) = \frac{{}_0P_{00}}{1-\alpha_z} \qquad \text{where } \alpha_z = \sum_{i=0}^{\infty}(\lambda z)^i\alpha_{i+1} \qquad 3.1$$

The Expected Number in the Queue

The mean number of tasks (L_q) in the queue is derived from the definition of the generating function. By definition the first derivative of the generating function evaluated at z=1 is the expected value of the distribution of the number of tasks waiting

$$F'(z) = \frac{d}{dz}F(z) = \frac{\frac{d}{dz}\alpha_z \; {}_0P_{00}}{(1-\alpha_z)^2} \qquad 3.2a$$

Since the term $(1-\alpha_z)$ for z=1 is equivalent to ${}_0P_{00}$ from equation c.1 of Appendix C, the derivation $F'(z)$ reduces to:

$$F'(z) = \frac{d}{dz}\alpha_z \; {}_0P_{00}{}^{-1} \tag{3.2b}$$

Substituting c.2 from Appendix C

Therefore,

$$L_q = F'(1) = \frac{\lambda \sum_{i=1}^{m} \left(\frac{k_i+1}{2k_i}\right) \frac{\lambda_i}{\mu_i^2}}{1-\sum_{i=1}^{m} \frac{\lambda_i}{\mu_i}} \tag{3.2}$$

Expected Number in the ~~System~~ queue

The mean number of tasks (L) in the system is derived also from the generating function of the distribution of the number of tasks in the system. By definition the generating function of the system:

$$E(z) = {}_0P_{00} + \sum_{i=0}^{\infty} z^{i+1} P_i \tag{3.3a}$$

Expanding equation 3.3a for all i's and simplifying gives:

$$E(z) = {}_0P_{00} + zP_0 + zF(z) - z\,{}_0P_{00} - zP_0$$

$$E(z) = (1-z)\,{}_0P_{00} + zF(z) \tag{3.3}$$

By definition the first derivative of the generating function evaluated at z=1 is the expected value. Therefore the derivative of equation 3.3 becomes:

$$E'(z) = \frac{d}{dz}E(z) = 1 - {}_0P_{00} + F'(z) \qquad 3.4a$$

Evaluation $E'(z)$ for $z=1$ and substituting equation 3.2 for $F'(1)$, the mean number in the system:

$$L = E'(1) = \sum_{i=1}^{m} \frac{\lambda_i}{\mu_i} + \frac{\lambda \sum_{i=1}^{m} \left(\frac{k_i+1}{2k_i}\right) \frac{\lambda_i}{\mu_i^2}}{1 - \sum_{i=1}^{m} \frac{\lambda_i}{\mu_i}} \qquad 3.4$$

CHAPTER IV

APPLICATION OF THE MODEL

In order to demonstrate the practical as well as the theoretical significance of the multiple input queuing model an actual service facility was observed. The characteristics of the facility are the same as the theoretical development given previously as Poisson arrivals, Erlang service, single channel facility, single queue with service as first-come first-serve. Each characteristic will be presented with its respective distribution function together with the mean and variance. The observed empirical function will be tested against the theoretical function assumed in the model to note significant difference.

Service Facility

The actual situation observed was a repair and modification facility of a large commercial airline. Component assemblies removed from operating aircraft, for reasons of malfunction and inoperative conditions, are repaired and tested in order that they may be made serviceable for re-installation.

The required service time to repair a given component was observed to compare favorably with the Erlang distribution function. This is justified in part by the service procedure which has a sequence of phases described as disassembly, repair as required, reassembly, test and final handling.

Because of the many types and varied construction of aircraft and engine components, each service facility is primarily a job shop. There is in general a classification and grouping of similar items requiring a common mechanical skill to repair. The work consists of a single repairman making serviceable a group of components of similar construction but differing in size and complexity.

The several components, requiring a similar repair skill, arrived in a random manner distributed as Poisson. They wait in a single queue and are serviced in the order of arrival.

System Parameters

The situation observed contained three components that required repair, each requiring a random amount of time to complete service. The service time is of the Erlang type. Table I lists the mean values of arrivals, service and k factors of the Erlang Service distribution.

The arrivals of tasks to the service facility was

TABLE I

SYSTEM PARAMETERS

i^{th} Item	λ_1	μ_i	μ_i^2	k_i	$\frac{\lambda_i}{\mu_i}$
1	1.19	1.77	3.13	10	.67
2	.41	3.39	11.49	4	.12
3	.41	2.42	5.85	4	.17
TOTAL	2.01				.96

found to be Poisson distributed. The chi-square test was utilized to determine the confidence of the observed (0_i) vs the theoretical (e_i) probabilities. The 95% confidence interval of the chi-square ($\chi^2_{.05}$) with degrees of freedom (ν) shown for each item type. Tables II through IV list the arrival data observed. Each table is displayed graphically, with the theoretical density function drawn and the observed probabilities shown as an ⊗.

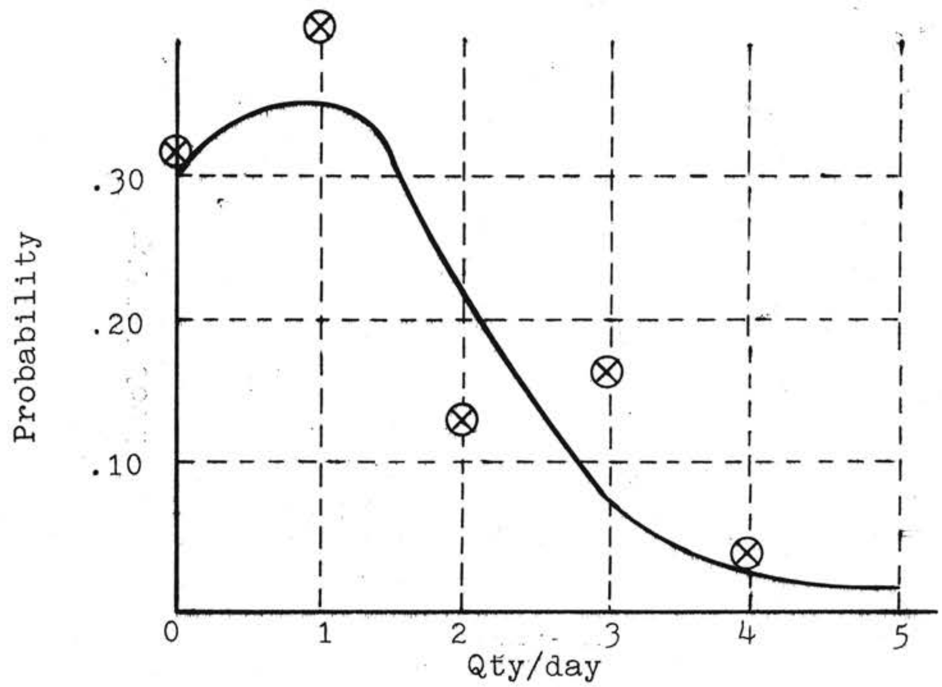


FIGURE 2

ARRIVAL DENSITY FUNCTION - ITEM 1

TABLE II

CHI-SQUARE TEST OF ARRIVAL DATA - ITEM 1

Qty/Day	Freq.	O_i	e_i	$(O_i-e_i)^2$	$\frac{(O_i-e_i)^2}{e_i}$
0	23	31%	30.1	.81	.0270
1	30	40.5	36.1	19.36	.5363
2	10	13.5	21.7	67.24	3.0986
3	11	14.9	8.7	38.44	4.4184
4	2	2.7	2.6	.01	.0038
	74				8.0841

For $\nu = 4$ $\chi^2_{.05} = 9.488$

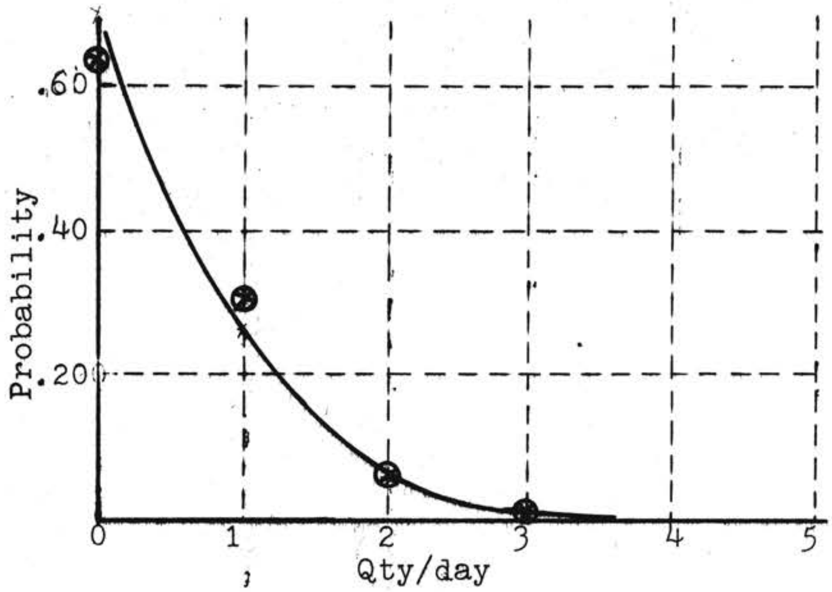


FIGURE 3

ARRIVAL DENSITY FUNCTION - ITEM 2

TABLE III

CHI-SQUARE TEST OF ARRIVAL DATA - ITEM 2

Qty/Day	Freq.	0_i	e_i	$(0_i - e_i)^2$	$\frac{(0_i - e_i)^2}{e_i}$
0	51	65.4%	67.0%	2.56	.0382
1	24	30.7	26.8	15.21	.5675
2	2	2.5	5.4	8.41	1.5572
3	0	0	.7	.49	.7000
4	1	.01	.07	.06	.0036
	78				2.9143

For $\nu = 3$ $\chi^2_{.05} = 7.815$

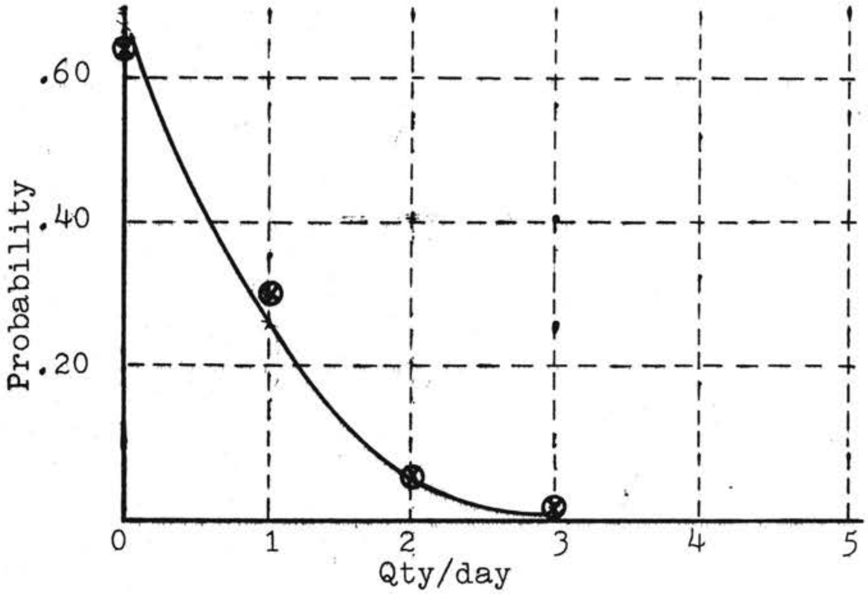


FIGURE 4

ARRIVAL DENSITY FUNCTION - ITEM 3

TABLE IV

CHI-SQUARE TEST OF ARRIVAL DATA - ITEM 3

Qty/Day	Freq.	0_i	e_i	$(0_i-e_i)^2$	$\frac{(0_i-e_i)^2}{e_i}$
0	50	64.0%	67.0%	9.0	.1343
1	24	30.7	26.8	15.2	.5671
2	4	5.1	5.4	.09	.0166
3	0	0	.7	.49	.7000
	78				1.418

For $\nu = 3$ $\chi^2_{.05} = 7.815$

The observed service times for the i^{th} item was found to be Erlang. Because of the small sample obtained, it was determined to evaluate the service times as ungrouped data. Tables V through VII lists the data at the points $\mu_i t$ with the respective chi-square coefficient. The ungrouped data is plotted as the $\frac{i}{n+1}$ probability interval[7] for Table V and as the median rank[8] for Table VI and Table VII. The observed values for the chi-square test was extracted from the "best fit" of the observed data.

The following table lists the chi-square values for the service time correlation with the respective degree's of freedom (ν) at the 95% confidence interval.

TABLE V

ERLANG FACTORS

i	ν	χ^2	$\chi^2_{.05}$
1	9	15.0133	16.919
2	8	3.0971	15.507
3	7	10.417	14.067

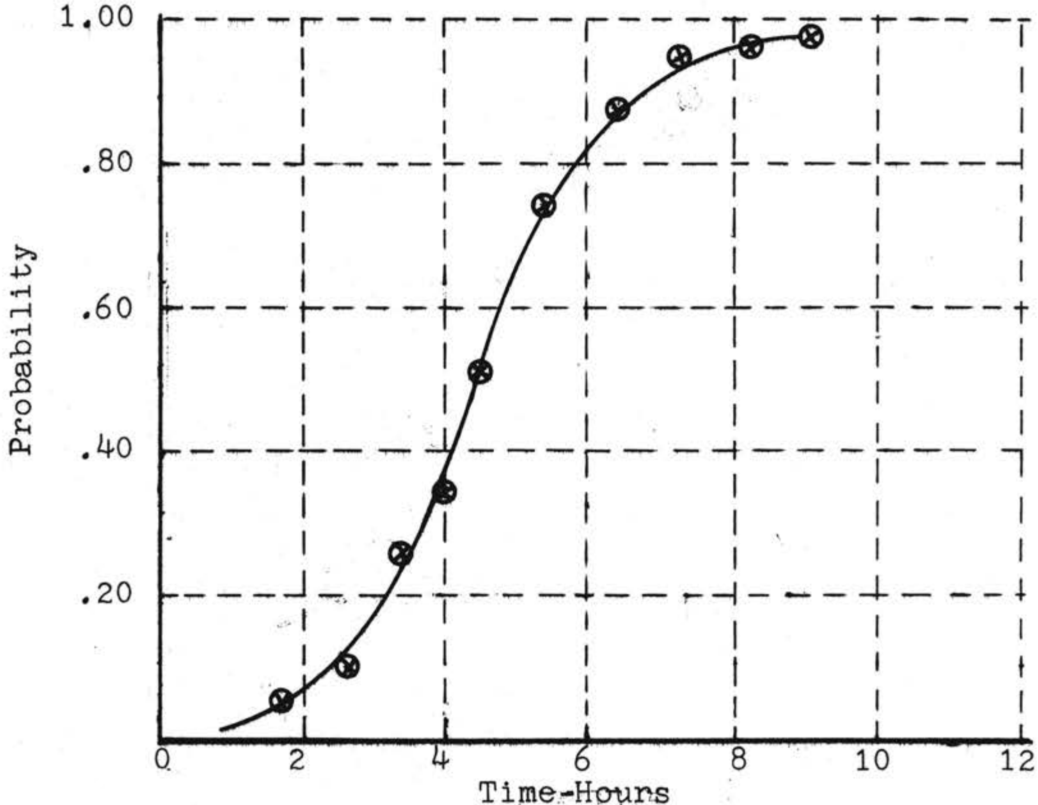


FIGURE 5

SERVICE DISTRIBUTION FUNCTION - ITEM 1

TABLE VI

CHI-SQUARE TEST OF SERVICE DATA - ITEM 1

TIME	O_i	e_i	$(O_i-e_i)^2$	$\frac{(O_i-e_i)^2}{e_i}$
1.8	4.0	.8	10.2400	12.8000
2.7	9.0	8.4	.3600	.0428
3.6	24.0	28.34	18.8356	.6646
4.0	35.0	41.26	39.1876	.9497
4.5	51.0	54.21	10.3041	.1900
5.4	76.0	75.76	.0576	.0007
6.3	88.0	89.06	1.1236	.1254
7.2	96.0	95.67	.1089	.0011
8.1	97.0	98.46	2.1316	.2164
9.0	98.0	99.50	2.2500	.0226
				15.0133

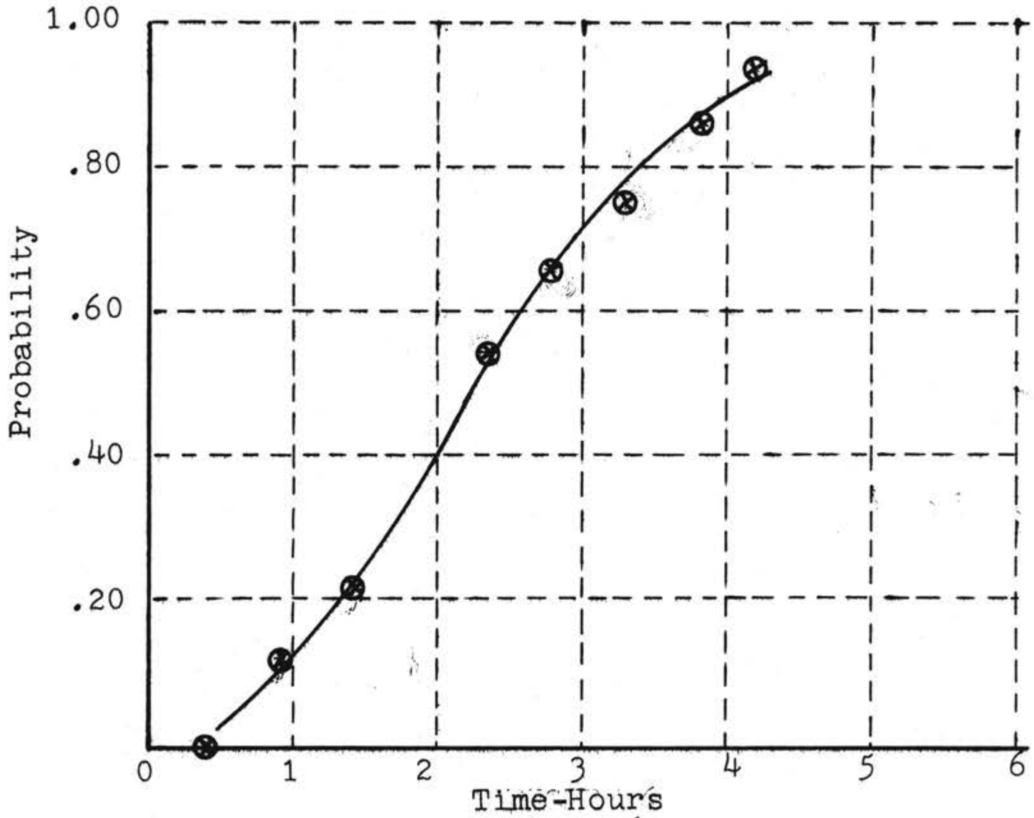


FIGURE 6
SERVICE DISTRIBUTION FUNCTION - ITEM 2

TABLE VII

CHI-SQUARE TEST OF SERVICE DATA - ITEM 2

TIME	O_i	e_i	$(O_i - e_i)^2$	$\frac{(O_i - e_i)^2}{e_i}$
.47	0	.91	.8281	.9100
.94	11.0	7.88	9.7344	1.2353
1.42	21.0	22.13	1.2769	.0576
1.88	40.0	39.75	.0625	.0015
2.36	55.0	56.65	2.7225	.0480
2.83	65.0	70.58	31.1364	.4411
3.30	76.0	80.94	24.4036	.3015
3.77	86.0	88.11	4.4521	.0505
4.25	95.0	92.81	4.7961	.0516
				3.0971

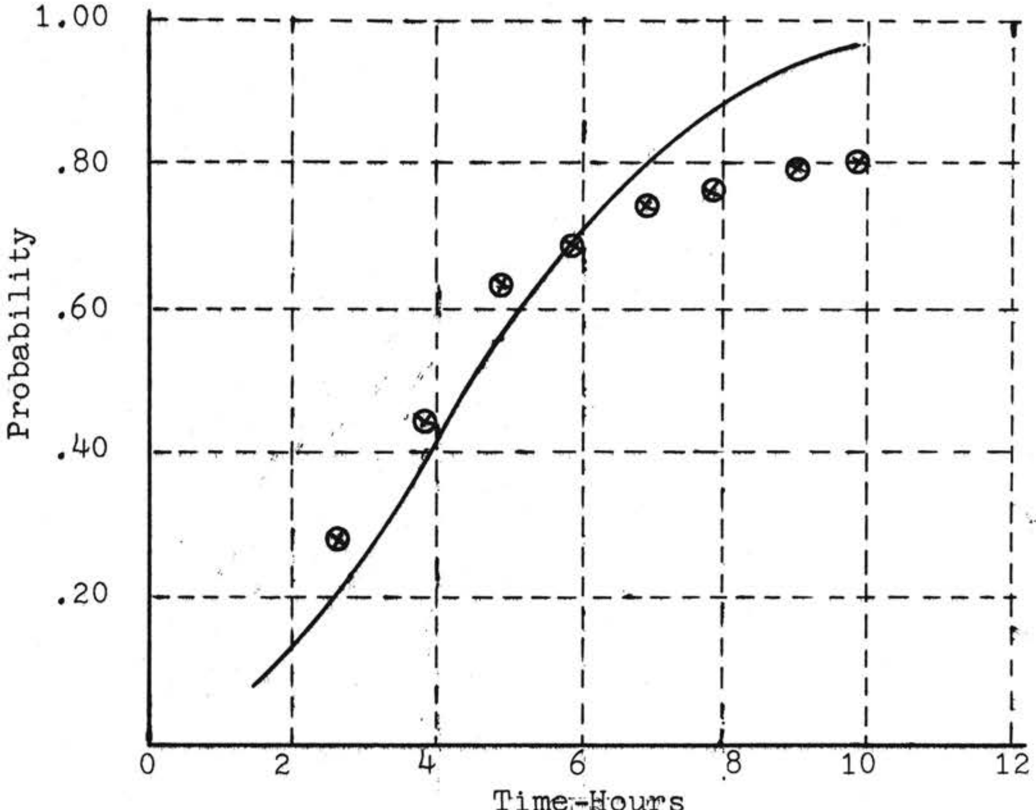


FIGURE 7

SERVICE DISTRIBUTION FUNCTION - ITEM 3

TABLE VIII

CHI-SQUARE TESTS OF SERVICE DATA - ITEM 3

TIME	0_i	e_i	$(0_i - e_i)^2$	$\frac{(0_i - e_i)^2}{e_i}$
2.97	28.0	24.13	34.457	1.557
3.97	43.0	39.75	10.562	.265
4.96	63.0	56.65	40.322	.7117
5.95	69.0	70.58	2.496	.0353
6.94	74.0	80.94	48.164	.5950
7.94	76.0	88.11	146.652	1.664
8.93	78.0	92.81	219.336	2.363
9.92	80.0	97.76	315.417	3.226
				10.417

System Evaluation

The evaluation of the observed system under review consisted of comparing the observed number with the expected number in the queue and in the service facility. Table VIII displays the distribution of the quantity per day for each item observed. The mean number ($\bar{x}$) and the standard deviation (s) of the empirical distribution are shown for the i^{th} item. Figure 9 displays the distribution of the total number of items observed in the system.

Shown below in table form is the theoretical and observed mean number of all items in the system. Very favorable results were obtained from the small sample observed.

Observed	Theoretical*
$\bar{x} = 14.615$	$L = 14.781$
$s = 6.951$	

The observed distribution of the mean number of all items in the system was found to be Erlang with a factor k=4. The theoretical distribution is shown in Figure 8.

*Equation 3.4

TABLE IX

DISTRIBUTION OF NUMBER IN SYSTEM

Qty/day	Frequency Item 1	Item 2	Item 3
0	1	0	1
1	2	9	13
2	5	9	9
3	6	13	14
4	13	9	14
5	10	6	0
6	13	2	1
7	11	7	7
8	9	2	3
9	2	6	5
10	4	4	4
11	1	3	3
12	0	1	1

TABLE X

NUMBER IN SYSTEM

i	1	2	3
$\bar{x}$	5.50	4.40	4.50
s	2.42	3.51	3.17

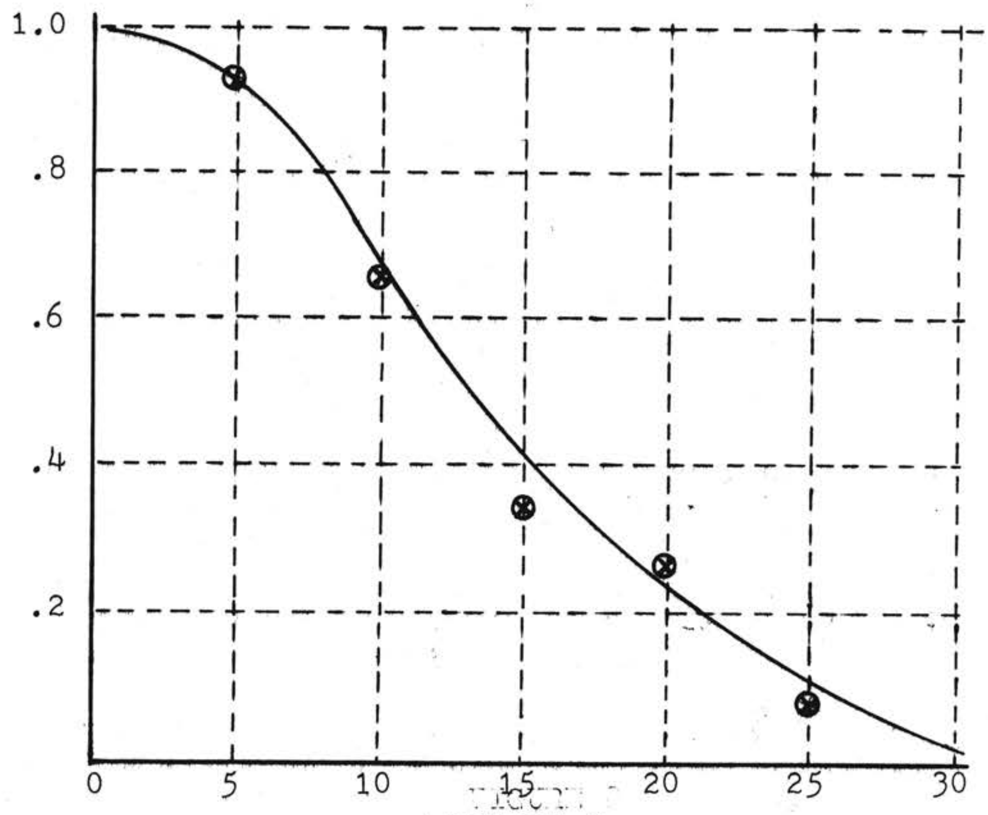


FIGURE 8
DISTRIBUTION OF NUMBER IN SYSTEM

The selected values from the observed mean number in the system plotted in Figure 8 are shown below:

TABLE XI

FREQUENCY OF i^{th} ITEM IN SYSTEM

Qty/Day	5	10	15	20	25	30
Probability (%)	95.0	63.7	35.0	26.3	7.5	1.2

CHAPTER V

SUMMARY OF RESULTS

A review of the available information for evaluation of a service facility described in this paper illustrates the variance in results to be obtained. The degree of error in assumptions such as exponential service and combined inputs when in reality multiple inputs and Erlang service is shown for the two previous results. In order to show the magnitude of the error, comparisons of the following available models will be made.

A - Ancher and Gafarian (5)

B - Morse's (4) Single Channel exponential service

C - Morse's (4) Single Channel Erlang service with a single combined input

D - Results of this paper

Table XII displays the formulas for evaluations of probability of zero units (P_0), mean number in the system (L) and the mean number in the queue (L_q) from the models shown above. Table XIII displays the values for each of the formulas when the data in Chapter IV was assumed to be of the desired type.

TABLE XII

FORMULA COMPARISON

Model	P_0	L	L_q
A	$1-\sum_i \frac{\lambda_i}{\mu_i}$	$\sum_i \frac{\lambda_i}{\mu_i} + L_q$	$\frac{\lambda \sum_i \frac{\lambda_i}{\mu_i^2}}{1 - P_0}$
B	$1-\rho$	$\frac{\rho}{1-\rho}$	$\frac{\rho^2}{1-\rho}$
C	$1-\rho$	$\frac{2k\rho-\rho^2(k-1)}{2k(1-\rho)}$	$\frac{\rho^2(k+1)}{2k(1-\rho)}$
D	$1-\sum_{i=1}^{m} \frac{\lambda_i}{\mu_i}$	$\sum_{i=1}^{m} \frac{\lambda_i}{\mu_i} + \frac{\lambda \sum_{i=1}^{m} \left(\frac{k_i+1}{2k_i}\right) \frac{\lambda_i}{\mu_1^2}}{1-\sum_{i=1}^{m} \frac{\lambda_i}{\mu_i}}$	$\frac{\lambda \sum_{i=1}^{m} \left(\frac{k_i+1}{2k_i}\right) \frac{\lambda_i}{\mu_i^2}}{1-\sum_{i=1}^{m} \frac{\lambda_i}{\mu_i}}$

TABLE XIII

VALUES OF SYSTEM CHARACTERISTICS

Model	ρ_0	L	L_q
A	.04	25.4	24.4
B	.21	3.8	3.2
C	.21	2.7	2.0
D	.04	14.8	13.8

The results of Table XIII indicate the variance in the system characteristics by the assumptions of the model. The value of the results of this study can be shown in the ratio of the three common system models. The factor displayed below is the ratio of the common model (A,B,C) to the value of the model (D) derived in this study. It is observed that an error of magnitude .14 ⟶ 1.8 is likely. The meaning is significant in considering the dollar value of inprocess inventory.

Model	P_0	L	L_q
A	1.0	1.70	1.80
B	5.3	.25	.23
C	5.3	.18	.14

A comparison of an analytical solution with empirical data requires careful scrutiny by the researcher. Several common factors influence empirical data. The number of items in the system can be composed of a quantity surplus to an effective operation. It is usually found that over-provisioning of in-process quantities causes an inflation of the observed number with little change in the service and arrival rates. The service rates are influenced by the performance of the manpower engaged in the activity. Where a time-standards program is in effect the researcher is able to adjust the empirical service times by the performance and utilization factors.

Miscellaneous factors causing interruption of the operation due to absentees, learning curve, work shifts and others should be reviewed for their effect on the observed data. Experience in analysis of empirical data and a review of the service operation enables the researcher to evaluate the effect of the influencing factors.

Further study of the multiple input model should develop the distribution of waiting time and the variance associated with the mean. Refinement of the technique described in this study could produce tables or graphs for ease of determining the theoretical results. Some consideration could be given to model the Markov process described in this study within matrix analysis for the study of the transient conditions.

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APPENDIX A

SOLUTION OF P_0

<u>First Step is to Find Set of Linear Equations for Initial Equations</u>

Sum the j's in Equation 2.2

$$\lambda\sum_j {}_jP_{0k} + \sum_j k_j\mu_j\ {}_jP_{0k} = \lambda_0 P_{00} + \sum_i k_i\mu_i\ {}_iP_{11} \qquad \text{A.1a}$$

Sum the j's in Equation 2.3 and substitute Equation 2.1

$$\sum_j k_j\mu_j\ {}_jP_{0k} = \lambda\sum_j \sum_{s=1}^{k-1} {}_jP_{0s} + \sum_j k_j\mu_j\ {}_jP_{01} =$$

$$= \lambda\sum_j \sum_{s=1}^{k-1} {}_jP_{0s} + \lambda_0 P_{00} \qquad \text{A.1b}$$

Add 2.6a and 2.6b

$$\lambda\sum_j {}_jP_{0k} + \lambda\sum_j \sum_{s=1}^{k-1} {}_jP_{0s} + \lambda\ {}_0P_{00} + \sum_j k_j\mu_j\ {}_jP_{0k} = \lambda\ {}_0P_{00} +$$

$$\sum_i k_i\mu_i\ {}_iP_{11} + \sum_j k_j\mu_j\ {}_jP_{0k} \qquad \text{A.1c}$$

Combine terms

$$\therefore\ \lambda\sum_j {}_jP_0 = \sum_i k_i\mu_i\ {}_iP_{11} \qquad 2.6$$

Substitute 2.6 in Equation 2.2

$$(\lambda+k_j\mu_j)\,{}_jP_{0k} = \lambda_j\ {}_0P_{00} + \frac{\lambda_j}{\lambda}\,\lambda\sum_j {}_jP_0 \qquad \text{2.7a}$$

Combine and Rearrange

$${}_0P_{00} = \left(\frac{\lambda+k_j\mu_j}{\lambda_j}\right){}_jP_{0k} - \sum_{i=1}^{m} {}_iP_0 \qquad 2.7$$

Next show that equation 2.6 holds for $n \geq 1$

Sum the s's in equation 2.4

$$(\lambda+k_j\mu_j)\sum_{s=1}^{k-1} {}_jP_{ns} = \lambda\sum_{s=1}^{k-1} {}_jP_{n-1,s} + k_j\mu_j\sum_{s=2}^{k} {}_jP_{ns} \qquad \text{A.2a}$$

Add equation 2.5 to 2.6a

$$(\lambda+k_j\mu_j)\sum_{s=1}^{k-1} {}_jP_{ns} + (\lambda+k_j\mu_j)\,{}_jP_{nk} = \lambda\sum_{s=1}^{k-1} {}_jP_{n-1,s} +$$

$$\lambda\,{}_jP_{n-1,k} + k_j\mu_j\sum_{s=2}^{k} {}_jP_{ns} + \frac{\lambda_j}{\lambda}\sum_i k_i\mu_i\ {}_iP_{n+1,1} \qquad \text{A.2b}$$

Sum j's and combine

$$\lambda\sum_j {}_jP_n + \sum_j k_j\mu_j\ {}_jP_n = \lambda\sum_j {}_jP_{n-1} + \sum_j\sum_{s=2}^{k} k_j\mu_j\ {}_jP_{ns} +$$

$$\sum_i k_i\mu_i\ {}_iP_{n+1,1} \qquad \text{A.2c}$$

Substitute 2.6 for n=1

$$\lambda\sum_j {}_jP_n + \sum_j k_j\mu_j\ {}_jP_n = \sum_i k_i\mu_i\ {}_iP_{n1} + \sum_j\sum_{s=2}^{k} k_j\mu_j\ {}_jP_{ns} +$$

$$\sum_i k_i\mu_i\ {}_iP_{n-1,1} \qquad \text{A.2d}$$

Combine terms

$$\therefore\ \lambda\sum_j {}_jP_n = \sum_i k_i\mu_i\ {}_iP_{n+1,1} \qquad \text{For } n\geq 1 \qquad 2.6$$

Substitute 2.6 in equation 2.5

$$(\lambda+k_j\mu_j)\,{}_jP_{nk} = \lambda\ {}_jP_{n-1,k} + \frac{\lambda_j}{\lambda}\ \lambda\sum_j {}_jP_n \quad n\geq 1 \qquad 2.9a$$

Combine and rearrange

$$\frac{\lambda}{\lambda_j}\,{}_jP_{n-1,k} = \left(\frac{\lambda+k_j\mu_j}{\lambda_j}\right){}_jP_{nk} - \sum_j {}_jP_n \qquad n\geq 1 \qquad 2.9$$

Expand Equation 2.3

$$(\lambda+k_j\mu_j)\,{}_jP_{01} = k_j\mu_j\;{}_jP_{02}$$

$$(\lambda+k_j\mu_j)\,{}_jP_{02} = k_j\mu_j\;{}_jP_{03}$$

$$\vdots \qquad\qquad \vdots$$

$$(\lambda+k_j\mu_j)\,{}_jP_{0s} = k_j\mu_j\;{}_jP_{0,s+1}$$

Repeated substitution

$${}_jP_{01} = \frac{k_j\mu_j}{\lambda+k_j\mu_j}\,{}_jP_{02} = \left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^2{}_jP_{03} = \left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^3{}_jP_{04} = \;.\;.\;.$$

Therefore

$${}_jP_{0r} = \left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^{s-r}{}_jP_{0s} \qquad \begin{matrix}1\leq r\leq s\\ 1\leq s\leq k\end{matrix} \qquad A.3a$$

Expand Equation 2.7

$${}_0P_{00} = \frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{0k} - \sum_i {}_iP_0$$

$${}_0P_{00} = \frac{\lambda+k_2\mu_2}{\lambda_2}\,{}_2P_{0k} - \sum_i {}_iP_0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$${}_0P_{00} = \frac{\lambda+k_j\mu_j}{\lambda_j}\,{}_jP_{0k} - \sum_i {}_iP_0$$

Solve simultaneously

$$\frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{0k} = \frac{\lambda+k_2\mu_2}{\lambda_2}\,{}_2P_{0k}$$

Since

$$\frac{\lambda+k_3\mu_3}{\lambda_3}\,{}_3P_{0k} = \frac{\lambda+k_2\mu_2}{\lambda_2}\,{}_2P_{0k} = \frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{0k}$$

Therefore

$$ {}_jP_{0k} = \left(\frac{\lambda+k_1\mu_1}{\lambda_1}\right)\left(\frac{\lambda_j}{\lambda+k_j\mu_j}\right){}_1P_{0k} \qquad \text{A.3b}$$

Let j=m and n=1

$$ {}_mP_{0k} = \left(\frac{\lambda_m}{\lambda+k_m\mu_m}\right)\left(\frac{\lambda+k_n\mu_n}{\lambda_n}\right){}_nP_{0k} \qquad \text{A.3c}$$

Substitute A.3a into A.3c

$$ {}_mP_{0r} = \left(\frac{k_m\mu_m}{\lambda+k_m\mu_m}\right)^{k-r}\left(\frac{\lambda_m}{\lambda+k_m\mu_m}\right)\left(\frac{\lambda+k_n\mu_n}{\lambda_n}\right){}_nP_{0k} \qquad \text{A.3}$$

Expand 2.7

$$\begin{aligned} {}_0P_{00} = \frac{\lambda+k_j\mu_j}{\lambda_j}\,{}_jP_{0k} - \Big[& {}_1P_{01} + {}_1P_{02} + {}_1P_{03} + \ldots + {}_1P_{0k} \\ & {}_2P_{01} + {}_2P_{02} + {}_1P_{03} + \ldots + {}_2P_{0k} \\ & \;\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ & + {}_mP_{01} + {}_mP_{02} + {}_mP_{03} + \ldots + {}_mP_{0k} \Big] \end{aligned}$$

Substitute A.3 into Expanded 2.7

For j=3

$$
{}_0P_{00} = \frac{\lambda+k_3\mu_3}{\lambda_3}\,{}_3P_{0k}\left\{1 - \left[\left(\frac{k_1\mu_1}{\lambda+k_1\mu_1}\right)^{k-1}\left(\frac{\lambda_1}{\lambda+k_1\mu_1}\right) + \left(\frac{k_1\mu_1}{\lambda+k_1\mu_1}\right)^{k-2}\left(\frac{\lambda_1}{\lambda+k_1\mu_1}\right)\cdots\right.\right.
$$

$$
+ \left(\frac{k_2\mu_2}{\lambda+k_2\mu_2}\right)^{k-1}\left(\frac{\lambda_2}{\lambda+k_2\mu_2}\right) + \left(\frac{k_2\mu_2}{\lambda+k_2\mu_2}\right)^{k-2}\left(\frac{\lambda_2}{\lambda+k_2\mu_2}\right)+\cdots+\left(\frac{\lambda_2}{\lambda+k_2\mu_2}\right)
$$

$$
+ \left(\frac{k_3\mu_3}{\lambda+k_3\mu_3}\right)^{k-1}\left(\frac{\lambda_3}{\lambda+k_3\mu_3}\right) + \left(\frac{k_3\mu_3}{\lambda+k_3\mu_3}\right)^{k-2}\left(\frac{\lambda_3}{\lambda+k_3\mu_3}\right)+\cdots+\left(\frac{\lambda_3}{\lambda+k_3\mu_3}\right)
$$

$$
\begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}
$$

$$
\left.\left. + \left(\frac{k_m\mu_m}{\lambda+k_m\mu_m}\right)^{k-1}\left(\frac{\lambda_m}{\lambda+k_m\mu_m}\right) + \cdots \cdots + \left(\frac{\lambda_m}{\lambda+k_m\mu_m}\right)\right]\right\}
$$

Rearrange

$$
{}_0P_{00} = \frac{\lambda+k_3\mu_3}{\lambda_3}\,{}_3P_{0k}\left\{1 - \left[\frac{\lambda_1}{\lambda+k_1\mu_1}\left(\left(\frac{k_1\mu_1}{\lambda+k_1\mu_1}\right)^{k-1} + \left(\frac{k_1\mu_1}{\lambda+k_1\mu_1}\right)^{k-2}+\cdots\right)\right.\right.
$$

$$
\left.\left. + \frac{\lambda_2}{n+k_2\mu_2}\left(\left(\frac{k_2\mu_2}{\lambda+k_2\mu_2}\right)^{k-1} + \left(\frac{k_2\mu_2}{\lambda+k_2\mu_2}\right)^{k-2} + \cdots\right)\right]\right\}
$$

Therefore

$$
{}_0P_{00} = \frac{\lambda+k_3\mu_3}{\lambda_3}\,{}_3P_{0k}\left[1 - \sum_{i=1}^{m}\sum_{r=1}^{k}\left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-r}\right]
$$

Let $\alpha = \sum_{i=1}^{m}\sum_{r=1}^{k}\left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-r}$

Then

$$ {}_{j}P_{0k} = \frac{\lambda_j}{(\lambda+k_j\mu_j)}(1-\alpha)\ {}_{0}P_{00} \qquad 3.8a $$

APPENDIX B

SOLUTION OF P_n

Expand Equation 2.4 $${}_jP_{ns} = \frac{\lambda}{\lambda+k_j\mu_j}\,{}_jP_{n-1,s} + \frac{k_j\mu_j}{\lambda+k_j\mu_j}\,{}_jP_{n,s+1}$$

$$1 \leq s < k \quad n \geq 1$$

For: $n=1 \quad s=1,2,3,\ldots$ Let: $a=\frac{\lambda}{\lambda+k_j\mu_j} \quad b=\frac{k_j\mu_j}{\lambda+k_j\mu_j}$

$${}_jP_{11} = a\,{}_jP_{01} + b\,{}_jP_{12} = ab^{k-1}\,{}_jP_{0k} + 4ab^{k-1}\,{}_jP_{0k} + b^5\,{}_jP_{16} =$$

$$5ab^{k-1}\,{}_jP_{0k} + b^5\,{}_jP_{16}$$

$${}_jP_{12} = a\,{}_jP_{02} + b\,{}_jP_{13} = ab^{k-2}\,{}_jP_{0k} + 3ab^{k-2}\,{}_jP_{0k} + b^4\,{}_jP_{16} =$$

$$4ab^{k-2}\,{}_jP_{0k} + b^4\,{}_jP_{16}$$

$${}_jP_{13} = a\,{}_jP_{03} + b\,{}_jP_{14} = ab^{k-3}\,{}_jP_{0k} + 2ab^{k-3}\,{}_jP_{0k} + b^3\,{}_jP_{16} =$$

$$3ab^{k-3}\,{}_jP_{0k} + b^3\,{}_jP_{16}$$

$${}_jP_{14} = a\,{}_jP_{04} + b\,{}_jP_{15} = ab^{k-4}\,{}_jP_{0k} + ab^{k-4}\,{}_jP_{0k} + b^2\,{}_jP_{16} =$$

$$2ab^{k-4}\,{}_jP_{0k} + b^2\,{}_jP_{16}$$

$${}_jP_{15} = a\,{}_jP_{05} + b\,{}_jP_{16} = ab^{k-5}\,{}_jP_{0k} + b\,{}_jP_{16}$$

Note: Use Equation A.3a to substitute ${}_jP_{0r} = \left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^{s-r}{}_jP_{0s}$

For: n=2

$$ {}_jP_{21} = a\,{}_jP_{11} + b\,{}_jP_{22} = 5a^2b^{k-1}\,{}_jP_{0k} + ab^5\,{}_jP_{1k} + 10a^2b^{k-1}\,{}_jP_{0k} $$

$$ + 4ab^5\,{}_jP_{1k} + b^5\,{}_jP_{26} = 15a^2b^{k-1}\,{}_jP_{0k} + 5ab^5\,{}_jP_{1k} + b^5\,{}_jP_{26} $$

$$ {}_jP_{22} = a\,{}_jP_{12} + b\,{}_jP_{23} = 4a^2b^{k-2}\,{}_jP_{0k} + ab^4\,{}_jP_{1k} + 6a^2b^{k-2}\,{}_jP_{0k} $$

$$ + 3ab^4\,{}_jP_{1k} + b^4\,{}_jP_{26} = 10a^2b^{k-2}\,{}_jP_{0k} + 4ab^4\,{}_jP_{1k} + b^4\,{}_jP_{26} $$

For: n=3

$$ {}_jP_{31} = 35a^3b^{k-1}\,{}_jP_{0k} + 15a^2b^5\,{}_jP_{1k} + 5ab^5\,{}_jP_{26} + b^5\,{}_jP_{36} $$

$$ {}_jP_{32} = 20a^3b^{k-2}\,{}_jP_{0k} + 10a^2b^4\,{}_jP_{1k} + 4ab^4\,{}_jP_{2k} + b^4\,{}_jP_{36} $$

Therefore:

$$ {}_jP_{ns} = \sum_{r=0}^{n} \binom{m+r-1}{m-1} \left(\frac{\lambda}{\lambda+k_j\mu_j}\right)^r \left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^m {}_jP_{n-r,s+m} \qquad 1 \le m \le k-s \tag{B.1} $$

Expand Equation 2.9

For: $j=1 \quad \frac{\lambda}{\lambda_1}\,{}_1P_{n-1,k} = \frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{nk} - \sum_j {}_jP_n$

$j=2 \quad \frac{\lambda}{\lambda_2}\,{}_2P_{n-1,k} = \frac{\lambda+k_2\mu_2}{\lambda_2}\,{}_2P_{nk} - \sum_j {}_jP_n$

$j=3 \quad \frac{\lambda}{\lambda_3}\,{}_3P_{n-1,k} = \frac{\lambda+k_3\mu_3}{\lambda_3}\,{}_3P_{nk} - \sum_j {}_jP_n$

Solve simultaneously

$$\frac{\lambda+k_2\mu_2}{\lambda_2}\,{}_2P_{nk} - \frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{nk} = \frac{\lambda}{\lambda_2}\,{}_2P_{n-1,k} - \frac{\lambda}{\lambda_1}\,{}_1P_{n-1,k}$$

$$\frac{\lambda+k_3\mu_3}{\lambda_3}\,{}_3P_{nk} - \frac{\lambda+k_2\mu_2}{\lambda_2}\,{}_2P_{nk} = \frac{\lambda}{\lambda_3}\,{}_3P_{n-1,k} - \frac{\lambda}{\lambda_2}\,{}_2P_{n-1,k}$$

Therefore:

$${}_jP_{nk} = \frac{\lambda_j}{\lambda+k_j\mu_j}\left[\frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{nk} + \frac{\lambda}{\lambda_j}\,{}_jP_{n-1,k} - \frac{\lambda}{\lambda_1}\,{}_1P_{n-1,k}\right] \qquad \text{B.2}$$

Substitute Equation B.2 into the term r=0 in Equation B.1

For example:

the r=0 term is: $\left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^m {}_jP_{nk} \qquad 1\le m\le k-s$

then

$$\left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^m {}_jP_{nk} = \left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^m \frac{\lambda_j}{\lambda+k_j\mu_j}\left[\frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{nk} + \frac{\lambda}{\lambda_j}\,{}_jP_{n-1,k} - \frac{\lambda}{\lambda_1}\,{}_1P_{n-1,k}\right]$$

$$= \left(\frac{\lambda}{\lambda+k_j\mu_j}\right)\left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^m {}_jP_{n-1,k} + \left(\frac{k_j\mu_j}{\lambda+k_j\mu_j}\right)^m\left(\frac{\lambda_j}{\lambda+k_j\mu_j}\right)\left[\frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{nk} - \frac{\lambda}{\lambda_1}\,{}_1P_{n-1,k}\right] \qquad \text{B.3}$$

Expand Equation 2.9 for values of n=1,2,3...

For n=1 j=1 $\quad \frac{\lambda}{\lambda_1}\,{}_1P_{0k} = \frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{1k} - \sum_i\left[{}_iP_{11}+{}_iP_{12}+{}_iP_{13}+\dots+{}_iP_{1k}\right]$

Substitute Equation B.1

$$\frac{\lambda}{\lambda_1}\,{}_1P_{0k} = \frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{11} -$$

$$\sum_i \Bigg[(k-1)\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-1} {}_iP_{0k} + \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-1} {}_iP_{1k}$$

$$+(k-2)\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-2} {}_iP_{0k} + \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-2} {}_iP_{1k}$$

$$+(k-3)\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-3} {}_iP_{0k} + \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-3} {}_iP_{1k}$$

$$\vdots$$

$$+ \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right) {}_iP_{0k} + \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right) {}_iP_{1k}$$

$$+ {}_iP_{1k} \Bigg] \qquad \text{B.4a}$$

Substitute Equation B.3 into B.4a

$$
\begin{aligned}
\frac{\lambda}{\lambda_1}\,{}_1P_{0k} = {} & \frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{1k} \\
& -\sum_i\Bigg[k\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-1}{}_iP_{0k} \\
& + \left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-1}\left(\frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{1k} - \frac{\lambda}{\lambda_1}\,{}_1P_{0k}\right) \\
& + (k-1)\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-2}{}_iP_{0k} \\
& + \left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-2}\left(\frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{1k} - \frac{\lambda}{\lambda_1}\,{}_1P_{0k}\right) \\
& \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
& + 2\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right){}_iP_{0k} \\
& + \left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)\left(\frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{1k} - \frac{\lambda}{\lambda_1}\,{}_1P_{0k}\right) \\
& + \frac{\lambda}{\lambda+k_i\mu_i}\,{}_iP_{0k} + \left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)\left(\frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{1k} - \frac{\lambda}{\lambda_1}\,{}_1P_{0k}\right)\Bigg]
\end{aligned}
\qquad \text{B.4b}
$$

Rearrange and combine terms

$$\frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{1k} - \frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{1k}\sum_i\sum_{m=0}^{k-1}\left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)\left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)^m =$$

$$\sum_i\sum_{m=0}^{k-1}\binom{m+1}{1}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{0k} + \frac{\lambda}{\lambda_1}\,{}_1P_{0k}$$

$$-\frac{\lambda}{\lambda_1}\sum_i\sum_{m=0}^{k-1}\left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_1P_{0k}$$

where $\binom{m+1}{1}$ is Binomial Coefficient B.4c

Therefore

Let $\alpha_n = \sum_{i=1}^{m}\sum_{m=0}^{k-1}\binom{m+n-1}{n-1}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^n\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m$

where $\binom{m+n-1}{n-1}$ is Binomial Coefficient

$$\frac{(\lambda+k_1\mu_1)(1-\alpha_1)}{\lambda_1}\,{}_1P_{1k} = \sum_i\sum_{m=0}^{k-1}\binom{m+1}{1}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{0k}$$

$$+ (1-\alpha_1)\frac{\lambda}{\lambda_1}\,{}_1P_{0k}$$

$${}_1P_{1k} = \frac{\lambda_1}{(\lambda+k_1\mu_1)(1-\alpha_1)}\sum_i\sum_{m=0}^{k-1}\binom{m+1}{1}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{0k}$$

$$+ (1-\alpha_1)\frac{\lambda}{\lambda_1}\,{}_1P_{0k} \qquad \text{B.4}$$

In a similar manner Equation B.2 For n=2 j=1

$$\frac{\lambda}{\lambda_1}\,{}_1P_{1k} = \frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{2k} - \sum_i\left[{}_iP_{21} + {}_iP_{22} + {}_iP_{23} + \dots + {}_iP_{2k}\right.$$

Substitute B.3 and B.4a

$$\frac{\lambda}{\lambda_1}\,{}_1P_{1k} = \frac{\lambda+k_1\mu_1}{\mu_1}\,{}_1P_{2k}$$

$$- \sum_i \Bigg[\frac{k(k-1)}{2!} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^2 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-1} {}_iP_{0k}$$

$$+ k \left(\frac{\lambda}{\lambda+k_i\mu_i}\right) \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-1} {}_iP_{1k}$$

$$+ \left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right) \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-1} \left(\frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{2k} - \frac{\lambda}{\lambda_1}\,{}_1P_{1k}\right)$$

$$+ \frac{(k-1)(k-2)}{2!} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^2 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-2} {}_iP_{0k}$$

$$+ (k-1) \left(\frac{\lambda}{\lambda+k_i\mu_i}\right) \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-2} {}_iP_{1k}$$

$$+ \left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right) \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k-2} \left(\frac{\lambda+k_i\mu_i}{\lambda_1}\,{}_1P_{2k} - \frac{\lambda}{\lambda_1}\,{}_1P_{1k}\right)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$+ \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^2 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right) {}_iP_{0k} + \left(\frac{\lambda}{\lambda+k_i\mu_i}\right) \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right) {}_iP_{1k}$$

$$+ \left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right) \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right) \left(\frac{\lambda+k_1\mu_i}{\lambda_1}\,{}_1P_{2k} - \frac{\lambda}{\lambda_1}\,{}_1P_{1k}\right)$$

$$+ \frac{\lambda}{\lambda+k_i\mu_i}\,{}_iP_{1k} + \frac{\lambda_i}{\lambda+k_i\mu_i} \left(\frac{\lambda+k_1\mu_1}{\lambda_1}\,{}_1P_{2k} - \frac{\lambda}{\lambda_1}\,{}_1P_{1k}\right) \Bigg]$$

Then

$$
{}_1P_{2k} = \frac{\lambda_1}{(\lambda+k_1\mu_1)(1-\alpha_1)} \sum_i \sum_{m=0}^{k-1} \Bigg[\binom{m+1}{1} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right) \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{1k} + \binom{m+1}{2} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^2 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{0k} \Bigg] + (1-\alpha_1)\frac{\lambda}{\lambda_1} {}_1P_{1k} \qquad \text{B.5}
$$

In a similar manner

for $n=3$ $j=1$

$$
{}_1P_{3k} = \frac{\lambda_1}{(\lambda+k_1\mu_1)(1-\alpha_1)} \sum_i \sum_{m=0}^{k-1} \Bigg[\binom{m+2}{3} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^3 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{0k} + \binom{m+1}{2} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^2 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{1k} + \binom{m+1}{1} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right) \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{2k} \Bigg] + (1-\alpha_1)\frac{\lambda}{\lambda_1} {}_1P_{2k} \qquad \text{B.6}
$$

Therefore by induction

$$
{}_jP_{nk} = \frac{\lambda_j}{(\lambda+k_j\mu_j)(1-\alpha_1)} \sum_i \Bigg[\sum_{r=1}^{r} \sum_{m=0}^{k-1} \binom{m+r-1}{r} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^r \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{n-r,k} + \left(\frac{\lambda}{\lambda+k_i\mu_i}\right) \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{n-1,k} + \delta_{ij}(1-\alpha_1)\frac{\lambda}{\lambda_j} {}_jP_{n-1,k} \Bigg]
$$

where $\delta_{ij} = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i=j \end{cases}$ 3.10d

Rearrange Equation 2.9

$$P_n = \sum_i {}_iP_n = \frac{\lambda+k_j\mu_j}{\lambda_j}\,{}_jP_{nk} - \frac{\lambda}{\lambda_j}\,{}_jP_{n-1,k}$$

Since $\sum_i {}_iP_n = P_n$

and substitute Equation 3.10b

$$P_n = \frac{1}{1-\alpha_1}\sum_i \sum_{r=1}^{n} \sum_{m=0}^{k-1}\Bigg[\binom{m+r-1}{r}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^r\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{n-r,k}$$

$$+ \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{n-1,k}\Bigg] \qquad \text{B.7}$$

Expand Equation 2.9 for n=1,2,3,...

For n=1 $\quad {}_jP_{1k} = \frac{\lambda_j}{\lambda+k_j\mu_j}P_1 + \frac{\lambda}{\lambda+k_j\mu_j}\,{}_jP_{0k}$

Substitute Equation 2.8a

$${}_jP_{1k} = \frac{\lambda_j}{\lambda+k_j\mu_j}P_1 + \frac{\lambda\lambda_j}{(\lambda+k_j\mu_j)^2}\,\frac{{}_0P_{00}}{1-\alpha_1}$$

For n=2 $\quad {}_jP_{2k} = \frac{\lambda_j}{\lambda+k_j\mu_j}P_2 + \frac{\lambda}{\lambda+k_j\mu_j}\,{}_jP_{1k}$

$$= \frac{\lambda_j}{\lambda+k_j\mu_j}P_2 + \frac{\lambda\lambda_j}{(\lambda+k_j\mu_j)^2}P_1 + \frac{\lambda^2\lambda_j}{(\lambda+k_j\mu_j)^3}\,\frac{{}_0P_{00}}{1-\alpha}$$

For n=3 $$ {}_jP_{3k} = \frac{\lambda_j}{\lambda+k_j\mu_j}P_3 + \frac{\lambda\lambda_j}{(\lambda+k_j\mu_j)^2}P_2 + \frac{\lambda^2\lambda_j}{(\lambda+k_j\mu_j)^3}P_1 + \frac{\lambda^2\lambda_j}{(\lambda+k_j\mu_j)^3}\,\frac{{}_0P_{00}}{1-\alpha_1} $$

Therefore

$$ {}_jP_{nk} = \frac{\lambda^{n-1}\lambda_j}{(\lambda+k_j\mu_j)^n}\,\frac{{}_0P_{00}}{1-\alpha_1} + \sum_{m=1}^{n} \frac{\lambda^{n-m}\lambda_j}{(\lambda+k_j\mu_j)^{n+1-m}}P_m \qquad 2.10d $$

Expand Equation 2.10b for n=1,2,3,...

For n=1 $$ P_1 = \frac{1}{1-\alpha_1}\sum_i\sum_{m=0}^{k-1}\left[\binom{m}{1}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{0k} + \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{0k}\right] $$

Substitute Equation 2.10d

$$ P_1 = \frac{1}{(1-\alpha_1)^2}\sum_i\sum_{m=0}^{k-1}\binom{m+1}{1}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^2\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_0P_{00} $$

For n=2

$$ P_2 = \frac{1}{1-\alpha_1}\sum_i\sum_{m=0}^{k-1}\left[\binom{m}{1}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{1k} + \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{1k} + \binom{m+1}{2}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^2\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{0k}\right] $$

Substitute 2.10d

$$P_2 = \frac{1}{1-\alpha_1}\sum_i \sum_{m=0}^{k-1}\left[\binom{m+1}{1}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^2 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m P_1\right.$$

$$+\binom{m+1}{1}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^3 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m \frac{{}_0P_{00}}{1-\alpha_1}$$

$$\left.+\binom{m+1}{2}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^3 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m \frac{{}_0P_{00}}{1-\alpha_1}\right]$$

$$P_2 = \frac{1}{1-\alpha_1}\sum_i \sum_{m=0}^{k-1}\left[\binom{m+1}{1}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^2 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m P_1\right.$$

$$\left.+\binom{m+2}{2}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^3 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m \frac{{}_0P_{00}}{1-\alpha_1}\right]$$

$$P_2 = \frac{\lambda\alpha_2}{1-\alpha_1}P_1 + \frac{\lambda^2\alpha_3}{(1-\alpha_1)^2}\,{}_0P_{00}$$

For n=3

$$P_3 = \frac{1}{1-\alpha_1}\sum_i \sum_{m=0}^{k=1}\left[\binom{m}{1}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right) \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{2k}\right.$$

$$+\binom{m+1}{2}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^2 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{1k}$$

$$+\binom{m+2}{3}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^3 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{0k}$$

$$\left.+\left(\frac{\lambda}{\lambda+k_i\mu_i}\right) \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m {}_iP_{2k}\right]$$

$$P_3 = \frac{1}{1-\alpha_1} \sum_i \sum_{m=0}^{k-1} \Bigg[\binom{m+1}{1} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^2 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m P_2$$

$$+ \binom{m+1}{1} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^3 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m P_1$$

$$+ \binom{m+1}{1} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^4 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m \frac{{}_0P_{00}}{1-\alpha_1}$$

$$+ \binom{m+1}{2} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^3 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m P_1$$

$$+ \binom{m+1}{2} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^4 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m \frac{{}_0P_{00}}{1-\alpha_1}$$

$$+ \binom{m+2}{3} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^4 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m \frac{{}_0P_{00}}{1-\alpha_1} \Bigg]$$

$$P_3 = \frac{1}{1-\alpha_1} \sum_i \sum_{m=0}^{k=i} \Bigg[\binom{m+1}{1} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^2 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m P_2$$

$$+ \binom{m+2}{2} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^3 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m P_1$$

$$+ \binom{m+3}{3} \left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^4 \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^m \frac{{}_0P_{00}}{1-\alpha_1} \Bigg]$$

$$P_3 = \frac{\lambda\alpha_2 P_2}{1-\alpha_1} + \frac{\lambda^2\alpha_3 P_1}{1-\alpha_1} + \frac{\lambda^3\alpha_{4\,0}P_{00}}{(1-\alpha_1)^2}$$

Therefore $$P_n = \frac{\lambda^r \alpha_{n+1\;0}P_{00}}{(1-\alpha_1)^2} + \sum_{r=1}^{n=1} \frac{\lambda^r \alpha_{r+1} P_{n-r}}{(1-\alpha_1)} \qquad 2.10$$

APPENDIX C

Solution of α_n

Equation 2.11c contains the infinite series $\sum_{n=0}^{\infty} \lambda^n \alpha_{n+1}$. In order to evaluate this infinite series it is necessary to expand the expression for $\alpha n (n=0,1,2,\ldots)$ and sum.

Since $$\sum_{n=0}^{\infty} \lambda^n \alpha_{n+1} = \alpha_1 + \lambda\alpha_2 + \lambda^2\alpha_3 + \ldots \qquad \text{C.1a}$$

and $$\lambda^n \alpha_{n+1} = \sum_i \frac{\lambda_i}{\lambda+k_i\mu_i}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^n \sum_{s=0}^{k-1}\binom{s+n-1}{n-1}\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^s \qquad \text{C.1b}$$

Expanding $\lambda^n\alpha_{n+1}$ for all values of (n)

$$\alpha_1 = \sum_i \frac{\lambda_i}{\lambda+k_i\mu_i}\left\{1+\frac{k_i\mu_i}{\lambda+k_i\mu_i}+\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^2+\ldots+\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k_i-1}\right\}$$

$$\lambda\alpha_2 = \sum_i \left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left\{1+2\frac{k_i\mu_i}{\lambda+k_i\mu_i}+3\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^2+\ldots + \binom{k}{k-1}\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k_i-1}\right\}$$

$$\lambda^2\alpha_3 = \sum_i \left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)^2\left\{1+3\frac{k_i\mu_i}{\lambda+k_i\mu_i}+6\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^2+\ldots + \binom{k+1}{k-1}\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k_i-1}\right\}$$

and etc.

Since $(1-x)^{-n} = 1+nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$

Summing like terms yields

$$\sum_{n=0}^{\infty} \lambda^n \alpha_{n+1} = \sum_i \frac{\lambda_i}{\lambda+k_i\mu_i} \left\{ \left(1-\frac{\lambda}{\lambda+k_i\mu_i}\right)^{-1} + \frac{k_i\mu_i}{\lambda+k_i\mu_i}\left(1-\frac{\lambda}{\lambda+k_i\mu_i}\right)^{-2} + \ldots \right.$$

$$\left. \ldots + \left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k_i-1}\left(1-\frac{\lambda}{\lambda+k_i\mu_i}\right)^{-k_i} \right\} \qquad \text{C.1c}$$

Since $1 - \frac{\lambda}{\lambda+k_i\mu_i} = \frac{k_i\mu_i}{\lambda+k_i\mu_i}$

and $\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{s-1}\left(1-\frac{\lambda}{\lambda+k_i\mu_i}\right)^{-s} = \frac{\lambda+k_i\mu_i}{k_i\mu_i}$ for $s=2,3,4\ldots$

Substituting the expression above and simplifying

$$\sum_{n=0}^{\infty} \lambda^n \alpha_{n+1} = \sum_i \left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right) k_i \left(\frac{\lambda+k_i\mu_i}{k_i\mu_i}\right) \qquad \text{C.1d}$$

Therefore

$$\alpha_n = \sum_{n=0}^{\infty} \lambda^n \alpha_{n+1} = \sum_i \frac{\lambda_i}{\mu_i} \qquad \text{C.1}$$

Solution of α_z

Equation 3.1 contains the infinite series $\sum_{i=0}^{\infty} (\lambda z)^i \alpha_{i+1}$. The solution of this series is accomplished in the same manner as α_n. Expand $(\lambda z)^i \alpha_{i+1}$ for all values of (i), change the order of summation and simplify. Since this series is not utilized in the solution of the system characteristic, it will not be derived. The derivative of this series is important for its use in equation 3.2 and 3.3.

The first derivative $\frac{d}{dz}\alpha_z$ becomes:

$$\frac{d}{dz}\alpha_z = \lambda \sum_{i=0}^{\infty} i(\lambda z)^{i-1}\alpha_{i+1} = \frac{1}{z} \sum_{i=0}^{\infty} i(\lambda z)^{i}\alpha_{i+1} \qquad \text{C.2a}$$

Expanding for i=0,1,2...as in equation C.1a

$$\lambda\alpha_2 = \sum_i \left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left\{1+2\frac{k_i\mu_i}{\lambda+k_i\mu_i} + 3\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^2 + \ldots\right.$$

$$2z\lambda^2\alpha_3 = \sum_i \left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)\left(\frac{2z\lambda^2}{\lambda+k_i\mu_i}\right)\left\{1+3\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)+6\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^2+\ldots\right.$$

and etc.

Evaluate at z=1 and change the order of summation yields:

$$\frac{d}{dz}\alpha_z = \sum_i \left(\frac{\lambda_i}{\lambda+k_i\mu_i}\right)\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left\{\left(1-\frac{\lambda}{\lambda+k_i\mu_i}\right)^{-2} + 2\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)\left(1-\frac{\lambda}{\lambda+k_i\mu_i}\right)^{-3} + \ldots + k_i\left(\frac{k_i\mu_i}{\lambda+k_i\mu_i}\right)^{k_i-1}\left(1-\frac{\lambda}{\lambda+k_i\mu_i}\right)^{k_i-2}\right\}$$

C.2b

Simplifying

$$\frac{d}{dz}\alpha_z = \sum_i \frac{\lambda_i}{\lambda+k_i\mu_i}\left(\frac{\lambda}{\lambda+k_i\mu_i}\right)\left(\frac{\lambda+k_i\mu_i}{k_i\mu_i}\right)^{2}\frac{(k_i+1)k_i}{2} \qquad \text{C.2c}$$

$$\frac{d}{dz}\alpha_z = \lambda\sum_i \left(\frac{k_i+1}{2k_i}\right)\frac{\lambda_i}{\mu_i^2} \qquad \text{For } z=1 \qquad \text{C.2}$$

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