# ANALYSIS OF ONE-WAY CONTINUOUS RECTANGULAR PLATE-BEAM STRUCTURE BY FLEXIBILITY METHODS

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# NOMENCLATURE

| | |
|---|---|
| $p$ | Load Intensity |
| $P$ | Concentrated Load |
| $h$ | Thickness of Plate |
| $b$ | Width of Edge Beam |
| $h'$ | Depth of Edge Beam |
| $\gamma$ | Specific Weight of Edge Beam |
| $E$ | Young's Modulus of Elasticity |
| $I$ | Moment of Inertia of Beam |
| $\mu$ | Poisson's Ratio |
| $D = \frac{Eh^3}{12(1-\mu^2)}$ | Flexural Rigidity of Plate |
| $EI$ | Flexural Rigidity of Edge Beam |
| $w$ | Vertical Deflection |
| $M$ | Bending Moment |
| $R$ | Edge Reaction |
| $X, Y$ | Rectangular Coordinates |
| $\Delta x, \Delta y, \Delta$ | Dimensions of Plate Element |
| $a, b, c, t, \lambda$ | Dimensionless Quantities |
| $i, j, k, m, n$ | Network of Points |
| $\theta, \beta$ | Angle of Rotation |
| $\delta$ | Displacement Load Function |
| $\tau$ | Angular Load Function |

| | |
|---|---|
| F | Angular Flexibility |
| G | Angular Carry-Over |
| Q | Angular-Displacement Carry-Over |
| D | Displacement Flexibility |
| E | Displacement Carry-Over |
| S | Displacement-Angular Flexibility |
| T | Displacement-Angular Carry-Over |
| U | Torsional Angular Flexibility, Free Edge |
| V | Torsional Angular Carry-Over, Free Edge |
| H | Torsional Angular Carry-Over, Simply Supported Boundary |
| $\nabla^2$ | Laplacian Differential Operator |
| $\eta$ | Deflection Influence Coefficient |

# CHAPTER I

# INTRODUCTION

1.1. Historical Review. There have been many solutions proposed during the past thirty-five years for problems involving continuous rectangular plates. Several of the earlier European publications on rectangular plates that are continuous in a single direction over rigid supports are those by Marcus (1)† and Galerkin. (2) A distribution procedure for the analysis of one-way continuous plates with two edges simply supported and continuous over rigid or flexible beams transverse to the simply supported edges was developed by Newmark. (3) Jensen (4) wrote an early publication about rectangular plates that are continuous in a single direction over rigid supports. A solution for rectangular plates that are continuous in a single direction over rigid supports and have stiffened edge beams continuous along the free edge was first presented in this country by Jensen. (5) In this particular paper, the author neglected the torsional resistance of the edge members. Using Jensen's method of analysis, Seely and Smith (6) have developed a series of tables for moments and deflections in single-span plates supported on elastic beams with varying length-width ratios.

Ang and Prescott, (7) in a recent article, have utilized the finite difference solution, including the torsional rigidity of an edge beam, to solve the problem of continuous rectangular plates simply supported

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†Note: Numbers in parentheses, after names, refer to numbered references in Selected Bibliography.

over flexible beams. In this presentation, the authors took the torsional moment of the edge member into account by writing a joint equilibrium equation for continuity between adjacent plates. The torsional stiffness term appeared in the deflection equation when written for points on or near the edge beam.

Ang and Newmark (8) presented a numerical procedure for the analysis of rectangular plates continuous in two directions utilizing finite differences and a stiffness method of approach.

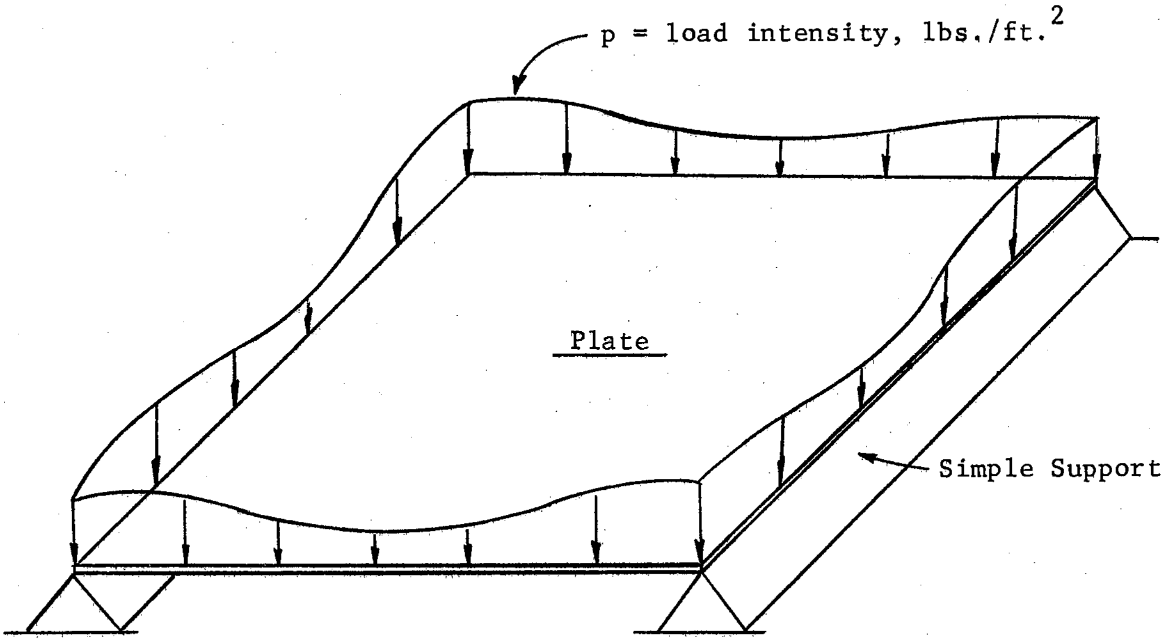


Figure 1.1

Basic Structure

1.2. Scope of Study. A method of analyzing thin rectangular plates of constant thickness, continuous in one direction over rigid supports, and supported longitudinally by elastic edge beams is described herein.

The flexibility method is employed. The basic structure is a rectangular plate with two parallel edges simply supported and the remaining two edges free to deflect and warp under a general system of applied loads (Figure 1.1, page 2).

Using finite differences, deflection equations are written for a 56-point set on the basic structure by applying a unit load at each point. The general differential equation for the deflected surface is modified to meet the free edge condition requirements of zero moment and shear. A set of influence coefficients is developed from the deflection equation employing the IBM 650 computer. The angular functions are defined in terms of the influence coefficients.

The magnitude of the inaccuracies which result from the substitution of a series of difference equations for the differential equation was made to diminish by increasing the number of equations, hence using as fine a network of points within the plate as the IBM 650 computer could feasibly handle.

The solution for a one-way continuous plate problem is derived from a single panel solution by accounting for the conditions of continuity between the adjacent panels and equal deflection of plate and edge beam on the free edges. A general moment and edge reaction equation is presented in matrix form.

A numerical example problem involving a two-span rectangular panel bridge is solved using the described theory.

# CHAPTER II

## DEFLECTION EQUATIONS

2.1. The Basic Plate Structure. Consider first the basic plate structure which has free edges in the X direction and is simply supported between adjacent panels in the Y direction (Figure 2.1).

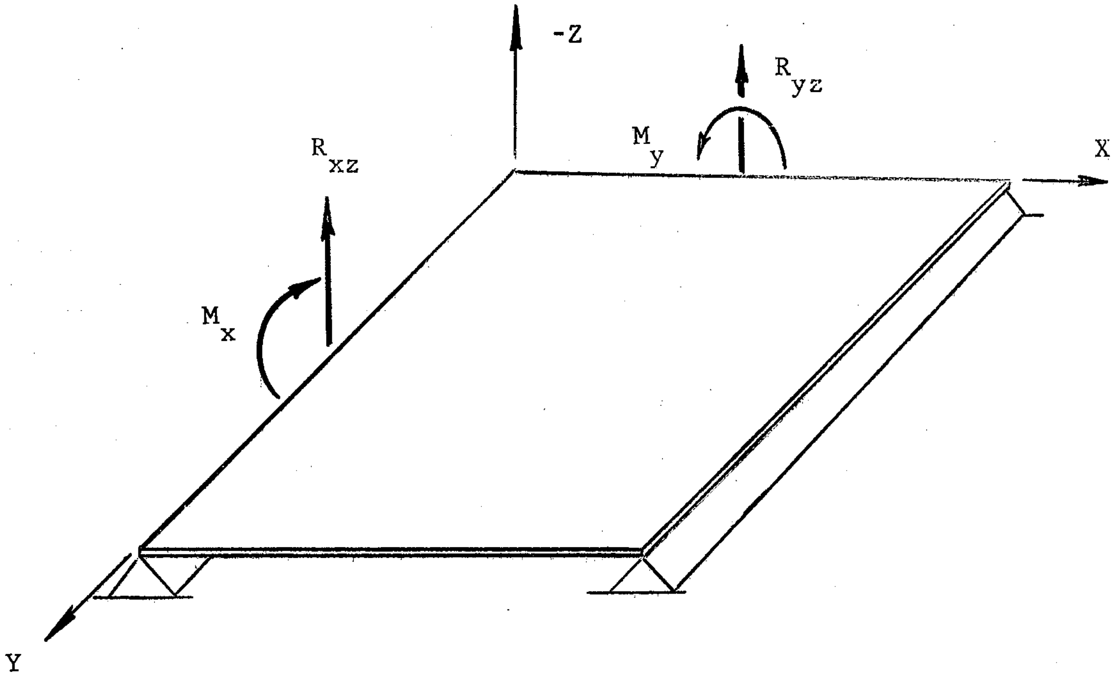


Figure 2.1

Basic Plate Structure with Edge Moments and Reactions

2.2. The Free Edge Conditions. Due to the condition of a free discontinuous edge, the moments and reactions are zero; hence,

$$M_y = 0 \qquad R_{yz} = 0$$

From equations (1-22) and (1-29), reference (9), the moment and reaction, respectively, at the free edge are:

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)$$

$$R_{yz} = -D\left(\frac{\partial^3 w}{\partial y^3} + (2-\mu) \frac{\partial^3 w}{\partial x^2 \partial y}\right)$$

where: D is the flexural rigidity of the plate.

Setting the moment at the free edge equal to zero

$$M_y = 0 = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)$$

one obtains the expression

$$\nabla^2 w = (1 - \mu) \frac{\partial^2 w}{\partial x^2} \quad . \tag{2.1}$$

Setting the reaction at the free edge equal to zero

$$R_{yz} = 0 = -D\left(\frac{\partial^3 w}{\partial y^3} + (2 - \mu) \frac{\partial^3 w}{\partial x^2 \partial y}\right)$$

one obtains the expression

$$\frac{\partial^3 w}{\partial y^3} + (2 - \mu) \frac{\partial^3 w}{\partial x^2 \partial y} = 0 \ . \tag{2.2}$$

### 2.3. The General Differential Equation of the Deflected Surface.

From equation (1-25), reference (9), the general differential equation of the deflected surface is:

$$\frac{\partial^4 w}{\partial x^2} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} \tag{2.3}$$

or, with simplified notation:

$$\nabla^2 \nabla^2 w = \nabla^4 w = \frac{p}{D} \tag{2.4}$$

where: $\nabla^2$ is the Laplacian differential operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ .

The deflection equation will always be applied to point i,j in the following network system. Points designated as i increase or decrease by single units as a function of the distance from the control point i,j in the X direction by increments of Δx. Points designated as j increase or decrease by single units as a function of the distance from the control point i,j in the Y direction by increments of Δy (Figure 2.2).

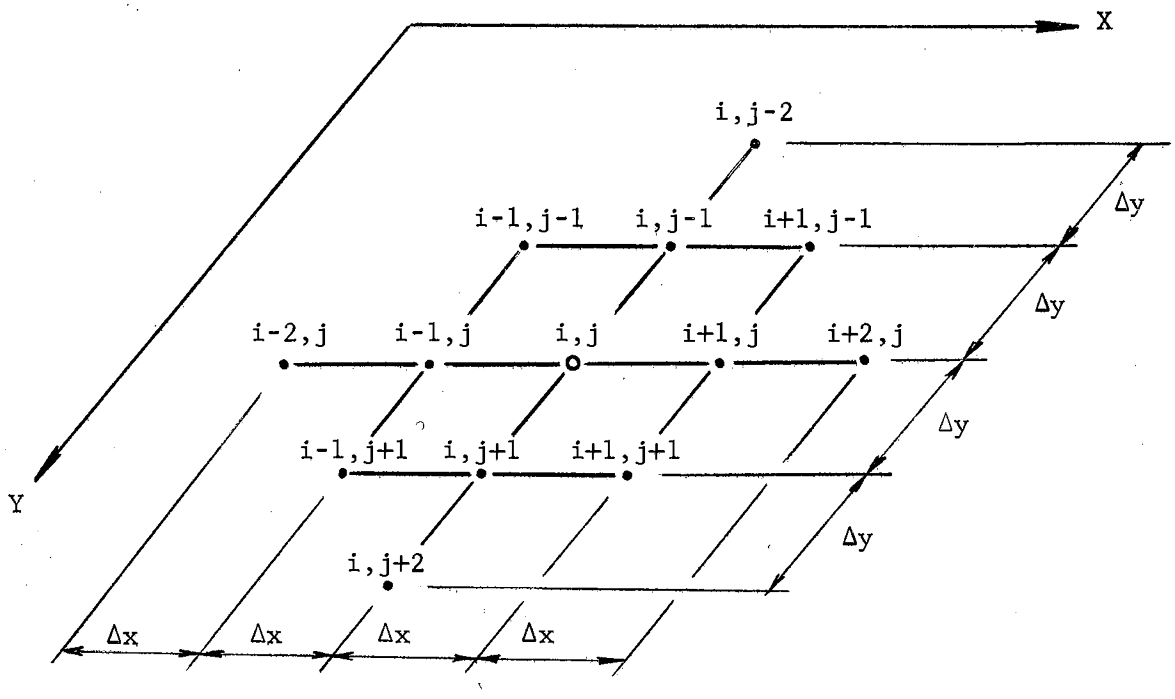


Figure 2.2

Control Points of Network System

Equation (2.1) can be rewritten to apply to a particular point i,j on the free edge of a plate; thus,

$$(\nabla^2 w)_{i,j} = (1 - \mu)\left(\frac{\partial^2 w}{\partial x^2}\right)_{i,j} \quad . \tag{2.1a}$$

By definition

$$(\nabla^2 w)_{i,j} = \left(\frac{\partial^2 w}{\partial x^2}\right)_{i,j} + \left(\frac{\partial^2 w}{\partial y^2}\right)_{i,j} \quad . \tag{2.5}$$

From Table 3-1, reference (9), the finite difference approximation for the expressions $\left(\frac{\partial^2 w}{\partial x^2}\right)_{i,j}$ and $\left(\frac{\partial^2 w}{\partial y^2}\right)_{i,j}$ are:

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_{i,j} = \frac{w_{i-1,j} - 2w_{i,j} + w_{i+1,j}}{\Delta x^2} \tag{2.6a}$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{i,j} = \frac{w_{i,j-1} + 2w_{i,j} + w_{i,j+1}}{\Delta y^2} \tag{2.6b}$$

Substituting equations (2.6a) and (2.6b) into equation (2.5) and simplifying

$$(\nabla^2 w)_{i,j} = \frac{2(1+t^2)}{\Delta x^2} \left[\begin{array}{ll} a(w_{i+1,j} + w_{i-1,j}) & \\ & - w_{i,j} \\ b(w_{i,j+1} + w_{i,j-1}) & \end{array}\right] \tag{2.7}$$

where:

$$t = \frac{\Delta x}{\Delta y} \qquad \text{or} \qquad t^2 = \frac{\Delta x^2}{\Delta y^2}$$

$$a = \frac{1}{2(1+t^2)}$$

$$b = \frac{t^2}{2(1+t^2)} \quad .$$

A further substitution of equation (2.7) into equation (2.4) will yield the general differential equation of the deflected surface.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \frac{2(1+t^2)}{\Delta x^2} \Big[a(w_{i+1,j} + w_{i-1,j})$$

$$+ b(w_{i,j+1} + w_{i,j-1}) - w_{i,j}\Big] = \frac{p}{D} .$$

Expanding and multiplying by $\frac{\Delta x^2}{2(1+t^2)}$ :

$$a\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)_{i+1,j} + \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)_{i-1,j}\right]$$

$$+ b\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)_{i,j+1} + \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)_{i,j-1}\right]$$

$$- \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)_{i,j} = \frac{p}{D} \frac{\Delta x^2}{2(1+t^2)} \cdot \frac{\Delta y}{\Delta y}$$

$$\left.\begin{array}{l} - (\nabla^2 w)_{i,j} + a\left[(\nabla^2 w)_{i+1,j} + (\nabla^2 w)_{i-1,j}\right] \\ \qquad + b\left[(\nabla^2 w)_{i,j+1} + (\nabla^2 w)_{i,j-1}\right] \end{array}\right. = \frac{p}{D} \lambda \Delta x \Delta y \quad (2.8)$$

where: $\lambda = \frac{t}{2(1+t^2)}$ .

2.4. Typical Interior Point - i,j. The interior point will be defined as any point on the plate for which equation (2.8) can be applied without the substitutions of special boundary conditions (Figure 2.3).

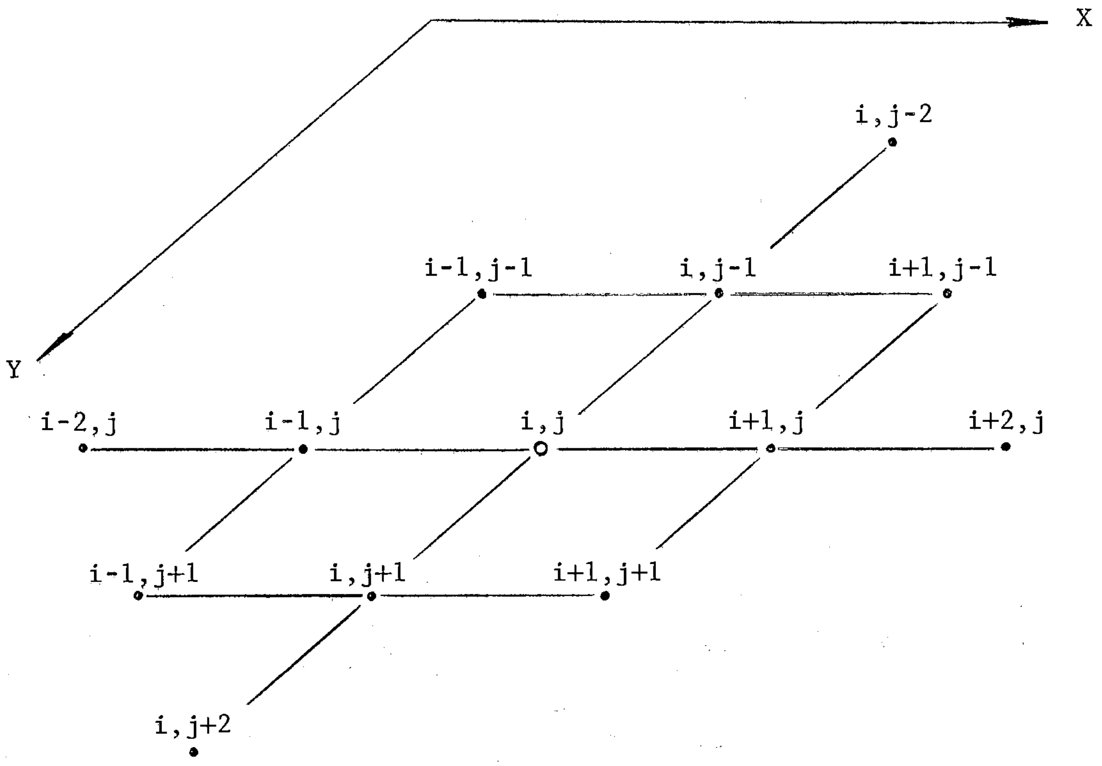
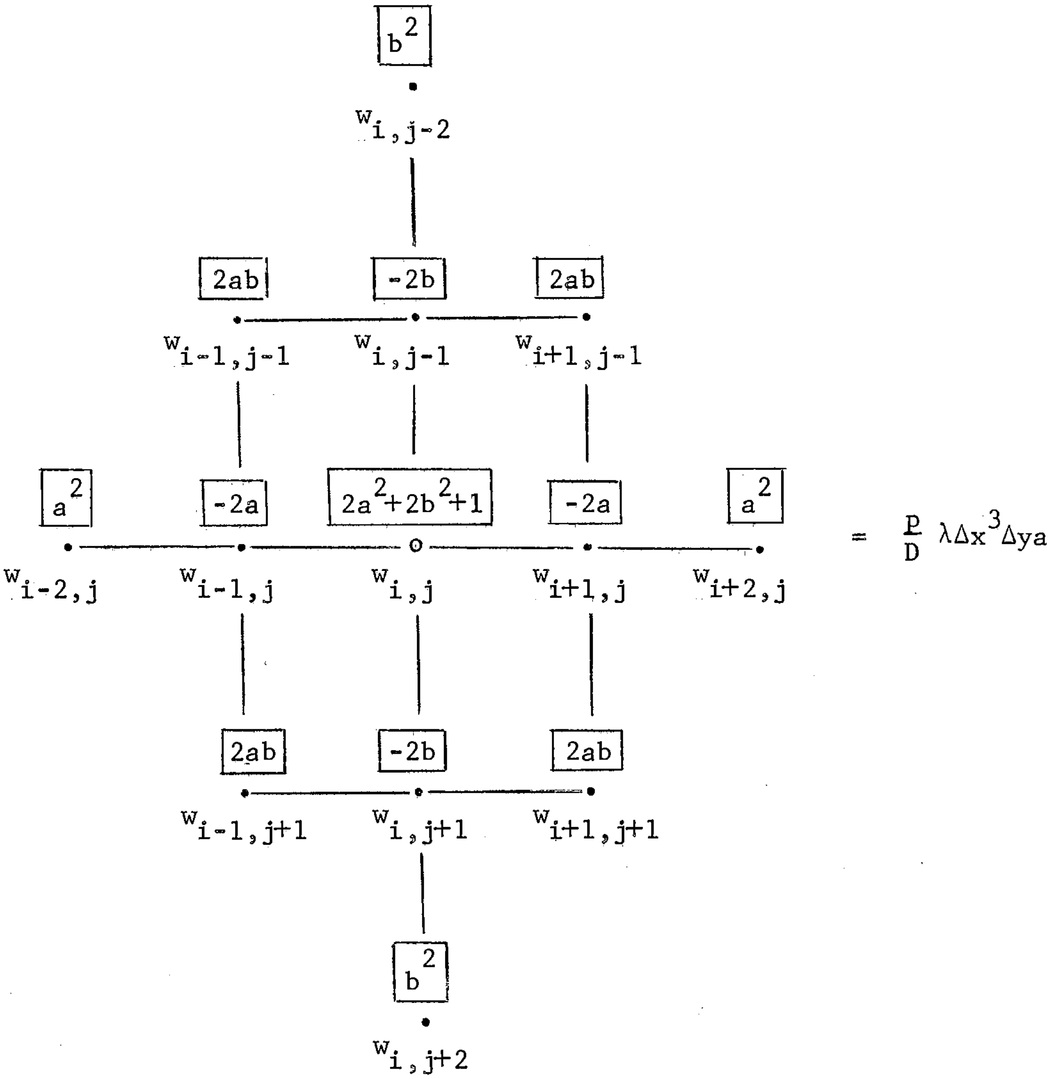


Figure 2.3

Typical Interior Point

Substituting the finite difference equation (2.7) for the $\nabla^2 w$ terms appearing in equation (2.8) and multiplying both sides by $a\Delta x^2$ will yield a finite difference solution to the fourth-order differential equation for a deflection at a typical interior point. This equation is illustrated in symbolic form in Figure 2.4.



(2.9)

Figure 2.4

Diagrammatical Form of the First-Order Finite Difference Operator of $\nabla^4 w$ at a Typical Interior Point

All subsequent equations developed in this chapter will be modifications of equation (2.9) to satisfy the free edge and/or simply supported edge boundary conditions.

2.5. Boundary Point - i,j. The boundary point will be defined as any point which is located on the free edge of the plate (Figure 2.5).

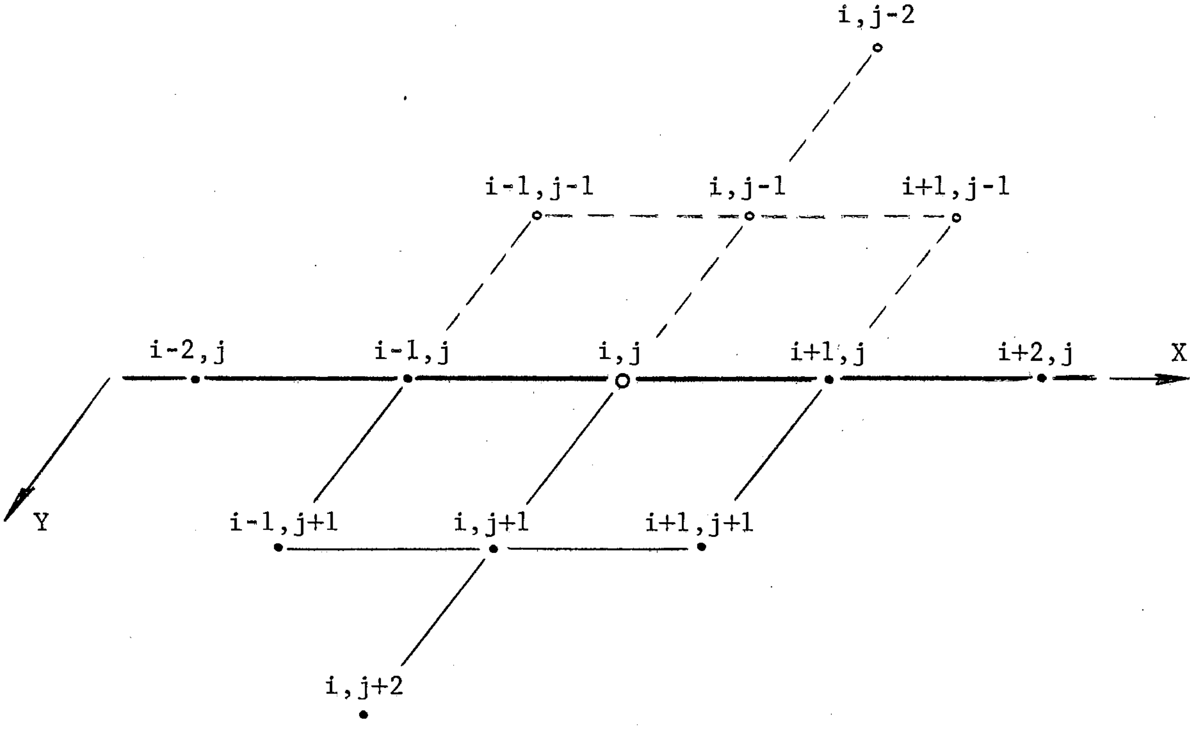


Figure 2.5

Boundary Point i,j on Free Edge

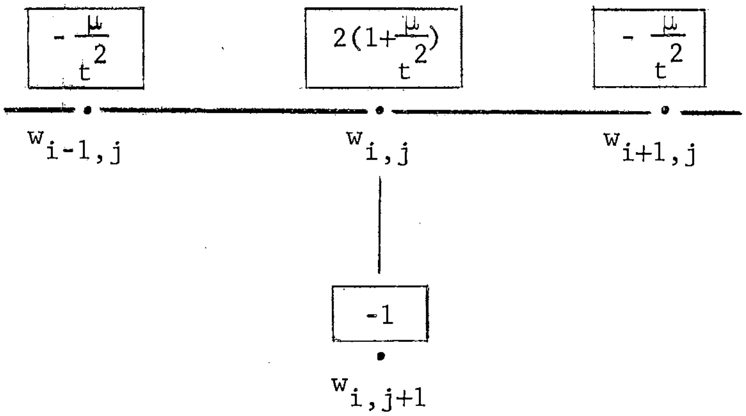
Points i+1,j-1, i,j-1, i-1,j-1, and i,j-2 are imaginary points outside the boundary of the plate. A modification of equation (2.9) must be made in such a manner as to eliminate the terms containing the imaginary points.

Solving equation (2.1) by substituting equation (2.5), the following expression is obtained:

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{i,j} = -\mu\left(\frac{\partial^2 w}{\partial x^2}\right)_{i,j} .$$

Substituting the finite difference equations (2.6a) and (2.6b) and solving for $w_{i,j-1}$:

$w_{i,j-1} =$



(2.10)

where: $\mu$ = Poisson's ratio

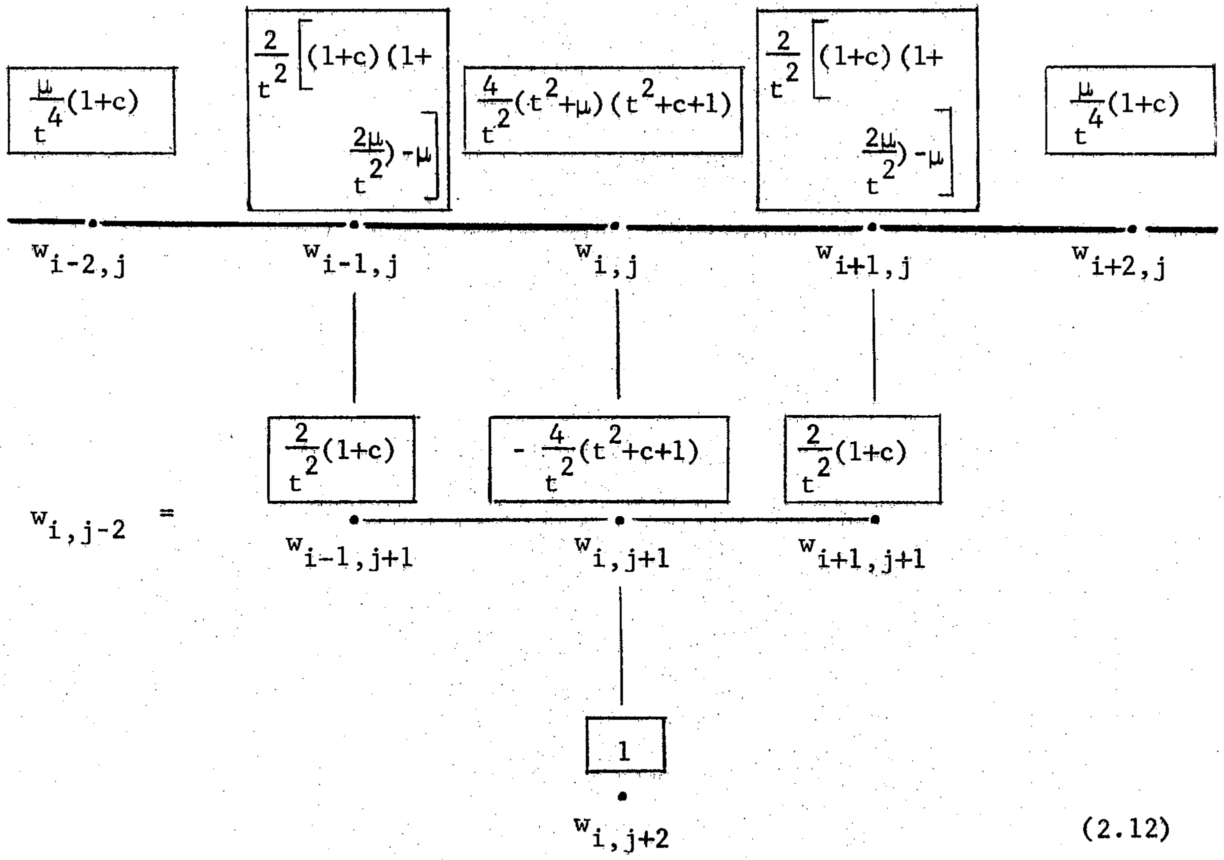
Equation (2.10) expresses the deflections at the imaginary points one row outside the free edge boundary in terms of points located on the plate.

From Table 3-1, reference (9), the finite difference approximation for the expressions appearing in equation (2.2) are:

$$\frac{\partial^3 w}{\partial y^3} = \frac{1}{2\Delta y^3}(w_{i,j+2} - 2w_{i,j+1} + 2w_{i,j-1} - w_{i,j-2}) \tag{2.11a}$$

$$\frac{\partial^3 w}{\partial x^2 \partial y} = \frac{1}{2\Delta x^2 \Delta y}(w_{i+1,j+1} - w_{i+j,j-1} - 2w_{i,j+1} + 2w_{i,j-1} - w_{i-1,j-1} + w_{i-1,j+1}) \tag{2.11b}$$

Substitute equations (2.11a) and (2.11b) into equation (2.2) and solve for $w_{i,j-2}$, thus

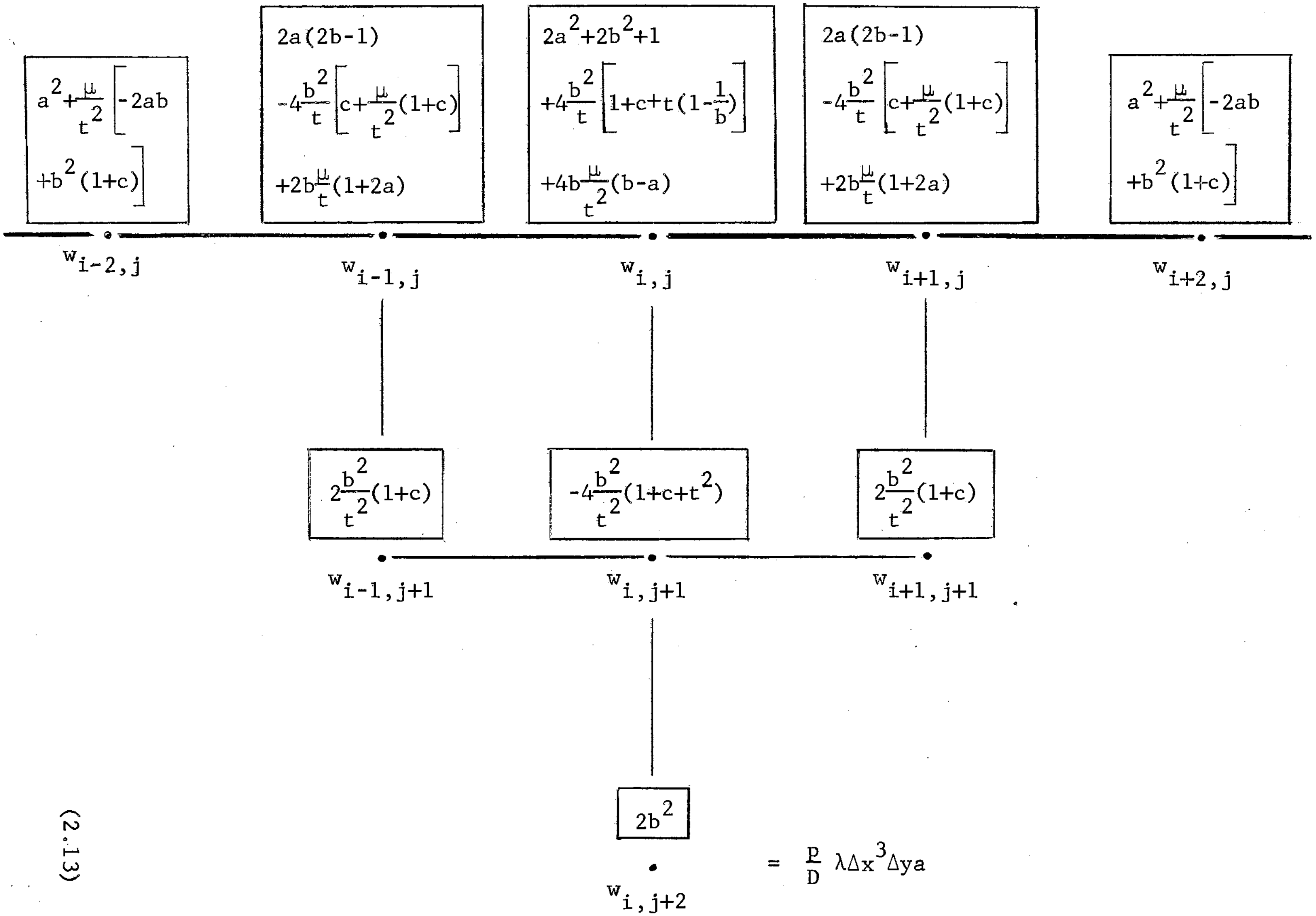


(2.12)

where: $c = 1 - \mu$

Equation (2.12) expresses the deflection of an imaginary point two rows outside the free edge boundary in terms of points located on the plate.

Substituting equation (2.12) for the point i,j-2 and equation (2.10) for the points i-1,j-1, i,j-1, and i+1,j-1 in equation (2.9), the resulting expression is presented in symbolic form in Figure 2.6.



(2.13)

Figure 2.6

Diagrammatical Form of First-Order Finite Difference Operator for $\nabla^4 w$ at a Boundary Point

2.6. <u>One Row Removed from Free Edge - Point i,j</u>. The point one row removed from the free edge will be any point lying on a row a distance $\Delta y$ from the free edge boundary (Figure 2.7).

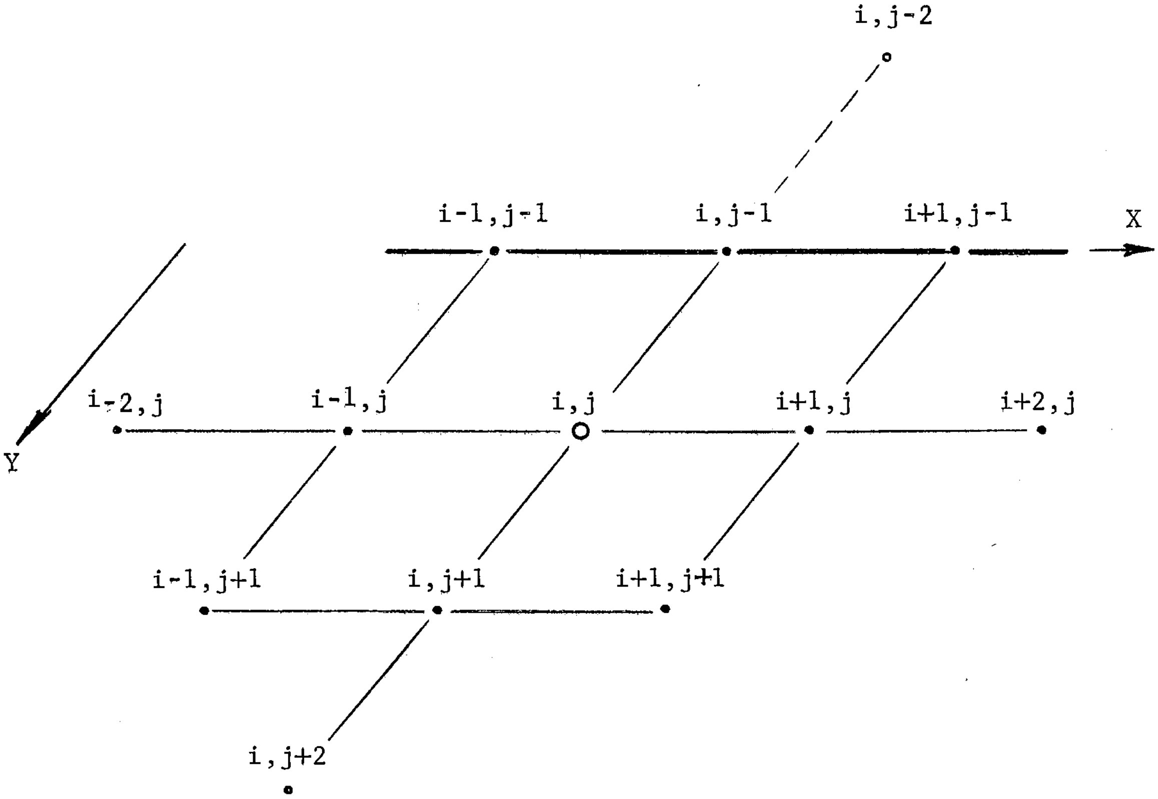
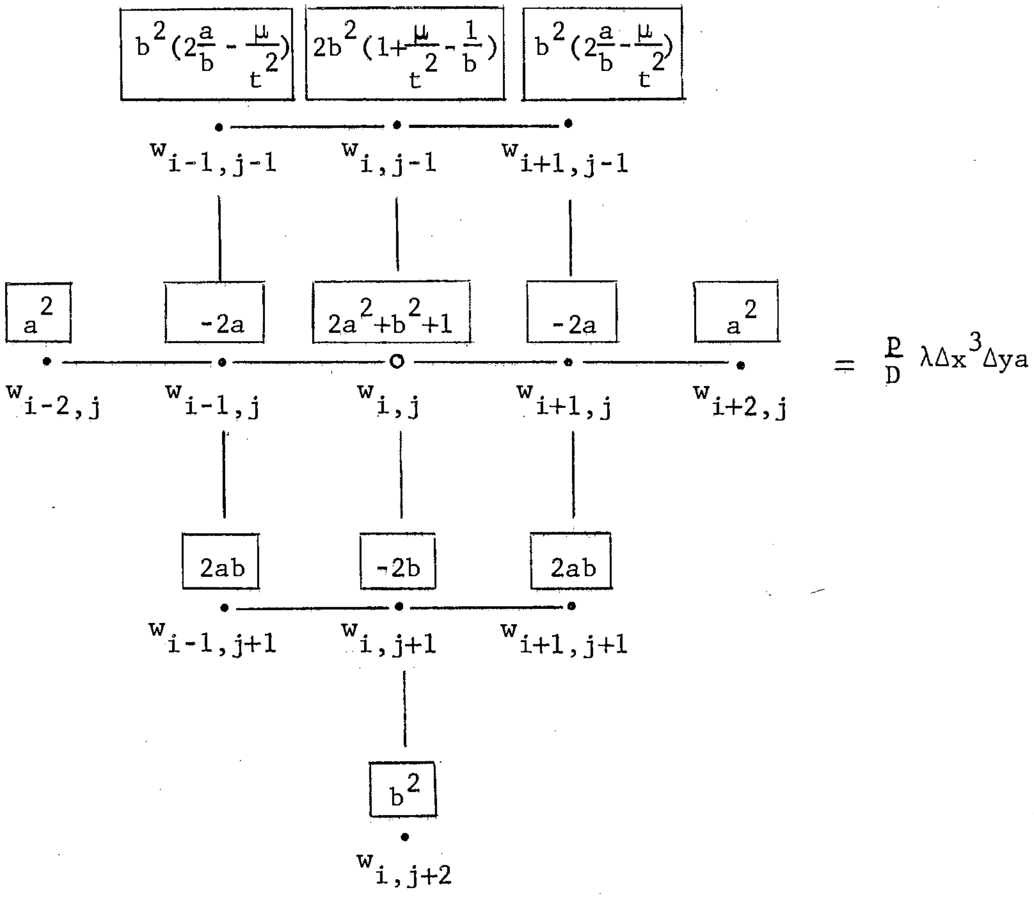


Figure 2.7

Any Point i,j One Row Removed from the Free Edge

Substituting equation (2.10) for the imaginary point i,j-2 in equation (2.9) yields the expression illustrated in Figure 2.8.



(2.14)

Figure 2.8

Diagrammatical Form of First-Order Finite Difference Operator for $\nabla^4 w$ at a Point One Row Removed from Free Edge

2.7. Corner Point - i,j. The corner point will be defined as a point positioned a distance Δx from the simply supported edge and lying on the free edge of the plate (Figure 2.9). Only four such points can exist in any one panel.

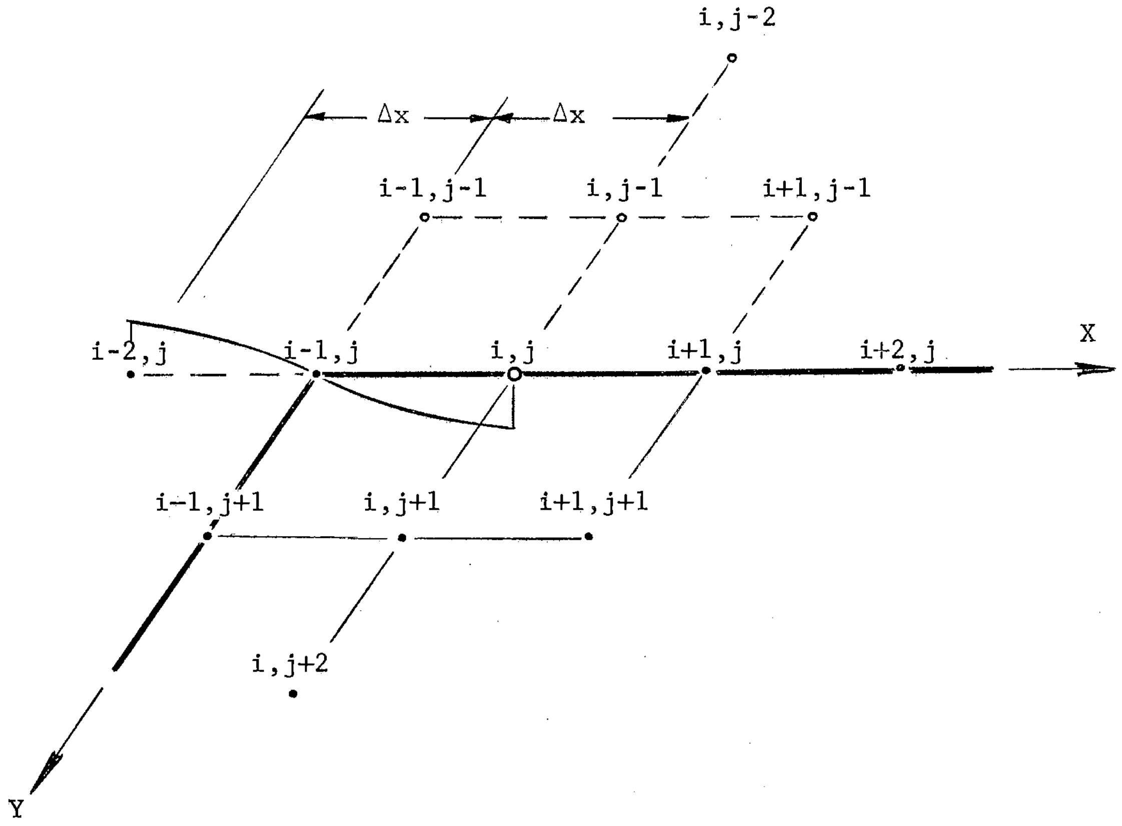


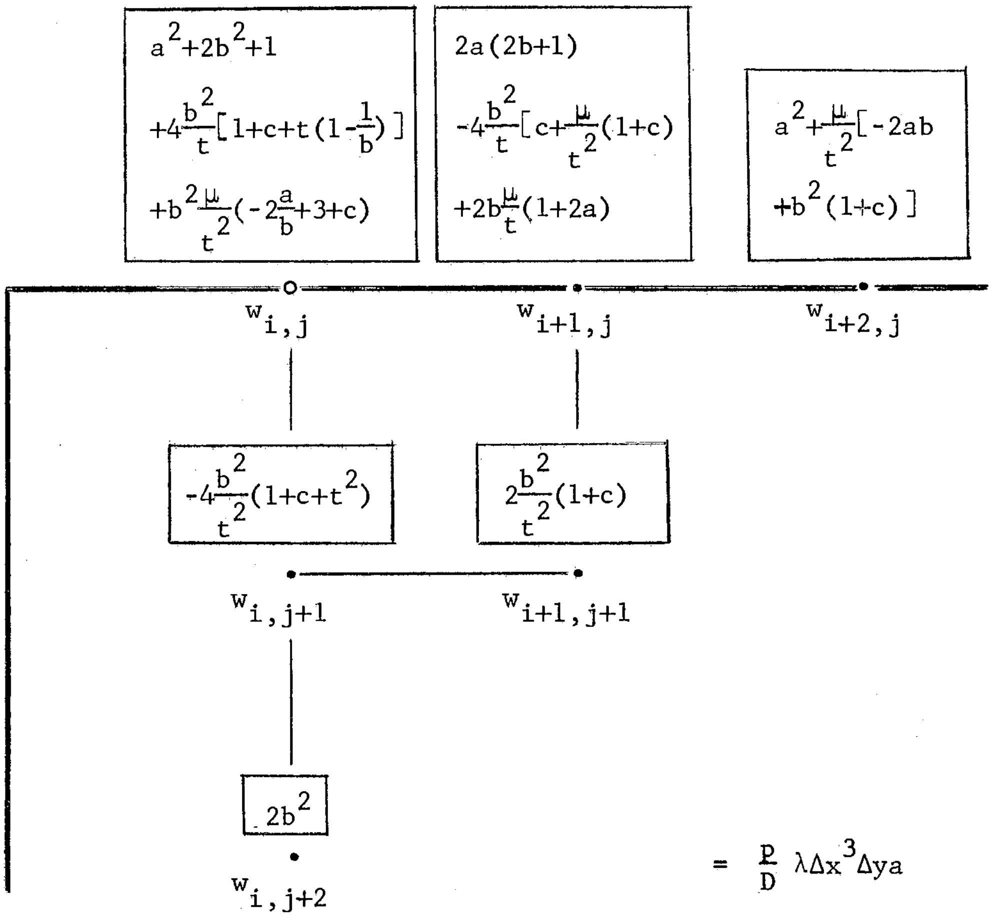
Figure 2.9

Corner Point i,j on Free Edge

Points i-1,j and i-1,j+1, located on the simply supported edge, will have zero deflection. Point i-2,j is similar to point i,j since they are both Δx distance from the simply supported edge. The relationship for an imaginary point across a simply supported boundary from a real point is:

$$-w_{i-1,j} = w_{i+1,j} \qquad (2.15)$$

Equation (2.13) is modified to meet the conditions of the corner point by applying equation (2.15) to the deflection at point i-2,j. The simplified expression is presented in symbolic form in Figure 2.10.



(2.16)

Figure 2.10

Diagrammatical Form of the First-Order Finite Difference Operator of $\nabla^4 w$ at a Corner Point

2.8. One Row Removed from Corner - Point i,j. A point one row removed from the corner will be defined as a point which is a distance $\Delta x$ from the simply supported boundary and a distance $\Delta y$ from the free edge boundary (Figure 2.11). Only four such points can exist in any one panel.

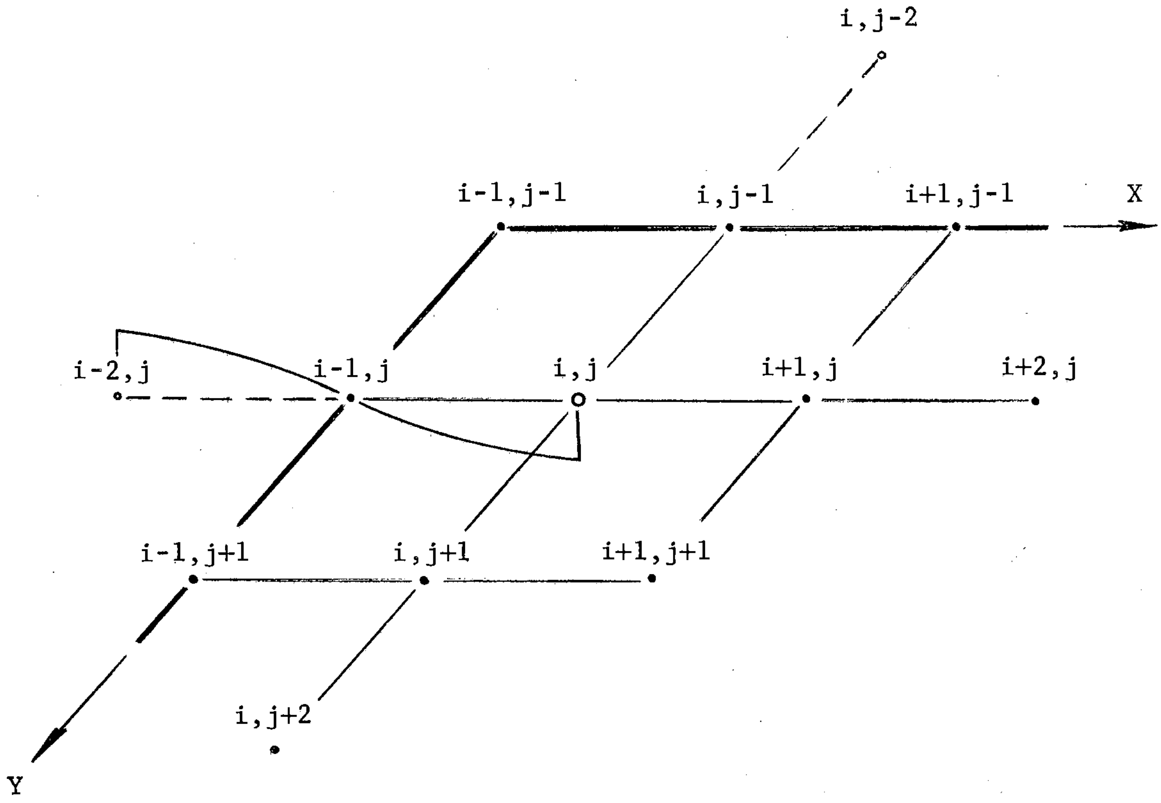
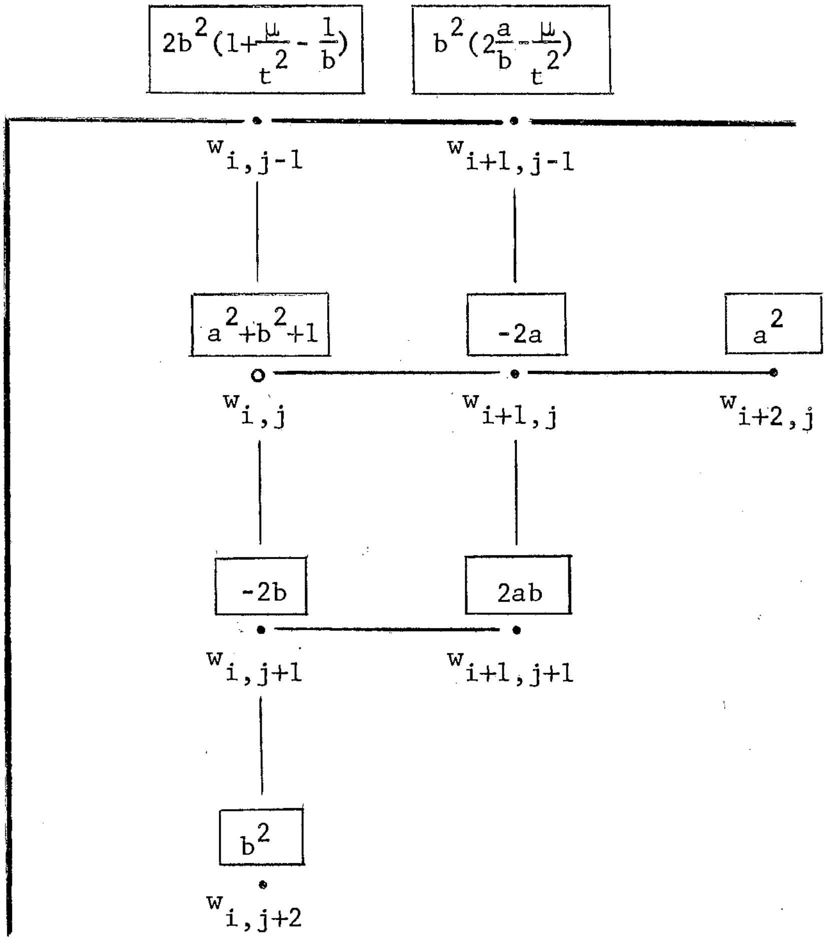


Figure 2.11

Point i,j - One Row Removed from Corner

Equation (2.14) will be modified to satisfy the conditions for a point one row removed from the corner by substituting equation (2.15) for the deflection at point i-2,j. This equation is illustrated in symbolic form in Figure 2.12.



$$= \frac{p}{D} \lambda \Delta x^3 \Delta y a \tag{2.17}$$

Figure 2.12

Diagrammatical Form of the First-Order Finite Difference Operator $\nabla^4 w$ at a Point One Row Removed from the Corner

2.9. <u>One Row Removed from Simply Supported Edge - Point i,j</u>. A point one row removed from the simply supported edge will be defined as a point which is at least $2\Delta y$ from a corner point and is a $\Delta x$ distance from the simply supported edge (Figure 2.13).

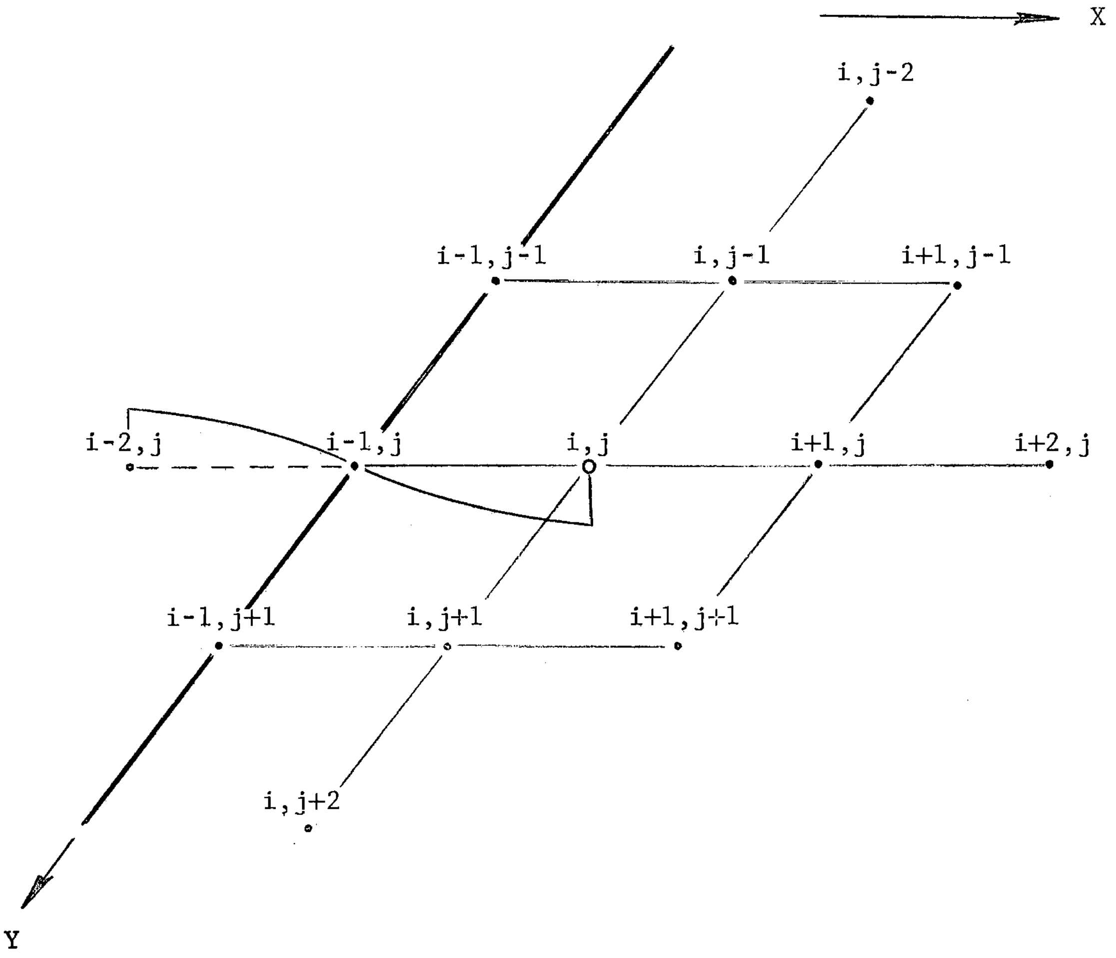
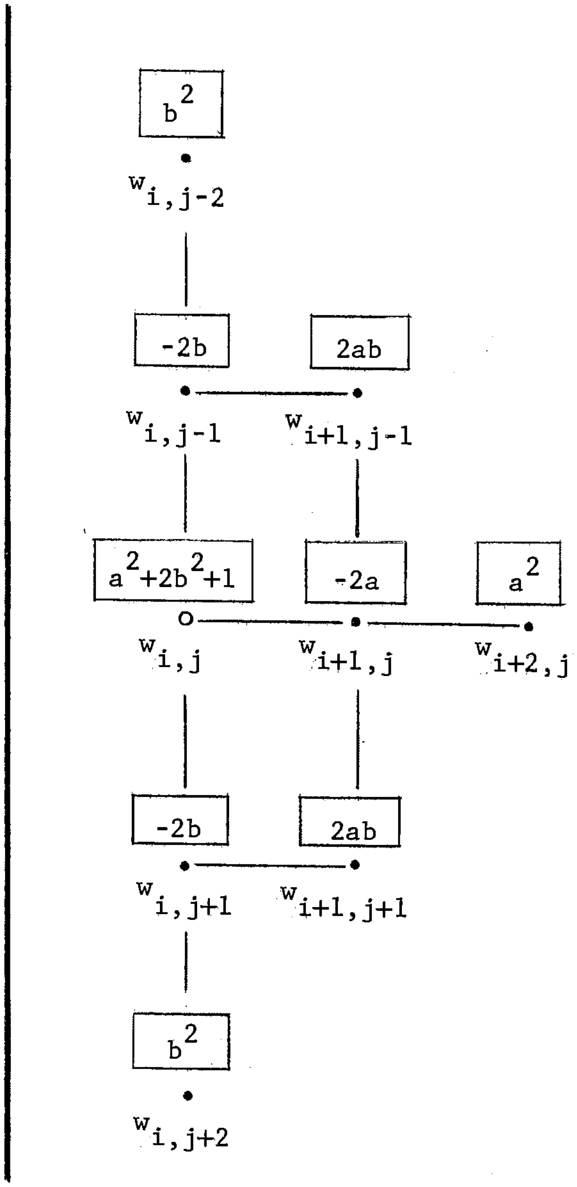


Figure 2.13

Point i,j - One Row Removed from Simply Supported Edge

Modifying equation (2.9) to meet the conditions of a point one row removed from a simply supported edge by substituting equation (2.15) for the deflection at point i-2,j, the following expression is illustrated in symbolic form in Figure 2.14.

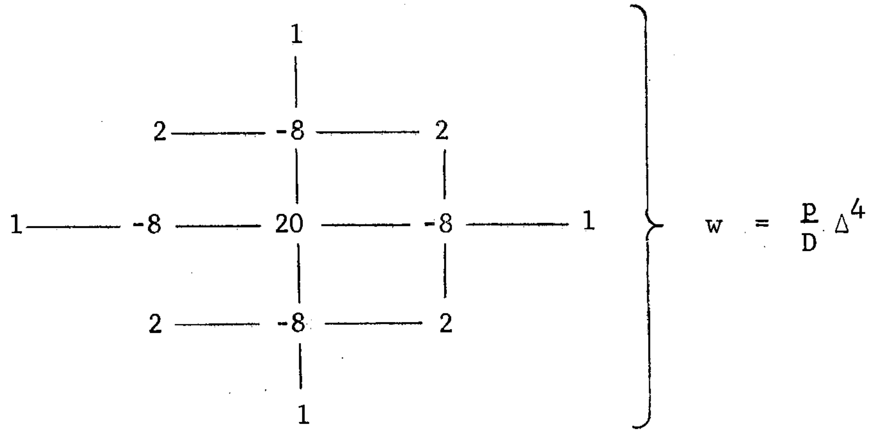


$$= \frac{p}{D} \lambda \Delta x^3 \Delta y a \qquad (2.18)$$

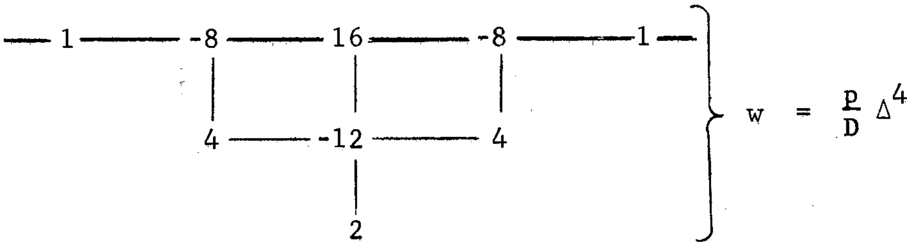
Figure 2.14

Diagrammatical Form of First-Order Finite Difference Operator for $\nabla^4 w$ at a Point One Row Removed from the Simply Supported Boundary

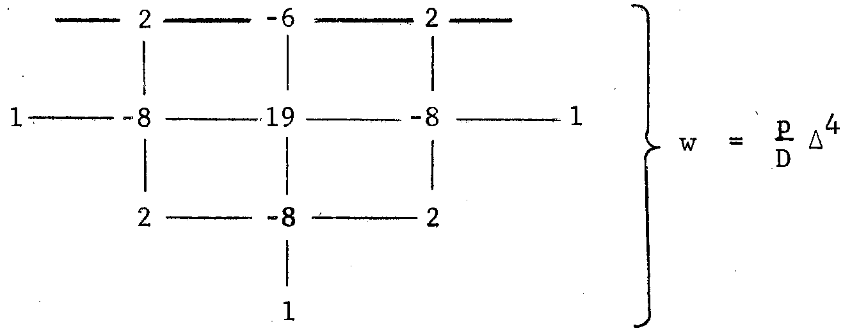
2.10. <u>Diagrammatical Form of the Deflection Equations</u>. Setting $\Delta x = \Delta y = \Delta$ and $\mu = 0$, the equations developed in this chapter may be presented as illustrated in Figures 2.15a and 2.15b.



Typical Interior Point



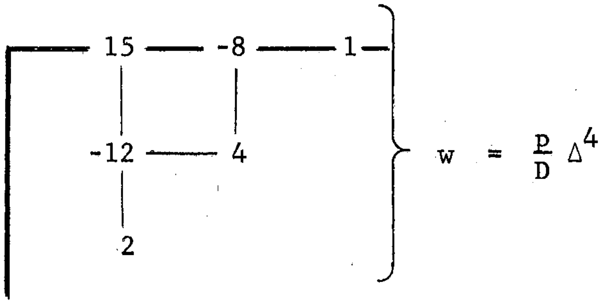
Boundary Point on the Free Edge



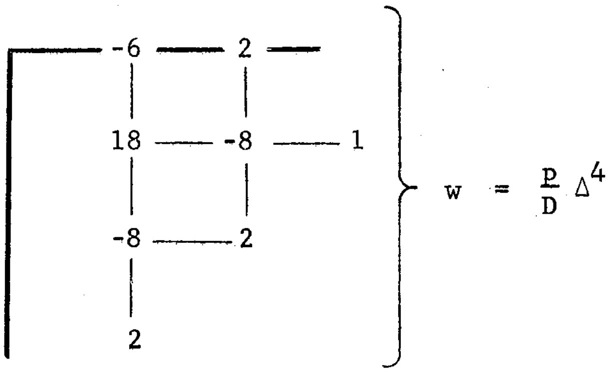
A Point One Row Removed from the Free Edge

Figure 2.15a

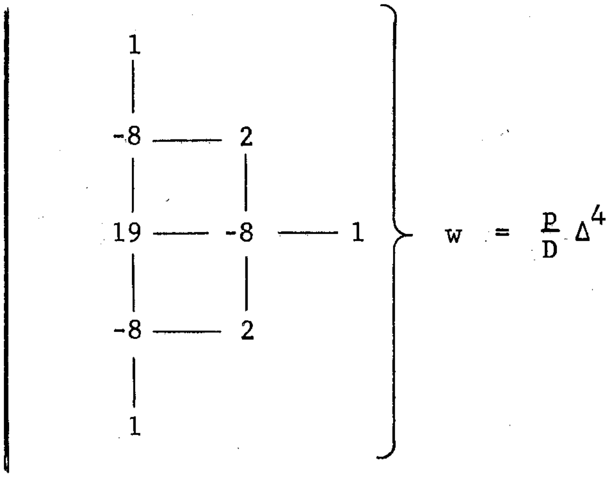
Diagrammatical Form of the Deflection Equations



Corner Point



A Point One Row Removed from the Corner Point



A Point One Row Removed from the Simply Supported Edge

Figure 2.15b

Diagrammatical Form of the Deflection Equations

# CHAPTER III

## INFLUENCE COEFFICIENTS FOR DEFLECTION

3.1. Introduction. From the expressions derived in Chapter II, a deflection equation can be written for any point on the plate. The coefficients (a, b, c) appearing in each term of the deflection equation can be evaluated for any ratio of $\Delta x$ to $\Delta y$.

Due to the laborious procedure to solve for the influence coefficients from the deflection equations by the use of a desk calculator, the IBM 650 electronic computer at the Oklahoma State University Computing Center was employed to perform the computational labor.

3.2. Procedure. The following definitions will be presented:

$P_{k,1}$ = a load at point k,1

$w_{i,j}^{k,1}$ = the deflection at point i,j due to a load at point k,1

$\eta_{i,j}^{k,1}$ = the influence coefficient for deflection at point i,j due to a load at point k,1

By application of the three definitions, the following expression is developed:

$$w_{i,j}^{k,1} = P_{k,1} \frac{\Delta x \Delta y}{D} \eta_{i,j}^{k,1} \qquad (3.1)$$

where: $P_{k,1} = p\Delta x\Delta y$

$\Delta x\Delta y$ = the area at point k,1

D = flexural rigidity of the plate

If a load p were applied to each point within the plate and if a deflection equation, as outlined in Chapter II, were written at each point, the following equations would be developed:

$$k_{1,1}w_1 + k_{2,1}w_1 + k_{3,1}w_3 + \ldots + k_{n,1}w_n = \frac{p_1}{D}\Delta^4$$

$$k_{1,2}w_1 + k_{2,2}w_2 + k_{3,2}w_3 + \ldots + k_{n,2}w_n = \frac{p_2}{D}\Delta^4$$

$$k_{1,3}w_1 + k_{2,3}w_3 + k_{3,3}w_3 + \ldots + k_{n,3}w_n = \frac{p_3}{D}\Delta^4$$

$$\vdots$$

$$k_{1,n}w_1 + k_{2,n}w_2 + k_{3,n}w_3 + \ldots + k_{n,n}w_n = \frac{p_n}{D}\Delta^4$$

where: k's are the evaluated coefficients (a, b, and c) of each deflection term;

$p_{1,2,\ldots n}$ is the load intensity in lbs. per sq. ft.

A matrix notation of the deflection equation may be written as:

$$\begin{bmatrix} k_{1,1} & k_{2,1} & k_{3,1} & \cdots & k_{n,1} \\ k_{1,2} & k_{2,2} & k_{3,2} & \cdots & k_{n,2} \\ k_{1,3} & k_{2,3} & k_{3,3} & \cdots & k_{n,3} \\ \vdots & & & & \\ k_{1,n} & k_{2,n} & k_{3,n} & \cdots & k_{n,n} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{bmatrix} \left(\frac{\Delta^4}{D}\right)$$

Simplifying the matrix notation to symbolic form, the following is obtained:

$$[K][w] = [p]\,\frac{\Delta^4}{D}$$

Multiplying both sides by the inverse of matrix K, $[K]^{-1}$,

$$[K][K]^{-1}[w] = [K]^{-1}[p] \frac{\Delta^4}{D}$$

where: $[K][K]^{-1} = I$, the identity matrix

$[p] \frac{\Delta^4}{D}$ = the load term matrix.

Solving for $[w]$,

$$[w] = [K]^{-1} [p] \frac{\Delta^4}{D} \tag{3.2}$$

Consider a unit load ($P = p\Delta x\Delta y = 1$) acting on a plate with unit flexural rigidity ($D = 1$) and over an area of unity ($\Delta x = \Delta y = 1$), then equation (3.1) becomes:

$$w^k_{i,j} = \eta^k_{i,j} \frac{\Delta x \Delta y}{D} \tag{3.1a}$$

Applying the same concept of unity to equation (3.2), the following is obtained:

$$[w] = [K]^{-1}\left(\frac{\Delta^2}{D}\right) \tag{3.2a}$$

The deflection at any point i,j contained in equation (3.2a) is equal to the product of the inverse coefficient of point i,j contained in the inverse matrix K multiplied by the unit load ($P_{k,1} = 1$) and the constants; thus the matrix would become

$$\begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ w_{i,j} \\ \cdot \\ \cdot \\ w_n \end{bmatrix} = \begin{bmatrix} k_{1,1} & k_{2,1} & \cdots & k_{i,1} & \cdots & k_{n,1} \\ \cdot & & & & & \\ \cdot & & & & & \\ k_{1,j} & k_{2,j} & \cdots & k_{i,j} & \cdots & k_{n,j} \\ \cdot & & & & & \\ \cdot & & & & & \\ k_{1,n} & k_{2,n} & \cdots & k_{i,n} & \cdots & k_{n,n} \end{bmatrix} \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ P_{k,1} = 1 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \left(\frac{\Delta^2}{D}\right)$$

where: $k_{1,1}$ to $k_{n,n}$ are the inverted matrix coefficients of deflections $w_1$ to $w_n$.

From the above matrix, one can conclude that:

$$w_1^{k,1} = k_{i,1} \frac{\Delta^2}{D}$$
$$w_{i,j}^{k,1} = k_{i,j} \frac{\Delta^2}{D} \qquad (3.3)$$
$$w_n^{k,1} = k_{i,n} \frac{\Delta^2}{D}$$

The particular value which is of interest to the discussion is $w_{i,j}^{k,1} = \frac{\Delta^2}{D} k_{i,j}$. Equation (3.1a) shows that $w_{i,j}^{k,1} = \eta_{i,j}^{k,1} \frac{\Delta x \Delta y}{D}$, and the conclusion may be reached that

$$\eta_{i,j}^{k,1} = k_{i,j} \qquad (3.3a)$$

i.e., the influence coefficient at point i,j due to a unit load at point k,l is equal to the inverse coefficient of point i,j of the inverted deflection equation matrix multiplied by a unit load applied at point k,1.

From the procedure outlined above, it is obvious that by applying a unit load at a point on the plate, influence coefficients can be determined at all other points and at the point of load application from the coefficients of the inverted deflection equation matrix. By applying a unit load at another point on the plate, an additional set of influence coefficients are determined, and so on until the unit load has been applied to all points on the plate.

3.3. Inversion of Deflection Equation Matrix for Influence Coefficients. Due to the geometry and the method of support of the basic plate structure, shown in Figure 1.1, the basic structure can be divided by two axes, each of which pass through the mid-point of opposite sides of the plate (Figure 3.1).

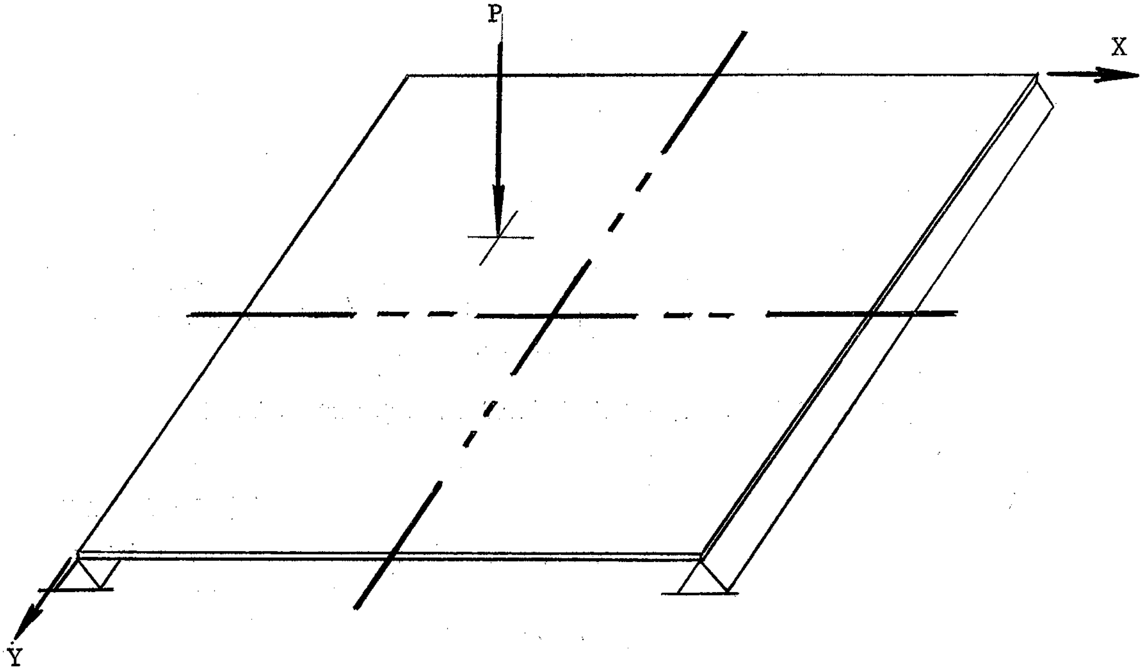


Figure 3.1

Basic Structure Divided by Axes of Symmetry

If a load (P) is applied to the plate at any point other than at the intersection of the axes, the system will be geometrically symmetrical but unsymmetrically loaded. In such a situation, the method of resolution and superposition greatly simplifies the load case. The theorem of resolution states that:

> Any geometrically symmetrical and unsymmetrically loaded two dimensional system may always be resolved into at least one symmetrically and three antisymmetrically loaded systems. (9)

A typical resolution is shown in Figure 3.2.

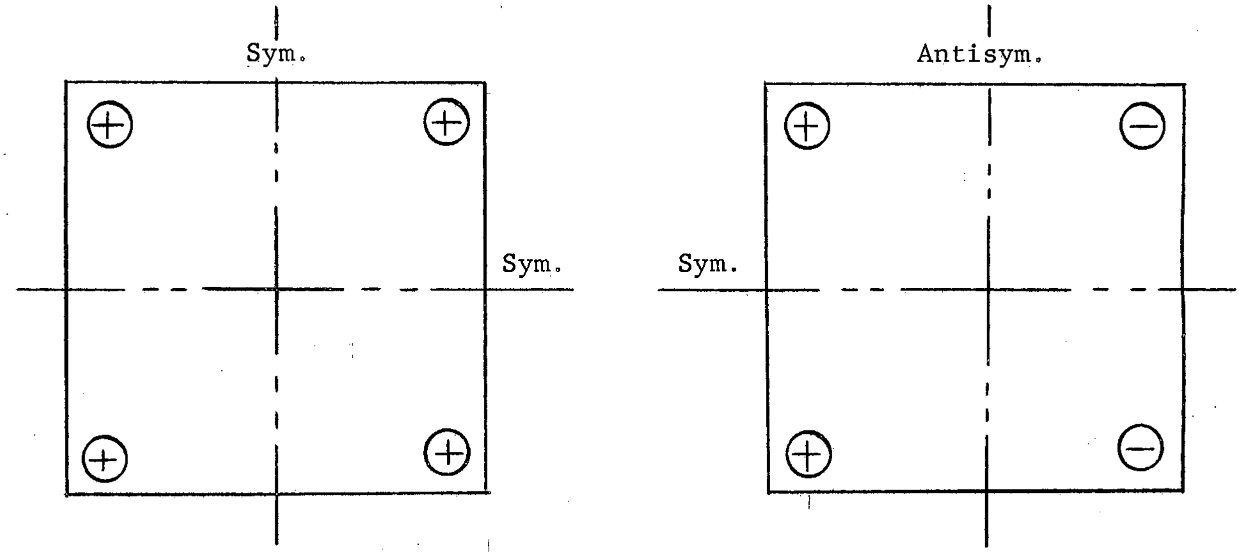


Figure 3.2a

CASE I

Figure 3.2b

CASE II

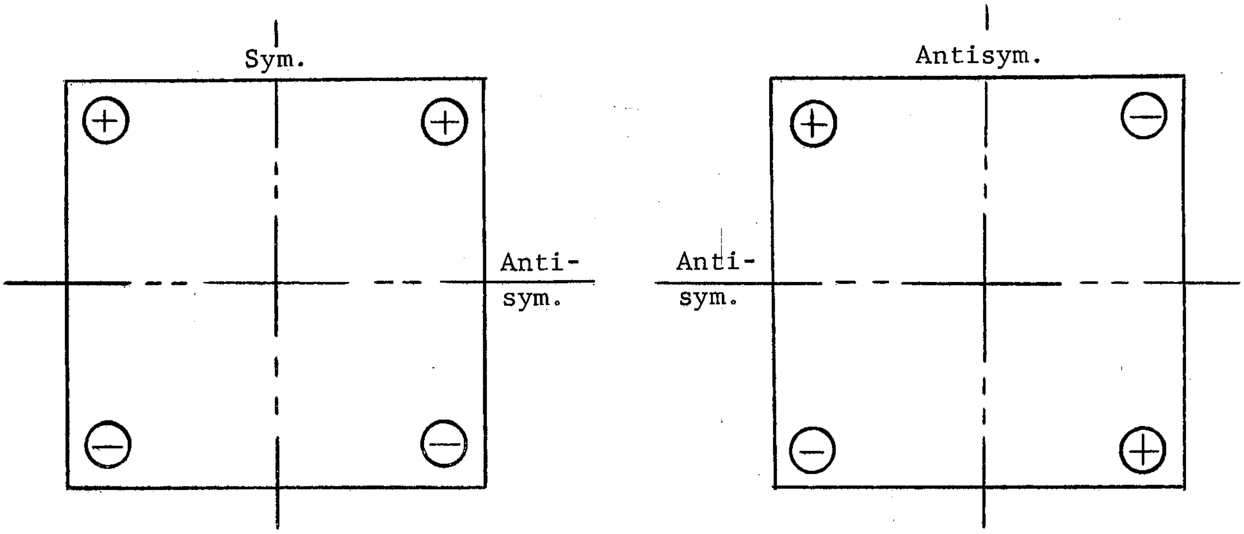


Figure 3.2c

CASE III

Figure 3.2d

CASE IV

Figure 3.2a,b,c,d

Resolution of a Two Dimensional System Loaded in the Upper-Left Quadrant into Symmetrical and Antisymmetrical Cases

The theorem of superposition states that:

> The algebraic sum of four or more results (each corresponding to one resolved system described above) at a certain point is always equal to the true result of the initial system at that point. (9)

The deflection equations derived in Chapter II will be applied to the basic structure containing a 56-point set (Figure 3.3).

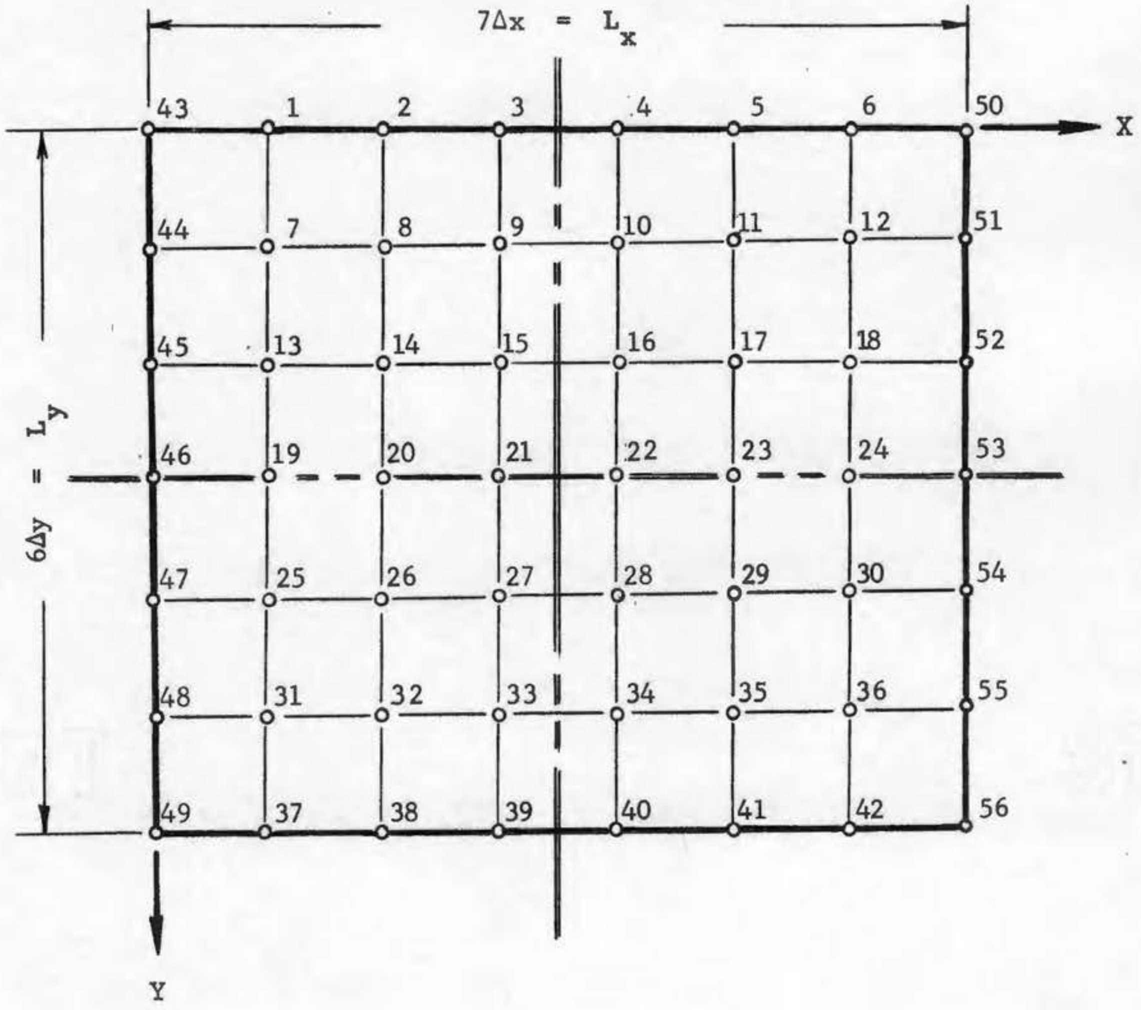


Figure 3.3

Fifty-Six Point Set

An examination of Figure 3.3 will indicate that there are 42 points (1 to 42) on the plate at which deflection will occur and there are 14 points (43 to 56) on the simply supported edge of the plate at which no deflection can occur. Also, it will be noted that there are 12 points in each quadrant, three of which are located on the axis of symmetry.

By proper use of the symmetrical and antisymmetrical cases illustrated in Figure 3.2, the deflection equations need to be written only for the points contained in the upper left quadrant. Special precaution must be exercised when the deflection equation is applied to a point near the symmetrical or antisymmetrical axis. For illustrative purposes, consider the application of equation (2.9) to the interior point for a solution to Case II loading (Figure 3.4).

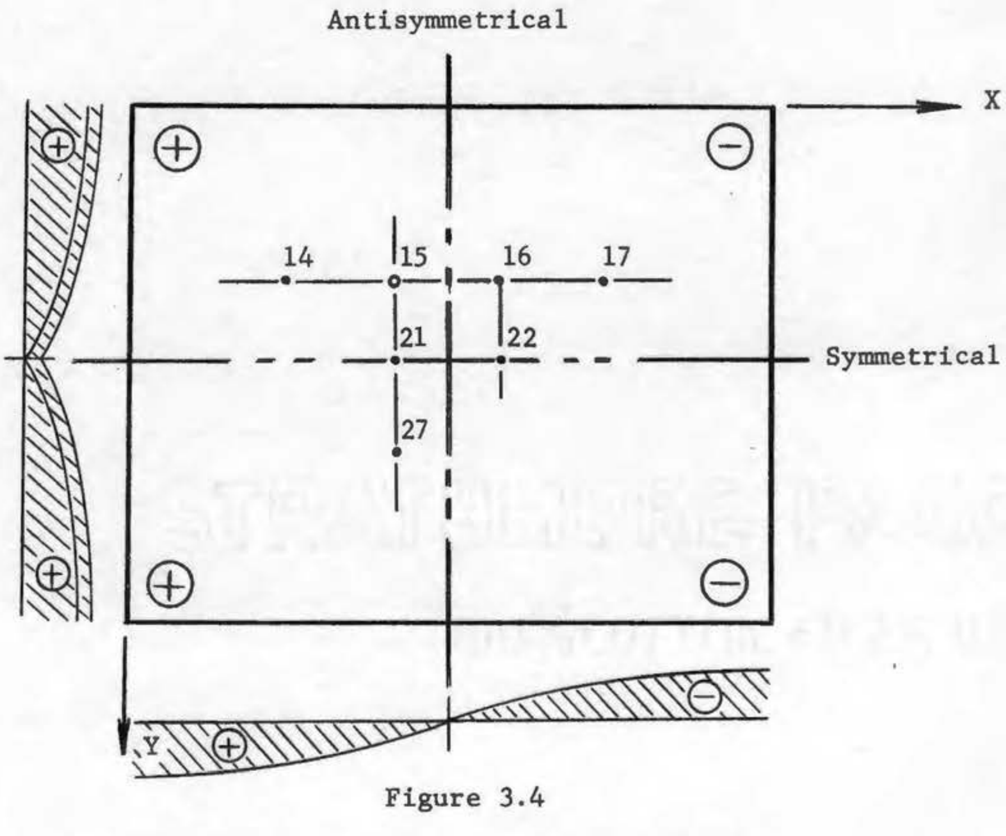


Figure 3.4

Point 15 - Case II, Illustrative Example

An examination of the points shown in Figure 3.4 will indicate that in the X direction, point 16 is similar to point 15, point 17 is similar to point 14, and in the Y direction point 27 is similar to point 15. Looking to the diagrammatical illustration of equation (2.9) in Figure 2.15a, one concludes that the coefficients of the deflections at the above-mentioned points are:

$$\ldots + w_{14}(-2a) + w_{15}(2a^2 + 2b^2 + 1) + w_{16}(-2a)$$
$$+ w_{17}(a^2) + w_{21}(-2b) + w_{22}(2ab) + w_{27}(b^2) + \ldots$$

Coefficients of points carried across an antisymmetrical axis will subtract from the coefficients of similar points. Conversely, the coefficients of points carried across a symmetrical axis will add to the coefficients of similar points. That portion of equation (2.9) containing points 14, 15, and 21 now becomes

$$\ldots + w_{14}(-2a - a^2) + w_{15}[2a^2 + 2b^2 + 1 - (-2a) + b^2]$$
$$+ w_{21}(-2b - 2ab) + \ldots$$

Again consider the application of equation (2.9) to the interior point 15 for a solution of Case III loading (Figure 3.2c). This particular load case is discussed to illustrate the condition of zero deflection coefficients for points 19, 20, and 21 (Figure 3.3) which are located on the antisymmetrical axis. Following the general outline of the above discussion for Case II loading, the portion of equation (2.9) containing points 14, 15, and 21 now becomes

$$\ldots w_{14}(-2a + a^2) + W_{15}[2a^2 + b^2 + 1 + (-2a) - b^2]$$
$$+ w_{21}\ (0) + \ldots$$

A proper substitution of the finite difference equations expressed in operator form (Figures 2.15a and 2.15b) is made at all points in the upper left quadrant to satisfy independently the load conditions of Cases I, II, III, IV. The IBM 650 electronic computer performed the inversion of each matrix corresponding to a load case.

The influence coefficients for deflections, $\eta$ values, for the twelve points contained in the upper-left quadrant are determined for unit loads applied to all interior points on the plate by a proper summation of values contained within the four inverted matrices. The inverted matrices corresponding to deflection equations written for points contained in the upper-left quadrant for load Cases I, II, III, and IV shall be referred to as inverted matrices 1, 2, 3, and 4, respectively. The following figures and discussion will outline the procedure used to obtain the influence coefficients.

The following sets of figures illustrate the theorem of resolution applied to a load placed in each quadrant.

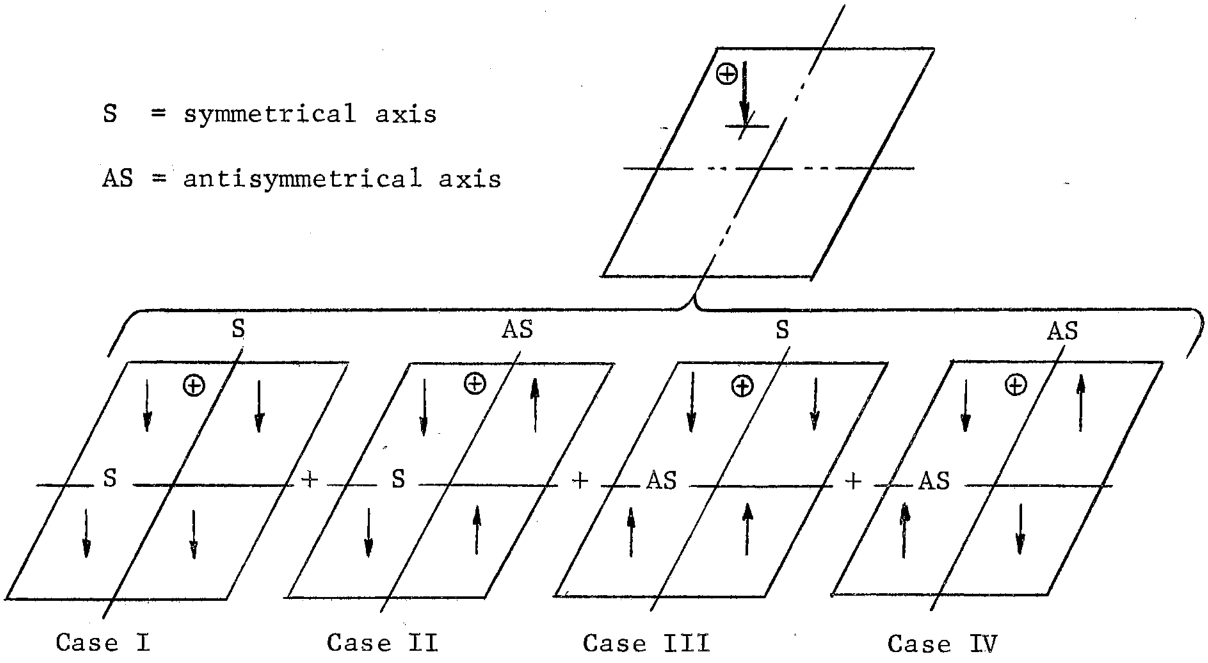


Figure 3.5

Resolution of Unit Load in Upper-Left Quadrant to Determine Influence Coefficients in Upper-Left Quadrant

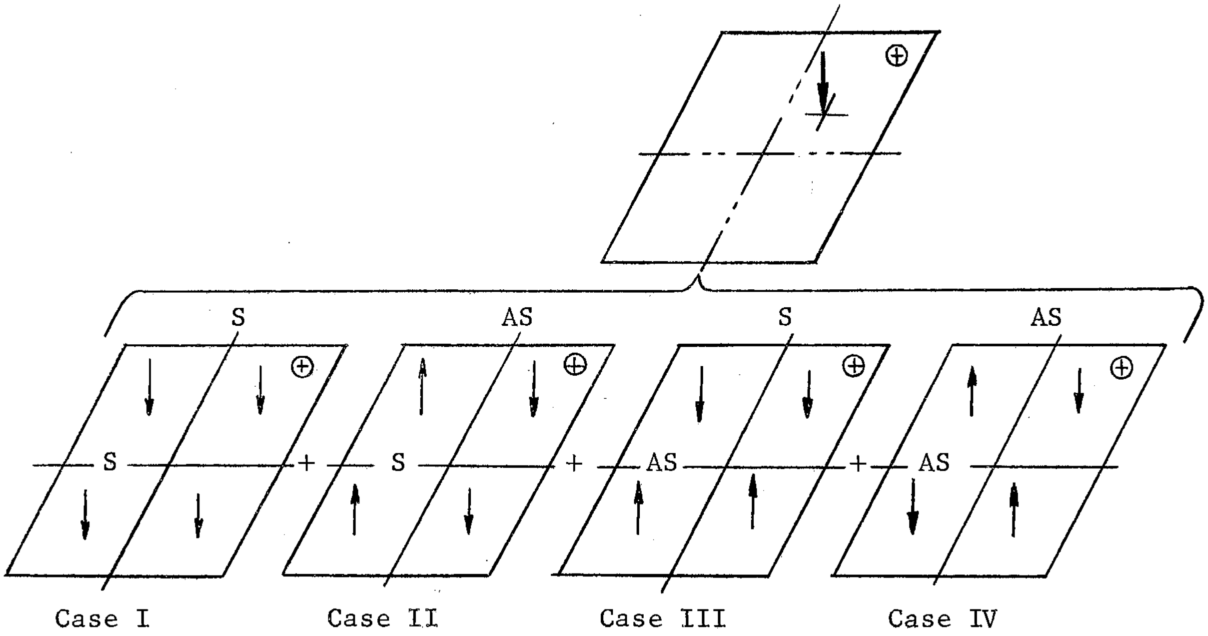


Figure 3.6

Resolution of Unit Load in Upper-Right Quadrant to Determine Influence Coefficients in Upper-Left Quadrant

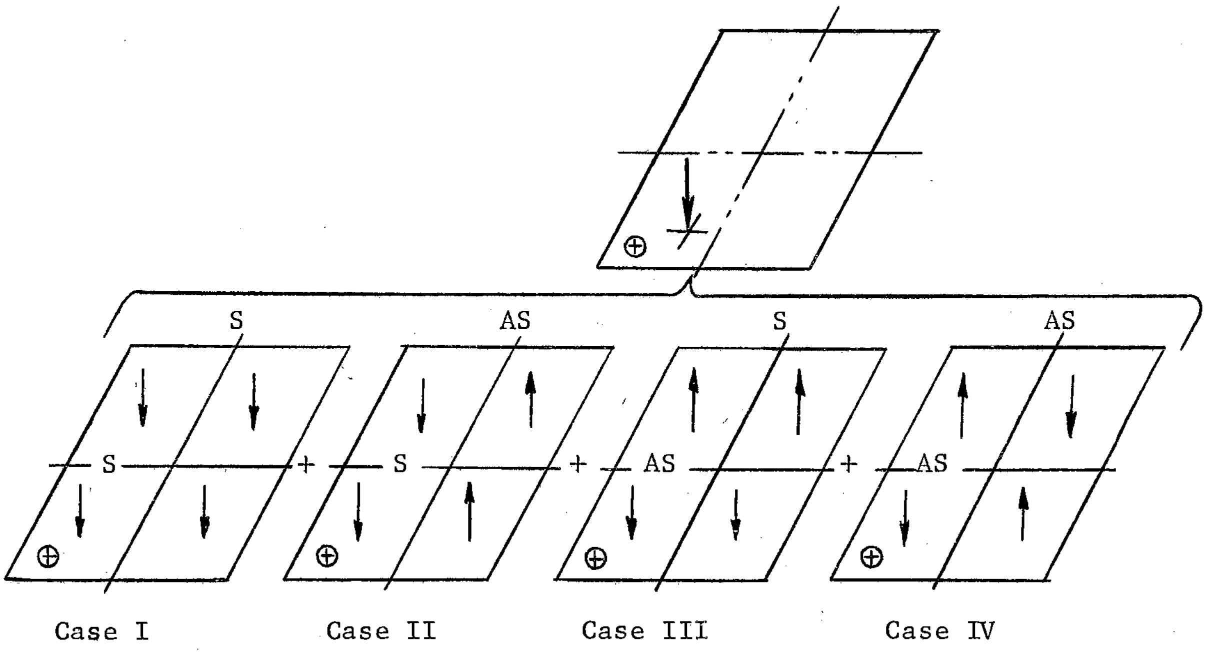


Figure 3.7

Resolution of Unit Load in Lower-Left Quadrant to Determine Influence Coefficients in Upper-Left Quadrant

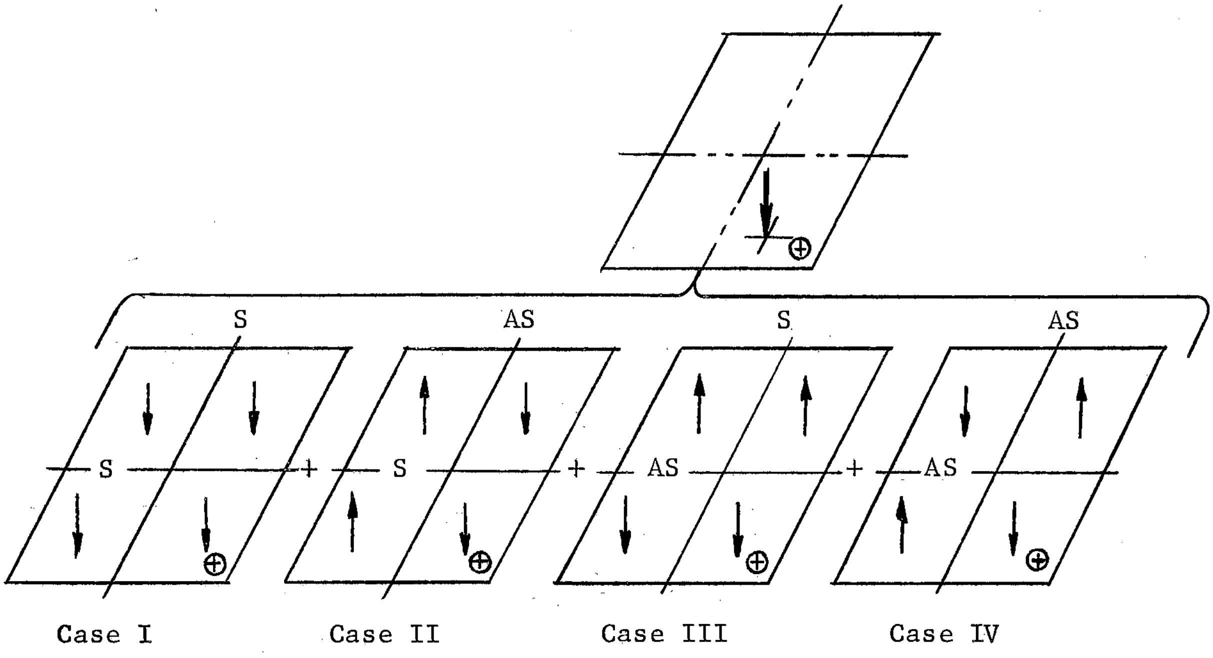


Figure 3.8

Resolution of Unit Load in Lower-Right Quadrant to Determine Influence Coefficients in Upper-Left Quadrant

Applying equation (3.3a) for a unit load at point k,l in a specific quadrant yields from each inverse matrix (1, 2, 3, 4) four inverse coefficients. Also, applying the theorem of superposition to each of the four load cases illustrated in the preceding figures, the influence coefficients for points located in the upper-left quadrants are found to be:

For unit load in upper-left quadrant:

$$\eta_{i,j}^{k,l} = \frac{1}{4} [(k_{i,j})_1 + (k_{i,j})_2 + (k_{i,j})_3 + (k_{i,j})_4]$$

For unit load in upper-right quadrant:

$$\eta_{i,j}^{k,l} = \frac{1}{4} [(k_{i,j})_1 - (k_{i,j})_2 + (k_{i,j})_3 - (k_{i,j})_4]$$

For unit load in lower-left quadrant:

$$\eta_{i,j}^{k,l} = \frac{1}{4} [(k_{i,j})_1 + (k_{i,j})_2 - (k_{i,j})_3 - (k_{i,j})_4]$$

For unit load in lower-right quadrant:

$$\eta_{i,j}^{k,l} = \frac{1}{4} [(k_{i,j})_1 - (k_{i,j})_2 - (k_{i,j})_3 + (k_{i,j})_4] .$$

The influence coefficients for points contained within the upper-right, lower-left, and lower-right quadrants are easily determined from the symmetrical condition of the basic structure. Figure 3.9 shows four typical points on the basic structure.

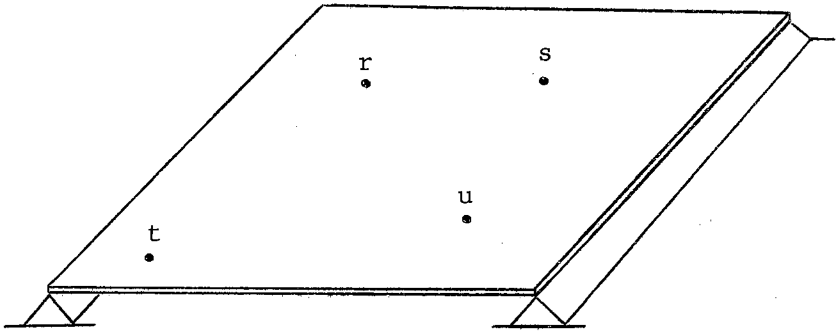


Figure 3.9

Four Typical Points on Basic Structure

The influence coefficient at point r due to a unit load at point u is equal to the influence coefficient at point u due to a unit load at point r; hence, the following equalities exist among the four typical points:

$$\eta_r^u = \eta_u^r \qquad \eta_t^s = \eta_s^t \qquad \eta_u^s = \eta_s^u$$

$$\eta_r^s = \eta_s^r \qquad \eta_t^u = \eta_u^t$$

$$\eta_r^t = \eta_t^r$$

By a similar procedure the following tables were completed to include influence coefficient values for all points on the plate due to a unit load applied at all points on the plate.

$\eta_{i,j}^{k,l}$ - Deflection Influence Coefficients for Basic Plate Structure

Unit Load Applied at Point

Deflection Influence Coefficient at Point

|  | k,l | 42 | 41 | 40 | 39 | 38 | 37 | 36 | 35 | 34 | 33 | 32 | 31 | 30 | 29 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| i,j | $_{i,j}^{k,l}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 42 | 1 | 0.745651 | 0.965342 | 0.957182 | 0.815400 | 0.587524 | 0.306528 | 0.505297 | 0.738024 | 0.774044 | 0.679202 | 0.497510 | 0.261801 | 0.346128 | 0.550358 |
| 41 | 2 | 0.965342 | 1.702833 | 1.780742 | 1.544706 | 1.121928 | 0.587524 | 0.738024 | 1.279341 | 1.417226 | 1.271554 | 0.941004 | 0.497510 | 0.550358 | 0.954867 |
| 40 | 3 | 0.957182 | 1.780742 | 2.295857 | 2.092790 | 1.544706 | 0.815400 | 0.774044 | 1.417226 | 1.776852 | 1.679028 | 1.271554 | 0.679202 | 0.608739 | 1.102142 |
| 39 | 4 | 0.815400 | 1.544706 | 2.092790 | 2.295857 | 1.780742 | 0.957182 | 0.679207 | 1.271554 | 1.679028 | 1.776852 | 1.417226 | 0.774044 | 0.551784 | 1.020799 |
| 38 | 5 | 0.587524 | 1.121928 | 1.544706 | 1.780742 | 1.702833 | 0.965342 | 0.497510 | 0.941004 | 1.271554 | 1.417226 | 1.279341 | 0.738024 | 0.412060 | 0.770860 |
| 37 | 6 | 0.306528 | 0.587524 | 0.815400 | 0.957182 | 0.965342 | 0.745651 | 0.261801 | 0.497510 | 0.679207 | 0.774044 | 0.738024 | 0.505297 | 0.219076 | 0.412060 |
| 36 | 7 | 0.505297 | 0.738024 | 0.774044 | 0.679207 | 0.497510 | 0.261801 | 0.493791 | 0.682001 | 0.705975 | 0.618680 | 0.453853 | 0.239178 | 0.374092 | 0.566046 |
| 35 | 8 | 0.738024 | 1.279341 | 1.417226 | 1.271554 | 0.941004 | 0.497510 | 0.682001 | 1.199766 | 1.300681 | 1.159829 | 0.857858 | 0.453853 | 0.566046 | 0.983728 |
| 34 | 9 | 0.774044 | 1.417226 | 1.776852 | 1.679028 | 1.271554 | 0.679207 | 0.705975 | 1.300681 | 1.653620 | 1.539860 | 1.159829 | 0.618680 | 0.609636 | 1.111155 |
| 33 | 10 | 0.679202 | 1.271554 | 1.679028 | 1.776852 | 1.417226 | 0.774044 | 0.618680 | 1.159829 | 1.539820 | 1.653620 | 1.300681 | 0.705975 | 0.545109 | 1.013953 |
| 32 | 11 | 0.497510 | 0.941004 | 1.271554 | 1.417226 | 1.279341 | 0.738024 | 0.453853 | 0.857858 | 1.159829 | 1.300681 | 1.199766 | 0.682001 | 0.404318 | 0.759399 |
| 31 | 12 | 0.261801 | 0.497510 | 0.679202 | 0.774044 | 0.738024 | 0.505297 | 0.239178 | 0.453853 | 0.618680 | 0.705975 | 0.682001 | 0.493791 | 0.214290 | 0.404318 |
| 30 | 13 | 0.346128 | 0.550358 | 0.608739 | 0.551784 | 0.412060 | 0.219076 | 0.374092 | 0.566046 | 0.609636 | 0.545109 | 0.404318 | 0.214290 | 0.413563 | 0.569530 |
| 29 | 14 | 0.550358 | 0.954867 | 1.102142 | 1.020799 | 0.770860 | 0.412060 | 0.566046 | 0.983728 | 1.111155 | 1.013953 | 0.759399 | 0.404318 | 0.569530 | 1.008629 |
| 28 | 15 | 0.608739 | 1.102142 | 1.366927 | 1.321217 | 1.020799 | 0.551784 | 0.609636 | 1.111155 | 1.388045 | 1.325445 | 1.013953 | 0.545109 | 0.595006 | 1.096486 |
| 27 | 16 | 0.551784 | 1.020799 | 1.321219 | 1.366929 | 1.102142 | 0.608739 | 0.545109 | 1.013953 | 1.325445 | 1.388045 | 1.111155 | 0.609636 | 0.526956 | 0.984766 |
| 26 | 17 | 0.412060 | 0.770860 | 1.020799 | 1.102142 | 0.954867 | 0.550358 | 0.404318 | 0.759399 | 1.013953 | 1.111155 | 0.983728 | 0.566046 | 0.389700 | 0.733334 |
| 25 | 18 | 0.219076 | 0.412060 | 0.551784 | 0.608739 | 0.550358 | 0.346129 | 0.214229 | 0.404318 | 0.545109 | 0.609636 | 0.566046 | 0.374092 | 0.206378 | 0.389700 |
| 24 | 19 | 0.246337 | 0.412585 | 0.475395 | 0.443426 | 0.337306 | 0.181180 | 0.278176 | 0.451798 | 0.508865 | 0.467602 | 0.352472 | 0.188411 | 0.330571 | 0.501240 |
| 23 | 20 | 0.412585 | 0.721732 | 0.856013 | 0.812700 | 0.624608 | 0.337306 | 0.451798 | 0.787040 | 0.919400 | 0.861336 | 0.656013 | 0.352472 | 0.501240 | 0.874030 |
| 22 | 21 | 0.475395 | 0.856013 | 1.059038 | 1.037193 | 0.812700 | 0.443428 | 0.508865 | 0.919400 | 1.139512 | 1.107811 | 0.861336 | 0.467602 | 0.543460 | 0.990678 |

Table 3.1

$\eta^{k,l}_{i,j}$ - Deflection Influence Coefficients for Basic Plate Structure

Unit Load Applied at Point

Deflection Influence Coefficient at Point

| | k,l | 28 | 27 | 26 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| i,j | k,l i,j | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| 42 | 1 | 0.608739 | 0.551784 | 0.412060 | 0.219076 | 0.246337 | 0.412585 | 0.475395 | 0.443428 | 0.337307 | 0.181180 | 0.183894 | 0.317376 | 0.376022 | 0.358409 |
| 41 | 2 | 1.102142 | 1.020799 | 0.770860 | 0.412060 | 0.412585 | 0.721732 | 0.856013 | 0.812701 | 0.624608 | 0.337307 | 0.317374 | 0.559917 | 0.675785 | 0.652742 |
| 40 | 3 | 1.366927 | 1.321219 | 0.020799 | 0.551784 | 0.475395 | 0.856013 | 1.059038 | 1.037194 | 0.812701 | 0.443428 | 0.376022 | 0.675785 | 0.836637 | 0.825705 |
| 39 | 4 | 1.321219 | 1.366927 | 1.102142 | 0.608739 | 0.443428 | 0.812701 | 1.037194 | 1.059038 | 0.856013 | 0.475395 | 0.358409 | 0.652742 | 0.825705 | 0.836637 |
| 38 | 5 | 1.020799 | 1.102142 | 0.954867 | 0.550358 | 0.337307 | 0.624608 | 0.812701 | 0.856013 | 0.721732 | 0.412585 | 0.276720 | 0.508328 | 0.652742 | 0.675785 |
| 37 | 6 | 0.551784 | 0.608739 | 0.550358 | 0.346128 | 0.181180 | 0.337307 | 0.443428 | 0.475395 | 0.412585 | 0.246337 | 0.149919 | 0.276720 | 0.358409 | 0.376022 |
| 36 | 7 | 0.609636 | 0.545109 | 0.404318 | 0.214290 | 0.278176 | 0.451798 | 0.508865 | 0.467602 | 0.352472 | 0.188411 | 0.213817 | 0.362554 | 0.422630 | 0.397832 |
| 35 | 8 | 1.111155 | 1.013953 | 0.759399 | 0.404318 | 0.451798 | 0.787040 | 0.919400 | 0.861336 | 0.656013 | 0.352472 | 0.362554 | 0.636446 | 0.760386 | 0.727178 |
| 34 | 9 | 1.388045 | 1.325445 | 1.013953 | 0.545109 | 0.508865 | 0.919400 | 1.139512 | 0.107811 | 0.861336 | 0.467602 | 0.422630 | 0.760386 | 0.940994 | 0.924574 |
| 33 | 10 | 1.325445 | 1.388045 | 1.111155 | 0.609636 | 0.467602 | 0.861336 | 1.107811 | 1.139512 | 0.919400 | 0.508865 | 0.397832 | 0.727178 | 0.924574 | 0.940994 |
| 32 | 11 | 1.013953 | 1.111155 | 0.983728 | 0.566046 | 0.352472 | 0.656013 | 0.861336 | 0.919400 | 0.787040 | 0.451798 | 0.304548 | 0.562020 | 0.727178 | 0.760386 |
| 31 | 12 | 0.545109 | 0.609636 | 0.566046 | 0.374092 | 0.188411 | 0.352472 | 0.467602 | 0.508865 | 0.451798 | 0.278176 | 0.164188 | 0.304548 | 0.397832 | 0.422630 |
| 30 | 13 | 0.595006 | 0.526956 | 0.389700 | 0.206378 | 0.330571 | 0.501240 | 0.543460 | 0.489438 | 0.365093 | 0.194178 | 0.260079 | 0.423886 | 0.479660 | 0.442686 |
| 29 | 14 | 1.096486 | 0.984766 | 0.733334 | 0.389700 | 0.501240 | 0.874030 | 0.990678 | 0.908552 | 0.683616 | 0.365093 | 0.423886 | 0.664738 | 0.866572 | 0.814449 |
| 28 | 15 | 1.398330 | 1.302864 | 0.984766 | 0.526956 | 0.543460 | 0.990678 | 1.239123 | 1.184856 | 0.908552 | 0.489438 | 0 479660 | 0.866572 | 1.074527 | 1.045887 |
| 27 | 16 | 1.302864 | 1.398330 | 1.096486 | 0.595006 | 0.489438 | 0.908552 | 1.184856 | 1.239123 | 0.990678 | 0.543460 | 0.442686 | 0.814449 | 1.045887 | 1.074527 |
| 26 | 17 | 0.984766 | 1.096486 | 1.008629 | 0.569530 | 0.365093 | 0.683616 | 0.908552 | 0.990678 | 0.874030 | 0.501240 | 0.334789 | 0.622001 | 0.814449 | 0.866572 |
| 25 | 18 | 0.526956 | 0.595006 | 0.569530 | 0.413563 | 0.194178 | 0.365093 | 0.489438 | 0.543460 | 0.501340 | 0.330571 | 0.179315 | 0.334789 | 0.442686 | 0.479660 |
| 24 | 19 | 0.543460 | 0.489438 | 0.365092 | 0.194178 | 0.393301 | 0.538461 | 0.562789 | 0.499609 | 0.370403 | 0.196488 | 0.330571 | 0.501240 | 0.543460 | 0.489438 |
| 23 | 20 | 0.990678 | 0.908552 | 0.683616 | 0.365092 | 0.538461 | 0.956090 | 1.038071 | 0.933192 | 0.696097 | 0.370403 | 0.501240 | 0.874030 | 0.990678 | 0.908552 |
| 22 | 21 | 1.239123 | 1.184856 | 0.908552 | 0.489438 | 0.562089 | 1.036062 | 1.326493 | 1.234559 | 0.933192 | 0.499609 | 0.543460 | 0.990678 | 1.239123 | 1.184856 |

Table 3.1 (Continued)

$\eta_{i,j}^{k,l}$ - Deflection Influence Coefficients for Basic Plate Structure

Unit Load Applied at Point

Deflection Influence Coefficient at Point

|  | k,l | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| i,j | k,l<br>i,j | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 |  |
| 42 | 1 | 0.276720 | 0.149919 | 0.144292 | 0.253006 | 0.304757 | 0.294542 | 0.229707 | 0.125197 | 0.116719 | 0.206209 | 0.250572 | 0.244113 | 0.191551 | 0.104799 | 16.725686 |
| 41 | 2 | 0.508328 | 0.276720 | 0.253006 | 0.449049 | 0.547548 | 0.534464 | 0.419739 | 0.229707 | 0.206209 | 0.367291 | 0.450322 | 0.442123 | 0.348912 | 0.191551 | 29.854553 |
| 40 | 3 | 0.652742 | 0.358409 | 0.304757 | 0.547548 | 0.678756 | 0.672745 | 0.534464 | 0.294542 | 0.250572 | 0.450322 | 0.565992 | 0.560621 | 0.442123 | 0.244113 | 37.011817 |
| 39 | 4 | 0.675785 | 0.376022 | 0.294542 | 0.534464 | 0.672745 | 0.678756 | 0.547548 | 0.304757 | 0.244113 | 0.442123 | 0.560621 | 0.565992 | 0.450322 | 0.250572 | 37.011817 |
| 38 | 5 | 0.559917 | 0.317374 | 0.229707 | 0.419739 | 0.534464 | 0.547548 | 0.449049 | 0.253006 | 0.191551 | 0.348912 | 0.442123 | 0.450322 | 0.367291 | 0.207209 | 29.854553 |
| 37 | 6 | 0.317376 | 0.183894 | 0.125197 | 0.229707 | 0.294542 | 0.304757 | 0.253006 | 0.144292 | 0.104799 | 0.191551 | 0.244113 | 0.250572 | 0.206209 | 0.116719 | 16.725686 |
| 36 | 7 | 0.304548 | 0.164188 | 0.172463 | 0.299209 | 0.356327 | 0.341024 | 0.264040 | 0.143281 | 0.144292 | 0.253006 | 0.304757 | 0.294542 | 0.229707 | 0.125197 | 16.403590 |
| 35 | 8 | 0.562020 | 0.304548 | 0.299209 | 0.528790 | 0.640234 | 0.620367 | 0.484305 | 0.264040 | 0.253006 | 0.449049 | 0.547548 | 0.534464 | 0.419739 | 0.229707 | 29.288895 |
| 34 | 9 | 0.727178 | 0.397832 | 0.356327 | 0.670234 | 0.792830 | 0.783515 | 0.620367 | 0.341024 | 0.304757 | 0.547548 | 0.678756 | 0.672745 | 0.534464 | 0.294542 | 36.335128 |
| 33 | 10 | 0.760386 | 0.422630 | 0.341024 | 0.620367 | 0.783515 | 0.792830 | 0.640234 | 0.356327 | 0.294542 | 0.534464 | 0.672745 | 0.678756 | 0.547548 | 0.304757 | 36.335128 |
| 32 | 11 | 0.636446 | 0.362554 | 0.264040 | 0.484305 | 0.620367 | 0.640234 | 0.528790 | 0.299209 | 0.229707 | 0.419739 | 0.534464 | 0.547548 | 0.449049 | 0.253006 | 29.288895 |
| 31 | 12 | 0.362554 | 0.213817 | 0.143281 | 0.264040 | 0.341024 | 0.356327 | 0.299209 | 0.172463 | 0.125197 | 0.229707 | 0.294542 | 0.304757 | 0.253006 | 0.144292 | 16.403590 |
| 30 | 13 | 0.334789 | 0.179315 | 0.213817 | 0.362554 | 0.422630 | 0.397832 | 0.304548 | 0.164188 | 0.183894 | 0.317376 | 0.376028 | 0.358409 | 0.276720 | 0.149919 | 16.165079 |
| 29 | 14 | 0.622001 | 0.334789 | 0.362554 | 0.636446 | 0.760386 | 0.727178 | 0.562020 | 0.304548 | 0.317376 | 0.559917 | 0.675783 | 0.652742 | 0.508328 | 0.276720 | 28.843772 |
| 28 | 15 | 0.814449 | 0.442686 | 0.422630 | 0.760386 | 0.940994 | 0.924574 | 0.727178 | 0.397832 | 0.376028 | 0.675783 | 0.836637 | 0.825704 | 0.652742 | 0.358409 | 35.848144 |
| 27 | 16 | 0.866572 | 0.479660 | 0.397832 | 0.727178 | 0.924574 | 0.940994 | 0.760386 | 0.422630 | 0.358409 | 0.652742 | 0.825704 | 0.836637 | 0.675783 | 0.376028 | 35.848144 |
| 26 | 17 | 0.664738 | 0.423886 | 0.304548 | 0.562020 | 0.727178 | 0.760386 | 0.636446 | 0.362554 | 0.276720 | 0.508328 | 0.652742 | 0.675783 | 0.559917 | 0.317376 | 28.843772 |
| 25 | 18 | 0.423886 | 0.260079 | 0.164188 | 0.304548 | 0.397832 | 0.422630 | 0.362554 | 0.213817 | 0.149919 | 0.276720 | 0.358409 | 0.376028 | 0.317376 | 0.183894 | 16.165079 |
| 24 | 19 | 0.365092 | 0.194178 | 0.278176 | 0.451798 | 0.508865 | 0.467602 | 0.352472 | 0.188411 | 0.246337 | 0.412585 | 0.475395 | 0.443428 | 0.337306 | 0.181180 | 16.096119 |
| 23 | 20 | 0.683616 | 0.365092 | 0.451798 | 0.787040 | 0.919400 | 0.816336 | 0.656013 | 0.352472 | 0.412585 | 0.721732 | 0.856013 | 0.812700 | 0.624608 | 0.337306 | 28.782736 |
| 22 | 21 | 0.908552 | 0.489438 | 0.508865 | 0.919400 | 1.139512 | 1.107811 | 0.861336 | 0.467602 | 0.475395 | 0.856013 | 1.059038 | 1.037193 | 0.812700 | 0.443428 | 35.680804 |

Table 3.1 (Continued)

3.4. Influence Coefficients for Deflection of Edge Beam. A set of influence coefficients for the deflection of the edge beam will be developed. Figure 3.3 shows that the free edge of the plate has a length equivalent to 7Δx. To improve the accuracy of the influence coefficients, the edge beam along the length $L_x$ will be considered to be divided into 14 strips (Figure 3.10).

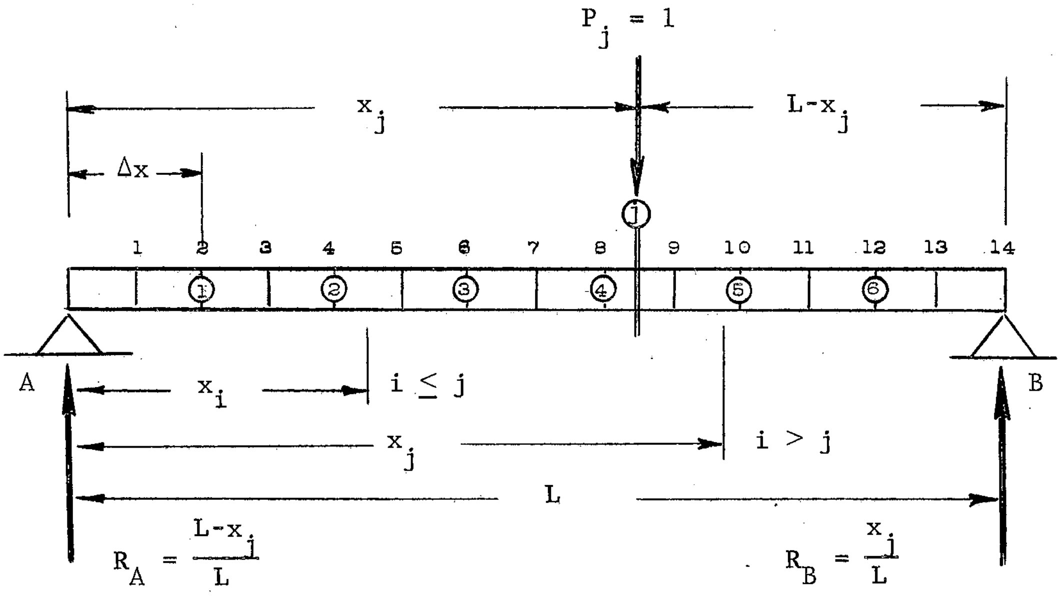


Figure 3.10

Edge Beam Divided into Fourteen Strips and
Loaded with a Unit Load

The influence coefficient, $Q_i^j$, is the influence of a unit load applied at point j on the moment, M, at point i. The moment at point i due to a unit load at point j is equal to the following expression:

$$M_i^j = Q_i^j P_j \Delta x \tag{3.4}$$

The influence coefficient, $\eta_i^j$, is the influence of a unit load applied at point j on the deflection, w, at point i in a beam of unit flexural stiffness. The deflection at point i due to a unit load at point j is equal to the following expression:

$$w_i^j = \eta_i^j P_j \frac{\Delta x^3}{EI} .$$

The influence of the moment at point k on the deflection of the beam at point i is given by the following expression:

$$w_i^{(k)j} = Q_i^k M_k^{P_k=1} \frac{\Delta x^2}{EI} .$$

Substituting equation (3.4) for $M_k^{P_j=1} = Q_k^j \Delta x$ the following is obtained:

$$w_i^{(k)j} = Q_i^k Q_k^j \frac{\Delta x^3}{EI} . \tag{3.5}$$

Rewriting equation (3.5) to include all points on the beam, the deflection at point i due to a unit load applied at point j is

$$w_i^{P_j=1} = \sum_{k=1}^{14} w_i^{(k)j} = \sum_{k=1}^{14} Q_i^k Q_k^j \frac{\Delta x^3}{EI}$$

or

$$w_i^{P_j=1} = \eta_i^j \frac{\Delta x^3}{EI} . \tag{3.5a}$$

Modifying equation (3.5a) to account for the deflection at point i due to any applied load at point j, the following equation is obtained:

$$w_i^j = P_j \frac{\Delta x^3}{EI} \eta_i^j \tag{3.5b}$$

where:

$$\eta_i^j = \sum_{k=1}^{14} Q_i^k Q_k^j . \tag{3.6}$$

The solution of equation (3.6) expressed in matrix form is presented below by allowing i, j, and k to vary from 1 to 14.

$$\begin{bmatrix} \eta_1^1 \; \eta_1^2 \; \cdots \; \eta_1^{14} \\ \eta_2^1 \; \eta_2^2 \; \cdots \; \eta_2^{14} \\ \cdot \\ \cdot \\ \cdot \\ \eta_{14}^1 \; \eta_{14}^2 \; \cdots \; \eta_{14}^{14} \end{bmatrix} = \begin{bmatrix} Q_1^1 \; Q_1^2 \; \cdots \; Q_1^{14} \\ Q_2^1 \; Q_2^2 \; \cdots \; Q_2^{14} \\ \cdot \\ \cdot \\ \cdot \\ Q_{14}^1 \; Q_{14}^2 \; \cdots \; Q_{14}^{14} \end{bmatrix} \begin{bmatrix} Q_1^1 \; Q_1^2 \; \cdots \; Q_1^{14} \\ Q_2^2 \; Q_2^2 \; \cdots \; Q_2^{14} \\ \cdot \\ \cdot \\ \cdot \\ Q_{14}^1 \; Q_{14}^2 \; \cdots \; Q_{14}^{14} \end{bmatrix} \qquad (3.6a)$$

Referring to Figure 3.4, the following values for $Q_i^j$ are obtained for $i \leq j$ and $i > j$.

Noting that: Number of strips = n

$$x_i = i\Delta x$$

$$x_j = j\Delta x$$

$$L = n\Delta x$$

For $i \leq j$:

$$M_i^{P_j=1} = x_i \frac{L-x_j}{L} = x_i \left(1 - \frac{x_j}{L}\right)$$

$$M_i^{P_j=1} = i \, \Delta x \, (1 - \frac{j}{n})$$

where:

$$Q_i^j = i \, (1 - \frac{j}{n}) \qquad (3.7a)$$

For $i > j$:

$$M_i^{P_j=1} = (L - x_i) \frac{x_j}{L} = x_j \left(1 - \frac{x_i}{L}\right)$$

$$M_i^{P_j=1} = j \, \Delta x \, (1 - \frac{i}{n})$$

where:

$$Q_i^j = j \, (1 - \frac{i}{n}) \qquad (3.7b)$$

Equations (3.7a) and (3.7b) can be used to determine the matrix formulation shown by equation (3.6a) and thus to evaluate the influence coefficients for deflection (Table 3.2).

$\eta_i^j$ - DEFLECTION INFLUENCE COEFFICIENTS FOR EDGE BEAM

| | | Unit Load at Point | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Influence Coefficient at Point | $i \backslash j$ | 1 | 2 | 3 | 4 | 5 | 6 | $\sum$ |
| | 1 | 1.71429 | 2.73810 | 3.04762 | 2.78571 | 2.09524 | 1.11905 | 13.50001 |
| | 2 | 2.73810 | 4.76100 | 5.52276 | 5.14188 | 3.90402 | 2.09524 | 24.16300 |
| | 3 | 3.04762 | 5.52276 | 6.85632 | 6.64206 | 5.14188 | 2.78571 | 29.99635 |
| | 4 | 2.78571 | 5.14188 | 6.64206 | 6.85632 | 5.52276 | 3.04762 | 29.99635 |
| | 5 | 2.09524 | 3.90402 | 5.14188 | 5.52276 | 4.76100 | 2.73810 | 24.16300 |
| | 6 | 1.11905 | 2.09524 | 2.78571 | 3.04762 | 2.73810 | 1.71429 | 13.50001 |

TABLE 3.2

# CHAPTER IV

# GENERAL MOMENT EQUATION

4.1. Explanation of Moment Equation. The plate-moment equation is developed from the four following basic considerations of compatibility and equilibrium.

1. Slope compatibility between adjacent panels in the direction of continuity.
2. Equality of deflections between the edge beam and the plate on the elastically supported edge.
3. Equality of rotations between the edge beam and the plate on the elastically supported edge.
4. Slope compatibility of the continuous edge beam over supports.

There are basically four unknown quantities which exist in a plate structure of the type considered herein, and they are as follows:

1. Moments in the continuous plate over supports.
2. Edge shears or reactive forces between plate and edge beam.
3. Torsional or twist moment in the edge beam.
4. Moments in the continuous edge beam over supports.

A moment-reaction equation will be developed in the succeeding articles which will actually be fourfold in that the unknown plate-moments will be determined by slope compatibility between adjacent panels, the unknown shears will be determined by equality of deflections between plate and edge beam, the unknown torsional moments will be determined by equality of rotations between plate and edge beam, and the unknown edge beam moments will be determined by slope compatibility at the point of simple support.

4.2. <u>Slope Compatibility - Plate-Moment Equation</u>. Consider a plate-beam structure which is continuous in the X direction over rigid supports and supported by flexible edge beams. The continuous plate structure is considered to be subjected to any system of normal loading as shown in Figure 4.1. The simple supports are assumed to be perfectly rigid and the flexural rigidity (D) of all panels is constant.

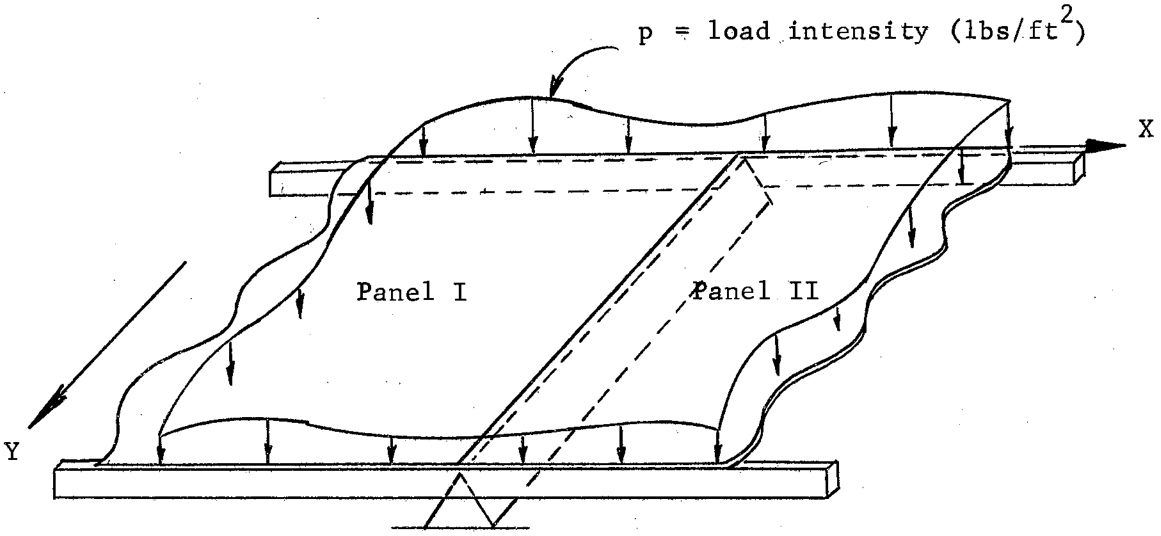


Figure 4.1

Generally Loaded Basic Structure

The sum of the rotation of the end slopes of adjacent panels, I and II, must be equal to zero for the satisfaction of compatibility requirements between panels, i.e.,

$$(\theta_i)_I + (\theta_i)_{II} = 0 \qquad (4.1)$$

where: $\theta_i$ is the rotation at any point i between panels. $(\theta_i)_I$ and $(\theta_i)_{II}$ are the rotations at i of panels I and II, respectively.

The slope compatibility between adjacent panels at the point of simple support is illustrated in Figure 4.2.

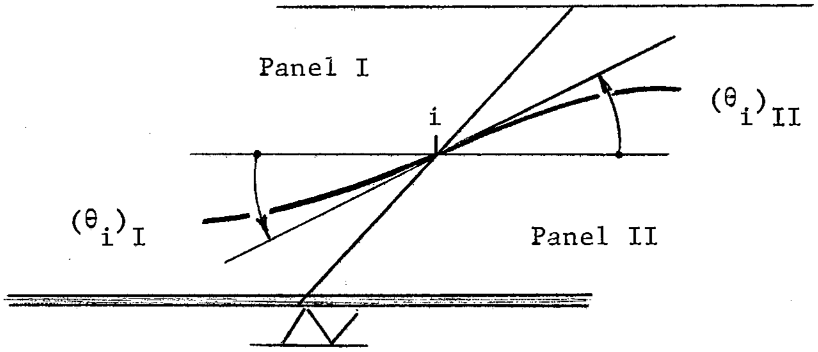


Figure 4.2

Slope Compatibility

The algebraic expressions for the slopes are:

$$(\theta_i)_I = \sum_{I} P_k \tau_{i,k} + M_i (F_{i,i})_I + \sum_{I} M_j G_{i,j} - \sum_{I} R_m Q_{i,m} + \sum_{I} M_m H_{i,m} \qquad (4.2)$$

$$(\theta_i)_{II} = \sum_{II} P_k \tau_{i,k} + M_i (F_{i,i})_{II} + \sum_{II} M_j G_{i,j} - \sum_{II} R_m Q_{i,m} + \sum_{II} M_m H_{i,m} \qquad (4.3)$$

The notations used in equations (4.2) and (4.3) are defined as follows:

The notation j corresponds to any point, other than point i, on the simply supported boundary of the basic plate structure.

The notation k corresponds to any interior point on the basic plate structure.

The notation m corresponds to any point, other than point n, on the free edge of the basic plate structure.

The angular load function, $\tau_{i,k}$, is the edge slope at point i due to a unit load applied at point k, considering the basic plate structure.

The angular flexibility, $(F_{i,i})_{I,II}$, is the edge slope at point i of panels I and II, respectively, due to a unit moment applied at point i, considering the basic plate structure.

The angular carry-over, $G_{i,j}$, is the edge slope at point i due to a unit moment applied at point j, considering the basic plate structure.

The displacement-angular carry-over, $Q_{i,m}$, is the edge slope at point i due to a unit reaction force applied at point m on the free edge of the basic plate structure.

The rotational-angular carry-over, $H_{i,m}$, is the edge slope at point i due to a unit torque applied at point m on the free edge of the basic plate structure.

The notation $P_k$ corresponds to a unit load applied at any typical point k on the basic plate structure.

The notations $M_i$ and $M_j$ correspond to unit moments applied at point i and point j, respectively, on the basic plate structure.

The notation $R_m$ corresponds to a unit reactive force applied at point m on the free edge of the basic plate structure.

The notation $M_m$ corresponds to a unit torsional moment applied at point m on the free edge of the basic plate structure.

Substituting equations (4.2) and (4.3) into equation (4.1),

$$\sum_{I} P_k \tau_{i,k} + \sum_{II} P_k \tau_{i,k} + M_i (F_{i,i})_{I} + M_i (F_{i,i})_{II}$$

$$+ \sum_{I} M_j G_{i,j} + \sum_{II} M_j G_{i,j} - \sum_{I} R_m Q_{i,m} - \sum_{II} R_m Q_{i,m}$$

$$+ \sum_{I} M_m H_{i,m} + \sum_{II} M_m H_{i,m} = 0 \qquad (4.4)$$

Rearranging the terms in equation (4.4), the general moment equation is:

$$\sum_{I,II} P_k \tau_{i,k} + M_i \sum_{I,II} F_{i,i} + \sum_{I,II} M_j G_{i,j}$$

$$- \sum_{I,II} R_m Q_{i,m} + \sum_{I,II} M_m H_{i,m} = 0 \qquad (4.5)$$

where: I, II beneath the summation sign indicates a summation over the specific simply supported boundary points or free edge points of both panels I and II.

4.3. Equality of Deflections - Reaction Equations. The deflection of the edge beam and the deflection at the free edge of the plate must be equal; thus,

$$(\Delta_m)_I^{\text{Plate}} = (\Delta_m)_I^{\text{Beam}}$$

or:

$$(\Delta_m)_I^{\text{Beam}} - (\Delta_m)_I^{\text{Plate}} = 0 \qquad (4.6)$$

where: $(\Delta_m)_I^{\text{Plate}}$ and $(\Delta_m)_I^{\text{Beam}}$ are the deflections at point m in panel I of the plate and edge beam, respectively.

The equality of deflections is illustrated in Figure 4.3.

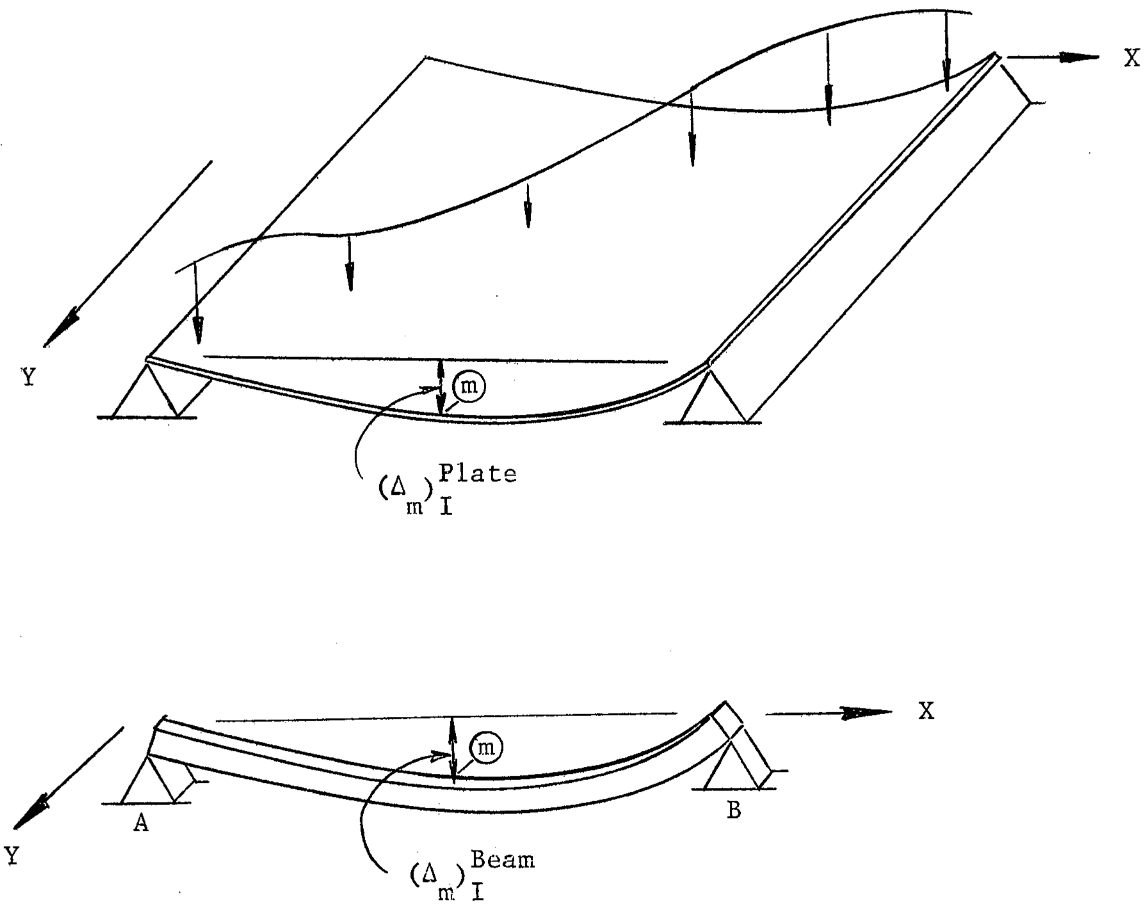


Figure 4.3

Equal Deflections of Edge Beam and Plate

A term having a subscript of either m or n or both m,m or m,n written together and also having a superscript word "Beam" is a term pertaining to the edge beam. Points m and n are points located on the edge beam and may also be considered to be superimposed upon the same points m and n located on the free edge of the plate.

The algebraic expressions for the deflection at point m on the plate and edge beam in panel I are as follows:

$$(\Delta_m)_I^{\text{Plate}} = \sum_I \delta_{m,k} P_k - R_m (D_{m,m})_I - \sum_I R_n E_{m,n} + \sum_I M_j Q_{m,j} + M_m (S_{m,m})_I + \sum_I M_n T_{m,n} \qquad (4.7)$$

$$(\Delta_m)_I^{\text{Beam}} = (\delta_m)_I^{\text{Beam}} + R_m (D_{m,m})_I^{\text{Beam}} + \sum_I R_n E_{m,n}^{\text{Beam}} + M_A^{\text{Beam}} Q_{m,A}^{\text{Beam}} + M_B^{\text{Beam}} Q_{m,B}^{\text{Beam}} \qquad (4.8$$

The notations used in equations (4.7) and (4.8) not previously defined are defined as follows:

Displacement Load Function, $\delta_{m,k}$, is the deflection of point m on the free edge of the basic plate structure due to a unit load applied at point k.

Displacement Flexibility, $D_{m,m}$, is the deflection of point m on the free edge of the basic plate structure due to a unit reactive force applied at point m.

Displacement Carry-Over, $E_{m,n}$, is the deflection of point m on the free edge of the basic plate structure due to a unit reactive force applied at point m.

Angular Displacement Carry-Over, $Q_{m,j}$, is the deflection of point m on the free edge of the basic plate structure due to a unit moment applied at point j.

Torsional Angular Displacement Flexibility, $S_{m,m}$, is the displacement of point m on the free edge of the basic plate structure due to a unit torque-moment applied at point m.

Torsional Angular Displacement Carry-Over, $T_{m,n}$, is the displacement of point m on the free edge of the basic plate structure due to a unit torque-moment applied at point n.

The notation $R_m$ corresponds to a unit reactive force applied at point m on the free edge of the basic plate structure.

The notation $M_m$ corresponds to a unit torsional moment applied at point m on the free edge of the basic plate structure.

Displacement Load Function, $\delta_m^{Beam}$, is the displacement of point m due to the dead load weight of the edge beam.

Displacement Flexibility, $D_{m,m}^{Beam}$, is the displacement of point m due to a unit reactive force applied at point m.

Displacement Carry-Over, $E_{m,n}^{Beam}$, is the displacement of point m due to a unit reactive force applied at point n.

Displacement Angular Carry-Overs, $Q_{m,A}^{Beam}$ and $Q_{m,B}^{Beam}$, are the displacements of point m due to unit moments applied at the simply supported ends A and B, respectively.

The notations $M_A^{Beam}$ and $M_B^{Beam}$ correspond to the end moments in the continuous edge beam at the simply supported ends A and B, respectively.

Substituting equations (4.7) and (4.8) into equation (4.6), the general reactive force equation is:

$$\begin{aligned}
&+\sum_{I} \delta_{m,k}P_k - (\delta_m)_I^{\text{Beam}} - R_m(+D_{m,m} + D_{m,m}^{\text{Beam}})_I \\
&-\sum_{I} R_n(+E_{m,n} + E_{m,n}^{\text{Beam}}) + \sum_{I} M_j Q_{m,j} \\
&+ M_m(S_{m,m})_I + \sum_{I} M_n T_{m,n} \\
&- M_A(Q_{m,A}^{\text{Beam}})_I - M_B(Q_{m,B}^{\text{Beam}})_I = 0 \qquad (4.9)
\end{aligned}$$

4.4. Equality of Rotations - Torque-Moment Equation. The rotations of the edge beam at a particular point must be equal to the slope on the edge of the plate at that same point; thus,

$$(\beta_m)_I^{Plate} = (\beta_m)_I^{Beam}$$

or:

$$-(\beta_m)_I^{Plate} + (\beta_m)_I^{Beam} = 0 \qquad (4.10)$$

where: $(\beta_m)_I^{Plate}$ and $(\beta_m)_I^{Beam}$ are the rotations at point m in panel I of the plate and edge beam, respectively.

The equality of free edge rotations is illustrated in Figure 4.4.

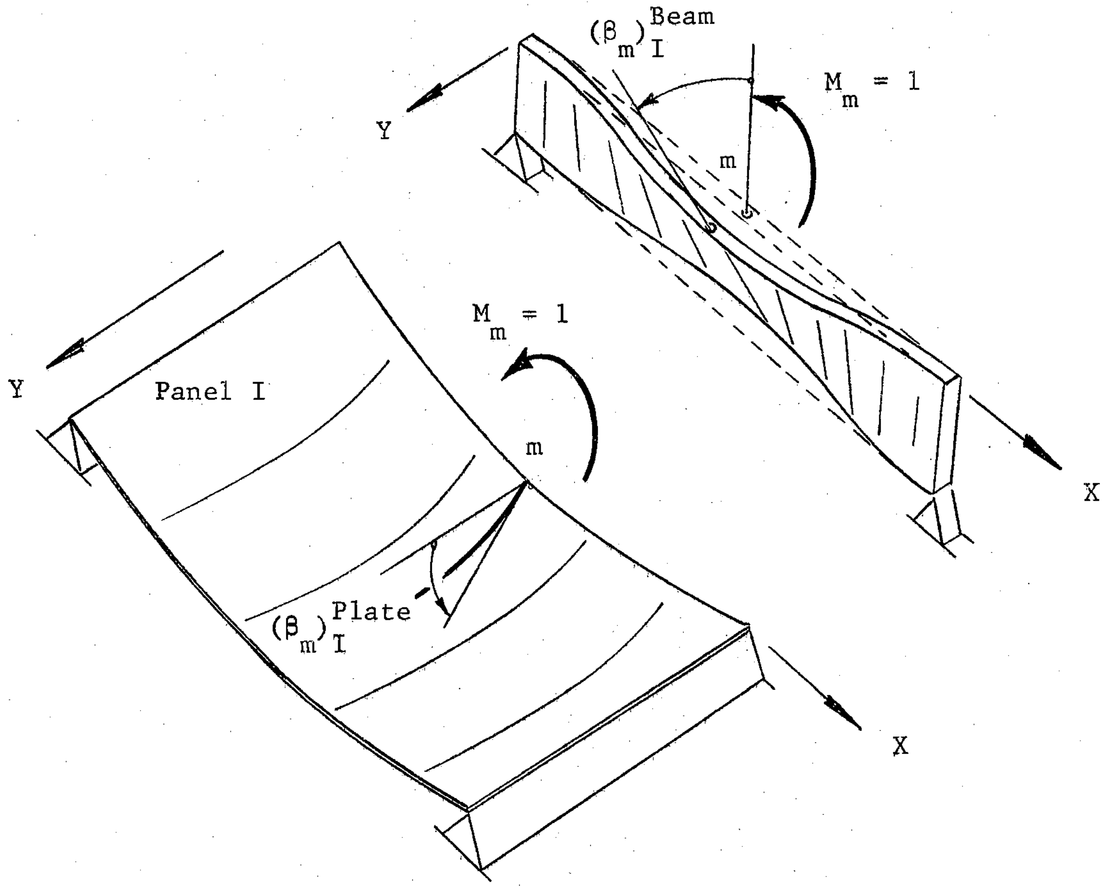


Figure 4.4

Equal Rotations of Edge Beam and Plate

The algebraic expressions for the rotation at point m on the plate and edge beam in panel I are as follow:

$$(\beta_m)_I^{Plate} = \sum_I P_k T_{m,k} - R_m(S_{m,m})_I - \sum_I R_n T_{m,n}$$

$$+ \sum_I M_j H_{m,j} + M_m(U_{m,m})_I + \sum_I M_n V_{m,n} \qquad (4.11)$$

$$(\beta_m)_I^{Beam} = M_m(U_{m,m})_I^{Beam} + \sum_I M_n V_{m,n}^{Beam} \qquad (4.12)$$

The notations used in equations (4.11) and (4.12) are defined as follow:

Angular Load Function on Free Edge, $T_{m,k}$, is the edge slope at point m on the free edge of the basic plate structure due to a unit load applied at point k.

Displacement Angular Flexibility, $S_{m,m}$, is the edge slope at point m on the free edge of the basic plate structure due to a unit reactive force applied at point m.

Displacement Angular Carry-Over, $T_{m,n}$, is the edge slope at point m on the free edge of the basic plate structure due to a unit reactive force applied at point n.

Angular Carry-Over on Free Edge, $H_{m,j}$, is the edge slope at point m on the free edge of the basic plate structure due to a unit moment applied at point j.

Torsional Angular Flexibility on Free Edge, $U_{m,m}$, is the edge slope at point m on the free edge of the basic plate structure due to a unit torque-moment applied at point m.

Torsional Angular Carry-Over on Free Edge, $V_{m,n}$, is the edge slope at point m on the free edge of the basic plate structure due to a unit torque-moment applied at point n.

Torsional Angular Flexibility, $U^{Beam}_{m,m}$, is the rotation of the edge beam at point m due to a unit torque-moment applied at point m.

Torsional Angular Carry-Over, $V^{Beam}_{m,n}$, is the rotation of the edge beam at point m due to a unit torque-moment applied at point n.

Substituting equations (4.11) and (4.12) into equation (4.10), the general torque-moment equation is:

$$
\begin{aligned}
&+ \sum_{I} P_k T_{m,k} - R_m (S_{m,m})_I - \sum_{I} R_n T_{m,n} \\
&+ \sum_{I} M_j H_{m,j} + M_m (+U_{m,m} - U^{Beam}_{m,m}) \\
&+ \sum_{I} M_n (+V_{m,n} - V^{Beam}_{m,n}) \qquad = 0 \qquad (4.13)
\end{aligned}
$$

4.5. Slope Compatibility - Beam-Moment Equation. The sum of the rotations of the end slopes at the point of simple support between panels I and II on the continuous edge beam must be equal to zero, thus

$$(\theta_A)_I^{Beam} + (\theta_A)_{II}^{Beam} = 0 \tag{4.14}$$

where: $(\theta_A)_I^{Beam}$ and $(\theta_A)_{II}^{Beam}$ are the end slopes at the simply supported point A in panels I and II, respectively.

The compatibility of the end slopes of the edge beam over a simple support is illustrated in Figure 4.5.

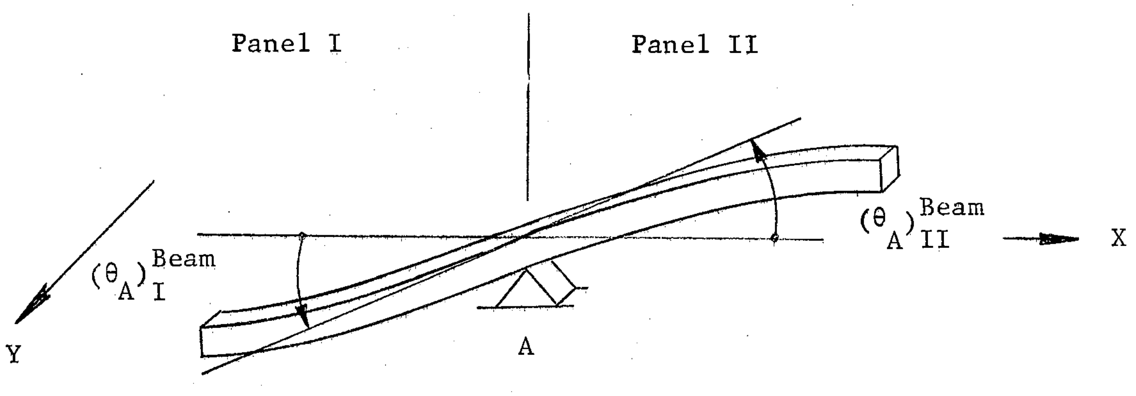


Figure 4.5

Slope Compatibility over Simple Support in Edge Beam

The algebraic expressions for the end slopes at the simply supported point A of the edge beam are as follow:

$$(\theta_A)_{I}^{Beam} = (\tau_A)_{I}^{Beam} + M_A(F_{A,A})_{I}^{Beam} + M_B(G_{A,B})_{I}^{Beam} + \sum_{I} R_m Q_{A,m}^{Beam} \tag{4.15}$$

$$(\theta_A)_{II}^{Beam} = (\tau_A)_{II}^{Beam} + M_A(F_{A,A})_{II}^{Beam} + M_B(G_{A,B})_{II}^{Beam} + \sum_{II} R_m Q_{A,m}^{Beam} \tag{4.16}$$

The notations used in equations (4.15) and (4.16) not previously defined are defined as follow:

<u>Angular Load Functions</u>, $(\tau_A)_{I}^{Beam}$ and $(\tau_A)_{II}^{Beam}$, are the end slopes at the simply supported end A due to the dead load weight of the edge beam in panels I and II, respectively.

<u>Angular Flexibilities</u>, $(F_{A,A})_{I}^{Beam}$ and $(F_{A,A})_{II}^{Beam}$, are the end slopes at the simply supported end A due to a unit moment applied at end A in panels I and II, respectively.

<u>Angular Carry-Overs</u>, $(G_{A,B})_{I}^{Beam}$ and $(G_{A,B})_{II}^{Beam}$, are the end slopes at the simply supported end A due to a unit moment applied at end B in panels I and II, respectively.

<u>Displacement Angular Carry-Overs</u>, $(Q_{A,m})_{I}^{Beam}$ and $(Q_{A,m})_{II}^{Beam}$, are the end slopes at the simply supported end A due to a unit reactive force applied at point m in panels I and II, respectively.

Substituting equations (4.15) and (4.16) into equation (4.14),

$$(\tau_A)_I^{Beam} + (\tau_A)_{II}^{Beam} + M_A[(F_{A,A})_I + (F_{A,A})_{II}]^{Beam}$$

$$+ M_B[(G_{A,B})_I + (G_{A,B})_{II}]^{Beam}$$

$$+ \sum_I R_m Q_{A,m}^{Beam} + \sum_{II} R_m Q_{A,m}^{Beam} \qquad = \quad 0 \qquad (4.17)$$

Rearranging the terms in equation (4.17), the general beam-moment equation is:

$$\sum_{I,II} \tau_A^{Beam} + M_A \sum_{I,II} F_{A,A}^{Beam}$$

$$+ \sum_{I,II} M_B G_{A,B}^{Beam} + \sum_{I,II} R_m Q_{A,m}^{Beam} \qquad = \quad 0 \qquad (4.18)$$

Rearranging the four equations developed thus far in such a manner so that the load terms are on the right side of the equations and the unknown quantities with their appropriate flexibility coefficients are on the left side of the equations, the following are obtained:

Plate-Moment Equation:

$$\begin{aligned} & M_i \sum_{I,II} F_{i,i} + \sum_{I,II} M_j G_{i,j} \\ & - \sum_{I,II} R_m Q_{i,m} + \sum_{I,II} M_m H_{i,m} \end{aligned} \quad = - \sum_{I,II} P_k \tau_{i,k} \tag{4.5}$$

Reactive Force Equation:

$$\begin{aligned} & - R_m \sum_{I,I} D_{m,m} - \sum_{I,I} R_n E_{m,n} \\ & + \sum_{I,I} M_j Q_{m,j} + M_m \sum_{I,I} S_{m,m} \\ & + \sum_{I,I} M_n T_{m,n} - M_A (Q_{m,A})_I^{\text{Beam}} \\ & - M_B (Q_{m,B})_I^{\text{Beam}} \end{aligned} \quad = - \sum_I \delta_{m,k} P_k + (\delta_m)_I^{\text{Beam}} \tag{4.9}$$

Torque-Moment Equation:

$$\begin{aligned} & M_m \sum_{I,I} U_{m,m} + \sum_{I,I} M_n V_{m,n} \\ & - R_m \sum_{I,I} S_{m,m} - \sum_{I,I} R_n T_{m,n} \\ & + \sum_{I,I} M_j H_{m,j} \end{aligned} \quad = - \sum_I P_k \tau_{m,k} \tag{4.13}$$

Beam-Moment Equation:

$$\begin{aligned} & - M_A \sum_{I,II} F_{A,A}^{\text{Beam}} - \sum_{I,II} M_B G_{A,B}^{\text{Beam}} \\ & - \sum_{I,II} R_m Q_{A,m}^{\text{Beam}} \end{aligned} \quad = + \sum_{I,II} \tau_A^{\text{Beam}} \tag{4.18}$$

4.6. <u>General Moment-Reaction Equation in Matrix Form</u>. A general moment-reaction equation written in matrix form will be given below and will be composed of equations (4.5), (4.9), (4.13), and (4.18). For simplicity, the even integers (2, 4, 6 ...., n) will represent points on the simply supported boundary of the plate, and the odd integers (1, 3, 5 .... n-1) will represent points on the free edge of the plate (Figure 4.6).

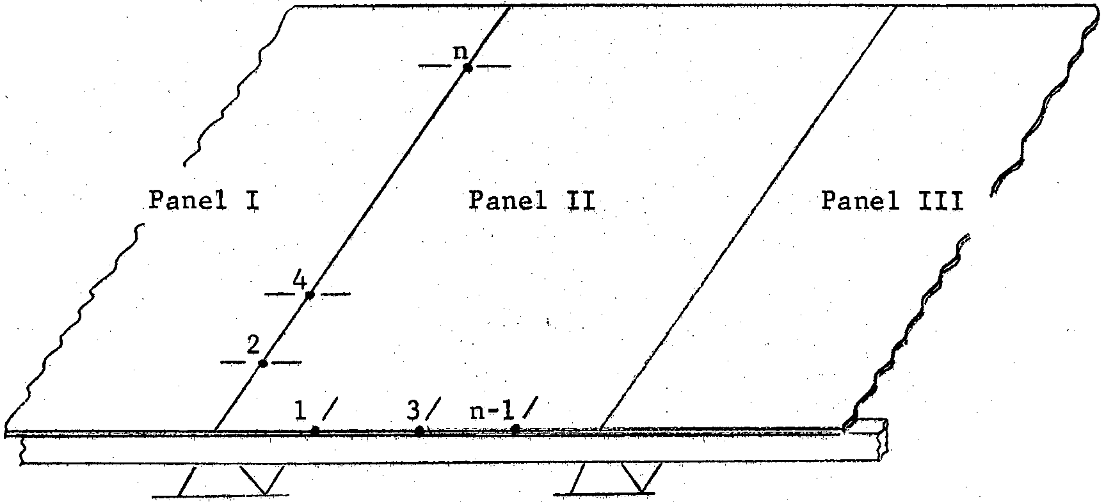


Figure 4.6

General Points on Basic Structure for Formulation of Matrix Equation

The matrix form of the general moment-reaction equation is presented in Figure 4.7.

$$
\begin{array}{c|ccccccccccc|c|c|c|}
 & \multicolumn{3}{c}{\text{Plate-Moment Flexibilities}} & \multicolumn{3}{c}{\text{Torque-Moment Flexibilities}} & \multicolumn{3}{c}{\text{Reaction Flexibilities}} & \multicolumn{2}{c}{\text{Edge-Beam End-Moment Flexibilities}} & \text{Unknowns} & & \text{Load Functions} \\
\text{Row 1} & \sum\limits_{I,II}F_{2,2} & (G_{2,4})_{I,II} & (G_{2,n})_{I,II} & (H_{2,1})_{I,II} & (H_{2,3})_{I,II} & (H_{2,n-1})_{I,II} & (Q_{2,1})_{I,II} & (Q_{2,3})_{I,II} & (Q_{2,n-1})_{I,II} & & & M_2 & & -\sum\limits_{I,II}\tau_{2,k}P_k \\
 & (G_{4,2})_{I,II} & \sum\limits_{I,II}F_{4,4} & (G_{4,n})_{I,II} & (H_{4,1})_{I,II} & (H_{4,3})_{I,II} & (H_{4,n-1})_{I,II} & (Q_{4,1})_{I,II} & (Q_{4,3})_{I,II} & (Q_{4,n-1})_{I,II} & & & M_4 & & -\sum\limits_{I,II}\tau_{4,k}P_k \\
 & (G_{n,2})_{I,II} & (G_{n,4})_{I,II} & \sum\limits_{I,II}F_{n,n} & (H_{n,1})_{I,II} & (H_{n,3})_{I,II} & (H_{n,n-1})_{I,II} & (Q_{n,1})_{I,II} & (Q_{n,3})_{I,II} & (Q_{n,n-1})_{I,II} & & & M_n & & -\sum\limits_{I,II}\tau_{n,k}P_k \\
\text{Row 2} & (H_{1,2})_{I,J} & (H_{1,4})_{I,J} & (H_{1,n})_{I,J} & \sum\limits_{I,J}U_{1,1} & (V_{1,3})_{I,J} & (V_{1,n-1})_{I,J} & (S_{1,1})_{I,J} & (T_{1,3})_{I,J} & (T_{1,n-1})_{I,J} & & & M_1 & & -\sum\limits_{I}\tau_{1,k}P_k \\
 & (H_{3,2})_{I,J} & (H_{3,4})_{I,J} & (H_{3,n})_{I,J} & (V_{3,1})_{I,J} & \sum\limits_{I,J}U_{3,3} & (V_{3,n-1})_{I,J} & (T_{3,1})_{I,J} & (S_{3,3})_{I,J} & (T_{3,n-1})_{I,J} & & & M_3 & & -\sum\limits_{I}\tau_{3,k}P_k \\
 & (H_{n-1,2})_{I,J} & (H_{n-1,4})_{I,J} & (H_{n-1,n})_{I,J} & (V_{n-1,1})_{I,J} & (V_{n-1,3})_{I,J} & \sum\limits_{I,J}U_{n-1,n-1} & (T_{n-1,1})_{I,J} & (T_{n-1,3})_{I,J} & (S_{n-1,n-1})_{I,J} & & & M_{n-1} & = & -\sum\limits_{I}\tau_{n-1,k}P_k \\
\text{Row 3} & (Q_{1,2})_{I,J} & (Q_{1,4})_{I,J} & (Q_{1,n})_{I,J} & (S_{1,1})_{I,J} & (T_{1,3})_{I,J} & (T_{1,n-1})_{I,J} & \sum\limits_{I,J}D_{1,1} & \sum\limits_{I,J}E_{1,3} & \sum\limits_{I,J}E_{1,n-1} & (Q^{Beam}_{1,A})_{I,J} & (Q^{Beam}_{1,B})_{I,J} & -R_1 & & -\sum\limits_{I}\delta_{1,k}P_k+(\delta_{1,k})^{Beam}_{I} \\
 & (Q_{3,2})_{I,J} & (Q_{3,4})_{I,J} & (Q_{3,n})_{I,J} & (T_{3,1})_{I,J} & (S_{3,3})_{I,J} & (T_{3,n-1})_{I,J} & \sum\limits_{I,J}E_{3,1} & \sum\limits_{I,J}D_{3,3} & \sum\limits_{I,J}E_{3,n-1} & (Q^{Beam}_{3,A})_{I,J} & (Q^{Beam}_{3,B})_{I,J} & -R_3 & & -\sum\limits_{I}\delta_{3,k}P_k+(\delta_{3,k})^{Beam}_{I} \\
 & (Q_{n-1,2})_{I,J} & (Q_{n-1,4})_{I,J} & (Q_{n-1,n})_{I,J} & (T_{n-1,1})_{I,J} & (T_{n-1,3})_{I,J} & (S_{n-1,n-1})_{I,J} & \sum\limits_{I,J}E_{n-1,1} & \sum\limits_{I,J}E_{n-1,3} & \sum\limits_{I,J}D_{n-1,n-1} & (Q^{Beam}_{n-1,A})_{I,J} & (Q^{Beam}_{n-1,B})_{I,J} & -R_{n-1} & & -\sum\limits_{I}\delta_{n-1,k}P_k+(\delta_{n-1,k})^{Beam}_{I} \\
\text{Row 4} & & & & & & & \sum\limits_{I,II}Q^{Beam}_{A,1} & \sum\limits_{I,II}Q^{Beam}_{A,3} & \sum\limits_{I,II}Q^{Beam}_{A,n-1} & \sum\limits_{I,II}F^{Beam}_{A,A} & \sum\limits_{I,II}G^{Beam}_{A,B} & -M_A & & +\sum\limits_{I,II}\tau^{Beam}_{A} \\
 & & & & & & & \sum\limits_{II,III}Q^{Beam}_{B,1} & \sum\limits_{II,III}Q^{Beam}_{B,3} & \sum\limits_{II,III}Q^{Beam}_{B,n-1} & \sum\limits_{II,III}G^{Beam}_{B,A} & \sum\limits_{I,III}F^{Beam}_{B,B} & -M_B & & +\sum\limits_{II,III}\tau^{Beam}_{B}
\end{array}
\tag{4.19}
$$

Figure 4.7

General Moment-Reaction Equation in Matrix Form

The matrix formulation is composed of four parts; each part represents one row and is developed from one of the four equations presented on page 61. A description of each row is as follows:

Row 1 - Rotations at 2, 4 .... n due to reactions, plate-moments, and torque-moments expressed by equation (4.5).

Row 2 - Rotations at 1, 3 .... n-1 due to reactions, plate-moments, and torque-moments expressed by equation (4.13).

Row 3 - Deflections at 1, 3 .... n-1 due to reactions, plate-moments, end-moments, and torque-moments expressed by equation (4.9).

Row 4 - Rotations at A and B due to reactions, torque-moments, and end-moments expressed by equation (4.18).

The following is a symbolic representation of the general moment-reaction equation.

$$\begin{bmatrix} \begin{bmatrix} G_{j,i} \end{bmatrix} & \begin{bmatrix} H_{i,j} \end{bmatrix} & \begin{bmatrix} Q_{i,m} \end{bmatrix} & \\ \begin{bmatrix} H_{m,i} \end{bmatrix} & \begin{bmatrix} V_{m,n} \end{bmatrix} & \begin{bmatrix} T_{m,n} \end{bmatrix} & \\ \begin{bmatrix} Q_{m,i} \end{bmatrix} & \begin{bmatrix} T_{n,m} \end{bmatrix} & \begin{bmatrix} E_{m,n} \end{bmatrix} & \begin{bmatrix} Q^{Beam}_{\substack{m,A \\ m,B}} \end{bmatrix} \\ & & \begin{bmatrix} Q^{Beam}_{\substack{A,m \\ B,m}} \end{bmatrix} & \begin{bmatrix} G^{Beam}_{B,A} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} M_{i} \end{bmatrix} \\ \begin{bmatrix} M_{m} \end{bmatrix} \\ -\begin{bmatrix} R_{m} \end{bmatrix} \\ -\begin{bmatrix} M^{Beam}_{\substack{A \\ B}} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -\tau_{i,k} \end{bmatrix} \\ \begin{bmatrix} -\tau_{m,k} \end{bmatrix} \\ \begin{bmatrix} -\delta_{m,k} + \delta^{Beam}_{m,k} \end{bmatrix} \\ \begin{bmatrix} T^{Beam}_{\substack{A \\ B}} \end{bmatrix} \end{bmatrix} \tag{4.19a}$$

If the torque-moments are neglected, it is a simple matter to modify the matrix formulation by eliminating the columns comprising the torque-moment flexibilities. The symbolic form of the resulting matrix is shown below.

$$\begin{bmatrix} \begin{bmatrix} G_{j,i} \end{bmatrix} & \begin{bmatrix} Q_{i,m} \end{bmatrix} & \\ \begin{bmatrix} Q_{m,i} \end{bmatrix} & \begin{bmatrix} E_{m,n} \end{bmatrix} & \begin{bmatrix} Q^{Beam}_{\substack{m,A\\m,B}} \end{bmatrix} \\ & \begin{bmatrix} Q^{Beam}_{\substack{A,m\\B,m}} \end{bmatrix} & \begin{bmatrix} G^{Beam}_{B,A} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} M_i \end{bmatrix} \\ -\begin{bmatrix} R_m \end{bmatrix} \\ -\begin{bmatrix} M^{Beam}_{\substack{A\\B}} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -\tau_{i,k} \end{bmatrix} \\ \begin{bmatrix} -\delta_{m,k} + \delta^{Beam}_{m,k} \end{bmatrix} \\ \begin{bmatrix} \tau^{Beam}_{\substack{A\\B}} \end{bmatrix} \end{bmatrix} \qquad (4.19b)$$

A solution to the above matrix formulation might be made by separating the last equation from the first two equations; thus,

$$\begin{bmatrix} \begin{bmatrix} G_{j,i} \end{bmatrix} & \begin{bmatrix} Q_{i,m} \end{bmatrix} & \\ \begin{bmatrix} Q_{m,i} \end{bmatrix} & \begin{bmatrix} E_{m,n} \end{bmatrix} & \begin{bmatrix} Q^{Beam}_{\substack{m,A\\m,B}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} M_i \end{bmatrix} \\ -\begin{bmatrix} R_m \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -\tau_{i,k} \end{bmatrix} \\ \begin{bmatrix} -\delta_{m,k} + \delta^{Beam}_{m,k} \end{bmatrix} \end{bmatrix}$$

$$-\begin{bmatrix} Q^{Beam}_{A,m} \end{bmatrix} \begin{bmatrix} R_m \end{bmatrix} - \begin{bmatrix} G^{Beam}_{A,B} \end{bmatrix} \begin{bmatrix} M^{Beam}_{\substack{A\\B}} \end{bmatrix} = \begin{bmatrix} \tau^{Beam}_{\substack{A\\B}} \end{bmatrix}$$

Multiplying the last equation by $\begin{bmatrix} G^{Beam}_{A,B} \end{bmatrix}^{-1}$ and noting that $\begin{bmatrix} G^{Beam}_{A,B} \end{bmatrix} \begin{bmatrix} G^{Beam}_{A,B} \end{bmatrix}^{-1} = \begin{bmatrix} I \end{bmatrix}$ (the identity matrix), a solution for the beam end-moments is found in terms of the reactive edge forces; thus,

$$\begin{bmatrix} M^{Beam}_{\substack{A\\B}} \end{bmatrix} = -\begin{bmatrix} G^{Beam}_{A,B} \end{bmatrix}^{-1} \begin{bmatrix} Q^{Beam}_{\substack{A,m\\B,m}} \end{bmatrix} \begin{bmatrix} R_m \end{bmatrix} - \begin{bmatrix} G^{Beam}_{A,B} \end{bmatrix}^{-1} \begin{bmatrix} \tau^{Beam}_{\substack{A\\B}} \end{bmatrix}$$

where: $\begin{bmatrix} G^{Beam}_{A,B} \end{bmatrix}^{-1}$ is the inverse matrix of $\begin{bmatrix} G^{Beam}_{A,B} \end{bmatrix}$.

The symbolic flexibility matrices presented in equation (4.19) will be defined as follows:

In Column One:

$\left[ G_{j,i} \right]$ is the (angular flexibility and carry-over) matrix which represents the slopes at the line of juncture between adjacent panels due to moments at the line of juncture between adjacent panels.

$\left[ H_{m,i} \right]$ is the (angular carry-over on free edge) matrix which represents the slopes at the free edge due to moments at the line of juncture between adjacent panels.

$\left[ Q_{m,i} \right]$ is the (angular-displacement carry-over) matrix which represents the displacements at the free edge due to moments at the line of juncture between adjacent panels.

In Column Two:

$\left[ H_{i,m} \right]$ is the (torsional angular carry-over on the simply supported boundary) matrix which represents the slopes at the line of juncture between adjacent panels due to torque-moments at the free edge.

$\left[ V_{m,n} \right]$ is the (torsional angular flexibility and carry-over on the free edge) matrix which represents the slopes at the free edge due to torque-moments at the free edge.

$\left[ T_{m,n} \right]$ is the (torsional angular-displacement flexibility and carry-over on the free edge) matrix which represents the displacements at the free edge due to torque-moments applied at the free edge.

In Column Three:

| | |
|---|---|
| $\begin{bmatrix} Q_{i,m} \end{bmatrix}$ | is the (displacement-angular carry-over on the simply supported boundary) matrix which represents the slopes at the line of juncture between adjacent panels due to reactive forces at the free edge. |
| $\begin{bmatrix} T_{m,n} \end{bmatrix}$ | is the (displacement-angular flexibility and carry-over on the free edge) matrix which represents the slopes at the free edge due to reactive forces at the free edge. |
| $\begin{bmatrix} E_{m,n} \end{bmatrix}$ | is the (displacement flexibility and carry-over) matrix which represents the displacement at the free edge due to reactive forces at the free edge. |
| $\begin{bmatrix} Q^{Beam}_{\substack{A,m\\B,m}} \end{bmatrix}$ | is the (displacement angular carry-over) matrix which represents the rotation at the ends of the beam due to reactive forces on the beam. |

In Column Four:

| | |
|---|---|
| $\begin{bmatrix} Q^{Beam}_{\substack{m,A\\m,B}} \end{bmatrix}$ | is the (angular-displacement carry-over) matrix which represents the displacement of the beam due to end-moments on the beam. |
| $\begin{bmatrix} G^{Beam}_{B,A} \end{bmatrix}$ | is the (angular flexibility and carry-over) matrix which represents the end slopes of the beam due to end-moments on the beam. |

In Column Five:

| | |
|---|---|
| $\begin{bmatrix} M_{i} \end{bmatrix}$ | is the plate-moment matrix which represents the moments at the line of juncture between adjacent panels. |
| $\begin{bmatrix} M_{m} \end{bmatrix}$ | is the torque-moment matrix which represents the moments at the free edge. |

| | |
|---|---|
| $\left[ R_m \right]$ | is the reactive force matrix which represents the reactive forces at the free edge. |
| $\left[ M^{Beam}_{\substack{A\\B}} \right]$ | is the edge-beam end-moment matrix which represents the end moments on the edge beam. |

On Right Side of Equation:

| | |
|---|---|
| $\left[ \tau_{i,k} \right]$ | is the simply-supported boundary load function matrix which represents the slopes at the line of juncture between adjacent panels due to loads. |
| $\left[ \tau_{m,k} \right]$ | is the free edge load function matrix which represents the slopes at the free edge due to loads. |
| $\left[ \sum \delta_{m,k} \right]$ | is the free edge displacement load function matrix which represents the displacement of the free edge due to loads. |
| $\left[ \tau^{Beam}_{\substack{A\\B}} \right]$ | is the simply-supported load function matrix which represents end slopes of the edge beam due to loads. |

The matrix formulation of the general moment equation may be presented in the following reduced symbolic form:

$$\begin{bmatrix} -\delta \\ -T \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} R \\ M \\ T \end{bmatrix} \tag{4.19c}$$

Premultiplying both sides by $\begin{bmatrix} A \end{bmatrix}^{-1}$:

$$\begin{bmatrix} -\delta \\ -T \end{bmatrix} \begin{bmatrix} A \end{bmatrix}^{-1} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} A \end{bmatrix}^{-1} \begin{bmatrix} R \\ M \\ T \end{bmatrix} \tag{4.19d}$$

where: $\begin{bmatrix} A \end{bmatrix}^{-1}$ is the inverse of the matrix $\begin{bmatrix} A \end{bmatrix}$.

$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} A \end{bmatrix}^{-1} = \begin{bmatrix} I \end{bmatrix}$, the identity matrix.

Equation (4.19d) becomes the symbolic solution to the general moment equation by applying the identity matrix concept; thus,

$$\begin{bmatrix} R \\ M \\ T \end{bmatrix} = \begin{bmatrix} A \end{bmatrix}^{-1} \begin{bmatrix} -\delta \\ -T \end{bmatrix} \tag{4.19e}$$

# CHAPTER V

# ANGULAR AND DISPLACEMENT FUNCTIONS OF THE BASIC STRUCTURE

5.1. Load Functions. The following is a brief summarization of the load function definitions pertaining to loads applied to the basic plate structure.

Angular Load Functions, $\tau_{i,k}$ and $\tau_{m,k}$, are the edge slopes at points i and m, respectively, due to a unit load at point k.

Displacement Load Function, $\delta_{m,k}$, is the displacement of point m on the free edge due to a unit load at point k.

Throughout this chapter, the location of various points on the basic plate structure will be as defined on page 49 in Chapter IV. Consider a rectangular plate simply supported at its boundaries in the Y direction and a free edge condition in the X direction to be acted upon by a unit load at point k (Figure 5.1).

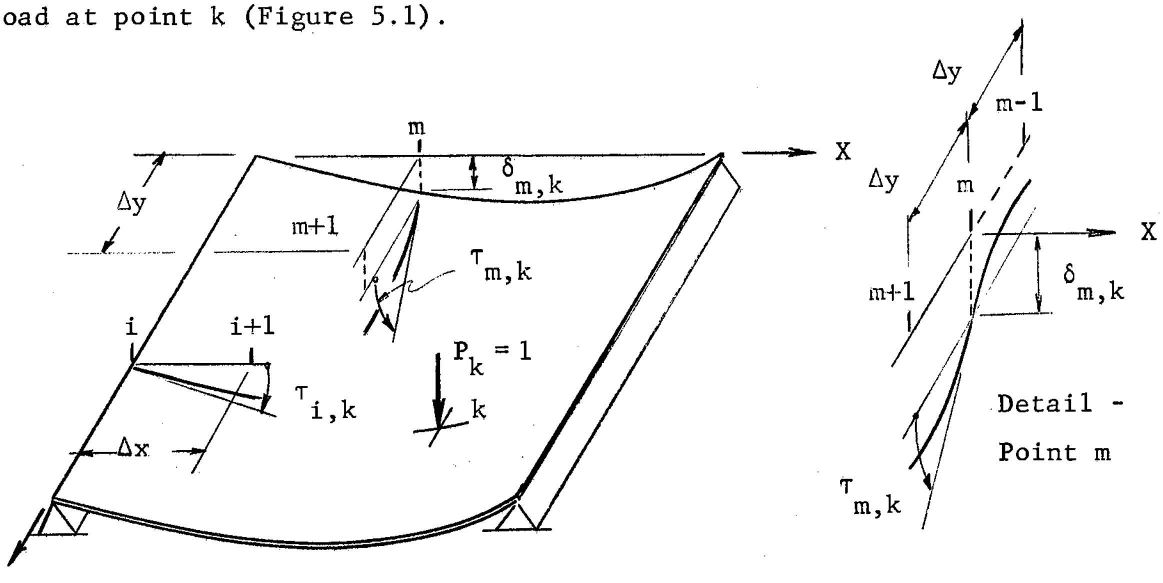


Figure 5.1

Angular and Displacement Load Functions due to $P_k = 1$

The plate is divided into an arbitrary number of equal size rectangular elements with dimensions $\Delta x$ and $\Delta y$ in the X and Y directions, respectively. The slope of the deflected curve can be approximated as a straight line between any two successive points as:

$$\theta_i = \frac{w_{i+1}}{\Delta x} \qquad \text{on the continuous edge} \qquad (5.1)$$

$$\theta_m = \frac{w_{m+1} - w_{m-1}}{2\Delta y} \qquad \text{on the free edge} \qquad (5.2)$$

where: $w_{i+1}$ is the vertical displacement of point i+1 with respect to point i.

$w_{m+1}$ and $w_{m-1}$ are the vertical deflections of point m+1 and the imaginary point m-1 with respect to point m.

From equation (7.3), reference (9), the displacement of point i+1 due to a unit load at point k is:

$$w_{i+1,k} = \frac{\Delta x \Delta y}{D} \eta^{k}_{i+1} \qquad (5.3)$$

where: D is the flexural rigidity of the plate.

$\eta^{k}_{i+1}$ is the influence coefficient for deflection at point i+1 due to a load at point k.

One should note that the equation for displacement of point i+1, or any other point, due to a load at any point k is:

$$w_{i+1,k} = \frac{\Delta x \Delta y}{D} P_k \eta^{k}_{i+1} \qquad (5.4)$$

where: $P_k$ is the load applied at point k.

Using equations (5.1) and (5.3), the angular load function on the simply supported continuous boundary is found to be

$$\tau_{i,k} = \frac{\frac{\Delta x \Delta y}{D} \eta^{k}_{i+1}}{\Delta x}$$

$$\tau_{i,k} = \frac{\Delta y}{D} \eta^{k}_{i+1} \qquad (5.5)$$

Substituting equation (2.10) into equation (5.2), the angular load function on the free edge is:

$$\tau_{m,k} = \frac{w_{m+1,k} - (2w_{m,k} - w_{m+1,k})}{2\Delta y}$$

$$\tau_{m,k} = \frac{w_{m+1,k} - w_{m,k}}{\Delta y}$$

Substituting equation (5.3) for the deflection terms in the above expressions,

$$\tau_{m,k} = \frac{1}{\Delta y}\left(\frac{\Delta x \Delta y}{D}\eta^k_{m+1} - \frac{\Delta x \Delta y}{D}\eta^k_m\right)$$

$$\tau_{m,k} = \frac{\Delta x}{D}\left(\eta^k_{m+1} - \eta^k_m\right) \qquad (5.7)$$

A direct substitution in equation (5.3) will yield the displacement load function; thus,

$$\delta_{m,k} = \frac{\Delta x \Delta y}{D}\eta^k_m \qquad (5.8)$$

5.2. Moment Flexibilities. The following is a brief summarization of moment function definitions.

Angular Flexibility, $F_{i,i}$, is the edge slope at point i due to a unit moment acting at point i.

Angular Carry-Over, $G_{j,i}$, is the edge slope at point j due to a unit moment acting at point i.

Angular Carry-Over on Free Edge, $H_{m,i}$, is the edge slope at point m on the free edge due to a unit moment acting at point i.

Angular-Displacement Carry-Over, $Q_{m,i}$, is the displacement of point m on the free edge due to a unit moment acting at point i.

Consider a rectangular plate simply supported at its boundary in the Y direction with a free edge condition in the X direction to be acted upon by a unit moment at point i (Figure 5.2).

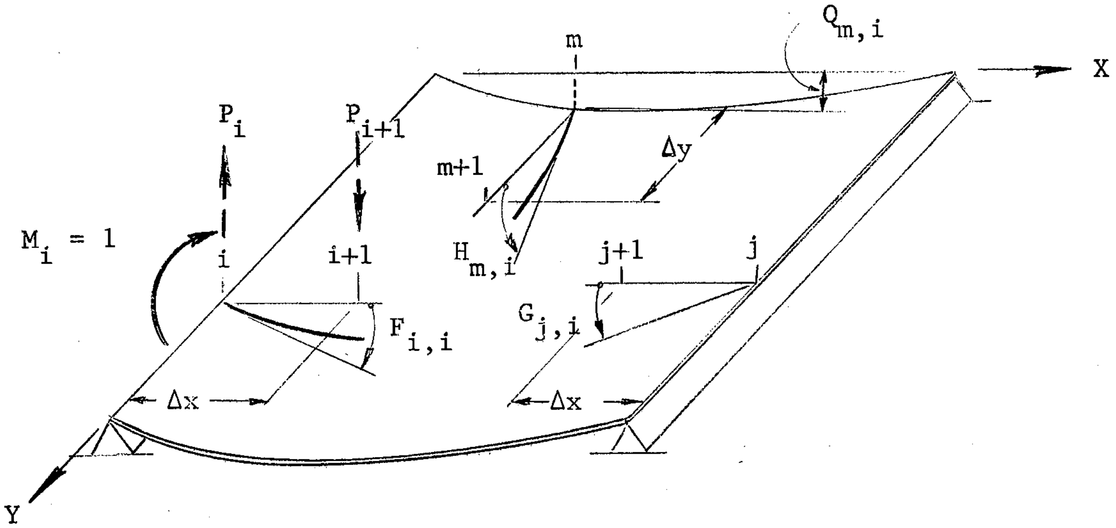


Figure 5.2

Angular and Displacement Moment Functions due to $M_i = 1$

Using equation (5.4), the deflections at point i+1 and j+1 are:

$$(w_{i+1})_{M_i=1} = w_{i+1,i+1} = \frac{\Delta x \Delta y}{D} P \, \eta_{i+1}^{i+1}$$

$$(w_{j+1})_{M_i=1} = w_{j+1,i+1} = \frac{\Delta x \Delta y}{D} P \, \eta_{j+1}^{i+1}$$

where: $P = \frac{M_i}{\Delta x} = \frac{1}{\Delta x}$

Thus,

$$w_{i+1,i+1} = \frac{\Delta y}{D} \eta_{i+1}^{i+1}$$

$$w_{j+1,i+1} = \frac{\Delta y}{D} \eta_{j+1}^{i+1}$$

Using equation (5.1) and the preceding expressions for deflections at points i+1 and j+1, the angular flexibility and carry-over factors are:

$$F_{i,i} = \frac{\Delta y}{\Delta x\ D}\ \eta_{i+1}^{i+1} \tag{5.9}$$

$$G_{j,i} = \frac{\Delta y}{\Delta x\ D}\ \eta_{j+1}^{i+1} \tag{5.10}$$

A direct substitution of equation (5.4) will yield the angular-displacement carry-over factor; thus,

$$Q_{m,i} = \frac{\Delta x \Delta y}{D}\ P\ \eta_{m}^{i+1}$$

$$Q_{m,i} = \frac{\Delta y}{D}\ \eta_{m}^{i+1} \tag{5.11}$$

Using equations (5.2) and (2.10) the angular carry-over on the free edge is:

$$H_{m,i} = \frac{w_{m+1,i+1} - w_{m,i+1}}{\Delta y}$$

Substituting equation (5.3) for the deflection terms in the above expression,

$$H_{m,i} = \frac{1}{\Delta y}\left(\frac{\Delta x \Delta y}{D}\ P\right)\left(\eta_{m+1}^{i+1} - \eta_{m}^{i+1}\right)$$

$$H_{m,i} = \frac{1}{D}\left(\eta_{m+1}^{i+1} - \eta_{m}^{i+1}\right) \tag{5.12}$$

5.3. Reaction Flexibilities. The following is a brief summarization of reaction function definitions.

Displacement Flexibility, $D_{m,m}$, is the displacement of point m on the free edge due to a unit reactive force at point m.

Displacement Carry-Over, $E_{n,m}$, is the displacement of point n on the free edge due to a unit reactive force at point m.

<u>Displacement-Angular Flexibility on Free Edge</u>, $S_{m,m}$, is the edge slope at point m due to a unit reactive force at point m.

<u>Displacement-Angular Carry-Over on Free Edge</u>, $T_{n,m}$, is the edge slope at point n due to a unit reactive force at point m.

<u>Displacement-Angular Carry-Over on Simply Supported Boundary</u>, $Q_{i,m}$, is the edge slope at point i due to a unit reactive force at point m.

Consider a rectangular plate simply supported at its boundary in the Y direction with a free edge condition in the X direction to be acted upon by a unit reactive force at point m as shown in Figure 5.3. The deflected surface of the plate is indicated to be in a downward direction, although the reactive force is being applied in an upward direction. The reaction flexibilities will be negative since the displacement and rotation will actually be upward.

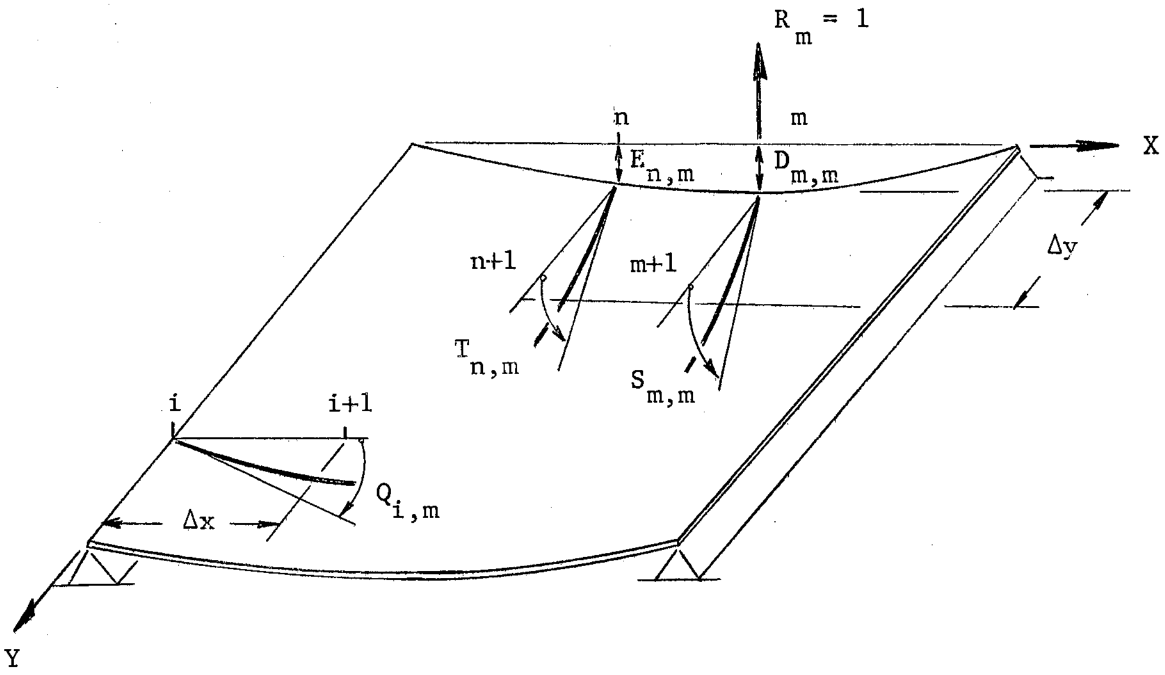


Figure 5.3

Displacement and Angular Reaction Functions due to $R_m = 1$

A direct substitution of equation (5.3) will yield the displacement flexibility and displacement carry-over factors; thus,

$$-D_{m,m} = \frac{\Delta x \Delta y}{D} \eta^{m}_{m} \tag{5.12}$$

$$-E_{n,m} = \frac{\Delta x \Delta y}{D} \eta^{m}_{n} \tag{5.13}$$

Using equation (5.3), the deflection at point i+1 is:

$$(w_{i+1})_{R_m=1} = w_{i+1,m} = \frac{\Delta x \Delta y}{D} \eta^{m}_{i+1}$$

Using equation (5.1) and the above expression for the deflection at point i+1, the displacement-angular carry-over factor is:

$$Q_{i,m} = \frac{\Delta y}{D} \eta^{m}_{i+1} \tag{5.14}$$

Using equations (5.2) and (2.10), the displacement-angular flexibility and the carry-over on the free edge are:

$$-S_{m,m} = \frac{w_{m+1,m} - w_{m,m}}{\Delta y}$$

$$-T_{n,m} = \frac{w_{n+1,m} - w_{n,m}}{\Delta y}$$

Substituting equation (5.3) for the deflection terms in the above expression for $S_{m,m}$ and $T_{n,m}$, respectively,

$$-S_{m,m} = \frac{1}{\Delta y}\left(\frac{\Delta x \Delta y}{D}\right)\left(\eta^{m}_{m+1} - \eta^{m}_{m}\right)$$

$$-T_{n,m} = \frac{1}{\Delta y}\left(\frac{\Delta x \Delta y}{D}\right)\left(\eta^{n}_{n+1} - \eta^{m}_{n}\right)$$

Simplifying,

$$-S_{m,m} = \frac{\Delta x}{D}\left(\eta^{m}_{m+1} - \eta^{m}_{m}\right) \tag{5.15}$$

$$-T_{n,m} = \frac{\Delta x}{D}\left(\eta^{m}_{n+1} - \eta^{m}_{n}\right) \tag{5.16}$$

5.4. Torsional-Moment Flexibilities - Free Edge Boundary. The following is a brief summarization of torsional-moment function.

Torsional Angular-Displacement Flexibility, $S_{m,m}$, is the edge displacement at point m due to a unit torque-moment applied at point m.

Torsional Angular-Displacement Carry-Over, $T_{n,m}$, is the edge displacement at point n due to a unit torque-moment applied at point m.

Torsional Angular Flexibility on Free Edge, $U_{m,m}$, is the edge slope at point m due to a unit torque-moment applied at point m.

Torsional Angular Carry-Over on Free Edge, $V_{n,m}$, is the edge slope at point n due to a unit torque-moment applied at point m.

Torsional Angular Carry-Over on Simply Supported Boundary, $H_{i,m}$, is the edge slope at point i due to a unit torque-moment applied at point m.

Consider a rectangular plate simply supported at its boundary in the Y direction with a free edge condition in the X direction to be acted upon by a unit torque-moment at point m (Figure 5.4).

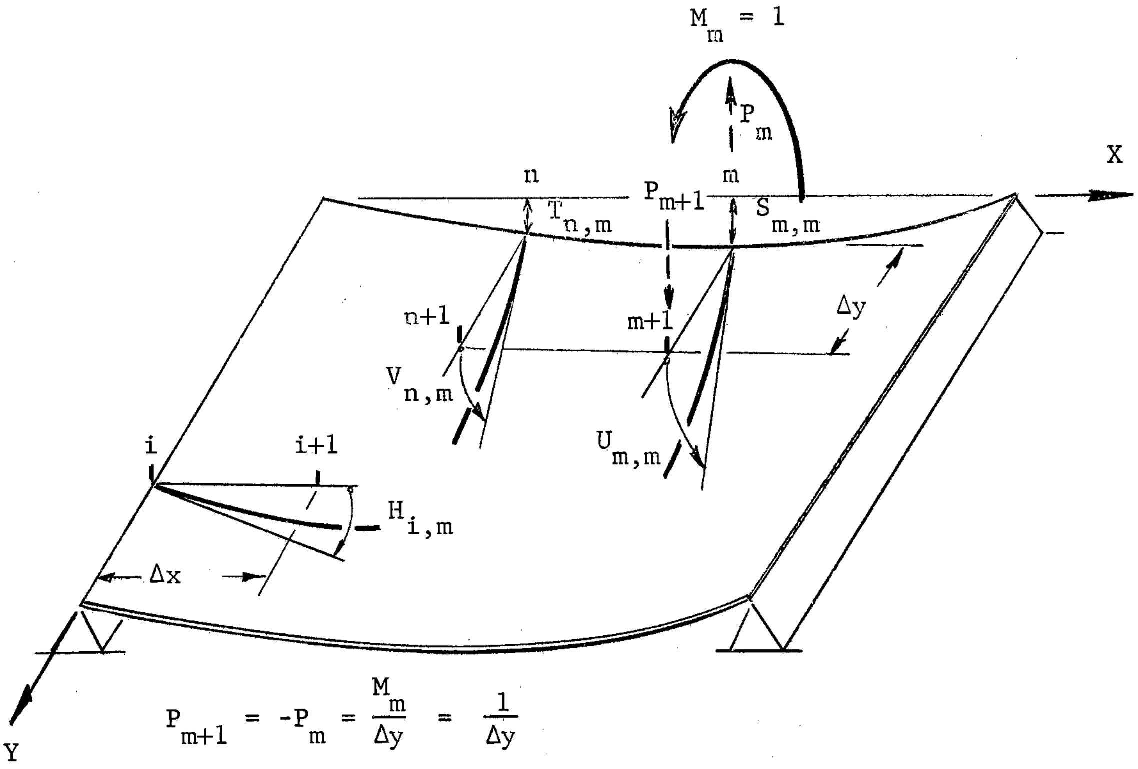


Figure 5.4

Displacement and Angular Torque-Moment Functions due to $M_m = 1$

The derivation of the displacement and angular torque-moment functions will be divided into two cases. Case I will pertain to the function containing $P_{m+1} = \frac{1}{\Delta y}$ in a downward direction causing positive rotations and deflections. Case II will pertain to the function containing $P_m = \frac{1}{\Delta y}$ in an upward direction causing negative rotations and deflections. The true value of the function will be a summation of Case I and Case II.

Substituting into equation (5.4) and noting that $P_m = \frac{1}{\Delta y}$, the torsional angular-displacement flexibility and carry-over factor for Case II loading are:

$$-(S_{m,m})_{II} = \frac{\Delta x}{D} \eta_m^m \tag{5.17a}$$

$$-(T_{n,m})_{II} = \frac{\Delta x}{D} \eta_n^m \tag{5.18a}$$

Substituting into equation (5.4) and noting that $P_{m+1} = \frac{1}{\Delta y}$ , the torsional angular-displacement flexibility and carry-over factor for Case I loading are:

$$(S_{m,m})_I = \frac{\Delta x}{D} \eta_m^{m+1} \tag{5.17b}$$

$$(T_{n,m})_I = \frac{\Delta x}{D} \eta_n^{m+1} \tag{5.18b}$$

Adding equations (5.17a) and (5.17b), the torsional angular-displacement flexibility is:

$$S_{m,m} = \frac{\Delta x}{D} \left( \eta_m^{m+1} - \eta_m^m \right) \tag{5.17}$$

Adding equations (5.18a) and (5.18b), the torsional angular-displacement carry-over is:

$$T_{n,m} = \frac{\Delta x}{D} \left( \eta_n^{m+1} - \eta_n^m \right) \tag{5.18}$$

Substituting equations (5.2) and (2.10) the torsional angular flexibility and carry-over factors are:

$$U_{m,m} = \frac{(w_{m+1} - w_m)_{T_m=1}}{\Delta y} \tag{5.19a}$$

$$V_{n,m} = \frac{(w_{n+1} - w_n)_{T_m=1}}{\Delta y} \tag{5.20b}$$

Substituting into equation (5.4) and noting that $P_m = \frac{1}{\Delta y}$ , the deflections at points m, m+1, n, and n+1 for load Case II are:

$$-(w_m)_{II} = \frac{\Delta x}{D} \eta_m^m \qquad -(w_n)_{II} = \frac{\Delta x}{D} \eta_n^m$$

$$-(w_{m+1})_{II} = \frac{\Delta x}{D} \eta_{m+1}^m \qquad -(w_{n+1})_{II} = \frac{\Delta x}{D} \eta_{n+1}^m$$

Substituting into equation (5.4) and noting that $P_{m+1} = \frac{1}{\Delta y}$, the deflections at points m, m+1, n, and n+1 for load Case I are:

$$(w_m)_I = \frac{\Delta x}{D} \eta_m^{m+1} \qquad (w_n)_I = \frac{\Delta x}{D} \eta_n^{m+1}$$

$$(w_{m+1})_I = \frac{\Delta x}{D} \eta_{m+1}^{m+1} \qquad (w_{n+1})_I = \frac{\Delta x}{D} \eta_{n+1}^{m+1}$$

Adding the appropriate deflection terms appearing in equation (5.19a) for Case I and Case II loading, the torsional angular flexibility factor on the free edge is:

$$U_{m,m} = \frac{1}{\Delta y} \frac{\Delta x}{D} \left[ \left( \eta_{m+1}^{m+1} - \eta_{m+1}^{m} \right) - \left( \eta_m^{m+1} - \eta_m^m \right) \right]$$

$$U_{m,m} = \frac{\Delta x}{\Delta y\ D} \left( \eta_{m+1}^{m+1} - \eta_{m+1}^{m} - \eta_m^{m+1} + \eta_m^m \right) \tag{5.19}$$

Adding the appropriate deflection terms appearing in equation (5.20a) for Case I and Case II loading, the torsional angular carry-over factor on the free edge is:

$$V_{n,m} = \frac{1}{\Delta y} \frac{\Delta x}{D} \left[ \left( \eta_{n+1}^{m+1} - \eta_{n+1}^{m} \right) - \left( \eta_n^{m+1} - \eta_n^m \right) \right]$$

$$V_{n,m} = \frac{\Delta x}{\Delta y\ D} \left( \eta_{n+1}^{m+1} - \eta_{n+1}^{m} - \eta_n^{m+1} + \eta_n^m \right) \tag{5.20}$$

Substituting into equation (5.1), the torsional angular carry-over factor on the simply supported boundary is:

$$H_{i,m} = \frac{(w_{i+1})_{T_m=1}}{\Delta x} \tag{5.21a}$$

Substituting into equation (5.4) and noting that $P_{m+1} = \frac{1}{\Delta y}$, the deflection at point i+1 for load Case I is:

$$(w_{i+1})_I = \frac{\Delta x}{D} \eta^{m+1}_{i+1}$$

Substituting into equation (5.4) and noting that $P_m = \frac{1}{\Delta y}$, the deflection at point i+1 for load Case II is:

$$-(w_{i+1})_{II} = \frac{\Delta x}{D} \eta^{m}_{i+1}$$

Adding the above deflection terms and substituting into equation (5.21a),

$$H_{i,m} = \frac{1}{D}\left(\eta^{m+1}_{i+1} - \eta^{m}_{i+1}\right) \tag{5.21}$$

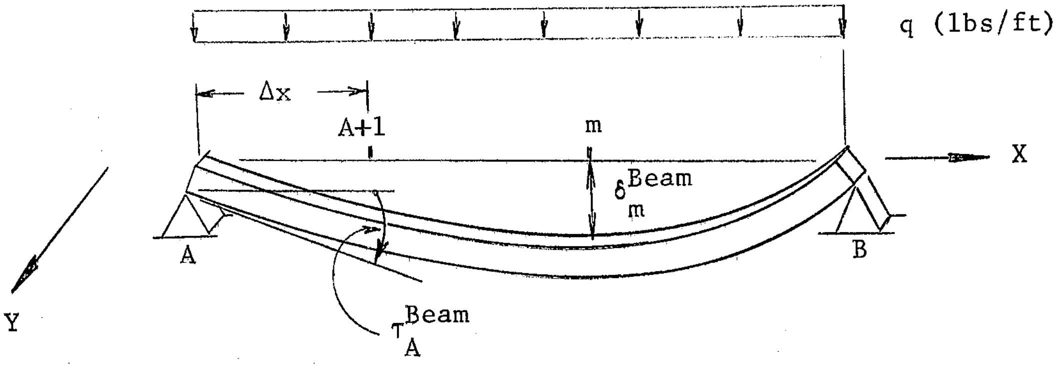
5.5. Beam Flexibilities - Dead Load. The edge beam will have a constant rectangular cross-sectional area and will be simply supported at the rigid panel points. Vertical deflection and twist of the edge beam will occur at all points on the edge beam except at the point of rigid, simple support. The edge beam is considered to be continuous in the X direction and is of adequate size to alter the free edge deflections of the basic plate structure.

The following is a brief summarization of beam flexibilities pertaining to the dead load forces.

Angular Load Function, $\tau^{Beam}_{A}$, is the end slope at A due to the weight of the beam.

Displacement Load Function, $\delta^{Beam}_{m}$, is the displacement at point m due to the weight of the beam.

Consider a constant cross-sectional, rectangular, simply supported edge beam to be loaded by a uniform dead load (Figure 5.5).



$q = \gamma bh$

$b$ = width of beam

$\gamma$ = specific weight of beam

$h'$ = height of beam

Figure 5.5

Edge Beam Flexibilities - Dead Load, q

Applying equation (5.1) for the edge rotation of a plate to the end rotation of an edge beam, the angular load function is:

$$\sum \tau_A^{Beam} = \frac{\sum w_{A+1}^q}{\Delta x}$$

where: $\sum w_{A+1}^q$ = the deflection at point A+1, which is a distance Δx from the simply supported end A, due to the uniform load q.

The uniform load, q, is divided into a series of equal concentrated loads, $P_m$, which are:

$$P_m = q\Delta x = \gamma bh'\Delta x$$

Applying equation (3.5b), the deflection at point A+1 due to concentrated loads at all stations on the edge beam Δx distance apart is:

$$w_{A+1}^q = P_m \frac{\Delta x^3}{EI} \sum_{n=1}^{7} \eta_{A+1}$$

Substituting for $P_m$:

$$w^q_{A+1} = \gamma b h' \frac{\Delta x^4}{EI} \sum_{n=1}^{7} \eta_{A+1}$$

Substituting the above expression into the expression for the angular load function, the following is obtained:

$$\sum \tau^{Beam}_A = \gamma b h' \frac{\Delta x^3}{EI} \sum_{n=1}^{7} \eta_{A+1} \tag{5.22}$$

Directly applying equation (3.5b) as a summation of concentrated loads at all stations on the edge beam Δx distance apart, the displacement load function is:

$$\sum \delta^{Beam}_m = P_m \frac{\Delta x^3}{EI} \sum_{n=1}^{7} \eta_m$$

$$\sum \delta^{Beam}_m = \gamma b h' \frac{\Delta x^4}{EI} \sum_{n=1}^{7} \eta_m \tag{5.23}$$

5.6. <u>Beam Flexibilities - End Moment</u>. The following is a brief summarization of beam flexibilities pertaining to an applied end moment.

<u>Angular Flexibility</u>, $F^{Beam}_{A,A}$, is the end slope at A due to a unit moment applied at A.

<u>Angular Carry-Over</u>, $G^{Beam}_{B,A}$, is the end slope at B due to a unit moment applied at A.

<u>Angular-Displacement Carry-Over</u>, $Q^{Beam}_{m;A}$, is the displacement at point m due to a unit moment applied at A.

Consider a constant cross-sectional, rectangular, simply supported edge beam to be acted upon by a unit moment applied at end A (Figure 5.6).

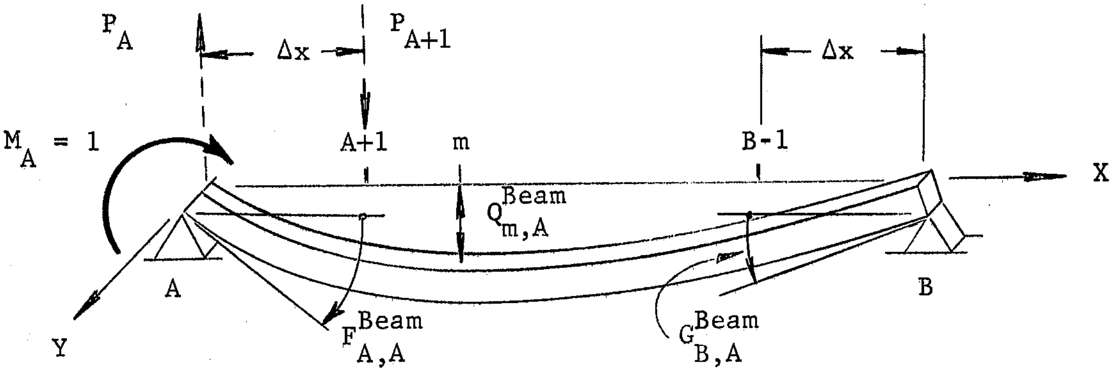


Figure 5.6

Edge Beam Flexibilities - End Moment, $M_A = 1$

Applying equation (5.1) to the end rotation of an edge beam, the angular flexibility and carry-over factors are:

$$F^{Beam}_{A,A} = \frac{(w_{A+1})_{M_A=1}}{\Delta x} = \frac{w_{A+1,A+1}}{\Delta x}$$

$$G^{Beam}_{B,A} = \frac{(w_{B-1})_{M_A=1}}{\Delta x} = \frac{w_{B-1,A+1}}{\Delta x}$$

Applying equation (3.5b), the deflections at points A+1, B-1, and m due to $P_{A+1} = \frac{1}{\Delta x}$ are:

$$w_{A+1,A+1} = P_{A+1} \frac{\Delta x^3}{EI} \eta^{A+1}_{A+1}$$

$$= \frac{\Delta x^2}{EI} \eta^{A+1}_{A+1}$$

$$w_{B-1,A+1} = P_{A+1} \frac{\Delta x^3}{EI} \eta^{A+1}_{B-1}$$

$$= \frac{\Delta x^2}{EI} \eta^{A+1}_{B-1}$$

$$w_{m,A+1} = Q^{Beam}_{m,A} = P_{A+1} \frac{\Delta x^3}{EI} \eta^{A+1}_{m}$$

$$Q^{Beam}_{m,A} = \frac{\Delta x^2}{EI} \eta^{A+1}_{m} \qquad (5.24)$$

Substituting the values for deflection at points A+1 and B-1 into the expressions for angular flexibility and carry-over, the following is obtained:

$$F^{Beam}_{A,A} = \frac{\Delta x}{EI} \eta^{A+1}_{A+1} \qquad (5.25)$$

$$G^{Beam}_{B,A} = \frac{\Delta x}{EI} \eta^{A+1}_{B-1} \qquad (5.26)$$

5.7. Beam Flexibilities - Reactive Force. The following is a brief summarization of beam flexibilities pertaining to a unit reactive force.

Displacement Flexibility, $D^{Beam}_{m,m}$, is the displacement at point m due to a unit reactive force applied at point m.

Displacement Carry-Over, $E^{Beam}_{n,m}$, is the displacement at point n due to a unit reactive force applied at point m.

Displacement Angular Carry-Over, $Q^{Beam}_{A,m}$, is the end rotation at A due to a unit reactive force applied at point m.

Consider a constant cross-sectional, rectangular, simply supported edge beam to be loaded with a unit reactive force at point m (Figure 5.7).

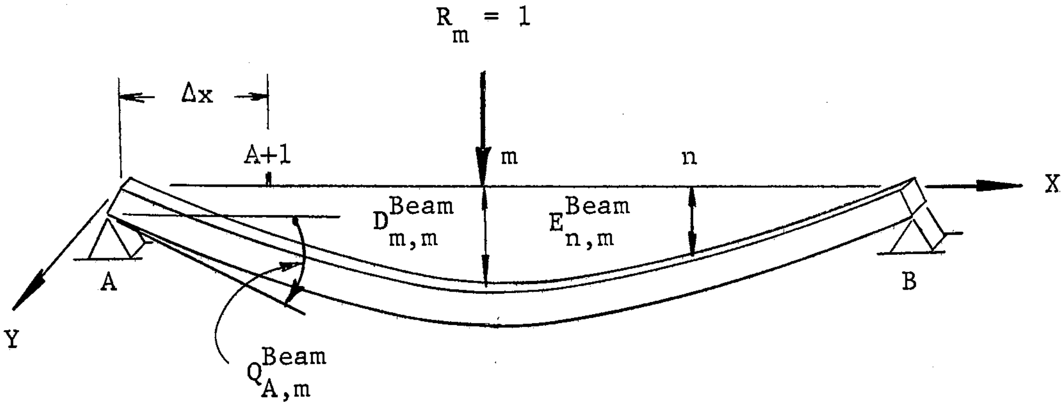


Figure 5.7

Edge Beam Flexibilities - Unit Reactive Force, $R_m = 1$

Applying equation (5.1) to the end rotation of an edge beam, the displacement angular carry-over factor is:

$$Q^{Beam}_{A,m} = \frac{(w_{A+1})_{R_m=1}}{\Delta x} = \frac{w_{A+1,m}}{\Delta x}$$

Applying equation (3.5b), the deflections at points A+1, m, and n due to $R_m = 1$ are:

$$w_{A+1,m} = \frac{\Delta x^3}{EI} \eta^m_{A+1}$$

$$w_{m,m} = D^{Beam}_{m,m} = \frac{\Delta x^3}{EI} \eta^m_m \qquad (5.27)$$

$$w_{n,m} = E^{Beam}_{n,m} = \frac{\Delta x^3}{EI} \eta^m_n \qquad (5.28)$$

Substituting the deflection at point A+1 into the above expression for the displacement angular carry-over factor, the following is obtained;

$$Q^{Beam}_{A,m} = \frac{\Delta x^2}{EI} \eta^m_{A+1} \tag{5.29}$$

5.8. Beam Flexibilities - Torsional Moment. The following is a brief summarization of beam flexibilities pertaining to an applied torque moment.

Torsional Angular Flexibility, $U^{Beam}_{m,m}$, is the rotation at point m due to a unit torsional moment applied at point m.

Torsional Angular Carry-Over, $V^{Beam}_{n,m}$, is the rotation at point n due to a unit torsional moment applied at point m.

Consider a constant cross-sectional, rectangular, simply supported edge beam to be acted upon by a unit torque-moment (Figure 5.8).

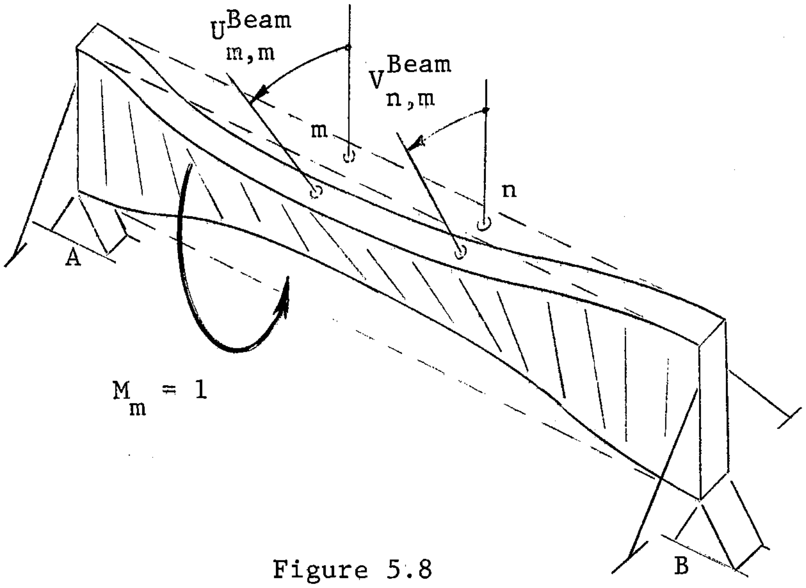


Figure 5.8

Beam Flexibilities - Torque-Moment, $M_m = 1$

The algebraic derivation of the torsional angular flexibility and carry-over factor will not be developed because of the minor influence which the rotation of an edge beam would effect on the over-all deflection of the plate structure.

# CHAPTER VI

## EXAMPLE PROBLEM

6.1. Statement of Problem. A uniformly loaded two-span plate structure of constant thickness, continuous in one direction over rigid supports, and with flexible longitudinal edge beams is considered (Figure 6.1). The uniform load (p) over the entire area of the plate is taken as 100 lbs. per sq. ft., and Poisson's ratio ($\mu$) is taken to be equal to zero. The plate is 6 inches thick, and each panel has a width of 30 feet and a length of 35 feet. The depth and width of the edge beam are 2 feet and 1 foot, respectively. The ratio of plate rigidity to beam rigidity, $\frac{D}{EI}$, is equal to $\frac{1}{64}$. In the solution of the problem all values, unless otherwise stated, are in units of kips, feet, and kip-feet. The torsional moment developed in the edge beam will be neglected.

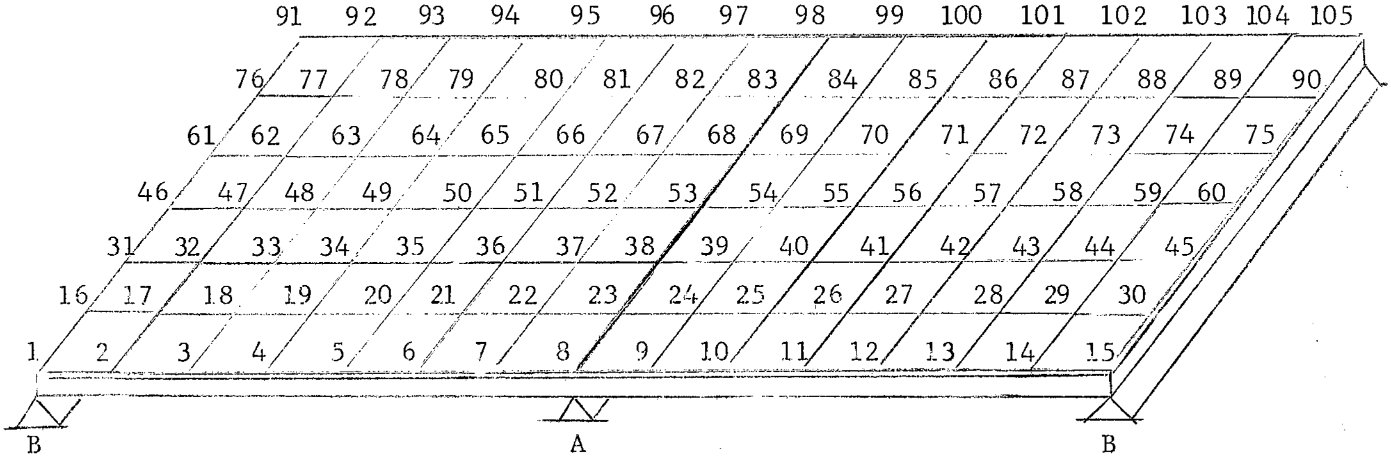


Figure 6.1

Two-Span Continuous Rectangular Plate

6.2. *Evaluation of Angular and Displacement Flexibilities and Load Functions*. The matrix form of the moment-reaction equation illustrated by equation (4.19b) will be used to formulate the problem. The equation(s) pertaining to each individual matrix element will be described in the following manner:

|  | POSITION |  |  |
|---|---|---|---|
| MATRIX ELEMENT | ROW | COLUMN | RELATING EQUATIONS |
| $\begin{bmatrix} G_{j,i} \end{bmatrix}$ | 1 | 1 | (5.9) and (5.10) |
| $\begin{bmatrix} Q_{i,m} \end{bmatrix}$ | 1 | 2 | (5.14) |
| $\begin{bmatrix} Q_{m,i} \end{bmatrix}$ | 2 | 1 | (5.11) |
| $\begin{bmatrix} E_{m,n} \end{bmatrix}$ | 2 | 2 | (5.12) and (5.13) |
| $\begin{bmatrix} Q^{\text{Beam}}_{\substack{m,A\\m,B}} \end{bmatrix}$ | 2 | 3 | (5.24) |
| $\begin{bmatrix} Q^{\text{Beam}}_{\substack{A,m\\B,m}} \end{bmatrix}$ | 3 | 2 | (5.29) |
| $\begin{bmatrix} G^{\text{Beam}}_{B,A} \end{bmatrix}$ | 3 | 3 | (5.25) and (5.26) |
| $\begin{bmatrix} -\tau_{i,k} \end{bmatrix}$ | 1 | 5 | (5.5) |
| $\begin{bmatrix} -\delta_{m,k} + \delta^{\text{Beam}}_{m,k} \end{bmatrix}$ | 2 | 5 | (5.8) and (5.23) |
| $\begin{bmatrix} \tau^{\text{Beam}}_{\substack{A\\B}} \end{bmatrix}$ | 3 | 5 | (5.22) |

| MATRIX ELEMENT | POSITION | | UNKNOWN QUANTITY |
|---|---|---|---|
| | ROW | COLUMN | |
| $\begin{bmatrix} M_i \end{bmatrix}$ | 1 | 4 | $M_8$, $M_{23}$, $M_{38}$, $M_{53}$ |
| $\begin{bmatrix} R_m \end{bmatrix}$ | 2 | 4 | $R_2$, $R_3$, $R_4$, $R_5$, $R_6$, $R_7$ |
| $\begin{bmatrix} M_A \\ B \end{bmatrix}$ | 3 | 4 | $M_A$ |

The following equalities exist between the unknown quantities due to the symmetry of the structure and the symmetrical loading:

$M_8 = M_{98}$

$M_{23} = M_{83}$

$M_{38} = M_{68}$

$R_2 = R_{92} = R_{14} = R_{104}$

$R_3 = R_{93} = R_{13} = R_{103}$

$R_4 = R_{94} = R_{12} = R_{102}$

$R_5 = R_{95} = R_{11} = R_{101}$

$R_6 = R_{96} = R_{10} = R_{100}$

$R_7 = R_{97} = R_9 = R_{99}$

6.3. *Matrix Moment-Reaction Solution Equation.* Figure 6.2 is the solution matrix formulation of the moment-reaction equation (4.19b) with each of the matrix elements evaluated.

$$
\left[\begin{array}{ccccccccccc}
1.608021 & 0.649589 & 0.530022 & 0.246337 & 4.113270 & 7.790750 & 10.595130 & 12.077540 & 11.715510 & 8.623700 & \\
0.649589 & 1.160045 & 0.587909 & 0.278176 & 3.869980 & 7.272170 & 9.737440 & 10.788010 & 9.910300 & 6.495890 & \\
0.530022 & 0.587909 & 1.087205 & 0.330571 & 3.689950 & 6.890200 & 9.101930 & 9.847610 & 8.677320 & 5.480220 & \\
0.492674 & 0.556352 & 0.661142 & 0.786602 & 3.623600 & 6.746140 & 8.868560 & 9.507900 & 8.251700 & 4.926740 & \\
2.056635 & 1.934990 & 1.844975 & 0.905900 & 25.907470 & 34.636625 & 36.146230 & 32.104445 & 23.569145 & 12.468815 & 0.43713 \\
3.895375 & 3.636085 & 3.443900 & 1.686535 & 34.636625 & 61.049140 & 66.563240 & 59.713455 & 44.396040 & 23.569145 & 0.81845 \\
5.297565 & 4.868720 & 4.550965 & 2.217140 & 36.146230 & 66.563240 & 84.937475 & 79.308045 & 59.713455 & 32.104445 & 1.08817 \\
6.038770 & 5.394005 & 4.923805 & 2.376975 & 32.104445 & 59.713455 & 79.308045 & 84.937475 & 66.563240 & 36.146230 & 1.19048 \\
5.857755 & 4.955150 & 4.338660 & 2.062925 & 23.569145 & 44.396040 & 59.713455 & 66.563240 & 61.049140 & 34.636625 & 1.06957 \\
4.311850 & 3.247945 & 2.740110 & 1.231685 & 12.468815 & 23.569145 & 32.104445 & 36.146230 & 34.636625 & 25.907470 & 0.66964 \\
 & & & & 0.87462 & 1.63690 & 2.17634 & 2.38096 & 2.13914 & 1.33928 & 0.26786
\end{array}\right]
\left[\begin{array}{c}
M_8 \\ M_{23} \\ M_{38} \\ M_{53} \\ -R_2 \\ -R_3 \\ -R_4 \\ -R_5 \\ -R_6 \\ -R_7 \\ -M_A
\end{array}\right]
=
\left[\begin{array}{r}
-\ 418.142 \\ -\ 410.090 \\ -\ 404.127 \\ -\ 402.403 \\ -1005.804 \\ -1795.120 \\ -2225.359 \\ -2225.359 \\ -1795.120 \\ -1005.804 \\ +\ 15.820
\end{array}\right]
$$

Figure 6.2

Solution Matrix for Moment-Reaction Equation

The solution of this matrix is in the form of equation (4.19e), and the values of the unknown quantities are as follows:

$M_8 = -122.4$

$M_{23} = -107.9$

$M_{38} = -\ 93.1$

$M_{53} = -\ 79.4$

$M_A = -359.7$

$R_2 = 10.0$

$R_3 = 12.5$

$R_4 = 15.0$

$R_5 = 17.6$

$R_6 = 12.2$

$R_7 = 10.7$

# CHAPTER VII

## SUMMARY AND CONCLUSIONS

7.1. Summary. An analysis by the flexibility method of a one-way continuous rectangular plate-structure over rigid supports with longitudinal flexible edge beams is presented in this thesis. The major steps comprising the discussion are as follows:

1. The continuous structure is reduced to a basic rectangular plate structure, and the support moments and edge reactions are selected as redundants. A general set of deflection equations are developed satisfying the boundary conditions pertaining to each particular point by utilizing the method of finite differences.
2. A table of deflection influence coefficients for the plate is developed from the deflection equations written at all points on the basic plate structure. Also, a table of beam influence coefficients is presented.
3. A set of general moment-reaction equations is written in terms of the plate and beam flexibilities by utilizing the conditions of compatibility of deformations over a continuous support and between the plate and edge beam. The moment-reaction equation written in matrix form yields a solution to the continuous plate-beam problem.
4. The angular and displacement flexibilities are expressed in terms of the deflection influence coefficients.

5. The use of the method in the analysis of a one-way continuous plate-beam structure is illustrated by an example problem.

7.2. Conclusions. The most significant conclusions drawn from this study can be stated as follows:

1. The deflection influence coefficients which are obtained from a fifty-six point set are of sufficient accuracy to solve most problems. The table of influence coefficients presented is for a length-width ratio of 1.17:1. Similar tables may be developed for any particular length-width ratio by inverting the coefficient matrix of the deflection equations.
2. Any conceivable system of loads can be applied to the structure and easily analyzed by use of the table of influence coefficients.
3. The flexibility method, when applied to a symmetrical structure, permits a significant reduction in the number of unknown quantities.
4. The general moment-reaction matrix can easily be modified to solve for any combination of redundants or to solve for support-moments separately.

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