DYNAMIC ANALYSIS OF BEAMS
BY THE FLEXIBILITY METHOD

By

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Stillwater, Oklahoma

Boen-Dar Liaw

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NOMENCLATURE

a_j	Length parameter of Segment $\overline{ij}$
b_j	Stiffness Parameter of Segment $\overline{ij}$
d_j	Length of Segment $\overline{ij}$
f_j	Natural Frequency of jth Mode
m_j	Mass Concentrated at Point j
p_j	Natural Circular Frequency of jth Mode
r	Number of Intermediate Supports
s	Number of Extreme End Restraints
t	Time Variable
y_j	Displacement of Mass m_j
F	Angular Flexibility
G	Carry-over Value
K	Linear Stiffness
L	Span Length
M	Moment
R_j	Reaction at Point j
V	Shear
Y	Shape Function
α	Phase Angle
ζ	Angular Flexibility Parameter
ρ	Mass per Unit Length of Beam
θ	Slope
ϕ_j	Angle Change at Point j
Ψ	Linear Flexibility
ω_{ij}	Slope of String Line $\overline{ij}$

CHAPTER I

INTRODUCTION

1.1 General

The vibration of a continuous elastic system may be conveniently analyzed by replacing the given system by a certain number of concentrated masses connected by massless elastic strings, therefore, reducing the number of degrees-of-freedom to a finite value.

Only pure bending is considered in this thesis; the effects of rotatory inertia and shearing deformation are neglected, although,these latter effects may be appreciable in the higher modes of vibration.

In the consideration of the free vibration of a beam in bending, the usual limitations of beam theory are utilized. The representation of the elastic curve of a straight beam as a differential string polygon was introduced by Mohr (1), in connection with his concept of elastic weights and the conjugate beam. Application to static analysis has been developed by Tuma and Oden (2), as the string polygon method.

The free vibration analysis of beams is one of solving a boundary value problem. The differential equations of motion are set up by equating the elastic restoring forces of the beam and the inertia forces of the concentrated masses.

The analysis of the problem falls mainly on the formulation of the matrix equation of motion for the "equivalent" system. Once the influence coefficient matrix is found, normal iteration techniques are applied to determine the modes and frequencies of the system. Orthogonallity conditions of the modes are employed to find the higher mode characteristics.

Basic theory is derived in the second chapter of this thesis. The general string polygon equation is used to relate the end moments of segments and the displacements of masses at the ends of the segments. Theory for the general case is derived, and is simplified for some special conditions.

The third chapter deals with the application of the general theory to some special end conditions. Tables, indicating the matrix formulation for single span beams of constant section, are presented. The procedure of analysis for both single span and multi-span beams is formulated.

In the fourth chapter, three numerical examples are solved to illustrate the theory presented in this thesis. Results are compared with those by other methods. Summary and conclusions of this study are included in the final chapter. Also, desirable future investigations are indicated.

1.2 Historical

"Exact" solutions for single span beams with constant cross section are easily obtained by solving the differential equations of motion. However, for the non-uniform section beam, no exact solution is available. Approximate solutions for the general case of a vibrating beam were developed by Rayleigh (3, 4) in 1877. In his treatise on the theory of sound, Lord Rayleigh calculated the fundamental natural frequency from an assumed shape for the dynamic deflection curve of the system. In 1921, Holzer (5) presented a method for solving the torsional vibration problem of a mechanical shaft by first assuming a frequency, and then calculating the deflection curve. In 1924, Stodola (6) extended Rayleigh's principle to the calculation of higher mode frequencies. Holzer's method was extended to the determination of undamped bending vibrations by Myklestad (7) in 1944, and is currently (1963) being applied in vibration analysis. In 1948, Young (8) presented a method to determine the natural frequencies of a composite system, which consisted of a uniform beam with a concentrated mass, spring and dashpot. The vibration of a non-uniform beam was analyzed employing matrix methods by Thomson (9) in 1950. In 1952, Lee and Saibel (10) developed an expression from which the frequency equation for vibration of a constrained beam with any combination of intermediate elastic or rigid supports can be found readily. In 1956, Ellington (11) presented a method of analyzing the free vibration of segmented beams. Vibration of stepped beams were analyzed from the theory of integral equations by Taleb (12) in 1961.

In 1921, Darney (13) found the natural frequencies of continuous beams using determinants. The application of the known natural frequencies and natural modes of ordinary beams to find the natural frequencies and modes of continuous beams was accomplished by Saibel (14) in 1944. In 1950, Ayre and Jacobsen (15) used a simple graphical network to determine the natural frequencies of bending vibration of a continuous beam having any number of spans of uniform length and section. In 1955, a method of calculating the natural frequencies of undamped flexural vibrations of continuous beams on rigid supports was presented by Veletsos and Newmark (16). A method for determining the natural frequencies of continuous beams on flexible supports was also presented by Veletsos and Newmark (17) in 1954.

CHAPTER II

DERIVATION OF THE BASIC THEORY

2.1 Statement of the Problem

A beam segment AB is considered (Fig. 2.1). The distributed mass system is simulated by an "equivalent" lumped mass system consisting of concentrated masses connected by elastic elements: (1, 2), (2, 3), ... (i, j), (j, k), ... (n-1, n).

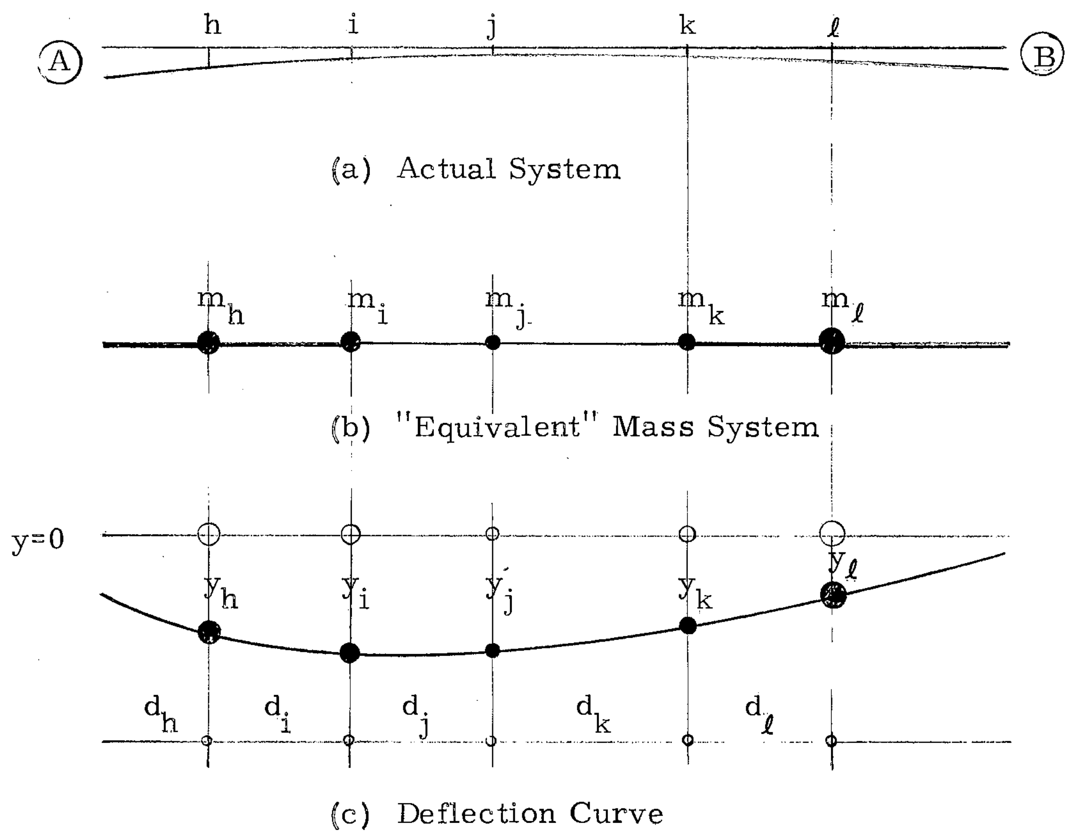


Fig. 2.1 Beam Segment

2.2 Moment-displacement Relation

Two adjacent segments ij and jk are considered (Fig. 2.2). Without intermediate loads on the segments ij and jk, the angle change ϕ_j at j can be expressed in the following form (2):

$$\phi_j = G_{ji} M_i + \Sigma F_j M_j + G_{jk} M_k \tag{2.1}$$

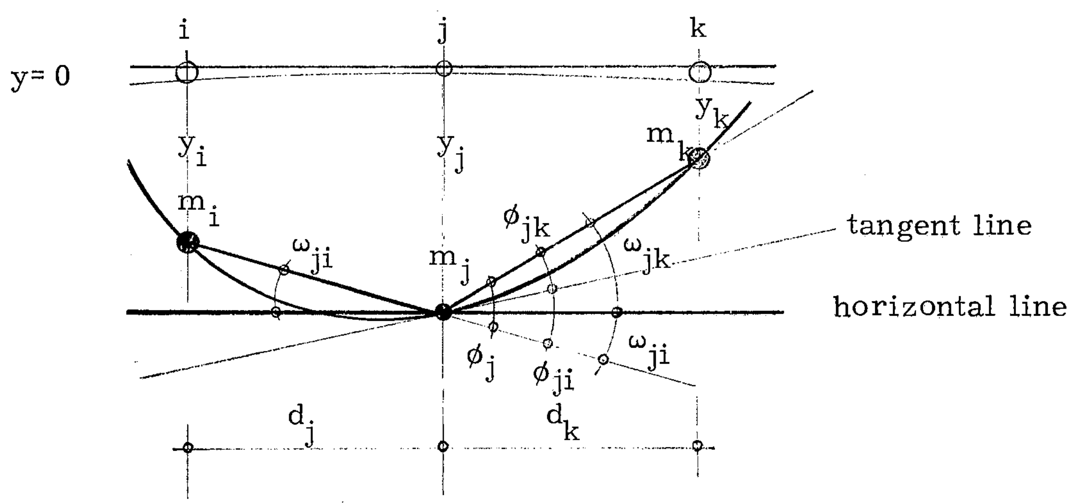


Fig. 2.2 Segment $\overline{ijk}$

From geometry,

$$\phi_j = \phi_{ji} + \phi_{jk}$$

also,

$$\phi_j = \omega_{ji} + \omega_{jk}$$

or, in terms of displacements

$$\phi_j = \frac{y_j - y_i}{d_j} + \frac{y_j - y_k}{d_k}$$

Let

d_o = length of the reference segment

$a_j = \dfrac{d_o}{d_j}$ = segmental ratio then,

$$\phi_j = \frac{1}{d_o}\left[-a_j\, y_i + (a_j + a_k)\, y_j - a_k\, y_k\right] \tag{2.2}$$

thus, the relation between moments and displacements can be stated as follows:

$$G_{ji}\, M_i + \Sigma F_j\, M_j + G_{jk}\, M_k$$

$$= \frac{1}{d_o}\left[-a_j\, y_i + (a_j + a_k)\, y_j - a_k\, y_k\right] \tag{2.3}$$

For a system composed of n concentrated masses, Eq. 2.3 can be written in the following form:

$$\begin{bmatrix} \cdots & \cdots & & & & & & & \\ \cdots & \cdots & \cdots & & & & & & \\ & \cdots & \cdots & \cdots & & & & & \\ & & G_{ih} & \Sigma F_i & G_{ij} & & & & \\ & & & G_{ji} & \Sigma F_j & G_{jk} & & & \\ & & & & G_{kj} & \Sigma F_k & G_{k\ell} & & \\ & & & & & \cdots & \cdots & \cdots & \\ & & & & & & \cdots & \cdots & \cdots \\ & & & & & & & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ M_h \\ M_i \\ M_j \\ M_k \\ M_\ell \\ \cdot \\ \cdot \end{bmatrix}$$

$$= \frac{1}{d_o} \begin{bmatrix} \cdots & \cdots & & & & & & & \\ \cdots & \cdots & \cdots & & & & & & \\ & \cdots & \cdots & \cdots & & & & & \\ & & -a_i & a_i + a_j & -a_j & & & & \\ & & & -a_j & a_j + a_k & -a_k & & & \\ & & & & -a_k & a_k + a_\ell & -a_\ell & & \\ & & & & & \cdots & \cdots & \cdots & \\ & & & & & & \cdots & \cdots & \cdots \\ & & & & & & & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ y_h \\ y_i \\ y_j \\ y_k \\ y_\ell \\ \cdot \\ \cdot \end{bmatrix},$$

or in the symbolic form

$$\left[F\right]\left[M\right] = \frac{1}{d_o}\left[a_1\right]\left[y\right] \tag{2.4}$$

from which

$$\left[M\right] = \frac{1}{d_o}\left[F\right]^{-1}\left[a_1\right]\left[y\right] \tag{2.5}$$

where

$\left[F\right]$ = angular flexibility matrix

$\left[a_1\right]$ = first segmental parametric matrix

F_{ij} = slope at end i of the simply supported segment ij due to a unit moment acting at end i.

G_{ij} = slope at end i of the simply supported segment ij due to a unit moment acting at end j.

For constant section throughout segments ij, jk,

$$F_{ij} = F_{ji} = \frac{d_j}{3EI_j} \quad \Bigg| \quad F_{jk} = F_{kj} = \frac{d_k}{3EI_k}$$

$$G_{ij} = G_{ji} = \frac{d_j}{6EI_j} \quad \Bigg| \quad G_{jk} = G_{kj} = \frac{d_k}{6EI_k}$$

2.3 Moment-acceleration Relation

The dynamic equilibrium of mass m_j, which is elastically restrained by strings ij and jk, is considered (Fig. 2.3). For transverse vibration, the mass is idealized to be displacing in the vertical direction only. The forces acting on the mass are the "inertia" force and the end shears, which can be expressed in terms of the moments at the ends of the strings.

Summation of forces in the vertical direction yields

$$V_{ji} + m_j \ddot{y}_j + V_{jk} = 0$$

but

$$V_{ji} = \frac{M_j - M_i}{d_j}, \quad V_{jk} = \frac{M_j - M_k}{d_k}$$

thus,

$$m_j \ddot{y}_j = - \frac{M_j - M_i}{d_j} - \frac{M_j - M_k}{d_k}$$

or

$$m_j \ddot{y}_j = - \frac{1}{d_o} \left[- a_j M_i + (a_j + a_k) M_j - a_k M_k \right] \tag{2.6}$$

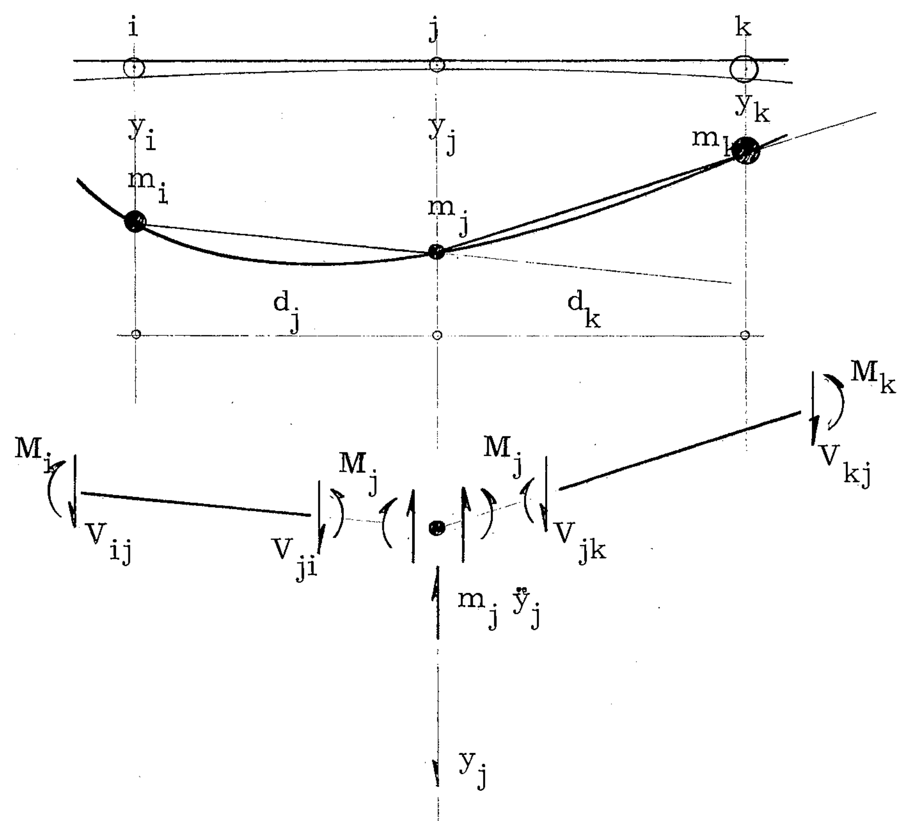


Fig. 2.3 Free-body Diagram of Mass m_j

For n masses, a set of equations can be written in matrix form;

$$\begin{bmatrix} \cdot & & & & & & & & \\ & \cdot & & & & & & & \\ & & \cdot & & & & & & \\ & & & m_i & & & & & \\ & & & & m_j & & & & \\ & & & & & m_k & & & \\ & & & & & & \cdot & & \\ & & & & & & & \cdot & \\ & & & & & & & & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \ddot{y}_i \\ \ddot{y}_j \\ \ddot{y}_k \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

$$= -\frac{1}{d_o} \begin{bmatrix} \cdots\cdots & \cdots & & & & & & \\ \cdots & \cdots\cdots & \cdots & & & & & \\ & \cdots & \cdots\cdots & \cdots & & & & \\ & & -a_i & a_i+a_j & -a_j & & & \\ & & & -a_j & a_j+a_k & -a_k & & \\ & & & & -a_k & a_k+a_\ell & -a_\ell & \\ & & & & & \cdots & \cdots\cdots & \cdots \\ & & & & & & \cdots & \cdots\cdots & \cdots \\ & & & & & & & \cdots & \cdots\cdots \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ M_h \\ M_i \\ M_j \\ M_k \\ M_\ell \\ \cdot \\ \cdot \end{bmatrix}$$

or

$$\left[m\right]\left[\ddot{y}\right] = -\frac{1}{d_o}\left[a_2\right]\left[M\right] \qquad (2.7)$$

where

$\left[m\right]$ = mass matrix and is a diagonal matrix

$\left[a_2\right]$ = second segmental parametric matrix.

The matrix $[a_2]$ is identical with the matrix $[a_1]$ in case the boundary conditions for the moments and displacements are similar. However, when moment and displacement boundary conditions are dissimilar, the matrix $[a_1]$ will be modified according to the displacement boundary conditions, while the matrix $[a_2]$ will be modified according to the moment boundary conditions.

2.4 Equation of Motion

Eliminating the moment matrix from Eq's. 2.5 and 2.7, the differential equations of motion for the concentrated masses will be in the form

$$\left[m\right]\left[\ddot{y}\right] = -\frac{1}{d_o^2}\left[a_2\right]\left[F\right]^{-1}\left[a_1\right]\left[y\right] \tag{2.8}$$

A linear stiffness matrix is defined such that

$$\left[K\right] = \frac{1}{d_o^2}\left[a_2\right]\left[F\right]^{-1}\left[a_1\right] \tag{2.9}$$

which may be expressed as follows;

$$\left[K\right] = \frac{6EI_o}{d_o^3}\left[a_2\right]\left[\zeta\right]^{-1}\left[a_1\right] \tag{2.10}$$

This matrix is always a square matrix; however, the matrices $[a_1]$ and $[a_2]$ may not be square, depending on the displacement and moment boundary conditions. A particular stiffness matrix corresponding to the boundary conditions is obtained for each case.

Since the approach is an approximation of an infinite degree-of-freedom system,it is desirable to find the linear flexibility matrix of the system so that the lowest mode frequency can be evaluated first by applying normal iteration techniques. The linear flexibility matrix is the inverse of the linear stiffness matrix;

$$\left[\Psi\right] = \left[K\right]^{-1} \tag{2.11}$$

For some cases, the matrices $\left[a_1\right]$ and $\left[a_2\right]$ are square,then the linear stiffness matrix may be defined directly in the following form:

$$\left[\Psi\right] = \frac{d_o^3}{6EI_o}\left[a_1\right]^{-1}\left[\zeta\right]\left[a_2\right]^{-1} \tag{2.12}$$

where the matrix $\left[\zeta\right]$ is always square and symmetrical due to the elastic reciprocal principle.

Investigating Eq. 2.9

$$\left[K\right] = \frac{1}{d_o^2}\left[a_2\right]\left[F\right]^{-1}\left[a_1\right] \tag{2.9}$$

the transpose of the matrix $\left[K\right]$ is

$$\left[K\right]^T = \frac{1}{d_o^2}\left[a_1\right]^T\left(\left[F\right]^{-1}\right)^T\left[a_2\right]^T \tag{2.13}$$

For an elastic system, the linear stiffness matrix $\left[K\right]$ and the angular flexibility matrix $\left[F\right]$ are symmetrical due to the reciprocal theorem, then,

$$\left[K\right] = \left[K\right]^T \tag{2.14}$$

and

$$\left[F\right]^{-1} = \left(\left[F\right]^{-1}\right)^T \tag{2.15}$$

thus,

$$\begin{aligned}\left[a_2\right] &= \left[a_1\right]^T \\ \left[a_1\right] &= \left[a_2\right]^T\end{aligned} \tag{2.16}$$

The transpose relation between matrices $\left[a_1\right]$ and $\left[a_2\right]$ also exists in case the intermediate boundary conditions are involved.

From Eq. 2.8, the equation of motion may be expressed as follows;

$$\left[m\right]\left[\ddot{y}\right] = -\left[K\right]\left[y\right] \tag{2.17}$$

A solution of periodic nature is assumed in the form

$$\left[y_i\right] = \left[Y_i\right] \sin\left(p_i t + \alpha_i\right)$$

where

$\left[y_i\right]$ = displacements of all vibrating masses in the ith mode.

$\left[Y_i\right]$ = shape function corresponding to the ith mode.

p_i = natural frequency of the ith mode

α_i = phase angle for the ith mode

then, substituting into Eq. 2.11,

$$\left[K\right]\left[Y_i\right] = p_i^2\left[m\right]\left[Y_i\right] \tag{2.18}$$

or

$$\left\{\left[K\right] - p_i^2\left[m\right]\right\}\left[Y_i\right] = 0 \tag{2.19}$$

Thus, the natural frequencies p_i of the system can be found by setting the determinant of the coefficient matrix

$$\left\{\left[K\right] - p_i^2\left[m\right]\right\}$$

equal to zero, and expressing as an nth order polynomial of p_i^2. However, when n is greater than three, the methods for solving this characteristic equation are tedious to apply, and it is therefore desirable to investigate other means of accomplishing that end. A method of successive approximations which has been widely applied in engineering problems is the method of matrix iteration (4). The application of iterative techniques is shown in Chapter IV.

2.5 Simplification of the Problem

For constant or stepped section members, no difficulty arises in the selection of concentrated mass locations or the determination of the angular flexibilities. However, when the cross section is continuously varying, some difficulty is encountered. Reasonable assumptions regarding the concentration of mass can usually be made, but since the angular flexibilities for each segment are of an integrable form (2), they are generally not easily calculated. Tables of angular flexibilities for certain types of section variation have been developed (18), and might be used where applicable.

For practical application, the variable section beam is considered to be replaced by a weightless string with elements of different stiffness carrying different size masses at the ends. Some advantage in formulating the problem and solution can be attained by utilizing constant length segments wherever feasible.

Consider the vibrating system AB (Fig. 2.4). Based on equal length segments

$$d_i = d_o , \quad i = 2, 3, \ldots n$$

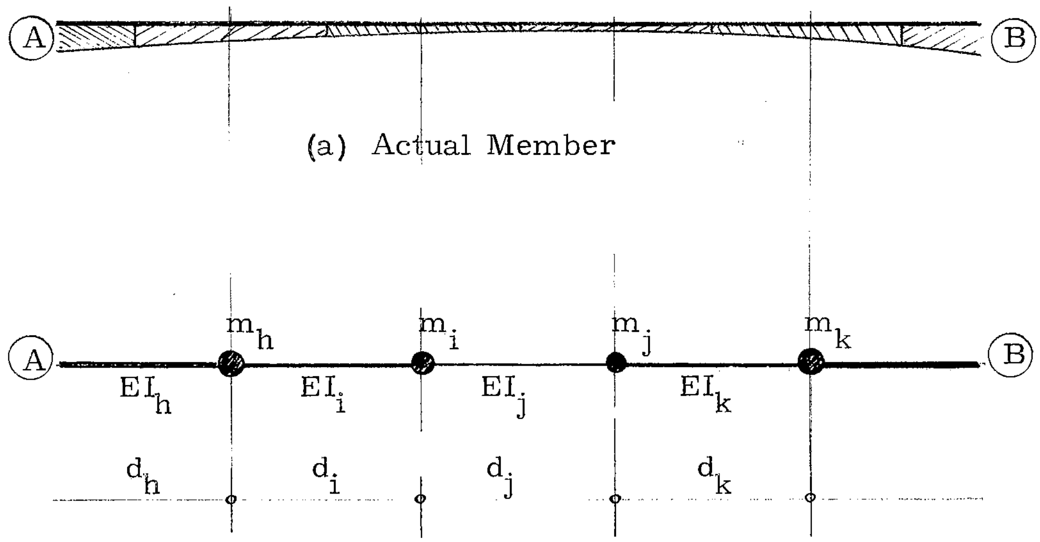
thus

$$a_i = \frac{d_o}{d_i} = 1 , \quad i = 2, 3, \ldots n$$

The following parameters are introduced

$$b_i = \frac{I_o}{I_i} \text{ , stiffness parameter}$$

$$\zeta_i = \frac{b_i}{a_i} \quad , \text{ angular flexibility parameter}$$



(a) Actual Member

(b) "Equivalent" Beam-mass System

Fig. 2.4 Vibrating System

Thus; the $\left[a\right]$ and $\left[F\right]$ matrices become

$$\left[a\right] = \begin{bmatrix} \cdot & \cdot & \cdot & & & & & & & \\ & \cdot & \cdot & \cdot & & & & & & \\ & & -1 & 2 & -1 & & & & & \\ & & & -1 & 2 & -1 & & & & \\ & & & & -1 & 2 & -1 & & & \\ & & & & & -1 & 2 & -1 & & \\ & & & & & & -1 & 2 & -1 & \\ & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & & \cdot & \cdot & \cdot \end{bmatrix}$$

$$[F] = \frac{d_o}{6EI_o} \begin{bmatrix} \cdots\cdots & & & & & & \\ \cdots & \cdots\cdots & \cdots & & & & \\ & \zeta_h & 2\zeta_h+2\zeta_i & \zeta_i & & & \\ & & \zeta_i & 2\zeta_i+2\zeta_j & \zeta_j & & \\ & & & \zeta_j & 2\zeta_j+2\zeta_k & \zeta_k & \\ & & & & \cdots & \cdots\cdots & \cdots \\ & & & & & \cdots & \cdots\cdots \end{bmatrix}$$

For constant cross section throughout the entire member, the angular flexibility parameter $\zeta_i = 1$; $i = 2, 3, \ldots n$; therefore, the angular flexibility matrix becomes

$$[F] = \frac{d_o}{6EI_o} \begin{bmatrix} \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & & & & \\ & 1 & 4 & 1 & & & \\ & & 1 & 4 & 1 & & \\ & & & 1 & 4 & 1 & \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot \end{bmatrix}$$

CHAPTER III

APPLICATION

3.1 Single Span Beams

The theory derived in Chapter II is applied to analyze several physical problems with different end conditions. The matrices are modified in accordance with the boundary conditions. For simplicity in showing the modification for end conditions, the beams are assumed to be of constant section and equal length segments are considered.

Equations of motion for a single span beam with several types of end supports are formulated and presented in the following tables (Tables 3.1, 2, 3, 4, 5, 6).

3.2 Continuous Beams

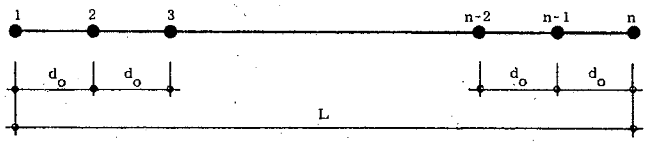
The basic philosophy discussed for the analysis of single span beams may be extended to the analysis of continuous beams by considering that some of the masses are restrained from displacing in the vertical direction by intermediate supports, as shown in Fig. 3.1.

Assuming that the one span beam $\overline{1\ n}$ has n masses and that (s+r) masses among these n masses are restrained from displacing, the n degree-of-freedom beam-mass system will be reduced to a (n-s-r) degree-of-freedom system.

The physical rigid supports provide the restraint conditions, that is, no physical displacements are allowed for these masses.

TABLE 3.1 MATRIX EQUATIONS. FREE-FREE BEAM

$M_1 = M_n = 0$



$d_i = d_o$

$EI_i = EI_o$

Free-free Beam-mass System

$$\left[M\right] = \frac{6EI_o}{d_o^2}\left[\zeta\right]^{-1}\left[a_1\right]\left[y\right]$$

$$\underset{(n-2)(1)}{\begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix}} = \frac{6EI_o}{d_o^2} \underset{(n-2)(n-2)}{\begin{bmatrix} 4 & 1 & & & & & & & \\ 1 & 4 & 1 & & & & & & \\ & 1 & 4 & 1 & & & & & \\ & & 1 & 4 & 1 & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & 1 & 4 & 1 & \\ & & & & & & 1 & 4 & 1 \\ & & & & & & & 1 & 4 & 1 \\ & & & & & & & & 1 & 4 \end{bmatrix}}^{-1} \underset{(n-2)(n)}{\begin{bmatrix} -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & 2 & -1 & & & & \\ & & & -1 & 2 & -1 & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \cdot & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 & -1 \\ & & & & & & & & -1 & 2 & -1 \end{bmatrix}} \underset{(n)(1)}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}}$$

$$\left[m\right]\left[\ddot{y}\right] = -\frac{1}{d_o}\left[a_2\right]\left[M\right]$$

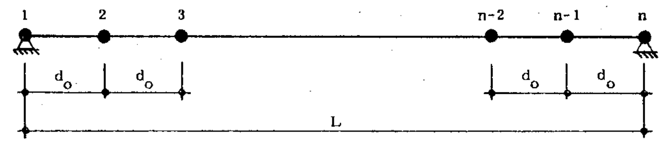
$$\underset{(n)(n)}{\begin{bmatrix} m_1 & & & & & & & \\ & m_2 & & & & & & \\ & & m_3 & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & m_{n-2} & & \\ & & & & & & m_{n-1} & \\ & & & & & & & m_n \end{bmatrix}} \underset{(n)(1)}{\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{y}_{n-2} \\ \ddot{y}_{n-1} \\ \ddot{y}_n \end{bmatrix}} = -\frac{1}{d_o} \underset{(n)(n-2)}{\begin{bmatrix} -1 & & & & & & \\ 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \\ & & & & & & & -1 \end{bmatrix}} \underset{(n-2)(1)}{\begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix}}$$

$$\left[K\right] = \frac{6EI_o}{d_o^3}\left[a_2\right]\left[\zeta\right]^{-1}\left[a_1\right], \qquad \left[m\right]\left[\ddot{y}\right] = -\left[K\right]\left[y\right]$$

TABLE 3.2 MATRIX EQUATIONS HINGED-HINGED BEAM

$y_1 = y_n = 0$

$M_1 = M_n = 0$



$d_i = d_o$

$EI_i = EI_o$

Hinged-hinged Beam-mass System

$$\left[M\right] = \frac{6EI_o}{d_o^2}\left[\zeta\right]^{-1}\left[a_1\right]\left[y\right]$$

$$\underset{(n-2)(1)}{\begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix}} = \frac{6EI_o}{d_o^2} \underset{(n-2)(n-2)}{\begin{bmatrix} 4 & 1 & & & & & & & & \\ 1 & 4 & 1 & & & & & & & \\ & 1 & 4 & 1 & & & & & & \\ & & 1 & 4 & 1 & & & & & \\ & & & \cdot & \cdot & \cdot & & & & \\ & & & & \cdot & \cdot & \cdot & & & \\ & & & & & 1 & 4 & 1 & & \\ & & & & & & 1 & 4 & 1 & \\ & & & & & & & 1 & 4 & 1 \\ & & & & & & & & 1 & 4 \end{bmatrix}}^{-1} \underset{(n-2)(n-2)}{\begin{bmatrix} 2 & -1 & & & & & & & & \\ -1 & 2 & -1 & & & & & & & \\ & -1 & 2 & -1 & & & & & & \\ & & -1 & 2 & -1 & & & & & \\ & & & \cdot & \cdot & \cdot & & & & \\ & & & & \cdot & \cdot & \cdot & & & \\ & & & & & -1 & 2 & -1 & & \\ & & & & & & -1 & 2 & -1 & \\ & & & & & & & -1 & 2 & -1 \\ & & & & & & & & -1 & 2 \end{bmatrix}} \underset{(n-2)(1)}{\begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \end{bmatrix}}$$

$$\left[m\right]\left[\ddot{y}\right] = -\frac{1}{d_o}\left[a_2\right]\left[M\right]$$

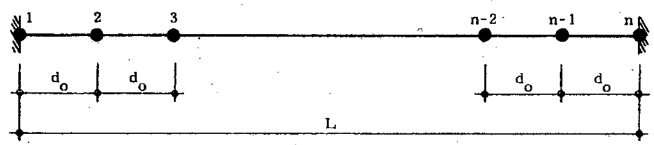
$$\underset{(n-2)(n-2)}{\begin{bmatrix} m_2 & & & & & & & & & \\ & m_3 & & & & & & & & \\ & & m_4 & & & & & & & \\ & & & \cdot & & & & & & \\ & & & & \cdot & & & & & \\ & & & & & \cdot & & & & \\ & & & & & & \cdot & & & \\ & & & & & & & m_{n-3} & & \\ & & & & & & & & m_{n-2} & \\ & & & & & & & & & m_{n-1} \end{bmatrix}} \underset{(n-2)(1)}{\begin{bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{y}_{n-3} \\ \ddot{y}_{n-2} \\ \ddot{y}_{n-1} \end{bmatrix}} = -\frac{1}{d_o} \underset{(n-2)(n-2)}{\begin{bmatrix} 2 & -1 & & & & & & & & \\ -1 & 2 & -1 & & & & & & & \\ & -1 & 2 & -1 & & & & & & \\ & & -1 & 2 & -1 & & & & & \\ & & & \cdot & \cdot & \cdot & & & & \\ & & & & \cdot & \cdot & \cdot & & & \\ & & & & & -1 & 2 & -1 & & \\ & & & & & & -1 & 2 & -1 & \\ & & & & & & & -1 & 2 & -1 \\ & & & & & & & & -1 & 2 \end{bmatrix}} \underset{(n-2)(1)}{\begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix}}$$

$$\left[K\right] = \frac{6EI_o}{d_o^3}\left[a_2\right]\left[\zeta\right]^{-1}\left[a_1\right] \quad , \quad \left[m\right]\left[\ddot{y}\right] = -\left[K\right]\left[y\right]$$

TABLE 3.3 MATRIX EQUATIONS FIXED-FIXED BEAM

$y_1 = y_n = 0$ $\quad$ $d_i = d_o$

$y'_1 = y'_n = 0$ $\quad$ $EI_i = EI_o$



Fixed-fixed Beam-mass System

$$\left[M\right] = \frac{6EI_o}{d_o^{\,2}}\left[\zeta\right]^{-1}\left[a_1\right]\left[y\right]$$

$$\underset{(n)(1)}{\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix}} = \frac{6EI_o}{d_o^{\,2}} \underset{(n)(n)}{\begin{bmatrix} 2 & 1 & & & & & & & \\ 1 & 4 & 1 & & & & & & \\ & 1 & 4 & 1 & & & & & \\ & & 1 & 4 & 1 & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & 1 & 4 & 1 & & \\ & & & & & & 1 & 4 & 1 & \\ & & & & & & & 1 & 4 & 1 \\ & & & & & & & & 1 & 2 \end{bmatrix}^{-1}} \underset{(n)(n-2)}{\begin{bmatrix} -1 & & & & & & \\ 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \\ & & & & & & -1 \end{bmatrix}} \underset{(n-2)(1)}{\begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \end{bmatrix}}$$

$$\left[m\right]\left[\ddot{y}\right] = -\frac{1}{d_o}\left[a_2\right]\left[M\right]$$

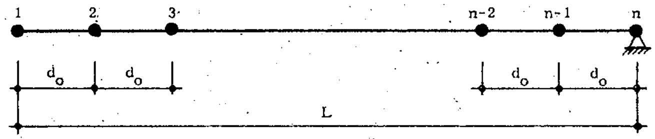
$$\underset{(n-2)(n-2)}{\begin{bmatrix} m_2 & & & & & & & & \\ & m_3 & & & & & & & \\ & & m_4 & & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & & & \cdot & & \\ & & & & & & & m_{n-3} & & \\ & & & & & & & & m_{n-2} & \\ & & & & & & & & & m_{n-1} \end{bmatrix}} \underset{(n-2)(1)}{\begin{bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{y}_{n-3} \\ \ddot{y}_{n-2} \\ \ddot{y}_{n-1} \end{bmatrix}} = -\frac{1}{d_o} \underset{(n-2)(n)}{\begin{bmatrix} -1 & 2 & -1 & & & & & & & \\ & -1 & 2 & -1 & & & & & & \\ & & -1 & 2 & -1 & & & & & \\ & & & -1 & 2 & -1 & & & & \\ & & & & \cdot & \cdot & \cdot & & & \\ & & & & & \cdot & \cdot & \cdot & & \\ & & & & & -1 & 2 & -1 & & \\ & & & & & & -1 & 2 & -1 & \\ & & & & & & & -1 & 2 & -1 \\ & & & & & & & & -1 & 2 & -1 \end{bmatrix}} \underset{(n)(1)}{\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix}}$$

$$\left[K\right] = \frac{6EI_o}{d_o^{\,3}}\left[a_2\right]\left[\zeta\right]^{-1}\left[a_1\right], \qquad \left[m\right]\left[\ddot{y}\right] = -\left[K\right]\left[y\right]$$

TABLE 3.4 MATRIX EQUATIONS FREE-HINGED BEAM

$M_1 = M_n = 0$ $\qquad$ $d_i = d_o$

$y_n = 0$ $\qquad$ $EI_i = EI_o$



Free-hinged Beam-mass System

$$\left[M\right] = \frac{6EI_o}{d_o^2}\left[\zeta\right]^{-1}\left[a_1\right]\left[y\right]$$

$$\underset{(n-2)(1)}{\begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix}} = \frac{6EI_o}{d_o^2} \underset{(n-2)(n-2)}{\begin{bmatrix} 4 & 1 & & & & & & & \\ 1 & 4 & 1 & & & & & & \\ & 1 & 4 & 1 & & & & & \\ & & 1 & 4 & 1 & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & 1 & 4 & 1 & \\ & & & & & & 1 & 4 & 1 \\ & & & & & & & 1 & 4 & 1 \\ & & & & & & & & 1 & 4 \end{bmatrix}}^{-1} \underset{(n-2((n-1)}{\begin{bmatrix} -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & \cdot \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 & -1 \\ & & & & & & & & -1 & 2 \end{bmatrix}} \underset{(n-1)(1)}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \end{bmatrix}}$$

$$\left[m\right]\left[\ddot{y}\right] = -\frac{1}{d_o}\left[a_2\right]\left[M\right]$$

$$\underset{(n-1)(n-1)}{\begin{bmatrix} m_1 & & & & & & & & \\ & m_2 & & & & & & & \\ & & m_3 & & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & & & \cdot & & \\ & & & & & & & m_{n-3} & & \\ & & & & & & & & m_{n-2} & \\ & & & & & & & & & m_{n-1} \end{bmatrix}} \underset{(n-1)(1)}{\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{y}_{n-3} \\ \ddot{y}_{n-2} \\ \ddot{y}_{n-1} \end{bmatrix}} = -\frac{1}{d_o} \underset{(n-1)(n-2)}{\begin{bmatrix} -1 & & & & & & \\ 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \end{bmatrix}} \underset{(n-2)(1)}{\begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix}}$$

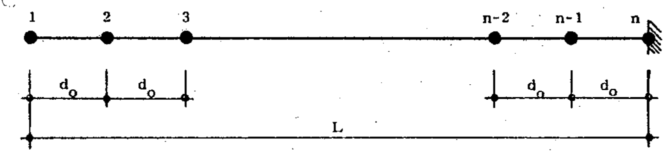
$$\left[K\right] = \frac{6EI_o}{d_o^3}\left[a_2\right]\left[\zeta\right]^{-1}\left[a_1\right] \quad , \quad \left[m\right]\left[\ddot{y}\right] = -\left[K\right]\left[y\right]$$

TABLE 3.5 MATRIX EQUATIONS FREE-FIXED BEAM

$y_n = 0$

$y_n' = 0$

$M_1 = 0$



$d_i = d_o$

$EI_i = EI_o$

Free-fixed Beam-mass System

$$\left[M\right] = \frac{6EI_o}{d_o^2}\left[\zeta\right]^{-1}\left[a_1\right]\left[y\right]$$

$$\underset{(n-1)(1)}{\begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix}} = \frac{6EI_o}{d_o^2} \underset{(n-1)(n-1)}{\begin{bmatrix} 4 & 1 & & & & & & \\ 1 & 4 & 1 & & & & & \\ & 1 & 4 & 1 & & & & \\ & & 1 & 4 & 1 & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & 1 & 4 & 1 & \\ & & & & & 1 & 4 & 1 \\ & & & & & & 1 & 4 & 1 \\ & & & & & & & 1 & 2 \end{bmatrix}}^{-1} \underset{(n-1)(n-1)}{\begin{bmatrix} -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & \cdot \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \\ & & & & & & & & -1 \end{bmatrix}} \underset{(n-1)(1)}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \end{bmatrix}}$$

$$\left[m\right]\left[\ddot{y}\right] = -\frac{1}{d_o}\left[a_2\right]\left[M\right]$$

$$\underset{(n-1)(n-1)}{\begin{bmatrix} m_1 & & & & & & & & \\ & m_2 & & & & & & & \\ & & m_3 & & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & & & m_{n-3} & & \\ & & & & & & & m_{n-2} & \\ & & & & & & & & m_{n-1} \end{bmatrix}} \underset{(n-1)(1)}{\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{y}_{n-3} \\ \ddot{y}_{n-2} \\ \ddot{y}_{n-1} \end{bmatrix}} = -\frac{1}{d_o} \underset{(n-1)(n-1)}{\begin{bmatrix} -1 & & & & & & & \\ 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 & -1 \end{bmatrix}} \underset{(n-1)(1)}{\begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix}}$$

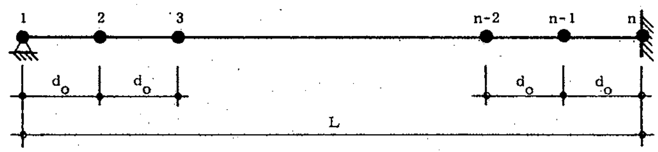
$$\left[K\right] = \frac{6EI_o}{d_o^3}\left[a_2\right]\left[\zeta\right]^{-1}\left[a_1\right] \quad , \quad \left[m\right]\left[\ddot{y}\right] = -\left[K\right]\left[y\right]$$

TABLE 3.6 MATRIX EQUATIONS HINGED-FIXED BEAM

$y_1 = y_n = 0$

$y'_n = 0$

$M_1 = 0$



$d_i = d_o$

$EI_i = EI_o$

Hinged-fixed Beam-mass System

$$\left[M\right] = \frac{6EI_o}{d_o^2}\left[\zeta\right]^{-1}\left[a_1\right]\left[y\right]$$

$$\underset{(n-1)(1)}{\begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix}} = \frac{6EI_o}{d_o^2} \underset{(n-1)(n-1)}{\begin{bmatrix} 4 & 1 & & & & & & & \\ 1 & 4 & 1 & & & & & & \\ & 1 & 4 & 1 & & & & & \\ & & 1 & 4 & 1 & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & 1 & 4 & 1 & \\ & & & & & & 1 & 4 & 1 \\ & & & & & & & 1 & 4 & 1 \\ & & & & & & & & 1 & 2 \end{bmatrix}}^{-1} \underset{(n-1)(n-2)}{\begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \\ & & & & & & & & -1 \end{bmatrix}} \underset{(n-2)(1)}{\begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \end{bmatrix}}$$

$$\left[m\right]\left[\ddot{y}\right] = -\frac{1}{d_o}\left[a_2\right]\left[M\right]$$

$$\underset{(n-2)(n-2)}{\begin{bmatrix} m_2 & & & & & & & & \\ & m_3 & & & & & & & \\ & & m_4 & & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & & & \cdot & & \\ & & & & & & & m_{n-3} & & \\ & & & & & & & & m_{n-2} & \\ & & & & & & & & & m_{n-1} \end{bmatrix}} \underset{(n-2)(1)}{\begin{bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{y}_{n-3} \\ \ddot{y}_{n-2} \\ \ddot{y}_{n-1} \end{bmatrix}} = -\frac{1}{d_o} \underset{(n-2)(n-1)}{\begin{bmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & 2 & -1 & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 & -1 \\ & & & & & & & & -1 & 2 & -1 \end{bmatrix}} \underset{(n-1)(1)}{\begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix}}$$

$$\left[K\right] = \frac{6EI_o}{d_o^3}\left[a_2\right]\left[\zeta\right]^{-1}\left[a_1\right] \quad , \quad \left[m\right]\left[\ddot{y}\right] = -\left[K\right]\left[y\right]$$

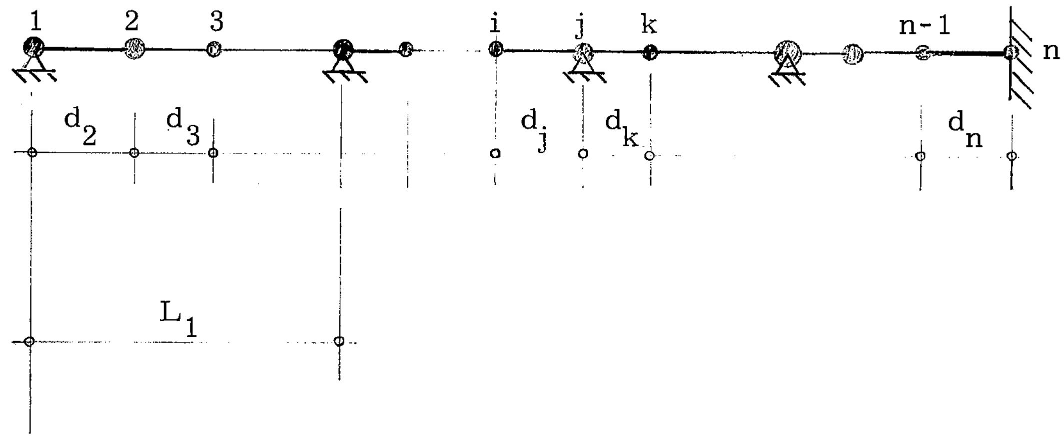


Fig. 3.1 Beam-mass System with Some Masses Restrained

The reaction at each support will be a time-dependent generalized force in the Lagrange's equation corresponding to each coordinate. Without any physical displacement, the shears transferred from adjacent masses will be completely absorbed by the reaction.

Taking away all intermediate supports, the continuous beam $\overline{1n}$ becomes a typical one span beam, which is taken as the basic structure in this analysis (Fig. 3.2).

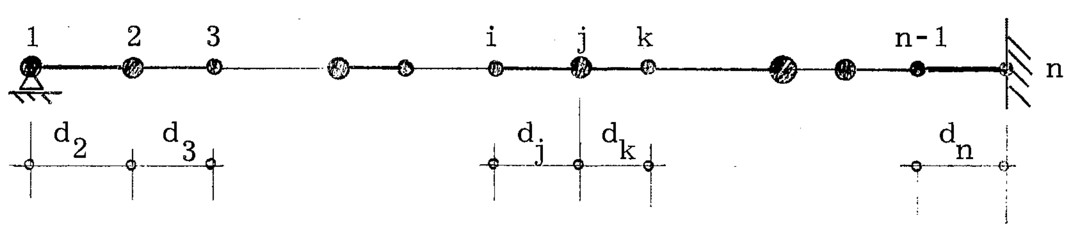
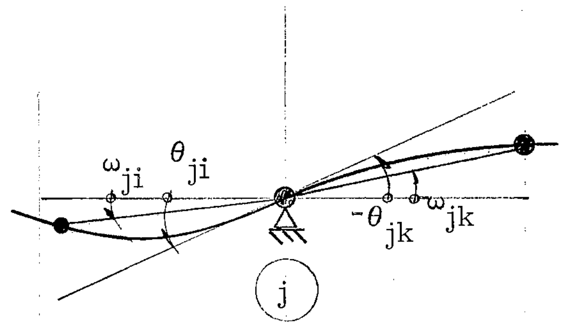


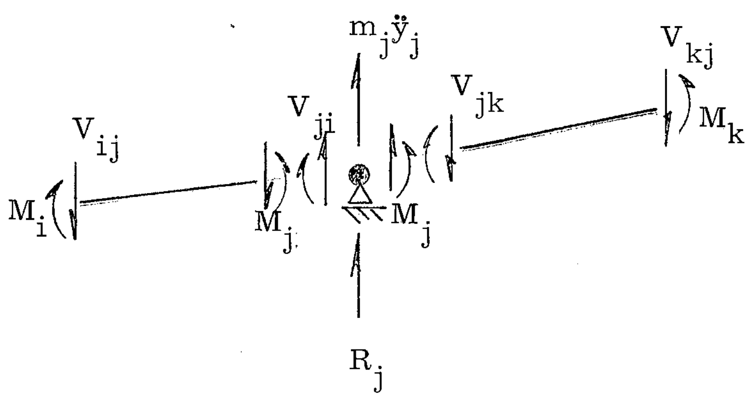
Fig. 3.2 Basic Structure of the Continuous Beam Shown in Fig. 3.1

A typical mass m_j on support j is considered (Fig. 3.3a). The condition of continuity over the rigid support is that the end slopes at j of two adjacent spans are equal, but in the opposite direction. Thus,

$$\theta_{ji} + \theta_{jk} = 0 \tag{3.1}$$



(a) Continuity Over Support j



(b) Free-body Diagram

Fig. 3.3 Equilibrium at Mass m_j

In Fig. 3.3a, the end slopes may be expressed as follows;

$$\theta_{ji} = -\frac{y_j - y_i}{d_j} + \left[F_{ji}\, M_j + G_{ji}\, M_i \right] \tag{3.2a}$$

$$\theta_{jk} = -\frac{y_j - y_k}{d_k} + \left[F_{jk}\, M_j + G_{jk}\, M_k \right] \tag{3.2b}$$

It is known that the displacement at support j is zero at any time for all modes;

$$y_j \equiv 0$$

Thus, from Eq. 3.1, the compatibility of slopes may be written as follows;

$$-a_j y_i - a_k y_k = d_o \left[G_{ji} M_i + \Sigma F_j M_j + G_{jk} M_k \right] \qquad (3.3)$$

This equation may be obtained directly by setting the additional boundary conditions due to intermediate supports into Eq. 2.3.

Summation of forces in y_j-direction (Fig. 3.3b) yields

$$\begin{aligned} R_j + m_j \ddot{y}_j &= -V_{ji} - V_{jk} \\ &= -\frac{M_j - M_i}{d_j} - \frac{M_j - M_k}{d_k} \end{aligned}$$

or

$$R_j + m_j \ddot{y}_j = -\frac{1}{d_o} \left[-a_j M_i + (a_j + a_k) M_j - a_k M_k \right] \qquad (3.4)$$

Since m_j is fixed on the rigid support, the displacement y_j and acceleration $\ddot{y}_j$ are zero for all time. The reaction R_j is a function depending upon mode and time, and may be expressed as

$$R_j = R_j (t, p_n) \qquad (3.5)$$

A fictitious displacement function for mass m_j is defined in the way such that

$$y_j^* = Y_j^* \sin (p_n t + \alpha_n) \qquad (3.6)$$

and

$$R_j = m_j \ddot{y}_j^* = - m_j p_n^2 Y_j^* \sin (p_n t + \alpha_j) \tag{3.7}$$

Thus, Eq. 3.4 becomes

$$m_j \ddot{y}_j^* = -\frac{1}{d_o} \left[- a_j M_i + (a_j + a_k) M_j - a_k M_k \right] \tag{3.8}$$

This expression will be used whenever the intermediate reaction R_j is required.

A general type of continuous beam is considered (Fig. 3.4). The basic structure is the one span beam with s restraints at two extreme ends.

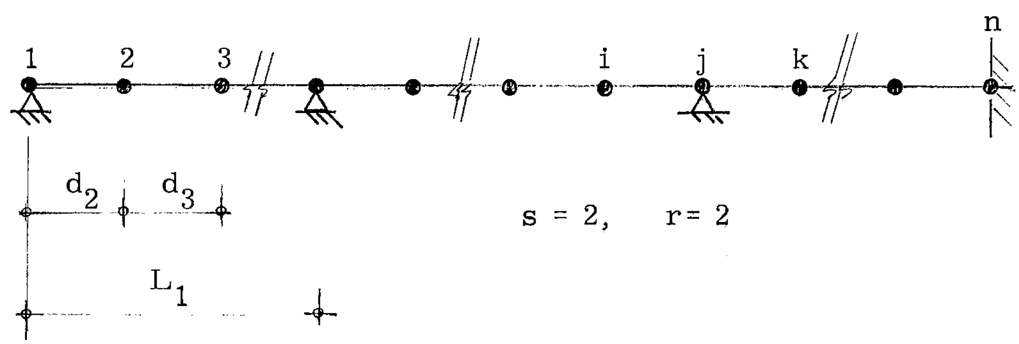


Fig. 3.4 Three Span Beam-mass System

In addition to the end conditions of the basic structure of the single span beam, r more restraints on masses m_r make the deflections at points r zero. Thus,

$$y_j = 0$$

where j indicates the point of intermediate support.

By Eq's. 2.5 and 3.3, the moments and displacements of the whole system can be related in the following form;

$$\left[M \right] = \frac{1}{d_o} \left[F \right]^{-1} \left[a_1 \right] \left[y \right]$$

where

$$\left[M \right] = \begin{bmatrix} . \\ . \\ . \\ M_i \\ M_j \\ M_k \\ . \\ . \\ . \end{bmatrix}_{(n-s)(1)} \qquad \left[y \right] = \begin{bmatrix} . \\ . \\ . \\ . \\ y_i \\ y_k \\ . \\ . \\ . \end{bmatrix}_{(n-s-r)(1)}$$

$$\left[F \right] = \begin{bmatrix} \cdots & \cdots & & & & & \\ \cdots & \cdots & \cdots & & & & \\ & G_{ih} & \Sigma F_i & G_{ij} & & & \\ & & G_{ji} & \Sigma F_j & G_{jk} & & \\ & & & G_{kj} & \Sigma F_k & G_{k\ell} & \\ & & & & \cdots & \cdots & \cdots \\ & & & & & \cdots & \cdots \end{bmatrix}_{(n-s)(n-s)}$$

This will be reduced into the form shown in Art. 2.5 when the beam is constant cross section throughout.

$$\left[a_1\right] = \begin{bmatrix} \cdots & \cdots\cdots & & & & & \\ & \cdots & \cdots\cdots & & & & \\ & & -a_i & a_i+a_j & & & \\ & & & -a_j & -a_k & & \\ & & & & a_k+a_\ell & -a_\ell & \\ & & & & \cdots & \cdots\cdots & \\ & & & & & \cdots & \cdots\cdots \end{bmatrix}_{(n-s)(n-s-r)}$$

From Eq. 2.7 and Eq. 3.8, the relation between moment and acceleration is stated in the matrix form;

$$\left[m\right]\left[\ddot{y}\right] = -\frac{1}{d_o}\left[a_2\right]\left[M\right]$$

where

$$\left[\ddot{y}\right] = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \ddot{y}_i \\ \ddot{y}_k \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}_{(n-s-r)(1)}$$

$$\left[m\right] = \begin{bmatrix} \cdot & & & & & \\ & \cdot & & & & \\ & & m_i & & & \\ & & & m_k & & \\ & & & & \cdot & \\ & & & & & \cdot \end{bmatrix}_{(n-s-r)\ \text{diag.}}$$

and

$$\left[a_2\right] = \begin{bmatrix} \cdots & \cdots\cdots & \cdots & & & & & & \\ & \cdots & \cdots\cdots & \cdots & & & & & \\ & & -a_i & a_i+a_j & -a_j & & & & \\ & & & & -a_k & a_k+a_\ell & -a_\ell & & \\ & & & & & \cdots & \cdots\cdots & \cdots & \\ & & & & & & \cdots & \cdots\cdots & \cdots \end{bmatrix}_{(n-s-r)(n-s)}$$

Then, the linear stiffness matrix is defined by Eq. 2.9, such that

$$\left[K\right]_{(n-s-r)(n-s-r)} = \frac{1}{d_o^2} \left[a_2\right]_{(n-s-r)(n-s)} \left[F\right]^{-1}_{(n-s)(n-s)} \left[a_1\right]_{(n-s)(n-s-r}$$

The linear flexibility matrix is the inverse of the linear stiffness matrix

$$\left[\Psi\right] = \left[K\right]^{-1}$$

Both are (n-s-r) square matrices.

CHAPTER IV

NUMERICAL EXAMPLES

4.1 General

Three numerical examples with various end conditions are solved in this chapter to illustrate the numerical procedures of the method studied in this thesis.

The first example is solved with a table calculator, while the last two are solved by the IBM 650 Electronic Computer. Results of each example have been compared with other available information.

4.2 Example No. 1

A fixed-fixed beam, as shown in Fig. 4.1, is analyzed. The section is constant throughout the entire beam, and the distributed mass is assumed to be lumped into five concentrated masses.

$$m_1 = m_5 = \frac{\rho L}{8} \;, \quad m_2 = m_3 = m_4 = \frac{\rho L}{4}$$

$$d_o = \frac{L}{4}$$

$$\rho = \text{mass per unit length}$$

The moment and displacement matrices are shown as follows:

$$[M] = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{bmatrix} \qquad [y] = \begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

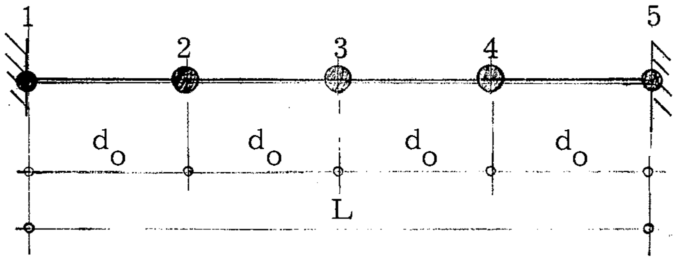


Fig. 4.1 Fixed-fixed Beam-mass System

Also,

$$\left[\zeta\right] = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix},$$

$$\left[\zeta\right]^{-1} = \frac{1}{168}\begin{bmatrix} 97 & -26 & 7 & -2 & 1 \\ -26 & 52 & -14 & 4 & -2 \\ 7 & -14 & 49 & -14 & 7 \\ -2 & 4 & -14 & 52 & -26 \\ 1 & -2 & 7 & -26 & 97 \end{bmatrix}$$

$$\left[a_2\right] = \begin{bmatrix} -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \end{bmatrix}$$

$$\left[a_1\right] = \begin{bmatrix} -1 & & \\ 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \\ & & -1 \end{bmatrix}$$

The linear stiffness matrix, as defined by Eq. 2.9, becomes

$$\left[K\right] = \frac{6EI_o}{d_o^{\,3}}\left[a_2\right]\left[\zeta\right]^{-1}\left[a_1\right]$$

thus,

$$\left[K\right] = \frac{6EI_o}{do^3} \begin{bmatrix} 3.143 & -2.000 & 0.857 \\ -2.000 & 2.500 & -2.000 \\ 0.857 & -2.000 & 3.143 \end{bmatrix}$$

The mass matrix is a 3x3 diagonal matrix

$$\left[m\right] = \begin{bmatrix} \rho d_o & & \\ & \rho d_o & \\ & & \rho d_o \end{bmatrix} = \rho d_o \left[I\right]_3$$

Using Eq. 2.13, the frequency equation becomes

$$\left| \left[K\right] - p_n^2 \left[m\right] \right| = 0$$

which can be obtained in the following form by substituting all values in terms of $EI_o / \rho L^4$.

$$\begin{vmatrix} 4828 \frac{EI_o}{\rho L^4} - p_n^2 & -3072 \frac{EI_o}{\rho L^4} & 1316 \frac{EI_o}{\rho L^4} \\ -3072 \frac{EI_o}{\rho L^4} & 3840 \frac{EI_o}{\rho L^4} - p_n^2 & -3072 \frac{EI_o}{\rho L^4} \\ 1316 \frac{EI_o}{\rho L^4} & -3072 \frac{EI_o}{\rho L^4} & 4828 \frac{EI_o}{\rho L^4} - p_n^2 \end{vmatrix} = 0$$

Solving for p_n^2, the following values are obtained,

$$p_1 = 3.5032\ (2\pi)\sqrt{\frac{EI_o}{\rho L^4}}$$

$$p_2 = 9.4500\ (2\pi)\sqrt{\frac{EI_o}{\rho L^4}}$$

$$p_3 = 15.5400\ (2\pi)\sqrt{\frac{EI_o}{\rho L^4}}$$

These results are compared with those of the "exact" solution (19) in Table 4.1.

TABLE 4.1 COMPARISON OF RESULTS

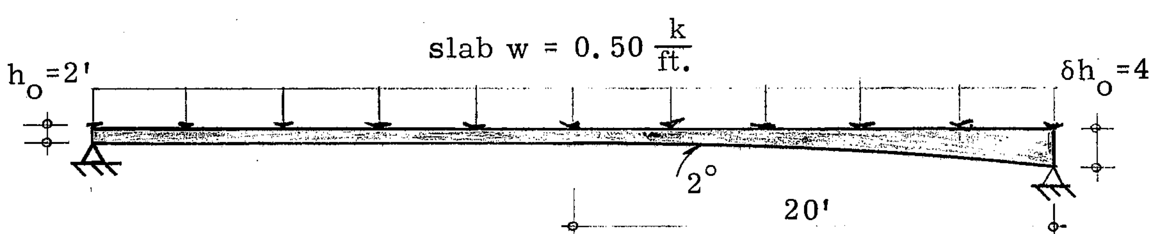
C_n	1st Mode	2nd Mode	3rd Mode
Author's Results	3.5032	9.4500	15.5400
Exact Solution (19)	3.56	9.82	19.20
Error	1.6 %	3.8 %	19.0 %

$$f_n = C_n \sqrt{\frac{EI_o}{\rho L^4}} \quad \frac{\text{cycles}}{\text{sec.}}$$

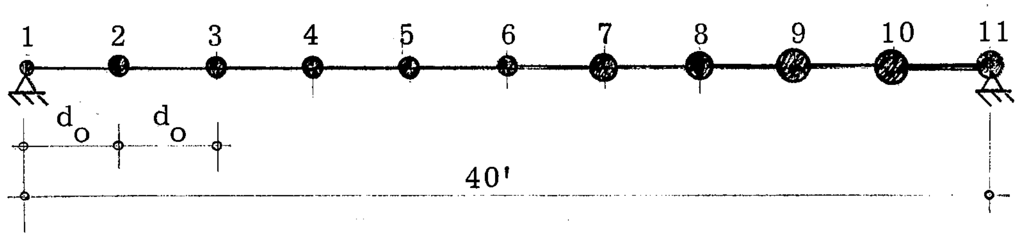
For engineering practice, the errors of first and second mode may be acceptable. However, if higher mode frequencies are required, the beam should be lumped into more masses.

4.3 Example No. 2

A simply supported reinforced concrete beam with a two-degree parabolic haunch is shown (Fig. 4.2). The beam is of constant width ($b=1.5'$) and is supporting a uniformly distributed slab load ($w=.50\frac{k}{ft}$). Considering ten equal length segments, the five lowest natural frequencies and corresponding mode shapes are computed.



(a) Actual Beam



(b) Simplified Beam-mass System

Fig. 4.2 Simple Beam with Variable Section

Data

$$E = 3 \times 10^3 \text{ ksi}$$

$$b = 1.5 \text{ ft.}, \quad d_o = 4 \text{ ft.}$$

$$w_c = 0.150 \frac{k}{ft^3}$$

$$EI_o = (3 \times 10^3)\left(\frac{1.5 \times 2^3}{12}\right)(144) = 432 \times 10^3 \quad k\text{-}ft^2$$

$$\rho_o = \left[(0.150)(2 \times 1.5) + 0.50\right]\frac{1}{32.2} = 0.0295 \frac{k\text{-}sec^2}{ft^2}$$

Parameters

$$a_i = \frac{d_o}{d_i} = 1$$

$$b_i = \frac{I_o}{I_i} = \left(\frac{h_o}{h_i}\right)^3 = \left(\frac{2}{h_i}\right)^3$$

$$\zeta_i = \frac{b_i}{a_i} = \left(\frac{2}{h_i}\right)^3$$

$$\rho_i = \rho_o\left[1 + 0.237\,(h_i - h_o)\right],$$

$$m_o = \rho_o d_o$$

Boundary Conditions

$$y_1 = y_{11} = 0$$

$$M_1 = M_{11} = 0$$

Then,

$$\left[m\right] = \begin{bmatrix} 1.000 & & & & & & & & \\ & 1.000 & & & & & & & \\ & & 1.000 & & & & & & \\ & & & 1.000 & & & & & \\ & & & & 1.000 & & & & \\ & & & & & 1.019 & & & \\ & & & & & & 1.076 & & \\ & & & & & & & 1.117 & \\ & & & & & & & & 1.304 \end{bmatrix} (\rho_o d_o)$$

This is the mass diagonal matrix. The elements on the main diagonal are the ratios of masses (m_2 to m_{10}) to the m_o value.

$$\left[\zeta\right] = \begin{bmatrix} 4.00 & 1.00 & & & & & & & \\ 1.00 & 4.00 & 1.00 & & & & & & \\ & 1.00 & 4.00 & 1.00 & & & & & \\ & & 1.00 & 4.00 & 1.00 & & & & \\ & & & 1.00 & 3.76 & 0.88 & & & \\ & & & & 0.88 & 3.04 & 0.64 & & \\ & & & & & 0.64 & 2.08 & 0.40 & \\ & & & & & & 0.40 & 1.26 & 0.23 \\ & & & & & & & 0.23 & 0.72 \end{bmatrix}$$

For the simply supported beam, the first parametric matrix and the second parametric matrix are identical.

$$\left[a_1\right] = \left[a_2\right] = \begin{bmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & 2 & -1 & & & & \\ & & & -1 & 2 & -1 & & & \\ & & & & -1 & 2 & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \end{bmatrix}$$

The linear stiffness matrix is defined by Eq. 2.9 such as;

$$\left[K\right] = \frac{6EI_o}{d_o^3}\left[a_2\right]\left[\zeta\right]^{-1}\left[a_1\right]$$

For simple beam, the parametric matrices are square and non-singular; the linear flexibility matrix has been defined as

$$\left[\Psi\right] = \left[K\right]^{-1} = \frac{d_o^3}{6EI_o}\left[a_1\right]^{-1}\left[\zeta\right]\left[a_2\right]^{-1} \qquad (2.12)$$

which has been calculated by the electronic computer.

The equation of motion can be obtained in the following form by premultiplying Eq. 2.17 by the linear flexibility matrix.

$$\left[\bar{\Psi}\right]\left[m\right]\left[Y\right] = \frac{1}{p_n^{\,2}}\left[Y\right] \qquad (4.2)$$

This is the form for evaluating the lowest frequency of the system first. By using the method of matrix iteration, the highest eigenvalue will be corresponding to the lowest natural frequency.

The lowest five frequencies have been calculated,by the IBM 650 Electronic Computer, as follows:

$$\left[\bar{\Psi}\right] = \frac{d_o^{\,3}}{6EI_o}\begin{bmatrix} 14.9210 & 25.4420 & 31.1630 & 32.6840 & 30.6050 & 26.0108 & 20.1374 & 13.6064 & 6.8091 \\ 25.4420 & 46.0840 & 58.1260 & 61.7680 & 58.2100 & 49.6216 & 38.4748 & 26.0128 & 13.0182 \\ 31.1630 & 58.1260 & 76.6890 & 83.6520 & 79.8150 & 68.4324 & 53.2122 & 36.0192 & 18.0273 \\ 32.6840 & 61.7680 & 83.6520 & 94.7360 & 92.4200 & 80.0432 & 62.5496 & 42.4256 & 21.2364 \\ 30.6050 & 58.2100 & 79.8150 & 92.4200 & 93.0250 & 82.0540 & 64.6870 & 44.0320 & 22.0455 \\ 26.0108 & 49.6216 & 68.4324 & 80.0432 & 82.0540 & 74.7336 & 59.8260 & 40.9728 & 20.5218 \\ 20.1374 & 38.4748 & 53.2122 & 62.5496 & 64.6870 & 59.8260 & 49.2266 & 34.0880 & 17.0853 \\ 13.6064 & 26.0128 & 36.0192 & 42.4256 & 44.0320 & 40.9728 & 34.0880 & 24.1504 & 12.1224 \\ 6.8091 & 13.0182 & 18.0273 & 21.2364 & 22.0455 & 20.5218 & 17.0853 & 12.1224 & 6.1028 \end{bmatrix}$$

$f_1 = 4.3446$ cycles per sec.

$f_2 = 19.4874$

$f_3 = 43.3362$

$f_4 = 78.8585$

$f_5 = 117.9504$

These results and their corresponding eigenvectors are tabulated in Table 4.2.

TABLE 4.2 NATURAL FREQUENCIES AND AMPLITUDE RATIOS

No. of Mode		1	2	3	4	5
Frequency $\frac{\text{cycles}}{\text{sec.}}$		4.3446	19.4874	43.3362	78.8585	117.9504
Amplitude Ratio	m_1	0	0	0	0	0
	m_2	0.3482	0.6582	-0.9485	1.0000	-0.8834
	m_3	0.6545	1.0000	-0.9131	0.2545	0.3531
	m_4	0.8815	0.8678	0.7339	-0.9386	0.7396
	m_5	1.0000	0.3454	1.0000	-0.5096	-0.6635
	m_6	0.9922	-0.2732	0.9380	0.7436	-0.5589
	m_7	0.8774	-0.7153	0.3018	0.8445	1.0000
	m_8	0.6958	-0.8246	-0.6989	-0.1872	0.3189
	m_9	0.4755	-0.6532	-0.7963	-0.7334	-0.6692
	m_{10}	0.2381	-0.3303	-0.4112	-0.3936	-0.3784
	m_{11}	0	0	0	0	0

By Stodola's method, the first mode natural frequency has been calculated as

$$f_1 = 4.3148$$

with a 0.685 per cent difference.

4.4 Example No. 3

A three span continuous beam is analyzed to illustrate the numerical application of the method. The cross section is constant throughout the entire beam, and the mass is uniformly distributed. The structure is considered as a beam-mass system with fourteen lumped masses loaded over three spans (Fig. 4.3).

$$EI = EI_o = \text{constant}$$

$$m = m_o = \rho\, d_o = \text{constant}$$

$$\rho = \text{mass per unit length}$$

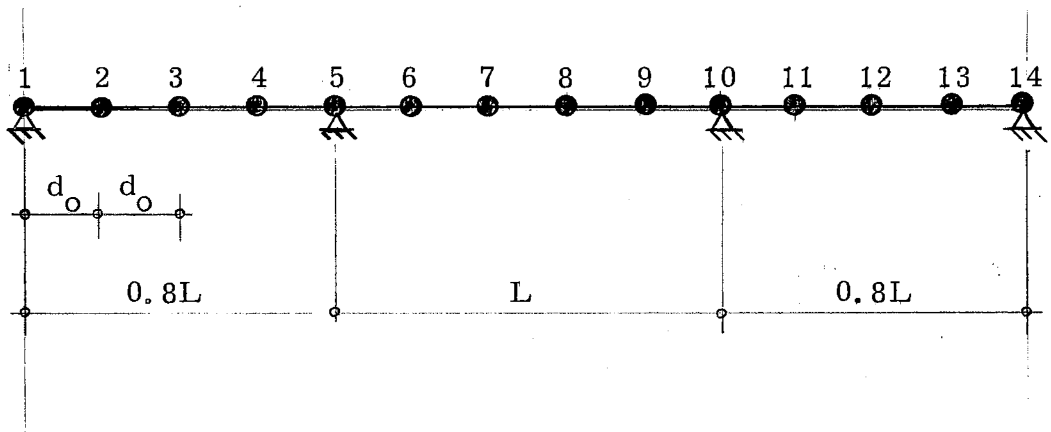


Fig. 4.3 Three-Span Continuous Beam-mass System

The mass matrix is

$$[m] = \rho d_o [I]_{10}$$

and the angular flexibility parametric matrix is

$$[\zeta] = \begin{bmatrix} 4 & 1 & & & & & & & & & & \\ 1 & 4 & 1 & & & & & & & & & \\ & 1 & 4 & 1 & & & & & & & & \\ & & 1 & 4 & 1 & & & & & & & \\ & & & 1 & 4 & 1 & & & & & & \\ & & & & 1 & 4 & 1 & & & & & \\ & & & & & 1 & 4 & 1 & & & & \\ & & & & & & 1 & 4 & 1 & & & \\ & & & & & & & 1 & 4 & 1 & & \\ & & & & & & & & 1 & 4 & 1 & \\ & & & & & & & & & 1 & 4 & 1 \\ & & & & & & & & & & 1 & 4 \end{bmatrix}$$

The first segmental parametric matrix is

$$
\left[a_1\right] = \begin{bmatrix}
2 & -1 & & & & & & & & \\
-1 & 2 & -1 & & & & & & & \\
 & -1 & 2 & & & & & & & \\
 & & -1 & -1 & & & & & & \\
 & & & 2 & -1 & & & & & \\
 & & & -1 & 2 & -1 & & & & \\
 & & & & -1 & 2 & -1 & & & \\
 & & & & & -1 & 2 & & & \\
 & & & & & & -1 & -1 & & \\
 & & & & & & & 2 & -1 & \\
 & & & & & & & -1 & 2 & -1 \\
 & & & & & & & & -1 & 2
\end{bmatrix}
$$

and the second segmental parametric matrix is

$$
\left[a_2\right] = \begin{bmatrix}
2 & -1 & & & & & & & & & & \\
-1 & 2 & -1 & & & & & & & & & \\
 & -1 & 2 & -1 & & & & & & & & \\
 & & & -1 & 2 & -1 & & & & & & \\
 & & & & -1 & 2 & -1 & & & & & \\
 & & & & & -1 & 2 & -1 & & & & \\
 & & & & & & -1 & 2 & -1 & & & \\
 & & & & & & & & -1 & 2 & -1 & \\
 & & & & & & & & & -1 & 2 & -1 \\
 & & & & & & & & & & -1 & 2
\end{bmatrix}
$$

Then, the linear stiffness matrix is

$$
\left[K\right] = \frac{6EI_o}{d_o^{\,3}} \left[a_2\right] \left[\zeta\right]^{-1} \left[a_1\right]
$$

which has the dimension (10 x 10). The linear flexibility matrix is obtained by inverting the K-matrix, as shown on the following page.

$$
[K] = \frac{6EI_o}{d_o^3}\begin{bmatrix}
+1.6461709 & -1.5846837 & +0.6925639 & +0.0497239 & -0.0133235 & +0.0035700 & -0.0009566 & -0.0000687 & +0.0000183 & -0.0000046 \\
-1.5846837 & +2.3387350 & -1.7702556 & -0.1988954 & +0.0532939 & -0.0142800 & +0.0038263 & +0.0002746 & -0.0000732 & +0.0000183 \\
+0.6925639 & -1.7702556 & +2.3884588 & +0.7458578 & -0.1998520 & +0.0535502 & -0.0143487 & -0.0010298 & +0.0002746 & -0.0000687 \\
+0.0497239 & -0.1988954 & +0.7458578 & +2.3922850 & -1.7846044 & +0.7461324 & -0.1999252 & -0.0143487 & +0.0038263 & -0.0009566 \\
-0.0133235 & +0.0532939 & -0.1998520 & -1.7846044 & +2.3923033 & -1.7846090 & +0.7461324 & +0.0535502 & -0.0142800 & +0.0035700 \\
+0.0035700 & -0.0142800 & +0.0535502 & +0.7461324 & -1.7846090 & +2.3923033 & -1.7846044 & -0.1998520 & +0.0532939 & -0.0133235 \\
-0.0009566 & +0.0038263 & -0.0143487 & -0.1999252 & +0.7461324 & -1.7846044 & +2.3922850 & +0.7458578 & -0.1988954 & +0.0497239 \\
-0.0000687 & +0.0002746 & -0.0010298 & -0.0143487 & +0.0535502 & -0.1998520 & +0.7458578 & +2.3884588 & -1.7702556 & +0.6925639 \\
+0.0000183 & -0.0000732 & +0.0002746 & +0.0038263 & -0.0142800 & +0.0532939 & -0.1988954 & -1.7702556 & +2.3387350 & -1.5846837 \\
-0.0000046 & +0.0000183 & -0.0000687 & -0.0009566 & +0.0035700 & -0.0133235 & +0.0497239 & +0.6925639 & -1.5846837 & +1.6461709
\end{bmatrix}
$$

$$
[\Psi] = \frac{d_o^3}{6EI_o}\begin{bmatrix}
+3.6534281 & +4.1454851 & +2.3147999 & -1.3244162 & -1.6404705 & -1.2943169 & -0.6321082 & +0.3292231 & +0.3762550 & +0.2351594 \\
+4.1454851 & +5.8327764 & +3.6036798 & -2.1190656 & -2.6247530 & -2.0709071 & -1.0113732 & +0.5267570 & +0.6020080 & +0.3762550 \\
+2.3147999 & +3.6036798 & +2.8407201 & -1.8541826 & -2.2966590 & -1.8120436 & -0.8849516 & +0.4609124 & +0.5267570 & +0.3292231 \\
-1.3244162 & -2.1190656 & -1.8541826 & +3.0480305 & +4.1935852 & +3.4151234 & +1.6911073 & -0.8849516 & -1.0113732 & -0.6321082 \\
-1.6404705 & -2.6247530 & -2.2966590 & +4.1935852 & +7.3011486 & +6.6119178 & +3.4151234 & -1.8120436 & -2.0709071 & -1.2943169 \\
-1.2943169 & -2.0709071 & -1.8120436 & +3.4151234 & +6.6119178 & +7.3011486 & +4.1935852 & -2.2966590 & -2.6247530 & -1.6404705 \\
-0.6321082 & -1.0113732 & -0.8849516 & +1.6911073 & +3.4151234 & +4.1935852 & +3.0480305 & -1.8541826 & -2.1190656 & -1.3244162 \\
+0.3292231 & +0.5267570 & +0.4609124 & -0.8849516 & -1.8120436 & -2.2966590 & -1.8541826 & +2.8407201 & +3.6036798 & +2.3147999 \\
+0.3762550 & +0.6020080 & +0.5267570 & -1.0113732 & -2.0709071 & -2.6247530 & -2.1190656 & +3.6036798 & +5.8327764 & +4.1454851 \\
+0.2351594 & +0.3762550 & +0.3292231 & -0.6321082 & -1.294369 & -1.6404705 & -1.3244162 & +2.3147999 & +4.1454851 & +3.6534281
\end{bmatrix}
$$

The problem becomes that of finding the eigenvalues and eigenvectors of the matrix equation.

$$[\Psi]\,[m]\,[y] = \frac{1}{p^2}\,[y]$$

The results are tabulated as follows (Table 4.3).

TABLE 4.3 NATURAL FREQUENCIES AND AMPLITUDE RATIOS

No. of Mode		1	2	3
Frequency $\frac{\text{cycles}}{\text{sec.}}$		$1.9875\sqrt{\frac{EI}{\rho L^4}}$	$3.0321\sqrt{\frac{EI}{\rho L^4}}$	$3.7308\sqrt{\frac{EI}{\rho L^4}}$
Amplitude Ratio	m_1	0	0	0
	m_2	-0.3894	-0.7547	+0.8322
	m_3	-0.5746	-1.0000	+1.0000
	m_4	-0.4399	-0.6153	+0.4735
	m_5	0	0	0
	m_6	+0.5892	+0.2599	+0.3603
	m_7	+1.0000	+0.1364	+0.8946
	m_8	+1.0000	-0.1364	+0.8946
	m_9	+0.5892	-0.2599	+0.3603
	m_{10}	0	0	0
	m_{11}	-0.4399	+0.6153	+0.4735
	m_{12}	-0.5746	+1.0000	+1.0000
	m_{13}	-0.3894	+0.7547	+0.8322
	m_{14}	0	0	0

Checking by the three-moment equation and Table A-1 of Ref. 19, the errors of the first and second mode are within 1%, while the third mode difference is 17%.

CHAPTER V

SUMMARY AND CONCLUSIONS

5.1 Summary

The dynamic analysis of beams is presented in this thesis. The actual distributed mass system is replaced by an "equivalent" lumped mass system; thus, reducing it from an infinite degree-of-freedom problem to one having a finite number of degrees-of-freedom.

The general theory is developed in matrix form utilizing the general string polygon equation for an elastic system. From the angular flexibility matrix, a linear stiffness matrix is derived. Elastic restoring forces for each segment are represented in terms of static angular functions and related to the "inertia" forces of the concentrated masses at the ends. Since the lower modes and corresponding natural frequencies are generally more important, a linear flexibility matrix is shown, for direct application to matrix iteration techniques for determining the eigenvalues and eigenvectors. These iteration methods rely on the orthogonality relations for evaluating higher mode characterictics.

Tables of matrix equations for single span beams having different boundary conditions are presented. These tables are for constant section members, and equal length segments are considered.

The general theory and results for single span beams are extended to continuous beams. The additional constraint conditions due to intermediate supports modify the general matrix equations presented for single span beams.

Illustrative examples are worked to show the application of the developed method.

5.2 Conclusions

Comparison of the results in Chapter IV with available information reveals that the procedure of analysis by the method presented is good for engineering practice. Obviously, the accuracy of the results obtained depend upon the number of mass concentrations. It appears that at least twice as many concentrated masses should be taken in a span as the number of natural frequencies desired.

The theory presented in this thesis has no restriction on the span length, support conditions, cross section and segmental length. However, there are some advantages in taking equal length segments.

It is necessary that one mass concentration be considered at each support so that the support boundary condition can be formulated into the matrix equation. In direct application to the analysis of multistory frames, mass should be concentrated at the floor levels and the deformation considered similar to that of the cantilever bar.

In the analysis of a variable section beam subjected to any kind of disturbance, some difficulties may arise in formulating an analytic solution. However, using the eigenvalues and eigenvectors obtained, the problem may be uncoupled by diagonalizing the linear stiffness matrix so that the system is transformed into a new space of normal coordinates; thus, each coordinate can be analyzed independently.

5.3 Extension

The method developed in this thesis may be extended directly to the dynamic analysis of continuous beams on elastic supports by taking into account the effect of the support elasticity. Further extension may be able to handle a general type of frame by considering the joint rotation and girder flexibility.

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