FIBONACCI NUMBERS

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CHAPTER I

INTRODUCTION AND STATEMENT OF THE PROBLEM

Introduction

In 1202 A. D., the mathematician Leonardo of Pisa, who was nicknamed Fibonacci, wrote one of the earliest treatments of arithmetic and algebra and gave his work the title Liber Abacci. Included in the book was a problem about the number of offspring of a pair of rabbits. The consideration of this problem led to a sequence of integers,

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots .$$

The mathematician Cantor regarded this as the first example of a recurring sequence to appear in a mathematical work. The obvious recurrence relation is

$$F_{n+2} = F_{n+1} + F_n, \; F_1 = 1, \; F_2 = 1.$$

This seemingly innocuous sequence has turned out to have so many remarkable and fascinating properties, ranging from elementary to sophisticated, that it and related recurring sequences have been investigated by some of the greatest mathematical minds.

The richness and the applications of the intriguing Fibonacci sequence have fascinated men through the centuries. It has been said that the research generated by it nearly amounts to the number of offspring produced by Leonardo's first rabbits.

Furthermore, interest has continued to the present with a notable increase in very recent years. So much renewed interest has been shown that in 1963 an organization, the Fibonacci Association, was formed and is devoted to the study of the special properties of integers.

Statement of the Problem

Often in mathematics courses, problem solving is approached in a wide range of patterns, from a routine application of miscellaneous techniques to artificially involved material, which is sometimes poorly motivated from the standpoint of the student. The routine applications combine the illustration of technique or principle with economy of time, while more involved and difficult material may produce more understanding and depth, provided sufficient effort is expended.

However, at the elementary level, the student's mathematical maturity generally is such that problems need to be challenging and worthwhile, yet not require inordinate background knowledge. A great many of the properties and problems relating to Fibonacci numbers are interesting, yet do not require elaborate previous knowledge, are not difficult, yet serve well to illustrate a number of types of proof, and are bona fide mathematical topics, yet have a natural fascination for the novice. In this regard, it almost seems as if the eminent mathematician and teacher, Polya, of Stanford University has the Fibonacci numbers in mind when he wrote:

> The trouble with the usual problem material of the high school textbooks is that they contain almost exclusively merely routine examples. A routine example is a short range example; it illustrates, and offers practice in the application of, just one isolated rule. Such routine examples may be useful and even necessary, I do not deny it, but they miss two important phases of learning: the exploratory phase and the phase of assimilation. ... In contrast with such routine problems, the high school should present

> more challenging problems, at least now and then, problems with a rich background that deserves further exploration, and problems which can give a foretaste of the scientist's work. [20][1]

The usefulness of this thesis stems from the fact that it provides interesting and challenging seminar, reading course, or enrichment material, suitable for undergraduate mathematics students. The mathematical content involved is most closely associated with algebra, number theory, and analysis. It will primarily be through these areas that this thesis can best serve with respect to problem solving. The Committee on the Undergraduate Program in Mathematics (CUPM) has recommended that a full course in number theory be in the curriculum for all future teachers of secondary mathematics. This thesis would be especially appropriate as a supplement to several topics in such a course.

The literature on Fibonacci numbers is usually only touched on in standard textbooks and many developments are available only in journals or notes. The purposes of this thesis are (1) to organize and bring together under one cover a contiguous body of appropriate and challenging enrichment material for the modern curriculum, (2) to make supplementary material associated with the Fibonacci sequence available for high school and undergraduate students, particularly prospective mathematics teachers, and (3) to provide an instrument to stimulate and encourage interest in mathematics, primarily at the level of the undergraduate and advanced high school student, directly from first-hand contact, and indirectly by broadening and rounding out the backgrounds of their teachers.

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[1] Numbers in brackets refer to references in bibliography.

Procedure

This thesis is intended as a mathematical research of expository nature devoted to the Fibonacci numbers and closely related developments. An exhaustive treatment would be beyond the scope of this work. Therefore, an extensive review of the literature with a careful analysis and selection was required.

The Mathematical Review, bibliographies of texts and published papers, and bibliographies of unpublished theses were used in locating material dealing with the Fibonacci sequence. A survey and analysis of available material was then made. Several articles were located in publications written in French and German.

The presentation is expository in nature and is presented in graduated levels of difficulty. Chapter II provides, in an informal manner, motivation and interest in the subject of the Fibonacci numbers. Most of the topics in this chapter reappear later as vehicles to demonstrate proof and develop mathematical skills in deductive reasoning. Chapters III and V constitute a more formal treatment and are intended to help develop the reader's skill and familiarity with certain basic algebraic concepts, particularly proof by mathematical induction. This type of proof, so widely used in all mathematics, is a very troublesome and elusive concept for too many students. One of the richer contributions of this thesis ought to be the instillation in the reader of an awareness of the utility and power of mathematical induction as an instrument for mathematical proof. Reinforcement of this effort is continued in the other chapters of the thesis; however, greater variety is introduced, which, in turn, makes somewhat greater demands on the reader. Knowledge of elementary calculus is necessary for some of the development, particularly in Chapters IV and VI. All the chapters are intended to contribute to the stimulation and motivation of the reader and to encourage him to find a measure of pleasantness and satisfaction in a mathematical setting, as well as increasing his mathematical maturity.

Expected Outcomes

Hopefully, the reading of this thesis will make the reader aware of the elegance, beauty, and charm of Fibonacci numbers, and, so, of mathematics itself. It is expected that high school teachers will be able to use this material to provide enrichment for their students and that undergraduates may be able to improve and broaden their knowledge of mathematics. Finally, it is worth noting that most secondary mathematics courses do not mention a single mathematician in the span between the great mathematicians of antiquity and the 16th century. All the better that one outcome might be to make more widely known this remarkable mathematician, Leonardo of Pisa.

CHAPTER II

INFORMAL INTRODUCTION

The Rabbit Problem and the Fibonacci Sequence

As was pointed out in the opening remarks of Chapter I, the consideration by Fibonacci of a certain rabbit problem led to a sequence of integers called the Fibonacci sequence. The problem discussed in what follows is similar to the one he introduced in his book, Liber Abacci. The problem is: Given a new pair of rabbits, find how many pairs will be on hand in a given number of months if each pair of rabbits gives birth to a new pair each month, starting with the second month of its life. If one considers this problem, the first month sees a total of one pair. In the second month there is still one pair since the original pair has not produced offspring. In the third month there are the original pair and the first pair of offspring, for a total of two pair. In the fourth month there is a total of three pair, the original pair and the two pair of offspring. But now a change commences, since some of the descendants themselves are producing offspring. For the fifth month, the original pair and their first descendants have offspring, plus those new offspring from the fourth month, also on hand, for a grand total of five pair. The sixth month there is a total of eight pair, the seventh month, nine pair, and so on, although keeping count has definitely become tedious. It proves to be helpful and suggestive to construct an array in which the number of pairs of rabbits two months or older are kept on one line and new ones on the line below. Assume that births occur at midnight on the last day of the month and that the census is taken on the first day of the new month.

TABLE I

RABBIT PAIR TOTALS FOR FIRST FEW MONTHS

Month	1	2	3	4	5	6	7	8	9	...
Old	1	1	1	2	3	5	8	13	21	...
New	0	0	1	1	2	3	5	8	13	...
Total	1	1	2	3	5	8	13	21	34	...

The total for the n^{th} month is the n^{th} Fibonacci number, and among the patterns noted in the chart, one may observe, omitting the first entry in the old line, that the old, new, and total rows each give the Fibonacci sequence, in staggered fashion. Indeed, this suggests the extremely simple formulation for the total in any month, namely, the sum of the totals for the two previous months. Stated more symbolically, if $F_1 = 1$, $F_2 = 1$, the totals for the first and second months respectively, then $F_3 = F_1 + F_2 = 1 + 1 = 2$, $F_4 = F_2 + F_3 = 1 + 2 = 3$, $F_5 = 2 + 3 = 5$, and so on, and in general, one has the recurrence rule of formulation for the Fibonacci numbers,

$$F_{n+2} = F_n + F_{n+1}.$$

Now, this sequence has interest in and of itself that goes far beyond the introductory rabbit problem from which it originated.

In this chapter, several of the interesting and surprising mathematical properties of the Fibonacci sequence are considered intuitively, and a brief comment is made on some of the remarkable connections to other areas.

For initial practice it might be helpful to calculate a few numbers in the Fibonacci sequence. Some initial results are shown in Table II, which will also be useful for later reference. One immediately appreciates the difficulty in tabulating large Fibonacci numbers, in that all the preceding terms must be known. This suggests that it might be desirable to be able to calculate F_n for any given n directly, and this problem is considered in a later chapter.

A perusal of Table II reveals several interesting properties. It appears by trial that any two consecutive Fibonacci numbers are relatively prime. Also, note, for example, that 4 is divisible by 1, 2, and 4, and $F_4 = 3$ is divisible by $F_1 = 1$, $F_2 = 1$, and $F_4 = 3$; 12 is divisible by 1, 2, 3, 4, 6, and 12, and $F_{12} = 144$ is divisible by $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_6 = 8$, and $F_{12} = 144$; and in general, it appears that whenever one index is divisible by another, the corresponding Fibonacci numbers possess the same divisibility properties. That this conjecture is true is proved in another chapter.

Another remarkable property that can be verified for some initial values using Table II is that the greatest common divisor of any two indexes is the index of the greatest common divisor of the two corresponding Fibonacci numbers. For instance, the greatest common divisor of $F_8 = 21$ and $F_{12} = 144$ is 3. The greatest common divisor of 8 and 12 is 4, and $F_4 = 3$. This conjecture is also proved in a following chapter. There are many other known relationships similar to these, and new ones are still being discovered.

TABLE II

PARTIAL LISTING OF FIBONACCI NUMBERS

n	F_n	n	F_n
0	0	24	46368
1	1	25	75025
2	1	26	121393
3	2	27	196418
4	3	28	317811
5	5	29	514229
6	8	30	832040
7	13	31	1346269
8	21	32	2178309
9	34	33	3524578
10	55	34	5702887
11	89	35	9227465
12	144	36	14930352
13	233	37	24157817
14	377	38	39088169
15	610	39	63245986
16	987	40	102334155
17	1597	50	12586269025
18	2584	60	1548008755920
19	4181	70	190392490709135
20	6765	80	23416728348467685
21	10946	90	2880067194370816120
22	17711	100	354224848179261915075
23	28657		

A natural property to investigate might be the initial sums of the Fibonacci numbers. In this connection, consider Table III, in which the third column contains the sums of the first n Fibonacci numbers.

TABLE III

SUMS OF FIBONACCI NUMBERS

n	F_n	sum of 1st n	sum of 1st n squares
1	1	1	1
2	1	2	2
3	2	4	6
4	3	7	15
5	5	12	40
6	8	20	104
7	13	33	273
8	21	54	714

Reflecting on the values in the sum column, one observes that each value is one less than a Fibonacci number. In particular, it is one less than the second Fibonacci number beyond the last F_n in the sum. In other words, the relation

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$$

is suggested. This identity does indeed turn out to be a valid one, and a proof of this is given in Chapter III. By considering respectively, the values of n which are odd, even, multiples of a given number, and so on, in Table III, it is possible to suggest other identities concerning sums.

An identity concerning the sum of the first n squares of Fibonacci numbers can be motivated by an interesting geometric device. Place two squares of side 1 next to each other. Construct next, a square of side 2 adjacent to the two unit squares, as shown in Figure 1, and continue constructing squares having dimensions equal to consecutive Fibonacci numbers.

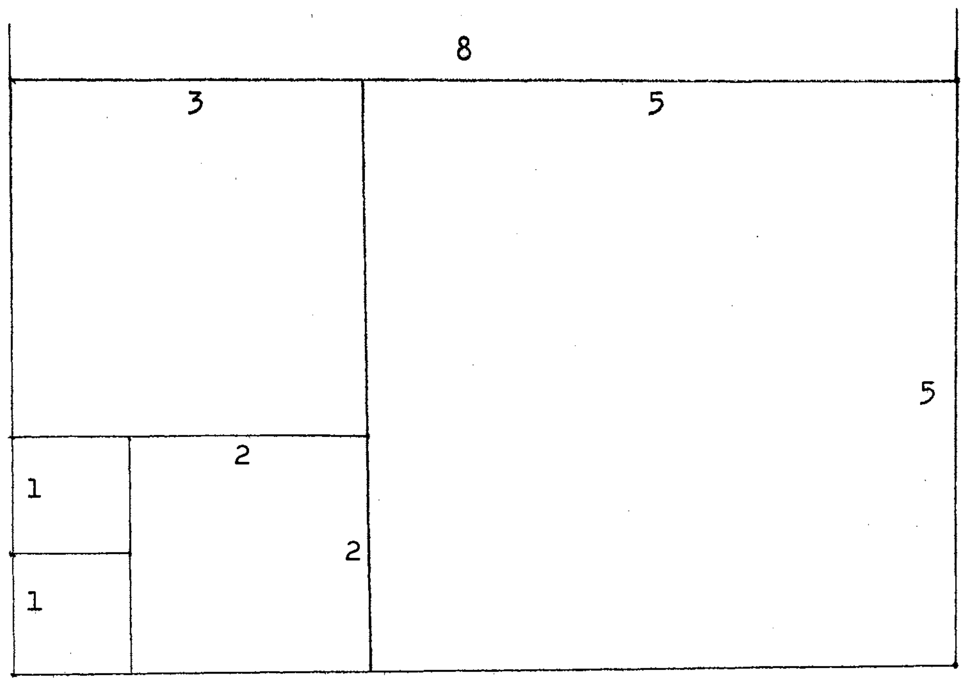

Figure 1. Squares With Dimensions Equal to Consecutive Fibonacci Numbers

Observe that the area of the individual squares is

$$F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 + F_6^2,$$

for the six squares shown in the figure. But this is the same as the total area of the rectangle formed, so that this quantity is apparently the same as $F_6 F_7$. One then generalizes

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1},$$

and this particular identity is proved later.

Every student of elementary algebra is familiar with the triangular array called Pascal's triangle. The use of this array to obtain the coefficients in the expansion of $(a + b)^n$ is well-known. Also, if

TABLE IV

PASCAL'S TRIANGLE

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

$$\vdots$$

horizontal rows are totaled, integer powers of two are obtained. It is not so widely known that this array can be used to obtain the Fibonacci numbers. Instead of running totals horizontally, one may take them along a 22.5° angle, following the diagonal lines in Table IV. In this manner the successive Fibonacci numbers are obtained. Making use of the notation C_k^n for the binomial coefficients leads to

$$F_n = C_0^n + C_1^{n-1} + C_2^{n-2} + \dots ,$$

with the convention that $C_k^n = 0$ for n less than k.

Other properties than those mentioned above are treated in detail in later chapters. In the final pages of Chapter III, an important result pertaining to the existence of a Fibonacci number divisible by an arbitrary integer m is established. By way of illustration, a study of Table II reveals that if m is taken as 7, 9, 13, or 14, then each of these respectively divides $F_8 = 21$, $F_{12} = 144$, $F_{14} = 377$, or $F_{24} = 46368$. It appears that not only is the existence of a Fibonacci number divisible by m guaranteed for an arbitrary integer m, but the first such number will not be extremely large.

Golden Ratio

The famous golden section involves the division of a given line segment into mean and extreme ratio, in other words, into two parts such that the longer is the mean proportional between the whole line and the shorter part. Figure 2 shows a line segment in which the two parts are

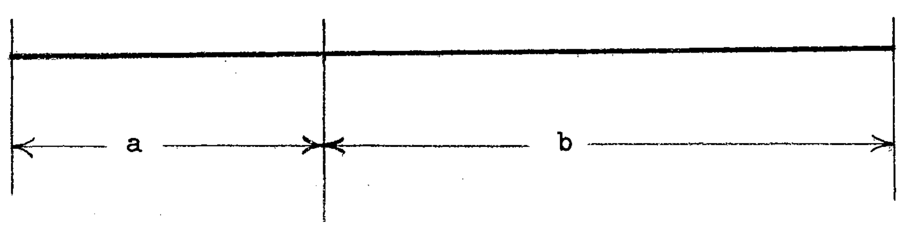

Figure 2. Golden Section

a and b. This condition may be expressed

$$\frac{a}{b} = \frac{b}{a+b}, \text{ a less than b.}$$

For $a = 1$, this proportion yields the quadratic equation

$$b^2 - b - 1 = 0.$$

The positive root is frequently denoted by ϕ, hence $\phi = (1 + \sqrt{5})/2$. The number ϕ was known to the ancients as the golden ratio.

It has been indicated by psychologists that a rectangle having sides in golden ratio is the rectangle having the most pleasing shape to behold. For instance, this is approximately the shape chosen for picture postcards and this shape is often seen in architecture. The golden section was mysterious and fascinating to the ancients. Indeed, it was referred to as De Divina Proportione. Even Kepler was awed by it as can be seen by this comment:

> Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line in extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel. [7]

Supernatural connotations have often been associated with this ratio. Even as recently as 1954, in a painting by Salvadore Dali, The Sacrament of the Last Supper, a portion of a regular polyhedron of twelve faces, a dodecahedron, is shown. The dodecahedron has regular pentagons for its faces, and the student of geometry is aware that this figure is intimately associated with the golden ratio.

Now one is never far from the Fibonacci numbers when in the presence of the golden ratio. Consider the large rectangle of Figure 3, with sides in the Golden Ratio, $\phi : 1$. Next, remove the large 1 x 1 square in the right portion of Figure 3. The remaining rectangle has its sides in the ratio of $1 : \phi - 1$. But ϕ is defined from the equation $\phi^2 - \phi - 1 = 0$, which may be written

$$\phi = \frac{1}{\phi - 1}$$

after a little manipulation. This indicates the upper left rectangle is similar to the original rectangle, and so the process may be repeated indefinitely, as is partially illustrated in Figure 3.

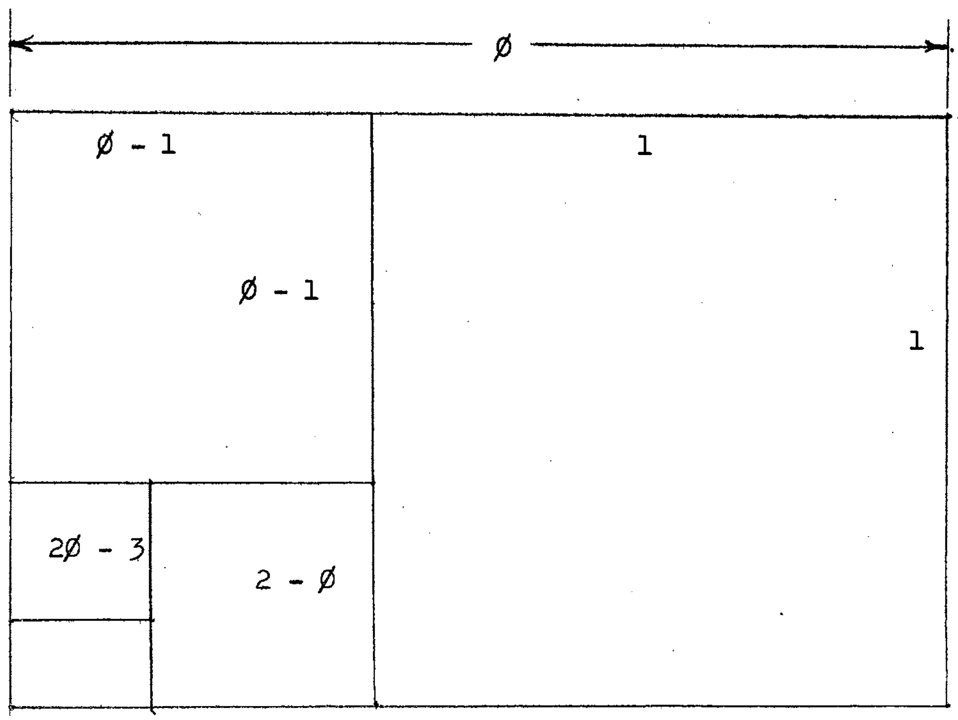

Figure 3. A Sequence of Golden Rectangles

At this point, Figures 1 and 3 should be compared. The two diagrams are considerably different in the lower left-hand corners; however, the larger parts are nearly alike. Furthermore, if both figures were increased by adjoining squares, the proportions would become increasingly more equal. The ratio of length to width of every rectangle of Figure 3 is $\emptyset$; the proportions of the rectangles of Figure 1 are the successive ratios of consecutive Fibonacci numbers, namely,

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots .$$

The further along in this sequence one goes, the more nearly the ratio

$F_{n+1} : F_n$ approximates $\emptyset$. This fact is verified for a few values in Table V. A proof of this property, which was first noted by Kepler, is presented in Chapter IV. Note that $\emptyset = 1.61803...$.

There are a number of facts regarding the Fibonacci numbers that have been discovered using modern computer equipment. F_{476} is the first 100 digit Fibonacci number. For each succeeding 100 digit level, the indexes are increased by 478 and 479 alternately, that is, F_{954} is the first Fibonacci number with 200 digits, F_{1433} has 300, F_{1911} has 400, and so on up to F_{19137} with 4000 digits.

TABLE V

QUOTIENTS OF CONSECUTIVE FIBONACCI NUMBERS

n	F_n	F_{n+1}	F_{n+1}/F_n
1	1	1	1
2	1	2	2
3	2	3	1.50
4	3	5	1.667
5	5	8	1.600
6	8	13	1.625
7	13	21	1.6154
8	21	34	1.6190
9	34	55	1.6176
10	55	89	1.6182
⋮	⋮	⋮	⋮

The last digits of the Fibonacci sequence repeat in cycles of 60. The Fibonacci numbers F_{11003} and F_{11004} have been calculated and are given by Berg. [4] Two pages are required for each number!

Phyllotaxis

The Fibonacci numbers are not confined exclusively to the mathematician's realm. They have a botanical connection in the phenomenon called phyllotaxis, meaning leaf arrangement. In some trees, such as the elm, the leaves along the twig seem to occur alternately on two opposite sides, and one speaks of ½-phyllotaxis. In the beech tree, the passage from one leaf to the next is given by a rotational displacement involving one third of a turn, that is 1/3-phyllotaxis. In similar fashion, the oak and apricot exhibit 2/5-phyllotaxis, the poplar and pear, 3/8-phyllotaxis, the willow and almond, 5/13-phyllotaxis, and so on. The Fibonacci numbers are conspicuously present, as these fractions are quotients of alternate Fibonacci numbers. Had the rotation been taken opposite to the above direction, the fractions would have been quotients of consecutive Fibonacci numbers. Thus, 8/13 could have replaced 5/13, for instance. It happens that if the leaves were arranged in precisely the ratio $1 : \emptyset$, instead of approximating this ratio, then no two leaves would ever be superposed. The biologists are not altogether decided as to the explanation of this and other phenomena related to the Fibonacci numbers, but the matter has a long history of study. Leaf arrangement has been explained as serving to let air pass between the leaves, keeping one from overshadowing another, and letting rain fall from one leaf onto the one below, and the phyllotaxis ratios represent the effort of the plant to seek a most beneficial arrangement.

Whatever the reason or explanation for these matters, this is a question to be answered by the biologist. However, observation demonstrates that numerous connections to the Fibonacci numbers cannot be denied. Kant has said that it was Nature herself, not the mathematician, who brings mathematics into natural philosophy.

Other manifestations of phyllotaxis are seen in the arrangement of the florets of a sunflower, or the scales of a fir cone, in spiral or helical whorls, which are referred to as parastichies. A comment is quoted from Thompson to illustrate further some of the relationships to Fibonacci numbers.

> Among other cases in which such spiral series are readily visible we have, for instance, the crowded leaves of the stone-crops and mesembryanthemums, and the crowded florets of the composites. Among these we may find plenty of examples in which the numbers of the serial rows are similar to those of the fir-cones; but in some cases, as in the daisy and others of the smaller composites, we shall be able to trace thirteen rows in one direction and twenty-one in the other, or perhaps twenty-one and thirty-four; while in a great big sunflower we may find (in one and the same species) thirty-four and fifty-five, fifty-five and eighty-nine, or even as many as eighty-nine and one hundred and forty-four. On the other hand, in an ordinary "pentamerous" flower, such as ranunculus, we may be able to trace, in the arrangement of its sepals, petals and stamens, shorter spiral series, three in one direction and two in the other; and the scales on the little cone of a Cypress show the same numerical simplicity. It will be at once observed that these arrangements manifest themselves in connection with very different things, in the orderly interspacing of single leaves and of entire florets, and among all kinds of leaf-like structures, foliage-leaves, bracts, cone-scales, and the various parts or members of the flower... the arrangements mentioned being set forth as follows (the fractional number used being simply an abbreviated symbol for the number of associated helices or parastichies which we can count running in opposite directions): 2/3, 3/5, 5/8, 8/13, 13/21, 21/34, 34/55, 55/89, 89/144. [21]

While there are irregularities in the manner in which some of these phenomena occur, the facts are born out too well to permit them to be dismissed as accidental. For instance, out of 505 cones of the Norway spruce, the American naturalist Beal found 92% in which the spirals were in five and eight rows; 6% were in four and seven, and 4% were in four and six rows. The relations involving the Fibonacci numbers are simply there, whatever the reasons may be.

The presence of Fibonacci numbers in phyllotaxis is not the only place in which these remarkable numbers seem to occur. Recent investigations have revealed their presence in electrical network theory, in music, and in nuclear physics, to mention a few, though nowhere do the relations seem so obvious as in phyllotaxis.

Leonardo of Pisa

It would not be proper to conclude this chapter without making some mention of the originator of the Fibonacci sequence. As was mentioned earlier, Leonardo of Pisa, nicknamed Fibonacci, a contraction of Filius Bonacci, son of Bonacci, created this sequence by considering a rabbit problem in his book, Liber Abacci. This book, written in 1202 A. D., was not published until 1857, near the time when it caught the attention of the French mathematician, Edouard Lucas. It was Lucas who did so much to revive and stimulate interest in the Fibonacci sequence, and who first applied Fibonacci's name to it.

Leonardo was a learned man, educated in Morocco, where his father was a clerk or dragoman to Pisan merchants. He travelled about the Mediterranean, met with scholars, and studied the various systems of arithmetic then in use. In so doing, he became convinced that the Hindu-Arabic system was superior and he consciously sought to promulgate this system in Italy. Largely for this reason he wrote Liber Abacci, the first thorough treatment of arithmetic and algebra written by a

Christian. While no copies of this book are available, copies of a second edition, which he wrote in 1228, exist today. He wrote several other works, notably Liber Quadratorum, which was a brilliant and original work. Without question, Leonardo was a true scholar, and is recognized as the outstanding mathematician between Diophantus and Fermat in analysis of certain types of equations of second degree.

The works of Leonardo Fibonacci are available in some universities in the United States through two volumes by the Italian historian of mathematics, Baldassarre Boncompagni, entitled Scritte di Leonardo Pisano, which have been published in Rome. The first volume contains the Liber Abacci and the second contains Patricia Geometrial, Flos, Epistola ad Magestrum Theodorum, and Liber Quadratorum.

With these remarks in tribute to Fibonacci, it is appropriate that this introductory chapter be drawn to a close. Many of the properties that were intuitively discussed are seen again in the remaining chapters, but are presented in a somewhat more formal setting. It is hoped that this formalization will serve to intensify those appealing qualities of the Fibonacci numbers that have been somewhat casually presented in the foregoing pages.

CHAPTER III

ELEMENTARY PROPERTIES

Proofs from the Definition

No doubt the reader is already familiar with many of the mathematical concepts of algebra, number theory, and analysis that appear in this thesis. Nevertheless, some of the basic definitions, notations, and operations that are used are included as the need arises.

Familiarity with the integers is assumed. When referring to the natural numbers, one means the positive integers $1, 2, 3, \ldots$. The stage is now set for formal definitions.

Definition 3.1. A recurring sequence is a sequence of numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ in which each term is defined as a function of the preceding terms.

Definition 3.2. The Fibonacci sequence is a recurring sequence $F_1, F_2, F_3, \ldots, F_n, \ldots$ such that $F_1 = 1, F_2 = 1,$ and with the recurrence formula $F_{n+2} = F_n + F_{n+1}$. Where convenient, one writes $F_0 = 0$.

Some simple identities involving Fibonacci numbers will now be stated in the following three theorems. These identities are proved using only the recurrence formula that defines the Fibonacci sequence.

Theorem 3.1. The Fibonacci numbers have the following properties for sums:

(1) $F_1 + F_2 + F_3 + \dots + F_n = F_{n+2} - 1,$

(2) $F_1 + F_3 + F_5 + \dots + F_{2n-1} = F_{2n},$

(3) $F_2 + F_4 + F_6 + \dots + F_{2n} = F_{2n+1} - 1,$

(4) $F_1 - F_2 + F_3 - F_4 + \dots + (-1)^{n+1}F_n = (-1)^{n+1}F_{n-1} + 1.$

Proof: From the recurrence formula, $F_{n+2} = F_n + F_{n+1}$,

$$\begin{aligned} F_1 &= F_3 - F_2, \\ F_2 &= F_4 - F_3, \\ F_3 &= F_5 - F_4, \\ &\vdots \\ F_{n-1} &= F_{n+1} - F_n, \\ F_n &= F_{n+2} - F_{n+1}. \end{aligned}$$

Adding left and right members of these equations yields

$$F_1 + F_2 + F_3 + \dots + F_n = F_{n+2} - F_2 = F_{n+2} - 1.$$

This proves (1).

For (2), one proceeds in similar fashion, noting that $F_1 = F_2 = 1$.

Hence,

$$\begin{aligned} F_1 &= F_2, \\ F_3 &= F_4 - F_2, \\ F_5 &= F_6 - F_4, \\ &\vdots \\ F_{2n-1} &= F_{2n} - F_{2n-2}. \end{aligned}$$

Again, adding left and right members yields (2).

A different type proof for (3) can be obtained using (1) and (2). Replacing n by $2n$ in (1) yields

$$F_1 + F_2 + F_3 + \ldots + F_{2n-1} + F_{2n} = F_{2n+2} - 1.$$

Subtracting corresponding terms of (2) from this yields

$$\begin{aligned} F_2 + F_4 + F_6 + \ldots + F_{2n} &= F_{2n+2} - 1 - F_{2n} \\ &= F_{2n+1} - 1, \end{aligned}$$

since $F_{2n+1} = F_{2n+2} - F_{2n}$.

In order to prove (4), one multiplies both members of (3) by -1 and adds termwise to (2). Then

$$\begin{aligned} F_1 - F_2 + F_3 - F_4 + \ldots + F_{2n-1} - F_{2n} &= F_{2n} - F_{2n+1} + 1 \\ &= -(F_{2n+1} - F_{2n}) + 1 \\ &= -F_{2n-1} + 1. \end{aligned}$$

This provides the desired result (4) when the last index is even. To treat the case when the last index is odd, one adds F_{2n+1} to both members of this expression to obtain

$$\begin{aligned} F_1 - F_2 + F_3 - F_4 + \ldots - F_{2n} + F_{2n+1} &= F_{2n+1} - F_{2n-1} + 1 \\ &= F_{2n} + 1. \end{aligned}$$

Hence, combining these last two expressions,

$$F_1 - F_2 + \ldots + (-1)^{n+1}F_n = (-1)^{n+1}F_{n-1} + 1.$$

This completes the proof of the theorem.

Challenge: Establish the identity (3), directly, without the use of the other identities.

In the following theorem, three identities involving the squares of Fibonacci numbers are proved.

Theorem 3.2. The Fibonacci numbers have the following properties:

(1) $$F_1^2 + F_2^2 + F_3^2 + \dots + F_n^2 = F_n F_{n+1},$$

(2) $$F_{n+1}^2 + F_{n+2}^2 = F_n F_{n+2} + F_{n+1} F_{n+3},$$

(3) $$2(F_n^2 + F_{n+1}^2) = F_{n-1}^2 + F_{n+2}^2.$$

Proof: For (1), note first that for k greater than 1,

$$F_k F_{k+1} - F_{k-1} F_k = F_k (F_{k+1} - F_{k-1}) = F_k^2.$$

Hence,

$$\begin{aligned} F_1^2 &= F_1 F_2, \\ F_2^2 &= F_2 F_3 - F_1 F_2, \\ F_3^2 &= F_3 F_4 - F_2 F_3, \\ &\vdots \\ F_n^2 &= F_n F_{n+1} - F_{n-1} F_n. \end{aligned}$$

Adding left and right members yields (1).

For (2), write

$$\begin{aligned} F_n F_{n+2} + F_{n+1} F_{n+3} &= (F_{n+2} - F_{n+1}) F_{n+2} + F_{n+1}(F_{n+1} + F_{n+2}) \\ &= F_{n+2}^2 + F_{n+1}^2. \end{aligned}$$

To prove (3), write

$$\begin{aligned} F_{n-1}^2 + F_{n+2}^2 &= (F_{n+1} - F_n)^2 + (F_n + F_{n+1})^2 \\ &= F_{n+1}^2 - 2F_n F_{n+1} + F_n^2 + F_n^2 + 2F_n F_{n+1} + F_{n+1}^2 \\ &= 2(F_n^2 + F_{n+1}^2). \end{aligned}$$

A problem sometimes faced by the teacher of elementary mathematics in formulating problems and test questions is the problem of finding integer solutions to the Pythagorean equation, $x^2 + y^2 = z^2$, other than the tried and true examples, (3, 4, 5), (5, 12, 13), and so forth. It is at least surprising that the Fibonacci sequence turns such solutions out in abundance, as seen from the next theorem due to Lairaut.

<u>Theorem 3.3</u>. If F_n, F_{n+1}, F_{n+2}, F_{n+3} are four consecutive Fibonacci numbers, then $x = F_nF_{n+3}$, $y = 2F_{n+1}F_{n+2}$, and $z = F_{n+1}^2 + F_{n+2}^2$ satisfy the equation $x^2 + y^2 = z^2$.

Proof: The proof is immediate since

$$\begin{aligned} x^2 + y^2 &= (F_nF_{n+3})^2 + (2F_{n+1}F_{n+2})^2 \\ &= ((F_{n+2} - F_{n+1})(F_{n+2} + F_{n+1}))^2 + 4F_{n+1}^2F_{n+2}^2 \\ &= (F_{n+2}^2 - F_{n+1}^2)^2 + 4F_{n+1}^2F_{n+2}^2 \\ &= (F_{n+2}^2 + F_{n+1}^2)^2 \\ &= z^2. \end{aligned}$$

<u>Corollary</u>. If F_n, F_{n+1}, F_{n+2}, F_{n+3} are four consecutive Fibonacci numbers, then $x = F_nF_{n+3}$, $y = 2F_{n+1}F_{n+2}$, and $z = F_nF_{n+2} + F_{n+1}F_{n+3}$ satisfy the equation $x^2 + y^2 = z^2$.

Proof: Apply identity (2) in Theorem 3.2. to the expression for z, and the corollary then follows from Theorem 3.3.

Using Mathematical Induction

So far, the results obtained have involved only the recurrence formula, $F_{n+2} = F_n + F_{n+1}$. As more mathematical tools are used, a wider selection of results may be obtained. In the sequel, these tools are brought out systematically, and their usefulness is demonstrated by proving various results concerning Fibonacci numbers. Probably the most useful instrument that is available for the type of work at hand is the Principle of Finite Mathematical Induction.

Principle of Finite Mathematical Induction. Let there be associated with each positive integer n a proposition $P(n)$ which is either true or false. If, firstly, $P(1)$ is true, and secondly, for all k, $P(k)$ implies $P(k+1)$, then $P(n)$ is true for all positive integers n.

Proof by induction is based on essentially two facets; namely, the existence of a first case ($P(1)$ is true), and the truth of the proposition in the $(k+1)$th case, whenever it is true in the kth case. The assumption $P(k)$ is true is often referred to as the induction hypothesis. It is unfortunate in a way that the name given to this principle uses the word induction, because proof by mathematical induction is in reality deduction, which is always the situation in mathematical proof.

The formula $n^2 - 79n + 1601$, n a positive integer, delivers prime numbers for all n up through $n = 79$. The scientist accustomed to empirical procedures would probably be content to risk a theory on far fewer than 79 experimental verifications, but this formula yields a composite for $n = 80$. In mathematics, neither seventy-nine nor a million and seventy-nine verifications constitute a proof.

The principle of mathematical induction is illustrated by proving a number of identities on Fibonacci numbers. Its nature and importance is such, however, that this type of proof appears frequently throughout this thesis. The identities stated in Theorem 3.1. and Theorem 3.2. could have been proved by induction.

As a demonstration, the identity

$$P(n):\quad F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1$$

of Theorem 3.1. will be proved by induction. Observe that P(1) is true since

$$F_2 = 1 = 2 - 1 = F_3 - 1 = F_{2+1} - 1.$$

For confidence, P(2), P(3), and so forth, could be verified, but this is not at all necessary. Instead, one makes the induction hypothesis that P(k) is true, namely,

$$F_2 + F_4 + F_6 + \cdots + F_{2k} = F_{2k+1} - 1.$$

Now add F_{2k+2} to both menbers. It follows that

$$\begin{aligned} F_2 + F_4 + F_6 + \cdots + F_{2k} + F_{2k+2} &= F_{2k+2} + F_{2k+1} - 1 \\ &= F_{2k+3} - 1. \end{aligned}$$

Regrouping the subscripts in left and right members

$$F_2 + F_4 + F_6 + \cdots + F_{2k} + F_{2(k+1)} = F_{2(k+1)+1} - 1,$$

which is precisely P(k+1). Since P(k) implies P(k+1) for any integer k, the proof by induction is complete.

<u>Theorem 3.4.</u> The Fibonacci numbers have the following properties:

(1) $F_1 + 2F_2 + 3F_3 + \cdots + nF_n = (n+1)F_{n+2} - F_{n+4} + 2,$

(2) $F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{2n-1}F_{2n} = F_{2n}^{\,2},$

(3) $F_{n+1}^{\,2} - F_nF_{n+2} = (-1)^n,$

(4) $F_{n+2}^{\,2} - 3F_{n+1}^{\,2} + F_n^{\,2} = 2(-1)^n.$

Proof: In (1), P(1) is true since

$$F_1 = 1 = (2)(2) - 5 + 2 = (1+1)F_3 - F_5 + 2.$$

Assume P(k) is true:

$$F_1 + 2F_2 + \cdots + kF_k = (k+1)F_{k+2} - F_{k+4} + 2.$$

Adding $(k+1)F_{k+1}$ to both members yields

$$\begin{aligned} F_1 + 2F_2 + \cdots + kF_k + (k+1)F_{k+1} &= (k+1)F_{k+2} - F_{k+4} + 2 + (k+1)F_{k+1} \\ &= (k+1)(F_{k+1} + F_{k+2}) - F_{k+4} + 2 \\ &= (k+1)F_{k+3} - F_{k+4} + 2 \\ &= (k+2)F_{k+3} - (F_{k+3} + F_{k+4}) + 2 \\ &= (k+2)F_{k+3} - F_{k+5} + 2 \\ &= ((k+1) + 1)F_{(k+1)+2} - F_{(k+1)+4} + 2. \end{aligned}$$

Hence, $P(k)$ true implies $P(k+1)$ true, and (1) is proved by induction.

For the proof of (2),

$$F_1F_2 = (1)(1) = 1 = 1^2 = F_2^2$$

demonstrates that $P(1)$ is true. Assume $P(k)$ is true:

$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{2k-1}F_{2k} = F_{2k}^2.$$

Add $F_{2k}F_{2k+1} + F_{2k+1}F_{2k+2}$ to the left and right members to obtain

$$\begin{aligned} &F_1F_2 + F_2F_3 + \cdots + F_{2k}F_{2k+1} + F_{2k+1}F_{2k+2} \\ &= F_{2k}^2 + F_{2k}F_{2k+1} + F_{2k+1}F_{2k+2} \\ &= (F_{2k+2} - F_{2k+1})^2 + F_{2k}F_{2k+1} + F_{2k+1}F_{2k+2} \\ &= F_{2k+2}^2 - 2F_{2k+2}F_{2k+1} + F_{2k+1}^2 + F_{2k}F_{2k+1} + F_{2k+1}F_{2k+2} \\ &= F_{2k+2}^2 + F_{2k+1}(-2F_{2k+2} + F_{2k+1} + F_{2k} + F_{2k+2}) \\ &= F_{2k+2}^2 + F_{2k+1}(-2F_{2k+2} + F_{2k+2} + F_{2k+2}) \\ &= F_{2k+2}^2 + 0. \end{aligned}$$

Hence,

$$F_1F_2 + F_2F_3 + \dots + F_{2(k+1)-1}F_{2(k+1)} = F_{2(k+1)}^2;$$

so P(k+1) follows from P(k), and the induction is complete.

It will be more convenient to prove (3) if it is written

$$F_{n+1}^2 = F_nF_{n+2} + (-1)^n.$$

One readily verifies P(1) by writing

$$F_2^2 = 1 = (1)(2) - 1 = F_1F_3 + (-1)^1.$$

The induction hypothesis is

$$F_{k+1}^2 = F_kF_{k+2} + (-1)^k.$$

Adding $F_{k+1}F_{k+2}$ to both members yields

$$F_{k+1}F_{k+2} + F_{k+1}^2 = F_{k+1}F_{k+2} + F_kF_{k+2} + (-1)^k,$$

which implies

$$F_{k+1}(F_{k+2} + F_{k+1}) = F_{k+2}(F_k + F_{k+1}) + (-1)^k,$$

or

$$F_{k+1}F_{k+3} = F_{k+2}^2 + (-1)^k.$$

This is the same as

$$F_{k+2}^2 = F_{(k+1)}F_{(k+2)+1} + (-1)^{k+1}.$$

Hence, P(k+1) is true.

Finally, to prove (4) one readily verifies P(1) is true. Assuming the truth of P(k) implies

$$F_{k+2}^2 - 3F_{k+1}^2 + F_k^2 = 2(-1)^k.$$

At this point it is not at all obvious what quantity ought to be added to both members of this equation to deduce P(k+1). However, one can resort to mathematical craftiness. The propositional statement of P(k+1) is

$$F_{(k+1)+2}^{\ 2} - 3F_{(k+1)+1}^{\ 2} + F_{k+1}^{\ 2} = 2(-1)^{k+1}.$$

Adding left and right members of this and the previous equation yields

$$F_{k+2}^{\ 2} - 3F_{k+1}^{\ 2} + F_k^{\ 2} + F_{k+3}^{\ 2} - 3F_{k+2}^{\ 2} + F_{k+1}^{\ 2} = 2(-1)^k + 2(-1)^{k+1} = 0.$$

Hence, $P(k+1)$ will be true if it can be shown that the left member reduces identically to zero. But

$$\begin{aligned} F_{k+2}^{\ 2} - 3F_{k+1}^{\ 2} + F_k^{\ 2} + F_{k+3}^{\ 2} - 3F_{k+2}^{\ 2} + F_{k+1}^{\ 2} \\ &= -2F_{k+2}^{\ 2} - 2F_{k+1}^{\ 2} + F_k^{\ 2} + F_{k+3}^{\ 2} \\ &= -2F_{k+2}^{\ 2} + 2F_{k+1}^{\ 2} + (F_{k+2} - F_{k+1})^2 + (F_{k+2} + F_{k+1})^2 \\ &= 0. \end{aligned}$$

Hence, the induction and the theorem are complete.

Challenge: Prove the identities of Theorem 3.1. by induction.

Challenge: Prove by induction:

(1) $nF_1 + (n-1)F_2 + (n-2)F_3 + \ldots + 2F_{n-1} + F_n = F_{n+4} - (n+3)$,

(2) $F_1F_2 + F_2F_3 + F_3F_4 + \ldots + F_{2n}F_{2n+1} = F_{2n+1}^{\ 2} - 1$,

(3) $F_{n+2}F_n - F_{n+3}F_{n-1} = 2(-1)^{n+1}$,

(4) $F_{n+1}^{\ 2} - F_{n+3}F_{n-1} = (-1)^{n-1}$.

The next theorem is a useful result that opens an avenue for proving several identities that involve more elaborate indexing than those obtained previously. It also brings to a close the section that emphasizes the inductive argument. It should not be assumed that no further need of finite induction will arise; however, its role will no longer be emphasized.

Theorem 3.5. The Fibonacci numbers have the property

$$F_{m+n} = F_{m+1}F_n + F_mF_{n-1}.$$

Proof: The proof is by induction on m. P(1) follows immediately, since

$$F_{1+n} = F_n + F_{n-1} = F_2F_n + F_1F_{n-1}$$

for every positive integer n. Assuming P(k) is true means that for all positive integers n,

$$F_{k+n} = F_{k+1}F_n + F_kF_{n-1}.$$

Using this relation for the case when n is replaced by n-1 implies

$$F_{k+n-1} = F_{k+1}F_{n-1} + F_kF_{n-2}.$$

Adding corresponding members of these two equalities yields

$$\begin{aligned} F_{(k+1)+n} = F_{(k+n)+1} &= F_{k+n} + F_{k+n-1} \\ &= F_{k+1}F_n + F_kF_{n-1} + F_{k+1}F_{n-1} + F_kF_{n-2} \\ &= F_{k+1}F_n + F_kF_{n-1} + F_k(F_n - F_{n-1}) + F_{k+1}F_{n-1} \\ &= F_{k+1}F_n + F_kF_n + F_{k+1}F_{n-1} \\ &= (F_{k+1} + F_k)F_n + F_{k+1}F_{n-1} \\ &= F_{(k+1)+1}F_n + F_{k+1}F_{n-1}. \end{aligned}$$

Hence, P(k) true implies P(k+1) is true, and the proof is complete.

<u>Corollary</u>. The property $F_{m+n} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1}$ holds for the Fibonacci numbers.

Proof: From the theorem

$$\begin{aligned} F_{m+n} &= F_{m+1}F_n + F_mF_{n-1} \\ &= F_{m+1}(F_{n+1} - F_{n-1}) + F_mF_{n-1} \\ &= F_{m+1}F_{n+1} - F_{m+1}F_{n-1} + F_mF_{n-1} \end{aligned}$$

$$= F_{m+1}F_{n+1} - F_{n-1}(F_{m+1} - F_m)$$

$$= F_{m+1}F_{n+1} - F_{n-1}F_{m-1},$$

which is the desired relation.

As a consequence of the preceding theorem, setting $m = n$ yields

$$F_{2n} = F_{n+1}F_n + F_nF_{n-1}$$

$$= F_n(F_{n+1} + F_{n-1}).$$

This indicates that the quotient of F_{2n} divided by F_n is an integer. A more general result of this nature is established later.

Another identity, $F_{2n+1} = F_{n+1}^2 + F_n^2$, is obtained from Theorem 3.5. by setting $n = m+1$, and then replacing the m with n. The following challenge illustrates some other identities that stem from this theorem.

Challenge: Establish the identities

(1) $$F_{2n-1} = F_n^2 + F_{n-1}^2,$$

(2) $$F_{2n} = F_{n+1}^2 - F_{n-1}^2.$$

The proofs by induction in the foregoing illustrate that the Fibonacci sequence provides an excellent vehicle to motivate and demonstrate mathematical proof. Further examples are yet to appear; however, more variety may be permitted if some additional definitions are introduced.

Arithmetical Properties of Fibonacci Numbers

The following definitions and theorems are stated from elementary algebra and number theory. For proofs, notation, and so forth, see Birkhoff and MacLane. [5]

Definition 3.3. An integer d, not zero, is said to divide an integer b, if there exists an integer c, such that $b = dc$. In this case d is called a divisor, or factor of b, and b is called a multiple of d. In symbols, one writes $d \mid b$, with the contrary expressed $d \nmid b$.

Definition 3.4. For two integers a and b, if $d \mid a$ and $d \mid b$, then d is a common divisor of a and b.

Theorem 3.A. Let a, b, c be integers.

(1) $b \mid a$ and $c \mid b$ implies $c \mid a$,

(2) $c \mid a$ and $c \mid a+b$ implies $c \mid b$,

(3) $c \mid a$ and $c \mid b$ implies $c \mid ma+nb$, for all integers m and n.

Definition 3.5. A positive integer p is prime if $p > 1$ and p has no positive divisors except 1 and p. A number greater than 1 and not prime is called composite.

Definition 3.6. A greatest common divisor of two integers a and b is a common divisor g, such that for any common divisor d of a and b, $d \mid g$. One writes $(a,b) = g$, for the positive greatest common divisor.

Definition 3.7. The integers a and b are relatively prime if, and only if, $(a,b) = 1$.

Theorem 3.B. The Division Algorithm. For given integers a and b, $b > 0$, there exist integers q and r such that

$$a = bq + r, \quad 0 \leq r < b.$$

It should be observed that the algorithm can be used to obtain the greatest common divisor of two integers. Applying the algorithm successively, under the hypothesis of Theorem 3.B.:

$$a = bq + r_1, \qquad 0 \leq r_1 < b,$$

$$b = r_1 q_1 + r_2, \qquad 0 \leq r_2 < r_1,$$

$$r_1 = r_2 q_2 + r_3, \qquad 0 \leq r_3 < r_2,$$

$$\vdots \qquad\qquad \vdots$$

$$r_{n-2} = r_{n-1} q_{n-1} + r_n, \qquad 0 \leq r_n < r_{n-1},$$

$$r_{n-1} = r_n q_n.$$

By the last line, $r_n \mid r_{n-1}$. Hence, the line before the last shows $r_n \mid r_{n-2}$. Continuing back to the first equation, one has $r_n \mid a$ and $r_n \mid b$, so that r_n is a common divisor. That r_n is the greatest common divisor follows by noting that for any common divisor d of a and b, $d \mid r_1$ by the first equation; hence, $d \mid r_2$ by the second, and continuing to the last, $d \mid r_n$.

<u>Theorem 3.C.</u> Let a, b, c be integers. Then,

(1) $(a,b) \mid (a,bc)$,

(2) $(ac,bc) = (a,b)c$,

(3) $b \mid a$ if, and only if, $(a,b) = b$,

(4) $(a,bc) = (a,b)$, if $(a,c) = 1$,

(5) $(a,b) = (a,b+c)$, if $a \mid c$,

(6) $ab \mid c$, if $(a,b) = 1$, $a \mid c$, and $b \mid c$.

A number of properties relating to divisibility can now be demonstrated for the Fibonacci numbers. Many of these properties show interesting and unusual connections between the Fibonacci numbers and their subscripts.

Theorem 3.6. Any two consecutive Fibonacci numbers are relatively prime. In symbols, $(F_n, F_{n+1}) = 1$.

Proof: The proof is by induction. For $n = 1$,

$$(F_1, F_2) = (1,1) = 1.$$

The induction hypothesis is $(F_k, F_{k+1}) = 1$.

The proof will be completed if it can be shown that this implies $(F_{k+1}, F_{k+2}) = 1$. But, using part (5) of Theorem 3.C. with $a = c = F_{k+1}$, one has $(F_{k+1}, F_{k+2}) = (F_{k+1}, F_k + F_{k+1}) = (F_{k+1}, F_k) = (F_k, F_{k+1}) = 1$.

Theorem 3.7. The Fibonacci numbers have the property $F_n \mid F_{rn}$, for every positive integer r.

Proof: The proof is by induction on r. For $r = 1$, the result is trivial. Assume $F_n \mid F_{kn}$, so that there is an integer h such that $F_{kn} = hF_n$. Hence, using Theorem 3.5.,

$$F_{(k+1)n} = F_{kn+n} = F_{kn+1}F_n + F_{kn}F_{n-1} = (F_{kn+1} + hF_{n-1})F_n.$$

Therefore, $F_n \mid F_{(k+1)n}$ and the proof is complete.

Lemma. $(F_n, F_{kn-1}) = 1$.

Proof: By Theorem 3.6., $(F_{kn}, F_{kn-1}) = 1$. Suppose $(F_n, F_{kn-1}) = d$ greater than 1. Then, $d \mid F_{kn-1}$ and $d \mid F_n$. Hence, $d \mid F_{kn-1}$ and $d \mid F_{kn}$ since $F_n \mid F_{kn}$ by Theorem 3.7. Hence, d is a divisor of F_{kn} and F_{kn-1}, therefore, $d \mid 1$, contrary to the supposition d greater than 1. The contradiction implies $(F_n, F_{kn-1}) = 1$.

Theorem 3.8. A Fibonacci number with subscript the greatest common divisor of the subscripts for any two Fibonacci numbers is itself the greatest common divisor of the two numbers. In symbols, $(F_n, F_m) = F_{(n,m)}$.

Proof: The proof is trivial when $m = n$. Suppose for definiteness that m is greater than n. Applying the division algorithm to m and n, and the successive remainders,

$$\begin{aligned} m &= nq_0 + r_1, & 0 &\leq r_1 < n, \\ n &= r_1 q_1 + r_2, & 0 &\leq r_2 < r_1, \\ r_1 &= r_2 q_2 + r_3, & 0 &\leq r_3 < r_2, \\ &\vdots & &\vdots \\ r_{t-2} &= r_{t-1} q_{t-1} + r_t, & 0 &\leq r_t < r_{t-1} \\ r_{t-1} &= r_t q_t, \end{aligned}$$

where r_t is the last remainder different from zero. Then $r_t = (m,n)$. Since $m = nq_0 + r_1$, it follows from Theorem 3.5. that

$$F_m = F_{r_1+nq_0} = F_{r_1+1}F_{nq_0} + F_{r_1}F_{nq_0-1}.$$

Hence, by Theorem 3.C., part (5),

$$\begin{aligned} (F_n, F_m) &= (F_n, F_{r_1+1}F_{nq_0} + F_{r_1}F_{nq_0-1}) \\ &= (F_n, F_{r_1}F_{nq_0-1}). \end{aligned}$$

Now, $(F_n, F_{nq_0-1}) = 1$ by the lemma. Hence,

$$(F_n, F_m) = (F_n, F_{r_1}F_{nq_0-1}) = (F_n, F_{r_1})$$

follows by Theorem 3.C., part (4). Proceeding in similar fashion,

$$(F_n,F_{r_1}) = (F_{r_1},F_{r_2})$$

$$(F_{r_1},F_{r_2}) = (F_{r_2},F_{r_3})$$

$$\vdots \qquad \vdots$$

$$(F_{r_{t-2}},F_{r_{t-1}}) = (F_{r_{t-1}},F_{r_t}).$$

Now, because of Theorem 3.7., $r_t \mid r_{t-1}$ implies $F_{r_t} \mid F_{r_{t-1}}$. Hence, combining the above results yields, with the aid of Theorem 3.C., part (3),

$$(F_n,F_m) = (F_{r_{t-1}},F_{r_t}) = F_{r_t} = F_{(n,m)}.$$

One interesting consequence of Theorem 3.8. is that no odd Fibonacci number is divisible by 17. To demonstrate this fact, suppose there does exist an odd Fibonacci number, say F_n, such that $17 \mid F_n$. Note that F_n odd implies $(2,F_n) = 1$. Then, with the aid of Theorem 3.C., part (4), and Theorem 3.8., it follows that

$$17 = (F_n,17) = (F_n,34) = (F_n,F_9) = F_{(n,9)}.$$

Since the only possible values of (n,9) are 1, 3, and 9, and $F_1 \neq 17$, $F_3 \neq 17$, and $F_9 \neq 17$, the supposition that $17 \mid F_n$ leads to a contradiction.

<u>Challenge</u>: Prove that if $(m,n) = 1$, then $F_mF_n \mid F_{mn}$. (Hint. Use Theorem 3.7., Theorem 3.8., and Theorem 3.C.)

<u>Theorem 3.9.</u> $F_m \mid F_n$ if, and only if, $m \mid n$.

Proof: If $m \mid n$, then there is an integer r such that $n = rm$, and

$F_m \mid F_{rm} = F_n$. To prove the converse, if $F_m \mid F_n$, then by Theorem 3.C., part (3) and Theorem 3.8., $F_m = (F_m, F_n) = F_{(m,n)}$. Hence, $m = (m,n)$, and so, $m \mid n$.

<u>Challenge</u>: Use Theorem 3.9. to show that a Fibonacci number is divisible by 3 if, and only if, its index is divisible by 4.

<u>Challenge</u>: Show that a Fibonacci number is divisible by 5 if, and only if, its index is divisible by 5.

The next and final theorem of this chapter deals with the question of whether, for an arbitrarily assigned integer m, there is some Fibonacci number that is divisible by m. The theorem shows that the first Fibonacci number divisible by m is not especially large.

<u>Theorem 3.10.</u> For any integer m there is at least one Fibonacci number among the first m^2 Fibonacci numbers that is divisible by m.

Proof: If $m = 1$, the proof is trivial. Let m 1 be a positive integer. For any positive integer r, the division algorithm implies

$$r = mq + r^*, \qquad 0 \leq r^* < m.$$

Note that the remainder r^* must be one of the m numbers $0, 1, 2, \ldots, m-1$. Now consider the sequence of pairs of such remainders

(1) $$(F_1^*:F_2^*),\ (F_2^*:F_3^*),\ (F_3^*:F_4^*),\ \ldots,\ (F_n^*:F_{n+1}^*),\ \ldots,$$

where for each n, F_n^* is the remainder on division of F_n by m. Since there are at most m different remainders possible on division by m, there can be at most m^2 pairs in the above sequence that have different first entries and different second entries. Therefore, for the first m^2+1 pairs in the sequence (1), at least two will be equal, in the sense that $(a:b) = (c:d)$ if, and only if, $a = c$ and $b = d$.

Let $(F^*_k:F^*_{k+1}) = (F^*_h:F^*_{h+1})$, $k < h \leq m^2+1$, be the first repeated pair in the sequence (1). Hence, $F^*_k = F^*_h$ and $F^*_{k+1} = F^*_{h+1}$. Now, suppose $k > 1$. Since $F_{h-1} = F_{h+1} - F_h$ and $F_{k-1} = F_{k+1} - F_k$, it follows that there exist integers q_{h-1}, q_{k-1} such that

(2) $$F_{h-1} = mq_{h-1} + F^*_{h-1}, \qquad 0 \leq F^*_{h-1} < m,$$

and

(3) $$F_{k-1} = mq_{k-1} + F^*_{k-1}, \qquad 0 \leq F^*_{k-1} < m.$$

Therefore, for suitable integers q_{h+1}, q_h, q_{k+1}, q_k,

$$\begin{aligned} F^*_{h-1} &= F_{h-1} - mq_{h-1} = F_{h+1} - F_h - mq_{h-1} \\ &= mq_{h+1} + F^*_{h+1} - mq_h - F^*_h - mq_{h-1}, \end{aligned}$$

and

$$\begin{aligned} F^*_{k-1} &= F_{k-1} - mq_{k-1} = F_{k+1} - F_k - mq_{k-1} \\ &= mq_{k+1} + F^*_{k+1} - mq_k - F^*_k - mq_{k-1}. \end{aligned}$$

Consider the difference $F^*_{h-1} - F^*_{k-1}$. From the immediately preceding equations,

$$F^*_{h-1} - F^*_{k-1} = mQ + (F^*_{h+1} - F^*_{k+1}) - (F^*_h - F^*_k) = mQ$$

where Q is an integer. Hence, m divides $\left|F^*_{h-1} - F^*_{k-1}\right|$.

Now, from the inequalities in (2) and (3), $0 \leq \left|F^*_{h-1} - F^*_{k-1}\right| < m$.

Because the positive integer m divides a non-negative integer smaller than itself, the integer must be zero, and so $F^*_{h-1} = F^*_{k-1}$. This means that $(F^*_{k-1}:F^*_k) = (F^*_{h-1}:F^*_h)$, $k-1 < k$. This contradicts the fact that $(F^*_k:F^*_{k+1})$ was the first pair in the sequence (1) to be repeated.

Therefore, the supposition $k > 1$ is false. Hence, $k = 1$ and $(F_k^*:F_{k+1}^*) = (F_1^*:F_2^*) = (1^*:1^*) = (1:1)$ is the first pair in sequence (1) to appear more than once, being repeated in the h^{th} position, $1 < h \leq m^2+1$. Now, $(F_h^*:F_{h+1}^*) = (1:1)$ implies $F_h = mq_h + F_h^* = mq_h + 1$ and $F_{h+1} = mq_{h+1} + F_{h+1}^* = mq_{h+1} + 1$. Therefore, $F_{h-1} = F_{h+1} - F_h = mq_{h+1} + 1 - (mq_h + 1) = m(q_{h+1} - q_h)$ implies $m \mid F_{h-1}$. In other words, the $(h-1)^{th}$ Fibonacci number is divisible by m and $1 \leq (h-1) \leq m^2$, proving the theorem.

It should be recognized that as a consequence of this theorem and Theorem 3.7., there must be infinitely many Fibonacci numbers divisible by a given integer m. Also, Theorem 3.10. indicates that the first Fibonacci number divisible by m will not be extremely large, though it gives no indication of how the number might be found.

CHAPTER IV

NON-RECURRENCE EXPRESSIONS FOR F_n

The Binet Formula

An important problem that needs resolution is the matter of determining a prescribed Fibonacci number as a function of its subscript, thus avoiding the necessity of a tedious calculation of all prior terms by the recurrence formula $F_{n+2} = F_{n+1} + F_n$. Obtaining such an expression also makes it possible to elicit further information about the Fibonacci numbers. The formula introduced in this section was actually known to Leonard Euler and David Bernoulli; however, it was rediscovered by J. P. M. Binet in 1843. Interestingly enough, the golden ratio occupies a prominent position in this formula.

It is unfortunate, but the proofs available at this stage of the exposition are not motivated particularly well. It should not be supposed that the proofs lack rigor. The difficulty lies in the fact that the reader may have cause to wonder why a certain approach is used. The only good answer, at this point, is that it brings about the desired result, which is not an especially satisfying reply. Because of these difficulties, the first theorem of this chapter is presented with two proofs. The first proof is the most elementary, but appears to be based on a very lucky guess. The second proof is presented at the close of this chapter and is based on complex variable theory. In Chapter VI the derivation of the Binet formula is given in a setting which removes the objections mentioned above.

Theorem 4.1. The n^{th} term of the Fibonacci sequence is given by the formula

$$F_n = 5^{-1/2}(r^n - s^n),$$

where $r = (1 + 5^{1/2})/2$ and $s = (1 - 5^{1/2})/2$.

Proof: Assume that the n^{th} term can be obtained from an expression of the form

$$F_n = cr^n + ds^n. \tag{1}$$

The proof is complete if it can be shown that this formula, for suitable values of r, s, c, and d, can be made to satisfy the recurrence formula

$$F_n = F_{n-1} + F_{n-2}, \tag{2}$$

with n greater than 2, and $F_1 = F_2 = 1$.

To determine r and s, substitute the assumed formula into the recurrence relation (2). Then

$$cr^n + ds^n = cr^{n-1} + ds^{n-1} + cr^{n-2} + ds^{n-2},$$

or,

$$r^{n-2}(r^2 - r - 1)c + s^{n-2}(s^2 - s - 1)d = 0.$$

This equation is satisfied if r and s are roots of the equation

$$p^2 - p - 1 = 0,$$

and for this choice (2) is satisfied whatever the values of c and d.

Now, choose c and d such that $F_1 = F_2 = 1$ in the formula (1); that is,

$$1 = cr + ds,$$

$$1 = cr^2 + ds^2.$$

This system will have a solution if $r \neq s$.

Let r be one of the roots of $p^2 - p - 1 = 0$ and s the other, so that $r \neq s$. Then, the system has the solution

$$c = \frac{s - 1}{r(s - r)}, \quad d = \frac{1 - r}{s(s - r)}.$$

For $r = (1 + 5^{1/2})/2$ and $s = (1 - 5^{1/2})/2$, a little manipulation yields

$$c = +\frac{1}{5^{1/2}}, \quad d = -\frac{1}{5^{1/2}}.$$

Hence, (1) becomes

$$F_n = \frac{1}{5^{1/2}}(r^n - s^n),$$

with $r = (1 + 5^{1/2})/2$ and $s = (1 - 5^{1/2})/2$, and the proof is complete.

The observation that $r = (1 + 5^{1/2})/2 = \emptyset$, the golden ratio, and $s = -1/\emptyset$ suggests the alternate form

$$F_n = \frac{\emptyset^n - (-\emptyset)^{-n}}{5^{1/2}}$$

for the formula of Theorem 4.1. The prominence of the golden ratio in this formula is perhaps not too surprising, in light of the discussion of Chapter II, but it is interesting, nevertheless.

The closed form expression for F_n given in the preceding theorem is quite useful in establishing identities for the Fibonacci numbers. For instance, the identity

$$F_{n+h}F_{h+k} - F_nF_{n+h+k} = (-1)^n F_h F_k$$

can be obtained by direct substitution from the formula in Theorem 4.1. and noting that $r^n s^n = (rs)^n = (-1)^n$. Incidentally, this identity is rather general and includes a number of others as special cases, some of which were proved in Chapter III. Examples would be the identities $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$, $F_{n-2}F_{n+2} - F_n^2 = (-1)^{n+1}$, and so on. In theory it should be possible to prove any identity for the Fibonacci numbers from the formula of Theorem 4.1., but to do so would often be tedious and inefficient.

It is worth noting that the Fibonacci numbers may be used as a tool for expanding

$$\left(\frac{1 + 5^{1/2}}{2}\right)^n,$$

which is laborious by the binomial theorem when n is not small. If the quadratic equation for which ϕ is a root is written

$$\phi^2 = \phi + 1,$$

and both members multiplied by ϕ, then

$$\begin{aligned} \phi^3 &= \phi^2 + \phi \\ &= (\phi + 1) + \phi \\ &= 2\phi + 1. \end{aligned}$$

Multiplying first and last members by ϕ again yields

$$\begin{aligned} \phi^4 &= 2\phi^2 + \phi \\ &= 2(\phi + 1) + \phi \\ &= 3\phi + 2. \end{aligned}$$

Similarly,

$$\begin{aligned} \phi^5 &= 3\phi^2 + 2\phi \\ &= 3(\phi + 1) + 2\phi \\ &= 5\phi + 3. \end{aligned}$$

This pattern suggests

$$\phi^n = F_n\phi + F_{n-1}, \quad n = 1, 2, 3, \ldots .$$

Proving the validity of this formula is accomplished by induction.

For $n = 1$,

$$F_1\phi + F_0 = 1(\phi) + 0 = \phi,$$

and it has already been verified above for $n = 2, 3, 4,$ and 5. For $n = k$, assume

$$\phi^k = F_k\phi + F_{k-1}.$$

Then, multiplying through by ϕ,

$$\begin{aligned}\phi^{k+1} &= F_k\phi^2 + F_{k-1}\phi \\ &= F_k(\phi + 1) + F_{k-1}\phi \\ &= (F_k + F_{k-1})\phi + F_k \\ &= F_{k+1}\phi + F_{(k+1)-1}.\end{aligned}$$

The computational advantage of this formula over a binomial expansion is striking.

Returning to the equation $\phi^2 - \phi - 1 = 0$, and arranging it as

$$\phi^{-1} = \phi - 1,$$

an efficient formula for calculating ϕ^{-n}, namely,

$$\phi^{-n} = (-1)^{n+1}(F_n\phi - F_{n+1}), \quad n = 1, 2, 3, \ldots ,$$

may be proved, and is left for the reader. Since,

$$((1 - 5^{1/2})/2)^n = (-2/(1 + 5^{1/2}))^n = (-1)^n\phi^{-n},$$

one then has a method of calculating $((1 - 5^{1/2})/2)^n$ which avoids tedious calculation by the binomial theorem. It might be conjectured that other expressions involving radicals could possibly be expanded in such a fashion. It is evident that different sequences would have to be used, or some more general type of recurring sequence introduced.

Quotients of Consecutive Fibonacci Numbers

Having Theorem 4.1. available makes it possible to prove the conjecture of Chapter II that ratios of consecutive Fibonacci numbers may be taken arbitrarily close to the golden ratio by choosing the terms sufficiently far out in the sequence. The following theorem establishes this fact.

Theorem 4.2. The Fibonacci numbers have the property

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi,$$

where $\phi = (1 + 5^{1/2})/2$, the golden ratio.

Proof: On application of the Binet formula to both numerator and denominator, it follows that

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \lim_{n \to \infty} \frac{5^{-1/2}(\phi^{n+1} - (-\phi)^{-n-1})}{5^{-1/2}(\phi^n - (-\phi)^{-n})}$$

$$= \lim_{n \to \infty} \left[\frac{\phi - \dfrac{1}{(-1)^{n+1}\phi^{2n+1}}}{1 - \dfrac{1}{(-1)^n \phi^{2n}}} \right]$$

$$= \phi,$$

since $\phi = (1 + 5^{1/2})/2 > 1$.

Theorem 4.2. was first proved by R. Simson using infinite continued fractions. For the purposes of this thesis it is preferable to deal with continued fractions very lightly, and only in order to point out the natural relationship between the quotients of consecutive Fibonacci numbers and the golden ratio ϕ. Dividing through the equation $x^2 - x - 1 = 0$ by x and rearranging gives

$$x = 1 + 1/x.$$

Using the right member as a formula for x and substituting into the denominator yields

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{x}}.$$

Continuing, one is led to consider the expression

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}},$$

which is an example of an infinite continued fraction. If the process is broken off and each fraction evaluated along the way, one obtains what are referred to as the convergents of the continued fraction. The convergents of this particular continued fraction are the quotients of consecutive Fibonacci numbers. Hence,

$$F_2/F_1 = 1 = 1,$$

$$F_3/F_2 = 2 = 1 + \frac{1}{1},$$

$$F_4/F_3 = \frac{3}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1}},$$

$$F_5/F_4 = \frac{5}{3} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}},$$

$$F_6/F_5 = \frac{8}{5} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}},$$

$$\vdots \qquad \vdots \qquad \vdots \quad .$$

In the theory of continued fractions, the value of the continued fraction is defined to be the limit of the sequence of convergents. Thus,

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}} .$$

A general property of convergent fractions may be noted by observing that the successive convergents bracket the limiting value of ϕ. That is, $F_2/F_1 = 1 < \phi$, $F_3/F_2 = 2 > \phi$, $F_4/F_5 = 1.5 < \phi$, and so on. This observation is apparent in Table V of Chapter II.

Another Non-recurrence Expression

The next theorem illustrates an unusual expression for computing Fibonacci numbers directly. In the quadratic equation $x^2 - x - 1 = 0$, the substitution $y = \frac{1}{x}$ yields $1 - y - y^2 = 0$. The latter equation has roots ϕ^{-1} and $-\phi$, whereas the roots of the former have been shown to be ϕ and $-\phi^{-1}$. Obviously, either equation is intimately related to the golden ratio and the Fibonacci numbers; however, for the purpose of obtaining the next result, the expression $1 - y - y^2$ is more tractable.

Theorem 4.3. The $(n + 1)^{\text{th}}$ Fibonacci number is related to the binomial coefficients by the expressions

$$F_{n+1} = C_0^n + C_1^{n-1} + C_2^{n-2} + \cdots + C_n^n, \text{ if } n \text{ is even,}$$

or

$$F_{n+1} = C_0^n + C_1^{n-1} + C_2^{n-2} + \cdots + C_n^{n+1}, \text{ if } n \text{ is odd,}$$

where

$$C_j^k = \frac{k!}{j!(k - j)!} .$$

Proof: By polynomial division,

$$\frac{1}{1 - y - y^2} = 1 + y + 2y^2 + 3y^3 + \dots + F_{n+1}y^n + \dots$$

$$= \sum_{n=0}^{\infty} F_{n+1}y^n.$$

which converges for $-\phi^{-1} < y < \phi^{-1}$ by the ratio test.

However,

$$\frac{1}{1 - y - y^2} = \frac{1}{1 - (y + y^2)} = \sum_{k=0}^{\infty} (y + y^2)^k, \quad \left|y + y^2\right| < 1,$$

from the formula for the sum of a geometric series. Hence, for y such that $-\phi^{-1} < y < \phi^{-1}$,

$$\sum_{n=0}^{\infty} F_{n+1}y^n = \frac{1}{1 - y - y^2}$$

$$= \sum_{k=0}^{\infty} (y + y^2)^k$$

$$= \sum_{k=0}^{\infty} y^k(1 + y)^k$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} C_j^k y^{k+j}$$

$$= \sum_{n=0}^{\infty} A_n y^n,$$

where

$$A_n = C_0^n + C_1^{n-1} + \dots + C_n^n, \text{ if } n \text{ even},$$

$$A_n = C_0^n + C_1^{n-1} + \dots + C_n^{n+1}, \text{ if } n \text{ odd}.$$

Equating coefficients of like powers of y yields $F_{n+1} = A_n$, which is the desired result.

Challenge: Verify through terms of 6th degree that

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} C_j^k y^{k+j} = \sum_{n=0}^{\infty} A_n y^n.$$

An Analytic Proof of the Formula for F_n

As a conclusion to this chapter, another proof of Theorem 4.1. is exhibited. This proof points out clearly how more sophisticated mathematical methods can be useful in proving even a simple theorem. By adding a touch of elegance it inspires interest. Furthermore, the proof, based on complex variable theory, demonstrates again the power of analytic techniques in the theory of numbers. For the reader unacquainted with complex analysis, the remainder of this chapter may be omitted without prejudice to later developments. The following proof is due to Hagis. [9]

As shown in the proof of Theorem 4.3., the generating function of the Fibonacci numbers is given by

$$f(z) = 1/(1 - z - z^2) = \sum_{n=0}^{\infty} F_{n+1} z^n.$$

If one considers z as a complex variable, then $f(z)$, being a rational function, is analytic except at those points where the denominator is zero. Hence, $f(z)$ has two singular points, namely, the simple poles $r = (-1 + 5^{1/2})/2$ and $s = (-1 - 5^{1/2})/2$. By Cauchy's integral theorem

$$F_{n+1} = f^{(n)}(0)/n! = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z^{n+1}},$$

where the contour C is the circle $|z| = \frac{1}{2}$. If Γ is any circle with center at the origin and radius greater than $|s| = \emptyset$, then by Cauchy's residue theorem

(1) $$F_{n+1} = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)dz}{z^{n+1}} - (R_r + R_s),$$

where R_r and R_s are the residues of $f(z)/z^{n+1}$ at the poles r and s, respectively. Now

$$R_r = \lim_{z \to r} (z - r)f(z)/z^{n+1} = \frac{1}{(s - r)r^{n+1}},$$

and

$$R_s = \lim_{z \to s} (z - s)f(z)/z^{n+1} = -\frac{1}{(s - r)s^{n+1}}.$$

Since $rs = -1$ and $r - s = 5^{\frac{1}{2}}$, it follows, after simplification, that

(2) $$-(R_r + R_s) = 5^{-\frac{1}{2}}\left\{\left(\frac{1 + 5^{\frac{1}{2}}}{2}\right)^{n+1} - \left(\frac{1 - 5^{\frac{1}{2}}}{2}\right)^{n+1}\right\}.$$

If Γ is the circle $|z| = k > \emptyset > 1$, then on Γ,

$$|f(z)| \leq \frac{1}{k^2 - k - 1}.$$

Hence,

(3) $$\left|\frac{1}{2\pi i}\int_\Gamma \frac{f(z)dz}{z^{n+1}}\right| \leq \frac{2\pi k}{2\pi k^{n+1}(k^2 - k - 1)}$$

$$\leq \frac{1}{k^n(k^2 - k - 1)}.$$

Since k may be taken arbitrarily large, it follows from (1), (2), and (3), that

$$F_{n+1} = 5^{-\frac{1}{2}}\left(\left(\frac{1 + 5^{\frac{1}{2}}}{2}\right)^{n+1} - \left(\frac{1 - 5^{\frac{1}{2}}}{2}\right)^{n+1}\right).$$

CHAPTER V

SOME GENERAL IDENTITIES

A General Summation Identity

In Chapter III, a few results concerning sums of Fibonacci numbers were obtained by induction in Theorem 3.1. The process of intuitive trials and proof by induction can be continued indefinitely in the attempt to obtain similar sum identities; however, more general results can sometimes be obtained through a broader attack. In particular, the following theorem includes all linear sums of Fibonacci numbers having subscripts in arithmetic progression.

<u>Theorem 5.1.</u> Let a, b, denote positive integers, b less than a. Then,

$$\sum_{k=1}^{n} F_{ak-b} = \frac{(-1)^a F_{an-b} - F_{a(n+1)-b} + (-1)^{a-b} F_b + F_{a-b}}{(-1)^a + 1 - (F_{a+1} + F_{a-1})} .$$

Proof: From Theorem 4.1.,

$$F_{ak-b} = 5^{-1/2}(r^{ak-b} - s^{ak-b}).$$

Hence,

$$\sum_{k=1}^{n} F_{ak-b} = \sum_{k=1}^{n} 5^{-1/2}(r^{ak-b} - s^{ak-b})$$

$$= 5^{-1/2} r^{a-b} \sum_{k=1}^{n} r^{a(k-1)} - 5^{-1/2} s^{a-b} \sum_{k=1}^{n} s^{a(k-1)}$$

$$= \frac{r^{a-b}(1 - r^{an})}{5^{1/2}(1 - r^a)} - \frac{s^{a-b}(1 - s^{an})}{5^{1/2}(1 - s^a)} ,$$

where the formula for the sum of a geometric series is used to obtain the last step. Simplifying, combining fractions, and grouping terms yields,

$$\sum_{k=1}^{n} F_{ak-b} = \frac{(rs)^a(r^{an-b}-s^{an-b})-(r^{an+a-b}-s^{an+a-b})+(rs)^{a-b}(r^b-s^b)+(r^{a-b}-s^{a-b})}{5^{1/2}(1 -(r^a + s^a) + (rs)^a)},$$

Using Theorem 4.1. and the relation $rs = - 1$, the right member may be further simplified to give

$$\sum_{k=1}^{n} F_{ak-b} = \frac{(-1)^a F_{an-b} - F_{a(n+1)-b} + (-1)^{a-b} F_b + F_{a-b}}{1 - (F_{a+1} + F_{a-1}) + (-1)^a} ,$$

and the proof is complete.

Challenge: Use Theorem 4.1. to show that

$$r^a + s^a = F_{a+1} + F_{a-1}.$$

Challenge: Obtain the sum formula of Theorem 3.1. part (3) as a special case of Theorem 5.1.

Challenge: With the aid of Table II of Chapter II, show that

$$\sum_{k=1}^{5} F_{7k-3} = 2256010.$$

A Matrix Approach

A number of Fibonacci identities can be obtained through matrix algebra and a certain 2 x 2 matrix. The necessary definitions and properties from matrix theory are reviewed as a preliminary development.

The 2 x 2 matrix A is an array of four numbers a, b, c, d, such that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

The zero matrix Z is obtained when $a = b = c = d = 0$. The identity matrix is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

The determinant of matrix A is

$$D(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The multiplication of A by a number q is defined by

$$qA = q\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} qa & qb \\ qc & qd \end{pmatrix} .$$

For any two matrices A, B with entries a, b, c, d and e, f, g, h respectively, the sum $A + B$ is defined by

$$A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} ,$$

and the product AB by

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} .$$

Also, $A = B$ if, and only if, $a = e$, $b = f$, $c = g$, and $d = h$. It is a simple exercise in algebra to prove that $D(AB) = D(A)D(B)$.

It is now possible to develop an effective instrument for proving a number of Fibonacci identities. This is accomplished through the next definition.

<u>Definition 5.1</u>. Let $Q = Q^1$ be the matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} .$$

Let $Q^0 = I$. Then $Q = Q^1 = Q^0Q = QQ^0$. In general, $Q^{n+1} = Q^nQ^1$ defines exponentiation inductively. Note that $D(Q) = -1$. Also,

$$Q^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} ,$$

$$Q^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_3 & F_2 \\ F_2 & F_1 \end{pmatrix} ,$$

$$Q^3 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} F_4 & F_3 \\ F_3 & F_2 \end{pmatrix} ,$$

$$Q^4 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} F_5 & F_4 \\ F_4 & F_3 \end{pmatrix} ,$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$Q^n = \begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_n + F_{n-1} & F_n \\ F_{n-1} + F_{n-2} & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} .$$

These statements include the essential steps for an inductive proof of the next important theorem.

<u>Theorem 5.2.</u>

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} .$$

Another inductive argument can be used to prove the next theorem, which is left to the reader.

<u>Theorem 5.3.</u> $D(Q^n) = (D(Q))^n = (-1)^n$.

The setting is now complete and several identities can be conveniently proved. Some of these identities were proved in Chapter III.

<u>Theorem 5.4.</u> The following identities hold for Fibonacci numbers:

(1) $$F_{n+1}F_{n-1} - F_n^{\,2} = (-1)^n,$$

(2) $$F_{2n+2} = F_{n+1}F_{n+2} + F_nF_{n+1},$$

(3) $$F_{2n+1} = F_{n+1}^{\,2} + F_n^{\,2},$$

(4) $$F_{2n+1} = F_nF_{n+2} + F_{n-1}F_{n+1},$$

(5) $$F_{2n} = F_n(F_{n+1} + F_{n-1}).$$

Proof: From the definition of determinant of a matrix,

$$D(Q^n) = \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = F_{n+1}F_{n-1} - F_n^{\,2}.$$

But, $D(Q^n) = (-1)^n$ by Theorem 5.3., hence (1) is proved. Identities (2), (3), (4), and (5) are all proved at once. Since

$$\begin{pmatrix} F_{2n+2} & F_{2n+1} \\ F_{2n+1} & F_{2n} \end{pmatrix} = Q^{2n+1} = Q^nQ^{n+1}$$

$$= \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix}$$

$$= \begin{pmatrix} F_{n+1}F_{n+2} + F_nF_{n+1} & F_{n+1}^{\,2} + F_n^{\,2} \\ F_nF_{n+2} + F_{n-1}F_{n+1} & F_nF_{n+1} + F_{n-1}F_n \end{pmatrix}.$$

Equating corresponding elements in accordance with the definition of equality for matrices yields the remaining identities of the theorem.

<u>Challenge</u>: Verify that $Q^2 - Q - I = Z$.

<u>Challenge</u>: Prove the identity $F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$ using the matrix Q.

Another General Identity

In the previous sections, indication was given that a more general approach in developing and proving identities could prove fruitful. This section is devoted to the development of one very general identity that includes dozens of others as special cases, including many of those discussed thus far in this thesis. It is well-known that mathematicians are marked by their desire to generalize. With regard to the reader for whom this work is intended, few more suitable examples could be given to demonstrate the importance of that characteristic.

The following theorem and definition are important preliminaries.

<u>Theorem 5.5.</u> The Binet formula of Theorem 4.1.,

$$F_n = 5^{-1/2}(r^n - s^n), \quad r = \emptyset, \quad s = -\emptyset,$$

is unique for the Fibonacci sequence.

Proof: The Fibonacci sequence is defined by

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1.$$

Suppose a sequence $\{F_n^*\}$ were to satisfy this definition. Then, the sequence $\{f_n\}$ such that $f_n = F_n - F_n^*$ has the property that $f_0 = f_1 = 0$, and $f_{n+2} = f_{n+1} + f_n$. Hence, it follows that $f_n = 0$, for every n, and therefore, $F_n = F_n^*$, for every n.

It proves convenient in what follows to define Fibonacci numbers with negative subscripts. The definition is prompted by considering the Binet formula for negative values of n. With this definition, the Fibonacci numbers are defined for any integer subscript.

<u>Definition 5.2.</u> For any integer n, $F_{-n} = (-1)^{n+1}F_n$.

Consider the function of n,

(1) $$S_0(n) = F_n + F_{n+1} - F_{n+2}.$$

It is immediate from the recurrence formula $F_{n+2} = F_{n+1} + F_n$ that

$S_0(n) = 0$ for all n. Next consider the function of m and n,

(2) $$S_1(m,n) = F_mF_n + F_{m+1}F_{n+1} - F_{m+n+1}.$$

Again, after appropriate use of the recurrence formula, it follows that, for any m and n,

(3) $$S_1(m+1,n) = S_1(m,n) + S_1(m-1,n).$$

Furthermore,

(4) $$S_1(0,n) = F_{n+1} - F_{n+1} = 0 \text{ and } S_1(1,n) = S_0(n) = 0.$$

Hence, for any integer n, the relations (3) and (4) infer by upward and downward induction on m that $S_1(m,n) = 0$ for all m. Therefore, $S_1(m,n) = 0$ for all m and n.

Next, consider

(5) $$S_2(t,m,n) = F_mF_n - (-1)^t(F_{m+t}F_{n+t} - F_tF_{m+n+t}).$$

On making the substitutions $F_n = F_{n+2} - F_{n+1}$, $F_{m+t+1} = F_{m+t} + F_{m+t-1}$, and $F_{t+1} = F_t + F_{t-1}$, manipulating, and taking a judicious arrangement of subscripts, one has

(6) $$S_2(t+1,m,n) = S_2(t-1,m,n+2) - S_2(t,m,n+1).$$

Since $F_0 = 0$, $F_1 = 1$, it follows from (5) and (2) that

(7) $$S_2(0,m,n) = 0 \text{ and } S_2(1,m,n) = S_1(m,n) = 0.$$

Again, using induction on t in (6), it follows that for all integers t, m, and n, $S_2(t,m,n) = 0$. Hence, this establishes the general identity

(8) $$F_mF_n = (-1)^t(F_{m+t}F_{n+t} - F_tF_{m+n+t}).$$

This identity includes several of those proved previously. For instance, when $t = 1$, (8) becomes the identity of Corollary 3.5. However, (8) is not the general identity promised. A relation even more general than (8) may yet be obtained.

For integers k and t such that $k \geq 0$, $t \neq 0$, consider the function

$$(9) \quad S_3(k,t,m,n) = F_m{}^k F_n - (-1)^{kt} \sum_{h=0}^{k} C_h^k (-1)^h F_t{}^h F_{m+t}{}^{k-h} F_{n+kt+hm}.$$

The recurrence relation $S_3(k+1,t,m,n) = F_m S_3(k,t,m,n)$ can be developed from (8), as follows:

$$(10) \quad S_3(k+1,t,m,n) = F_m{}^{k+1} F_n - (-1)^{(k+1)t} \sum_{h=0}^{k+1} C_h^{k+1} (-1)^h F_t{}^h F_{m+t}{}^{k+1-h} F_{n+(k+1)t+hm}.$$

Consider the second term of the right member.

$$(-1)^{(k+1)t} \sum_{h=0}^{k+1} C_h^{k+1} (-1)^h F_t{}^h F_{m+t}{}^{k+1-h} F_{n+(k+1)t+hm}$$

$$= (-1)^{kt}(-1)^t \left\{ \sum_{h=0}^{k} C_h^k (-1)^h F_t{}^h F_{m+t}{}^{k+1-h} F_{n+kt+hm+t} + \sum_{h=1}^{k+1} C_{h-1}^k (-1)^h F_t{}^h F_{m+t}{}^{k+1-h} F_{n+kt+hm+t} \right\}$$

$$= (-1)^{kt}(-1)^t \left\{ \sum_{h=0}^{k} C_h^k (-1)^h F_t{}^h F_{m+t}{}^{k+1-h} F_{n+kt+hm+t} + \sum_{h=0}^{k} C_h^k (-1)^{h+1} F_t{}^{h+1} F_{m+t}{}^{k-h} F_{n+kt+hm+m+t} \right\}$$

$$= (-1)^{kt} \sum_{h=0}^{k} C_h^k (-1)^h F_t{}^h F_{m+t}{}^{k-h} \left[(-1)^t (F_{m+t} F_{(n+kt+hm)+t} - F_t F_{m+(n+kt+hm)+t}) \right]$$

$$= (-1)^{kt} \sum_{h=0}^{k} C_h^k (-1)^h F_t{}^h F_{m+t}{}^{k-h} F_{n+kt+hm} F_m,$$

where use has been made of properties of the binomial coefficient, an adjustment of the dummy index h, and (8). Hence, factoring F_m from the right side of (10),

(11) $$S_3(k+1,t,m,n) = F_m S_3(k,t,m,n).$$

Also, from (9), $S_3(0,t,m,n) = F_n - F_n = 0$, and $S_3(1,t,m,n) = S_2(t,m,n) = 0$. Hence, by induction on k, one has, for all integers $t \neq 0$, m, n, and all integers $k \geq 0$, $S_3(k,t,m,n) = 0$. The foregoing proves the following theorem.

Theorem 5.6. For all integers $t \neq 0$, m, n, and all integers $k \geq 0$,

$$F_m{}^k F_n = (-1)^{kt} \sum_{h=0}^{k} C_h^k (-1)^h F_t{}^h F_{m+t}{}^{k-h} F_{n+kt+hm}.$$

With the occasional aid of definition 5.2. and various choices of k, t, m, n, it is possible to obtain dozens of identities as special cases of Theorem 5.6. Note that for $k = 1$, and any t, m, n, the identity (8) is obtained. Therefore, Theorem 5.6. gives all the identities that could be obtained as special cases of identity (8). Also, using the four-tuple (k,t,m,n) to identify the assigned values of k, t, m, and n,

(12) $$(1,t,a,a-t): \quad F_t F_{2a} = F_{t+a} F_a - (-1)^t F_a F_{a-t},$$

and

(13) $$(1,1,a,a-1): \quad F_{2a} = F_a (F_{a+1} + F_{a-1}).$$

Substituting from (13) into (12), dividing out F_a, and using definition 5.2.,

$$(F_{a+1} + F_{a-1})F_t = F_{t+a} - (-1)^t F_{a-t} = F_{t+a} + (-1)^a F_{t-a}.$$

Subtracting $F_t + (-1)^a F_t$ from both sides, this becomes

$$(F_{a+1} + F_{a-1} - 1 - (-1)^a)F_t = (F_{t+a} - F_t) + (-1)^a(F_{t-a} - F_t).$$

Set $t = ah-b$, and sum from $h = 1$ to $h = n$. The right member telescopes to yield

$$(F_{a+1}+F_{a-1}-1-(-1)^a) \sum_{h=1}^{n} F_{ah-b} = F_{a(n+1)\ -b} - F_{a-b} + (-1)^a(F_{-b} - F_{an-b}).$$

Since $(-1)^a F_{-b} = (-1)^a(-1)^{b+1}F_b = -(-1)^a(-1)^{-b}F_b = -(-1)^{a-b}F_{-b}$ in the third term of the right member, division by the coefficient of the sum $\sum_{h=1}^{n} F_{ah-b}$ yields

$$(14) \qquad \sum_{h=1}^{n} F_{ah-b} = \frac{(-1)^a F_{an-b} - F_{a(n+1)-b} + (-1)^{a-b}F_b + F_{a-b}}{1 + (-1)^a - (F_{a+1} + F_{a-1})},$$

which is the identity of Theorem 5.1. Hence, all the special identities from Theorem 5.1. can be included under Theorem 5.6.

For $a = 2$, $b = 1$, and $n = t$, in (14),

$$\sum_{h=1}^{t} F_{2h-1} = - F_{2t-1} + F_{2t+1} = F_{2t}.$$

For $k = 1$, $t = -1$, and $n = m$, in Theorem 5.6.,

$$F_{2m-1} = F_m^{\ 2} + F_{m-1}^{\ 2}.$$

Combining these two identities,

$$F_{2t} = \sum_{m=1}^{t} F_{2m-1} = \sum_{m=1}^{t} F_m^{\ 2} + \sum_{m=1}^{t-1} F_m^{\ 2}.$$

Hence, using (13),

$$2\sum_{m=1}^{t} F_m^2 = F_{2t} + F_t^2 = F_t(F_{t+1} + F_{t-1}) + F_t^2$$

$$= F_t(F_{t+1} + F_{t-1} + F_t)$$

$$= 2F_tF_{t+1}.$$

Cancelling the 2 yields the identity for the sum of the first t squares of Theorem 3.2.

As a final illustration of the generality of Theorem 5.6., consider (14) for $a = 1$, $b = -s$, and $n = t$. Hence,

$$\sum_{h=1}^{t} F_{h+s} = F_{t+s+2} - F_{s+2}.$$

Sum both members of this equation, with $t = w-s$, from $s = 0$ to $s = w-1$. Thus, with u replacing $h+s$,

$$\sum_{s=0}^{w-1} \sum_{u=s+1}^{w} F_u = \sum_{s=0}^{w-1} (F_{w+2} - F_{s+2}).$$

Since changing the order of summation yields the same result,

$$\sum_{u=1}^{w} \sum_{s=0}^{u-1} F_u = \sum_{u=1}^{w} uF_u = \sum_{s=0}^{w-1} (F_{w+2} - F_{s+2}).$$

Hence, using Theorem 3.1., part (1),

$$\sum_{u=1}^{w} uF_u = wF_{w+2} - \sum_{s=0}^{w-1} F_{s+2}$$

$$= wF_{w+2} - F_{w+3} + 2$$

$$= (w+1)F_{w+2} - F_{w+4} + 2.$$

This is the identity of Theorem 3.4., part (1).

Obviously, the identity of Theorem 5.6. is very general indeed, and includes a large body of identities as special cases. Identities may, of course, be obtained directly from Theorem 5.6. as well as through procedures similar to those used above. For an extended listing of identities which arise directly from Theorem 5.6., consult Halton. [10] A brief listing follows:

$$(1,1,m,n-1):\quad F_{m+n} = F_{m+1}F_n + F_mF_{n-1},$$

$$(1,2,m-1,m-1):\quad F_{2m} = F_{m+1}^{\,2} - F_{m-1}^{\,2},$$

$$(1,1,m,-m):\quad F_m^{\,2} - F_{m+1}F_{m-1} = (-1)^{m-1},$$

$$(k,1,1,-nk):\quad F_{nk} = \sum_{h=0}^{k} C_h^k F_{(n-1)k-h},$$

$$(k,t,m,0):\quad \sum_{h=0}^{k} C_h^k (-1)^h F_t^{\,h} F_{t+m}^{\,k-h} F_{kt+hm} = 0.$$

Identities by a Finite Difference Technique

The final section of this chapter is devoted to a specialized method of obtaining identities involving sums of Fibonacci numbers, and is based on the calculus of finite differences. In order to facilitate the presentation, a brief review of some pertinent facts from finite differences is given. The reader who is interested in a more complete treatment of finite differences may refer to any standard textbook on finite differences, such as Miller. [17]

The ordinary derivative of elementary calculus is, for a function f, defined on an interval a,b , is

$$Df(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The symbol D is sometimes referred to as the differentiation operator. In the calculus of finite differences, the difference quotient $(f(x+h) - f(x))/h$ is considered as in ordinary calculus; however, h is held fixed and limits are not taken. It is customary, in fact, to let $h=1$, and use the difference operator Δ to indicate the finite difference

$$\Delta f(x) = f(x+1) - f(x).$$

The parallel between Δ and D is striking. For instance, the difference formulas (c = constant)

$$\Delta c = 0,$$

$$\Delta cf(x) = c\Delta f(x),$$

$$\Delta(f(x) + g(x)) = \Delta f(x) + \Delta g(x),$$

$$\Delta(f(x)g(x)) = g(x)\Delta f(x) + f(x+1)\Delta g(x),$$

$$\Delta\frac{f(x)}{g(x)} = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+1)}, \quad g(x)g(x+1) \neq 0,$$

all have their obvious counterpart in differential calculus. The third of these formulas may also be written

(1) $$\Delta(f(x)g(x)) = f(x)\Delta g(x) + g(x+1)\Delta f(x).$$

The second difference is $\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta(f(x+1) - f(x)) = f(x+2) - f(x+1) - (f(x+1) - f(x)) = f(x+2) - 2f(x+1) + f(x)$; and recursively, one has $\Delta^n f(x) = \Delta(\Delta^{n-1} f(x))$.

Just as one may consider the anti-derivative D^{-1} in calculus, so also the anti-difference Δ^{-1} has an analogous interpretation in finite differences. That is, given a function F defined for all x, the difference $\Delta F(x) = f(x)$ invites consideration of the converse;

namely, given f, can F be found such that $\Delta F(x) = f(x)$? If so, one writes $F(x) = \Delta^{-1}f(x)$, or $F(x) = \sum f(x)$, where the sigma sign stands for indefinite summation, analogous to the indefinite integral. Hence, Δ and $\sum$ are inverse difference operators, just as D and $\int$ are inverse operators in ordinary calculus.

If there exist two functions F and G such that $F(x) = \sum f(x)$ and $G(x) = \sum f(x)$, then

$$\Delta(F(x) - G(x)) = \Delta F(x) - \Delta G(x) = f(x) - f(x) = 0.$$

Let $F(x) - G(x) = P(x)$. Hence, $\Delta P(x) = 0$ implies $P(x+1) = P(x)$. A function enjoying this property is called a periodic constant and plays the same role in the theory of summation as the constant of integration plays in the theory of integration. General formulas analogous to those of integral calculus occur:

$$\sum cf(x) = c\sum f(x),$$

$$\sum(f(x) + g(x)) = \sum f(x) + \sum g(x),$$

$$\sum f(x)\Delta g(x) = f(x)g(x) - \sum g(x+1)\Delta f(x).$$

This last formula is referred to as summation by parts and is derived from difference formula (1) for products. Summation by parts is useful in proving certain Fibonacci sum identities, however, it is first necessary to discuss definite summation.

The fundamental theorem of integral calculus states that if f is continuous on $[a,b]$, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = D^{-1}f(x)\Big|_a^b,$$

where $F'(x) = f(x)$. For the parallel formula for the sum, $\sum_{x=a}^{a+n} f(x)$, let $F(x) = \sum f(x)$. Then $f(x) = \Delta F(x) = F(x+1) - F(x)$, and hence,

$$\begin{aligned} F(a+1) - F(a) &= f(a), \\ F(a+2) - F(a+1) &= f(a+1), \\ &\vdots \\ F(a+n) - F(a+n-1) &= f(a+n-1), \\ F(a+n+1) - F(a+n) &= f(a+n). \end{aligned}$$

Adding left and right members in these equations yields

$$\sum_{x=a}^{a+n} f(x) = F(a+n+1) - F(a) = \sum f(x)\Big|_a^{a+n+1} = \Delta^{-1} f(x)\Big|_a^{a+n+1} .$$

Periodic constants may be ignored here in the same manner as constants of integration with respect to definite integration.

With these tools at hand, it is possible to prove additional Fibonacci identities. The x's above are replaced by k's to emphasize that the domain of the functions considered is the set of positive integers. The key formula for this effort is the summation by parts expression for definite sums

$$(2) \qquad \sum_{k=a}^{a+n} f(k)\Delta g(k) = f(k)g(k)\Big|_a^{a+n+1} - \sum_{k=a}^{a+n} g(k+1)\Delta f(k).$$

Success in using summation by parts is contingent on a judicious choice for $f(k)$ and $\Delta g(k)$, just as is the case in integration by parts. For proving Fibonacci identities this means having at hand some already established identities. A few examples are given as theorems for illustration.

Theorem 5.7.

$$\sum_{k=0}^{n} kF_k = (n+1)F_{n+2} - F_{n+4} + 2.$$

Proof: In formula (2) let $a=0$, $f(k) = k$, and $\Delta g(k) = F_k$.

Then, $\Delta f(k) = (k+1) - k = 1$, and

$$g(k) = \sum_{j=0}^{k-1} F_j = F_{k+1} - 1,$$

by Theorem 3.1., part (1). Therefore,

$$\sum_{k=0}^{n} kF_k = k(F_{k+1} - 1)\Big|_0^{n+1} - \sum_{k=0}^{n} (1)(F_{k+2} - 1)$$

$$= (n+1)F_{(n+1)+1} - (n+1) - \sum_{k=0}^{n} F_{k+2} + (n+1)$$

$$= (n+1)F_{n+2} - (F_1 + \sum_{k=0}^{n} F_{k+2}) + F_1$$

$$= (n+1)F_{n+2} - (F_{n+4} - 1) + F_1$$

$$= (n+1)F_{n+2} - F_{n+4} + 2,$$

where Theorem 3.1., part (1), has been used again.

Theorem 5.8.

$$\sum_{k=0}^{n} kF_{2k} = (n+1)F_{2n+1} - F_{2n+2}.$$

Proof: In formula (2), let $a = 0$, $f(k) = k$, $\Delta g(k) = F_{2k}$.

Then, $\Delta f(k) = 1$, and

$$g(k) = \sum_{j=0}^{k-1} F_{2j} = F_{2k-1} - 1,$$

by Theorem 3.1., part (3). Hence,

$$\sum_{k=0}^{n} kF_{2k} = k(F_{2k-1} - 1)\Big|_{0}^{n+1} - \sum_{k=0}^{n}(F_{2(k+1)-1} - 1)$$

$$= (n+1)(F_{2n+1} - 1) - \sum_{k=0}^{n} F_{2k+1} + (n+1)$$

$$= (n+1)F_{2n+1} - F_{2n+2},$$

using Theorem 3.1., part (2).

Theorem 5.9.

$$\sum_{k=0}^{n} k^2 F_k = (n^2+2)F_{n+2} - (2n-3)F_{n+3} - 8.$$

Proof: In formula (2), let $a = 0$, $f(k) = k^2$, $\Delta g(k) = F_k$.

Then, $\Delta f(k) = (k+1)^2 - k^2 = 2k+1$ and

$$g(k) = \sum_{j=0}^{k-1} F_j = F_{k+1} - 1,$$

by Theorem 3.1., part (1). Discarding the -1,

$$\sum_{k=0}^{n} k^2 F_k = k^2 F_{k+1}\Big|_{0}^{n+1} - \sum_{k=0}^{n}(2k+1)F_{k+2}.$$

$$= (n+1)^2 F_{n+2} - \sum_{k=0}^{n}(2k+1)F_{k+2}.$$

Summation by parts is required again on the last term. Let $f(k) = 2k+1$, $\Delta g(k) = F_{k+2}$. Then, $\Delta f(k) = 2$ and

$$g(k) = \sum_{j=0}^{k-1} F_{j+2} = F_{k+3} - 2.$$

Ignoring the constant -2,

$$\sum_{k=0}^{n} k^2F_k = (n+1)^2F_{n+2} - \left\{(2k+1)F_{k+3}\Big|_0^{n+1} - \sum_{k=0}^{n}(2)F_{k+4}\right.$$

$$= (n+1)^2F_{n+2} - (2n+3)F_{n+4} + F_3 + 2F_{n+6} - 2(1+F_1+F_2+F_3)$$

$$= (n^2+2)F_{n+2} - (2n-3)F_{n+3} - 8.$$

Theorem 5.10.

$$\sum_{k=0}^{n}\sum_{i=0}^{k}F_i = F_{n+4} - (n+3).$$

Proof: In formula (2), let

$$f(k) = \sum_{i=0}^{k}F_i = F_{k+2} - 1 \quad \text{and} \quad \Delta g(k) = 1.$$

Then, $\Delta f(k) = F_{k+3} - F_{k+2} = F_{k+1}$ and $g(k) = k$. Hence,

$$\sum_{k=0}^{n}\sum_{i=0}^{k}F_i = k(F_{k+2} - 1)\Big|_0^{n+1} - \sum_{k=0}^{n}(k+1)F_{k+1}$$

$$= (n+1)(F_{n+3} - 1) - \sum_{k=0}^{n}(k+1)F_{k+1}.$$

Now apply summation by parts to the last term, letting $f(k) = k+1$, and $\Delta g(k) = F_{k+1}$. Then, $\Delta f(k) = 1$ and

$$g(k) = \sum_{j=0}^{k-1}F_{j+1} = F_{k+2} - 1.$$

Therefore, ignoring the constant -1,

$$\sum_{k=0}^{n}\sum_{i=0}^{k}F_i = (n+1)(F_{n+3} - 1) - \left\{(k+1)F_{k+2}\Big|_0^{n+1} - \sum_{k=0}^{n}F_{k+3}\right\}$$

$$= (n+1)(F_{n+3} - 1) - (n+2)F_{n+3} + F_2 + F_{n+5} - 3$$

$$= F_{n+4} - (n+3).$$

Theorem 5.11.

$$\sum_{k=0}^{n} F_k^{\,2} = F_n F_{n+1}.$$

Proof: Consider $\sum_{k=0}^{n-1} F_k^{\,2}$. In formula (2), let $f(k) = F_k$ and

$\Delta g(k) = F_k$. Then, $\Delta f(k) = F_{k+1} - F_k$ and $g(k) = \sum_{j=0}^{k-1} F_j = F_{k+1} - 1$.

Then,

$$\sum_{k=0}^{n-1} F_k^{\,2} = F_k F_{k+1}\Big|_0^{(n-1)+1} - \sum_{k=0}^{n-1} (F_{k+1} - F_k)(F_{k+2})$$

$$= F_n F_{n+1} - \sum_{k=0}^{n-1} (F_{k+1} - F_k)(F_{k+1} + F_k)$$

$$= F_n F_{n+1} - \sum_{k=0}^{n-1} F_{k+1}^{\,2} + \sum_{k=0}^{n-1} F_k^{\,2}.$$

Cancelling $\sum_{k=0}^{n-1} F_k^{\,2}$ from both members yields,

$$F_n F_{n+1} = \sum_{k=0}^{n-1} F_{k+1}^{\,2} = \sum_{k=0}^{n} F_k^{\,2}.$$

Any number of identities may be derived by following and extending the methods used above. A few additional ones are listed in the challenges below.

<u>Challenge</u>: With the help of Theorem 3.1., parts (2) and (3), show that

$$\sum_{k=0}^{n} kF_{2k+1} = (n+1)F_{2n+2} - F_{2n+3} + 1,$$

using summation by parts.

<u>Challenge</u>: Use summation by parts to show:

(1) $$\sum_{k=0}^{n} k^2F_{2k} = (n^2+2)F_{2n+1} - (2n+1)F_{2n} - 2,$$

(2) $$\sum_{k=0}^{n} F_k{}^2F_{k+1} = F_nF_{n+1}F_{n+2}/2.$$

CHAPTER VI

A GENERALIZED FIBONACCI SEQUENCE

The recurrence formula for the Fibonacci sequence,

$$F_{n+2} - F_{n+1} - F_n = 0,$$

is an example of a linear difference equation of order two. Indeed, the natural setting for recurring sequences in general, of which the Fibonacci sequence is one special case, is within the framework of finite difference equations. Difference equations are analogous in many respects to differential equations, continuing many of the parallels observed in the last section of Chapter V. The study of recurring sequences is, in reality, included within the subject of difference equations. In order to discuss generalizations of the Fibonacci sequence, it would seem, therefore, that a general study of difference equations should be made. However, such a completely general approach is beyond the scope of this thesis, and, therefore, the generalizations dealt with here are restricted. Attention is focused on homogeneous linear difference equations, primarily those of order two. The necessary definitions and theorems pertinent to this discourse are stated without proof, though many times the proofs are simpler than the analogous result for ordinary differential equations. To find proofs or further details, the reader may consult Miller. [17]

<u>Definition 6.1</u>. Let $p_0(x)$, $p_1(x)$, ..., $p_n(x)$, and $r(x)$ be defined

for all x in J^*, the set of non-negative integers, with the property that $p_0(x) \cdot p_n(x) \neq 0$, for all x in J^*. Then

$$(1) \qquad p_0(x)y(x+n) + p_1(x)y(x+n-1) + \ldots + p_n(x)y(x) = r(x)$$

is a linear difference equation of order n. A function g not identically zero that satisfies (1) over J^* is said to be a solution of (1).

The existence and uniqueness of solutions are assumed without further discussion. Equation (1) is called homogeneous if $r(x)$ is identically zero. It is clear that if $g_1(x)$ and $g_2(x)$ are solutions of (1), then $P_1(x)g_1(x) + P_2(x)g_2(x)$ is also a solution, where $P_1(x)$ and $P_2(x)$ are periodic constants. For the purpose at hand, the periodic constants will be treated simply as ordinary constants. Within this context, the concept of linear independence of a set of functions is now defined.

Definition 6.2. Let $g_1(x), g_2(x), \ldots, g_n(x)$ be n functions defined on the set J^* of non-negative integers. Then $g_1(x), g_2(x), \ldots, g_n(x)$ are linearly independent if, for constants $c_1, c_2, \ldots, c_n$,

$$c_1g_1(x) + c_2g_2(x) + \ldots + c_ng_n(x) = 0$$

for all x in J^*, implies $c_1 = c_2 = \ldots = c_n = 0$. In the contrary case, the functions are said to be linearly dependent.

Theorem 6.A. Consider the homogeneous linear difference equation

$$(2) \qquad p_0(x)y(x+n) + p_1(x)y(x+n-1) + \ldots + p_n(x)y(x) = 0,$$

where the coefficients $p_0(x), p_1(x), \ldots, p_n(x)$ are defined on J^*, and $p_0(x) \cdot p_n(x) \neq 0$, for all x in J^*. Then there exist n linearly independent solutions of (2). Furthermore, any n solutions, $g_1(x)$, $g_2(x)$, ..., $g_n(x)$, of (2) are linearly independent if, and only if,

$$\begin{vmatrix} g_1(x) & g_2(x) & \cdots & g_n(x) \\ g_1(x+1) & g_2(x+1) & \cdots & g_n(x+1) \\ \vdots & \vdots & \vdots & \vdots \\ g_1(x+n-1) & g_2(x+n-1) & \cdots & g_n(x+n-1) \end{vmatrix} \neq 0,$$

for all values of x.

The determinant in the preceding theorem is called the Casorati of the n solutions $g_1(x)$, $g_2(x)$, ..., $g_n(x)$. It plays a role in difference equations similar to the Wronskian of differential equations.

Theorem 6.B. Let $g_1(x)$, $g_2(x)$, ..., $g_n(x)$ be a linearly independent set of solutions of (2). Let $G(x)$ be a solution of (2). Then there exist constants c_1, c_2, ..., c_n such that

$$G(x) = c_1 g_1(x) + c_2 g_2(x) + \cdots + c_n g_n(x).$$

This theorem states that a linearly independent set of n solutions of an n^{th} order homogeneous linear difference equation provides the most general solution to that difference equation in the sense that any solution may be obtained as a linear combination of the n solutions.

It is now desirable to return to the generalization of the Fibonacci sequence. There are a number of ways in which this may be done. Consider the recurrence formula

(3) $$p_0 Y_{x+n} = -p_1 Y_{x+n-1} - p_2 Y_{x+n-2} - \cdots - p_n Y_x,$$

where p_0, p_1, p_2, ..., p_n are constants, $p_0 p_n \neq 0$, and $x \geq 1$ in J^*.

Converting to function notation, (3) becomes

(4) $$p_0y(x+n) + p_1y(x+n-1) + \dots + p_ny(x) = 0.$$

Clearly, (4) is an n^{th} order homogeneous linear difference equation with constant coefficients. Then in accordance with Theorem 6.B., any solution of (4) can be expressed, for suitable constants $c_1, c_2, \dots, c_n$, as

(5) $$y(x) = c_1g_1(x) + c_2g_2(x) + \dots + c_ng_n(x),$$

where $g_1(x), g_2(x), \dots, g_n(x)$ is a set of n linearly independent solutions of (4). To determine a particular solution would require a choice of n initial or boundary conditions, just as in the case for ordinary differential equations. Stated in terms of recurring sequences and following the notation in (3), this means the first n initial values would need to be prescribed, and then a particular recurring sequence would be obtained from (3). Imposing these initial conditions on (5) would produce a generalized Binet formula, a generalization of Theorem 4.1. for the Fibonacci sequence. It should be noted that additional generality could be obtained if (4) were not homogeneous. However, it is not the intention to give a full discussion of recurring sequences here. In fact, (4) will be restricted to the case $n = 2$ in what follows.

Suppose (3) is considered subject to the restriction $n = 2$. Hence, (3) becomes

(6) $$p_0Y_{x+2} = - p_1Y_{x+1} - p_2Y_x,$$

which leads to the difference equation

(7) $$p_0y(x+2) + p_1y(x+1) + p_2y(x) = 0.$$

In order to find the general solution of this equation it is necessary to find two linearly independent solutions. By analogy to the situation for ordinary differential equations, one is led to assume a solution of the form $g(x) = m^x$. Then, (7) becomes

$$p_0 m^{x+2} + p_1 m^{x+1} + p_2 m^x = 0,$$

or

$$m^x(p_0 m^2 + p_1 m + p_2) = 0.$$

If $g(x) = m^x$ is to be a solution of (7), then m must be a root of the algebraic equation $p_0 m^2 + p_1 m + p_2 = 0$. This leads to three possibilities, namely, real and distinct roots, imaginary roots, and real and equal roots. All three cases may be handled in a manner similar to the situation in ordinary differential equations. For this occasion, it is preferable to restrict attention to the case where $p_0 = 1$, $p_1 = -1$, $p_2 = -1$. Then, the recurrence relation (6) is

(8) $$Y_{x+2} = Y_{x+1} + Y_x,$$

and the difference equation (7) is

(9) $$y(x+2) - y(x+1) - y(x) = 0.$$

The roots of $m^2 - m - 1 = 0$ are $m_1 = (1+5^{1/2})/2 = r$ and $m_2 = (1-5^{1/2})/2 = s$, which are real and distinct. Then, $g_1(x) = r^x$ and $g_2(x) = s^x$ are solutions of the difference equation (9). Since the Casorati of r^x and s^x, is not zero, that is

$$\begin{vmatrix} r^x & s^x \\ r^{x+1} & s^{x+1} \end{vmatrix} = (rs)^x(s-r) \neq 0,$$

it follows that $g_1(x) = r^x$ and $g_2(x) = s^x$ are two linearly independent solutions of (9). Hence, the general solution of (9) is $y(x) = c_1 r^x + c_2 s^x$. In terms of the recurring sequence (8), one has

(10) $$Y_x = c_1 r^x + c_2 s^x.$$

In order to obtain a specific sequence, it is now necessary to impose initial or boundary conditions. It is customary, in dealing with recurring sequences, to specify the first terms of the sequence. For $Y_1 = 1$ and $Y_2 = 1$ in (10), the system of equations

$$1 = c_1 r + c_2 s$$

$$1 = c_1 r^2 + c_2 s^2$$

imply that $c_1 = 5^{\frac{1}{2}}$, $c_2 = -5^{\frac{1}{2}}$. In this case (10) is simply the Binet formula of Theorem 4.1., and the sequence obtained in this case is the Fibonacci sequence itself. Other sequences may be obtained, of course, by varying the initial terms. The corresponding Binet formula is then determined from (10). For instance, if $Y_1 = 1$ and $Y_2 = 3$, solving the system of equations obtained from (10) gives $c_1 = c_2 = 1$, and for this case, (10) becomes $Y_x = r^x + s^x$. This yields the sequence of Lucas

$$1, 3, 4, 7, 11, 18, \ldots ,$$

which possesses a number of remarkable properties similar to many of those pertaining to the Fibonacci sequence.

Rather than applying various conditions to (10) and investigating the corresponding sequence in an isolated fashion, why not hold back the explicit values of the initial terms and study the sequence in general? Suppose, therefore, that (8) and (10) are retained and the conditions imposed are $Y_1 = p$ and $Y_2 = p+q$, where p and q are arbitrary integers. The sequence so obtained is

$$p, p+q, 2p+q, 3p+2q, 5p+3q, 8p+5q, \ldots,$$

and it may be observed that

(11) $$Y_{x+2} = pF_x + (p+q)F_{x+1},$$

where F_x, F_{x+1} are the x^{th} and $(x+1)^{th}$ Fibonacci numbers.

It follows from (10) that

$$p = c_1 r + c_2 s$$

$$p+q = c_1 r^2 + c_2 s^2.$$

Solving this system and substituting into (10) yields

$$Y_x = 5^{-1/2}((p-sq)r^x + (rq-p)s^x).$$

It is clear from either (11) or (12) that $p = 1$, $q = 0$ is that specialization which yields the Fibonacci sequence. It may be noted that any choice in which p and q are consecutive Fibonacci numbers, say F_x and F_{x-1}, produces the Fibonacci sequence with the first $x - 1$ terms missing. But a question should be raised. What properties of the Fibonacci sequence are carried over to the generalized sequence? Answering this question involves an investigation of considerable magnitude. A number of such results may be found in Horadam. [13] For instance, Y_{x+1}/Y_x approaches the golden ratio $\emptyset$, just as was the case for the Fibonacci numbers. The identities

$$Y_{x-1}Y_{x+1} - Y_x^{\,2} = (p^2 - pq - q^2)(-1)^x,$$

$$Y_{x+1}^{\,2} + (p^2 - pq - q^2)F_x^{\,2} = pY_{2x+1},$$

have their respective counterparts in the Fibonacci identities

$$F_{x-1}F_{x+1} - F_x^{\,2} = (-1)^x,$$

$$F_{x+1}^{\,2} + F_x^{\,2} = F_{2x+1}.$$

There is also a useful expression for Pythagorean triples similar to that developed in Chapter III for Fibonacci numbers, namely,

$$\left[2Y_{x+1}Y_{x+2}\right]^2 + \left[Y_xY_{x+3}\right]^2 = \left[2Y_{x+1}Y_{x+2} + Y_x^2\right]^2.$$

The work of this chapter indicates some of the directions in which generalizations of the Fibonacci sequence may be taken. Certainly, the variety of ways for so doing are extensive and the reader is invited to investigate and explore the possibilities on his own. The opportunities may be nearly as abundant as the offspring of Fibonacci's mythical pair of rabbits!

CHAPTER VII

SUMMARY AND EDUCATIONAL IMPLICATIONS

The presentation in this thesis makes material concerning the Fibonacci sequence readily available to the undergraduate mathematics student. It illustrates how a variety of techniques and mathematical tools, drawn from several areas of mathematics, can be used to prove theorems about the Fibonacci numbers.

Summary

In Chapter I the statement of the problem, scope of the thesis, methods and procedures, and expected outcomes are given. Chapter II provides an informal discussion of some properties of the Fibonacci numbers. Chapter III includes a formal definition of the Fibonacci sequence and several theorems illustrating proof from the definition. Additional theorems illustrating mathematical induction also appear, and some arithmetical properties of the Fibonacci numbers are proved. Chapter IV is a continuation of Chapter III but requires that the reader possess a knowledge of limits. The important Theorem 4.1. provides a formula for the direct calculation of a given Fibonacci number as a function of its subscript. Chapter V deals with recent research in obtaining generalized identities. The identity of Theorem 5.6. is the most striking result of this chapter. The connection between the Fibonacci sequence and the calculus of finite differences is initially established and is further broadened in Chapter VI, where the Fibonacci sequence is viewed in its natural setting as a homogeneous linear finite difference equation.

Educational Implications

Much of the material included concerning the Fibonacci numbers can be readily understood by secondary school students, particularly in the initial chapters, and is designed to supplement the undergraduate curriculum at both the upper and lower division. This thesis serves to consolidate research and present well motivated problem material, fashioned around a topic interesting to a number of students.

As a result of reading this thesis, the student should gain an awareness of several facets of mathematics, including an acquaintance with current and past research that has been done in connection with the Fibonacci numbers. It is also of significance that the reader who is a potential teacher at either the public school or college level may find motivational material for his pupils and will perhaps enlarge on some of the ideas presented.

Without question, the most significant outcome of this thesis lies in the experience that the investigator gained in its preparation.

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