A COMPARISON OF TWO METHODS OF TEACHING MULTIPLICATION: REPEATED-ADDITION AND RATIO-TO-ONE

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A COMPARISON OF TWO METHODS OF TEACHING

MULTIPLICATION: REPEATED-ADDITION

AND RATIO-TO-ONE

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PREFACE

Educators must make decisions as to what instructional procedures are most effective for the teaching of multiplication to elementary school children. Making these decisions would be facilitated by knowledge of instructional procedures that have been validated by research. Because of the recent changes in the arithmetic curriculum of the elementary school, this knowledge of effective instructional procedures is limited.

Instructional procedure was the subject of this study. The relationship between selected approaches and pupil acquisition, retention, and understanding of multiplication, as measured by mean scores on the post-test, was studied.

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TABLE OF CONTENTS

LIST OF TABLES

Table		Page

CHAPTER I

INTRODUCTION

The Problem

From the time the Committee of Seven recommended optimum grade placement of mathematical topics until a decade ago the content of the arithmetic curriculum has remained relatively constant even though its purposes and objectives have changed from time to time. However, during the last ten years many aspects of the arithmetic curriculum have been revised.

Recent innovations in the method of teaching basic multiplication facts reflect changes in the arithmetic curriculum. These methods emphasize understanding of multiplication as a concept prior to memorization of the facts and operations. Multiplication has been defined in the elementary school as a special kind of counting; that is, counting by equal sized groups. Instructional procedures based upon this definition of multiplication have been used to develop an understanding of mathematical principles and relationships that apply to multiplication.

Although any multiplication problem with whole numbers can be solved by either adding or counting, multiplying by a proper fraction does not result in a product larger than the multiplicand. Neither does multiplying by one result in a product larger than the multiplicand. Therefore, placing emphasis upon the rationalization that multiplication is repeated addition does not develop understanding of all aspects of the multiplication concept. Such an emphasis fails to develop the idea that multiplication is an extension of Cartesian cross-product of sets. The cross-product is thought of as the new set consisting of ordered pairs. The new set being generated by pairing each member of the first set with each member of the second set.

Multiplication may also imply a ratio-to-one idea. This second meaning merits attention if understanding of the multiplication concept is to be fully achieved. On page 62 Wren (52) referred to this second meaning in his definition of multiplication, "... is the process of finding a third number relating to one of two given numbers in the same ratio as the second is related to one."

Although this second idea has been neglected during the initial study of multiplication, children have been requested to solve problems that involve this idea. There has been little conclusive evidence as to the means and advisability of using instructional material based upon the ratio-to-one idea.

Thus, this research was designed to compare two introductory approaches to the teaching of multiplication: one based on the repeated addition idea, the other based on the ratio-to-one idea. Two questions were considered in the study. The first, can understanding of basic mathematical concepts such as commutativity, associativity, distributivity, closure, and multiplicative identity be developed; and second, can mastery of the basic facts be achieved equally well by utilizing the ratio-to-one idea as by the more commonly used repeated addition idea?

Review of the Literature

The arithmetic curriculum in the elementary school is vastly different today from what it was even a decade ago. The major change has been in the content of the arithmetic program. More mathematical content has been introduced. Emphasis has been placed on the study of mathematical structure. The change in content has been accompanied by changes in recommended instructional procedures.

Three questions need to be answered before the significance of this study can be seen in its proper setting. First of all, what particular aspects of multiplication are being emphasized in the "new" curriculum? Second, what instructional procedures have been introduced to develop understanding of the multiplication concept? And third, what studies have been conducted to compare instructional procedures used to introduce the multiplication process?

What particular aspects of multiplication are being emphasized? On page 191 Swenson (42) stated that five idea are contained in the concept of multiplication. These concepts are listed as: (1) multiplication is a special form of addition, (2) multiplication is based on a special form of counting, (3) multiplication is a ratio-to-one idea, (4) multiplication is a rectangular-array idea, and (5) a statement of multiplication is a statement of equality. Although five ideas have been given emphasis in the arithmetic curriculum of the elementary school has been placed on developing the first, second, and fourth ideas. Neglect of the ratio-to-one idea and the equality idea has led to confusion on the part of the student when understanding of a problem required the use of these ideas.

Dienes (12) stated on page 183 that pupils were confused as to the basic meaning of multiplication because it was generally ignored in the teaching of the multiplication concept. He continued by stating that this confusion was not apparent until an attempt to teach mathematical properties was made. Dienes defined multiplication in terms of sets.

On page 127 Ward (48) defined multiplication as the operation of finding the product of two numbers. He also defined the product of two numbers in terms of sets.

> If A and B are sets and if n(A) = a, and n(B) = b, then the product of a and b is the number of the Cartesian product of A and B.

$$a \times b = n(A \times B)$$

According to Ward certain properties of the operation of multiplication follow directly from its definition. These properties are: (1) the set of numbers is closed under the operation of multiplication, (2) multiplication is an associative operation, (3) multiplication is a commutative operation, (4) multiplication has the cancellation property, and (5) the identity element for multiplication is one. Later, Ward noted that the distributive property of multiplication with respect to addition allows either factor to be renamed as the sum of two numbers, and the other factor distributed over these addends.

From the preceding definition of multiplication, it is apparent that the understanding of the multiplication concept would necessitate the study of mathematical properties related to multiplication. The writer's survey of prominent arithmetic series (7, 17, 47) revealed that mathematical structure is being included in the newer editions. The survey also revealed that the ratio-to-one idea is infrequently taught.

The second question to be answered was in regard to instructional procedures that have been introduced to develop understanding of the multiplication concept. The Greater Cleveland Mathematics Program (27), often referred to as SRA, defined multiplication as an operation on sets to find a cardinal number of a set formed from a number of equivalent disjoint sets. However, mathematical properties that applied to multiplication were taught by means of an array that emphasized repeated addition. On page 119 a departure was made from the array approach to demonstrate the following problem:

> Mrs. Murray bought two books about space travel. Last week both of these books were read by five children, Tom, Dick, Harry, Betty, and Sue. How many times were the books read?

Teachers were requested to draw a given illustration on the board. The writer noted that even SRA did not suggest that teachers illustrate the idea of a Cartesian cross-product.

Buswell's (7) introductory approach parallelled those previously cited. On page 84 multiplication was defined as a special case of addition and it was stated that multiplication may be employed instead of addition under special conditions, namely, when the group of numbers to be combined in finding the total are equal in size. However, the following excerpt from page 159 of Buswell's fourth grade text indicated that Buswell did not recommend the exclusive use of the array to illustrate the multiplication concept.

> With five Indians in each canoe, find how many Indians Jane will put in five canoes. Hint: Cover all but five canoes. Count by fives as you touch the canoes. Five 5's = ___________.

From the preceding information and the survey of prominent arithmetic series, the writer concluded that the authors of arithmetic texts for the elementary school have rejected, at least for the present, the introduction of multiplication by means of rate pairs and graphs in the first quadrant of a Cartesian plane.

These statements are not to imply that the use of an array is not a good instructional procedure. However, the array must be modified considerably when used in the study of the ratio-to-one idea and the multiplication of fractions. It appeared that the coordinate system, the physical referent for the ratio-to-one approach might be applicable to the study of fractions.

Research has been done using either instructional procedures suggested by authors of arithmetic books for the elementary school or manipulative devices to teach multiplication. Such research related to this study was reviewed.

Lucow (49) examined the difference in achievement of Manitoba children in the third grade. One group used the Cuisenaire rods and the other used the regular Manitoba curriculum during the introductory teaching of multiplication. The Cuisenaire rods were judged to be effective; however, they did not appear to be superior to other technique.

Mastian (29) compared the effectiveness in developing mathematical reasoning, computational efficiency, understanding of structure, and attitudes toward mathematics on the part of fourth grade students. The experimental group used the School Mathematics Study Group, often referred to as SMSG, text, Mathematics for Elementary School - Grade 4. The control group used a regular text. Mastian found that the experimental group did slightly better on the measurement of mathematical comprehension and reasoning. There was a significant difference favoring the experimental group in understanding of structure. Mastian concluded that fourth grade pupils of all ability levels can understand principles and properties of mathematics.

Banghart (49) compared achievement of children who used a programmed fourth grade text, including some contemporary content, with pupils who used a conventional text. Banghart concluded that the difference in achievement was significantly in favor of the programmed text group for comprehension and total achievement. However, there was no significant difference between the groups for problem-solving.

All of the research in regard to how children acquire mathematical understanding is not in agreement. Suppes (40) stated that his aim was to contribute to the development of a scientific theory of concept formation. This aim stemmed from two concerns as opposed to perfection of rote learning by the makers of the revised mathematical curriculum. According to Suppes, this distinction is banal because the advocates of the new curriculum do not indicate what is meant by developing understanding, do not identify overt behavior indicating understanding, and do not have measures of that overt behavior.

Suppes (40) concluded from his experiments that: (1) incidental learning does not appear to be effective; (2) the formation of simple mathematical concepts by young children is approximately an all-or-none process; (3) learning is more efficient if the error is corrected in the presence of the stimulus; and (4) contiguity of response, stimulus, and reinforcement enhance learning.

It is apparent from the preceding information that a variety of instructional procedures have been introduced into the arithmetic curriculum. It is equally as evident that the problem of what instructional procedures are most effective has not been studied to any extent. Consequently, textbook writers do not agree as to the most effective instructional procedures.

The third question was in regard to studies that have been conducted to compare instructional procedures used to introduce the multiplication concept. During the past five years two doctoral candidates have directed their studies toward investigating the effectiveness of instructional procedures used to teach introductory multiplication.

Gray (24) did a study to determine how a method of teaching introductory multiplication that stressed development of an understanding of the distributive property would relate to pupil growth as measured in terms of arithmetic achievement, transfer of knowledge, retention, and progress toward maturity of understanding the multiplication concept. Gray defined the distributive property as an element of the structure of mathematics.

Gray conducted his research at the third grade level. Two sets of experimental lessons were devised. The experimental lessons provided for the teaching of introductory multiplication in terms of understanding the distributive property. Only the combinations involving 2, 3, and 4 were used. Data were analyzed for conclusions relative to the merits of teaching for understanding. Although there appeared to be some difference favoring the experimental group in regard to arithmetic achievement, it was not significant. Gray did find a significant difference favoring the experimental group in the retention and transfer test. In addition, results from the interview test indicated that the experimental group was superior in various aspects. These results were: (1) the experimental group differed significantly from the control group on test items requiring application of untaught procedures, (2) the experimental group differed significantly from the control group on the use of the distributive property, and (3) subjects giving distributive property responses were generally superior in intelligence quotient and arithmetic reasoning.

Gray concluded that: (1) a program of arithmetic instruction that introduced multiplication by a method stressing understanding of the distributive property produced results superior to the current method; (2) knowledge of the distributive property appeared to enable children to proceed independently in finding products; (3) children appeared not to be able to develop an understanding of the distributive property unless it is specifically taught; and (4) in as far as the distributive property is an element of the structure of mathematics, the findings tend to support the assumption that teaching for an understanding of structure can provide superior results in terms of pupil growth.

Schell (38) was concerned with two aspects of the initial teaching of multiplication of whole numbers to third grade pupils. These aspects were: (1) the use of illustrations, particularly arrays, to represent multiplication, and (2) pupil learning of the distributive property of multiplication over addition.

Schell used two instructional methods. One was referred to as the Variety approach, and the other as the Array approach. Data were analyzed for conclusions relative to the merits of using arrays exclusively when illustrating multiplication at the introductory stage. Although Schell found that there was no significant difference between the two groups when all items of the final test were compared, he did find that there was a significant difference between the groups in favor of the Array group as to the general understanding of multiplication as measured by specific items on the test.

Two conclusions might be drawn from Schell's study that are pertinent to this study. First, use of an array exclusively to illustrate multiplications seemingly has several limitations. One of these limitations was that subjects who had only the array with which to represent multiplication had more difficulty in discriminating correctly between addition and/or subtraction problems and multiplication problems than did subjects who had been taught to use a variety of illustrations. Secondly, findings in Schell's study seemed to indicate that pupils may rotely manipulate illustrations as well as rotely manipulate numbers. There was no assurance that any better understanding occurred when an array was used. And finally, there appeared to be two distinct levels of functioning in arithmetic. Schell referred to these levels as the "computational" level and the "understanding" level. He did not find that much overlap necessarily existed between the two. According to Schell, a correct illustration might indicate that the child grasped the concept of the computational procedure but it did not necessarily mean that the child grasped the relationship between the problem and/or written fact and the drawing.

Theoretical Background

Considerable interest has been expressed in regard to the advisability, as well as possibility, of including in the arithmetic curriculum for the elementary school abstract mathematical concepts. Davis (10), director of the Madison Project, explored the possibility of teaching intermediate grade students identities and quadratic equations. Suppes (40) and his associates at Stanford University prepared programs for teaching logic to intermediate grade students and geometry to primary grade pupils. Bruner and Dienes (5) at the Center for Cognitive Studies have been conducting research to determine whether or not children in the elementary school are able to learn mathematical concepts such as the associative, commutative, and distributive properties.

Although many of the studies have reported success, they have not all been uniformly successful. Further research should provide more information in regard to what mathematical concepts young children are able to learn. All of the studies have one characteristic in common. Each study was based on the assumption that abstract mathematical concepts might be introduced earlier in the arithmetic curriculum of the elementary school. If the learner can understand a basic mathematical principle at an earlier age, he will have at his command a conceptual tool that will help him progress academically faster and more efficiently. The abstract mathematical concepts, therefore, should be acquired by the learner as early in his school experience as possible.

Bruner (4) on page 6 stated "... these studies have stimulated a renewed interest in complex learning ... learning designed to produce general understanding of the structure of a subject." He continued by stating:

> Grasping the structure of a subject is understanding it in a way that permits many other things to be related to it meaningfully. To learn structure, in short, is to learn how things are related.

Bruner (4) stated that the fundamental structure of mathematics needs to be taught because: (1) an understanding of the fundamental structure makes the subject more comprehensible; (2) research indicates that if details of a subject are not placed in a structural pattern, they are rapidly forgotten: (3) the understanding of fundamental principles and ideas appears to be the primary apparatus of "transfer of training," and (4) by constant re-examination of material taught at all levels for its fundamental character, the gap between "advanced" and "elementary" knowledge of mathematics is narrowed.

The psychological investigation of Piaget, a Swiss psychologist, and his collaborators at the University of Geneva has served as one source, directly or indirectly, from which research activities in regard to the intellectual processes of childred stemmed.

Piaget (36) stated on page 176, "... an operation is thus the essence of knowledge; it is an interiorized action which modifies the object of knowledge." He maintains that the development of knowledge passes through four main stages whose order is constant, but whose time of appearance may vary with the individual and with the culture. Each stage represents a new coherence and a new structuring of elements which until that time have not been systematically related to each other. The first stage is known as the sensory-motor or preverbal stage which extends from birth to approximately two years of age. The fundamental beginnings of an operation of reversibility and associativity are to be noted in the motor behavior of the child in space.

The period of pre-operational representation-the beginning of language, and therefore of thought, extends from two to seven years of age. The child has not as yet acquired any concept of conservation nor is he able to deal reversible operations. For example, to the child the amount of liquid changes according to the shape of the container. The child's judgment of transitivity is lacking, also. He may recognize that A and B are equal, and B and C are equal, yet he is unable to reach the conclusion that A is equal to C.

According to Piaget, the first operation appears during the third stage. This stage, known as "concrete operation," extends from seven to eleven years of age. The child is now able to deal with objects in ways that indicate an understanding of reversibility. Piaget (36) commented on page 177:

> ... children operate on objects, and not yet on verbally expressed hypotheses. ... there are the operations of classification, ordering, the construction of the idea of number, spatial and temporal operations, and all the fundamental operations of elementary logic of classes and relations, of elementary mathematics, of elementary geometry, and even of elementary physics.

About this same time systems having multiplicative character begin to develop. The child is able to classify an object according to two properties, e.g., size and shape. The child's thoughts are still restricted because the operations are still related to concrete objects. Thus, at this stage the child is unable to do formal logic.

The fourth stage is known as that of formal or hypotheticdeductive operations. It begins at about age eleven. Not until this stage is the child able to reason on hypothesis and to draw logical conclusions from hypothetical data. Now, the child no longer need rely on concrete objects. Piaget (36) concluded on page 178, "... he constructs new operations, operations of propositional logic, and not simply the operations of classes, relations, and numbers." The child is able to draw implications from various statements and to synthesize these implications.

Coxford (9) on page 119 quoted Piaget as stating that children by the time they are six to seven and a half years of age are able to understand the concept of number. In regard to the concept of multiplication Piaget's experiments indicate children attain the understanding of one-to-one correspondence by age four and a half to five. Gradual awareness of the multiplication property of one-to-one correspondence is attained by age five or six. Immediate grasp of the multiplication properties of many-to-one and fraction-to-one are attained by age six to seven and a half. These levels do not refer to the abstraction associated with symbols but to their concrete counterparts.

Piaget, then, has determined age levels for the attainment of the concept of number. His findings have two implications for the teaching of arithmetic: (1) the age of attainment gives some indication of when a child may have an understanding of a concept and (2) the analysis of the developmental process indicates what material and procedures might be appropriate in aiding the child in concept formation.

Another inference made from Piaget's studies is that children should be taught the underlying principles of a content area, after which they should be able to relate specific learnings to the general structure. The learning situation should be structured so that the child, by participating actively, is able to develop an understanding of the mathematical concept.

Bruner (4) pointed out on page 82 that it is possible to present fundamental structure in a sequence such that the child is guided to discover structure for himself. Many of the new approaches to the teaching of arithmetic utilize this discovery approach to the teaching of mathematical concepts.

Teaching by a discovery approach is not easy. There are no manuals listing the steps to follow. Children neither learn at the same rate nor in the same way. Therefore, the teacher must provide learning situations that allow for these differences. One approach to the problem of individual differences might be the use of programmed material. Programmed material might be effective in helping students achieve specific objectives in the area of mathematics.

No matter what procedure is used, the efforts of the various curriculum study groups have sought to make school mathematics more a science of numbers and less a set of drills. Glennon (22) stated on page 355 that prior to this century two theories determined the content of the school mathematics programs. The first, the need of society for mathematics training on the part of the citizenry, the sociological approach; the second, the need for the subject to be taught as a system of related ideas, the logical approach. As defined by Morton (32) on page 21, "... the logical approach is concerned with the structure and organization of arithmetic as a science; while, the social approach is concerned with the usefulness of arithmetic in life's affairs."

However, with the accumulation of knowledge regarding the condition necessary for effective learning, the nature of child growth and development, and the importance of good mental health, educators need to consider a third criterion when determining what ought to be included in the elementary school mathematics programs. Referring to this criterion on page 22, as the psychological approach, Morton (32) defined it by stating, "... in mathematics learning proceeds from an awareness of quantity concept and relationship; to abstraction of symbolic manipulation; and finally, to greater understanding and skill in the application of newly acquired concepts and processes in social situations in life."

Educators are aware of the fact that children can learn more content than can possibly be taught during the time they attend school. The use of Morton's three criteria will enable educators to select from all that can be learned that which is of the greatest importance.

Studies (49) have indicated that there is some agreement on such things as the need to build fundamental understandings, the use of spaced practice to assure mastery, and the establishment of sequential learning experiences. The search now needs to center on optimum procedures that will enable desired mathematical goals to be reached within the framework of a good teaching-learning situation.

This study was based on certain postulates presented in the preceding material. First, if children can be taught abstract mathematical ideas at an early age, then all fourth grade pupils should be of the age to understand the mathematical properties related to multiplication. Second, if children of the fourth grade level are at a stage of intellectual development necessitating concrete referents, then the search for the most effective physical referment should be continued. Third, if the use of programmed instructional material provides for individual differences, then the use of programmed material would provide for the individual needs of fourth grade pupils. And last, if the most encompassing concept should be used to direct instruction on a topic, then the instructional approach utilizing an equivalent ratio idea of multiplication with a graph in the first quadrant will be as effective or more effective in terms of computational proficiency and understanding of mathematical principles as an approach utilizing an array.

Delimitations of the Study

Does the method of introducing multiplication facts affect the learner's understanding of the mathematical concepts applicable to multiplication of whole numbers and mastery of multiplication facts?

In an attempt to answer the question, this study was designed to use two different approaches to introduce the multiplication facts: the Repeated-Addition approach and the Ratio-to-One approach. Thus, the independent variable in the study was the approach used to introduce the multiplication facts.

The dependent variable was the scores on the post-test. This test was constructed to test mastery and understanding of the mathematical concepts that applied to multiplication of whole numbers.

Motivation and interest are difficult to control. Both of these could have been intervening variables. However, for this study these variables were considered to have a negligible effect due to the random assignment of each group to one of the two approaches. As programmed material was used for the lessons, the teacher variable was considered to have a negligible effect, also.

The population of this study was limited to fourth grade pupils in the area served by Wisconsin State University-River Falls. Thus, any inferences drawn from this study may only be done in regard to this population.

The scope of the study was limited to (1) mastery of the basic multiplication facts and (2) understanding of five basic properties of a number system: the commutative property with respect to multiplication, the associative property with respect to multiplication, the distributive property with respect to multiplication over addition, closure, and the identity element of one in multiplication of whole numbers.

Definition of Terms

In order to clarify meanings of terms used in the study, the following list of terms and definitions was compiled :

Coordinate System. A method of labeling points in a plane by pairs of numberals denoting distance along two intersecting perpendicular rays called axes. This coordinate system is similar to the cartesian coordinate in a plane except that use is made of only the first quadrant.

Ordered Pair. An ordered pair consists of a pair of numerals written in a prescribed way.

Programmed Material. Learning material organized step by step so that: (1) frequent response is required, (2) immediate reinforcement is provided, (3) opportunity is furnished for discovery, and (4) allowance is made for the student to work individually at his own rate.

Ratio-to-One Approach. The approach that used as a model a coordinate system with ordered pairs to represent the multiplication properties and facts.

Repeated-Addition. Addition of equal addends.

Repeated-Addition Approach. An array, based upon repeated addition was used to represent the multiplication properties and facts.

Specific Hypotheses

While most authors of elementary arithmetic series recommend that multiplication be introduced as a repeated addition idea, others noted that the ratio-to-one idea had been neglected. However, whether the basic mathematical concepts relating to multiplication could be developed at the fourth grade level as well by the Ratio-to-One approach as by the Repeated-Addition approach had not been determined.

This study was designed to determine whether there was a significant difference in student achievement and understanding when different approaches were taken in teaching introductory multiplication.

The null form of the hypotheses tested are given as follows:

1. There is no significant difference between the post-test mean scores of those fourth grade students who had been introduced to multiplication by the Repeated-Addition approach and those who had been introduced to multiplication by the Ratio-to-One approach.

2. There is no significant difference between the understanding of mathematical properties of high-achieving fourth grade students who had been introduced to these properties as related to multiplication by the Repeated-Addition approach and those who had been introduced to these properties by the Ratio-to-One approach.

3. There is no significant difference between the understanding of mathematical properties of the middle-achieving fourth grade student who had been introduced to these properties by the Repeated-Addition approach and those who had been introduced to these properties by the Ratio-to-One approach.

4. There is no significant difference between the understanding of mathematical properties of the low-achieving fourth grade students who had been introduced to these properties as related to multiplication by the Repeated-Addition approach and those who had been introduced to these properties by the Ratio-to-One approach.

The t-test based on 0.05 level of confidence was used to test the null hypotheses that there is no significant difference between the mean scores of the two groups on the total post-test as well as the difference between the mean scores for the respective achievement levels.

Analysis of variance was run for each of the mathematical properties according to achievement levels.

And last, item analysis was done to determine the level of difficulty of each item on the post-test. Comparisons were made between the level of difficulty found for the high and low groups.

CHAPTER II

PROCEDURE

Research Design

Since 1960 mathematical concepts and terminology have been introduced into the arithmetic curriculum of the elementary school at earlier levels. This has resulted in experimentation to determine how these concepts could best be introduced to children at these levels. One instructional aid, the array, has been used in various experimental studies as a representation of the multiplication process. However, the array has its limitations when applied to multiplication of numbers other than whole numbers. The question arose as to what limitations would exist if another physical referent were used during the teaching of introductory multiplication. Therefore, experimentation was done using another representation, a coordinate system, to provide information relative to the teaching of multiplication.

This study was designed to determine whether there are significant differences in student mastery of the multiplication facts and understanding of those mathematical principles applicable to the multiplication process when two different approaches were taken in teaching introductory multiplication. Four classes in the River Falls School system and four classes in the New Richmond School system were randomly assigned to the two approaches. Each approach was used in four classrooms, two in each school system.

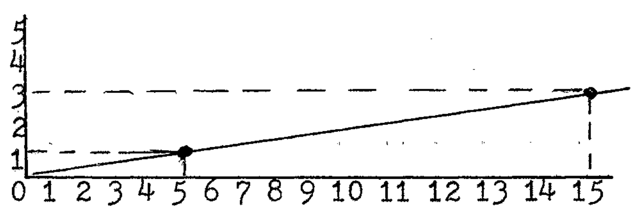
One approach, to be known as the Repeated-Addition approach, used the array as the physical referent to emphasize the repeated addition idea in regard to multiplication. The other approach, to be known as the Ratio-to-One approach, used a coordinate system and ordered pairs of numbers as the physical referent to emphasize the ratio idea in regard to multiplication.

Upon completion of fifteen programmed lessons, each group was administered a post-test by the writer. The retention test was administered four weeks later. No multiplication was taught during the time between the administering of the post-test and the retention test. This period of time also included the regular two week Christmas vacation.

This study was begun in October of 1967 and completed in January of 1968. Each experimental group used fifteen programmed lessons designed specifically for that group. Upon reaching page 46 in Seeing Through Arithmetic, Grade 4, the adopted text, the first programmed lesson was introduced. Succeeding lessons coincided with the text's introduction of specific multiplication facts. Immediately upon the completion of the final lesson, designed to be used with page 110, the post-test was administered.

Instructional Material

Multiplication may be defined as a special case of addition or in terms of ratio. In the first sense, multiplication may be thought of as repeated addition of a given quantity. Thus, 3×5 may be considered as $5 + 5 + 5$, which is to be thought of as joining together of equivalent groups. A common representation of this meaning is the array, a rectangular arrangement of equivalent groups. In the second sense, multiplication may be considered as the association of a number to a third number in the same ratio as a second number is associated to one. In this sense, 3 x 5 may be considered as $5/1 = \square/3$. In this study the representation used for this second definition was ordered pairs of numbers on a line in a coordinate system. The product of 3 x 5 was designated by the ordered pairing of numbers distributed along the coordinate axes. The product of 3 x 5 was that number on the horizontal axis that is associated with three on the vertical axis in the same ratio as five is associated to one as the following example indicates:



Although both definitions of multiplication described above were developed to some degree in the textbooks (7, 17, 47) surveyed by the writer, only the repeated addition definition using the array representation, was commonly stressed. However, in no textbook was the coordinate system used in just the way it was described above as a representation of multiplication.

In spite of the representation used, however, if multiplication is to be taught in terms of how the structure of mathematics is related to it, emphasis must be given to the development and understanding of the multiplication process. A number of basic mathematical properties apply to the multiplication of whole numbers. The commutative, associative, and distributive properties are considered to hold for multiplication of whole numbers. That is, a x b = b x a, and a x (b x c) = (a x b) x c, and a x (b + c) = (a x b) + (a x c) respectively. The set of whole numbers used in the multiplication operation is closed. That is, any whole number when multiplied by another whole number results in a product that is also a whole number. The set of whole numbers contains an identity element for multiplication. The identity element is that number which when multiplied by a second whole number always results in the second whole number as a product. That is, 1 x n = n.

However, in order to determine whether there was any difference in achievement and understanding in the specific aspects of multiplication of whole numbers between pupils taught to illustrate multiplication facts by the use of ordered pairs of numbers on a line in a coordinate system and pupils taught to use an array for representation, it was necessary to construct two sets of instructional material. For purposes of identification, one set was referred to as the Repeated-Addition approach and the other set as the Ratio-to-One approach. The same general outline was followed for the sets as a whole and in the format for corresponding lessons in the two sets. Fifteen programmed lessons were constructed for each experimental approach. Copies of these lessons may be found in Appendix B and Appendix C respectively.

Before final development of the material used in the main part of the study, Lessons 1, 2, 3, 4, 6, 7, and 13 of the Ratio-to-One approach were used in a pilot study with a group of children from the fourth grade of the J. H. Ames Laboratory School. The writer observed the children as they used the programmed material. As a result of the observation, several changes were made in the lessons. These were made prior to the beginning of the main part of the experiment.

In general, the introduction of the multiplication facts followed the pattern suggested by Hartung, Van Engen, and Knowles in Seeing Through Arithmetic, Grade 4, the textbook used by all of the participating classes. Because the study was not on the developmental approach suggested in the pupils' textbook, liberal changes were made to fit the lessons to the desired emphasis upon the mathematical concepts. The facts used in the lessons were from 1 x 1 = 1 to, and including, 9 x 9 = 81.

Since corresponding programmed lessons in each set had almost identical content, one description will suffice for both sets. A brief description of the main emphasis of each lesson is given below:

Lesson 1: Introduction To Multiplication

The meanings of multiplication were investigated. Ways of illustrating multiplication were considered.

Lesson 2: Closure Property

The set of whole numbers was reviewed prior to the introduction of the closure property.

Lesson 3: Specific Representation To Be Used

Emphasis was placed on a specific representation, array or ordered pairs of numbers on a line in a coordinate system. The fact that multiplication of whole numbers is a binary operation was introduced. Terms such as "factor" and "product" were also introduced.

Lesson 4: Commutative Property

This lesson dealt with the introduction to and practice in writing and illustrating "pairs" of multiplication facts.

Lesson 5: Commutative Property Continued

Further practice was provided in the use of the commutative property of multiplication.

Lesson 6: Multiplicative Identity

This lesson dealt with the introduction to and practice in illustrating and writing the multiplicative identity.

Lesson 7: Distributive Property

The rationale as well as ways of illustrating the distributive property of multiplication of whole numbers was studied.

Lesson 8: Distributive Property Continued

A variety of examples dealing with the distributive property was introduced.

Lesson 9: Practice

A variety of examples dealing with the various aspects of multiplication studied in preceding lessons was provided.

Lesson 10: Distributive Property Continued

Different ways of expressing the distributive property were examined and practice in using it was given.

Lesson 11: Commutative Property

A variety of examples dealing with the commutative property was provided.

Lesson 12: Introduction Of Two Difficult Multiplication Facts

Emphasis was placed on the learning of two multiplication facts, 7 x 8 = 56 and 6 x 9 = 54.

Lesson 13: Associative Property

The concept of associativity as it relates to multiplication was introduced. Different ways of illustrating and expressing the property were examined. Practice in using it was given.

Lesson 14: Practice

A variety of examples dealing with all aspects of multiplication studied in previous lessons was provided.

Lesson 15: Review

A variety of examples was used to review all aspects of multiplication studied. Chief among these was a property identification exercise.

Both the Repeated-Addition and the Ratio-to-One sets of programmed material were made as alike as possible. The same multiplication facts, the same terminology, the same number of examples, and the same format were used as much as was feasible. The only difference was in the representation used and the wording of the introductory word problems. The Ratio-to-One group made no use of any representation other than ordered pairs of numbers on a line in a coordinate system, the Repeated-Addition group used an array exclusively. The word problems used to introduce each mathematical concept for the Repeated-Addition group dealt exclusively with objects that could be arranged in rows and columns. The Ratio-to-One group's word problems dealt exclusively with situations that could be illustrated as a ratio. However, even though the two approaches often dealt with different items in the word problems, the answers to the problems were numerically identical.

Prior to the use of the programmed material, a meeting of the participating teachers, the elementary school principal, and the writer was held in each school. At this time, an explanation of the study was presented and examples of the programmed material were examined and discussed. Questions were answered and the teachers determined the approach each would use by a flip of a coin.

The writer introduced a coordinate system and ordered pairs of numbers to two classes in each school system before the study began. These classes had been designated to use the Ratio-to-One approach.

Personal contact with the teachers during the study was maintained through three sources: distribution of the material, observation of the lessons, and discussion with the participating teachers. It was decided to deliver the material to the teachers three lessons at one time. Thus, the writer had an opportunity to come into closer contact with the participating teachers through five successive deliveries of material. This, also, provided opportunities to observe the children as they worked on a lesson, to discuss the material with the teachers, and to record pertinent comments relative to the study.

During the study the writer visited each classroom at least four times. Additionally, the writer administered the post-test to all eight classes. During the visits the writer was able to note points of difficulty as well as pupil reaction to the material.

Measuring Instrument

A 47 question test, consisting of 61 items, was designed to measure understanding of specific aspects of multiplication presented during the study. Objective-type, generally multiple-choice items, were prepared because of the objectivity of scoring. Each item was written so that it could be scored and analyzed separately as well as with the other items measuring understanding or mastery of the same aspect of multiplication.

The components of the test were:

Number of Test Question	Aspect Measured
1-24	Mastery
22, 24, 27, 30, 40, 45, 47	Commutative Property
21, 23, 31, 41, 42, 45, 47	Associative Property
19, 32, 43, 44, 45, 47	Distributive Property
25, 35, 47	Closure
5, 8, 13, 15, 28, 36, 41, 47	Multiplicative Identity
1-47	Overall knowledge

Preliminary test forms were prepared and used with children in the fifth grade of the J. H. Ames Laboratory School. Items were deleted or added to the final form of the test upon the evaluation of the test items on the preliminary forms.

A copy of the test, titled, "Multiplication," is in Appendix D.

The test was used for both the post-test and the retention test. The post-test was administered in December, 1961 and the retention test was administered four weeks later.

Selection of Subjects

A few of the school systems in the Wisconsin State University-River Falls area have used one or another of the newer mathematics programs for a few years. Some of the programs provided for the introduction of all the multiplication facts prior to the fourth grade as well as stressed mathematical properties that are central to this study. It became evident, then, that children who had this type of experience could not be considered as part of the population for the present study. Therefore, school systems using the newer arithmetic programs were excluded from the study. The population from which the sample was drawn had to be limited further to those schools whose arithmetic curriculum called for the completion of the multiplication facts during the fourth grade.

In May of 1967 a form letter was mailed to twenty-nine school systems in the area served by Wisconsin State University-River Falls requesting information in regard to the willingness to participate in the research project, the number of available classes, and the arithmetic text to be used during the 1967-68 school year. A copy of the form letter is in Appendix A. Fourteen of the nineteen schools responding indicated a willingness to participate in the project. From the fourteen participating school systems, two were randomly selected whose fourth grade classes would make up the representative sample. These school systems were New Richmond and River Falls. Both of these communities are representative of the socio-economic levels found within a typical rural Wisconsin community.

The elementary school population in rural Wisconsin communities is relatively stable. As a result few children would have transferred from another district. The randomized selection of the sample; therefore, makes it possible to assume that these children were typical fourth grade pupils in this area of Wisconsin. It seems probable, though, that some children are included in the sample who are transfer pupils. It is further assumed, however, that the effect of the presence of such pupils is randomized throughout both treatment groups and, therefore, does not materially influence the findings of the study.

The fourth grade population for this study consisted of sixty-four classes in fourteen school systems. A total of two hundred twenty-two children in eight classrooms constituted the sample, one hundred twenty-one in River Falls and one hundred one in New Richmond. Ten subjects did not complete all fifteen programmed lessons or did not take the post-test because of absence from school during the study. As a result of this factor, complete data was available for a total of two hundred fourteen subjects in the eight classes.

As all of the children in the fourth grades of each school system took part in the study, it was assumed that they constituted a representative sample of the population. During the first meeting between the writer and the participating teachers, a coin was flipped to determine the approach to be used by each class. Two classes in each school system were thus randomly assigned to the Repeated-Addition approach and two classes in each school system were randomly assigned to the Ratio-to-One approach. Four classes used the array programmed material and four classes used the ordered pairs of numbers on a line in a coordinate system programmed material.

Treatment of Data

The reliability coefficient of the post-test was obtained by the split-half method. To correct the correlation found by this method, the Spearman-Brown formula was used.

The t-test (0.05) was used to determine whether there was any significant difference between the means of scores on the total test, on items one through twenty-four, and on items twenty-five through forty-seven, respectively of the Repeated-Addition group and the Ratio-to-One group.

The subjects in each approach were then divided into three levels on the basis of their post-test scores. These levels were referred to as high, middle, and low. The high level was composed of subjects having post-test scores more than one standard deviation above the mean in their respective groups. The middle level was composed of those subjects having scores located between one standard deviation below the mean and one standard deviation above the mean of their respective groups. The low level was composed of subjects having scores more than one standard deviation below the mean of their respective groups. The t-test (0.05) was used to determine whether the mean score of each of these levels using the Repeated-Addition approach differed significantly from its corresponding level using the Ratio-to-One approach on the various components of the post-test.

Comparisons were made between the corresponding levels of the two approaches in regard to the number of correct items for each mathematical property introduced during the study. An analysis of variance was run for each mathematical property.

An item analysis was done to determine the difficulty level of of the items in the post-test. Comparisons were made between the mean level of difficulty for the high-achieving pupils and that of the low-achieving pupils.

CHAPTER III

RESULTS

Introduction

This study involved the analysis of three separate aspects of the teaching of multiplication of whole numbers to fourth grade pupils: (1) the use of the array versus the use of a coordinate system to illustrate multiplication, (2) pupil learning of mathematical properties related to multiplication of whole numbers, and (3) pupil mastery of the multiplication facts.

Test Reliability and Validity

The reliability of the post-test was determined by using the results from the split-half method, odd and even numbered items, from the 214 pupil responses for the post-test. The Pearson product-moment coefficient of correlation was computed. The coefficient obtained by the split-half method was $r = 0.81$. To estimate the coefficient of reliability of the test if the full-length test had been used instead of split-halves, the Spearman-Brown formula was used. The corrected coefficient was $r = 0.89$. The reliability appeared to be sufficiently high to justify the use of the items in the post-test for this study.

Two aspects of multiplication were emphasized in the programmed material for the study, mastery of multiplication facts and understanding of mathematical properties applicable to multiplication.

The post-test contained (1) twenty-four items to measure mastery of the multiplication facts, (2) eight items to measure understanding and use of the commutative property, (3) seven items to measure understanding and use of the associative property, (4) five items to measure understanding and use of the distributive property, (5) three items to measure understanding and use of the closure property, and (6) eight items to measure understanding and use of the multiplicative identity. Thus, on the basis of subjective evaluation, it appeared that the post-test had content validity.

Testing the Hypotheses

The results of comparisons involving groups relating to the independent variables, taken one at a time (page 19), are presented below. Three independent variables were used in making comparisons with the achievement test data for a given hypothesis. Three tables will be related to each hypothesis. All of the tables will report the t-ratio using the total or subtest scores of the post-test for specific approaches.

Comparisons Involving Total Groups

The first hypothesis was concerned with the mean scores of the total group on (1) the test as a whole, (2) the mastery measuring items, and (3) the items measuring understanding of the mathematical properties applicable to the multiplication of whole numbers. Summaries of the data appear in Tables I, II, and III respectively.

The data for applying the t-test to Hypothesis One is found in Table I. The mean of the Repeated-Addition group for the total test was 36.14 and that of the Ratio-to-One group was 34.80. The difference between means of 1.26 (36.14-34.80) was in favor of the Repeated-Addition approach. An F-test applied to the data demonstrated no significant difference in variance. The t-ratio for the difference between means of 1.11 for 212 degrees of freedom was not significant at the 0.05 level. A t-ratio of approximately 1.97 would be necessary for the null hypothesis to be rejected at the 0.05 level, the level arbitrarily selected as the value for accepting or rejecting each hypothesis. As the t-ratio did not approach that magnitude, the null hypothesis of no difference between means was not rejected. Throughout the study this finding should be interpreted as meaning that a difference of this magnitude would occur more than one time in twenty if only chance factors are operating.

TABLE I

TOTAL GROUP MEAN SCORES AND t-RATIO FOR ALL ITEMS OF THE POST-TEST

Approach	N	Mean	SD	df	t-Ratio
Repeated-Addition	111	36.14	8.13	110	
Ratio-to-One	102	34.80	9.36	102	
				212	1.11

The data for applying the t-test to the mean scores for mastery of the multiplication facts, as measured by the first twenty-four items of the test, is found in Table II. The mean of the Repeated-Addition group was 17.11 and that of the Ratio-to-One group was 16.39. The difference between means of 0.72 (17.11-16.39) was in favor of the

Repeated-Addition approach. An F-test applied to the data demonstrated no significant difference in variance. The t-ratio for the difference between means of 1.16 for 212 degrees of freedom was not significant at the 0.05 level. The null hypothesis of no difference between the means was not rejected.

TABLE II

TOTAL GROUP MEAN SCORES AND t-RATIO FOR ITEMS MEASURING MASTERY OF THE MULTIPLICATION FACTS: ITEMS 1-24

Approach	N	Mean	SD	df	t-Ratio
Repeated-Addition	111	17.11	3.78	110	
Ratio-to-One	103	16.39	5.18	102	
				212	1.16

The data for applying the t-test to the mean scores for understanding of mathematical properties applicable to multiplication of whole numbers, as measured by items twenty-five through forty-seven, is found in Table III. The mean of the Repeated-Addition total group was 18.95 and that of the Ratio-to-One group was 18.42. The difference between means of 0.53 (18.95-18.42) was in favor of the Repeated-Addition approach. An F-test applied to the data demonstrated no significant difference in variance. The t-ratio for the difference between means of 0.75 for 212 degrees of freedom was not significant at the 0.05 level. Thus, the null hypothesis of no difference between the means was not rejected.

These data indicate that neither approach has been demonstrated to be more effective than the other for the mastery of multiplication facts, understanding of mathematical properties applicable to multiplication of whole numbers, and the total test for the total group of the population.

TABLE III

TOTAL GROUP MEAN SCORES AND t-RATIO FOR ITEMS MEASURING UNDERSTANDING OF MATHEMATICAL PROPERTIES: ITEMS 25-47

Approach	N	Mean	SD	df	t-Ratio
Repeated-Addition	111	18.95	5.25	110	
Ratio-to-One	103	18.42	5.00	102	
				212	0.75

Establishing Levels

The mean score and standard deviation of each approach were used as the basis to determine the respective groups by levels. A summary of the data has been presented in Table IV.

The Repeated-Addition group consisted of 111 pupils. The mean was 36.14 and the standard deviation was 8.13 on a 61 item test. Twenty pupils' scores were located more than one standard deviation above the mean. This group has been designated as the Repeated-Addition high group. Sixty-seven pupils' scores were located between one standard deviation below the mean and one standard deviation above the mean. This group has been designated as the Repeated-Addition middle group. Twenty-four pupils' scores were located more than one standard deviation below the mean. This group has been designated as the Repeated-Addition low group.

The Ratio-to-One group consisted of 103 pupils. The mean was 34.80 and the standard deviation was 9.36 on a 61 item test. Fifteen pupils' scores were located more than one standard deviation above the mean. This group has been designated as the Ratio-to-One high group. Seventy-four pupils' scores were located between one standard deviation below the mean and one standard deviation above the mean. This group has been designated as the Ratio-to-One middle group. Fifteen pupils" scores were located more than one standard deviation below the mean. This group was designated as the Ratio-to-One low group.

Note that the mean score was greater for the Repeated-Addition group than for the Ratio-to-One group. It may further be noted that the standard deviation for the Ratio-to-One group was slightly greater than that for the Repeated-Addition group.

TABLE IV

SUMMARY OF DATA TO DETERMINE ACHIEVEMENT LEVELS

Repeated-Addition				Ratio-to-One			
Level	N	Mean	SD	Level	N	Mean	SD
High	20	47.75	2.68	High	15	48.00	1.26
Middle	67	36.67	4.31	Middle	74	35.19	5.50
Low	24	25.00	3.30	Low	14	18.64	5.91
Total	111	36.14	8.13	Total	103	34.80	9.36

Comparisons Involving High Levels

The second hypothesis dealt with comparisons involving high levels. Comparisons were made between the mean scores for (1) the test as a whole, (2) the mastery measuring items, and (3) the items measuring understanding of the mathematical properties applicable to the multiplication of whole numbers. Summaries of the data appear in Tables V, VI, and VII respectively.

The data for applying the t-test to the mean scores for the total test is found in Table V. The mean of the Repeated-Addition high group was 47.75 and that of the Ratio-to-One group was 48.00. The difference between means of 0.25 (48.00-47.75) was in favor of the Ratio-to-One approach. An F-test applied to the data demonstrated a significant difference in variance. The obtained ratio was $F_{19,14} = 4.49$. The t-ratio for the difference between means of 0.32 for 33 degrees of freedom was not significant at the 0.05 level. A t-ratio of approximately 2.03 would be necessary for the null hypothesis to be rejected at the 0.05 level. As the t-ratio did not approach that magnitude, the null hypothesis of no difference between means was not rejected.

TABLE V

HIGH GROUP MEAN SCORES AND t-RATIO FOR THE TOTAL TEST

Approach	N	Mean	SD	df	t-Ratio
Repeated-Addition	20	47.75	2.68	19	
Ratio-to-One	15	48.00	1.26	14	
				33	0.32

The data for applying the t-test to the mean scores for mastery of the multiplication facts, as measured by items one through twenty-four, is found in Table VI. The mean of the Repeated-Addition high group was 21.70 and that of the Ratio-to-One high group was 22.13. The difference between means of 0.43 (22.13-21.70) was in favor of the Ratio-to-One approach. An F-test applied to the data demonstrated no significant difference in variance. The t-ratio for the difference between means of 0.82 for 33 degrees of freedom was not significant at the 0.05 level. The null hypothesis of no difference between the means was not rejected.

TABLE VI

HIGH GROUP MEAN SCORES AND t RATIO FOR MASTERY OF MULTIPLICATION FACTS: ITEMS 1-24

Approach	N	Mean	SD	df	t-Ratio
Repeated-Addition	20	21.70	1.52	19	
Ratio-to-One	15	22.13	2.25	14	
				33	0.82

The data for applying the t-test to Hypothesis Two is found in Table VII. The mean of the Repeated-Addition high group in regard to understanding of mathematical properties applicable to multiplication of whole numbers, as measured by items twenty-five through forty-seven, was 25.55 and that of the Ratio-to-One high group was 25.87. The difference between the means of 0.32 (25.87-25.55) was in favor of the Ratio-to-One approach. An F-test applied to the data demonstrated a significant difference in variance. The obtained ratio was

$F_{19,14} = 9.73$. The t-ratio for the difference between means of 0.32 for 33 degrees of freedom was not significant at the 0.05 level. The null hypothesis of no difference between the means was not rejected.

These data indicated that neither approach has been demonstrated to be more effective than the other for the mastery of multiplication facts, the total test, and the understanding of mathematical properties applicable to multiplication of whole numbers for the high group of the population.

TABLE VII

HIGH GROUP MEAN SCORES AND t-RATIO FOR UNDERSTANDING OF MATHEMATICAL PROPERTIES: ITEMS 25-47

Approach	N	Mean	SD	df	t-Ratio
Repeated-Addition	20	25.55	3.58	19	
Ratio-to-One	15	25.87	1.32	14	
				33	0.32

Comparisons Involving Middle Levels

The third hypothesis dealt with comparisons involving middle levels. Comparisons were made between the mean scores for (1) the test as a whole, (2) the mastery measuring items, and (3) the items measuring understanding of the mathematical properties applicable to the multiplication of whole numbers. Summaries of the data appear in Tables VIII, IX, and X respectively.

The data for applying the t-test to the mean scores for the total test is found in Table VIII. The mean of the Repeated-Addition middle group was 36.67 and that of the Ratio-to-One middle group was 35.19. The difference between means of 1.48 (36.67-35.19) was in favor of the Repeated-Addition approach. An F-test applied to the data demonstrated no significant difference in variance. The t-ratio for the difference between means of 1.76 for 139 degrees of freedom was not significant at the 0.05 level. However, this ratio was almost the magnitude necessary for the rejection of the null hypothesis, 1.98 at the 0.05 level. But, as the t-ratio did not reach this magnitude, the null hypothesis of no difference between means was not rejected.

TABLE VIII

MIDDLE GROUP MEAN SCORES AND t-RATIO FOR THE TOTAL TEST

Approach	N	Mean	SD	df	t-Ratio
Repeated-Addition	67	36.67	4.31	66	
Ratio-to-One	74	35.19	5.50	73	
				139	1.76

The data for applying the t-test to the mean scores for mastery of multiplication facts, as measured by items one through twenty-four, is found in Table IX. The mean of the Repeated-Addition middle group was 17.43 and that of the Ratio-to-One middle group was 17.08. The difference between means of 0.35 (17.43-17.08) was in favor of the Repeated-Addition approach. An F-test applied to the data demonstrated no significant difference in variance. The t-ratio for the difference between means of 0.71 for 139 degrees was not significant at the 0.05 level. Thus, the hypothesis of no difference between the means was not rejected.

TABLE IX

MIDDLE GROUP MEAN SCORES AND t-RATIO FOR MASTERY OF MULTIPLICATION FACTS: ITEMS 1-24

Approach	N	Mean	SD	df	t-Ratio
Repeated-Addition	67	17.43	2.62	66	
Ratio-to-One	74	17.08	3.13	73	
				139	0.71

The data for applying the t-test to Hypothesis Three is found in Table X. The mean of the Repeated-Addition middle group in regard to understanding of mathematical properties applicable to multiplication of whole numbers, as measured by items twenty-five through forty-seven, was 19.25 and that of the Ratio-to-One middle group was 18.10. The difference between means of 1.15 (19.25-18.10) was in favor of the Repeated-Addition approach. An F-test applied to the data demonstrated a significant difference in variance. The obtained ratio was $F_{66,73} = 1.49$, slightly above the significant level of 1.48. The t-ratio for the difference between means of 1.99 for 139 degrees of freedom was significant at the 0.05 level. A t-ratio of 1.98 would be necessary for the null hypothesis to be rejected at the 0.05 level. Thus, the null hypothesis of no difference between the means was rejected. The difference was in favor of the Repeated-Addition approach.

These data indicate that neither approach has been demonstrated to be more effective than the other for the mastery of multiplication facts for the middle group of the population. However, these data indicate that the Repeated-Addition approach has been demonstrated to be more effective than the Ratio-to-One approach for the understanding of mathematical properties for the middle group of the population.

TABLE X

MIDDLE GROUP MEAN SCORES AND t-RATIO FOR UNDERSTANDING OF MATHEMATICAL PROPERTIES: ITEMS 25-47

Approach	N	Mean	SD	df	t-Ratio
Repeated Addition	67	19.25	3.00	66	
Ratio-to-One	74	18.10	3.68	73	
				139	1.99

Comparisons Involving Low Levels

The fourth hypothesis dealt with comparisons involving low levels. Comparisons were made between the mean scores for (1) the test as a whole, (2) the mastery measuring items, and (3) the items measuring understanding of the mathematical properties applicable to the multiplication of whole numbers. Summaries of the data appear in Tables XI, XII, and XIII respectively.

The data for applying the t-test to the mean scores for the total test is found in Table XI. The mean of the Repeated-Addition low group was 25.00 and that of the Ratio-to-One low group was 18.64. The difference between the means of 6.36 (25.00-18.64) was in favor of the Repeated-Addition approach. An F-test applied to the data demonstrated a significant difference in variance. The obtained ratio was

$F_{13,23} = 3.11$. The t-ratio for the difference between means of 4.14 for 36 degrees of freedom was significant at the 0.05 level. A t-ratio of approximately 2.03 would be necessary for the null hypothesis to be rejected at the 0.05 level. Thus, the null hypothesis of no difference between the means of the low groups was rejected. The difference was in favor of the Repeated-Addition approach.

TABLE XI

LOW GROUP MEAN SCORES AND t-RATIO FOR THE TOTAL TEST

Approach	N	Mean	SD	df	t-Ratio
Repeated Addition	24	25.00	3.30	23	
Ratio-to-One	14	18.64	5.91	13	
				36	4.14

The data for applying the t-test to the mean scores for mastery of the multiplication facts, as measured by items one through twenty-four, is found in Table XII. The mean of the Repeated-Addition low group was 12.42 and that of the Ratio-to-One low group was 6.64. The difference between means of 5.78 (12.42-6.64) was in favor of the Repeated-Addition approach. An F-test applied to the data demonstrated no significant difference in variance. The t-ratio for the difference between means of 6.21 for 36 degrees of freedom was significant at the 0.05 level. A t-ratio of approximately 2.03 would be necessary for the null hypothesis to be rejected. As the t-ratio was greater than that magnitude, the null hypothesis of no difference between the means was rejected. The difference was in favor of the

Repeated-Addition approach.

TABLE XII

LOW GROUP MEAN SCORES AND t-RATIO FOR MASTERY OF THE MULTIPLICATION FACTS: ITEMS 1-24

Approach	N	Mean	SD	df	t-Ratio
Repeated-Addition	24	12.42	2.29	23	
Ratio-to-One	14	6.64	3.26	13	
				36	6.21

The data for applying the t-test to Hypothesis Four is found in Table XIII. The mean of the Repeated-Addition low group in regard to understanding of mathematical properties applicable to multiplication of whole numbers, as measured by items twenty-five through forty-seven, was 12.58 and that of the Ratio-to-One low group was 12.07. The difference between means of 0.51 (12.58-12.07) was in favor of the Repeated-Addition approach. An F-test applied to the data demonstrated no significant difference in variance. The t-ratio of 0.41 for 36 degrees of freedom was not significant at the 0.05 level. The null hypothesis of no difference between the means was not rejected.

This data indicated that the Repeated-Addition approach has been demonstrated to be more effective than the Ratio-to-One approach for the total test and the mastery of multiplication facts for the low group. However, these data did not indicate that either approach was more effective than the other for the developing of understanding of mathematical properties applicable to the multiplication of whole numbers for the low group of the population.

TABLE XIII

LOW GROUP MEAN SCORES AND t-RATIO FOR UNDERSTANDING MATHEMATICAL PROPERTIES: ITEMS 25-47

Approach	N	Mean	SD	df	t-Ratio
Repeated-Addition	24	12.58	3.81	23	
Ratio-to-One	14	12.07	3.26	13	
				36	0.41

Comparison Involving Parts of the Test

Five mathematical properties applicable to the multiplication of whole numbers were included in the post-test. These mathematical properties were (1) the commutative property, (2) the associative property, (3) the distributive property, (4) the closure property, and (5) the multiplicative identity. The hypothesis that there is no difference between the means will be assumed for each comparison. Three tables will be related to each of the properties. The first will report the analysis of variance using the low group for the two approaches and the mean scores for items measuring understanding of the commutative property. The second will report the analysis of variance using the middle group for the two approaches and the mean scores for items measuring understanding of the commutative property. The third will report the analysis of variance for the high group for the two approaches and the mean scores for the items measuring understanding of the commutative property. Then the pattern of tables is repeated for each of the remaining mathematical properties measured in the post-test.

Nine items dealt with various aspects of the commutative property. Summaries of the data appear in Tables XIV, XV, and XVI respectively.

Summary of the analysis of variance data for the low group is found in Table XIV. The F-ratio for approach of 0.07 for 1 and 36 degrees of freedom was not significant at the 0.05 level. An F-ratio of 4.11 would be necessary for the null hypothesis to be rejected at the 0.05 level. As the ratio did not approach that magnitude, the conclusion gave support to the practical consideration of the limited significance of the difference between the mean scores.

TABLE XIV

SUMMARY TABLE FOR LOW GROUP OF ANALYSIS OF VARIANCE OF MEANS FOR COMMUTATIVE PROPERTY ITEMS

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	0.28	0.28	1	
Within	137.92	3.83	36	
Total	138.20		37	0.07

$F_{1,36} = 0.07$

Summary of the analysis of variance data for the middle group is found in Table XV. The F-ratio for approach of 0.45 for 1 and 139 degrees of freedom was not significant at the 0.05 level. An F-ratio of 3.92 would be necessary for the null hypothesis to be rejected at the 0.05 level. As the ratio did not approach that magnitude, the conclusion gave support to the practical consideration of the limited significance of the difference between the mean scores.

TABLE XV

SUMMARY TABLE FOR MIDDLE GROUP OF ANALYSIS OF VARIANCE OF MEANS FOR COMMUTATIVE PROPERTY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	1.44	1.44	1	
Within	453.26	3.26	139	
Total	454.70		140	0.45

$F_{1,139} = 0.45$

Summary of the analysis of variance data for the high group is found in Table XVI. The F-ratio for approach of 4.11 for 1 and 33 degrees of freedom was not significant at the 0.05 level. An F-ratio of approximately 4.14 would be necessary for the null hypothesis to be rejected at the 0.05 level. Note that the obtained F-ratio was approaching that magnitude.

TABLE XVI

SUMMARY TABLE FOR HIGH GROUP OF ANALYSIS OF VARIANCE OF MEANS FOR COMMUTATIVE PROPERTY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	11.33	11.33	1	
Within	90.95	2.75	33	
Total	102.28		34	4.11

$F_{1,33} = 4.11$

These data indicated that neither approach has been demonstrated to be more effective than the other for the developing of understanding of the commutative property.

Eight items dealt with various aspects of the associative property. Summaries of the data appear in Tables XVII, XVIII, and XIX respectively.

Summary of the analysis of variance data for the low group is found in Table XVII. The F-ratio for approach of 0.04 for 1 and 36 degrees of freedom is not significant at the 0.05 level. An F-ratio of 4.11 would be necessary for the null hypothesis to be rejected at the 0.05 level. As the ratio did not approach that magnitude, the conclusion gave support to the practical consideration of the limited significance of the difference between mean scores.

TABLE XVII

SUMMARY TABLE FOR LOW GROUP OF ANALYSIS OF VARIANCE OF MEANS FOR ASSOCIATIVE PROPERTY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	0.10	0.10	1	
Within	85.71	2.38	36	
Total	85.81		37	0.04

$F_{1,36} = 0.04$

Summary of the analysis of variance data for the middle group is found in Table XVIII. The F-ratio for approach of 0.05 for 1 and 139 degrees of freedom was not significant at the 0.05 level. An F-ratio of 3.92 would be necessary for the null hypothesis to be rejected at the 0.05 level. As the ratio did not approach that magnitude, the conclusion gave support to the practical consideration of the limited significance of the difference between the mean scores.

TABLE XVIII

SUMMARY TABLE FOR THE MIDDLE GROUP OF ANALYSIS OF VARIANCE OF MEANS FOR THE ASSOCIATIVE PROPERTY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	0.13	0.13	1	
Within	305.61	2.19	139	
Total	305.74		140	0.05

$F_{1,139} = 0.05$

Summary of the analysis of variance data for the high group is found in Table XIX. The F-ratio for approach of 0.01 for 1 and 33 degrees of freedom was not significant at the 0.05 level. An F-ratio of approximately 4.14 would be necessary for the null hypothesis to be rejected at the 0.05 level.

Five items dealt with various aspects of the distributive property. Summaries of the data appear in Tables XX, XXI, and XXII respectively.

Summary of the analysis of variance data for the low group is found in Table XX. The F-ratio for approach of 0.17 for 1 and 36 degrees of freedom is not significant at the 0.05 level. An F-ratio of 4.11 would be necessary for the null hypothesis to be rejected at the 0.05 level.

TABLE XIX

SUMMARY TABLE FOR HIGH GROUP OF ANALYSIS OF VARIANCE OF MEANS FOR ASSOCIATIVE PROPERTY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	0.02	0.02	1	
Within	50.95	1.54	33	
Total	50.97		34	0.01

$F_{1,33} = 0.01$

TABLE XX

SUMMARY TABLE FOR LOW GROUP OF ANALYSIS OF VARIANCE OF MEANS FOR DISTRIBUTIVE PROPERTY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	0.38	0.38	1	
Within	80.45	2.23	36	
Total	80.83		37	0.17

$F_{1,36} = 0.17$

Summary of the analysis of variance data for the middle group is found in Table XXI. The F-ratio for approach of 0.72 for 1 and 139 degrees of freedom is not significant at the 0.05 level. An F-ratio of 3.92 would be necessary for the null hypothesis to be rejected at the 0.05 level.

TABLE XXI

SUMMARY TABLE FOR MIDDLE GROUP OF ANALYSIS OF VARIANCE OF MEANS FOR DISTRIBUTIVE PROPERTY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	0.77	0.77	1	
Within	149.36	1.07	139	
Total	150.13		140	0.72

$F_{1,139} = 0.72$

Summary of the analysis of variance data for the high group is found in Table XXII. The F-ratio for approach of 2.86 for 1 and 33 degrees of freedom is not significant at the 0.05 level. An F-ratio of approximately 4.14 would be necessary for the null hypothesis to be rejected at the 0.05 level.

These data indicated that neither approach has been demonstrated to be more effective than the other for the developing of understanding of the distributive property.

Three items dealt with various aspects of the closure property. Summaries of the data appear in Tables XXIII, XXIV, and XXV respectively.

Summary of the analysis of variance data for the low group is

TABLE XXII

SUMMARY TABLE FOR HIGH GROUP OF ANALYSIS OF VARIANCE
OF MEANS FOR DISTRIBUTIVE PROPERTY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	3.05	3.05	1	
Within	35.60	1.07	33	
Total	38.65		34	2.86

$F_{1,33} = 2.86$

found in Table XXIII. The F-ratio for approach of 0.27 for 1 and 36 degrees of freedom is not significant at the 0.05 level. An F-ratio of 4.11 would be necessary for the null hypothesis to be rejected at the 0.05 level. As the ratio did not approach that magnitude, the conclusion gave support to the practical consideration of the limited significance of the difference between the mean scores.

TABLE XXIII

SUMMARY TABLE FOR LOW GROUP OF ANALYSIS OF VARIANCE
OF MEANS FOR CLOSURE PROPERTY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	0.21	0.21	1	
Within	27.26	0.75	36	
Total	27.47		37	0.27

$F_{1,36} = 0.27$

Summary of the analysis of variance data for the middle group is found in Table XXIV. The F-ratio for approach of 0.45 for 1 and 139 degrees of freedom was not significant at the 0.05 level.

Summary of the analysis of variance data for the high group is found in Table XXV. The F-ratio for approach of 0.31 for 1 and 33 degrees of freedom is not significant at the 0.05 level. An F-ratio of approximately 4.14 would be necessary for the null hypothesis to be rejected at the 0.05 level.

TABLE XXIV

SUMMARY TABLE FOR MIDDLE GROUP OF ANALYSIS OF VARIANCE OF MEANS FOR CLOSURE PROPERTY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	0.37	0.37	1	
Within	114.44	0.82	139	
Total	114.81		140	0.45

$F_{1,139} = 0.45$

As none of the F-ratios approached the magnitude necessary for rejection of the null hypothesis, the conclusion gave support to the practical consideration of the limited significance of the difference between mean scores. Thus, these data indicated that neither approach has been demonstrated to be more effective than the other for the development of understanding of the closure property.

Four items dealt with various aspects of the multiplicative identity concept. Summary of the data appear in Tables XXVI, XXVII, and XXVIII respectively.

TABLE XXV

SUMMARY TABLE FOR HIGH GROUP OF ANALYSIS OF VARIANCE OF MEANS FOR CLOSURE PROPERTY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	0.12	0.12	1	
Within	12.28	0.37	33	
Total	12.40		34	0.31

$F_{1,33} = 0.31$

Summary of the analysis of variance data for the low group is found in Table XXVI. The F-ratio for approach of 5.62 for 1 and 36 degrees of freedom was significant at the 0.05 level. An F-ratio of 4.11 would be necessary for the null hypothesis to be rejected at the 0.05 level. As the F-ratio was greater than that magnitude, the conclusion gave support of significant difference between the mean scores. The difference was in favor of the Repeated-Addition approach.

Summary of the analysis of variance data for the middle group is found in Table XXVII. The F-ratio for approach of 3.04 for 1 and 139 degrees of freedom is not significant at the 0.05 level. An F-ratio of 3.92 would be necessary for the null hypothesis to be rejected at the 0.05 level. Note, however, that the F-ratio was approaching that magnitude.

Summary of the analysis of variance data for the high group is found in Table XXVIII. The F-ratio for approach of 4.06 for 1 and 33 degrees of freedom is not significant at the 0.05 level. An F-ratio of approximately 4.14 would be necessary for the null hypothesis

TABLE XXVI

SUMMARY TABLE FOR LOW GROUP OF ANALYSIS OF VARIANCE
OF MEANS FOR THE MULTIPLICATIVE IDENTITY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	5.21	5.21	1	
Within	33.33	0.92	36	
Total	38.54		37	5.62

$F_{1,36} = 5.62$

TABLE XXVII

SUMMARY TABLE FOR MIDDLE GROUP OF ANALYSIS OF VARIANCE
OF MEANS FOR THE MULTIPLICATIVE IDENTITY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	3.16	3.16	1	
Within	144.53	1.03	139	
Total	147.69		140	3.04

$F_{1,139} = 3.04$

to be rejected at the 0.05 level.

These data indicated that there was a significant difference in favor of the Repeated-Addition approach for the developing of understanding of the multiplicative identity for the low group. However, the same was not true for the middle and high group.

TABLE XXVIII

SUMMARY TABLE FOR HIGH GROUP OF ANALYSIS OF VARIANCE OF MEANS FOR THE MULTIPLICATIVE IDENTITY

Source of Variance	Sum of Squares	Mean Square	df	F-ratio
Approach	3.80	3.80	1	
Within	30.93	0.94	33	
Total	34.73		34	4.06

$F_{1,33} = 4.06$

Through an item analysis of the test, much information was gained that was not amenable to statistical comparisons without stating extreme limitations in regard to significance of results. This information will now be discussed.

Comparisons of Drawings For Item Forty-Three

The results of an analysis of the drawings made by the pupils in each approach to illustrate that multiplication distributes over addition are given in Table XXIX. One of the most enlightening facts revealed in this analysis was the frequency with which the Ratio-to-One group used the array. Of the 83 judgeable drawing completed by the Ratio-to-One group, 41 per cent were arrays; of the 28 correct drawings, 71 per cent were arrays. The Ratio-to-One group had not been presented with an array to be used as the physical referent in any of the programmed lessons. Of the 103 judgeable drawings done by the Repeated-Addition group, 81 per cent were arrays; of the 31 correct drawings, 93 per cent were arrays. A second enlightening fact revealed was that no pupil in the Ratio-to-One group used a coordinate system and ordered pair of numbers. Another informative aspect of the analysis was that although 75 per cent of the pupils in the Repeated-Addition group drew arrays, only 35 per cent of the arrays were correct. The fourth informative fact of the analysis was that the Ratio-to-One group used an array correctly to illustrate the distributive property almost as frequently as the Repeated-Addition group.

TABLE XXIX

ANALYSIS OF DRAWINGS FOR ITEM 43 OF THE POST-TEST

Approach	N	Judgeable			Correct	
		Total	Coordinate System	Array	Array	Others
Repeated Addition	111	103	0	83	29	2
Ratio-to-One	103	83	0	34	20	8

Two contrasts are noticeable in Table XXIX. One was that no matter whether the illustration was correct or not, pupils in the Repeated-Addition group tended to use an array to illustrate the problem. For example, 54 pupils drew an incorrect array to illustrate the distributive property. Thus, 65 percent of the Repeated-Addition group's arrays were incorrect. In contrast only 14 pupils, 41 per cent of the drawings, in the Ratio-to-One group committed a similar error.

The other contrasting point is that a relatively large number of the Repeated-Addition group attempted to illustrate the example. A noticeable smaller number of pupils in the Ratio-to-One group made an unsuccessful attempt to illustrate the example. The pupils in this group were more prone to refrain from trying than to be unsuccessful.

Comparison Among Parts of the Test

A second part of the study was concerned with pupil mastery of multiplication facts and understanding of the mathematical properties applicable to the multiplication of whole numbers. This part of the chapter will discuss (1) the difficulty level of the items and (2) how the high-scoring pupils compared to the low-scoring pupils in regard to the items measuring the various mathematical properties.

The analysis of the data attempted to ascertain whether the pupils seemingly had more difficulty learning mathematical properties than multiplication facts will be discussed first.

To determine this, the proportion of correct responses to items that require knowledge of the multiplication facts and items that require understanding of the mathematical properties applicable to multiplication of whole numbers are shown in Tables XXX through XXXV respectively. The proportion of correct responses was found for all pupils irrespective of the approach used. This proportion gave the level of difficulty for each item. When summed for all items and divided by the total number of items in a specific section, the result was the mean level of difficulty for each section.

Summary of the data for the correct responses to items measuring mastery of the multiplication facts is found in Table XXX. The item having the greatest proportion of correct responses was item six. The item having the least proportion of correct responses was item eleven. The mean level of difficulty for the mastery section of the final test was 0.73.

TABLE XXX

PROPORTION OF CORRECT RESPONSES FOR MASTERY OF MULTIPLICATION FACTS: ITEMS 1-18

Item	Proportion Correct	Item	Proportion Correct	Item	Proportion Correct
1	.91	7	.45	13	.92
2	.74	8	.83	14	.59
3	.54	9	.92	15	.87
4	.51	10	.91	16	.58
5	.97	11	.38	17	.87
6	.85	12	.59	18	.76

Summary of data for the proportion of correct responses to the commutative property items is in Table XXXI. Note, except for item 27, the proportion of correct responses to the commutative items is quite high. Item 27 required the pupils to apply their knowledge of the commutative property to a problem. The mean level of difficulty was .67.

The mean level of difficulty for the distributive property items was .38. Summary of the data for the correct response to these items is found in Table XXXII. Item 47 appeared to be the least difficult. This item called for recognition of the distributive property.

TABLE XXXI

PROPORTION OF CORRECT RESPONSES FOR THE COMMUTATIVE PROPERTY ITEMS

Item	Proportion Correct	Item	Proportion Correct	Item	Proportion Correct
22	.98	30	.76	45(a)	.81
24	.92	40(a)	.64	47(a)	.57
27	.39	40(b)	.62	47(d)	.64

TABLE XXXII

PROPORTION OF CORRECT RESPONSES FOR DISTRIBUTIVE PROPERTY ITEMS

Item	Proportion Correct	Item	Proportion Correct	Item	Proportion Correct
19	.30	44	.41	47(f)	.71
32	.22	45(b)	.21		

Summary of the data for the correct response to the associative items is found in Table XXXIII. Item 42 seemed to be the least difficult. The mean level of difficulty for the associative property items was .59.

The proportion of correct responses for each item of the closure property was almost equivalent. Summary of the data pertaining to this property is in Table XXXIV. The mean level of difficulty was .65.

TABLE XXXIII

PROPORTION OF CORRECT RESPONSES FOR ASSOCIATIVE PROPERTY ITEMS

Item	Proportion Correct	Item	Proportion Correct	Item	Proportion Correct
21	.56	41(b)	.44	45(c)	.50
23	.53	42(a)	.89	47(a)	.58
31	.48	42(b)	.77		

TABLE XXXIV

PROPORTION OF CORRECT RESPONSES FOR THE CLOSURE PROPERTY ITEMS

Item	Proportion Correct	Item	Proportion Correct	Item	Proportion Correct
25	.68	35	.62	47(e)	.64

Summary of the data in regard to the correct responses for the multiplicative identity items is in Table XXXV. Item 28 appeared to be the most difficult. The pupils were required to apply their understanding of this property in an example that contained no numerals. The mean level of difficulty was .67.

Summary of the mean level of difficulty for each of the mathematical properties included in the post-test is in Table XXXVI. The mean level of difficulty for the distributive property items was .38. This seemingly indicated that the pupils found these items more difficult than those for the other mathematical properties.

TABLE XXXV

PROPORTION OF CORRECT RESPONSES FOR THE MULTIPLICATIVE IDENTITY ITEMS

Items	Proportion Correct	Item	Proportion Correct	Item	Proportion Correct
5	.97	15	.86	41(a)	.47
8	.83	28	.29	47(c)	.77
13	.93	36	.47		

TABLE XXXVI

MEAN LEVEL OF DIFFICULTY FOR THE VARIOUS ASPECTS OF THE POST-TEST

Aspect	Mean Level	Aspect	Mean Level
Mastery	.73	Commutative	.67
Associative	.59	Distributive	.38
Multiplicative Identity	.67	Closure	.66

Summary of the data for the high and low groups in regard to the level of difficulty for the various parts of the test is in Table XXXVII. Both groups seemingly found the mastery items the least difficult and the distributive items the most difficult.

TABLE XXXVII

MEAN LEVEL OF DIFFICULTY FOR HIGH AND LOW GROUPS FOR THE VARIOUS ASPECTS OF THE POST-TEST

Aspect	Mean Level of Difficulty		
	Total Group	High Group	Low Group
Mastery	.73	.94	.49
Commutative	.67	.83	.38
Associative	.59	.78	.36
Distributive	.38	.50	.23
Closure	.66	.82	.32
Multiplicative Identity	.67	.67	.36

These data indicate that the low group consistently found the various aspects of the post-test more difficult than the high group did.

Retention Test Results

The results of comparisons involving retention test mean scores are presented below. Three tables will be related to the results. All of the tables will report the t-ratio using the total or subtest scores of the retention test for specific approaches. The comparisons were concerned with the mean scores of the total group on (1) the test as a whole, (2) the items measuring understanding of the mathematical properties applicable to the multiplication of whole numbers, and (3) the mastery measuring items. Summaries of the data appear in Tables XXXVIII, XXXIX, and XL respectively.

The data for applying the t-test to the total test means for the retention test is in Table XXXVIII. The mean of the Repeated-Addition group for the total test was 35.79 and that of the Ratio-to-One group was 35.12. The difference between means of 0.66 (35.79-35.12) was in favor of the Repeated-Addition approach. However, note that the mean of the Ratio-to-One group was slightly higher than that obtained on the post-test, 34.80. Additionally, the difference between the means of 0.66 is 0.60 less than it was for the post-test. The t-ratio for the difference between means of 0.59 for 212 degrees of freedom was not significant at the 0.05 level. The null hypothesis of no difference between the retention test mean scores was not rejected.

The data for applying the t-test to the retention test mean scores for the mastery of the multiplication facts is found in Table XXXIX. The mean of the Repeated-Addition group was 18.71 and that of the Ratio-to-One group was 18.26. The difference between means of 0.44 (18.70-18.26) was in favor of the Repeated-Addition approach. The t-ratio for the difference between means of 0.79 for 212 degrees of freedom was not significant at the 0.05 level. The null hypothesis of no difference between the retention test means in regard to mastery of the multiplication facts was not rejected.

The data for applying the t-test to the retention test mean scores for understanding of mathematical properties applicable to multiplication of whole numbers is found in Table XL. The mean of the Repeated-Addition group was 17.01 and that of the Ratio-to-One group was 16.84. The difference between means of 0.17 (17.01-16.84) was in favor of the Repeated-Addition approach. The t-ratio for the difference between the retention test means of 0.23 for 212 degrees of freedom was not significant at the 0.05 level. Thus, the null hypothesis of no difference between the means was not rejected.

TABLE XXXVIII

TOTAL GROUP MEAN SCORES AND t-RATIO FOR ALL ITEMS FOR THE RETENTION TEST

Approach	N	Mean	SD	df	t-Ratio
Repeated-Addition	111	35.79	7.37	110	
Ratio-to-One	103	35.12	9.03	102	
				212	0.59

TABLE XXXIX

TOTAL GROUP MEAN SCORES AND t-RATIO FOR ITEMS MEASURING MASTERY OF THE MULTIPLICATION FACTS FOR THE RETENTION TEST

Approach	N	Mean	SD	df	t-Ratio
Repeated-Addition	111	18.70	13.79	110	
Ratio-to-One	103	18.26	19.55	102	
				212	0.79

These data indicated that neither approach has been demonstrated to be more effective than the other for the retention of multiplication facts or understanding of mathematical properties applicable to the multiplication of whole numbers.

TABLE XL

TOTAL GROUP MEAN SCORES AND t-RATIO FOR ITEMS MEASURING UNDERSTANDING OF MATHEMATICAL PROPERTIES FOR THE RETENTION TEST

Approach	N	Mean	SD	df	t-Ratio
Repeated-Addition	111	17.01	4.71	110	
Ratio-to-One	103	16.84	4.98	102	
				212	0.23

Summary

The results of the statistical test of the hypotheses are summarized below. When the mean score for the total Repeated-Addition group was compared to the mean score for the total Ratio-to-One group, no significant difference was found at the 0.05 level.

The second hypothesis dealt with the comparison of the mean scores of the Repeated-Addition high group and the Ratio-to-One high group. Again, there was no significant difference between the mean scores on any of the three aspects measured.

However, the t-ratio obtained when the mean scores for the mathematical properties for the Ratio-to-One middle group and the Repeated-Addition middle group were compared was slightly higher than the ratio needed for the third null hypothesis to be rejected. The difference was in favor of the Repeated-Addition group.

The fourth hypothesis dealt with the comparison of the mean scores for the low groups. The statistical test resulted in a significant difference between the mean scores on the total test in favor of the Repeated-Addition group. When the two parts of the test, mastery of multiplication facts and mathematical properties, were investigated, it was noted that this significant difference appeared to be located in the mastery of multiplication part of the post-test. No significant difference was found between the mean scores for the two groups for the items measuring understanding of the mathematical properties.

The results of an analysis of the drawings made to illustrate that multiplication distributes over addition revealed some enlightening facts. One of these was that the Ratio-to-One group drew almost as many correct arrays for illustration as did the Repeated-Addition group in spite of the fact that the array was not introduced in the Ratio-to-One approach. A second fact noted was that no pupil used a coordinate system to illustrate that multiplication distributes over addition.

When the data were analyzed to determine whether the pupils seemingly had more difficulty with learning mathematical properties than they did the mastery of multiplication facts, it was found that the mean level of difficulty indicated greater difficulty with the mathematical properties than with the mastery items. Seemingly, the low group had more difficulty with items pretaining to the mathematical properties than did the high group.

When the data from the retention test were analyzed, no significant difference was found at the 0.05 level. These data indicated that neither approach was more effective for the retention of multiplication facts and understanding of mathematical properties than the other.

CHAPTER IV

IMPLICATIONS

Introduction

The present study dealt with (1) the effectiveness of using two different physical referents to illustrate multiplication and (2) the relation between physical referent and pupil learning of multiplication as measured by mean achievement for the following parts of the post-test:

1. Total test
2. Multiplication facts
3. Mathematical properties
4. Commutative property
5. Associative property
6. Distributive property
7. Closure property
8. Multiplicative identity
9. Illustration of the distributive property
10. Retention test

A random sample of 214 fourth grade pupils in eight classes for the 1967-68 school year in the River Falls Public Schools and the New Richmond Public Schools took part in the study. Each class was randomly assigned to one of two approaches. Two sets of programmed instructional material, fifteen lesson each, were constructed by the writer. The corresponding lessons in each set were as alike as possible. The only major difference being in the illustrations being used. One group used the array exclusively as the physical referent to illustrate multiplication; the other used a coordinate system exclusively.

The results of the post-test, constructed by the writer, were subjected to statistical analysis.

Conclusions

The study was designed to test four hypotheses: (1) there is no significant difference between the post-test mean scores of those fourth grade pupils who had been introduced to multiplication by the Repeated Addition approach and those who had been introduced to multiplication by the Ratio-to-One approach, (2) there is no significant difference between the mean scores for understanding of mathematical properties of the high level fourth grade pupils who had been introduced to these properties as related to multiplication by the Repeated Addition approach and those who had been introduced to these properties by the Ratio-to-One approach, (3) there is no significant difference between the mean scores for the understanding of mathematical properties of the middle level fourth grade pupils who had been introduced to these properties as related to multiplication by the Repeated Addition approach and those who had been introduced to these properties by the Ratio-to-One approach, and (4) there is no significant difference between the mean scores for the understanding of mathematical properties as related to multiplication of the low level fourth grade pupils who had been introduced to these properties by the

Repeated-Addition approach and those who had been introduced by the Ratio-to-One approach.

From the available evidence several conclusions seemed warranted.

The first set of conclusions pertained to the first hypothesis. It was found that there was no significant difference between the mean scores for the two approaches when all items on the post-test were compared. Neither was there any significant difference between the mean scores for the two approaches for items measuring mastery of the multiplication facts. When the two approaches were compared as to the understanding of mathematical properties, no significant difference was found between the mean scores.

From the above data one interpretation was made. The interpretation was that during the introductory teaching of multiplication to the total group one approach was not demonstrated to be more effective than the other for the population.

The second set of conclusions pertained to the second hypothesis. There was no significant difference between the mean scores for the Repeated-Addition high group and the Ratio-to-One high group for all items on the post-test. There was no significant difference between the mean scores of the two groups for items measuring mastery of the multiplication facts. Neither was there any significant difference between the mean scores for the two groups for items measuring understanding of mathematical properties.

The third set of conclusions pertained to the third hypothesis. There was no significant difference between the mean scores for the Repeated-Addition middle group and the Ratio-to-One middle group for all items on the post-test and for items measuring mastery of the multiplication facts. Although slight, a significant difference was found in favor of the Repeated-Addition group when the mean scores for the understanding of mathematical properties items were compared. Such a slight difference caused the writer to wonder if the use of a larger number of items might have caused a change in this conclusion.

The fourth set of conclusions pertained to the fourth hypothesis. There was a significant difference between the mean scores of the Repeated-Addition low group and the Ratio-to-One low group, in favor of the Repeated-Addition approach, for the total test. When the two parts of the post-test, mastery of multiplication facts and understanding of mathematical properties were investigated, it was noted that this significant difference appeared to be located in the mastery of multiplication facts part of the post-test. The t-ratio for this part verified that this assumption was true. Although there was a significant difference between the mean scores of the low groups for items measuring mastery of multiplication facts, no significant difference was found between the mean scores for the items measuring understanding of the mathematical properties. From a difference this large, it was assumed that the use of the Repeated-Addition approach might be more effective than the use of the Ratio-to-One approach for the mastery of multiplication facts for the low group of the population.

The data indicate that neither approach appeared to be more effective than the other for the developing of understanding of the (1) commutative property, (2) associative property, (3) distributive property, and (4) closure property for any of the achievement levels. However, the data did demonstrate that the Repeated-Addition approach seemingly was more effective than the Ratio-to-One approach for the developing of understanding and use of the multiplicative identity for the low group of the population.

The illustration most frequently used, both correctly and incorrectly, to illustrate that multiplication distributes over addition, item 43, was the array. The Repeated-Addition group used the array as the physical referent almost exclusively. Similarly, it was used more frequently by the Ratio-to-One group than any other referent. As the array had not been introduced to this group, the frequency of its use seemed unusual. The coordinate system was not used by any pupil to illustrate that multiplication distributes over addition even though it was used exclusively in the programmed material for the Ratio-to-One group. However, only a small proportion of the pupils in both groups illustrated the distributive property correctly. Apparently, either they did not visualize the problem correctly or did not understand what the exercise required. The pupils in the Ratio-to-One group produced noticeably fewer drawings to illustrate the distributive property than did the Repeated-Addition group.

Overall, those items dealing with the mathematical properties were seemingly more difficult than those items dealing with the mastery of multiplication facts. Data for the mean level of difficulty of the items seemingly indicate that the pupils found the distributive property items the most difficult.

The comparisons made between the mean level of difficulty of items for the high and low groups seemingly indicated that the low group consistently found the items more difficult than did the high group.

Only one of the four hypotheses was rejected. However, when the final test scores were divided into three levels, there were two other statistically significant differences found between groups. All of these were in favor or the Repeated-Addition approach. Although the significant difference found between the middle groups was slight, there was one between the mean scores for the understanding of mathematical properties as measured by items 23 through 47. It seemed plausible to question its importance because of this slight difference. On the other hand, the statistical difference found between the two low groups as to the total score and mastery items scores was great.

One possible interpretation of these findings is that these three significant differences are directly attributable to the difference in physical referent used to illustrate multiplication as this was the only manipulated variable in the study. An acceptance of this interpretation would mean that the association between multiplication fact and the array as a referent operated to facilitate the acquisition, retention, and understanding of multiplication facts more than the association between the coordinate system as a referent and the multiplication fact.

However, this interpretation should not be accepted without taking into account certain factors. If the association between referent and multiplication fact did facilitate acquisition and learning of the facts for the low achieving group, then it seemed reasonable that it should have facilitated the understanding of mathematical properties applicable to the multiplication of whole numbers for the group. It also seemed probable that if the Repeated-Addition middle group were superior in regard to mean scores for the mathematical properties items because of the use of the array as a physical referent, then the practice provided for multiplication facts while studying the mathematical properties should have had some bearing on the mean score for the mastery of multiplication facts items. If this were true, it did not provide a significant difference in mean scores between the two middle groups.

However, this was not the case. The use of the array as a referent was shown to be superior only for mastery of the multiplication facts for the low group and for understanding of mathematical properties for the middle group.

Next, there appeared a confounding effect. The second element that seemingly did not support the conclusion that the association between physical referent and multiplication fact facilitated learning, retention, and understanding of multiplication is that the Ratio-to-One group used arrays to illustrate that multiplication distributes over addition, item 43, although they had not been introduced to it in the programmed material. Not a single pupil in the Ratio-to-One group used a coordinate system as the physical referent to illustrate the same item.

The third element that seemingly fails to support the conclusion that association between physical referent and multiplication fact facilitated learning, retention, and understanding of multiplication is that the Repeated-Addition group used an array incorrectly approximately twice as frequently as the group used it correctly to illustrate item 43. In fact, few of the incorrect arrays for items 43 even illustrated the basic multiplication fact, $6 \times 4 = 24$, being used in the exercise.

These inconsistencies suggest that the conclusion that the use of a single physical referent to illustrate multiplication would facilitate acquisition, retention, and understanding of multiplication was not definitely demonstrated by the study.

However, before the conclusion can be definitely accepted or rejected, further research is needed to clarify whatever a relationship may exist between the physical referent used and the multiplication fact or mathematical property being taught.

The findings also showed that pupils were able to use an array to illustrate multiplication even though it was not specifically used during the introductory teaching of multiplication. On the post-test 19 per cent of the pupils in the Ratio-to-One group used the array correctly to illustrate that multiplication distributes over addition even though they had not had experience with the array during the study.

The use of any physical referent exclusively for all achievement levels seemingly has limitations. First, there was some evidence that pupils who had been exposed to the array exclusively used it indiscriminately. On the other hand, pupils who had been exposed to the coordinate system exclusively often failed to make any illustrations unless confident of success. The Repeated-Addition group made 84 arrays to illustrate item 43; out of which only 29 were correct. Secondly, the Repeated-Addition low-achieving group performed significantly better on those items measuring mastery of multiplication facts; yet, the same was not true for those items measuring understanding of mathematical properties. This might suggest that for low-achieving pupils the array is a simple and easily understood referent of the multiplication facts.

In conclusion, then, the findings seem to indicate three major points. First, the learning of the mathematical properties, especially the distributive property, was more difficult than mastery of the multiplication facts for all levels. Second, the use of different physical referents to teach introductory multiplication might be chosen according to purpose and achievement level. And last, the choice of physical referent might be dependent upon the meaning of multiplication to be emphasized.

Theoretical Implications

One of the postulates stated that pupils who are in fourth grade are of an age to be able to learn mathematical properties applicable to multiplication of whole numbers as well as multiplication facts. The mean level of difficulty found for mastery items and that for mathematical properties items did not unequivocally support the postulate. Instead, the findings seemingly indicated that pupils differed as to ability to learn mathematical properties related to multiplication. The achievement levels differed, also, as to their ability to learn the mathematical properties applicable to multiplication of whole numbers. In addition, the findings indicated that the total group found the distributive items more difficult than any other mathematical property items.

Another postulate stated that pupils at the fourth grade level of attainment were aided in the development of intuitive thinking by the use of physical referents and that some physical referents might be more effective than others. No significant difference was found between the total group using the Repeated-Addition approach and the total group using the Ratio-to-One approach. Thus, this study produced no evidence of a significant relationship between physical referent used and the learning of multiplication for the population.

However, when the various achievement levels using the Repeated-Addition approach were compared to their respective levels using the Ratio-to-One approach, a relationship was found for the low-achieving level. The mean score of the Repeated-Addition low group was significantly superior to that of the Ratio-to-One low group. This study provided some evidence of a significant relationship between physical referent and learning of multiplication facts for the low group of the population.

The third postulate stated that programmed instructional material might be effective for all pupils. This study gives little support to this postulate. Low-achieving pupils found items requiring reading of words more difficult than those items requiring reading of numerals; however, the same difference did not apply to the high-achieving pupils. Thus, it might be assumed that programmed material for the learning of mathematical properties was not equally as effective for all levels of the population.

The above statements that no evidence was found to verify the postulate that all fourth grade pupils are able to learn mathematical properties applicable to multiplication of whole numbers to some degree do not mean that the postulate is not true. Instead, they mean that this study did not produce any evidence that it was true for all fourth grade pupils in the population. However, such statements cast doubt on the validity of the postulate for the teaching of introductory multiplication to all fourth grade pupils.

The results of this study were in line with the results of other studies that young children many times do have difficulty with learn- the mathematical properties applicable to multiplication of whole numbers. And secondly, however, that many fourth grade pupils are able to understand those mathematical properties applicable to multiplication of whole numbers.

Implication for Future Research

Effort to determine what physical referents are most effective for the teaching of the various mathematical properties applicable to multiplication of whole numbers to the various achievement levels needs to be continued. The results of the present study are not justification for the cessation of effort to determine what physical referents are most effective for learning multiplication by fourth grade pupils.

The portion of the study that revealed a significant relationship between the use of an array and the learning of mathematical properties but not between the use of an array and the learning of multiplication facts for the middle group was surprising. It would seem reasonable that the finding should of held true for both aspects. Thus, further research should be done in regard to this portion of the study.

In addition, the portion of the study that revealed a significant relationship between the use of an array and the learning of multiplication facts but not between the use of an array and the learning of mathematical properties for low-achieving pupils was also surprising. It, also, would seem reasonable that the findings should of held true for both aspects. Thus, further research should be done in regard to this portion of the study.

Because of the findings cited above, it seems reasonable that further research should be done to determine the effectiveness of varying the physical referent according to the aspect of multiplication being taught.

And finally, further research needs to be conducted to determine the effectiveness of the use of programmed instructional material to teach introductory multiplication to low-achieving fourth grade pupils.

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APPENDIXES

APPENDIX A

May 10, 1967

Dear Sir:

During the fall term of the 1967-68 school year, a study in regard to the teaching of introductory multiplication is to be conducted. Elementary school systems located within a sixty mile radius of Wisconsin State University-River Falls are being contacted to see if they are willing to take part in the study.

The research will be conducted at the fourth grade level. Programmed material that coincides with the subject content of the textbook is to be distributed to the participating classes. Teachers will be requested to use the material whenever a new multiplication fact is introduced.

In order that the research might be done with a random sampling from the area, would you answer the accompanying questionaire and mail it to me. A self-addressed envelope has been included for your convenience.

Thank you.

Sincerely yours,

Mrs. Naunda Tietz
502 West Maple
Stillwater, Oklahoma 74074

RESEARCH QUESTIONNAIRE

____
School System

____
Administrator

Would you be willing for your school system to be part of the population from which the sample group is drawn?

Please check____ ____
Yes No

If you answer "yes" to the previous question, the following information is requested.

1. How many fourth grade classes are there in your school system from which to select participating classes? ________

2. What arithmetic text will be used in your school system during the 1967-68 school year? ________

APPENDIX B

REPEATED ADDITION

Lesson 1:

1. Bill's mother made breakfast for the campers. She needed eggs for five boys. Each boy was to have two eggs. Bill's mother thought, "I need to boil____eggs."

10

Think of some ways Bill's mother could find out how many eggs she needed to boil. Check to see if any of your answers agree with the ideas suggested below.

One fourth grade class suggested three different ways of finding how many eggs Bill's mother needed. These are given in exercises 2, 3, and 4. Read each exercise, study the drawing, and try to answer the question.

2. Jill said, "Bill's mother could have made a drawing that showed five rows of eggs with two eggs in each row. Then she could have counted.

Bill's mother could get the right answer by c________.

1 2
0 0
3 0 0 4
5 0 0 6
7 0 0 8
9 0 0 10

---

counting

3. John said that he thought Bill's mother could have added the number of eggs in each row.

0	0	2
0	0	2
0	0	2
0	0	2
0	0	2
		10

4. Bob said, "Bill's mother could look at the drawing and think to herself, 'Five twos are 10.' she would know right away that she needed to boil ten eggs.

She could look at the array, notice the number of eggs in each row, and notice the number of rows.

```
   1 2
1 |0 0
2 |0 0
3 |0 0
4 |0 0
5 |0 0
```

Bill's mother could get the right answer. She thought_____2's are 10.

---

five

When you think with numbers such as 5 and 2 and get 10, you are multiplying.

5. Look at these drawings. What two numbers can you think with to get the total number in the drawing? Write and circle the numerals. The first one is done for you.

```
      1 2 3 4
a.  1|* * * *        b. |0 0 0        c. |0
    2|* * * *           |0 0 0           |
    3|* * * *           |0 0 0           |0
```

---

b. 3,3 c. 2,1

6. A drawing like Jill, John, and Bob made is called an array. Each drawing for exercise #6 was an array, too. Do you think that you know what an array is?

Draw a line around each drawing below that is an array.

```
a. X X X X    b.   0 0 0 0   c.  # #    d. +----+----+
                    0 0 0       # # #      |    |    |
   X X X X       0 0 0                     +----+----+
                                           |    |    |
                                           +----+----+
```

---

circle a and d

7. Mary looked at this array. She said, "Five and five and five are fifteen. I can add with the help of an array as well as using it to help me learn to multiply.

```
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
```

8. Jim looked at the same array. He said, "Three fives are fifteen." Was Jim adding or multiplying?

He was ______________.

---

multiplying

9. Mary said, "Jim found the answer faster by multiplying than I did by adding." Maybe ______________ is often faster than adding.

---

multiplying

Jim could have written three fives are fifteen like this

$$3 \times 5 = 15$$

10. Look at each array. For each one what two numbers do you think with to get the total number in the drawing. Write and circle the numerals. How many things are there in each drawing? The first one is done for you.

```
a.      (4)
     * * * *
  (2)* * * *

b.  X X X X
    X X X X
    X X X X
 __ X X X X

c.  o o o
    o o o
    o o o
    o o o
 __ o o o

d.  00
    00
    00
    00
    00
 __ 00
```

2 x 4 = 8 ___ x ___ = ___ ___ x ___ = ___ ___ x ___ = ___

---

b. 4, 4
4 x 4 = 16

c. 5, 3
5 x 3 = 15

d. 6, 2
6 x 2 = 12

REPEATED ADDITION

Lesson 2:

1. Probably, since you can remember playing games you have been using counting numbers. The set of counting numbers can be written using numerals as { 1, 2, 3, 4, . . . }. The three dots, . . . , mean to continue the numerals on and on. The numeral 5 represents a counting number. The numeral 35 represents a counting n___________.

---

number

2. If to the set of counting numbers you add zero (0), you will have the set of whole numbers. To get the set of whole numbers one needs to add _________ to the set of counting numbers.

0

3. The set of numbers written as { 0, 1, 2, 3, 4, . . . } represents a set of ____________ numbers.

---

whole

4. Using whole numbers we can do certain mathematical operations. An operation is a way of associating with two numbers a third number called the result. The operation called addition was used when Jim associated the numbers two (2) and three (3) and got the result five (5).

   When doing the operation addition, Bill associated the numbers two (2) and four (4) and got the result __________.

---

six

5. Using whole numbers we can also do an operation called multiplication. If Mary used the multiplication operation to associate the numbers two (2) and three (3), she would get the result six (6).

5. continued

Next, Mary associated the numbers two (2) and four (4) and got the result eight (8). Mary was using the__________operation.

---

multiplication

6. The results for associating the same numbers in the operations of multiplication and addition_________________the same.
were were not

---

were not

7. When we associate the number eight (8) with the two numbers two (2) and four (4), we are using the___________called multiplication.

---

operation

8. When we associate the number six (6) with the two numbers two (2) and four (4), we are using the operation called_____________.

---

addition

9. Although we used <u>the same two numbers</u> in #7 and #8, the results <u>were not</u> the same. We got a <u>unique result</u> for each operation. The <u>unique</u> result means that in a given operation there is only one right number to associate with any two numbers (pair of numbers).

Jane used the operation multiplication. She associated the numbers three (3) and three (3) with the result_____________.

---

nine

10. Bill used the multiplication operation to associate the numbers three and two. He got the unique number_____________.

---

six

11. Bill's unique number six (6) and Janes's unique number (9) are in the set of whole n_____________.

---

numbers

12. The pairs of numbers used in #8 and #9 are w________numbers.

---

whole

13. The result from associating the two numbers (pair of numbers) in #8 and #9 was a w__________ number.

---

whole

14. In mathematical operation whenever two whole numbers are associated and the result is a whole number, we say that the closure property is true for that operation.

When we use an operation in which the result is always the same kind of number as the pair of numbers, the __________ property is true for that operation.

---

closure

15. Peter said, "The closure property must be true for addition because when I add two whole numbers, I get a whole number."

Peter was right. The closure property is true for addition of w__________ numbers.

---

whole

16. Tom said, "The same thing is true for multiplication of whole numbers. When I multiply a whole number by another whole number, I get a whole number for the answer. The__________ property is true for the multiplication of whole numbers."

---

closure

17. Work this exercise. Is Tom's statement true every time? Did you get a whole number for the answer when you multiplied a whole number by a whole number?

a. 2 x 2 = ______ c. 2 x 1 = ______

b. 3 x 3 = ______ d. 4 x 2 = ______

---

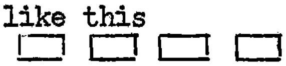
a. 2 x 2 = 4 b. 3 x 3 = 9 c. 2 x 1 = 2 d. 4 x 2 = 8

18. We didn't work very many examples to check Tom's statement. But, it is true that the closure property holds for the multiplication of whole numbers. Sometimes we say it like this. The multiplication of whole numbers is closed.

REPEATED ADDITION

Lesson 3:

1. Mary and Jane were arranging some stamps to be placed in the stamp book. Mary arranged hers in a row like this



Jane arranged here in a column like this

2. If the stamps were arranged so that each stamp was placed next to the one ahead of it, the stamps were arranged in a__________

---

row

3. If the stamps were arranged so that each stamp was placed below or above the stamps already on the table, the stamps were arranged in a __________.

---

column

4. It is important in our work in multiplication today that we write the numeral representing the number of rows first in a multiplication fact and the numeral representing the number of columns second.

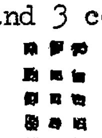
In this array there are______rows and 4 columns.

```
     4
   0000
   0000
   0000
3  0000
```

3

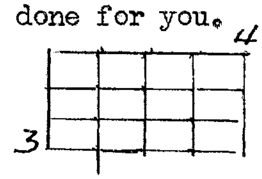
5. Mary looked at the stamps in her stamp book. She said, "The stamps are arranged in 5 rows and 3 columns. This shows that five 3's equals 15.

$5 \times 3 = 15$

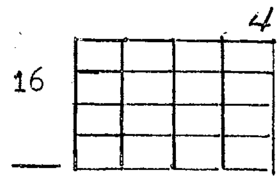


6. Describing these arrays as Mary did. Show that your answer is correct by adding. The first one is done for you.

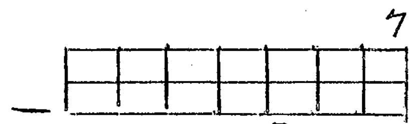
a. Three 4's equals 12.
4 + 4 + 4 = 12



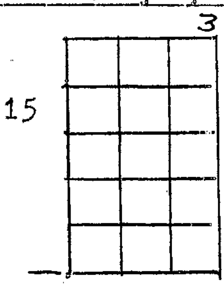
b. ______4's equals 16.
4 + 4 + 4 + ____ = 16



c. ______7's equals 14.
7 + ____ = 14



d. ______3's equals 15.
3 + 3 + 3 + 3 + ____ = 15



---

b. four
4

c. two
7

d. five
3

7. Jill said, "It is easier to write the addition example, four and four and four equals twelve with numerals like this, 4 + 4 + 4 = 12, than it is to write it with words. I can write the multiplication fact, 'three fours equals twelve' with numerals, too. It is written like this using only numerals and signs.

3 x 4 = 12.

8. When written as in the exercise above, a multiplication fact is read, "three fours equals 12."

Using numerals as Jill did, describe the following multiplication facts.

a. Three fours equals twelve. ______ __ ______ = ______

b. Six two's equals twelve. ______ __ ______ = ______

c. Two seven's equals fourteen. ______ __ ______ = ______

---

a. 3 x 4 = 12 b. 6 x 2 = 12 c. 2 x 7 = 14

9. In the multiplication fact, 2 x 7 = 14, each of the numerals 2 and 7 is called a <u>factor</u>.

9. continued

In the multiplication fact 4 x 4 = 16, each of the 4's is a f________.

---

factor

10. In the multiplication fact 6 x 2 = 12, ________ and ________ are factors.

6, 2

11. The answer in the fact 2 x 7 = 14 is called the product.

In the multiplication fact 4 x 4 = 16, the product is________.

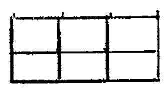
16

12. In the multiplication fact 6 x 2 = 12, ________ is the product.

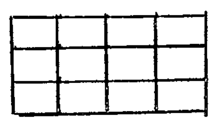
12

13. Study the arithmetic facts given below. Decide which ones are multiplication facts. Circle the product of each multiplication fact.

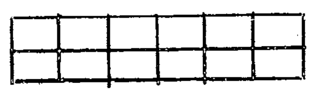
a. 3 + 3 = 6



b. 3 x 4 = 12



c. 2 x 6 = 12



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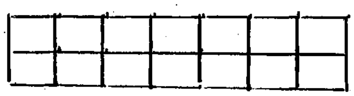
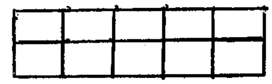
a. No b. (12) c. (12)

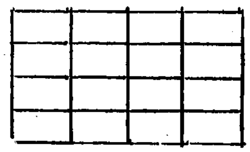
14. Using numerals to write the multiplication facts for the arrays.

a. 

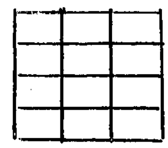
__2__ x __7__ = 14

b. 

______ x ______ = __

c. 

______ x ______ = __

d. 

______ x ______ = __

---

b. 2 x 5 = 10 c. 4 x 4 = 16 d. 4 x 3 = 12

15. The operation of multiplication can be performed on just a pair (two whole numbers) at one time. An operation that is done on just two numbers is called a binary operation.

Multiplication is a b ______operation because I can only multiply two number at one time.

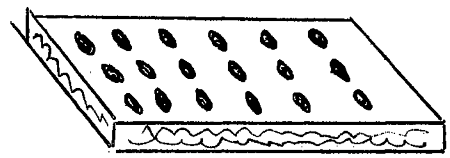
---

binary

REPEATED ADDITION

Lesson 4:

1. Tom bought a box of candy. He said, "If I look at the box one way there are three 6's. If I look at the box another way, there are six 3's. But, there are always 18 pieces of candy. Is Tom right? ____________
yes no



---

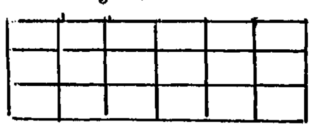
yes

2. Three 6's equal ____________ 3's.

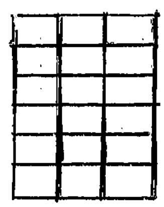
---

1. yes 2. six

3. Mary illustrated that three 6's equals six 3's. She drew these arrays. What facts would Mary write under each array?



a. _____ x _____ = 18



b. _____ x _____ = 18

---

a. 3 x 6 = 18 b. 6 x 3 = 18

4. Tom and Mary showed that __________ multiplication facts might use the same factors. one two

---

two

5. You might have heard these two multiplication facts called pairs of facts. Write two pairs of facts you could use to illustrate a dozen eggs in an egg carton.

a. 3 x _____ = 12

4 x _____ = 12

b. 2 x _____ = 12

_____ x 2 = 12

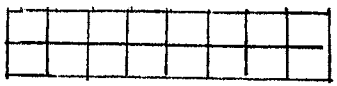
---

a. 4
3

b. 6
6

6. Write the pairs of facts which each array below may be used to illustrate. The first pair is done for you. If you don't know the answer, you may need to count.

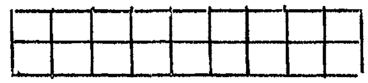
a.



2 x 8 = 16

8 x 2 = 16

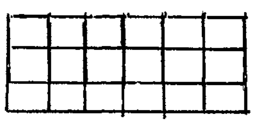
b.



_____ x _____ = _____

_____ x _____ = _____

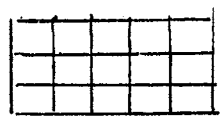
c.



_____ x _____ = _____

_____ x _____ = _____

d.



_____ x _____ = _____

_____ x _____ = _____

---

b. 2 x 9 = 18
9 x 2 = 18

c. 3 x 6 = 18
6 x 3 = 18

d. 3 x 5 = 15
5 x 3 = 15

REPEATED ADDITION

Lesson 5:

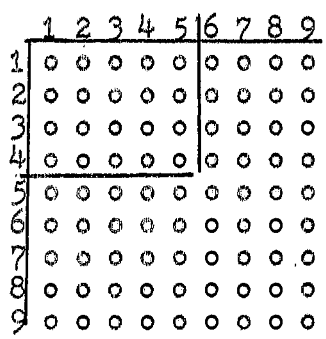
1. If two multiplication facts have the same factors, their products are_____________________.
   equal   not equal

---

equal

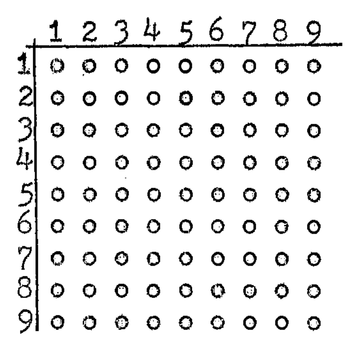
Jane did not want to draw a new array each time she found the product of two factors. She used an array like the one shown in this exercise to find the product of the pair of multiplication facts in which 4 and 5 are factors. She drew the line to help her. Then she counted the number of dots inside the lines. Finally she wrote this pair of facts:

$4 \times 5 = 20$ $\qquad$ $5 \times 4 = 20$



2. Write the pairs of facts Jane discovered using the factors given below (If you use a different colored crayon for each pair of facts, you can use one chart).

   a. for 3 and 7   b. for 3 and 8



2. continued

c. for 4 and 6

_____ x _____ = _____

_____ x _____ = _____

d. for 5 and 1

_____ x _____ = _____

_____ x _____ = _____

---

a. $3 \times 7 = 21$ b. $3 \times 8 = 24$ c. $4 \times 6 = 24$ d. $5 \times 1 = 5$
$7 \times 3 = 21$ $8 \times 3 = 24$ $6 \times 4 = 24$ $1 \times 5 = 5$

3. In all of the exercises that Jane did, the number of dots for each pair of facts (i.e., $4 \times 5 = 20$ and $5 \times 4 = 20$). This shows that multiplication of whole numbers is <u>commutative</u>. An operation is commutative if the order of the factors may be changed without changing the result.

Jane could change the order of the factors in the multiplication fact $3 \times 8 = 24$ to $8 \times 3 = 24$. The result was not changed. Multiplication of whole numbers is c__________.

---

commutative

4. Do the following facts show that multiplication of whole numbers is commutative?

a. $3 \times 7 \stackrel{?}{=} 7 \times 3$ __________ yes no

b. $3 \times 8 \stackrel{?}{=} 8 \times 3$ __________ yes no

c. $4 \times 6 \stackrel{?}{=} 4 \times 6$ __________ yes no

d. $5 \times 1 \stackrel{?}{=} 1 \times 5$ __________ yes no

---

a. yes b. yes c. no d. yes

5. Multiplication of whole numbers is c__________ because the order of the factors may be changed without changing the result.

---

commutative

REPEATED ADDITION

<u>Lesson 6:</u>

1. John said, "I think that the product in a multiplication example is always larger than either of the factors." He drew these arrays and wrote the multiplication fact for each array.

```
a. * * * * *      b. * * * * * *      c. * * * * * * *
   * * * * *         * * * * * *         * * * * * * *
   * * * * *         * * * * * *         * * * * * * *
   * * * * *         * * * * * *
```

a. ____ x ____ = ____ b. ____ x ____ = ____ c. ____ x ____ = ____

```
d.                e.      *          f.
   * * * * *              *             * * * *
                          *
                          *
                          *
                          *
```

d. ____ x ____ = ____ e. ____ x ____ = ____ f. ____ x ____ = ____

a. 4 x 5 = 20   b. 4 x 6 = 24   c. 3 x 7 = 21   d. 1 x 5 = 5

e. 6 x 1 = 6   f. 1 x 4 = 4

John was ________________.
right   wrong

---

wrong

2. The product for a multiplication example __________ always
is   is not
larger than either of the factors.

---

is not

3. When John multiplied 5 x 1, he got the product__________.

---

4. The answer for 5 x 1 was 5. This is one of the factors, too. So when John multiplied 5 x 1, the answer was the same as the factor ________.

5

5. When John multiplied 1 x 6, the answer was ______. Six was one of the factors. The other factor was 1.

6

6. If John had multiplied 9 x 1, he would have got the answer of ________. Nine is one of the factors. The other factor is ________.

9, 1

7. Everytime John multiplied when one of the factors was 1, the product was the same as the other f____________.

---

factor

8. Work these examples. Check to see if what factor you have written in #7 is true.

a. 2 x 1 = _____ b. 3 x 1 = _____ c. 4 x 1 = _____

1 x 2 = _____ 1 x 3 = _____ 1 x 4 = _____

---

a. 2 b. 3 c. 4

2 3 4

d. 5 x 1 = _____ e. 6 x 1 = _____ f. 7 x 1 = _____

1 x 6 = _____ 1 x 6 = _____ 1 x 7 = _____

g. 8 x 1 = _____ h. 9 x 1 = _____

1 x 8 = _____ 1 x 9 = _____

---

d. 5 e. 6 f. 7 g. 8 h. 9

5 6 7 8 9

9. Both John and you have discovered the <u>multiplicative</u> <u>identity.</u> One is the multiplicative identity. In other words when a number is multiplied by one, the answer is <u>always</u> the same as the other number. (i.e., 5 x 1 = 5)

The numeral __________ represents the multiplicative identity.

1

REPEATED ADDITION

Lesson 7:

1. Ben was not sure that his answer to this multiplication fact was correct. $4 \times 8 = 32$

   He drew these arrays to check his answer.

```
     4              4
  |X X X X       |X X X X
  |X X X X       |X X X X
  |X X X X       |X X X X
 4|X X X X      4|X X X X

     16      +      16   =  32
```

   Each of the arrays shows that _____ x _____ = 16.

$4 \times 4 = 16$

2. Ben added the product from the first array to the product from the second array. 16 + _____ = 32.

16

3. Alice said, "I know that $4 \times 3 = 12$ and $4 \times 5 = 20$. If I add 12 and 20, I get 32. So I have shown that $4 \times 8 = 32$. I will draw the arrays to illustrate my work. I renamed the factor 8."

```
       3                 5
    |X X X         |X X X X X
    |X X X         |X X X X X
    |X X X         |X X X X X
 4  |X X X       4 |X X X X X
        12       +          20
```

4. Both Ben and Alice renamed the factor 8. Ben renamed the factor 8 to (4 and _____).

4

5. Alice renamed the factor 8 to (3 + ____).

5

6. Imagine that each of the numerals below represents the second factor in a multiplication fact. Can you rename each of the factors? The first one is done for you.

a. 5

1 + 4

2 + 3

b. 3

______

c. 9

______

______

______

______

---

b. either of these

1 + 2

2 + 1

c. any of these

1 + 8

8 + 1

2 + 7

7 + 2

3 + 6

6 + 3

7. Terry said, "I didn't rename 8 like either Ben or Alice. I renamed 8 as (2 + 6). Ben's example will now look like this."

4 x 8 =

4 x (2 + 6) =

"I used the parenthesis ( ) so that I knew what numerals were used in the renaming of ____________."

8

8. Peggy didn't believe that Terry could get the right answer so Terry drew these arrays to show her.

```
         2              6
      |0 0     |0 0 0 0 0 0
      |0 0     |0 0 0 0 0 0
      |0 0     |0 0 0 0 0 0
    4 |0 0   4 |0 0 0 0 0 0
         8  +  |            24
```

4 x 2 = 8   4 x 6 = 24   8 + 24 = 32

24

9. Peggy said, "Yes, Terry can get the right answer by renaming 8 as (2 + 6)."

8

10. John said, "Let's try another example and see how we can rename the second factor. Let's use 4 x 7. I will draw the array that shows 4 x 7. John drew this array.

4 x 7 = ______

```
                  7
      |0 0 0 0 0 0 0
      |0 0 0 0 0 0 0
      |0 0 0 0 0 0 0
    4 |0 0 0 0 0 0 0
                    28
```

28

11. Andy said, "I'm going to rename 7 as (4 + 3). Now the array will look like this."

```
          4           3
     |0 0 0 0     |0 0 0
     |0 0 0 0     |0 0 0
     |0 0 0 0     |0 0 0
    4|0 0 0 0   4 |0 0 0
```

a. ____ + 12

Show with numerals how Andy renamed 7.

b. 4 x 7

4 x (4 + ____)

---

a. 16 b. 3

12. Andy continued by saying, "First I multiplied 4 x 4 and then 4 x 3. Then I added the two products."

12. continued

Can you complete the following exercise to show Andy's work?

a. 4 x 7 = 4 x (4 + ____)

b. = (4 x 4) + (4 x ____)

c. = ____ + 12

d. = 28

---

a. 3 b. 3 c. 16

13. Jerry looked at Andy's work and said, "You renamed 7 as (4 + 3). I am going to rename 7 as (5 + 2). My work will look like this."

```
        5            2
  |0 0 0 0 0     |0 0
  |0 0 0 0 0     |0 0
  |0 0 0 0 0     |0 0
 4|0 0 0 0 0    4|0 0
         20    +    8
```

a. 4 x 7 = 4 x (5 + ____)

b. = (4 x ____) + (4 x ____)

c. = 20 + 8

d. = ____

---

a. 2 b. 5, 2 d. 28

14. John laughed. "Both Jerry and Andy are right. But I see another way the factor 7 can be renamed. We can rename 7 like this."

a. 4 x 7 = 4 x (1 + 6)

b. = (4 x 1) + (4 x ____)

c. = 4 + ____

d. = 28

---

b. 6 c. 24

The children had discovered something new about multiplication of whole numbers. By the time you have the next lesson will you have discovered what it is?

REPEATED ADDITION

Lesson 8:

1. Did you discover what the new property of multiplication of whole numbers is? If not let's see if Alice's work will help you.

2. Alice illustrated the multiplication fact 3 x 9 by this array first. Later she renamed the factor 9 and drew different arrays.

```
              9
  |0 0 0 0 0 0 0 0 0
  |0 0 0 0 0 0 0 0 0
3 |0 0 0 0 0 0 0 0 0
                 27
```

Alice's array showed that

3 x ________ = 27.

9

3. This is the second illustration that Alice made.

```
        5                    4
  |0 0 0 0 0          |0 0 0 0
  |0 0 0 0 0          |0 0 0 0
3 |0 0 0 0 0        3 |0 0 0 0
             15    +            12
```

a. Alice said, "I renamed the factor 9 as (5 + ________).

b. I multiplied 3 x 5 = ________.

c. I multiplied 3 x 4 = ________.

d. I added 15 and 12 and got the sum of ______."

---

a. 4 b. 15 c. 12 d. 27

4. Alice got the same result in #2 and #3. In #2 she showed that 3 x 9 = ____________.

In #3 she renamed the factor 9 to (5 + 4). She then said that 3 x (5 + 4) = 27.

Alice worked the example like this

4. continued

$$3 \times (5 + 4) = (3 \times 5) + (3 \times 4)$$
$$= 15 + \_\_\_\_$$
$$= 27$$

---

27 12

5. Alice got the same result when she added first (5 + 4) to get the factor 9 and then multiplied as she did when she multiplied 3 x 5 and 3 x 4 first and then added the products.

6. Alice had discovered the important property. It is known as the distributive law. We say that multiplication distributes over addition when it doesn't make any difference whether you add to get the factor first or you multiply each part first and then add.

7. Mary used the distributive law to show that her answers were correct.

a. $5 \times 5 = \_\_\_\_$

$$5 \times (3 + 2) = (5 \times 3) + (5 \times \_\_\_\_)$$
$$= \_\_\_\_ + 10$$
$$= \_\_\_\_$$

---

25 2 15 25

b. $5 \times 6 = \_\_\_\_$

$$5 \times (4 + 2) = (5 \times \_\_\_\_) + (5 \times 2)$$
$$= 20 + \_\_\_\_$$
$$= 30$$

---

30 4 10

8. Paul used the distributive law to do this exercise.

$$7 \times 4 = (\_\_\_\_ \times 1) + (7 \times 3)$$
$$= 7 + \_\_\_\_$$
$$= 28$$

---

7 21

9. Mary said, "If I can say 8 x 4 can be renamed 8 x (2+2), then I can say that 8 x (2+2) can be renamed as 8 x 4."

Was Mary right? ______________
yes no

---

yes

10. Can you fill in the answers to this exercise?

a. 9 x 3 = ________

b. (9 x 1) + (9 x 2) = 9 + ______ = 27

c. (8 x (2 + 2) = (8 x _____) + (8 x 2) = 16 + 16 = 32

d. 8 x 4 = __________

---

a. 27 b. 18 c. 2 d. 32

11. The distributive property holds for the multiplication of whole n______________.

---

numbers

REPEATED ADDITION

Lesson 9:

1. Jane said, "Now that we have learned about the distributive property for whole numbers, we won't have any trouble finding out how much four 9's will be. We know that we can rename the factor 9. I am going to rename 9 as (5 + 4). I know that 4 x 5 = 20. I also know that 4 x 4 = 16. Then if I add 20 and 16, I get 36." The answer is ______
right wrong

---

right

Check the work with these arrays.

```
        5               4
   |X X X X X       |X X X X
   |X X X X X       |X X X X
   |X X X X X       |X X X X
  4|X X X X X      4|X X X X

      ____     +      16
```

20

2. When Jane added 20 and 16, she got 36. Her answer was ______.
right wrong

---

right

3. Mary said, "If 4 x 9 = 36, then ____ x 4 = 36 because multiplication of whole numbers is commutative."

9

If Jane had used the multiplication fact 9 x 4, she could have renamed the factor 4 as (1 + 3) or(2 + ____).

2

4. Peter asked, "Who can look at these arrays and tell me the multiplication fact?"

```
   1                 6
  |0           |0 0 0 0 0 0
  |0           |0 0 0 0 0 0
  |0           |0 0 0 0 0 0
  |0           |0 0 0 0 0 0
 5|0          5|0 0 0 0 0 0
```

$\underline{5}$ + ____ = 35

30

5. "That's easier. All of us can do it," remarked Gary.

a. $(5 \times 1) + (5 \times 6) =$ ____ + ____.

b. $(5 \times 1) + (5 \times 6) = 5 \times (1 +$ ____).

$5 \times (1 + 6) = 5 \times 7$

---

a. 5, 30 b. 6

6. "Gary, did you notice that we used the multiplicative identity?" asked Peter.

"Yes," answered Gary. "The multiplicative identity was one in the multiplication fact $5 \times 1 =$ ____."

5

7. Mary reminded the class that multiplication of whole numbers was c__________ because 5×7 equals 7×5.

---

commutative

"Today we have one more new multiplication fact to learn," commented Miss Brown. "The multiplication factors are 6×6. Does anyone think he know the answer?"

8. Henry said, "I think that $6 \times 6 = 36$. We have learned that $6 \times 2 = 12$. We have learned that $6 \times 4 = 24$. So if I ______________ 12 and 24, I get 36."
add multiply

---

add

9. Helen said, " I can prove Henry is right by renaming the factor 6 as (1 + 5). I know that 6 x 1 = ____. I know that 6 x 5 = 30. When I add 6 and 30, I get the same answer as Henry. The answer is 36."

6

10. Can you complete this exercise?

a. 6 x 6 = ______

b. 5 x ______ = 35

c. ______ x 9 = 36

---

a. 36 b. 7 c. 4

REPEATED ADDITION

Lesson 10:

1. Henry said, "I have learned to check the product to a multiplication fact by renaming the fact, I will do it like this."

   a. Multiplication fact 4 x 9 = ______

   b. Rename one factor 4 x (4 + ____)

   c. Multiply each numeral of the factor by the other factor.

   (4 x 4) and (4 x 5)

   d. Add the two products 16 + 20 = ______

   e. This way of working the example could be called "multiplying twice and adding."

---

a. 36 b. 5 d. 36

2. Henry has learned that multiplication distributes over a______________.

---

addition

3. Henry______________check the product of a multiplication fact
   can can not
   by making use of his knowledge about the distributive property.

---

can

Miss Brown said, "We have some new multiplication facts to learn today. Let's use Henry's method to check our answers."

4. Peggy drew a new array on the board.

She said, "I have shown that five ________ are 40.

```
  |0 0 0 0 0 0 0 0
  |0 0 0 0 0 0 0 0
  |0 0 0 0 0 0 0 0
  |0 0 0 0 0 0 0 0
5 |0 0 0 0 0 0 0 0
```

8's

5. Paul checked Peggy's work by drawing lines to separate Peggy's array like this

```
     4         4
  o o o o | o o o o
  o o o o | o o o o
  o o o o | o o o o
  o o o o | o o o o
5 o o o o | o o o o
```

"I will use Henry's idea to check Peggy's work," said Paul.

a. Multiplication fact
   5 x 8 = 40

b. Rename one factor
   5 x (4 + ________)

c. Multiply
   5 x 4 = 20
   and
   5 x 4 = 20

d. Add the answers
   20 + 20 = ________

Peggy's answer is right.

---

b. 4 d. 40

6. If 5 x 8 = 40, then 8 x 5 = 40 because multiplication of whole numbers is c____________________.

---

commutative

7. Some of the children were not sure that Peter had written the answer on the board for this multiplication fact 6 x 7 = 42. They decided to draw an array and check using Henry's method.

8. Karen looked at this array and said, I am going to rename the factor 7 as (2 + 5)." Draw lines to show what Karen did.

```
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
```

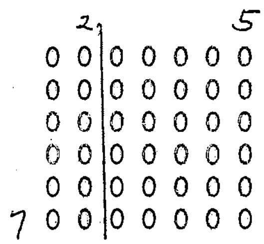
Karen thought.

a. 6 x 7 = (6 x 2) + (6 x 5)

b. = ________ + 30

c. = 42

8. Continued



b. 12

9. Write the answers to this exercise.

a. 6 x 7 = ____________

b. 7 x 6 = ____________

c. 5 x 8 = ____________

c. 8 x 5 = ____________

a. 42 b. 42 c. 40 d. 40

REPEATED ADDITION

<u>Lesson 11:</u>

1. We know that an array can be used to illustrate more than one multiplication fact. This array could be used to illustrate two multiplication facts. If I look at it one way, it illustrates that

   a. 5 x 9 = ______

   If I look at it another way it illustrates that

   b. 9 x 5 = ______

```
0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0
```

---

a. 45 b. 45

2. Sometimes the array has the same number of ______ as it has columns. In that case there is only one multiplication fact to learn.

---

rows

3. This array shows that there are ______ rows and ______ columns.

```
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
```

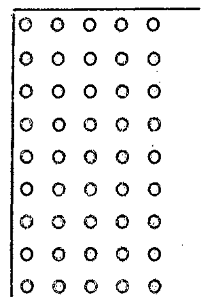
---

7 rows 7 columns

4. The array in #3 illustrated that 7 x 7 = ______.

49

5. Jim was absent from school nine weeks. The children wanted to know how many days Jim was absent. Bob knew that there were five school days each week. So they decided to draw this array.



a. Jim was absent ______ days.

b. 9 x ______ = 45.

---

a. 45 b. 5

6. In order to play a card game, each player needed eight (8) cards. Tom said, "We need to have ______ cards if six (6) of us are going to play."

Tom knew that five 8's was equal to 40. He said, "I will add one more set of 8. That will make the result 48.

48

7. a. 6 x 8 = ______

b. 8 x _____ = 48

---

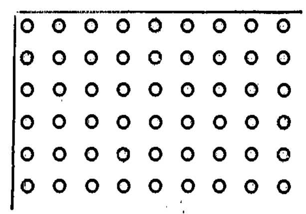
a. 48 b. 6

REPEATED ADDITION

Lesson 12:

1. Mary said, "We have 9 girls in each Girl Scout troup. There are six troups in this school. I wonder how many girls are Girl Scouts. If I draw an array with 9 girls in each row, I will need to draw ______ rows." Mary's array looked like this

6 x 9 = ______



6, 54

2. Jane said, "If we had 6 girls in each Girl Scout troup and had 9 troups, we would have had ______ Girl Scouts, also.

54

3. Miss Brown said, "The other day we learned that six 8's was equal to ______. Today we want to find out how much seven 8's would be. Does anyone know how we can find the answer?"

48

4. Tom said, "All we need to do is add one more set of 8's to the answer for six 8's. We know that six 8's was equal to 48. If we add 48 and 8, we get 56. Seven 8's must be equal to 56."

a. 6 x 8 = ______

b. 7 x 8 = ______

a. 48 b. 56

5. What multiplication fact does each of the arrays illustrate?

a.

```
oooooooo
oooooooo
oooooooo
oooooooo
oooooooo
oooooooo
oooooooo
```

______ x ______ = 56

b.

```
ooooooo
ooooooo
ooooooo
ooooooo
ooooooo
ooooooo
ooooooo
ooooooo
```

______ x ______ = 56

---

a. 7 x 8 = 56 b. 8 x 7 = 56

6. Complete this exercise.

a. 6 x 8 = ______

b. ______ x 6 = 48

c. ______ x 7 = 56

d. 7 x 8 = ______

---

a. 48 b. 8 c. 8 d. 56

REPEATED ADDITION

Lesson 13:

1. Miss Brown began the arithmetic class by saying, "There is one more property of multiplication of whole numbers that we can discover with our multiplication facts. Do you remember that we do the operation within the parenthesis first? i.e., 4 x (2 + 5) = 4 x 7. The symbol of + within the parenthesis ( ) indicated that we add before we multiply by 4."

   In the example 4 + (2 x 3), the symbol within the parenthesis tells us to ________________ before we add the 4.
   add multiply

---

multiply

2. Before we try to discover the new multiplication of whole numbers' property, we had better check to see if we can get the correct answers for this exercise.

   a. 12 + (3 + 4) = 12 + ________ = 19

   b. 6 + (2 + 5) = 6 + 7 = _____

   c. 3 x (1 + 2) = 3 x _______ = 9

   d. 3 x (3 x 2) = 3 x _______ = 18

   e. 4 x (4 + 4) = 4 x _______ = 32

   f. (2 x 3) + 6 = ______ + 6 = 12

---

a. 7 b. 13 c. 3 d. 6 e. 8 f. 6

The children played a game at Mary's birthday party. For one game Mary put 4 plates with 2 cups on each plate on the table. In each cup she put 3 peanuts. How many peanuts did she need for the game?

3. John said, "We need to __________ to find the answer the easiest way."
add multiply

---

multiply

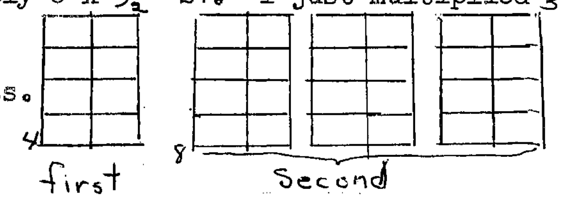
4. "How can we multiply three numbers?" asked John. We learned that multiplication was a binary operation so we can only multiply __________ factors at one time."

---

two

5. John said, "Watch me. That is what I will do. I will multiply just two factors at one time." I will use the factors 4, 2, and 3. First, I will multiply 4 x 2 = 8. Then I will use 8 as a factor. I will multiply 8 x 3 = 24. I just multiplied two factors at one time.

Mary needs __________ peanuts.



24

6. Here is how we can write John's example so that we know which two factors to multiply.

(4 x 2) x 3 =

_____ x 3 = 24

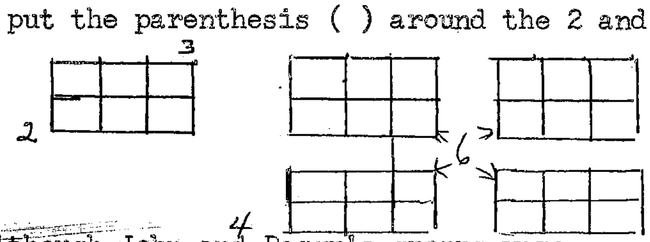
8

7. "I think that we should have found the number of peanuts in the two cups on each plate first. After we find the number of peanuts on each plate, we can find how many are needed for the 4 plates," commented Peggy.

"I would just need to put the parenthesis ( ) around the 2 and 3 like this."

4 x (2 x 3) =

4 x _______ = 24



Did you notice that although John and Peggy's groups were different they got the same result?

6

8. Do you think that this always happens when we multiply whole numbers?

Miss Brown asked the children to work the following exercise to see if they could discover the new property.

a. 3 x (2 x 4) = 3 x (______) = 24

(3 x 2) x 4 = _____ x 4 = 24

b. (5 x 1) x 2 = _____ x 2 = 10

5 x (1 x 2) = 5 x _____ = 10

c. 3 x (3 x 2) = 3 x _____ = 18

(3 x 3) x 2 = _____ x 2 = 18

d. 4 x (2 x 3) = 4 x _____ = 24

(4 x 2) x 3 = _____ x 3 = 24

---

a. 8 6 b. 5 2 c. 6 9 d. 6 8

9. Bill said, "I know what the new property is. The factors were kept in the same order. The parenthesis was around two of the f________. One time it was around the first two factors. The next time it was around the last two factors. We multiplied the numerals within the parenthesis first. Then we multiplied the product by the other factor. The order of the factors ______________ change."

did did not

The groups did change," said Mary.

---

factors did not

(4 x 2) x 3 = ______ x 3 = 24

4 x (2 x 3) = 4 x ______ = 24

---

8 6

10. Bill had discovered the associative property for multiplication of whole numbers.

If one has three factors, the associative property for the multiplication of whole numbers means that the <u>order</u> of the factors will remain the same but the groups will change. The result is the ______________ when the associative property holds. same different

---

same

11. When using only the associative property, I ___________ change the order of the factors. do do not

---

do not

12. When using only the associative property. I ___________ change the grouping. do do not

---

do

REPEATED ADDITION

Lesson 14:

1. Chuck bought seven (7) arrows at nine cents (9¢) an arrow. How much did he have to pay for the arrows?

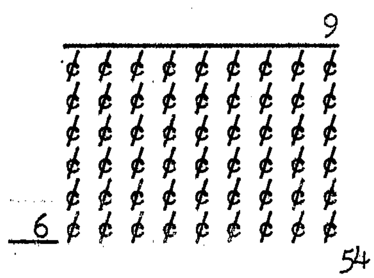
   Chuck knew that 6 arrows would cost ______cents.

   6 x 9 = 54¢

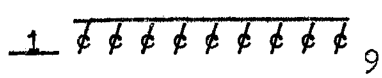
54¢

2. Dick said, "Just add another 9¢ on the 54¢ and you will have how much the arrows cost."

   He used this array to show Chuck how much the arrows cost.



   Now, add one more row



   a. 7 x 9 = ______ b. 9 x 7 = ______

---

a. 63 b. 63

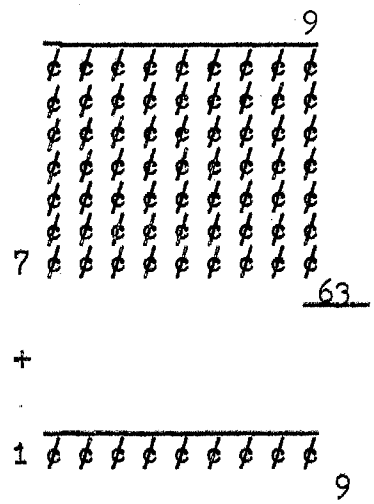
3. Because the commutative property is true for multiplication of whole numbers, if 7 x 9 = 63, then the class knew that 9 x ______ = 63.

7

4. "Now that we know that 7 x 9 = 63, how can we find out how much 8 x 9 would be?" asked Peggy.

Jane said, "Dick just added one more row of 9's to the six 9's to get seven 9's. Why can't we add one more row of 9's to the seven 9's to get 8 x 9?"

It is like this



a. 8 x 9 = ______ b. If 8 x 9 = 72, then 9 x ______ = 72

---

a. 72 b. 8

5. In lessons 7, 8, and 9 we learned that the distributive property was very useful. This property was true for the multiplication of whole numbers. This means that if I don't know a multiplication fact such as 6 x 9, I could rename one of the factors, multiply each part of the renamed factor, and then add the two products together.

Suppose that you did not know that 6 x 9 = 54. Would you get the same answer if you renamed the factor 6 as (4 + 2)? Check to see.

a. (4 + 2) x 9 = ______

b. (4 x 9) + (2 x ______) = ______

c. ______ + 18 = 54

---

a. 54 b. 9, 54 c. 36

6. We didn't need to rename the factor 6 as (4 + 2). We could have renamed it as (3 + 3). Then:

a. (3 + 3) x 9 = ______

6. continued

b. (3 x 9) + (_____ x 9) = _____

c. _____ + 27 = 54

It didn't make any difference what we renamed ths six. We just had to be sure that the two parts could be added together to be six.

---

b. 3 c. 27

7. Today we also learned that 8 x 9 = 72. Rename one of the factors to see if the distributive property works for this multiplication fact. Hint: You might decide to rename 8 as (7 + 1), (6 + 2), (5 + 3), or (4 + 4). Let's use one of the suggestions.

a. (5 + _____) x 9 =

b. (_____ x 9) + (3 x 9) =

c. _____ + 27 = 72

Did our choice work? _______________
yes no

---

a. 3 b. 5 c. 45 d. yes

8. Of course, you could have decided to rename the factor 9 instead of the factor 8. You might have renamed 9 as (8 + 1), (7 + 2), (6 + 3), or (5 + 4). Would you have found the same product of 72? Let's try one of the choices.

a. 8 x (8 +1) =

b. (8 x _____) + (8 x 1) =

c. _____ + 8 = 72

---

b. 8 c. 64

9. In lessons 4 and 5, we learned about the commutative property. The commutative property says that the order of the factors in a multiplication fact will not change the product.

a. If 6 x 9¢ = 54¢, then 9¢ x 6 = _____

b. If 8 x 9 = 72, then 9 x _____ = 72

---

a. 54 b. 8

10. When we learned about the associative property, we learned that the order of the factors remained the same but the groups changed. This means that if I have three factors, 3 x 2 x 4, I can say:

a. (3 x 2) x 4 =

b. ______ x 4 = 24

or I can say:

c. 3 x (2 x 4) =

d. 3 x ______ = 24

---

b. 6 d. 8

The product was the same for both ways of grouping the factors.

11. Does the associative property work for these factors? 2 x 3 x 3 =

a. I can group like this: (2 x 3) x 3 =

or ______ x 3 = 18

b. I can group like this: 2 x (3 x 3) =

2 x ______ = 18

---

a. 6 b. 9

12. Do you remember about the closure property? If both factors are whole numbers, the product must be a whole number in order that the closure property is true for multiplication of whole numbers.

Both of the factors in the multiplication fact 8 x 9 = 72 are whole numbers. The product is 72. We know that 72 is a whole number. The closure property is true for the multiplication of whole numbers.

a. 6 x 9 = ______

b. Six (6) is a whole number.

c. Nine (9) is a w______ number.

d. The product seventy two (72) is a w______ number.

12. continued

e. Therefore, the cl___________ property is true for the multiplication of whole numbers.

---

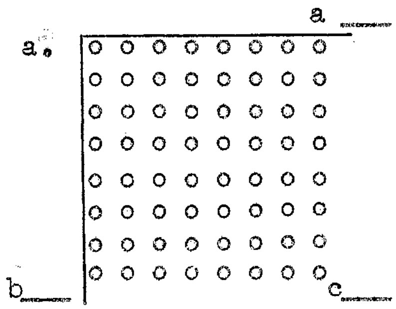
a. 54 c. whole d. whole e. closure

REPEATED ADDITION

Lesson 15:

1. "There are just two multiplication facts left to learn," said Miss Brown. Look at the two arrays and see if you can write the fact for each array.

a.



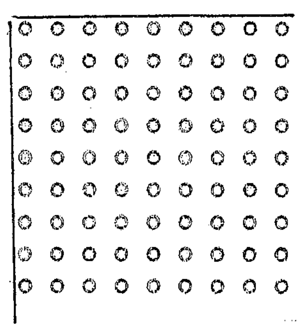
d ______ x ______ = ______

---

a. 8 b. 8 c. 64 d. 8 x 8 = 64

b. This multiplication fact is just as easy as 8 x 8 = 64. It is like this:

9 x _____ = _____



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9 x 9 = 81

2. The following exercise contains some of the more difficult facts. Check to see if you know of these facts.

a. 7 x 6 = ____

b. 7 x 9 = ____

c. 8 x 7 = ____

d. 8 x 8 = ____

e. 8 x 7 = ____

f. 6 x 8 = ____

---

a. 42 b. 63 c. 56 d. 64 e. 56 f. 48

During the lessons we have learned that certain things are true about the multiplication of whole numbers. As this is the last lesson, see if you can choose the word that names the property I am illustrating.

3. One of the multiplication facts in today's lesson was 8 x 8 = 64. All of the numbers used in this multiplication fact were whole numbers. The example 8 x 8 = 64 shows the ________ property for the multiplication of whole numbers. closure associative

---

closure

4. The fact that 6 x 9 = 54 makes us sure that 9 x 6 = 54, also. This is an example to show the ________
associative commutative
for the multiplication of whole numbers.

---

commutative

5. Do you remember something about the number one (1)? It acts very odd in a multiplication fact. Whenever one is a factor in a multiplication fact, the product is the same as the other factor. For example, 6 x 1 = 6. In this case the number one is called the ________.
closure multiplicative identity

---

multiplicative identity

6. Sometimes when we don't know the product for a combination, we use one of the properties we have studied. If I don't know that 9 x 9 = 81, I could use this property. I would rename one of the factors, multiply each part, and add the products.

Example:

$9 \times (5 + 4) =$

$(9 \times 5) + (9 \times 4) =$

$45 \quad + \quad 36 \quad = 81$

In this example the ______________________ property was
associative distributive
illustrated.

---

distributive

7. We studied just one more property. We had to be careful to keep the factors in the same order when we used this property. We changed the group though.

Example: $4 \times 2 \times 3 =$ OR $4 \times (2 \times 3) =$

$(4 \times 2) \times 3 =$ $4 \times 6 = 24$

$8 \quad \times 3 = 24$

This example illustrates the ______________________
associative distributive
property.

---

associative

How did you do in naming the properties? Just think ---- even if you had only one right, that was one more than you knew when you started the lessons.

APPENDIX C

RATIO TO ONE

Lesson 1:

1. Bill's mother made breakfast for the campers. She needed eggs for five boys. Each boy was to have two eggs. Bill's mother thought, "I need to boil__________eggs."

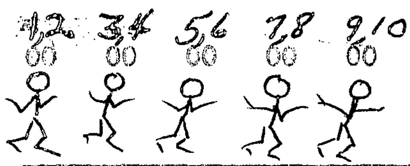
10

Think of some ways Bill's mother could find out how many eggs she needed to boil. Check to see if any of your answers agree with the ideas suggested below.

One fourth grade class suggested three different ways of finding how many eggs Bill's mother needed. These are given in #2, #3, and #4.

2. Jill said, "Bill's mother could have made a drawing that showed five boys with two eggs for each boy. Then she could have counted."

The drawing would look like this



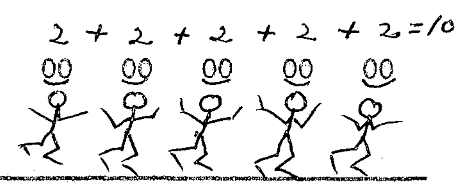
Bill's mother could get the right answer by c__________.

---

counting

3. John said that he thought Bill's mother could have added the number of eggs each boy would have.

Bill's mother could get the right answer by a__________the number of eggs each boy was to have.

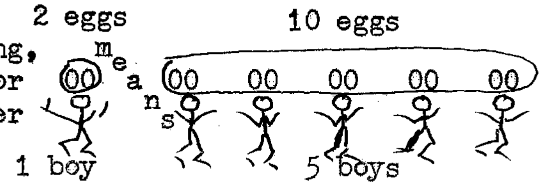


---

adding

4. Bob said, "Bill's mother could have looked at the drawing and said to herself that two for one boy means that I need ten for five boys.

She could look at the drawing, notice the number of eggs for one boy and notice the number of eggs for five boys.



Bill's mother could get the right answer. She thought two eggs for one boy means ten eggs for____________boys.

---

five

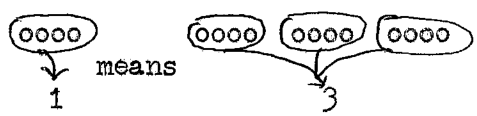
She could have written the example like this

$$\frac{2}{1} = \frac{10}{5}$$

$$5 \times 2 = 10$$

When you think with numbers such as five and two and get ten, you are multiplying.

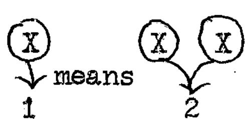
5. Look at these drawings. Did you see how to get the total number in the drawing? The first one is done for you.

a. 

$$\frac{4}{1} = \frac{12}{3}$$

b. 

$$\frac{3}{1} = \frac{\square}{3}$$

c. 

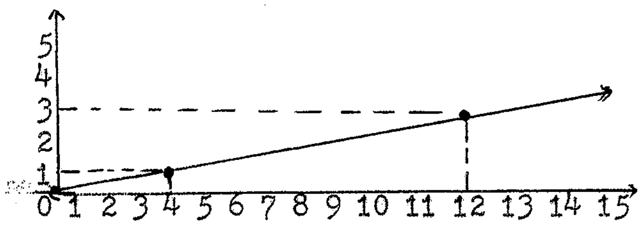
$$\frac{1}{1} = \frac{\square}{2}$$

---

b. 9 c. 2

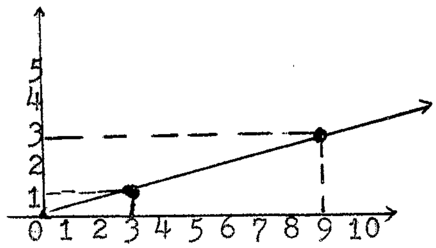
6. Instead of using all the different drawings to show how many we need, we could use two number lines like this

$\frac{4}{1} = \frac{12}{3}$



12

7. $\frac{3}{1} = \frac{9}{3}$



3 x 3 = ___

---

a. 12 b. 9

Did you notice that the numeral 0 was the beginning place for each of the number lines?

8. Mary looked at drawing and said, "The origin of both lines is 0 because the numeral is the beginning numeral for each line. The line that goes up and down on the page is a vertical line. The line that goes across the page is a horizontal line."

Beneath each line write horizontal or vertical.

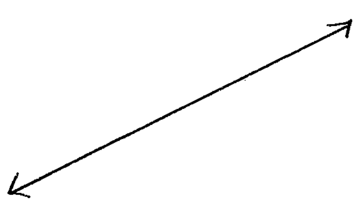
a.  b.  c. 

h________ v________ v________

---

a. horizontal b. vertical c. vertical

9. Paul said, "I can draw a line that is not either horizontal or vertical.



Paul's line__________
was was not
either horizontal or vertical

9. continued

The line that Paul drew was a diagonal line.

---

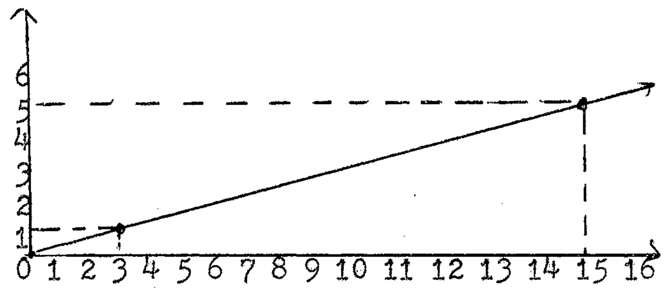
was not

Multiplication can be written like this

3 x 5 = 15

10. Jim said, "I see how we use the number lines to find the answer to the multiplication fact '5 x 3 = 15.' It is like this

First, I find the 3 on the horizontal line. I know that I am multiplying by 3 because when I compare 3 to 1, I got 3.



Next, I found 5 on the vertical line. I followed a line drawn across the page to the right of 5 until it met the diagonal line. Then I just drew a line down to the numeral.

The numeral was 15. That was the answer.

5 x 3 = ______

because $\frac{3}{1} = \frac{\square}{5}$

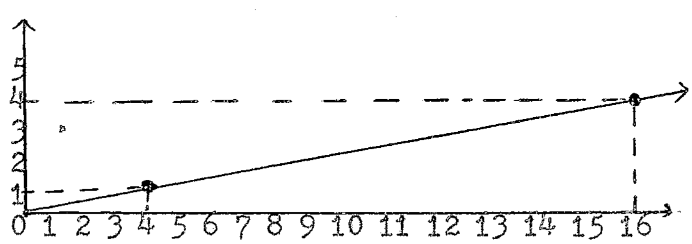
---

15 15

11. Alice said, "In this drawing the numeral 4 is compared to 1.

The diagram shows that 4 x 4 = ____

$\frac{4}{1} = \frac{16}{4}$

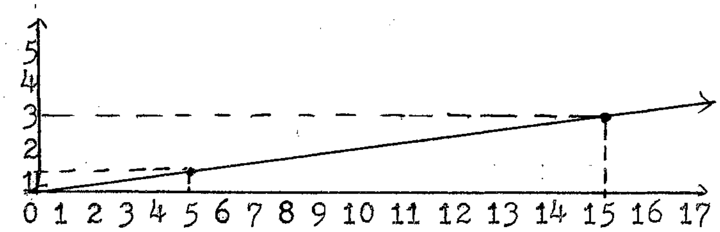


16

12. Can you write the multiplication facts that these diagrams show?

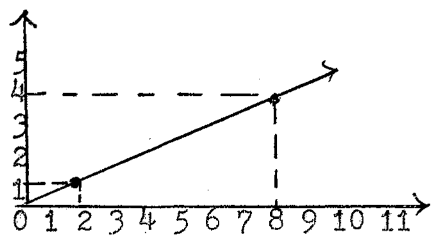
a. ___ x ___ = ___

$\frac{5}{1} = \frac{\square}{3}$



b. ___ x ___ = ___

$\frac{2}{1} = \frac{\square}{4}$



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a. 3 x 5 = 15

$\frac{5}{1} = \frac{15}{3}$

b. 4 x 2 = 8

$\frac{2}{1} = \frac{8}{4}$

RATIO TO ONE

Lesson 2:

1. Probably, since you can remember playing games, you have been using counting numbers. The set of counting numbers can be written using numerals as $\{1, 2, 3, \ldots\}$. The three dots ... mean to continue the numerals on and on. The numeral 5 represents a counting number. The numeral 35 represents a counting n________.

---

number

2. If to the set of counting numbers you add zero (0), you will have the set of whole numbers. To get the set of whole numbers one needs to add________to the set of counting numbers.

0

3. The set of numbers written as $\{0, 1, 2, 3, 4, \ldots\}$ represents a set of w________-numbers.

---

whole

4. Using whole numbers we can do certain mathematical operations. An operation is a way of associating with two numbers a third number called the result. The operation called addition was used when Jim associated the numbers two (2) and three (3) and got the result five (5).

   When doing the operation addition, Bill associated the numbers two (2) and four (4) and got the result__________.

---

six (6)

5. When using whole numbers, we can also do an operation called multiplication. If Mary used the multiplication operation to associate the numbers two (2) and three (3), she would get the result six (6).

5. continued

Mary associated the numbers two (2) and four (4) and got the result eight (8). Mary was using the m___________ operation.

---

multiplication

6. The results for associating the same numbers in the operations of multiplication and addition _______________ the same.
were were not

---

were not

7. When we associate the number eight (8) with the numbers two (2) and four (4), we are using the operation called ___________ addition or ___________________.
multiplication

---

multiplication

8. When we associate the number six (6) with the two numbers two (2) and four (4), we are using the O_______________ called addition.

---

operation

Although we used the <u>same two numbers</u> in #7 and #8, the results <u>were not</u> the same. We got a <u>unique</u> result for each operation. The unique result means that in a given operation there is only one right number to associate with any two numbers (pair of numbers).

9. Jane used the operation multiplication. She associated the numbers three (3) and three (3) with the result _____________.

---

nine (9)

10. Bill's unique number six (6) and Jane's unique number nine (9) are in the set of whole n_________________.

---

numbers

11. The pairs of numbers used in #9 and #10 are w________ numbers.

---

whole

12. The result from associating the two numbers (pair of numbers in #8 and #9 was a w__________ number.

---

whole

13. In mathematical operations whenever two whole numbers are associated and the result is a whole number, we say that the closure property is true for a certain operation.

When we use an operation in which the result is always the same kind of number as the pair of numbers, the cl__________ property is true.

---

closure

Peter said, "The closure property must be true for addition because when I add two whole numbers, I get a whole number."

14. Tom said, "I think that the closure property is true for multiplication of whole numbers. When I multiply a whole number by a whole number, I get a whole number for the answer. The cl__________ property is true for the multiplication of whole numbers."

---

closure

15. Work this exercise. Is Tom's statement true everytime? Did a whole number result when you multiplied one whole number by another whole number?

a. 2 x 2 = ____

$\frac{2}{1} = \frac{4}{2}$

b. 1 x 2 = ____

$\frac{2}{1} = \frac{2}{1}$

c. 3 x 3 = ____

$\frac{3}{1} = \frac{9}{3}$

d. 2 x 4 = ____

$\frac{4}{1} = \frac{8}{2}$

---

a. 4 b. 2 c. 9 d. 8

16. The closure property seems to be________________for the
true or false
multiplication of whole numbers. As we continue to work with the combinations, we will check to see if it holds everytime. We will try to find out if the multiplication of whole numbers is closed.

---

true

RATIO TO ONE

Lesson 3:

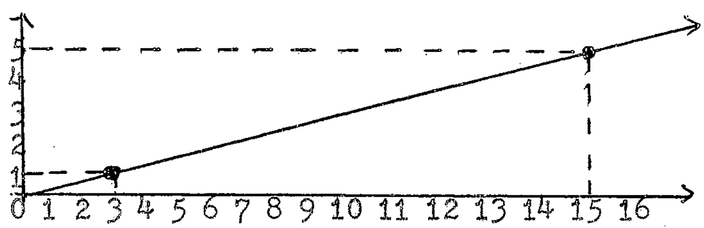
1. Mary and Jane were arranging their stamps to be placed in the stamp book. Mary arranged hers like this.

She had four for one row.
She could write it as 4 for 1 or as $\frac{4}{1}$.

2. Jane looked at the arrangement of stamps in the book. She said, "There are only 3 stamps in each row. But I can see five rows. This shows that 3 is to 1 as ______ is to 5." It could be represented like this.

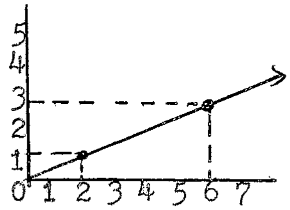
$\frac{3}{1} = \frac{15}{5}$

$5 \times 3 = 15$



15

3. It is important in our work that we look for the second numeral of the multiplication fact on the horizontal line and for the first numeral of the multiplication fact on the vertical line. This diagram shows that one looks for the answer on the ______________ line.
horizontal or vertical



$\frac{2}{1} = \frac{6}{3}$

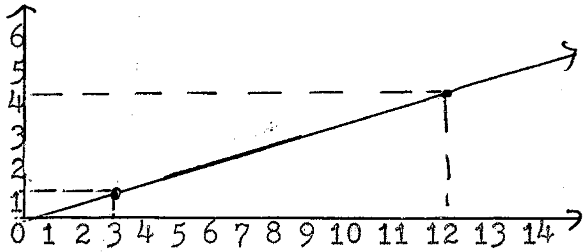
$3 \times 2 = 6$

---

horizontal

4. Look at these diagrams. Can you fill in the missing numerals?

a.

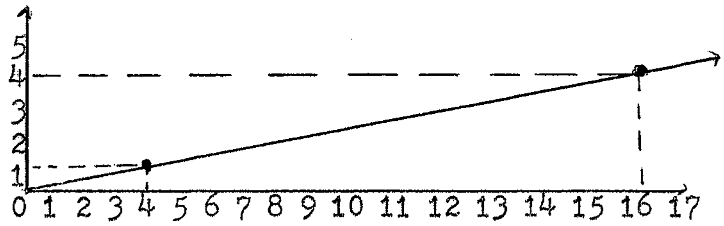


$\frac{3}{1} = \frac{\square}{4}$

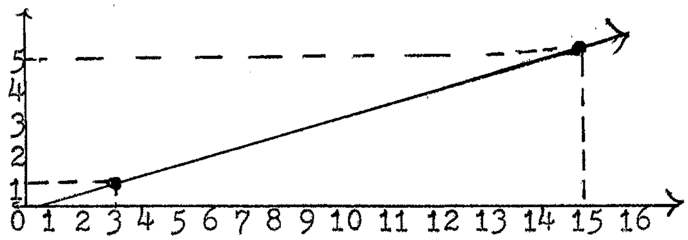
4 x 3 = ____

b. 4 x ____ = 16

$\frac{4}{1} = \frac{\square}{4}$



c.

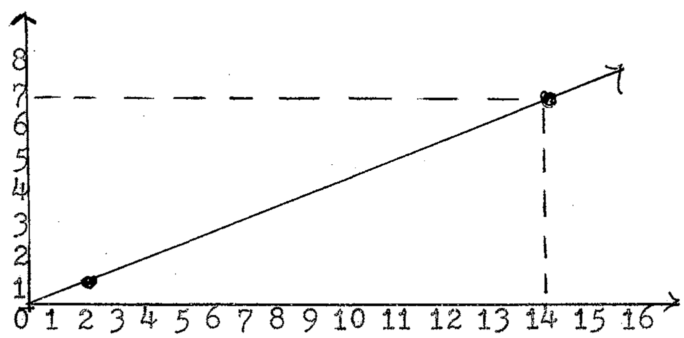


$\frac{3}{1} = \frac{15}{\square}$

____ x 3 = 15

d. $\frac{2}{1} = \frac{\square}{7}$

____ x 2 = 14



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a. 12	b. 4	c. 5	d. 14
12	16	5	7

5. Jill said, "It is easier to write the multiplication fact, 'three is to one as twelve is to four' with numerals." It means that $\frac{3}{1} = \frac{12}{4}$

One can say it with numerals like this

4 x 3 = 12

6. Using numerals as Jill did, describe the following multiplication facts. The first one is done for you.

   a. Three is to one as twelve is to four. 4 x 3 = 12

   b. Two is to one as twelve is to six. ____________

   c. Seven is to one as fourteen is to two. ____________

---

b. 6 x 2 = 12 c. 2 x 7 = 14

7. In the multiplication fact '2 x 7 = 14' each of the numerals 2 and 7 is called a factor.

8. In the multiplication fact '4 x 4 = 16' each of the 4's is a f__________.

---

factor

9. In the multiplication fact '6 x 2 = 12' the factors are _____ and _______.

---

6 2

10. The answer to the multiplication example '2 x 7 = 14' is called the product.

11. In the multiplication fact '4 x 4 = 16' the product is_______.

16

12. In the multiplication fact '6 x 2 = 12'________is the product.

12

13. Study the arithmetic facts given below. Decide which ones are multiplication facts.

    a. 3 + 3 = 6 ____________
       yes no

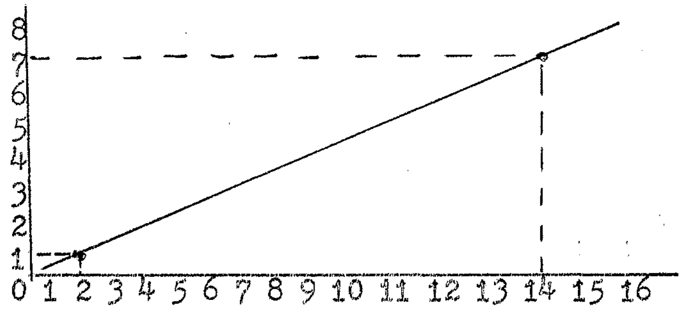
    b. 3 x 4 = 12 ____________
       yes no

    c. 2 x 6 = 12____________
       yes no

---

a. no b. yes c. yes

14. Use numerals to write the multiplication facts for these diagrams. When you use diagrams as appear below, you are finding your answer in a coordinate system. The first one is done for you.

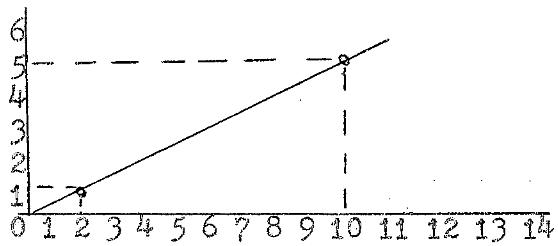
a. 

The diagram shows that $\frac{2}{1} = \frac{14}{7}$

The multiplication fact is $7 \times 2 = 14$

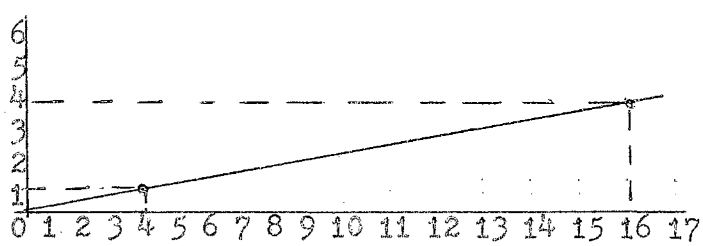
b. The following diagram shows that $\frac{2}{1} = \frac{10}{5}$

The multiplication fact is _____ x _____ = _____



c. The following diagram shows that $\frac{4}{1} = \frac{16}{4}$

The multiplication fact is _____ x _____ = 16

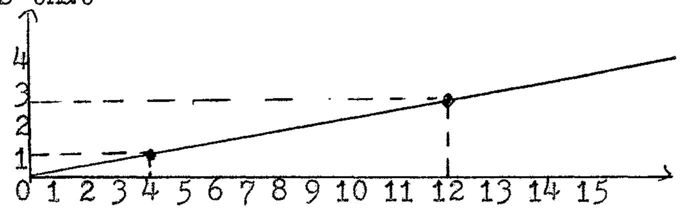


---

b. 5 x 2 = 10 c. 4 x 4 =16

d. The diagram shows that

$\frac{4}{1} = \frac{12}{3}$



The multiplication fact is
_______ x ______ = ______

---

3 x 4 = 12

15. The operation of multiplication can be performed on just a pair (two numbers) of whole numbers at one time. An operation that is done on just two numbers at one time is called a binary operation.

16. Multiplication of whole numbers is a b________ operation because one multiplied only two numbers at one time.

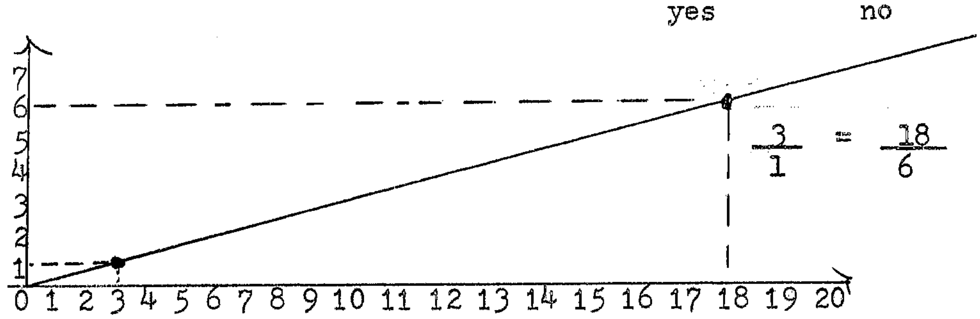
---

binary

RATIO TO ONE

Lesson 4:

1. Tom and Jim both bought some candy in packages. Tom said, "I got 3 pieces in each sack. My package contained six sacks. I got 18 pieces of candy." Is Tom right? ____________
yes no

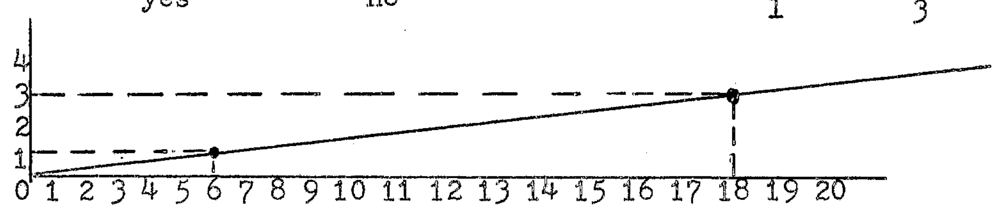


$\frac{3}{1} = \frac{18}{6}$

____
yes

2. Jim said, "I got 18 pieces, too. My package contained only 3 sacks. But each sack had 6 pieces of candy in it." Is Jim right? ____________
yes no

$\frac{6}{1} = \frac{18}{3}$



____
yes

3. Mary said, "Both Jim and Tom used the same factors. They changed the order of the factors." Was Mary right?

____
yes no

____
yes

4. Tom's multiplication fact looked like this

6 x 3 = ________

18

5. Jim's multiplication fact looked like this

3 x 6 = ________

18

6. Mary had seen that ________ multiplication facts might use
one two
the same factors but in different order.

---

two

7. You might have heard these two multiplication facts called pairs of facts. Write two pairs of facts you could use to illustrate a dozen eggs.

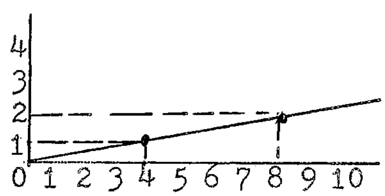
a. 3 x ________ = 12 b. 2 x ________ = 12

4 x ________ = 12 ________ x 2 = 12

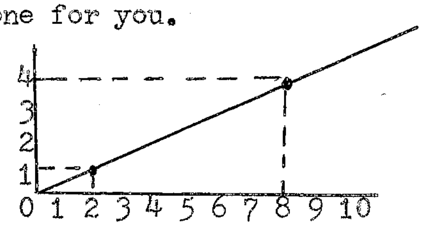
---

a. 4
3
b. 6
6

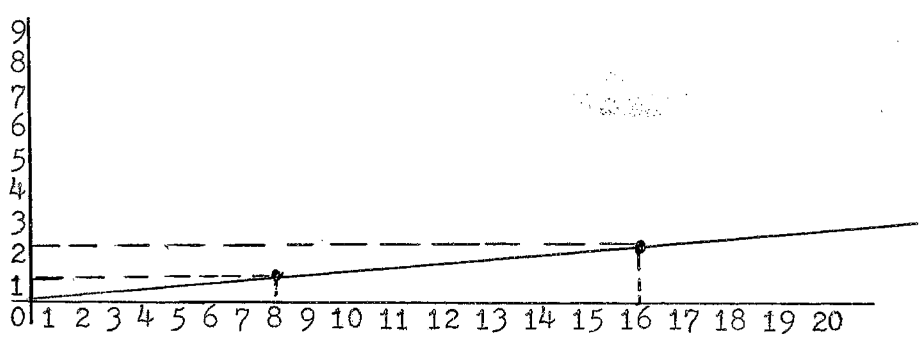
8. Look at the following sets of coordinate systems. Write the multiplication fact which each coordinate system of the pair shows. The first pair is done for you.



2 x 4 = 8

b. 

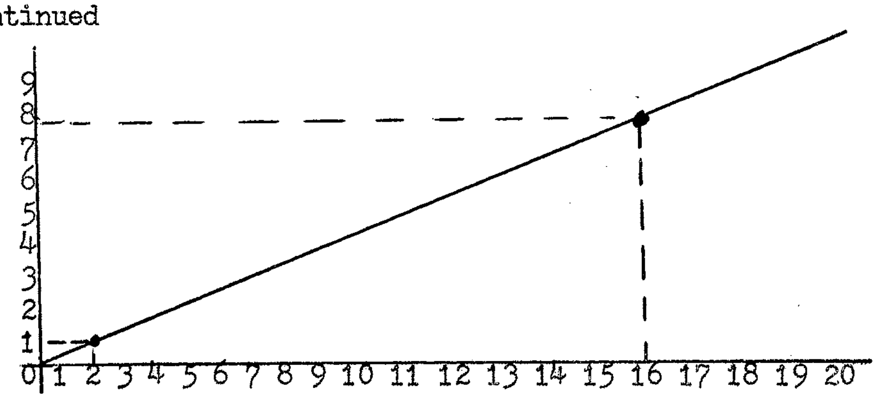
4 x 2 = 8

9. a. 

________ x ________ = ________

9. continued

b.



_______ x _______ = _______

---

a. 2 x 8 = 16 b. 8 x 2 = 16

10. In #9 the two multiplication facts were

2 x 8 = _______

8 x 2 = _______

---

16 16

11. Can you complete the following exercise so that you have pairs of facts?

a. 2 x 9 = _______

9 x _______ = 18

b. 3 x 6 = _______

6 x _______ = 18

c. 3 x 5 = _______

_______ x 3 = 15

---

a. 18
2

b. 18
3

c. 15
5

RATIO TO ONE

Lesson 5:

1. If two multiplication facts have the same factors, their products are ______________.
   equal   not equal

---

equal

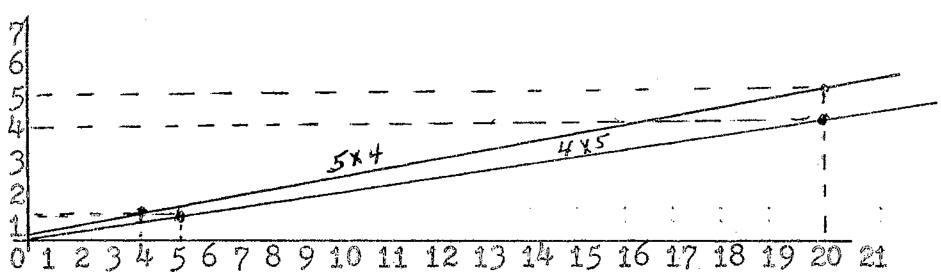
2. Jane did not want to draw a new coordinate system each time she found the product of two factors. She used the same coordinate system but changed the shape of the line for each fact. In this exercise Jane wanted to find the product of the pair of multiplication facts in which 4 and 5 are factors.

   The two multiplication facts for the factors (4, 5) are

   4 x 5 = 20

   5 x 4 = 20

   Look at the coordinate system below and notice how Jane showed both multiplication facts for the pair of factors (4, 5).



4 x 5 = ______ $\frac{5}{1} = \frac{20}{4}$

5 x 4 = ______ $\frac{4}{1} = \frac{20}{5}$

---

20   20

3. Write the pairs of facts Jane discovered using the pair of factors given below. The first is done for you. <u>The next sheet</u> contains some coordinate systems for you to use if you desire to check your answers.

a. (3, 7)

3 x 7 = 21

7 x 3 = 21

b. (3, 8)

____ x ____ = ____

____ x ____ = ____

c. (4, 6)

____ x ____ = ____

____ x ____ = ____

d. (5, 1)

____ x ____ = ____

____ x ____ = ____

---

b. 3 x 8 = 24
8 x 3 = 24

c. 4 x 6 = 24
6 x 4 = 24

d. 5 x 1 = 5
1 x 5 = 5

4. In all of the exercises that Jane did, the product for both multiplication facts was the same. (i.e., 4 x 5 = 20 and 5 x 4 = 20). This shows that multiplication of whole numbers is commutative. An operation is commutative if the order of the factors may be changed without changing the results.

Jane could change the order of the factors in the multiplication fact 3 x 8 = 24 to 8 x 3 = 24. The result was not changed. Multiplication of whole numbers is c__________.

---

commutative

5. Do the following facts show that multiplication of whole numbers is commutative?

a. 3 x 7 ≟ 7 x 3 ______ yes no

b. 3 x 8 ≟ 8 x 3 ______ yes no

c. 4 x 6 ≟ 4 x 6 ______ yes no

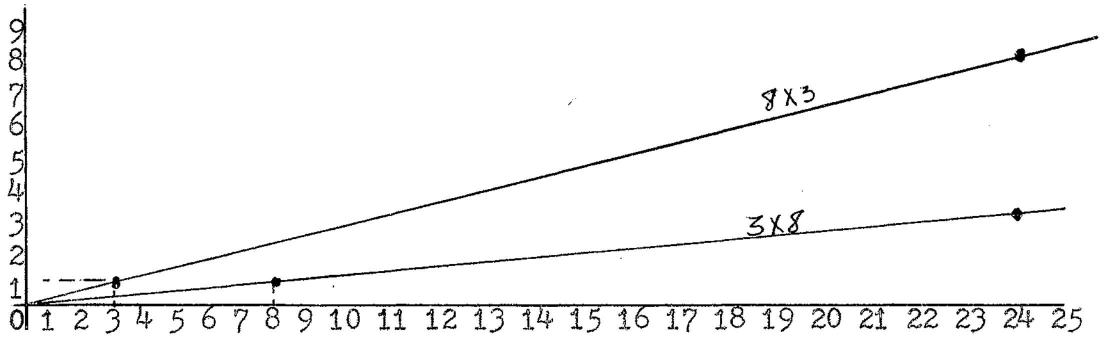
d. 5 x 1 ≟ 1 x 5 ______ yes no

---

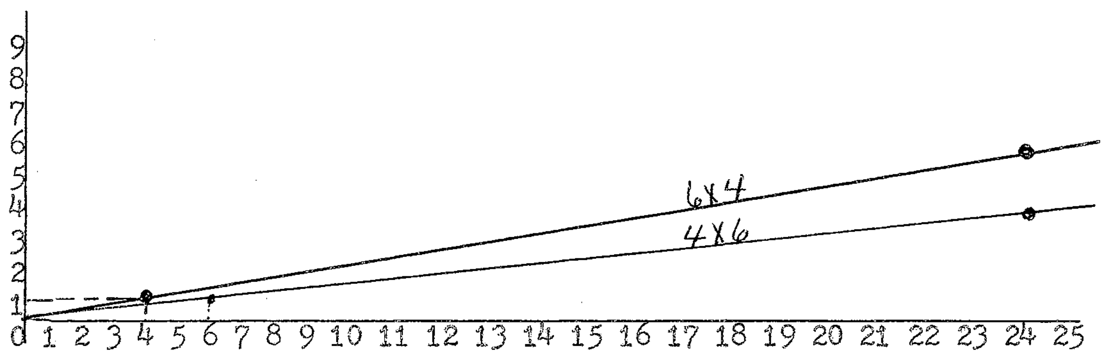
a. yes b. yes c. no d. yes

Worksheet

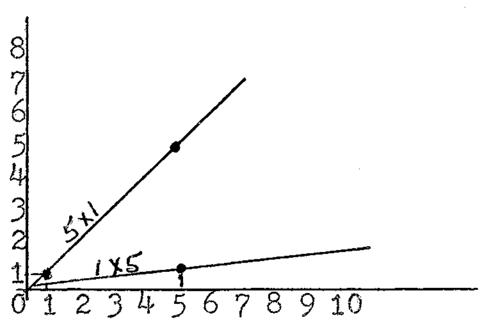
Use for b. (3, 8)



Use for c. (4, 6)



Use for d. (5, 1)



6. Multiplication of whole numbers is c______________ because the order of the factors may be changed without changing the result.

---

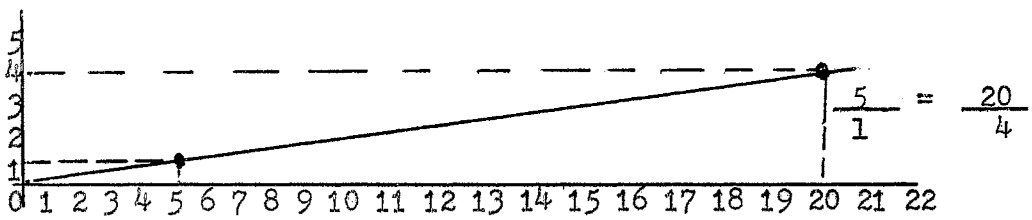
commutative

RATIO TO ONE

Lesson 6:

1. John said, "I think the product in a multiplication example is always larger than either of the factors." He drew these co-ordinate systems to illustrate his multiplication fact for each.

a.

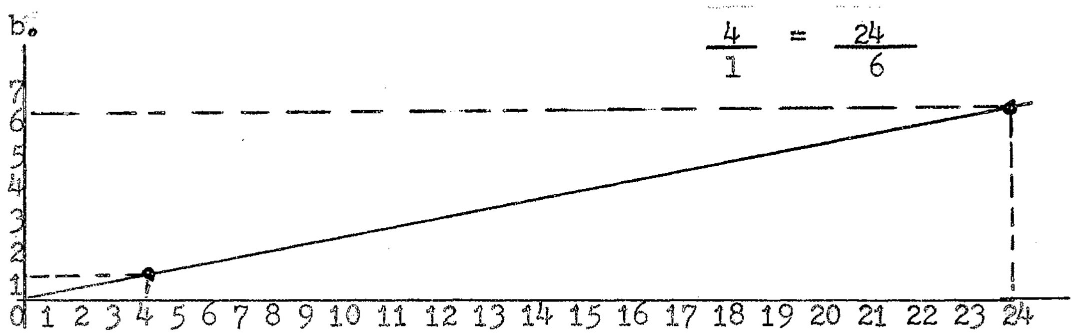


________ x ________ = ________

____

4 x 5 = 20

b.

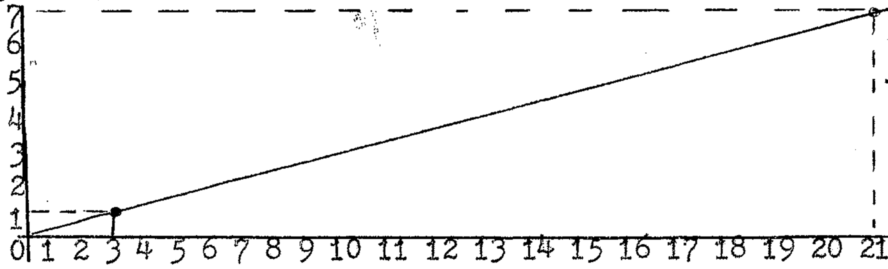


________ x ________ = ________

____

6 x 4 = 24

1. continued

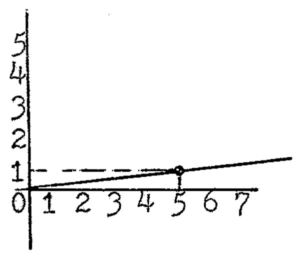


$\frac{3}{1} = \frac{21}{7}$

______ x ______ = ______

7 x 3 = 21

d.

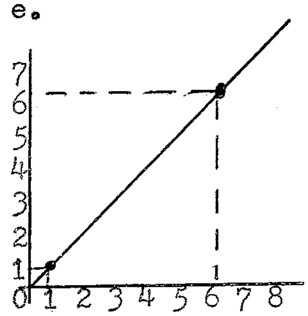


$\frac{5}{1} = \frac{5}{1}$

______ x ______ = ______

1 x 5 = 5

e.



$\frac{1}{1} = \frac{6}{6}$

______ x ______ = ______

6 x 1 = 6

2. The product for a multiplication fact ______________ always larger than both.
is   is not

is not

3. When John multiplied 5 x 1, he got the product ________.

5

4. The answer for 5 x 1 was 5. This is one of the factors, too. So when John multiplied 5 x 1, the answer was the same as one of the factors. It was the same as the factor ________.

5

5. When John multiplied 1 x 6, the answer was ______. Six is one of the factors. The other factor was 1.

6

6. If John multiplied 9 x 1, he would have got the answer ______. Nine is one of the factors. The other factor is ______.

---

9 1

7. Everytime John multiplied when one of the factor was 1, the product was the same as the other________.

---

factor

8. Work these examples. Check to see if what you have written in #7 is true.

a. 2 x 1 = ____ b. 3 x 1 = ____ c. 4 x 1 = ____

1 x 2 = ____ 1 x 3 = ____ 1 x 4 = ____

d. 5 x 1 = ____ e. 6 x 1 = ____ f. 7 x 1 = ____

1 x 5 = ____ 1 x 6 = ____ 1 x 7 = ____

g. 8 x 1 = ____ h. 9 x 1 = ____

1 x 8 = ____ 1 x 9 = ____

---

a. 2 b. 3 c. 4 d. 5 e. 6 f. 7 g. 8 h. 9
2 3 4 5 6 7 8 9

9. Both John and you have discovered the multiplicative identity. <u>One</u> is the multiplicative identity for whole numbers. In other words when a number is multiplied by <u>one</u>, the answer is always the same as the other number. (i.e., 5 x 1 = 5)

10. The numeral______represents the multiplicative identity.

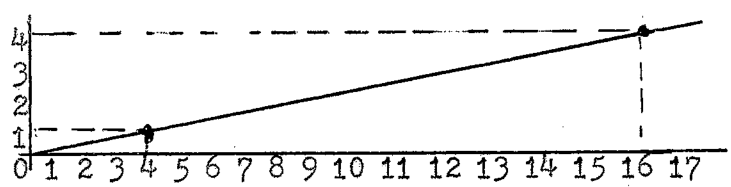
1

RATIO TO ONE

Lesson 7:

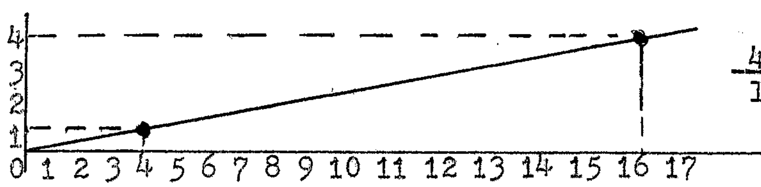
1. Ben was not sure that his answer to this multiplication fact was correct. 4 x 8 = 32

   He drew these diagrams to check his work

a. 

$\frac{4}{1} = \frac{16}{4}$

4 x 4 = ______

b. 

$\frac{4}{1} = \frac{16}{4}$

4 x 4 = ______

---

a. 16 b. 16

2. Each of the diagrams shows that ______ x ______ = 16.

---

4 x 4

3. Ben added the product from the first diagram to the product from the second diagram. He said, "16 + ______ = 32."

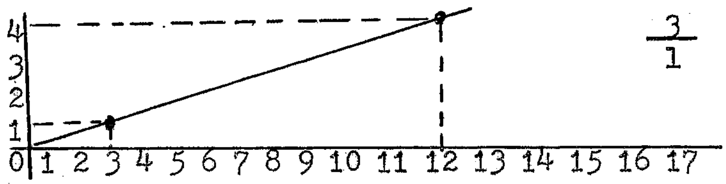
16

4. Alice said, "I know that 4 x 3 = 12 and 4 x 5 = 20. If I add 12 and 20, I get 32. I will draw the diagrams to illustrate

4. continued

my work. I renamed the factor 8."

a.

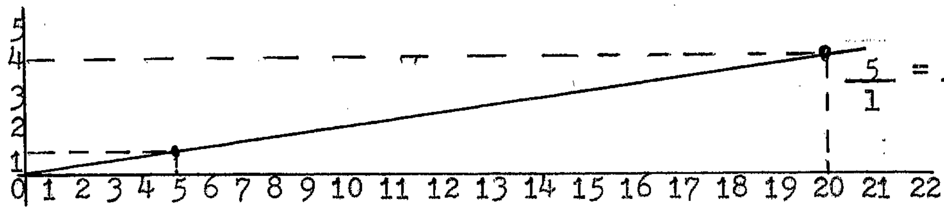


$\frac{3}{1} = \frac{12}{4}$

4 x 3 = ______

12

b.



$\frac{5}{1} = \frac{20}{4}$

4 x 5 = ______

20

5. Both Ben and Alice renamed the factor 8. Ben renamed the factor 8 to (4 + ____).

4

6. Alice renamed the factor 8 to (3 + ____).

5

7. Imagine that each of the numerals below represent the second factor in a multiplication fact. Can you rename each of the factors. The first one is done for you.

a. 5   b. 3   c. 9

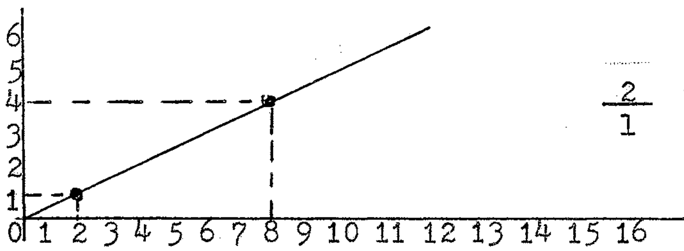
1 + 4   ______   ______ ______

2 + 3   ______   ______ ______

b. 2 + 1   c. 8 + 1   7 + 2
1 + 2   4 + 5   6 + 3

8. Terry said, "I didn't rename 8 like either Ben or Alice. I renamed 8 as (2 + 6). Ben's example will look like this."

a.

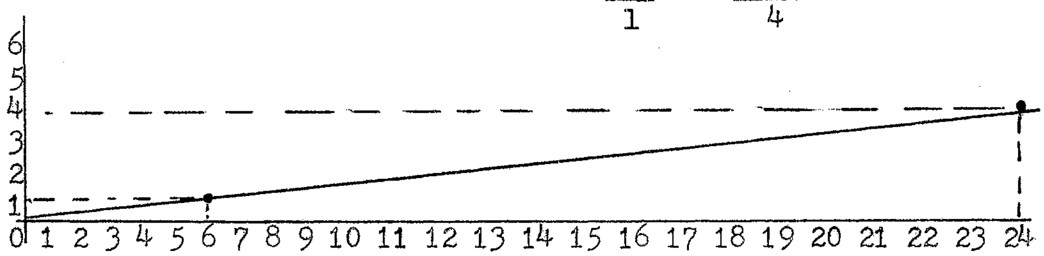


$\frac{2}{1} = \frac{8}{4}$

4 x 2 = ______

b.

$\frac{6}{1} = \frac{24}{4}$



4 x 6 = ______

---

a. 8 b. 24

9. Peggy said, "First, Terry multiplied 4 x 2 = 8 and then he multiplied 4 x 6 = 24. Then he added 8 and 24."

Terry's work looked like this:

4 x 2 = 8 and 4 x 6 = 24

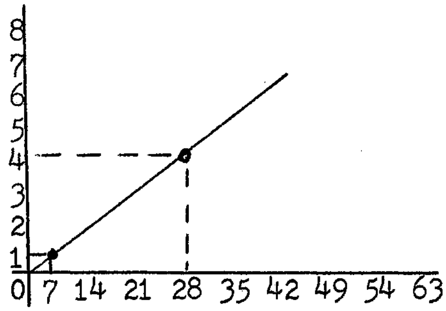
8 + 24 = 32

Yes, Terry can get the right answer by renaming______ as (2 + 6).

8

10. John said, "Let's try another example and see how we can rename the second factor. Let's try 4 x 7. I will draw the diagram that shows 4 x 7."

It looked like this:



$\frac{7}{1} = \frac{28}{4}$

4 x 7 ____

28

11. Andy said, "I'm going to rename 7 as (4 + 3). Now the example should say this."

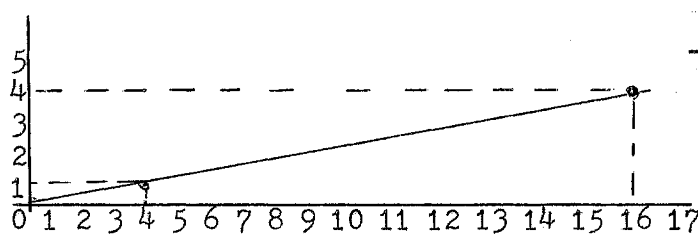
4 x 7 = 28

4 x (4 + ____) = 28

3

My work looks like this

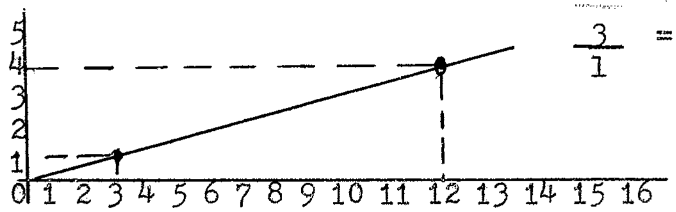
a.



$\frac{4}{1} = \frac{16}{4}$

4 x 4 = ____

11. continued

b. 

$\frac{3}{1} = \frac{12}{4}$

$4 \times 3 =$ ______

c. $16 + 12 =$ ______

---

a. 16 b. 12 c. 28

12. Andy continued by saying, "First, I multiplied $4 \times 4 =$ ______. Then I multiplied $4 \times 3 =$ ______. Then I added the two products."

---

16 12

13. Can you complete the following exercise to show Andy's work?

$4 \times 7 = 4 \times (4 +$ ______$)$

$= (4 \times 4) + (4 \times 3)$

$=$ ______ $+ 12$

---

3 16

14. Jerry looked at Andy's work and said, "You renamed 7 as (4 + 3). I am going to rename 7 as (5 + 2). My work will look like this."

$4 \times 7 = 4 \times (5 +$ ______$)$

$= (4 \times 5) + (4 \times 2)$

$=$ ______ $+ 8$

$= 28$

---

2 20

15. John laughed. "Both Jerry and Andy are right. But I see another way the factor 7 can be renamed. We can rename 7 like this."

$4 \times 7 = 4 \times (1 + 6)$

$= (4 \times 1) + (4 \times \_\_\_\_)$

$= \_\_\_\_ + 24$

$= 28$

---

6 4

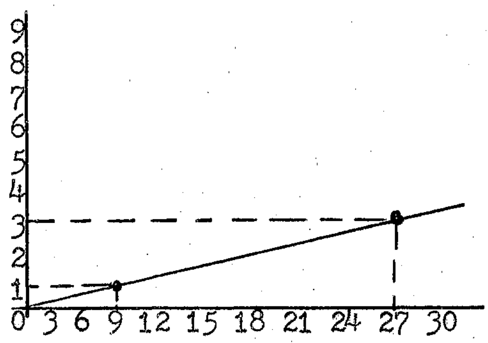
16. The children had discovered something new about multiplication of whole numbers. By the time you have the next lesson will you have discovered what it is?

RATIO TO ONE

Lesson 8:

1. Did you discover what the new property of multiplication of whole numbers is? If not, let's see if Alic's work will help you.

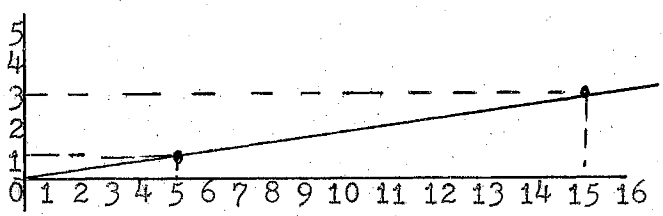
2. Alice illustrated the multiplication fact 3 x 9 by this illustration. She said that she could do it two ways. First, she would illustrate 3 x 9. Then she would rename the factor 9 as (5 + 4) and illustrate her work.



$\frac{9}{1} = \frac{27}{3}$

3 x 9 = ______

3. Here is Alice's work when she renamed the factor 9 as (5 + 4).

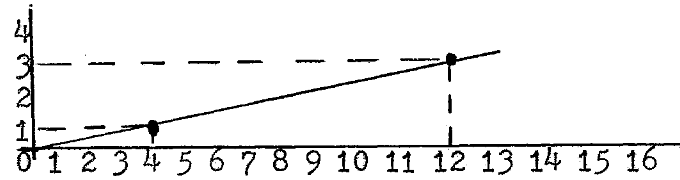
a. 

$\frac{5}{1} = \frac{15}{3}$

3 x 5 = ______

3. continued.

b.



$\frac{4}{1} = \frac{12}{3}$

$3 \times 4 =$ ______

---

a. 15 b. 12

4. When Alice multiplied 3×5, the product was______.
When Alice multiplied 3×4, the product was______.

To find the answer for 3×9, Alice needed to add______ + 12.
When she added 15 + 12, she got the answer ______.

---

15 12 15 27

5. Alice showed that the multiplication fact $3 \times 9 =$ ______.

27

In #3 and #4 Alice showed that she could rename the factor 9 as (5 + 4). She said that

$$3 \times (5 + 4) = (3 \times 5) + (3 \times 4)$$
$$= 15 + \_\_\_\_$$
$$= 27$$

12

Alice got the same result when she added first (5 + 4) to get the factor 9 and then multiplied as she did when she multiplied 3 x 5 and 3 x 4 first and then added the products.

6. Alice had discovered the important property. It is known as the <u>distributive</u> property. We say that multiplication distributes over addition when it doesn't make any difference whether you add to get the factor first or you multiply each part first and then add.

7. Mary used the distributive property to show that her answers were correct.

a. $5 \times 5 = 25$

$$5 \times (3 + 2) = (5 \times 3) + (5 \times \_\_\_\_)$$
$$= \_\_\_\_ + 10$$
$$= \_\_\_\_$$

7. continued.

b. $5 \times 6 =$ ____
$5 \times (4 + 2) = (5 \times$ ____ $) + (5 \times 2)$
$=$ 20 + ____
$=$ 30

---

a. 2, 15, 25 b. 30, 4, 10

8. Paul used the distributive property to do this exercise.

$7 \times 4 = ($ ____ $\times 1) + (7 \times 3)$
$=$ 7 + ____
$=$ 28

---

7 28

9. Mary said, "If I can say that 8 x 4 can be renamed as 8 x (2 + 2), then I can say that 8 x (2 + 2) can be renamed as 8 x 4."

Was Mary right? ________
yes no

---

yes

10. Can you fill in the answers to this exercise?

a. $9 \times 3 =$ ____

b. $(9 \times 1) + (9 \times 2) = 9 +$ ____ $= 27$

c. $8 \times (2 + 2) = (8 \times 2) + (8 \times$ ___ $) = 16 + 16 = 32$

d. $8 \times 4 =$ ____

---

a. 27 b. 18 c. 2 d. 32

11. The distributive property holds for the multiplication of whole ________.

---

numbers

RATIO TO ONE

Lesson 9:

1. Did you notice that sometimes when you counted you counted by 2's instead of by ones. This means that you say 2, 4, 6, ____, 10, and on and on. If you count by two's, you don't need to say or write as many numerals.

8

2. I can count by other groups besides by groups of 2's. Look at each row of numerals. See if you can tell how large each group is.

   a. 2, 4, 6, 8, ... (the 3 dots mean that I can keep going on and on)

      I was counting by groups of ____.

   b. 3, 6, 9, 12, 15, ...

      This time I counted by groups of ____.

   c. 5, 10, 15, 20, 25, ...

      I counted by groups of ____.

---

a. 2 b. 3 c. 5

Sometimes you will see the horizontal line marked off in 2's, 3's, or 5's. (It is necessary to use these groups so that the illustration will fit on the page).

3. Jane said, "Now that we have learned about the distributive property for multiplication of whole numbers, we won't have any trouble finding out how much four 9's will be. We know that we can rename the factor 9. I am going to rename 9 as (5 + 4). I know that 4 x 5 = 20. I also know that 4 x 4 = 16.

3. continued.

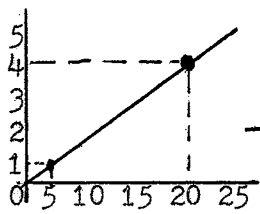
Then, if I add 20 and 16, I get 36." Jane's answer is right.

yes no

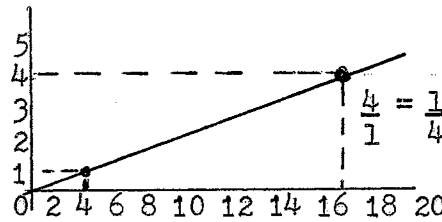
---

yes

Check the work with these illustrations. Remember that I am going to count by groups for the horizontal lines.



$\frac{5}{1} = \frac{20}{4}$



$\frac{4}{1} = \frac{16}{4}$

4 x 5 = ______ 4 x ______ = 16

---

20 4

4. When Jane added 20 and 16, she got 36. Her answer was

right wrong

---

right

5. Mary said, "If 4 x 9 = 36, then ______ x 4 = 36 because multiplication of whole numbers is cummutative.

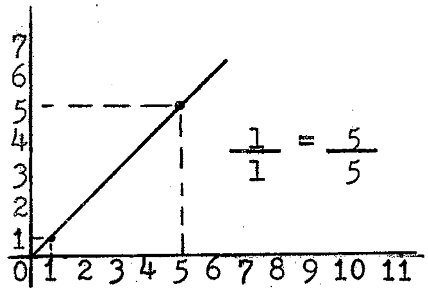
9

6. If Jane had used the multiplication fact 9 x 4, she could have renamed the factor 4 as (1 + 3) or as (2 + ______).

2

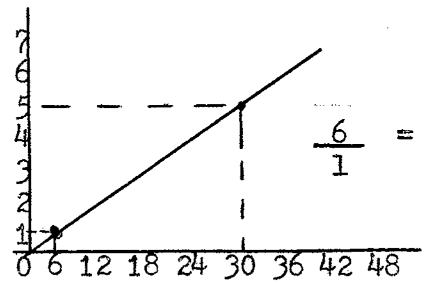
7. Peter asked, "Who can look at these illustrations and tell me the multiplication fact?"

7. continued.



$\frac{1}{1} = \frac{5}{5}$

5 x 1 = ______



$\frac{6}{1} = \frac{30}{5}$

5 x 6 = ______

---

5 30

8. "That's easy. All of us can do it." remarked Gary.

a. (5 x 1) + (5 x 6) = ______ + 30.

= 35

or

(5 x 1) + (5 x 6) = 5 x (1 + ______).

= 5 x (1 + 6) = 5 x 7

6

9. "Gary, did you notice that we used the multiplicative identity?" asked Peter.

"Yes," answered Gary. "The multiplicative identity was one in the multiplication fact 5 x 1 = ______."

5

10. Mary reminded the class that multiplication of whole numbers was c________________ because 5 x 7 = 7 x 5.

---

commutative

11. "Today, we have one more new multiplication fact to learn," commented Miss Brown. "The multiplication factors are 6 x 6. Does anyone think that he knows the answer?"

12. Henry said, "I think that 6 x 6 = 36. We learned that 6 x 2 = 12. We learned that 6 x 4 = 24. So if I ______________ 12
add multiply
and 24, I get 36."

---

add

13. Helen said, "I can prove that Henry is right by renaming the factor 6 as (1 + 5). I know that 6 x 1 = ______. I know that 6 x 5 = 30. When I add 6 and 30, I get the same answer as Henry. The answer is 36."

6

14. Can you fill in the blanks for these multiplication facts?

a. 6 x 6 = ______ b. 5 x ______ = 35 c. ______ x 9 = 36

---

a. 36 b. 7 c. 4

RATIO TO ONE

Lesson 10:

1. Henry said, "I have learned to check the product to a multiplication fact by renaming one of the factors. I will do it like this."

   a. Multiplication fact — 4 x 9 = ______

   b. Rename one factor — 4 x (4 + ______)

   c. Multiply each numeral of the factor by the other factor — (4 x 4) and (4 x 5)

   d. Add the two product — 16 + 20 = ______

   e. This way of working the example could be called "multiplying twice and adding."

---

a. 36 b. 5 c. 36

2. Henry has learned that multiplication distributes over ______________.

   addition multiplication

---

addition

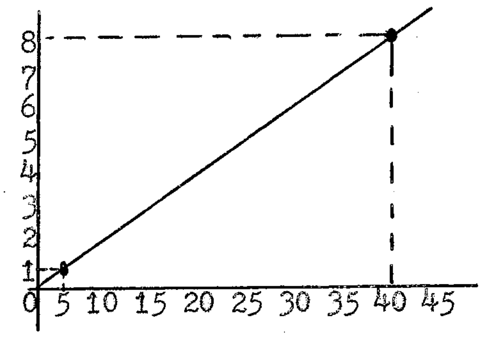
3. Henry______________check the product of a multiplication fact
   can can not
   by making use of his knowledge about the distributive property.

---

can

4. Miss Brown said, "We have some new multiplication facts to learn today. Let's use Henry's method to check the answers."

5. Peggy drew this illustration on the board.



$\frac{5}{1} = \frac{40}{8}$

She said, "I have shown that 8 x ______ = 40."

5

6. Paul checked Peggy's work. He said, "According to Henry, I can rename a factor. I will do it like this."

a.	Multiplication fact	8 x 5 = 40
b.	Rename the factor 5	8 x (3 + 2)
c.	Multiply	(8 x 3) + (8 x ______)
d.	Add	______ + 16
e.	The answer is	______

Peggy's answer was right.

---

c. 2 d. 24 e. 40

7. If 8 x 5 = 40, then 5 x 8 = 40 because multiplication of whole numbers is c______________.

---

commutative

8. Some of the children were not sure that Peter had written the correct answer on the board for this multiplication fact 6 x 7 = 42. They decided to check Peter's work by renaming the factor 7 as (2 + 5) and then working the example.

8. continued.

a. 6 x 7 = ______

b. 6 x 7 = 6 x (2 + 5)

c. = (6 x ______) + (6 x 5)

d. = 12 + ______

e. = ______

---

a. 42 c. 2 d. 30 e. 42

9. Write the answers to this exercise.

a. 6 x 7 = ______

b. 7 x 6 = ______

c. 5 x 8 = ______

d. 8 x 5 = ______

---

a. 42 b. 42 d. 40 e. 40

RATIO TO ONE

Lesson 11:

1. Look at these multiplication facts. What do you see that is alike in all of them?

   a. $4 \times 4 = 16$

   b. $6 \times 6 = 36$

   c. $5 \times 5 = 25$

   In each example both fa________ were represented by the same numeral.

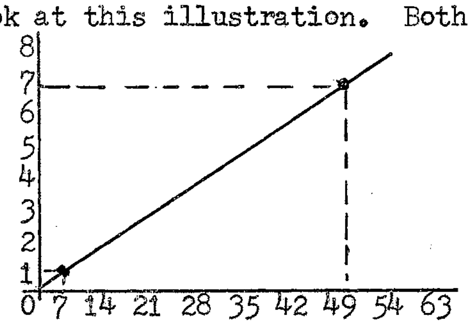
---

factors

When both factors are the same number, we only need to learn one multiplication combinations. i.e., $6 \times 6 = 36$

2. Look at this illustration. Both factors are ________.

   a.



$$\frac{7}{1} = \frac{\square}{7}$$

   b. The horizontal line was numbered as groups of ________.

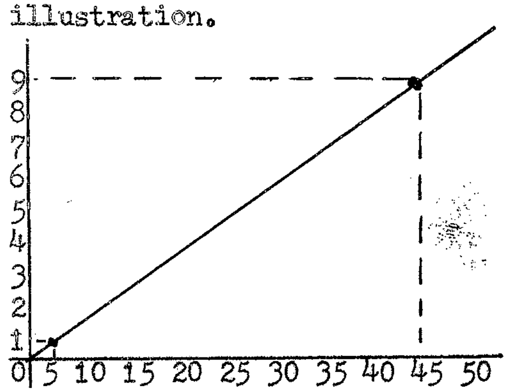
---

a. 7, 7 b. 7's

3. The illustration in #2 shows that $7 \times 7 =$ ________.

$7 \times 7 = 49$

4. Jim was absent from school nine weeks. The children wanted to know how many days Jim was absent. Bob knew that there were five school days each week. So they decided to draw this illustration.



a. Bob said that $\frac{5}{1}$ as $\frac{\square}{9}$

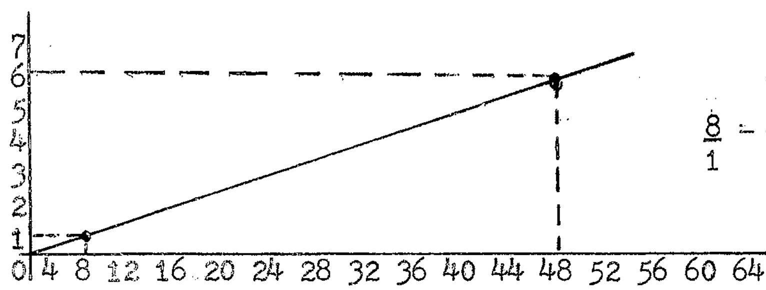
b. so 9 x 5 = ________.

---

a. $\frac{5}{1}$ as $\frac{45}{9}$ b. 9 x 5 = 45

The horizontal line in 4 was numbered in groups of 5's.

5. In order to play a card game, each player needed eight (8) cards. Tom said, "We need to have ________ cards if six (6) of us are going to play."



$\frac{8}{1} = \frac{\square}{6}$

Tom said that he knew 5 x 8 = 40 therefore 6 x 8 must be just 8 more. The answer just had to be 48.

6 x 8 = ________

8 x 6 = ________

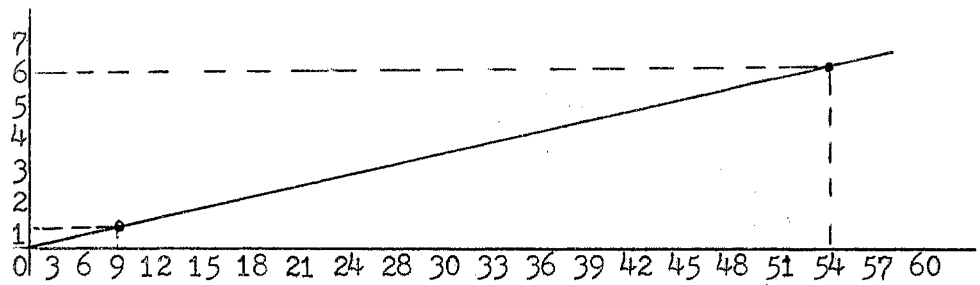
---

48 48 48 48

RATIO TO ONE

Lesson 12:

1. Mary said, "We have 9 girls in each Girl Scout troup. There are six troups in this school. I wonder how many girls are Girl Scouts. If I draw an illustration showing 9 girls for every troup, it will look like this."



a. $\frac{9}{1} = \frac{\square}{6}$

b. 9 x 6 = ___

---

a. 54 b. 54

2. Jane said, "If we had 6 girls in each Girl Scout troup and had 9 troups, we would have had__________Girl Scouts."

54

3. Miss Brown said, "The other day we learned that 6 x 8 = _____. Today, we want to know how much 7 x 8 equals. Does anyone know how we can find the answer?

48

4. Tom said, "All we need to do is add one more 8."

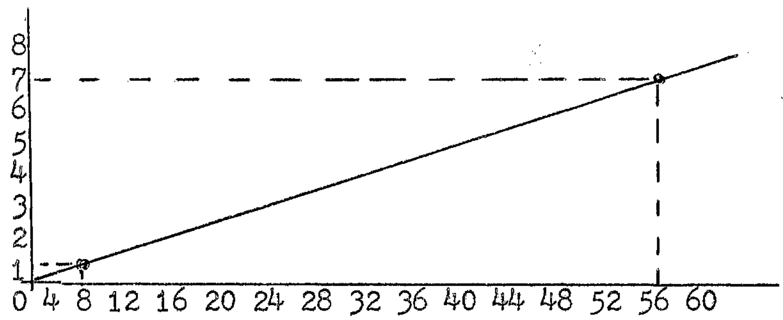
a. 6 x 8 = ___

b. 7 x 8 = ___

---

a. 48 b. 56

5. What multiplication fact does this illustration show?



$\frac{8}{1} = \frac{\square}{7}$

---

7 x 8 = 56

6. Complete this exercise.

a. 6 x 8 = ___ b. 8 x ___ = 48

c. ___ x 7 = 56 d. 7 x 8 = ___

---

a. 48 b. 6 c. 8 d. 56

RATIO TO ONE

Lesson 13:

1. Miss Brown began the arithmetic class by saying, "There is one more property of multiplication of whole numbers that we can discover with our multiplication facts. Do you remember that we do the operation indicated within the parenthesis before we do the other operation? i.e., 4 x (2 + 5) = 4 x 7. The symbol of + within the parenthesis ( ) indicated that we add before we multiply by 4."

2. In the example 4 + (2 x 3), the symbol within the parenthesis tells us to ______________ before we add the 4.
   add   multiply

---

multiply

3. Before we try to discover the new multiplication of whole numbers' property, we had better check to see if we can get the correct answers for this exercise.

   a. 12 + (3 + 4) = 12 + _____ = 19

   b. 6 + (2 + 5) = 6 + _____ = 13

   c. 3 x (1 + 2) = 3 x _____ = 9

   d. 3 x (3 x 2) = 3 x _____ = 18

   e. 4 x (4 x 2) = 4 x _____ = 32

   f. (2 x 3) + 6 = _____ + 6 = 12

---

a. 7   b. 7   c. 3   d. 6   c. 8   f. 6

The children played a game at Mary's birthday party. For each game Mary put 4 plates with 2 cups on each plate on the table. In each cup she put 3 peanuts. How many peanuts did she need for the game?

4. John said, "We need to ______________ to find the answer
add multiply
the easiest way."

---

multiply

5. "How can we multiply three numbers?" asked Jane. "We learned that multiplication was a binary operation so we can only multiply ________ factors at one time."

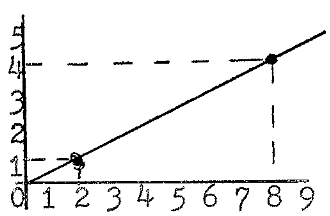
2

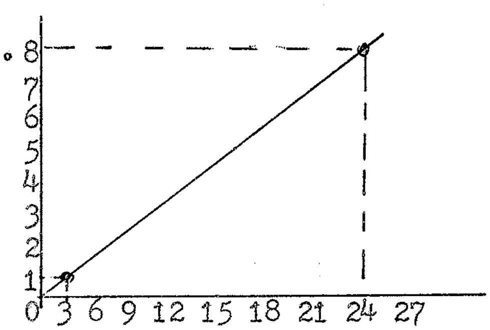
6. John said, "Watch me. This is what I will do. I will multiply just two factors at one time. I will use the factors 4, 2, and 3. First I will multiply 4 x 2 = 8. Then I will use 8 as a factor. I will multiply 8 x 3 = 24. See, I just multiplied 2 factors at one time."

Mary needs__________ peanuts.

24

7. Here is how we can write John's example so that we know which two factors to multiply. (4 x 2) x 3 = 24

1. 

2. 

$\frac{2}{1} = \frac{8}{4}$ $\frac{3}{1} = \frac{24}{8}$

4 x 2 = 8 8 x 3 = 24

First 4 x 2 = 8 Then 8 x 3 = 24

8. To work these examples in #5 we needed two illustrations because multiplication is a binary operation and that means that we can only multiply _____ factors at one time.

---

two

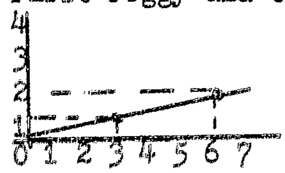
9. "I think that we should have found the number of peanuts in the two cups on each plate first. After we find the number of peanuts on each plate, we can find how many are needed for the 4 plates," commented Peggy.

Peggy needed to put the parenthesis ( ) around the 2 and 3 like this.

4 x (2 x 3) =

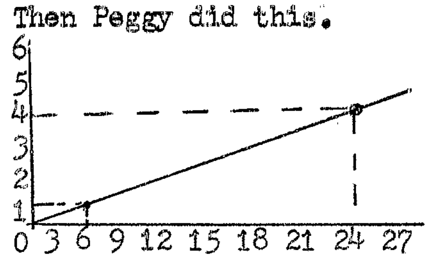
4 x ____ = 24

1. First Peggy did this.



$\frac{3}{1} = \frac{6}{2}$

2. Then Peggy did this.



$\frac{6}{1} = \frac{24}{4}$

4 x 6 = ______

---

6 4 x 6 = 24

10. Did you notice that although John's and Peggy's groups were different they got the same result. Both Peggy and John got ________ for the answer.

24

Do you think that this always happens when we multiply whole numbers?

11. Miss Brown asked the children to work the following exercise to see if they could discover the new property.

a. 3 x (2 x 4) = 3 x _____ = 24

(3 x 2) x 4 = _____ x 4 = 24

b. (5 x 1) x 2 = _____ x 2 = 10

5 x (1 x 2) = 5 x _____ = 10

c. 3 x (3 x 2) = 3 x _____ = 18

(3 x 3) x 2 = _____ x 2 = 18

d. $4 \times (2 \times 3) = 4 \times$ ______ $= 24$

$(4 \times 2) \times 3 =$ _____ $\times 3 = 24$

---

a. 8
6

b. 5
2

c. 6
9

d. 6
8

12. Bill said, "I know what the new property is. The factors were kept in the same order. The parenthesis was around two of the ____________. One time it was around the first two factors. The next time it was around the last two factors. We multiplied the numerals within the parenthesis first. The order of the factors ______________ change."
did did not

---

factors did not

The groups did change.

a. $(4 \times 2) \times 3 =$ ______ $\times 3 = 24$

b. $4 \times (2 \times 3) = 4 \times$ _____ $= 24$

---

a. 8 b. 6

13. Bill had discovered the associative property for the multiplication of whole numbers.

If one has three factors, the associative property for the multiplication of whole numbers means that the order of the factors will remain the same but that the groups will change. The result is the ________________ when the associative property is true. same different

---

same

14. When using only the associative property, I _____________ change the order of the factors. do do not

---

do not

15. When using only the associative property, I _____________change the grouping. do do not

---

do

RATIO TO ONE

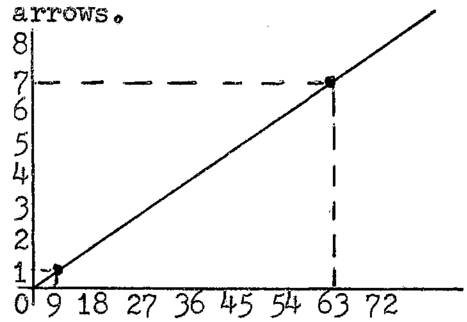
Lesson 14:

1. Chuck bought seven (7) arrows at nine cents (9¢) an arrow. How much did he have to pay for the arrows:

   a. Chuck said,"It costs 9¢ for one arrow, so it will cost ________ for 7 arrows.

   b. $\frac{9¢}{1} = \frac{\square\ ¢}{7}$

   c. 7 x 9 = ______



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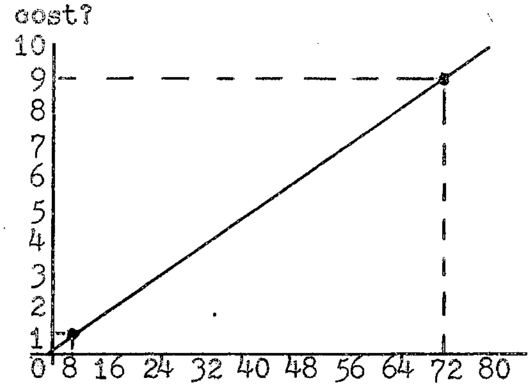
a. 63¢ b. 63 c. 63

2. If 9 x 7 = 63, then 7 x 9 = ________.

3. Mary bought 9 balloons for her party. Each ballon cost 8¢. How much did the balloons cost?

   a. $\frac{8¢}{1} = \frac{\square\ ¢}{9}$

   b. 9 x 8 = ________



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a. 72¢ b. 72

4. Because the commutative property is true for the multiplication of whole numbers, if 9 x 8 = 72 then 8 x 9 = ___.

72

5. In lessons 7, 8, and 9 we learned that the distributive property was very useful. This property was true for the multiplication of whole numbers. This means that if I don't know a multiplication fact such as 6 x 9, I could rename one of the factors, multiply each part of the renamed factor, and then add the two products together.

Suppose that you did not know that 6 x 9 = 54. Would you get the same answer if you renamed the factor 6 as (4 + 2)? Check to see.

a. (4 + 2) x 9 =

b. (4 x 9) + (2 x ______) =

c. ______ + 18 = 54

---

b. 9 c. 36

6. We didn't need to rename the factor 6 as (4 + 2). We could have renamed it as (3 + 3). Then:

a. (3 + 3) x 9 =

b. (3 x 9) + (____ x 9) =

c. ______ + 27 = 54

---

b. 3 c. 27

It didn't make any difference what we renamed the six. We just had to be sure that the two parts could be added together to be six.

7. Today we also learned that 8 x 9 = 72. Rename one of the factors to see if the distributive property works for this multiplication fact. Hint: You might decide to rename 8 as (7 + 1), (6 + 2), (5 + 3), or (4 + 4). Let's use one of the suggestions.

a. (5 + ______) x 9 =

b. (_____ x 9) + (3 x 9) =

c. ______ + 27 = 72

Did our choice work? ____________
yes no

---

a. 3 b. 5 c. 45 yes

8. Of course, you could have decided to rename the factor 9 instead of the factor 8. You might have renamed 9 as (8 + 1), (7 + 2), (6 + 3), or (5 + 4). Would you have found the same products of 72? Let's try one of the choices.

a. 8 x (8 + 1) =

b. (8 x ____) + (8 x 1) =

---

b. 8 c. 64

9. In lessons 4 and 5, we learned about the commutative property. The commutative property says that the order of the factors is a multiplication fact will not change the product.

a. If 6 x 9¢ = 54¢, then 9¢ x 6 = ______

b. If 8 x 9 = 72, then 9 x ____ = 72

---

a. 54¢ b. 8

10. When we learned about the associative property, we learned that the order of the factors remained the same but the groups changed. This means that if I have three factors, 3 x 2 x 4, I can say:

a. (3 x 2) x 4 =

______ x 4 = 24

or I can say

b. 3 x (2 x 4) =

3 x ______ = 24

The product was the same for both ways of grouping the factors.

---

a. 6 b. 8

11. Does the associative property work for these factors?

2 x 3 x 3 =

a. I can group like this: (2 x 3) x 3 =

______ x 3 = 18

or

b. I can group like this: 2 x (3 x 3) =

2 x ______ = 18

---

a. 6 b. 9

12. Do you remember about the closure property? If both factors are whole numbers, the product must be a whole number if the closure property is true for multiplication of whole numbers.

Both of the factors in the multiplication fact $8 \times 9 = 72$ are whole numbers. The product is 72. We know that 72 is a whole number. The closure property holds for multiplication of whole numbers.

a. $6 \times 9 =$ ________

b. Six (6) is a whole number.

c. Nine (9) is a w______ number.

d. The product seventy two (72) is a w________ number.

e. Therefore, the cl__________ property is true for the multiplication of whole numbers.

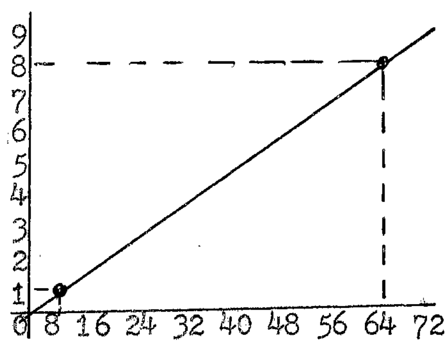
---

a. 54 c. whole d. whole e. closure

RATIO TO ONE

Lesson 15:

1. "There are just two multiplication facts left to learn," said Miss Brown. "Look at the two diagrams. See if you can write the fact for each."



a. $\frac{8}{1} = \frac{\square}{8}$

b. 8 x 8 = ____

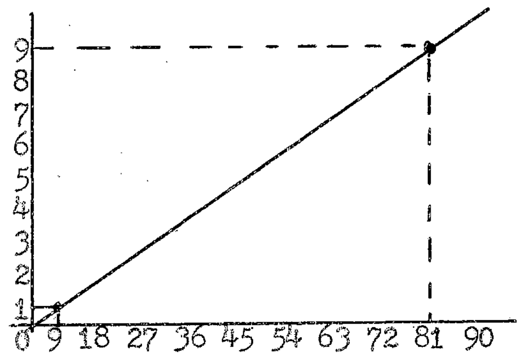
---

a. $\frac{8}{1} = \frac{64}{8}$ b. 64

2.

a. $\frac{9}{1} = \frac{\square}{9}$

b. 9 x 9 = ____



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a. $\frac{9}{1} = \frac{81}{9}$ b. 81

3. The following exercise contains some of the more difficult facts that you have studied. Check to see if you know all of these facts.

a. 7 x 6 = ____

b. 7 x 9 = ____

c. 8 x 7 = ____

3. continued

d. 8 x 8 = ______ e. 6 x 9 = ______

---

a. 42 b. 63 c. 56 d. 64 e. 54

During the lessons we have learned that certain things are true about the multiplication of whole numbers. As this is the last lesson, see if you can choose the word that names the property I am illustrating.

4. One of the multiplication facts in today's lesson was 8 x 8 = 64. All of the numbers used in this multiplication fact were whole numbers. The example 8 x 8 = 64 shows the ____________
closure associative
property for the multiplication of whole numbers.

---

closure

5. The fact that 6 x 9 = 54 makes us sure that 9 x 6 = 54, also. This is an example to show the ____________ property
associative commutative
for the multiplication of whole numbers.

---

commutative

6. Do you remember something about the number one (1)? It acts very different from other multiplication factors. Whenever one is a factor in a multiplication fact, the product is the same as the other factor. For example, 6 x 1 = 6. In this case the number one is called the ____________
closure multiplicative identity

---

multiplicative identity

7. Sometimes when we don't know the product for a combination, we use one of the properties we have studied. If I don't know that 9 x 9 = 81, I could use this property. I would rename one of the factors, multiply each part, and add the products.

Example: 9 x (5 + 4) =
(9 x 5) + (9 x 4) =
45 + 36 = 81

In this example the ____________ property
associative distributive
illustrated.

---

distributive

8. We studied just one more property. We had to be careful to keep the factors in the same order when we used this property. We changed the groups though.

Example:	$4 \times 2 \times 3 =$		$4 \times 2 \times 3 =$
	$(4 \times 2) \times 3 =$	OR	$4 \times (2 \times 3) =$
	$8 \times 3 = 24$		$4 \times 6 = 24$

This example illustrates the ______________ property.
associative distributive

---

associative

How did you do in naming the properties? Just think---even if you had only one right, that was one more than you knew when you started the lessons.

APPENDIX D

Approach________________

MULTIPLICATION

Name______________________ Date______________

Write the Missing Factor or Product

1. 2 x 9 = ___
2. 5 x 6 = ___
3. 4 x 0 = ___
4. 9 x 6 = ___
5. 6 x 1 = ___
6. 7 x 3 = ___
7. 8 x 7 = ___
8. 1 x 9 = ___
9. 8 twos are ____
10. 8 ones are ____
11. 6 sevens are __
12. 8 x 0 = ___
13. 1 x 5 = ___
14. ___ x 9 = 0
15. 9 x ___ = 0
16. 9 x ___ = 72
17. 5 x ___ = 20
18. 7 x ___ = 28
19. 6 x 2 = (3 x 2) x 2 = ___
20. 7 x ___ = 56
21. (3 x 2) x 2 = ___
22. 4 x 6 = ___ x 4
23. 4 x 4 = (2 x ___) x 4
24. Since 4 x 6 = 24, we know that 6 x 4 = ___.

For each statement you have four possible answers. Circle the letter that you find before the correct answer.

25. If □ and △ are whold numbers, the product of □ and △ is

a. a whole number
b. one more than either □ or △
c. any number
d. all of the above are correct

26. Which of the following could be used to show that 4 x 3 = 3 x 4?

a. 3 + 3 + 3 + 3 = 4 + 4 + 4
b. 3 x 3 x 3 x 3 = 4 x 4 x 4
c. 4 x 4 x 4 x 4 = 3 x 3 x 3
d. 4 + 4 + 4 + 4 = 3 + 3 + 3

27. Nancy had 8 bags of jacks with 6 jacks in each bag. She gave all of these jacks to Mary. Mary took 6 bags in which to put the jacks. She put the same number of jacks in each of the six bags. Mary put____jacks into each bag.

a. 6
b. 8
c. 42
d. 48

28. The product of a specific whole number and one is

a. that whole number
b. always one
c. never one
d. it is impossible to say without knowing the number

29. Suppose that your teacher read the following example for you to do mentally. She said, "Four times one, (pause) times three, (pause) times zero." Your answer would have been.

a. zero
b. four
c. eight
d. twelve

30. Miss White has 24 boxes of pencils. There are 15 pencils in each box. This is a total of 360 pencils. Without doing any multiplying can you answer this question? If Miss White had 15 boxes with 24 pencils in each box, she would have ________ pencils.

a. 320
b. 340
c. 360
d. 380

31. Mary saw 4 sets of 2 cups each on the table. She noticed that each cup had 3 walnuts in it. Mary decided that there were 24 walnuts in all. Which of the following shows that when Mary worked the problem she found the number of cups first?

a. (4 x 2) x 3 = 24
b. 4 x (2 x 3) = 24
c. 3 x 4 x 2 = 24
d. 4 x 2 x 3 = 24

32. Without doing any multiplying which of the following shows a correct solution for 16 x 8?

a. (9 x 8) + (7 x 8) = 16 x 8
b. (9 x 8) x (7 x 8) = 16 x 8
c. (9 x 4) + (7 x 4) = 16 x 8
d. (9 x 4) x (7 x 4) = 16 x 8

33. Multiplication of whole numbers is

a. a binary operation
b. a mathematical property
c. an array
d. a number system

34. Which of the following is a whole number?

a. 4,648
b. 46/48
c. 4.468
d. -4648

35. If □ and △ are whole numbers and □ x △ = ○, the number that ○ represents must be in which set?

a. {1/2, 1/4, 2/4, 1/3, 2/3, ...}
b. {0, 1, 2, 3, 4, 5, 6, 7, ...}
c. {-1, -2, -3, -4, -5, -6, ...}
d. any of the above sets

36. If in a multiplication fact one (1) is a factor, then the product is

a. never 1
b. always larger than 1
c. always the other factor
d. all of the above are correct

37. Multiplying five by seven can be thought of as

a. increase five by seven
b. add five sevens
c. increase five by twelve
d. add seven fives

38. Which of the following addition examples could you have worked by multiplying?

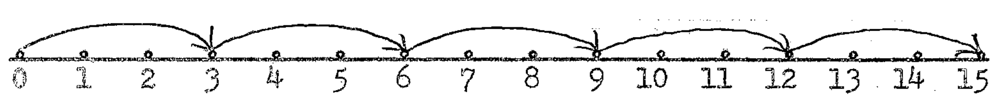
a. 42 + 14 + 23 =
b. 42 + 42 + 42 =
c. 42 + 42 + 42 + 59 =
d. none of the above

39. Zero times any number is

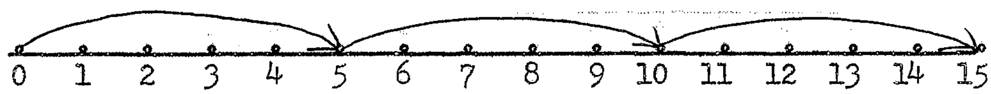
a. always zero
b. never zero
c. larger than zero
d. the other factor

40 Look at these number lines. Think about the arrows and write the multiplication fact for each number line.

a.



a. ____________________
fact

b. 

b. ______________
fact

41. In front of each statement write t (always true), s (sometimes true), or f (never true).

_____ a. The product of a whole number times 1 is 1.

_____ b. If a, b, c, are whole numbers, the product of these numbers may be obtained by grouping any two of the whole numbers together, multiplying, and then multiplying the answer by the remaining whole number.

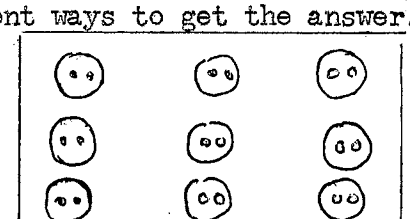
_____ c. Zero times any whole number is zero.

_____ d. To multiply seven times three means to add three sevens.

42. Look at this card of buttons. We want to find the total number of holes in the buttons. Put parentheses ( ) in the following examples to show two different ways to get the answer.

a. $2 \times 3 \times 3 = 18$

b. $2 \times 3 \times 3 = 18$



43. Draw a picture or diagram to show that $4 \times 6 = (4 \times 4) + (4 \times 2)$.

44. Mrs. Brown packed 5 boxes of cupcakes to give to her friends. Each box contained 2 chocolate cupcakes and 4 vanilla cupcakes. How many cupcakes did Mrs. Brown pack for her friends? Which of the following shows how Mrs. Brown could find out how many cupcakes she needed to pack?

a. $(5 \times 2) + (5 \times 4) = 30$
b. $(5 \times 2) \times (5 \times 4) = 30$
c. $(2 \times 4) + (4 \times 5) = 30$
d. $(2 \times 4) \times (4 \times 5) = 30$

45. What number is represented by $\square$ so that each of these statements is true?

a. If $7 \times \square = 8 \times 7$, then $\square$ is _____.

b. If 8 x (10 + [ ]) = (8 x 10) + (8 x 9), then [ ] is ______.

c. If 6 x (3 x 7) = (6 x 3) x [ ], then [ ] is ______.

46. Study this chart. Write the numerals in the blanks that make the statements true.

Boys	3	6	__	12	__	18	21	__	27
Tents	1	2	3	__	5	__	__	8	__

a. There are ______ boys for 3 tents.

b. There are ______ boys for 8 tents.

c. There are ______ boys for 5 tents.

47. Without doing the multiplying indicate the mathematical property that is being illustrated in each example. Write the correct numeral that you find in front of the name of the property in the left-hand column. A property may be used more than one time.

1. The commutative property
2. The associative property
3. The distributive property
4. The multiplicative identity
5. Closure for multiplication of whole numbers

Property	Equation
a. ____________	a. 12 x (6 x 9) = (12 x 6) x 9
b. ____________	b. 376 x 892 = 892 x 376
c. ____________	c. 65932 x 1 = 65932
d. ____________	d. 8 x (9 x 7) = (9 x 7) x 8
e. ____________	e. 43 x 20 = 860
f. ____________	f. 57 x (10 + 6) = (57 x 10) + (57 x 6)

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