# EXTREMAL STRUCTURE OF STAR-SHAPED SETS

By

FREDDIE EUGENE TIDMORE

Bachelor of Science
Hardin-Simmons University
Abilene, Texas
1962

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1963

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Thesis Approved:

E. K. McLachlan
Thesis Adviser

Hiroshi Uehara

Jeanne Agnew

Richard L. Cummins

N Durham
Dean of the Graduate College

## PREFACE

The basic problem of this thesis is the study of the relationships that exist in a star-shaped set S between the extremal structure of S and the convex kernel of S. The extremal structure considered not only includes the familiar extreme point, but also involves a generalization of extreme point, an $\alpha$-extreme point, and relative extreme point. The results give additional information on the characterization of star-shaped sets. Most of the topics discussed are illustrated by examples and counterexamples.

Chapter I gives the background associated with the problem and introduces the notation and terminology that is used throughout the study. Chapter II deals with the topic of $\alpha$-extreme points of star-shaped sets. It is shown that the convex kernel of a compact star-shaped subset S of a locally convex space L is completely determined by the $\alpha$-extreme points of S. The cardinality of the set of $\alpha$-extreme points is determined for a compact star-shaped set in a locally convex space of dimension greater than two. Also given is the result that any compact star-shaped subset S of a normed linear space L contains a countable set of $\alpha$-extreme points which determines the convex kernel of S.

In a star-shaped set S the points which are extreme relative to the convex kernel of S are used in Chapter III to give a result similar to the Krein-Milman theorem. This result shows that a compact star-shaped set S in a locally convex space L is completely determined by the convex kernel of S and the subset of points of S that are extreme relative to the convex kernel of S. Chapter IV introduces the polyhedral star-shaped set, the star-shaped set analogous to the convex polytope in the setting of convex sets. The polyhedral star-shaped set is discussed because of the simplicity of its extremal structure. Sufficient conditions are given for a set to be a polyhedral star-shaped set in the linear space $E_2$. The setting for Chapter V is the metric space of compact subsets of a normed linear space L. It is shown that any compact star-shaped subset S of a normed linear space L can be approximated by a polyhedral star-shaped set. This approximation makes some of the advantages of the simple extremal structure of polyhedral star-shaped sets available for the study of more general star-shaped sets. Sufficient conditions are given for the sequence of convex kernels $\{ck(A_i)\}$ to converge to the convex kernel of A if the sequence $\{A_i\}$ converges to A. It is shown that for any compact star-shaped subset S of a normed linear space L there exists a sequence of polyhedral star-shaped sets which converges to S such that the associated sequence of convex kernels converges to the convex kernel of S. A constructive procedure is given for finding a polyhedral star-shaped set which approximates a compact star-shaped subset of $L_n$.

Recognition is due numerous individuals for their assistance in the graduate work that preceded this study. Professor E. K. McLachlan merits special recognition for his valuable guidance and encouragement throughout the preparation of this thesis. Indebtedness is expressed to the National Science Foundation for its financial support through a Cooperative Graduate Fellowship and a Summer Research Assistantship.

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# CHAPTER I

## INTRODUCTION

This study is primarily concerned with the combination of two basic topics which are part of the area of study known as convexity. The first of these topics is centered around the discussion of extremal elements of convex sets; for some time this particular subject has been a fruitful area of research which has produced findings by Klee [7], Price [11], Krein and Milman [10], and numerous others. The second of these topics deals with the study of star-shaped sets. Although star-shaped sets have been of considerable interest for some time, only recently has work in this area been very widespread. Among those investigators in this area are Valentine [14], Hare and Kenelly [6].

Valentine provided much of the motivation for this study when in [13] he suggested that the convex kernel might be a basis for characterizing star-shaped sets. The observation of numerous examples led to the finding of various relationships that exist between the convex kernel of a star-shaped set S and the extremal structure of S. Thus, the study of the extremal structure of star-shaped sets is used to investigate the convex kernel of star-shaped sets.

The setting for each discussion throughout this study is some real linear space. In some cases a topology on the linear space is needed; in such cases the space will be a linear topological space, which will always have a Hausdorff topology.

Much of the notation that will be used throughout the discussion comes from [13]. The convex kernel of a star-shaped set S will be denoted by ckS, the line segment $\{\alpha x + (1-\alpha)y: \alpha \in [0,1]\}$ will be denoted by xy, the ray $\{\beta y + (1-\beta)x: \beta \geq 1\}$ will be denoted by $xy^\infty$ and L(x,y) will denote the line containing x and y, $x \neq y$. The convex hull of a set S will be denoted by conv S. The notation intv S will denote the interior of S relative to the minimal flat that contains S. Euclidean n-space will be denoted by $E_n$ and $L_n$ will denote an n-dimensional Minkowski space. The set of real numbers will be denoted by R. The set $\{x: f(x) = \alpha\}$, where f is a linear functional, will be denoted by $[f:\alpha]$. The set A\B is the collection of points that belong to A and do not belong to B. The interior of S will be denoted by int S, $\overline{S}$ will be the closure of S and bd S will denote the boundary of S.

## CHAPTER II

## GENERALIZED EXTREME POINTS

In studying sets of points, quite often it is possible to find classes of sets such that the structure of a set in the class is dictated to a great extent by that of some proper subset of the set. Convex sets are an example. In particular, if a convex set S in a linear topological space L is compact, the collection of extreme points of such a set are of considerable assistance in describing that set. A number of authors, among them Price, Klee, Krein and Milman, have examined the extremal structure of convex sets. Perhaps the most notable result is the Krein-Milman theorem [10].

The purpose of this chapter is to examine the relationships that exist in a compact star-shaped set between the extremal structure of that set and its convex kernel. Even if it is known that a given set in a linear space L is star-shaped, it is no small task in many cases to determine the convex kernel of S. The extremal structure of star-shaped sets will first be used to assist in determining the convex kernel of a star-shaped set of a somewhat general nature.

The definition of extreme point of a convex set is readily adapted to the setting of star-shaped sets; hence, one can use that concept for a study of these sets.

Definition 2.1: If S is a star-shaped set in a linear space L, then a point $x \in S$ is an extreme point of S if there is no nondegenerate line segment in S which contains x in its relative interior.

Asplund [2] has generalized the idea of extreme point of a convex set and has extended some of the results of Klee which deal with extreme points. The following definition extends the class of sets on which such points are defined to include all star-shaped sets.

Definition 2.2: If S is a star-shaped set in a linear topological space L, then a point $x \in S$ is an $\alpha$-extreme point of S if there does not exist an $\alpha$-dimensional flat F and a neighborhood U of x such that $x \in F \cap U \subset S$. The collection of $\alpha$-extreme points of S will be denoted by $\text{ext}_\alpha S$.

Unless stated otherwise in the discussion of $\text{ext}_\alpha S$ in some linear space L, it will be assumed that $\alpha$ is the dimension of the hyperplanes in the space L. If $\alpha$ and $\beta$ are cardinals, $\alpha \leq \beta$, then $\text{ext}_\alpha S \subset \text{ext}_\beta S$. The above definition can be made completely algebraic in nature, particularly for finite-dimensional sets, by using an n-simplex instead of an n-flat and neighborhoods. However, the given definition is more convenient if no distinction is to be made on the dimension of the sets involved.

The following example illustrates the two previous definitions.

Example 2.1: Consider the linear space $E_3$ and its natural basis $\{e_1, e_2, e_3\}$. Let $S = \text{conv}\,\{0, e_1, e_2, e_3\}$ (cf. Fig. 2.1). Then ext S $= \{0, e_1, e_2, e_3\}$, and $\text{ext}_2 S$ is made up of the following line segments: $e_1e_2$, $e_1e_3$, $e_2e_3$, $0e_1$, $0e_2$, $0e_3$.

Krasnosel'skii has made use of the concept of an x-star of a set in the proof of his now famous theorem on sufficiency conditions for a set

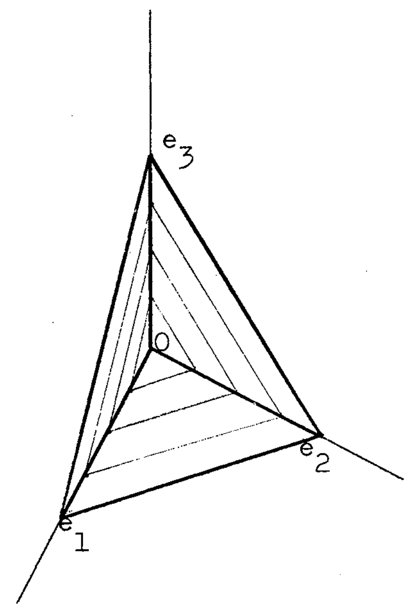


Figure 2.1

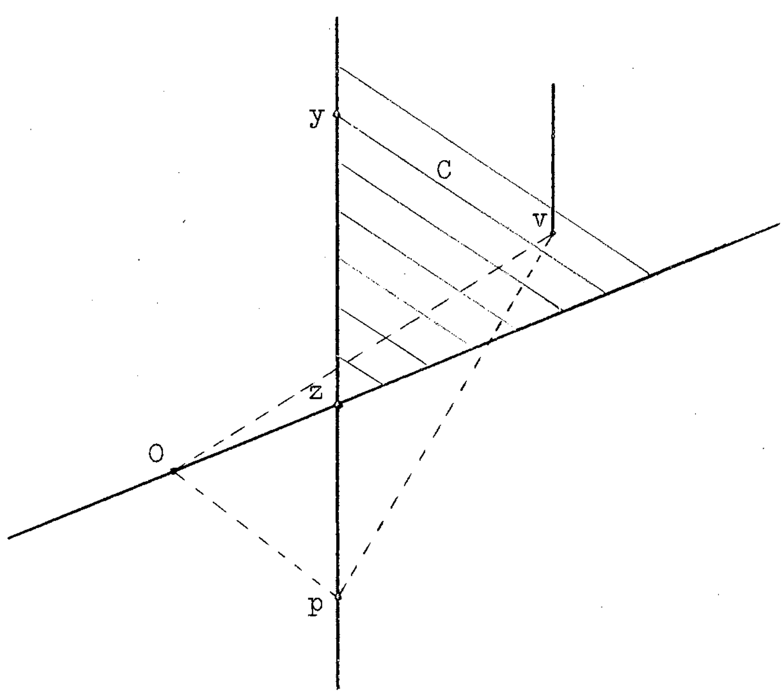


Figure 2.2

to be star-shaped [9]. For completeness, the definition of this set is given below.

Definition 2.3: Let x be a point of S, a subset of a linear space L; the set of all points y such that $xy \subset S$ is the x-star of S, and will be denoted by $S_x$.

The following lemmas reveal properties of x-stars which will be of considerable value in subsequent proofs.

Lemma 2.1: If S is a closed subset of a linear topological space L, then for any $x \in S$, $S_x$ is a closed set.

Proof: Let $x \in S$, and let q be a limit point of $S_x$. For an arbitrary $\alpha \in (0,1)$ consider any neighborhood U of $\alpha q + (1-\alpha)x$. Then

$$\frac{1}{\alpha} U + \frac{\alpha-1}{\alpha} x$$

is a neighborhood of q. Since q is a limit point of $S_x$ there exists a point y which belongs to

$$S_x \cap \left(\frac{1}{\alpha} U + \frac{\alpha-1}{\alpha} x\right),$$

which implies that

$$y = \frac{1}{\alpha} u + \frac{\alpha-1}{\alpha} x$$

for some $u \in U$. But $\alpha y + (1-\alpha)x \in S$ since $y \in S_x$, and

$$\alpha y + (1-\alpha)x = \alpha\left(\frac{1}{\alpha} u + \frac{\alpha-1}{\alpha} x\right) + (1-\alpha)x = u,$$

which belongs to $S \cap U$. Thus, every neighborhood of $\alpha q + (1-\alpha)x$ contains a point of S, which implies that $\alpha q + (1-\alpha)x$ is a limit point of

S. The fact that S is closed implies that $\alpha q + (1-\alpha)x \in S$. Clearly, $q \in S$, and since $\alpha$ was arbitrary, $qx \subset S$. This yields the fact that $q \in S_x$, so that the set $S_x$ is closed.

<u>Lemma 2.2</u>: If S is a compact subset of a linear topological space L, then for any $x \in S$, $S_x$ is a compact set.

<u>Proof</u>: Since L is a Hausdorff space, S is closed. For any $x \in S$ the above lemma gives the fact that $S_x$ is closed. Then $S_x$ is compact, since any closed subset of a compact set is compact.

If S in a linear space L is a star-shaped set, then it is clearly true that

$$\text{ckS} = \bigcap_{x \in S} S_x.$$

That is, a point p belongs to the convex kernel of S if, and only if, xp is contained in S for all $x \in S$. The latter statement is true if, and only if, p belongs to $S_x$ for all $x \in S$. The previous identity suggests the following definition.

<u>Definition 2.4</u>: In a linear space L a subset T of a star-shaped set is said to star-generate the convex kernel of S if

$$\text{ckS} = \bigcap_{x \in T} S_x.$$

Such a subset T is said to be a star-generating set for ckS.

As noted above, a star-shaped set S star-generates its convex kernel, or equivalently, is itself a star-generating set of its convex kernel. A question of interest now is the possibility of finding proper subsets of a star-shaped set that star-generate the convex kernel, and indeed, the possibility of finding such sets that are minimal. It is at this point that use is made of the extremal structure in the determination of the convex kernel of a star-shaped set.

Theorem 2.1: Let L be a locally convex space and S a compact star-shaped subset of L. Then

$$\text{ckS} = \bigcap_{x \in A} S_x,$$

where $A = \text{ext}_\alpha S$.

Proof: It may be assumed without loss of generality that $0 \in$ ckS. Let $p \in S \backslash$ckS. Then there exists a point $y \in S$ such that $py \not\subset S$. Since S is compact, y may be chosen such that if $u = \lambda y + (1-\lambda)p$, $\lambda > 1$, then $u \notin S$. Since $py \not\subset S$, there exists a point $z \in$ intv py such that $z \notin S$. Consider the convex cone $C = \{\alpha y + (\beta-\alpha+1)z: \alpha, \beta \geq 0\}$, which has vertex z and is contained in the subspace L' with basis $\{p,y\}$ (cf. Fig. 2.2). If $y \in A$ then since $p \notin S_y$,

$$p \notin \bigcap_{x \in A} S_x.$$

If $y \notin A$, then there exists a hyperplane H' and a neighborhood U of y such that $y \in H' \cap U \subset S$. Since $S \cap \text{intv } py^\infty = \emptyset$, H' must intersect L' in some line other than L(p,y). Thus, $H' \cap L'$ is a line which contains a segment that is contained in (intv C) $\cap$ S, which implies that (intv C) $\cap S \neq \emptyset$. Since L is locally convex there exists a continuous linear functional f defined on L such that $f(q) = 1$ for every $q \in L(p,y)$; clearly, $0 \notin L(p,y)$ since $py \not\subset S$ and $0 \in$ ckS. The cone C is closed and S is compact; hence, $C \cap S$ is compact. Then the continuous linear functional f', the restriction of f to L', attains a maximum at some point

$w \in C \cap S$; since $S \cap Oz^{\infty} = \emptyset$ and $f'(q) = 1$ for $q \in L(q,y)$, then $f'(w) > 1$ and $w \in \text{intv } C$. Let $H = [f:f(w)]$. Since $H \cap C \cap S$ is a compact subset of the 1-dimensional set $H \cap L'$, there exists a minimal closed line segment in intv C which contains $H \cap C \cap S$. This segment is contained in intv C since $Oz^{\infty} \cap S = \emptyset$ and $(H \cap C) \cap pz^{\infty} = \emptyset$. Each end-point of this line segment, which may be degenerate, must be a point of A. For, if not, the argument used above to show that $(\text{intv } C) \cap S \neq \emptyset$ when $y \notin A$ can be used to show that $f'$ does not attain its maximum on $C \cap S$ at w. Let v be one of these end points. The points p, y, z and v are all contained in $L'$. If $pv \subset S$, then the fact that $O \in \text{ck}S$ implies that $\text{conv}\{O, p, v\} \subset S$; since $z \in \text{conv}\{O, p, v\}$, $z \in S$, a contradiction. Hence, $pv \not\subset S$ and

$$p \notin \bigcap_{x \in A} S_x.$$

It then follows that

$$S \setminus \text{ck}S \subset S \setminus \bigcap_{x \in A} S_x$$

and that

$$\bigcap_{x \in A} S_x \subset \text{ck}S.$$

This conclusion, along with obvious set inclusion

$$\text{ck}S \subset \bigcap_{x \in A} S_x,$$

implies that

$$\text{ck}S = \bigcap_{x \in A} S_x.$$

A valid question that might be asked about the above theorem is the following: Is it necessary to use the larger set $\text{ext}_\alpha S$ rather than the set ext S? Since ext S = $\text{ext}_\alpha S$ for S in a 2-dimensional space, this question is of interest in linear spaces of dimension greater than two. The following example shows that it is not always sufficient to consider only ext S as a star-generating set for the convex kernel of a compact star-shaped set S if the dimension of the set is greater than two.

Example 2.2: Let the linear space L be in $E_3$, and let $\{e_1, e_2, e_3\}$ be its natural basis. Let

$$S = \bigcup_{i \neq j} \text{conv}\{0, e_i, e_j\},$$

i, j $\in$ {1, 2, 3} (cf. Fig. 2.1). Clearly, ckS = {0}, ext S = $\{0, e_1, e_2, e_3\}$, but

$$\bigcap_{x \in \text{ext } S} S_x = \bigcup_{i=1}^{3} 0e_i,$$

which is not ckS. The set of 2-extreme points for the above set is the same as for Example 2.1.

It can be readily verified in the above example that the 2-extreme points actually do star-generate the convex kernel of S. A close examination of that example reveals several questions concerning the existence of proper subsets of $\text{ext}_\alpha S$ that star-generate the convex kernel of S. First, does there always exist a proper subset? Second, can anything be said about the cardinality of these star-generating subsets? It may be noted in Example 2.2 that a finite subset of $\text{ext}_2 S$ exists which star-generates ckS; one such subset is $\{\frac{1}{2}(e_1 + e_2), \frac{1}{2}(e_1 + e_3), \frac{1}{2}(e_2 + e_3)\}$.

The following example provides an answer for the first question; in some cases the collection of $\alpha$-extreme points is a minimal star-generating set for the convex kernel of a star-shaped set.

Example 2.3: In $E_2$ let $S = \text{conv}\,\{(1,1), (1,-1), (-1,-1), (-1,1)\} \cup \text{conv}\,\{(0,0), (-2,2), (2,2)\} \cup \text{conv}\,\{(0,0), (2,-2), (-2,-2)\}$ (cf. Fig. 2.3). It can be shown with little difficulty that for any proper subset T of ext $S = \{(2,2), (2,-2), (-2,-2), (-2,2)\}$ that

$$\{(0,0)\} = \text{ck}S \subsetneq \bigcap_{x \in T} S_x.$$

The next example is in response to the second question posed earlier. It was noted in Example 2.2 that for a particular set S in $E_3$ there exists a finite subset of $\text{ext}_2 S$ which star-generates the convex kernel of S. A compact star-shaped set S in $E_3$ will now be given such that any star-generating subset of $\text{ext}_2 S$ must be infinite.

Example 2.4: In the linear space $E_3$ consider the points $p_n = (1-2^{-n}, 2^{-n}, 0)$, $q_n = 2p_n$ and $r_n = (0,0,2^{-n})$, where $n = 0, 1, 2, \cdots$. If $q = (2,0,0)$ and $O = (0,0,0)$, then let $S' = \text{conv}\,\{q, r_0, O\}$, $S_n = \text{conv}\,\{q_n, r_0, O\}$, where $n = 0, 1, 2, \cdots$, and let $T_n = \text{conv}\,\{p_{n-1}, p_n, r_n, O\}$, for $n = 1, 2, 3, \cdots$ (cf. Fig. 2.4). Each $T_n$ is a tetrahedron; each pair $T_n$ and $T_{n+1}$ is separated by the 2-simplex $S_n$, $n = 1, 2, \cdots$. Then let

$$S = S' \cup \left( \bigcup_{n=0}^{\infty} S_n \right) \cup \left( \bigcup_{n=1}^{\infty} T_n \right).$$

For this set S, $\text{ext}_2 S$ is made up of the following line segments: $Or_0$, $Oq$, $Oq_0$, $r_0q_0$, $r_0q_1$, $r_0q_2$, $\cdots$; $p_0q_0$, $p_1q_1$, $p_2q_2$, $\cdots$; $p_0p$, where $2p = q$. If T is any nonempty subset of $(\text{ext}_2 S) \cap (S' \cup S_0 \cup S_1 \cup \cdots)$, then

$$\text{ck}S = \{O\} \subsetneq Or_0 \subset \bigcap_{x \in T} S_x.$$

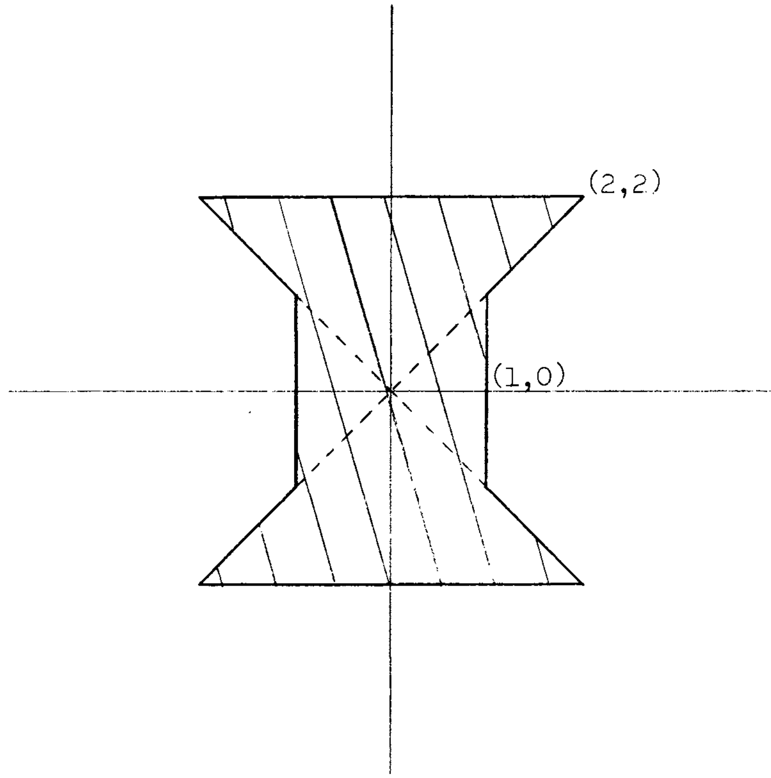


Figure 2.3

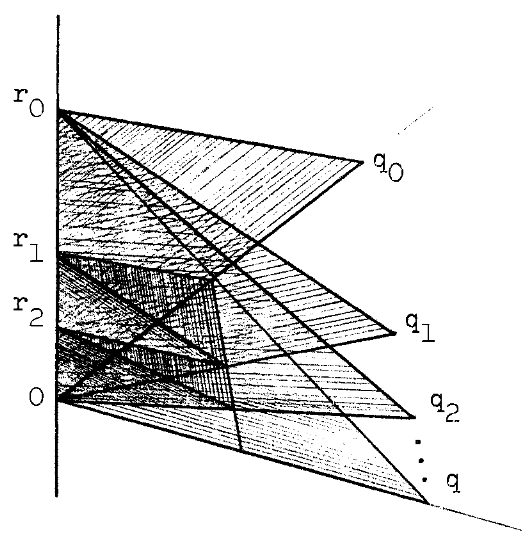


Figure 2.4

If T' is a finite subset of $(\text{ext}_2 S) \cap (T_1 \cup T_2 \cup \cdots)$, then there exists a maximum positive integer k such that $T' \cap (T_k \setminus Or_0) \neq \emptyset$. Thus,

$$\text{ck}S = \{0\} \subsetneq Or_k \subset \bigcap_{x \in T'} S_x.$$

Hence, any subset of $\text{ext}_2 S$ which star-generates ckS must be infinite.

It may be noted in the three examples given above from $E_3$ that in each case $\text{ext}_2 S$ is uncountable. The next theorem shows that this is indicative of the general situation in linear spaces of sufficiently large dimension.

Theorem 2.2: If S is a compact star-shaped set in a locally convex space L, and $\dim(S) \geq 3$, then $\text{ext}_\alpha S$ is uncountable.

Proof: Without loss of generality it can be assumed that $0 \in \text{ck}S$. Since $\dim(S) \geq 3$ there exists some point $x \in S$, $x \neq 0$. If $\beta x \in \text{ext}_\alpha S$ for every $\beta \in (0,1)$, then $\text{ext}_\alpha S$ is uncountable. If there are only a countable number of points of 0x that belong to $\text{ext}_\alpha S$, consider some $w = \beta x$ such that $w \notin \text{ext}_\alpha S$, $\beta \in (0,1)$. Then there exists a hyperplane H and a neighborhood U of w such that $w \in H \cap U \subset S$. Since $\dim(H) \geq 2$, $H \cap U \cap S$ contains a nondegenerate line segment zw such that $z \neq \tau x$ for any $\tau \in R$. The fact that L is locally convex implies that there exists a continuous linear functional f defined on L such that $f(w) = f(z) = 1$. There exists a point $z' \in [f:1]$ such that $\{z, z', w\}$ is an affinely independent set. Since $0 \notin [f:1]$, $\{0, z, z', w\}$ is an affinely independent set, which implies that $\{z, z', w\}$ is linearly independent, as is the set $\{y, w, z\}$, where $y = z' - w$. Clearly, $y \in [f:0]$. For any $\lambda \in [0,1]$ consider the subspace $L_\lambda$ of L with basis $\{y, \lambda z + (1-\lambda)w\}$. Let $f_\lambda$ be the restriction of f to $L_\lambda$. The set $L_\lambda \cap S$ is compact; hence, $f_\lambda$

attains a maximum on $L_\lambda \cap S$ at some point u, $f_\lambda(u) \geq 1$. Since $\dim(L_\lambda \cap [f:f(u)]) = 1$ and $L_\lambda \cap S \cap [f:f(u)]$ is compact, there exists a minimal closed line segment in $L_\lambda$ which contains $L_\lambda \cap [f:f(u)] \cap S$. This line segment must have at least one end point; each end point that exists must belong to $\text{ext}_\alpha S$ (cf. Fig. 2.5).

For each pair of distinct real numbers $\lambda, \mu$ in $[0,1]$, $L_\lambda \cap L_\mu = \{p: p = \tau y, \tau \in R\} \subset [f:0]$. As was shown above, for each $\lambda \in [0,1]$ there exists a point $p_\lambda \in L_\lambda \cap \text{ext}_\alpha S$, where $f(p_\lambda) \geq 1$; hence, for any two distinct real numbers $\lambda, \mu \in [0,1]$ the associated points $p_\lambda$ and $p_\mu$ must be distinct. This then implies that the set $\text{ext}_\alpha S$ is uncountable.

It has now been shown that in some cases subsets of $\text{ext}_\alpha S$ which star-generate the convex kernel of a compact star-shaped set S must be infinite. The question of cardinality of such subsets can be approached by considering conditions sufficient to yield countable subsets of the $\alpha$-extreme points which star-generate the convex kernel of a compact star-shaped set. Such conditions are now of considerable interest since it is known that $\text{ext}_\alpha S$ is uncountable for compact star-shaped sets of dimension greater than two in locally convex spaces. The next theorem is a step toward finding such sufficient conditions.

<u>Theorem 2.3</u>: Let S be a closed subset of a linear topological space L and let T be a subset of S that star-generates ckS, which may be empty. Then if M is a subset of T such that $T \subset \overline{M}$, M star-generates ckS.

<u>Proof</u>: Since $M \subset T$ then clearly

$$\text{ckS} = \bigcap_{x \in T} S_x \subset \bigcap_{x \in M} S_x.$$

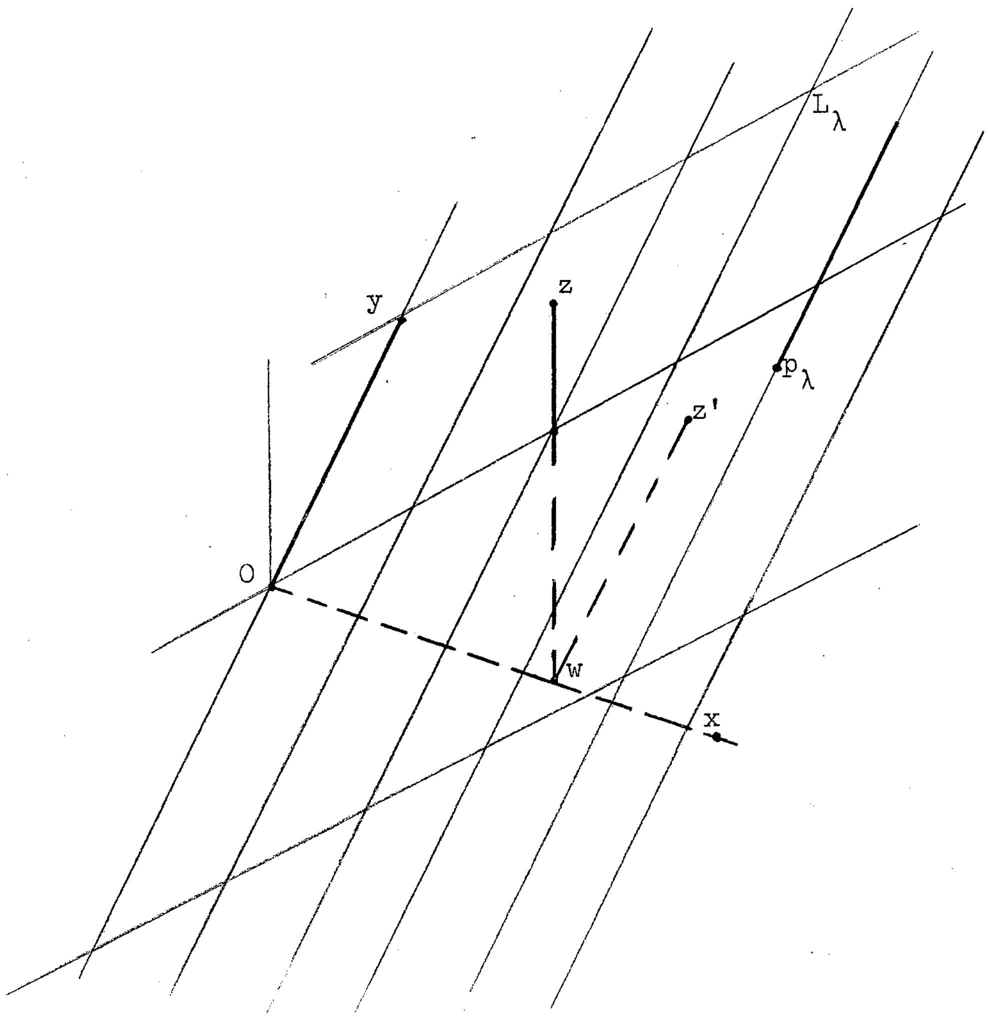


Figure 2.5

Suppose that M is a proper subset of T and ckS is a proper subset of

$$\bigcap_{x \in M} S_x.$$

Then there exists a point q which belongs to

$$\bigcap_{x \in M} S_x \setminus \bigcap_{x \in T} S_x.$$

But

$$\bigcap_{x \in T} S_x = (\bigcap_{x \in M} S_x) \cap (\bigcap_{x \in T \setminus M} S_x);$$

thus,

$$q \notin \bigcap_{x \in T \setminus M} S_x.$$

This implies that $q \notin S_{x_0}$ for some $x_0 \in T \setminus M$. Since

$$q \in \bigcap_{x \in M} S_x,$$

$M \subset S_q$, which is closed as a result of Lemma 2.1. Hence, $x_0 \in T \subset \overline{M} \subset S_q$, which yields the fact that $x_0 q \subset S$. But if $x_0 q \subset S$, then $q \in S_{x_0}$, a contradiction. Therefore,

$$ckS = \bigcap_{x \in M} S_x.$$

In the above theorem the set M is dense in T. The concept of dense subsets leads to the idea of separable spaces, which is the basis for the next theorem and the desired countable star-generating sets.

Theorem 2.4: If S is a compact star-shaped subset of a normed linear space L, then any subset T of S which star-generates the convex kernel of S contains a countable subset M which also star-generates the convex kernel of S.

Proof: The norm of L induces a metric on L. Hence, the compact set S can be considered as a compact metric space, where space is now used in the topological sense. The compact metric space S is separable, which implies that S is second countable [5]. Any nonempty subset T of S is a second countable topological space with the relative topology, which implies that T is separable. Therefore, there exists a countable subset M of T such that $T \subset \overline{M}$. Theorem 2.3 implies that M star-generates ckS and the theorem is proved.

Although the preceding result states the existence of a countable star-generating set for each set in a class of compact star-shaped sets, it seems unlikely that such a countable subset could be described further in general to any great extent. A close examination of the set from $E_2$ given in Example 2.5 indicates that any countable star-generating set for the convex kernel must be chosen with considerable care.

Example 2.5: In $E_2$ let $B(1) = \{p: \|p\| \leq 1\}$ and let $B_u(1) = B(1) \cap \{(x_1,x_2): x_2 \geq 0\}$, $B_d(1) = B(1) \cap \{(x_1,x_2): x_2 \leq 0\}$, $B_l(1) = B(1) \cap \{(x_1, x_2): x_1 \leq 0\}$ and $B_r(1) = B(1) \cap \{(x_1,x_2): x_1 \geq 0\}$. Then $S = \text{conv } \{(-2,8), (0,8), (2,-8), (0,-8)\} \cup \text{conv } \{(-8,-2), (-8,0), (8,2), (8,0)\} \cup (B_u(1) + (-1,8)) \cup (B_r(1) + (8,1)) \cup (B_d(1) + (1,-8)) \cup (B_l(1) + (-8,-1))$ (cf. Fig. 2.6). The set ext S is made up of the four circular arcs in bd S. It is not difficult to show that $ckS = \{(0,0)\}$. Any subset T of ext S that star-generates ckS must contain an infinite number of points from each of the circular arcs; in fact, each arc A must contain an infinite sequence from ext S which converges to $A \cap \{(0,8), (8,0), (0,-8), (-8,0)\}$.

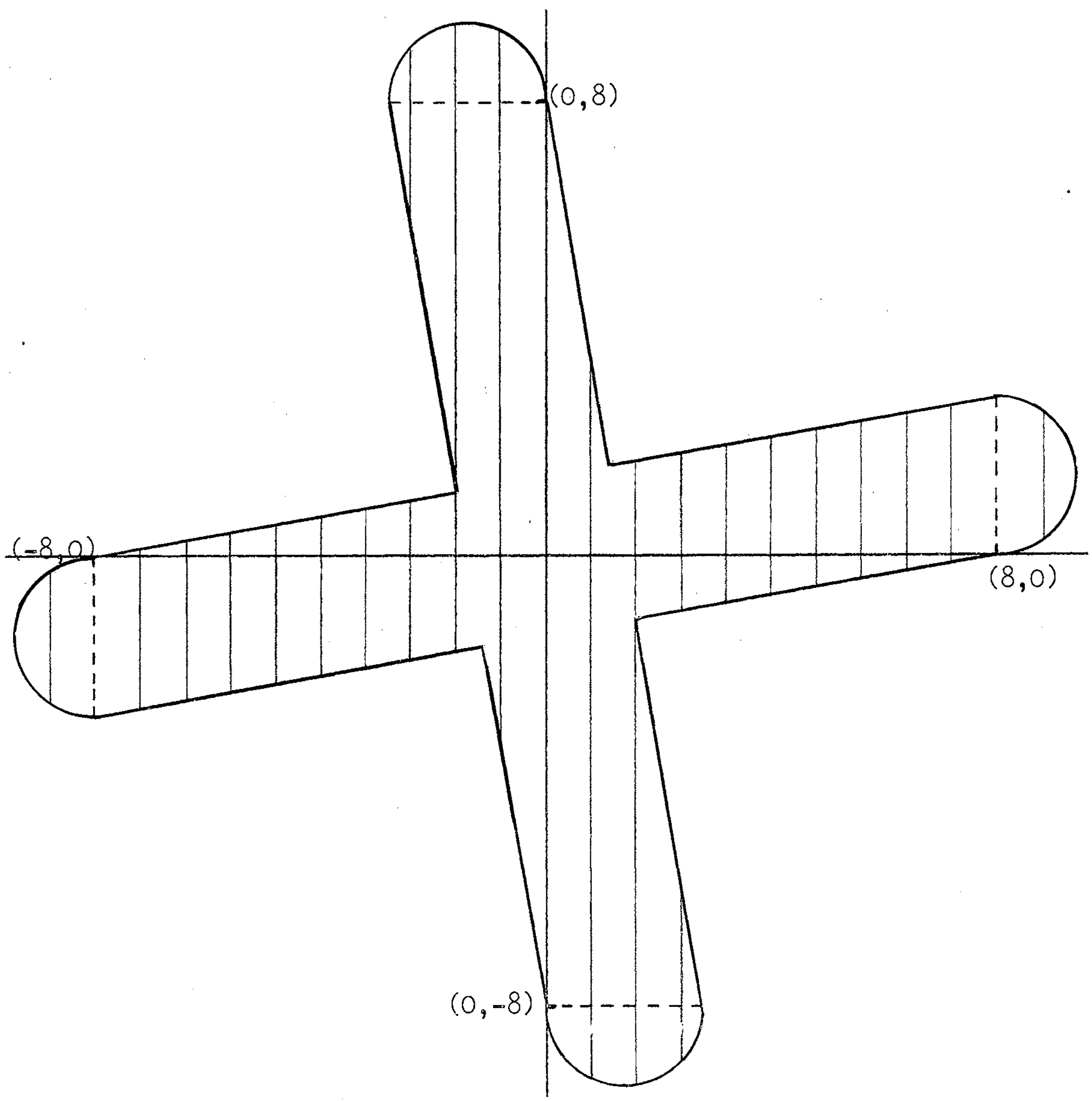


Figure 2.6

## CHAPTER III

## RELATIVE EXTREME POINTS

Klee [8] introduced the concept of relative extreme point. The following definition is a result of the paper mentioned above.

Definition 3.1: If S and T are subsets of a linear space L then $x \in S$ is said to be extreme in S relative to T if there do not exist points $y \in S$, $z \in T$ such that $x \in$ intv yz.

The relative extreme points of star-shaped sets that will be of particular interest are the points of a star-shaped set S that are extreme relative to ckS. The collection of such relative extreme points in S will be denoted by exk S. The set of points in exk S which are not in ckS will be denoted by $E_S$. The following example points out some of the relationships that exist between exk S and ext S, as well as with $E_S$.

Example 3.1: In $E_2$ let S = conv $\{(0,0), (-3,6), (0,3)\} \cup$ conv $\{(0,0), (0,3), (3,6)\}$ (cf. Fig. 3.1). Then ckS = conv $\{(0,0), (-1,2), (0,3), (1,2)\}$, exk S = ext S = $\{(0,0), (-3,6), (3,6)\}$ and $E_S = \{(-3,6), (3,6)\}$.

Let S' = conv $\{(1,2), (1,-2), (-1,-2), (-1,2)\} \cup$ conv $\{(2,1), (2,-1), (-2,-1), (-2,1)\}$ (cf. Fig. 3.2). Then ckS' = conv $\{(1,1), (1,-1), (-1,-1), (-1,1)\}$, ext S' = $\{(1,2), (1,-2), (-1,-2), (-1,2), (2,1), (2,-1), (-2,-1), (-2,1)\}$ and exk S' = $E_{S'}$ = conv $\{(2,1), (2,-1)\} \cup$ conv

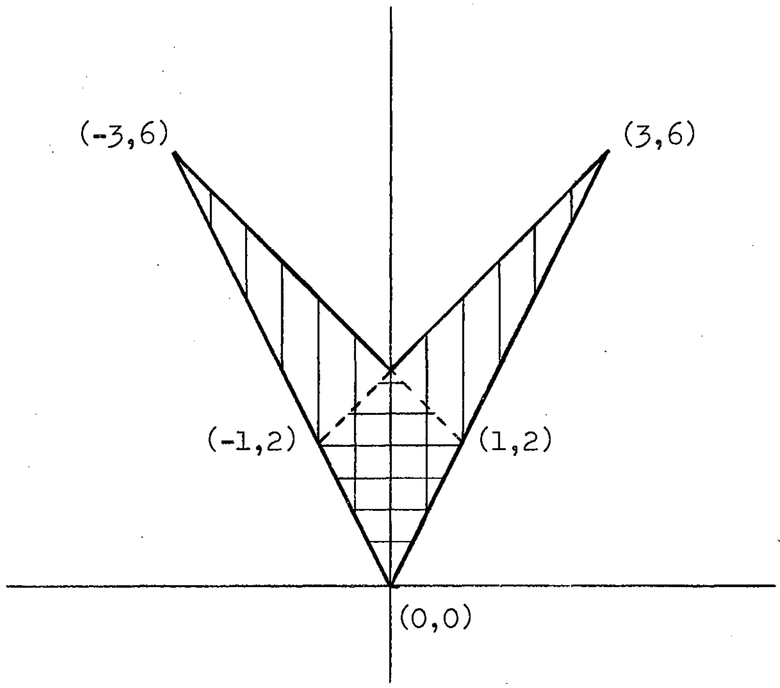


Figure 3.1

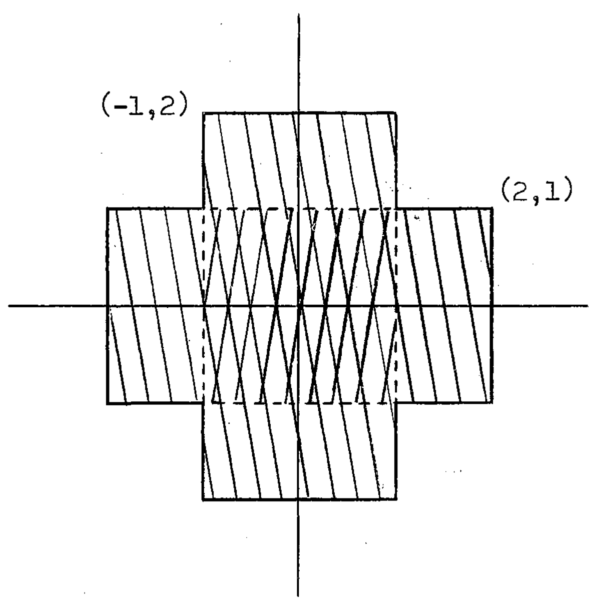


Figure 3.2

$\{(1,-2), (-1,-2)\} \cup \text{conv} \{(-2,-1), (-2,1)\} \cup \text{conv} \{(-1,2), (1,2)\}$.

The following lemma is included for completeness; its proof is omitted since the result is quite well known [3].

Lemma 3.1: If L is a linear topological space and C is a compact convex subset of L such that $0 \notin C$, then $C^* = \{\lambda x: x \in C, \lambda \geq 0\}$ is a closed convex cone with vertex the origin.

Theorem 3.1 gives a decomposition of a compact star-shaped set by the use of relative extreme points. Any such set can be represented in terms of its convex kernel and the extreme points relative to the convex kernel. The convex kernel of a compact star-shaped set S is a compact convex set. If such a set S is a subset of a locally convex space, the Krein-Milman theorem gives the fact that ckS is the closed convex hull of its extreme points, ext (ckS). For example, if ext (ckS) is finite and $p \in$ ckS, there exists a subset $\{x_1, x_2, \ldots, x_k\}$ of ext (ckS) such that

$$p = \sum_{i=1}^{k} \lambda_i x_i, \quad \sum_{i=1}^{k} \lambda_i = 1, \; \lambda_i \geq 0.$$

To further illustrate in this special case, Theorem 3.1 implies that if $p \in S$, a compact star-shaped subset of a locally convex space, there exists a subset $\{y_1, y_2, \ldots, y_n\}$ of ext (ckS) and a point $q \in E_S$ such that

$$p = \alpha q + (1-\alpha) \sum_{i=1}^{n} \beta_i y_i, \quad \sum_{i=1}^{n} \beta_i = 1, \; \beta_i \geq 0, \; 0 \leq \alpha \leq 1.$$

These results are now given in Theorem 3.1.

Theorem 3.1: Let S be a compact nonconvex star-shaped set in a locally convex space L. Then

$$S = \bigcup_{y \in E_S} \text{conv}\,(\text{ck}S \cup \{y\}).$$

Proof: Since $E_S \subset S$, conv $(\text{ck}S \cup \{y\}) \subset S$ for every $y \in E_S$. Thus,

$$\bigcup_{y \in E_S} \text{conv}\,(\text{ck}S \cup \{y\}) \subset S.$$

Consider any $z \in S$; if $z \in \text{ck}S \cup \text{exk}\,S$, then since $E_S \neq \emptyset$, as shown below,

$$z \in \bigcup_{y \in E_S} \text{conv}\,(\text{ck}S \cup \{y\}).$$

Let $C = \text{ck}S$. Suppose $z \in S \backslash (\text{ck}S \cup \text{exk}\,S)$ and without loss of generality suppose that $z = 0$. Since $C$ is compact and convex, the above lemma yields the result that $C^*$ and $-C^*$ are closed convex cones with $z$, the origin, as the vertex of each. Since $z \notin \text{exk}\,S$ there exist points $x \in C$ and $w \in S$ such that $0 \in \text{intv}\,xw$. Clearly, $w \in -C^* \backslash 0$, $S \cap (-C^* \backslash 0) \neq \emptyset$ and $S \cap (-C^*)$ is compact. Let $u$ be an arbitrary point in $-C^* \backslash 0$; since $L$ is locally convex and $C^*$ is closed and convex, there exists a closed hyperplane $H = [f:f(u)]$ such that $u \in H$ and $H \cap C^* = \emptyset$, where $f$ is a continuous linear functional. It can be assumed that $f(C^*) \leq 0$, which implies that $f(u) > 0$. The function $f$ then attains a maximum on $S \cap (-C^*)$ at some point $v_0 \in S \cap (-C^*)$. Suppose that $v_0 \notin \text{exk}\,S$. Then there exist points $p \in \text{ck}S$, $q \in S$ such that $v_0 \in \text{intv}\,pq$. Since $v_0 \in -C^*$, $v_0 = -\lambda p'$, $p' \in \text{ck}S$, $\lambda > 0$, and $v_0 = \alpha p + (1-\alpha)q$, $0 < \alpha < 1$. Therefore, $v_0 = -\lambda p' = \alpha p + (1-\alpha)q$ and

$$\begin{aligned} q &= \frac{1}{\alpha-1}\,(\lambda p' + \alpha p) \\ &= \frac{\lambda+\alpha}{\alpha-1}\left(\frac{\lambda}{\lambda+\alpha}\,p' + \frac{\alpha}{\lambda+\alpha}\,p\right) \\ &= \tau q', \end{aligned}$$

where $\tau < 0$ and $q' \in \text{ck}S$. Thus, $q \in -C^*$, so that $q \in S \cap (-C^*)$. But

$$f(q) = f(\frac{1}{1-\alpha} v_0 - \frac{\alpha}{1-\alpha} p)$$

$$= \frac{1}{1-\alpha} f(v_0) - \frac{\alpha}{1-\alpha} f(p)$$

$$> \frac{1}{1-\alpha} f(v_0) - \frac{\alpha}{1-\alpha} f(v_0)$$

$$= f(v_0).$$

This contradicts the fact that $f(v_0) \geq f(x)$ for all $x \in S \cap (-C^*)$. Hence, $v_0 \in (\text{exk } S) \cap (-C^*)$ and

$$0 \in \text{conv } (\text{ckS} \cup \{v_0\}) \subset \bigcup_{y \in E_S} \text{conv } (\text{ckS} \cup \{y\}).$$

Therefore,

$$S = \bigcup_{y \in E_S} \text{conv } (\text{ckS} \cup \{y\}).$$

Earlier it was discovered that in the case of Theorem 2.1 there will often exist proper subsets T of $\text{ext}_\alpha S$ which star-generate the convex kernel of S. The following result shows that the set E is minimal in its use in Theorem 3.1.

Theorem 3.2: Let S be a compact star-shaped set in a locally convex space L. If T is a proper subset of $E_S$ then

$$\bigcup_{y \in T} \text{conv } (\text{ckS} \cup \{y\})$$

is a proper subset of S.

Proof: Consider any proper subset T of $E_S$; then there exists some point $x_0 \in E_S \setminus T$. If

$$x_0 \in \bigcup_{y \in T} \text{conv } (\text{ckS} \cup \{y\})$$

there exists some $y_0 \in T$ such that $x_0 \in \text{conv } (\text{ckS} \cup \{y_0\})$. Hence, $x_0 =$

$\lambda x + (1-\lambda)y_0$, where $\lambda \in [0,1]$, $x \in$ ckS; but $\lambda \in (0,1)$ since $x_0 \notin$ ckS $\cup$ T. This expression for $x_0$ implies that $x_0 \notin$ exk S, a contradiction. Thus, $x_0$ does not belong to

$$\bigcup_{y \in T} \text{conv } (\text{ckS} \cup \{y\}),$$

which must be a proper subset of S.

If S is a compact star-shaped set in a locally convex space L, then the procedure employed in the proof of Theorem 3.1 can be used to locate points of S\ckS which are extreme relative to ckS. For any such set S consider an arbitrary point $y \in$ bd S. If $y \notin$ exk S and $y \notin$ ckS, then consider the closed convex cone $C = \{\lambda x\colon x \in y - \text{ckS},\ \lambda \geq 0\} + y$, which has vertex y. Since $y \notin$ exk S, $C \cap S \neq \emptyset$; hence, there exists a point $z \in C \cap S$ and a continuous linear functional defined on L such that $f(w) < f(z)$ for every $w \in -C + 2y$. The functional f will attain a maximum on $C \cap S$ at some point u, which must belong to exk S.

## CHAPTER IV

## POLYHEDRAL STAR-SHAPED SETS

Convex polytopes are of considerable interest in several areas of study. Certain systems of linear inequalities have solution sets which are convex polytopes. For example, consider the following system of inequalities in two variables:

$$x + y \leq 1$$

$$-2x + y \leq 2$$

$$x - y \leq 1$$

The solution set in the (x,y)-plane will be conv $\{(1,0), (-1/3,4/3), (-3,-4)\}$, a convex polytope (cf. Fig. 4.1).

Locating optimal values for a real function f with the constraints of such a system of linear inequalities involves examining the collection of extreme points of the convex polytope which is the solution set for that system. This process is aided by the fact that there are only a finite number of extreme points in a convex polytope. Since the structure of a convex polytope is simple to describe analytically, convex polytopes have been used to approximate more general convex sets. It seems quite natural to extend the idea of convex polytope to a similar set in the class of star-shaped sets. The analogous entity, a polyhedral star-shaped set, will bear much the same relationships to star-shaped sets as do convex polytopes to convex sets.

Because the extremal structure of convex polytopes and polyhedral

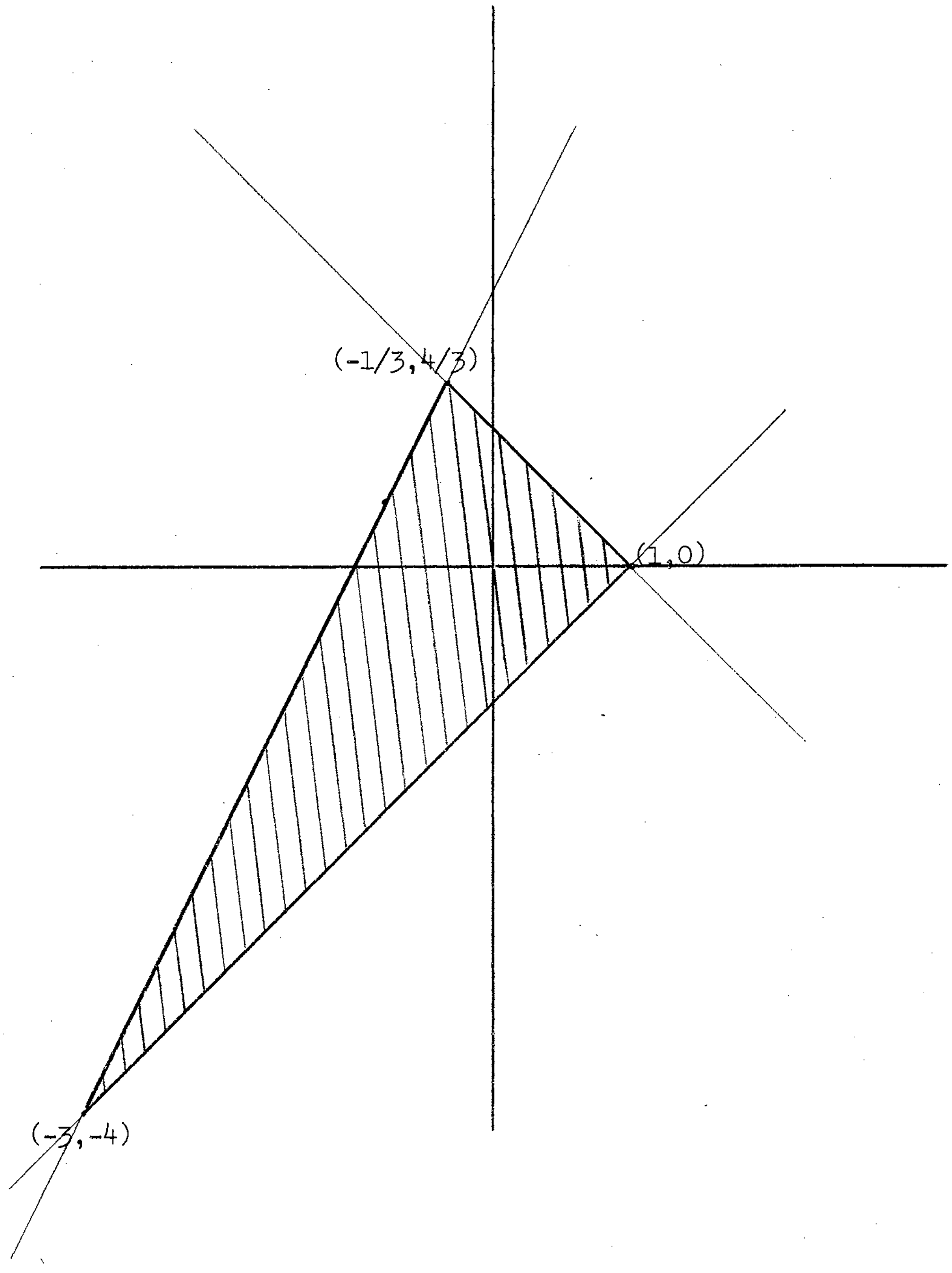


Figure 4.1

star-shaped sets is simpler than that of convex sets and star-shaped sets in general, the results of the previous chapters will be particularly applicable to such sets. In the next chapter the relative ease of applying such results to polyhedral star-shaped sets will be made available to sets in a more general class through a process of approximation similar to that of convex polytopes for convex sets.

The former of the two definitions that follow is not new [4]; it is included to complement the latter definition. To the knowledge of the writer the latter definition is new.

Definition 4.1: A subset C of a linear space S is a convex polytope if C is the convex hull of a finite number of points.

Definition 4.2: A subset S of a linear space L is a polyhedral star-shaped set if

$$S = \bigcup_{i=1}^{n} C_i,$$

where each $C_i$ is a convex polytope and

$$\bigcap_{i=1}^{n} C_i \neq \emptyset.$$

The first consideration of polyhedral star-shaped sets will be focused upon the linear space $E_2$. Part of the reason for this is the fact that sharper results can be obtained here than in more general spaces. Too, polyhedral star-shaped sets have several applications in this space. Since a set in $E_2$ can be described in complex notation, the study of star-shaped sets in $E_2$ is of interest in complex analysis and other areas of mathematics. For example, if one is studying boundary value problems in the plane associated with some partial differential equation, the setting lends itself to seeking a solution by use of conformal mappings. Any special class of domains for which a conformal map to some more familiar set is readily found for each of its member sets is of considerable value. The class of polyhedral star-shaped sets in $E_2$ with convex kernels which are convex bodies form a class of sets for which the Schwarz-Christoffel transformation can be used to find a conformal map.

The following sequence of results gives sufficient conditions for a set to be a polyhedral star-shaped set in $E_2$.

Lemma 4.1: Let S be a compact star-shaped subset of $E_2$ with a finite number of extreme points. If for $x, y \in S$, $xy \cap \text{ck}S = \{x\}$, then there exists a point $p \in \text{ext } S$ such that $S_p \cap xy = \{x\}$.

Proof: Consider any such pair $x, y \in S$ and the sequence $\{x_n\}$, where $x_n = 2^{-n}(y-x) + x$, which converges to x. Since $x_n \notin \text{ck}S$ for each $n > 0$, Theorem 2.1 implies that there exists a point $p_n \in \text{ext } S$ such that $x_n p_n \not\subset S$. If $x_k p_n \subset S$, then $x_m p_n \subset \text{conv } \{x, p_n, x_k\} \subset S$ for $1 \leq m \leq k$ since $x \in \text{ck}S$ and $x_m \in x_k x$. Thus if $x_n p_n \not\subset S$, then $x_k p_n \not\subset S$ for $1 \leq k \leq n$. Since ext S is a finite set, some $p \in \text{ext } S$ must appear as a $p_n$ an infinite number of times; that is, for every $N > 0$ there exists an $m > N$ such that $p = p_m$, which implies that $px_k \not\subset S$ for any k such that $1 \leq k \leq m$. Hence, $px_k \not\subset S$ for $k = 1, 2, \cdots$. If there exists some $z \in xy \setminus \{x\}$ such that $pz \subset S$, then for any $w \in xz$, $pw \subset \text{conv } \{x, z, p\} \subset S$. Since there exists a point $x_n \in xz$ for any such z, $pz \not\subset S$. This implies that $S_p \cap xy = \{x\}$.

The following lemma defines a correspondence between ext S and a subset of ext (ckS) which aids in determining the cardinality of ext (ckS).

Lemma 4.2: Let S be a compact star-shaped subset of $E_2$ which has a finite number of extreme points. If dim(ckS) = 2, then for every point $x \in$ ext (ckS)\ext S there exists a point $p \in$ ext S such that L(x,p) supports ckS at x.

Proof: Without loss of generality suppose that $0 \in$ int (ckS). If $x \in$ ext (ckS)\ext S then there exists a closed hyperplane of support to ckS at x since ckS is a convex body and $x \in$ bd(ckS). Let

$$C = \bigcap_{\alpha \in \Lambda} H_\alpha,$$

where $\{H_\alpha\}_{\alpha \in \Lambda}$ is the collection of all closed halfspaces which contain ckS and have x as a boundary point. Then C is a closed convex cone with vertex at x and with two closed rays emanating from x forming the boundary of C. If there is only one such halfspace, say $H_1$, then it may happen that $S \subset H_1$. Since $x \notin$ ext S there exists points y, $z \in S$ such that $x \in$ intv yz. Since we are assuming that $S \subset H_1$, $yz \subset H'$, the hyperplane which determines the halfspace $H_1$. There exists a minimal closed line segment in H' which contains $S \cap H'$; each endpoint of this segment must be in ext S. Such a point p and the line H' give the desired conclusion. If there is only one such halfspace $H_1$ and $S \not\subset H_1$, then there exists a point $w \in (E_2\backslash H_1) \cap S$. There exists linear functional f and a real number $\lambda > 0$ such that $[f:\lambda] = H'$, where H' is the hyperplane which determines $H_1$. Since $0 \in H_1$, $f(w) > \lambda$. The functional f attains a maximum $\tau > \lambda$ at some point $u \in (E_2\backslash H_1) \cap S$. The compact set $S \cap [f:\tau]$ must be contained in some minimal closed line segment, each end point of which must belong to ext S. Thus, ext $S \cap (E_2\backslash H_1) \neq \emptyset$. The point $x \in$ ext (cks), which implies that ckS $\cap$ intv $0x^\infty = \emptyset$ since $0 \in$ ckS. Lemma 4.1 implies that there exists some point $p \in$ ext S such that $S_p \cap$ intv $0x^\infty = \emptyset$. If this point p belongs to H', the desired conclusion follows. Consider any point $q \in (E_2 \backslash H_1) \cap \text{ext } S$. If $O \in L(q,x)$, then clearly $S_q \cap \text{intv } Ox^\infty \neq \emptyset$. Suppose that $O \notin L(q,x)$. The line $L(q,x)$ does not bound ckS since H' is the only such line in the case under consideration. Hence, there exists a point $y \in \text{ckS} \cap H^-$, where $H^-$ is the closed halfspace determined by $L(q,x)$ which does not contain O. Thus, $x \in \text{int } (\text{conv } \{O, q, y\}) \subset \text{conv } \{O,q,y\} \subset S$, which implies that $S_q \cap \text{intv } Ox^\infty \neq \emptyset$. The point p sought to give the result of Lemma 4.1 cannot belong to $E_2 \backslash H_1$.

Now consider any point $q' \in \text{ext } S \cap \text{int } H_1$. If $O \in L(q',x)$, then $S_{q'} \cap \text{intv } Ox^\infty \neq \emptyset$ since $O \in \text{ckS}$. If $O \notin L(q',x)$, then since $L(q',x)$ does not bound ckS, there exists a point $y' \in \text{ckS} \cap H^*$, where $H^*$ is the closed halfspace determined by $L(q',x)$ which does not contain O (cf. Fig. 4.2). Since $\{O, x, y'\} \subset \text{ckS}$ and $q' \in S$, $\text{conv } \{O, x, y, q'\} \subset S$. For a point $q \in (E_2 \backslash H_1) \cap \text{ext } S$ it has been shown that $x \in \text{int } (\text{conv } \{O, q, x'\}) \subset S$. It can then be easily shown that $S_{q'} \cap \text{intv } Ox^\infty \neq \emptyset$. Thus, the desired point p cannot belong to int $H_1$, which implies that $p \in H'$. The point p gives the desired conclusion.

If there exist more than one halfspace $H_\alpha$, let $H_1$ and $H_2$ be the two halfspaces containing ckS which are determined by the two hyperplanes $H_1'$ and $H_2'$ which contain x and the boundary of C. The two lines $H_1'$ and $H_2'$ determine four convex cones, each with vertex x. Let $H_1^-$ and $H_2^-$ be the complementary closed halfspaces of $H_1$ and $H_2$, respectively (cf. Fig. 4.3). If there exists a point $p \in \text{ext } S \cap [(H_1 \cap H_2^-) \cup (H_1^- \cap H_2)]$, then this point gives the desired conclusion. On the other hand, suppose that no such point p exists. The fact that $x \notin \text{ext } S$ implies that $S \cap (E_2 \backslash (H_1 \cap H_2)) \neq \emptyset$ since $H_1 \neq H_2$. Then $S \cap (E_2 \backslash H_1) \neq \emptyset$ or $S \cap (E_2 \backslash H_2) \neq \emptyset$. Without loss of generality suppose that the former is true.

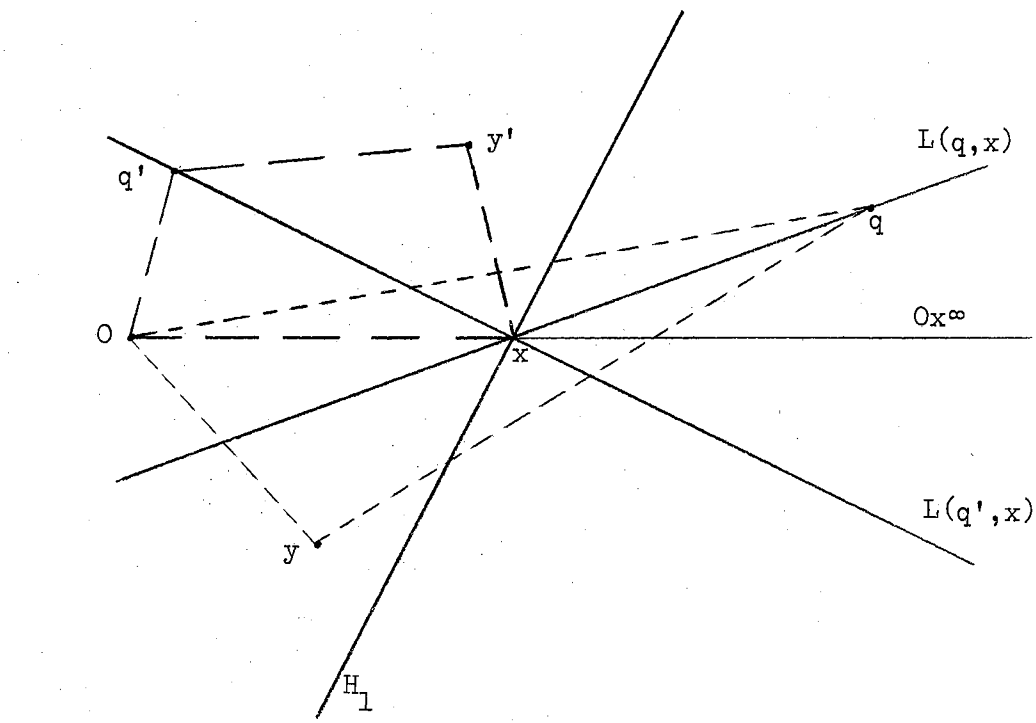


Figure 4.2

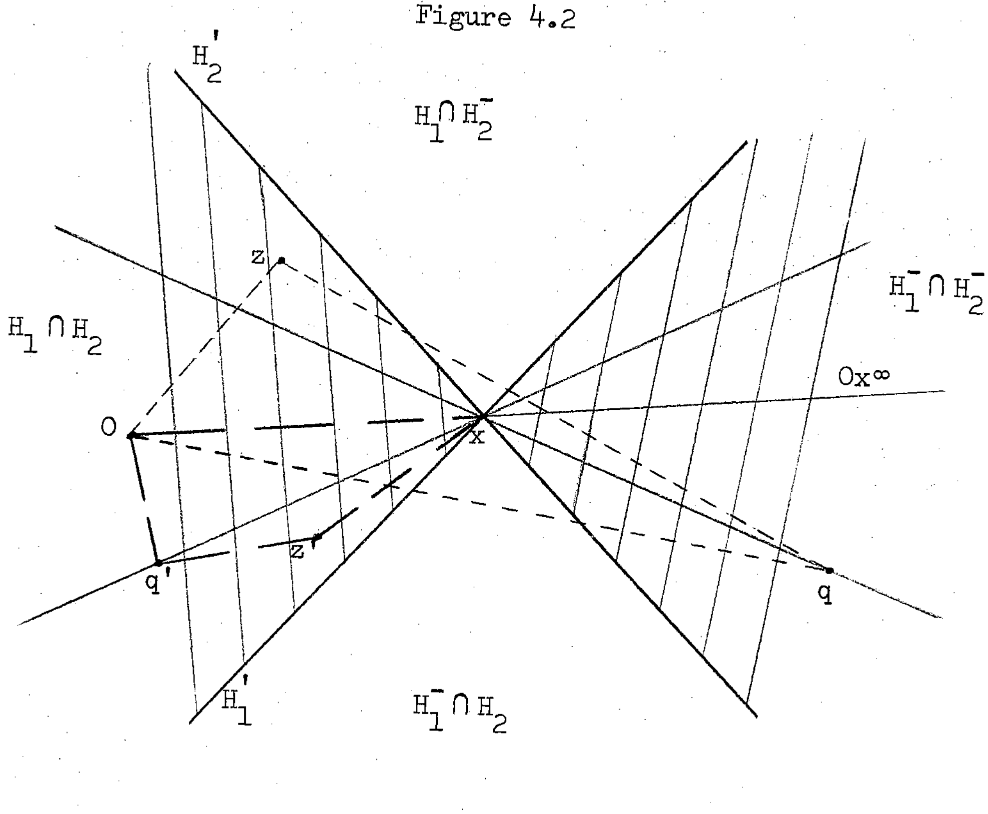


Figure 4.3

There exists a linear fuctional g and a real number $\beta > 0$ such that $H_1' = [g:\beta]$. The linear functional g attains a maximum $\alpha > \beta$ on $S \cap H_1^-$ at some point $v \in S \cap (E_2 \setminus H_1)$. The compact set $S \cap [g:\alpha]$ is contained in a minimal closed line segment, each end point of which must belong to ext S. Since it has been assumed that ext $S \cap (H_1^- \cap H_2) = \emptyset$, these end points must belong to int $(H_1^- \cap H_2^-)$. Therefore, ext $S \cap$ int $(H_1^- \cap H_2^-) \neq \emptyset$. Consider any point $q \in$ ext $S \cap$ int $(H_1^- \cap H_2^-)$. If $0 \in L(q,x)$, then $S_q \cap$ intv $0x^\infty \neq \emptyset$ since $0 \in$ ckS. Suppose that $0 \notin L(q,x)$. The fact that $q \in$ int $(H_1^- \cap H_2^-)$ implies that $L(q,x)$ does not bound ckS. There exists a point $z \in$ ckS $\cap H_3^-$, where $H_3^-$ is the closed halfspace determined by $L(q,x)$ which does not contain 0. Then $x \in$ int (conv $\{q, z, 0\}) \subset$ conv $\{q, z, 0\} \subset S$, which implies that $S_q \cap$ intv $0x^\infty \neq \emptyset$.

If $q' \in$ ext $S \cap$ int $(H_1 \cap H_2)$ and $0 \in L(q',x)$, then $S_{q'} \cap$ intv $0x^\infty \neq \emptyset$. Suppose that $0 \notin L(q',x)$. Since $q' \in$ int $(H_1 \cap H_2)$, $L(q',x)$ does not bound ckS. There then exists a point $z' \in$ ckS $\cap H_4^-$, where $H_4^-$ is the closed halfspace determined by $L(q',x)$ which does not contain 0. Since $\{0, x, z'\} \subset$ ckS and $q' \in S$, conv $\{0, x, z', q'\} \subset S$, and as before, it can be readily shown that $S_{q'} \cap$ intv $0x^\infty \neq \emptyset$. The point p needed to give the conclusion of Lemma 4.1 cannot belong to int $(H_1 \cap H_2) \cup$ int $(H_1^- \cap H_2^-)$. Hence, ext $S \cap [(H_1 \cap H_2^-) \cup (H_1^- \cap H_2)] \neq \emptyset$ and the conclusion follows.

The next lemma shows that the convex kernel of a compact star-shaped subset of $E_2$ with a finite number of extreme points is a convex polytope.

<u>Lemma 4.3</u>: If S is a compact star-shaped subset of $E_2$ with a finite number of extreme points, then the convex kernel of S has a finite

number of extreme points.

Proof: The conclusion follows immediately if dim(ckS) < 2. Suppose that dim(ckS) = 2. For any $x \in$ ext (ckS)\ext S there exists a point $p \in$ ext S such that L(x,p) supports ckS at x, a result of Lemma 4.2. If $p \in$ ext S, then $p \notin$ ckS or $p \in$ bd S. In either case, consider the intersection of all closed halfspaces which contain ckS and contain p in the bounding hyperplane. This intersection of halfspaces is a closed convex cone with vertex p. The boundary of this cone is the union of two closed rays emanating from p. Each such ray is contained in exactly one line through p. These lines are the only hyperplanes which contain p and support ckS. Each such hyperplane intersects ckS in a closed line segment, which contains at most two points from ext (ckS). Thus, for any $p \in$ ext S there exists at most four points in ext (ckS)\ext S with which p might be associated as above. This implies that ext (ckS)\ext S is a finite set, as is ext S $\cap$ ext (ckS), which implies that the convex kernel of S has a finite number of extreme points.

The theorem that follows is the principal result of this chapter.

Theorem 4.1: Let S be a compact star-shaped subset of $E_2$ which has a finite number of extreme points. If for every $x \in E_S$ there exists points y, $z \in$ ext S such that $x \in yz \subset S$, then S is a polyhedral star-shaped set.

Proof: Consider the set T of all pairs of points y, $z \in$ ext S such that $yz \subset S$. Clearly there exist a finite number of such pairs, and for every $x \in E_S$ there exists a pair y, $z \in$ ext S such that $x \in yz \subset S$. Also, for any such pair y, z, conv (ckS $\cup$ {y,z}) $\subset S$; with the aid of Lemma 4.3

it can be readily shown that conv $(\text{ck}S \cup \{y,z\})$ is a convex polytope. But

$$S = \bigcup_{x \in E_S} \text{conv } (\text{ck}S \cup \{x\}),$$

and for each $x \in E_S$, conv $(\text{ck}S \cup \{x\}) \subset$ conv $(\text{ck}S \cup \{y,z\})$ for some pair $y, z \in$ ext S. Thus,

$$S = \bigcup_{y,z \in T} \text{conv } (\text{ck}S \cup \{y,z\}),$$

and S is a polyhedral star-shaped set.

Lemma 4.3 and Theorem 4.1 are not necessarily true in linear spaces of dimension greater than two. The example that follows will bear this out.

Example 4.1: In the linear space $E_3$ let

$$D = \{(x,y,0): x^2 + y^2 \leq 1\},$$

$$S = \text{conv } (D \cup \{(0,0,4)\}) \cup \text{conv } \{(2,2,0), (2,-2,0), (-2,-2,0), (-2,2,0)\}.$$

Then ext S = $\{(0,0,4), (2,2,0), (2,-2,0), (-2,-2,0)\ (-2,2,0)\}$, which is a finite set (cf. Fig. 4.4).

The hypothesis of Lemma 4.3 is satisfied, but ckS = D and ext (ckS) is an uncountable set. Similarly, the hypothesis of Theorem 4.1 is satisfied, but S is not a polyhedral star-shaped set.

By strengthening the hypothesis of Theorem 4.1, an extension to more general spaces can be obtained. This extension is given in Theorem 4.2.

Theorem 4.2: Let S be a compact star-shaped subset of L, a locally convex space; suppose that S has a finite number of extreme points. If

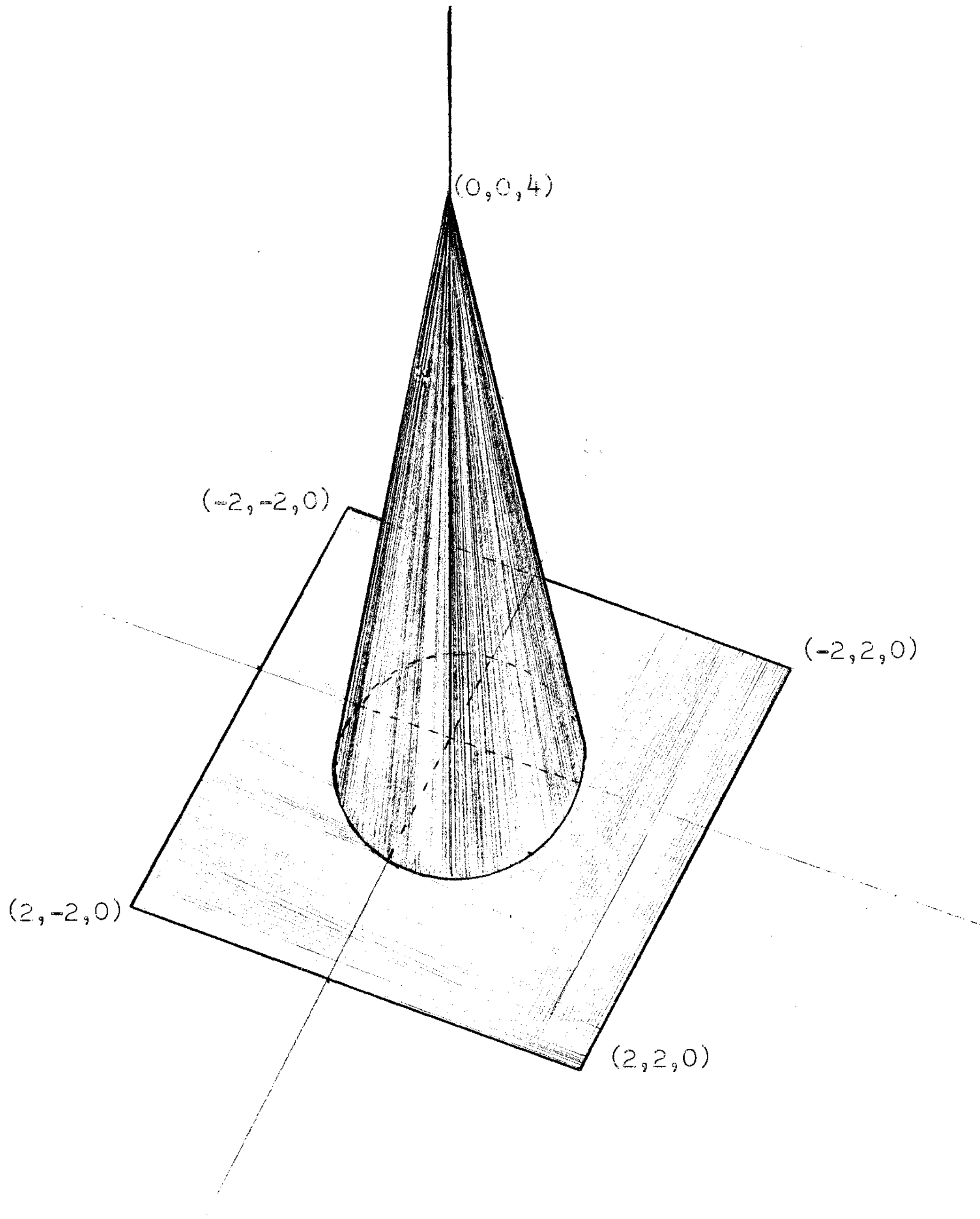


Figure 4.4

ckS has a finite number of extreme points and if for every $x \in E_S$ there exists a subset T of ext S such that $x \in \text{conv } T \subset S$, then S is a polyhedral star-shaped set.

<u>Proof</u>: Let $\mathcal{E}$ be the collection of all subsets A of ext S such that $\text{conv } A \subset S$. Since ext S is a finite set, there are a finite number of sets in $\mathcal{E}$. For each $A \in \mathcal{E}$, conv (conv A $\cup$ ckS) $\subset$ S and since conv A and ckS are convex polytopes, so is conv (conv A $\cup$ ckS). Hence,

$$\bigcup_{A \in \mathcal{E}} \text{conv } (\text{conv } A \cup \text{ckS})$$

is a polyhedral star-shaped subset of S. Theorem 3.1 implies that

$$S = \bigcup_{y \in E_S} \text{conv } (\text{ckS} \cup \{y\}).$$

The hypothesis states that for every $y \in E_S$ there exists a set $A_y \in \mathcal{E}$ such that $y \in \text{conv } A_y \subset S$. Therefore,

$$S = \bigcup_{y \in E_S} \text{conv } (\text{ckS} \cup \{y\}) \subset \bigcup_{A \in \mathcal{E}} \text{conv } (\text{ckS} \cup \text{conv } A),$$

which leads to the conclusion that S is a polyhedral star-shaped set.

Earlier it was mentioned that convex polytopes often occur as solution sets to systems of linear equalities. Such a set is the solution set common to each of the inequalities in a given system. On the other hand, suppose that several systems of linear inequalities are given, and a solution is sought for any one of the systems. Then the solution set may then be a polyhedral star-shaped set. The following example illustrates two such systems.

<u>Example 4.2</u>: Consider the following two systems of inequalities in two variables:

$$\begin{aligned} x + y &\le 1 \\ x \quad &\le 1 \\ -4x - y &\le -1 \end{aligned} \qquad\qquad \begin{aligned} x + y &\le 1 \\ -2x + y &\le 1 \\ x - 2y &\le 1 \end{aligned}$$

Then the set of points in the (x,y)-plane that satisfy one system or the other will be the set $S = \text{conv}\,\{(1,0),\ (0,1),\ (1,-3)\} \cup \text{conv}\,\{(1,0),\ (0,1),\ (-1,-1)\}$, a polyhedral star-shaped set (cf. Fig. 4.5).

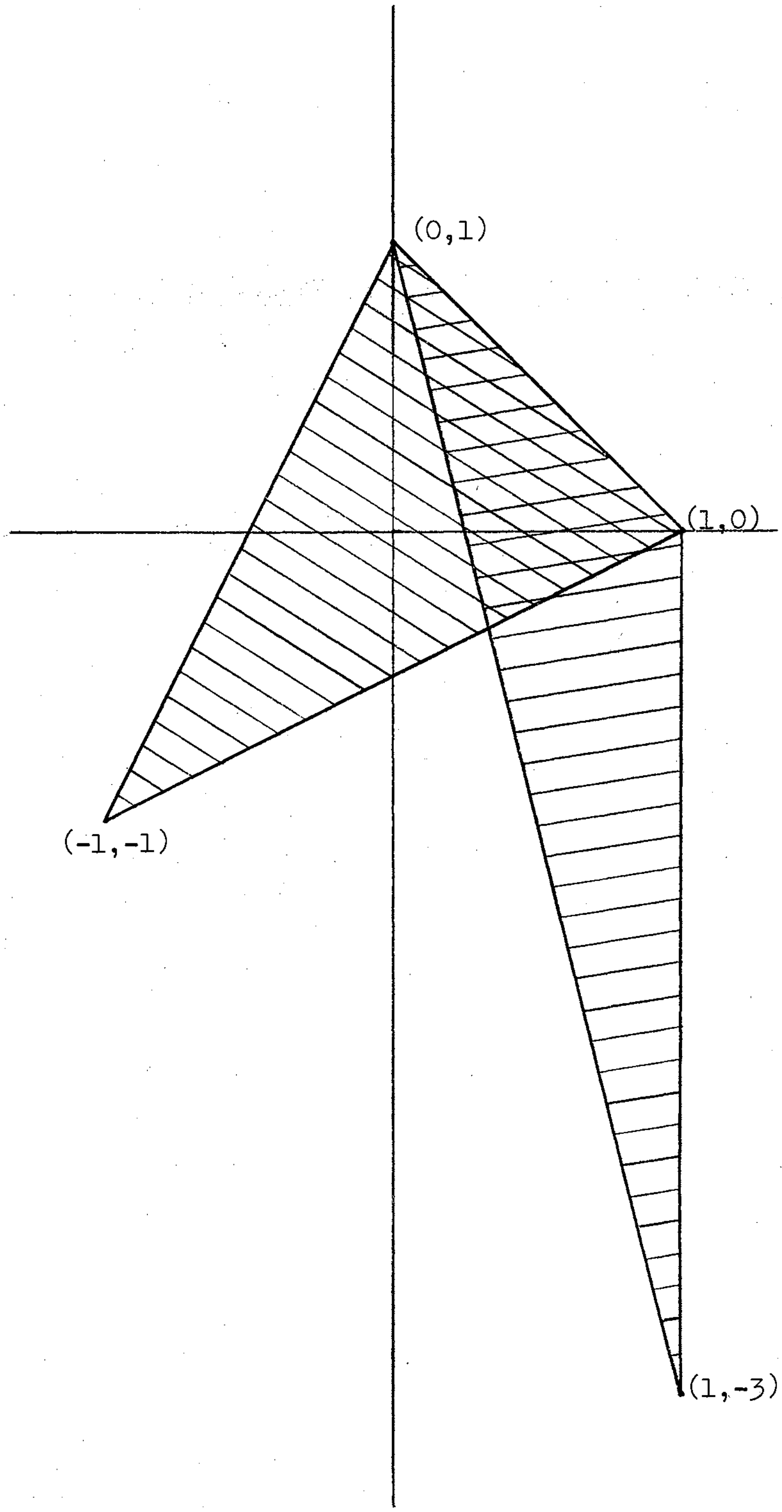


Figure 4.5

## CHAPTER V

## APPROXIMATION OF COMPACT STAR-SHAPED SETS

The setting for this approximation will be a normed linear space L. The compact sets of such a space can be considered as the elements of a metric space by defining a distance function $\Delta$ on the collection of such sets. If $\| \ \|$ is the norm on L, then for any compact set S in L let $S_\epsilon = S + \{x: \|x\| < \epsilon\}$. For any two compact sets A and B in L the distance between A and B will be defined to be $\inf \{\epsilon: A \subset B_\epsilon \text{ and } B \subset A_\epsilon\}$ and will be denoted by $\Delta(A,B)$. If $\{A_i\}$ is a sequence of compact sets in L then

$$\lim_{i \to \infty} A_i = A$$

if, and only if,

$$\lim_{i \to \infty} (A_i, A) = 0.$$

The collection of subsets of L that will be of interest to this study is the collection of compact star-shaped sets, which will be denoted by $\mathcal{S}$. The first result shows that the collection $\mathcal{P}$ of all polyhedral star-shaped sets in L is dense in $\mathcal{S}$.

<u>Theorem 5.1</u>: Let S be a compact star-shaped set in a normed linear space L. Then for any $\epsilon > 0$ there exists a polyhedral star-shaped set P such that $P \subset S \subset P_\epsilon$.

<u>Proof</u>: The set ckS is compact. Consider the collection $\mathfrak{B}_1$ of all

$\epsilon$-balls with centers in ckS. There exists a finite subcollection $\mathfrak{B}_1'$ of $\mathfrak{B}_1$ which covers ckS. Let $T = \{x_1, x_2, \cdots, x_m\}$ be the associated collection of centers, which will be a subset of ckS. Now consider the collection $\mathfrak{B}_2$ of all $\epsilon$-balls with centers in S\ckS. Since $\mathfrak{B}_1' \cup \mathfrak{B}_2$ covers S, which is compact, there exists a finite subcollection $\mathfrak{B}_2'$ of $\mathfrak{B}_2$ such that $\mathfrak{B}_1' \cup \mathfrak{B}_2'$ covers S. Let $T' = \{y_1, y_2, \cdots, y_n\}$ be the set of centers associated with the $\epsilon$-balls in $\mathfrak{B}_2'$. Let $\mathcal{E}$ be the collection of subsets A of T' such that conv $A \subset S$. Since T' is a finite set there is at most a finite number of sets in $\mathcal{E}$. Since ckS is convex, conv $T \subset$ ckS $\subset S$. For any $A \in \mathcal{E}$

$$\text{conv}\ (\text{conv}\ T \cup \text{conv}\ A) = \{\alpha x + (1-\alpha)y\colon \alpha \in [0,1],\ x \in \text{conv}\ T,\ y \in \text{conv}\ A\},$$

and since conv $T \subset$ ckS, conv $A \subset S$, then

$$\text{conv}\ (\text{conv}\ T \cup \text{conv}\ A) \subset S.$$

Thus, if

$$P = \bigcup_{A \in \mathcal{E}} \text{conv}\ (\text{conv}\ T \cup \text{conv}\ A),$$

then $P \subset S$. Since, for each $A \in \mathcal{E}$, conv A and conv T is each a convex polytope, then so is conv (conv T $\cup$ conv A). Then P is the union of a finite number of convex polytopes; clearly conv T is a subset of each convex polytope of this union. The set P is then by definition a polyhedral star-shaped set.

Let $y \in S$. Then there exists an $\epsilon$-ball B in $\mathfrak{B}_1' \cup \mathfrak{B}_2'$ such that $y \in B$. There exists some $x_i \in T$ or some $y_j \in T'$ which is the center of B. In either case $\|y-x_i\| < \epsilon$ or $\|y-y_j\| < \epsilon$, and since $x_i \in P$, $y_j \in P$, it follows that $y \in P_\epsilon$. Hence, $P \subset S \subset P_\epsilon$.

Theorem 5.1 provides the basic step needed to obtain for every compact star-shaped subset S of a normed linear space L a sequence of polyhedral star-shaped sets which converges to S.

<u>Corollary 5.1</u>: If S is a compact star-shaped set in a normed linear space L, then there exists a sequence $\{P_i\}$ of polyhedral star-shaped sets which converges to S.

<u>Proof</u>: For each positive integer n there exists a polyhedral star-shaped set $P_n$ such that $\Delta(S,P_n) < n^{-1}$. Let $\epsilon > 0$; there exists a positive integer N such that $N^{-1} < \epsilon$. If $n > N$ then $n^{-1} < N^{-1} < \epsilon$, which implies that if $n > N$, $|\Delta(S,P_n) - 0| = \Delta(S,P_n) < \epsilon$, so that

$$\lim_{n\to\infty} \Delta(S,P_n) = 0.$$

This implies that the sequence $\{P_i\}$ converges to S.

Since the convex kernel of a star-shaped set S is of fundamental importance to the study of S, it would be of considerable value to know that there is a relationship between the convex kernel of the star-shaped sets of a sequence $\{A_i\}$ and the convex kernel of A, where $\{A_i\}$ converges to the star-shaped set A. The following examples show that restrictions must be placed upon the star-shaped sets of the sequence $\{A_i\}$ if the associated sequence of convex kernels $\{ckA_i\}$ is to converge to ckA.

The first such example shows that there exists in $E_2$ a sequence $\{A_i\}$ of compact star-shaped sets which converges to a star-shaped set A, but the sequence $\{ckA_i\}$ does not converge.

<u>Example 5.1</u>: For each odd positive integer n let $A_n = \text{conv}\ \{(1,1), (1,-1), (-1,-1), (-1,1)\}$. For each even positive integer n let $A_n$ =

$A_1 \cup \text{conv}\ \{(0,0),\ (1+n^{-1},1+n^{-1}),\ (-(1+n^{-1}),1+n^{-1})\} \cup \text{conv}\ \{(0,0),\ (1+n^{-1}, -(1+n^{-1})),\ (-(1+n^{-1}),-(1+n^{-1}))\}$ (cf. Fig. 5.1).

Each $A_n$ is convex when n is odd; thus $ckA_n = A_n$ in that case. If n is even, then $ckA_n = \{(0,0)\}$. The sequence $\{ck(A_i)\}$ obviously does not converge. It can be readily shown that the sequence $\{A_i\}$ converges to $A_1$.

The problem now at hand is to determine sufficient conditions for the limit of the sequence of convex kernels to be the convex kernel of the limit of the sequence. The first restriction to be considered is monotonicity of the sequence $\{ck(A_i)\}$. The oscillation of the terms of the sequence of convex kernels in the previous example prevented its convergence to any set. The next examples show that monotonicity alone for the sequence $\{ck(A_i)\}$ is not enough to assure that this sequence converges to ckA.

First, there exists in $E_2$ a sequence of compact star-shaped sets $\{A_i\}$ such that $\{A_i\}$ and $\{ck(A_i)\}$ both converge, and $\{ck(A_i)\}$ is monotone decreasing, but

$$\lim_{i\to\infty} (ckA_i) \neq ck(\lim_{i\to\infty} A_i).$$

<u>Example 5.2</u>: For each positive integer n let

$$A_n = \text{conv}\ \{(0,-1),\ (0,1),\ (2^{-(n+1)},1),\ (2^{-(n+1)},-1)\} \cup$$
$$\text{conv}\ \{(2^{-n},0),\ (0,2^{-n}),\ (0,-2^{-n})\}.$$

For each such n

$$ck(A_n) = \text{conv}\ \{(0,2^{-n}),\ (0,-2^{-n}),\ (2^{-(n+1)},\ -2^{-(n+1)}),\ (2^{-(n+1)},\ 2^{-(n+1)})\}.$$

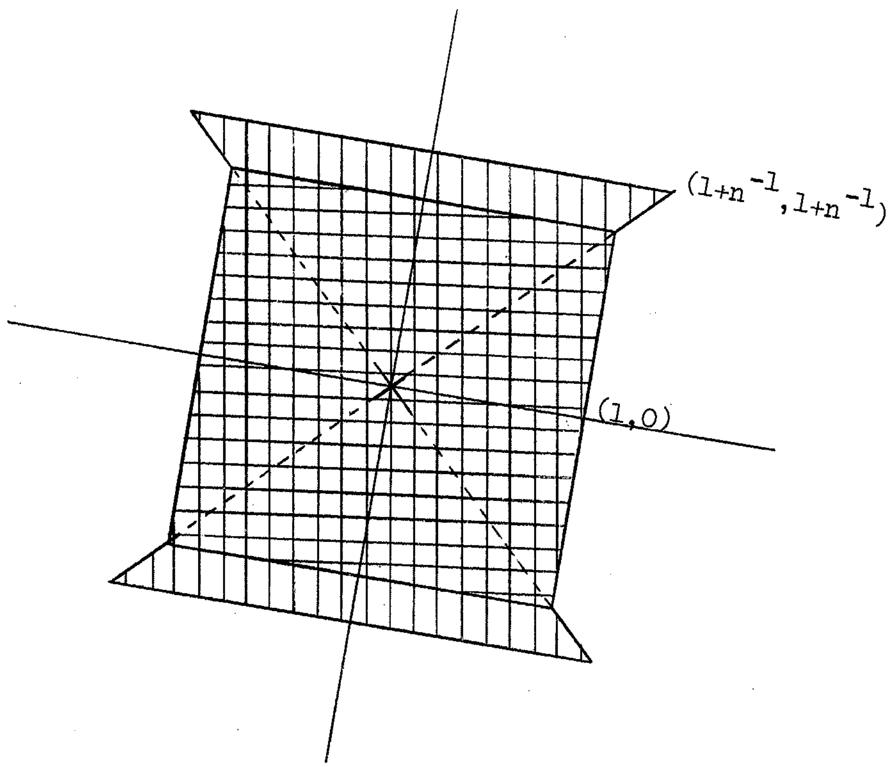


Figure 5.1

The sequence $\{A_i\}$ converges to the convex set $A = \text{conv}\ \{(0,1), (0,-1)\} = \text{ck}A$, but the sequence $\{\text{ck}(A_i)\}$ converges to the set $\{(0,0)\} \neq \text{ck}A$ (cf. Fig. 5.2).

Also, in $E_2$ there exists a sequence of compact star-shaped sets $\{A_i\}$ such that $\{A_i\}$ and $\{\text{ck}(A_i)\}$ both converge, and $\{\text{ck}(A_i)\}$ is monotone increasing, but

$$\lim_{i\to\infty} (\text{ck}(A_i)) \neq \text{ck}(\lim_{i\to\infty} A_i).$$

The following example indicates such a case.

Example 5.3: For each positive integer k let

$$B_k = \{(0,0), (3,0)\} \cup$$

$$\text{conv}\ \{(1+(k+2)^{-1},0),\ (1+(k+2)^{-1},k^{-1}),\ (2-(k+2)^{-1},k^{-1}),\ (2-(k+2)^{-1},0)\}.$$

For each positive integer n let

$$A_n = \bigcup_{k=n}^{\infty} B_k.$$

For each such integer n

$$\text{ck}(A_n) = \text{ck}(B_n) = \text{conv}\ \{(1+(n+2)^{-1},0),\ (2-(n+2)^{-1},0)\} \subset \text{ck}(A_{n+1}).$$

The sequence $\{A_i\}$ converges to the convex set $A = \text{conv}\ \{(0,0), (3,0)\} = \text{ck}A$, but the sequence $\{\text{ck}(A_i)\}$ converges to the set $\text{conv}\ \{(1,0), (2,0)\} \neq \text{ck}A$ (cf. Fig. 5.3).

It may be noted in Example 5.3 that for each $n > 0$, $\text{ck}(A_n) \subset \text{ck}A$. The next theorem shows that if the condition that

$$\text{ck}(\lim_{i\to\infty} A_i) \subset \text{ck}(A_n)$$

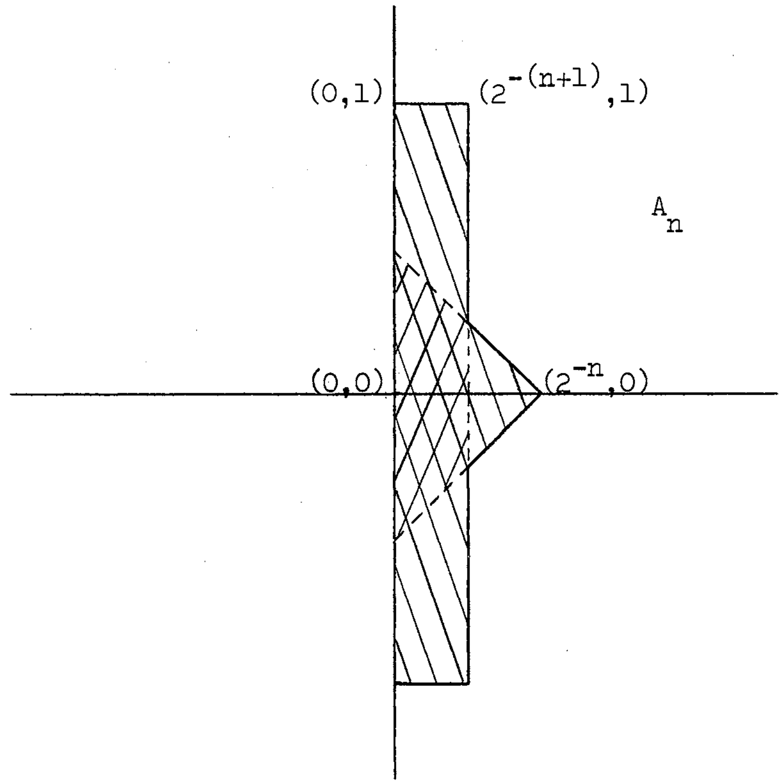


Figure 5.2

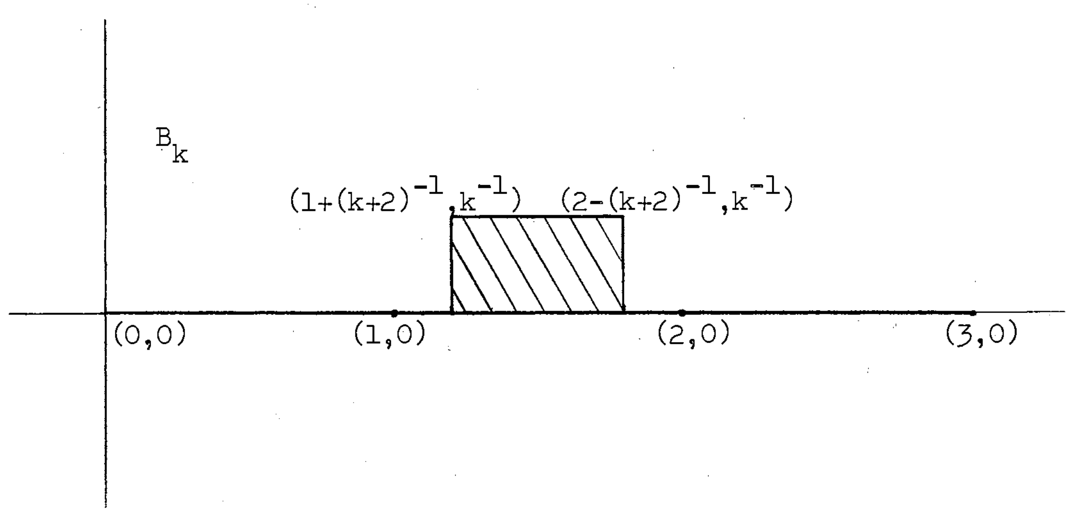


Figure 5.3

for each n is imposed upon the sequence $\{A_i\}$ then the desired convergence of the associated sequence of convex kernels can be obtained.

Theorem 5.2: Let $\{A_i\}$ be a sequence of compact star-shaped sets in a normed linear space L such that $\{A_i\}$ converges to a star-shaped set A. If the sequence $\{ck(A_i)\}$ converges and $ckA \subset ck(A_i)$ for each $i > 0$, then

$$\lim_{i\to\infty} (ck(A_i)) = ckA.$$

Proof: Let $ck(A_i) = K_i$ and let $x \in K$, where

$$K = \lim_{i\to\infty} K_i.$$

Then there exists a smallest integer $N_1$ such that if $n \geq N_1$, $K_n \cap S_1(x) \neq \emptyset$, where $S_1(x)$ is given by $S_\varepsilon(x) = \{y: \|y-x\| < \varepsilon\}$ with $\varepsilon = 1$. Choose a point

$$y_{N_1} \in K_{N_1} \cap S_1(x);$$

for $i < N_1$, choose $y_i$ from $K_i$. There exists a smallest integer $N_2 > N_1$ such that if $n \geq N_2$, $K_n \cap S_{1/2}(x) \neq \emptyset$. Choose a point

$$y_{N_2} \in K_{N_2} \cap S_{1/2}(x);$$

for any i such that $N_1 < i < N_2$, choose $y_i$ from $K_i \cap S_1(x)$, which is nonempty since $i > N_1$. Suppose that for some $m > 1$ the smallest $N_m > N_{m-1}$ has been found such that if $n \geq N_m$, $K_n \cap S_{1/m}(x) \neq \emptyset$. Furthermore, suppose that for all $i \leq N_m$, $y_i$ has been chosen as follows: if $i = N_j$ for some j such that $1 \leq j \leq m$, then $y_i \in K_i \cap S_{1/j}(x)$; if $N_j < i < N_{j+1}$ for some j such that $1 \leq j \leq m-1$, then $y_i \in K_i \cap S_{1/j}(x)$. For $i < N_1$, $y_i$ is defined as described above. Then there exists a smallest $N_{m+1} > N_m$ such that if $n \geq N_{m+1}$, then $K_n \cap S_{1/(m+1)}(x) \neq \emptyset$. Choose a point

$$y_{N_{m+1}} \in K_{N_{m+1}} \cap S_{1/(m+1)}(x);$$

for any i such that $N_m < i < N_{m+1}$ choose $y_i$ from $K_i \cap S_{1/m}(x)$, which is nonempty since $i > N_m$. A sequence of points $\{y_i\}$ has now been defined inductively such that for any $i > 0$, $y_i \in K_i$.

Consider any $\epsilon > 0$; there exists an integer $r > 0$ such that $r^{-1} < \epsilon$. Let $n > N_r$, where $N_r$ is defined as in the above procedure. Then $n = N_p$ for some $p > r$ or $N_q < n < N_{q+1}$ for some $q > r$. If $n = N_p$, then $y_n \in K_n \cap S_{1/p}(x) \subset S_{1/r}(x) \subset S_\epsilon(x)$. If $N_q < n < N_{q+1}$, then $y_n \in K_n \cap S_{1/q}(x) \subset S_{1/r}(x) \subset S_\epsilon(x)$. In either case, $y_n \in S_\epsilon(x)$, which implies that $\|y_n - x\| < \epsilon$ for any $n > N_r$. Thus,

$$\lim_{i \to \infty} y_n = x.$$

Theorem 40 of Allen [1] implies that $x \in \text{ck}A$; hence, $K \subset \text{ck}A$.

Suppose that there exists a point $y \in \text{ck}A \setminus K$. If $y \notin K$ then there exists a real number $\beta > 0$ such that for any $N > 0$ there exists some $n > N$ such that $\Delta(K_n, K \cup \{y\}) \geq \beta$, which implies that $K_n \not\subset (K \cup \{y\})_{\beta/2}$ or that $K \cup \{y\} \not\subset (K_n)_{\beta/2}$. But the fact that $\{K_i\}$ converges to K implies that there exists an integer $N_\beta > 0$ such that if $n > N_\beta$, then $\Delta(K_n, K) < \beta/2$, that is, $K_n \subset K_{\beta/2}$ and $K \subset (K_n)_{\beta/2}$. Since $K_{\beta/2} \subseteq (K \cup \{y\})_{\beta/2}$, there must exist some $n > N_\beta$ such that $K \cup \{y\} \not\subset (K_n)_{\beta/2}$, and since $K \subset (K_n)_{\beta/2}$, $y \notin (K_n)_{\beta/2}$; thus, $y \notin K_n$. But $y \in \text{ck}A \subset K_n$ for any $n > 0$. This contradiction implies that $\text{ck}A \subset K$, and the above result yields the fact

$$\lim_{i \to \infty} K_i = K = \text{ck}A.$$

There are several questions that might be raised concerning the hypothesis of Theorem 5.2. First, if $\{A_i\}$ is a convergent sequence of compact star-shaped sets in a normed linear space L, then under what conditions will $\{A_i\}$ converge to a star-shaped set A? The next theorem gives a sufficient condition for this set A to be star-shaped.

Theorem 5.3: Let $\{A_i\}$ be a sequence of compact star-shaped sets in the linear space $L_n$. If

$$\lim_{i\to\infty} A_i = A,$$

then A is a star-shaped set.

Proof: Consider any $\epsilon > 0$. There exists an integer $N > 0$ such that if $m > N$ then $\Delta(A_m, A) < \epsilon$, which implies that $A_m \subset A_\epsilon$ and $A \subset (A_m)_\epsilon$. Since $A_m$ is compact for each m, it is bounded, as is $(A_m)_\epsilon$. The set A $\subset (A_m)_\epsilon$ for $m > N$; thus, A is bounded, as is $A_\epsilon$. The fact that $A_m \subset A_\epsilon$ for all but a finite number of integers $m > 0$ implies that $\{A_i\}$ is uniformly bounded, since each $A_i$ is bounded. Choose a sequence of points $\{y_i\}$ such that $y_i \in ck(A_i)$ for each i. The set of points $\{y_i\}$ is bounded; therefore, there exists a subsequence $\{y_j'\}$ which converges to some point $y \in L_n$. But Theorem 40 of Allen [1] gives the result that

$$y \in ck(\lim_{j\to\infty} A_j'),$$

where $y_j' \in A_j'$. But

$$\lim_{j\to\infty} \Delta(A_j', A) = 0,$$

and

$$\lim_{j\to\infty} A_j' = A = \lim_{i\to\infty} A_i,$$

so that $y \in ckA$, which implies that A is star-shaped.

This result shows that part of the hypothesis of Theorem 5.2 is redundant in the linear space $L_n$.

Another point of interest in the hypothesis of Theorem 5.2 is the existence of the limit of the sequence $\{ck(A_i)\}$. The theorem that follows shows that in the linear space $L_n$ this limit exists.

<u>Theorem 5.4</u>: Let $\{A_i\}$ be a sequence of compact star-shaped sets in $L_n$. If the sequence $\{A_i\}$ converges to A and ckA $\subset$ ck$(A_i)$ for any $i > 0$, then

$$\lim_{i\to\infty} (ck(A_i))$$

exists.

<u>Proof</u>: The previous proof shows that $\{ck(A_i)\}$ is uniformly bounded. Consider any subsequence of $\{ck(A_i)\}$; if this subsequence contains only a finite number of distinct sets then there will exist a constant subsequence that converges. If the number of distinct sets is infinite, then the Blaschke selection theorem gives the convergence of some subsequence. In either case, Theorem 5.2 implies that the convergent subsequence converges to the convex kernel of A. If the sequence $\{ck(A_i)\}$ does not converge, then it does not converge to the set ckA, which is nonempty as a result of Theorem 5.3. There exists a real number $\epsilon_0 > 0$ such that for any $N > 0$ there exists some $m > N$ such that $\Delta(ck(A_m), ckA) \geq \epsilon_0$. However, the subsequence of $\{ck(A_i)\}$ so obtained must contain a subsequence which converges to ckA, a contradiction since no such subsequence can exist. Thus,

$$\lim_{i\to\infty} (ck(A_i))$$

must exist.

The ultimate aim is to show that for any compact star-shaped set S in a normed linear space L there exists a sequence of polyhedral star-shaped sets $\{P_i\}$ which converges to S such that

$$\lim_{i\to\infty} (ck(P_i)) = ckS.$$

Theorem 5.5 gives the basic step needed to show the existence of such a sequence. However, before considering that theorem we first need to prove the following lemma.

Lemma 5.1: Let C be a compact convex set in a normed linear space L. For any $\epsilon > 0$, if $x \notin C$ then $S_\epsilon(x) \not\subset C_\epsilon$.

Proof: Without loss of generality suppose that $x = 0$. Since $0 \notin C$, which is closed, there exists a real number $\alpha > 0$ such that $S_\alpha(0) \cap C = \emptyset$. Since C is compact and $\|\ \|$, a particular norm on L, is continuous, there exists a point $p \in C$ such that $\|p\| = \inf \{\|y\| : y \in C\}$ and $\|p\| > 0$. For any $\lambda > 0$ consider the point $-\lambda p$. Suppose that there exists a point $x_1 \in C$ such that $\|x_1 + \lambda p\| < \|p + \lambda p\|$. The scalar $\lambda(1+\lambda)^{-1} \in (0,1)$ since $\lambda > 0$. Thus, $\lambda(1+\lambda)^{-1}p + (1+\lambda)^{-1}x_1 \in C$, and

$$\begin{aligned}\left\| \frac{\lambda}{1+\lambda} p + \frac{1}{1+\lambda} x_1 \right\| &= \frac{1}{1+\lambda} \|\lambda p + x_1\| \\ &< \frac{1}{1+\lambda} \|p + \lambda p\| \\ &= \frac{1}{1+\lambda}(1+\lambda)\|p\| \\ &= \|p\|,\end{aligned}$$

a contradiction. Therefore, $\|p + \lambda p\| \leq \|y + \lambda p\|$ for any $y \in C$.

Consider any $\epsilon > 0$. If $\|p\| \geq \epsilon$, then $0 \notin C_\epsilon$ and $S_\epsilon(0) \not\subset C_\epsilon$. Suppose that $\|p\| < \epsilon$. Then $\epsilon(\|p\|)^{-1} > 1$ and $\epsilon(\|p\|)^{-1} - \frac{1}{2} > 0$, so that

$$\left\|\left(\frac{\epsilon}{\|p\|} - \frac{1}{2}\right)p\right\| = \left(\frac{\epsilon}{\|p\|} - \frac{1}{2}\right)\|p\|$$

$$= \epsilon - \frac{\|p\|}{2}$$

$$< \epsilon,$$

which implies that

$$\left(\frac{1}{2} - \frac{\epsilon}{\|p\|}\right)p \in S_\epsilon(0).$$

But

$$\left\|p + \left(\frac{\epsilon}{\|p\|} - \frac{1}{2}\right)p\right\| = \left(\frac{\epsilon}{\|p\|} + \frac{1}{2}\right)\|p\|$$

$$= \epsilon + \frac{\|p\|}{2}$$

$$> \epsilon,$$

and since

$$\left\|y + \left(\frac{\epsilon}{\|p\|} - \frac{1}{2}\right)p\right\| \geq \left\|p + \left(\frac{\epsilon}{\|p\|} - \frac{1}{2}\right)p\right\| > \epsilon$$

for any $y \in C$,

$$\left(\frac{1}{2} - \frac{\epsilon}{\|p\|}\right)p \notin C_\epsilon.$$

Therefore, $S_\epsilon(0) \not\subset C_\epsilon$.

<u>Theorem 5.5</u>: Let S be a compact star-shaped set in $L_n$. Then for any $\epsilon > 0$ there exists a polyhedral star-shaped set P such that $ckS \subset ckP$ and $\Delta(S,P) < \epsilon$.

<u>Proof:</u> For any $\epsilon > 0$, $B(\epsilon) = \{x: \|x\| \leq \epsilon\} \subset L_n$ is compact. Thus, $S + B(\epsilon)$ is compact and star-shaped. Furthermore, $ckS + B(\epsilon) \subset ck(S + B(\epsilon))$. For, let $y \in ckS + B(\epsilon)$. Then $y = x + z$, where $x \in ckS$ and $z \in B(\epsilon)$, which implies that $\|y - x\| = \|z\| \leq \epsilon$. Let $w \in S + B(\epsilon)$, that is, $w = u + v$, where $u \in S$, $v \in B(\epsilon)$; as before, $\|w - u\| = \|v\| \leq \epsilon$. For any $\alpha \in [0,1]$ consider $\alpha y + (1-\alpha)w$. Clearly $\alpha x + (1-\alpha)u \in S$, and

$\|\alpha y + (1-\alpha)w - (\alpha x + (1-\alpha)u)\| \leq \alpha\|y - x\| + (1-\alpha)\|w - u\| \leq \epsilon$. Therefore, $\alpha y + (1-\alpha)w - (\alpha x + (1-\alpha)u) \in B(\epsilon)$, which implies that $\alpha y + (1-\alpha)w \in S + B(\epsilon)$. Since $\alpha$ was arbitrary, $y \in ck(S + B(\epsilon))$; hence, $ckS + B(\epsilon) \subset ck(S + B(\epsilon))$.

Consider the collection $\mathfrak{B}_1$ of all $\epsilon/2$-balls with centers in $ckS + B(\epsilon/2)$. Since $ckS + B(\epsilon/2)$ is compact there exists a finite subcollection $\mathfrak{B}_1'$ of $\mathfrak{B}_1$ which covers $ckS + B(\epsilon/2)$; denote the collection of centers by $T = \{y_1, y_2, \cdots, y_m\}$. Consider the collection $\mathfrak{B}_2$ of all $\epsilon/2$-balls with centers in $(S + B(\epsilon/2))\backslash ckS + B(\epsilon/2)$. There exists a finite subcollection of $\mathfrak{B}_2'$ of $\mathfrak{B}_2$ such that $\mathfrak{B}_1' \cup \mathfrak{B}_2'$ covers $S + B(\epsilon/2)$; denote the collection of centers associated with $\mathfrak{B}_2'$ by $T' = \{x_1, x_2, \cdots, x_n\}$. Let $\mathcal{E} = \{A : A \subset T' \text{ and conv } A \subset S + B(\epsilon/2)\}$. Since $T \subset ck + B(\epsilon/2) \subset ck(S + B(\epsilon/2))$, $\text{conv } T \subset ck(S + B(\epsilon/2))$. If

$$P = \bigcup_{A \in \mathcal{E}} \text{conv } (\text{conv } T \cup \text{conv } A),$$

then $P \subset S + B(\epsilon/2) \subset P_{\epsilon/2}$ and $P$ is a polyhedral star-shaped set. Clearly, $P \subset S + B(\epsilon/2) \subset S_{3\epsilon/4}$ and $S \subset S + B(\epsilon/2) \subset P_{\epsilon/2}$, which implies that $\Delta(P,S) < \epsilon$.

Let $C = \text{conv } T$; it can be readily shown that $C \subset ckS + B(\epsilon/2) \subset C_{\epsilon/2}$. Suppose that there exists a point $x \in ckS\backslash C$. Since $x \notin C$, and since $C$ is a compact convex set in $L_n$, Lemma 5.1 implies that $S_{\epsilon/2}(x) \not\subset C_{\epsilon/2}$. But $S_{\epsilon/2}(x) \subset x + B(\epsilon/2) \subset ckS + B(\epsilon/2) \subset C_{\epsilon/2}$, a contradiction. Thus, $ckS \subset C \subset ckP$.

<u>Theorem 5.6</u>: Let $S$ be a compact star-shaped set in $L_n$. Then there exists a sequence $\{P_i\}$ of polyhedral star-shaped sets which converges to $S$ such that

$$\lim_{i\to\infty} (ck(P_i) = ckS.$$

Proof: Theorem 5.5 implies that for every $n > 0$ there exists a polyhedral star-shaped set $P_n$ such that $\Delta(S,P_n) < n^{-1}$ and $ckS \subset ck(P_n)$. Since $0 \le \Delta(S,P_n) < n^{-1}$ for each $n > 0$ and

$$\lim_{n\to\infty} n^{-1} = 0,$$

then

$$\lim_{n\to\infty} \Delta(S,P_n) = 0,$$

which implies that $\{P_n\}$ converges to S. Theorem 5.4 gives the result that $\{ck(P_i)\}$ converges and Theorem 5.2 implies that

$$\lim_{i\to\infty} (ck(P_i)) = ckS.$$

For any compact star-shaped set S in a normed linear space L, and for any $\epsilon > 0$, Theorem 5.1 gives the existence of a polyhedral star-shaped set P which approximates S such that $\Delta(S,P) < \epsilon$. A constructive procedure will now be given for finding such a polyhedral star-shaped set in the linear space $L_n$.

Let S be a compact star-shaped subset of $L_n$ and let $\epsilon > 0$. Let $\|\ \|$ be any norm on $L_n$. Then there exists a basis $\{b_1, b_2, \cdots, b_n\}$ of $L_n$ such that $\|b_i\| = 1$ for $i = 1, 2, \cdots, n$. Consider the set

$$T(\epsilon) = \frac{\epsilon}{2n}\{x: x = \sum_{i=1}^{n} \lambda_i b_i,\ 0 \le \lambda_i \le 1,\ i = 1, 2, \cdots, n\}.$$

If $x, y \in T(\epsilon)$, then

$$x = \frac{\epsilon}{2n}\sum_{i=1}^{n} \lambda_i b_i,\quad y = \frac{\epsilon}{2n}\sum_{i=1}^{n} \mu_i b_i,\ 0 \le \lambda_i \le 1,\ 0 \le \mu_i \le 1,\ 1 \le i \le n;$$

hence,

$$\|x - y\| = \frac{\epsilon}{2n}\left\|\sum_{i=1}^{n}(\lambda_i - \mu_i)b_i\right\| \le \frac{\epsilon}{2n}\sum_{i=1}^{n}|\lambda_i - \mu_i|\,\|b_i\| \le \frac{\epsilon}{2n}\sum_{i=1}^{n}\|b_i\| = \epsilon/2 < \epsilon,$$

since $|\lambda_i - \mu_i| \leq 1$. Consider the set M of all n-tuples of integers $(m_1, m_2, \cdots, m_n)$. Let

$$G = \{\frac{\epsilon}{2n}\sum_{i=1}^{n} m_i b_i : (m_1, m_2, \cdots, m_n) \in M\}$$

and let $\mathfrak{M}_\epsilon = \{x + T(\epsilon): x \in G\}$. If $y \in L_n$, then

$$y = \sum_{i=1}^{n} \beta_i b_i.$$

There exists an n-tuple of integers $(k_1, k_2, \cdots, k_n) \in M$ such that

$$\frac{\epsilon}{2n} k_i \leq \beta_i \leq \frac{\epsilon}{2n}(k_i + 1),$$

which implies that

$$0 \leq \frac{2n}{\epsilon}\beta_i - k_i \leq 1.$$

Thus,

$$\frac{\epsilon}{2n}\sum_{i=1}^{n}(\frac{2n}{\epsilon}\beta_i - k_i)b_i = \sum_{i=1}^{n}\beta_i b_i - \frac{\epsilon}{2n}\sum_{i=1}^{n} k_i b_i$$

belongs to $T(\epsilon)$, so that $y \in x + T(\epsilon) \in \mathfrak{M}_\epsilon$ for

$$x = \frac{\epsilon}{2n}\sum_{i=1}^{n} k_i b_i \in G.$$

Therefore, $\mathfrak{M}$ covers $L_n$ and the set S.

For any $\lambda > 0$ the set

$$X_\lambda = \{p: p = \sum_{i=1}^{n} \alpha_i b_i, \ |\alpha_i| < \lambda\}$$

is open in $L_n$. Any such $X_\lambda$ intersects at most a finite number of sets in $\mathfrak{M}_\epsilon$. Since S is compact, S is bounded, so that $S \subset X_\lambda$ for some $\lambda > 0$. Hence, there are only a finite number of sets A in $\mathfrak{M}_\epsilon$ such that $A \cap S \neq \emptyset$. Let $\{A_1, A_2, \cdots, A_r\}$ be the collection of sets in $\mathfrak{M}_\epsilon$ that intersect ckS. Let $\{A_{r+1}, A_{r+2}, \cdots, A_k\}$ be the collection of sets in $\mathfrak{M}_\epsilon$ that intersect S but do not intersect ckS. For each i such that $1 \le i \le r$ choose $y_i \in A_i \cap \text{ckS}$; for each i such that $r + 1 \le i \le k$, choose $y_i \in A_i \cap S$. Since $\{y_1, y_2, \cdots, y_n\} = D \subset \text{ckS}$, conv $D \subset \text{ckS}$. If $D' = \{y_{r+1}, y_{r+2}, \cdots, y_k\}$ let $\mathcal{E} = \{E: E \subset D', \text{conv } E \subset S\}$. Then let

$$P = \bigcup_{E \in \mathcal{E}} \text{conv } (\text{conv } D \cup \text{conv } E);$$

$P \subset S$ and P is a polyhedral star-shaped set. If $y \in S$ there exists some i, $1 \le i \le k$, such that $y \in A_i$. Then $\|y - y_i\| < \epsilon$ since $y, y_i \in A_i = x + T(\epsilon)$ for some $x \in G$, and the fact that $y_i \in P$ implies that $y \in P_\epsilon$; thus, $P \subset S \subset P_\epsilon$.

The constructive procedure given above provides a method of finding a polyhedral star-shaped set P which is known to exist by Theorem 5.1. A similar procedure may be applied to obtain a polyhedral star-shaped set P which will satisfy the demands of Theorem 5.5.

The following example is given to illustrate the previously defined procedure.

<u>Example 5.4</u>: Consider the linear space $E_2$ with its natural basis. Let $B(1) = \{p: \|p\| \le 1\}$, where $\| \ \|$ is the Euclidean norm. Then let $S = B(1) + \{(1,3/4), (1,-3/4)\}$ (cf. Fig. 5.4). For $\epsilon = 1$, the parallelepipeds in $\mathfrak{M}_1$ are squares with sides of length 1/4. It can be shown analytically that ckS = conv $\{(1 + \sqrt{7}/4, 0), (1 - \sqrt{7}/4, 0), (1, 7/12), (1, -7/12)\}$. The points of D and D', as well as the polyhedral star-shaped set P, are shown in Figure 5.4. The points of D are denoted by the symbol "o".

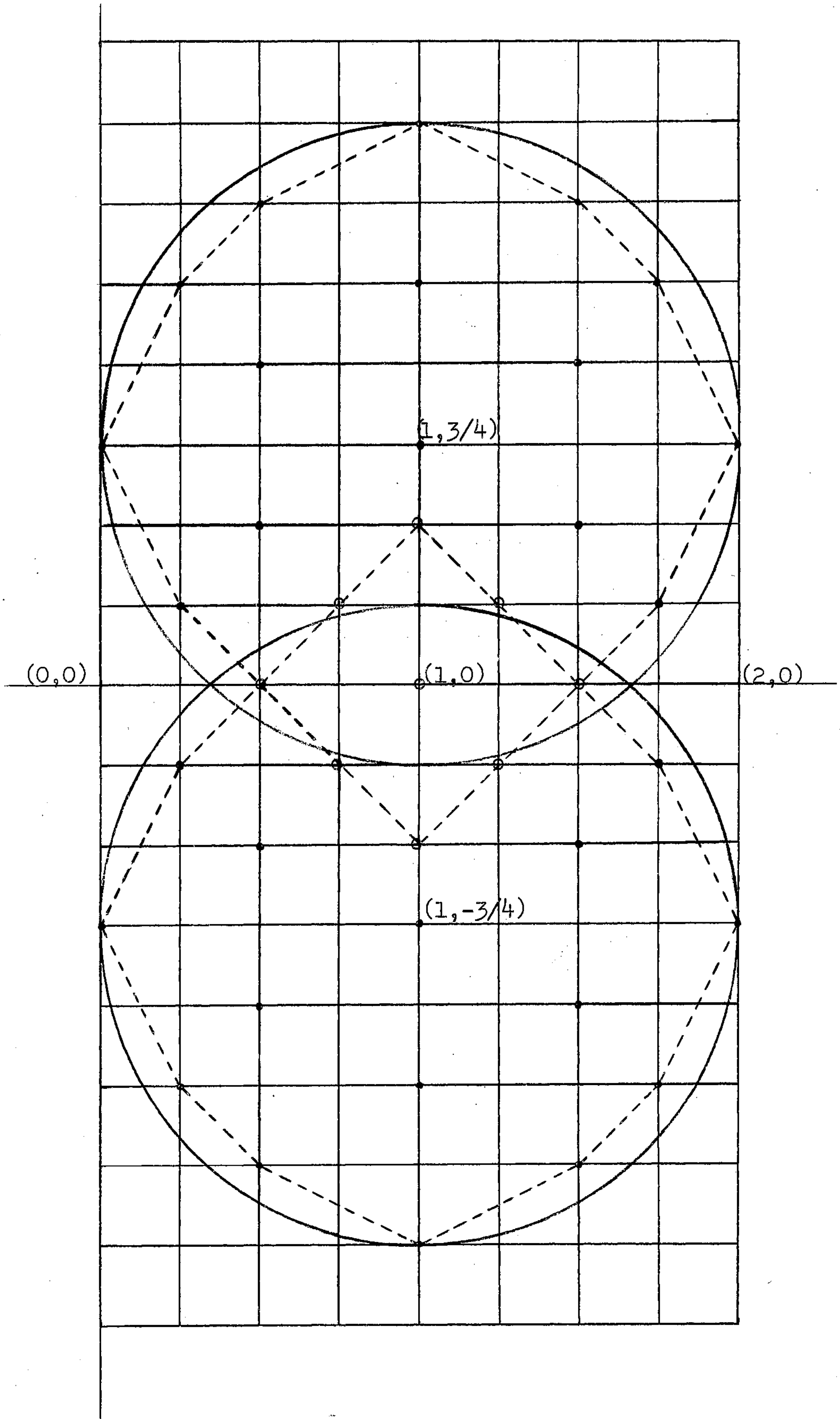


Figure 5.4

# CHAPTER VI

## SUMMARY AND CONCLUSIONS

The basic purpose of this study has been to examine the extremal structure of star-shaped sets and to determine the relationships that exist between this structure and the convex kernel. The first approach was made by considering $\alpha$-extreme points of compact star-shaped sets. It was found that such points can be used to actually determine the convex kernel of a compact star-shaped set in a locally convex space. Further investigation revealed properties concerning the cardinality of the set of $\alpha$-extreme points in a compact star-shaped set of dimension greater than two. It was discovered that in a normed linear space any compact star-shaped set has a countable subset which star-generates its convex kernel. The next topic to be examined was relative extreme points of star-shaped sets. It was shown that in a locally convex space any compact star-shaped set is completely determined by its convex kernel and the subset of points that are extreme points relative to the convex kernel. This representation of a compact star-shaped set resembles that of the Krein-Milman Theorem for compact convex sets. A class of compact star-shaped sets, called polyhedral star-shaped sets, was defined since the extremal structure of each set in this class is simpler than that of star-shaped sets in general. Sufficient conditions were given for a subset of $E_2$ to be a polyhedral star-shaped set. A metric was defined on the collection of compact star-shaped sets in a normed linear space and it was shown that any such set can be approximated by a polyhedral star-shaped set. Sufficient conditions were given for the sequence of convex kernels of the sets in a convergent sequence of compact star-shaped sets to converge to the convex kernel of the limit set. It was shown that for any compact star-shaped set S in a normed linear space, a sequence of compact star-shaped sets can be found which converges to S such that the associated sequence of convex kernels converges to the convex kernel of S. A constructive procedure was given for finding a polyhedral star-shaped set which approximates a compact star-shaped subset of $L_n$.

There are several problems which have been raised by this study which would be of interest for further consideration.

One such problem is the characterization of compact star-shaped sets according to the cardinality of minimal star-generating subsets, particularly those sets with finite star-generating subsets.

Sufficient conditions were given for a set to be a polyhedral star-shaped set, but necessary conditions were not found.

Theorem 5.2 would be of more value in applications if the sufficient conditions were independent of the convex kernel of A.

It would be desirable to extend Theorem 5.3 and Theorem 5.4 to infinite dimensional linear spaces or give a counterexample to show that it is not possible.

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# VITA

Freddie Eugene Tidmore

Candidate for the Degree of

Doctor of Philosophy

Thesis: EXTREMAL STRUCTURE OF STAR-SHAPED SETS

Major Field: Mathematics

Biographical:

Personal Data: Born in Ballinger, Texas, September 1, 1940, the son of Freddie A. and Ruth E. Tidmore.

Education: Attended grade and high school in Norton, Texas, and was graduated from Norton High School in 1958; received the Bachelor of Science degree from Hardin-Simmons University, Abilene, Texas, with a major in mathematics, in May, 1962; received the Master of Science degree in mathematics from Oklahoma State University, Stillwater, Oklahoma, in August, 1963; completed requirements for the Doctor of Philosophy degree in mathematics from Oklahoma State University in May, 1968.

Professional Experience: Assistant Professor of Mathematics, Baylor University, 1963-1965.

Professional Organizations: Institutional member of American Mathematical Society, member of Mathematical Association of America.