VIBRATION ANALYSIS OF PLANAR FRAMES

BY THE STRING POLYGON METHOD

By

RAMESHCHANDRA KAPILRAM MUNSHI

Bachelor of Engineering (Civil)
University of Bombay
Bombay, India
1953

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1961

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
July, 1968

VIBRATION ANALYSIS OF PLANAR FRAMES

BY THE STRING POLYGON METHOD

Thesis Approved:

Jim Weller

Thesis Adviser

Robert L. Janes

Ahmed Dalan

E. K. McFachlan

D. D. Durham

Dean of the Graduate College

PREFACE

Dynamic analysis of structures is becoming more and more important in these days of rapidly advancing technology. One of the most important characteristic of a physical system on which its dynamic behavior depends is its natural frequency of vibration. In this dissertation, free and forced harmonic vibartions of planar frames are investigated by the String Polygon method.

Though the basic idea of the string polygon is about a century old, its application to the analysis of frames and other structural systems was first proposed by Professor Jan J. Tuma in 1960-61 in his lectures at Oklahoma State University and in numerous publications thereafter of himself and his associates. This dissertation is an outgrowth of those ideas and is possibly the first to investigate the application of the String Polygon method to dynamic analysis of structures.

The author wishes to take this opportunity to express his gratitude and indebtedness to the following individuals and organizations without whose assistance this work could not have been completed.

To Professor Jan J. Tuma for being chiefly responsible for the author's graduate training in Structural Engineering, for his friendship, advice and incessant encouragement throughout the author's graduate program;

To the members of his advisory committee Drs. R. L. Janes, E. K. McLachlan and A. E. Salama for their understanding and guidance;

To his former teachers Dr. K. S. Havner, Dr. J. W. Gillespie and

Prof. N. R. Tembe for their excellent instructions in the author's major field;

To his former colleagues Dr. M. N. Reddy, Dr. J. T. Oden, Dr. S. Bart Childs, Mr. James D. Ramey and Dr. Charles Lindbergh for their friendship and encouragement;

To Mr. Ashok Nain and Mr. Brij Kishor for helping check some of the computations;

To Mr. Eldon Hardy and Mr. Yohannes Woldemariam and numerous other friends for their friendship and confidence;

To the entire staff of the Oklahoma State University Computing Center for their invaluable assistance at all times;

And finally to the most valuable people in his life--the members of his family--his wife Donna Jean, his mother Mrs. Chatura K. Munshi, all his brothers and sisters, and also Mr. and Mrs. Mugatlal M. Vakil and Mr. and Mrs. Harold L. Davis--for their sacrifices, understanding, encouragement and most of all their faith in the author's ability to complete this undertaking.

Thanks are indeed also due Mrs. Donna DeFrain for an excellent job of typing for a very impatient author.

R. K. Munshi

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES (Continued)

NOMENCLATURE

a, b	Constant multipliers
i, j	End points of a general bar
k.	Frequency parameter = $(mp^2/AE)^{\frac{1}{2}}$
m.	Mass per unit length
p.	Angular frequency in rad/sec.
t.	Time variable
u.	Axial displacement
v.	Transverse displacement
w.	Load intensity
x, y, z.	Position variables
A.	Cross-sectional area
A, B, C.	Constants
A', B', C'	Constants
$\bar{C}$.	Elastic couple
E.	Modulus of elasticity
$\{F\}$.	Force Vector $\{N', V', M'\}$
F1	Sin λL Sinh λL
F3	Cos λL Cosh λL - 1
F5	Cos λL Sinh λL - Sin λL Cosh λL
F6	Cos λL Sinh λL + Sin λL Cosh λL
F7	Sin λL + Sinh λL
F8	Sin λL - Sinh λL
F9	Cos λL + Cosh λL

F10. . . . . Cos λL - Cosh λL

G1 . . . . 2 Cos kL

G2 . . . . 2 Sin kL

I. . . . . Moment of inertia

[I]. . . . . Unit matrix

[J]. . . . . Matrix defined on page 29

L. . . . . Length of a member

[L]. . . . . Matrix relating the member elastic functions to its end deformations

M. . . . . Bending moment

M' . . . . $M/\lambda_o^2 EI$

N. . . . . Axial force

N' . . . . $N/\lambda_o^3 EI$

P. . . . . Concentrated load

$\overline{P}$. . . . . Elastic load

$\{\overline{P}_j\}$ . . . . Elastic load vector of the jth member

R. . . . . $(A/I)^{\frac{1}{2}}$

{S}. . . . . Force-displacement vector $\{N', V', M', \Delta_x, \Delta_y, \theta'\}$

T. . . . . Function of time variable

[T]. . . . . Transport matrix

V. . . . . Shear force

V' . . . . $V/\lambda_o^3 EI$

X. . . . . Function of position variable

{X}. . . . . Vector of unknowns

α. . . . . Ratio of masses or cross-sectional areas

β. . . . . Ratio of moments of inertia

$\{\delta\}$. . . . . Displacement vector $\{\Delta_x, \Delta_y, \theta'\}$

Δ . . . . Axial or transverse displacement

ϵ . . . . Normal strain

θ . . . . End slope of a member

θ' . . . . θ/λ_o

λ . . . . Frequency parameter = $(mp^2/EI)^{\frac{1}{4}}$

ξ . . . . Position variable

$[\pi]$ . . . . Angular transformation matrix

$\{\tau\}$ . . . . Vector of load functions of a free-free bar

ϕ . . . . End slope of a string line

ω . . . . Slope of a member or angular frequency of applied loads in rad/sec

$[\omega]$ . . . . Angular transformation matrix

Subscripts

i . . . . Refers to the ith member or joint

ij . . . . Refers to the end i of the member ij

o . . . . Refers to the reference member

x' . . . . Refers to a general section at a distance x'

x, y . . . . Refer to the direction x, y

Superscript

M . . . . Refers to the member reference system

O . . . . Refers to the basic reference system

CHAPTER I

INTRODUCTION

1.1 Statement of the Problem

Free, as well as forced, harmonic, in-plane vibrations of planar rigid-jointed frames (Figure 1) are investigated using the String Polygon method. A minimum number of unknown forces and deformations are chosen as primary unknowns in any given frame. All end forces and end deformations for each member are expressed in terms of the primary unknowns by using transport matrices. The end elastic weights and the elastic moment for each member are derived from its end deformations. These elastic loads are then used to establish the elasto-static equations for all conjugate panels of the given frame.

The resulting set of simultaneous equations provides the solution for the primary unknowns. In case of free vibrations, the criterion of singularity of the coefficient matrix of the final equations gives the natural frequencies of the frames. For forced vibrations the unknowns are solved for in terms of the applied loads.

1.2 Scope of the Problem

This investigation concerns vibration analysis of planar frames vibrating in-plane. Free vibrations are studied for the natural frequencies of the frames whereas forced vibrations are studied for the response of the frame due to any applied harmonic loads. The

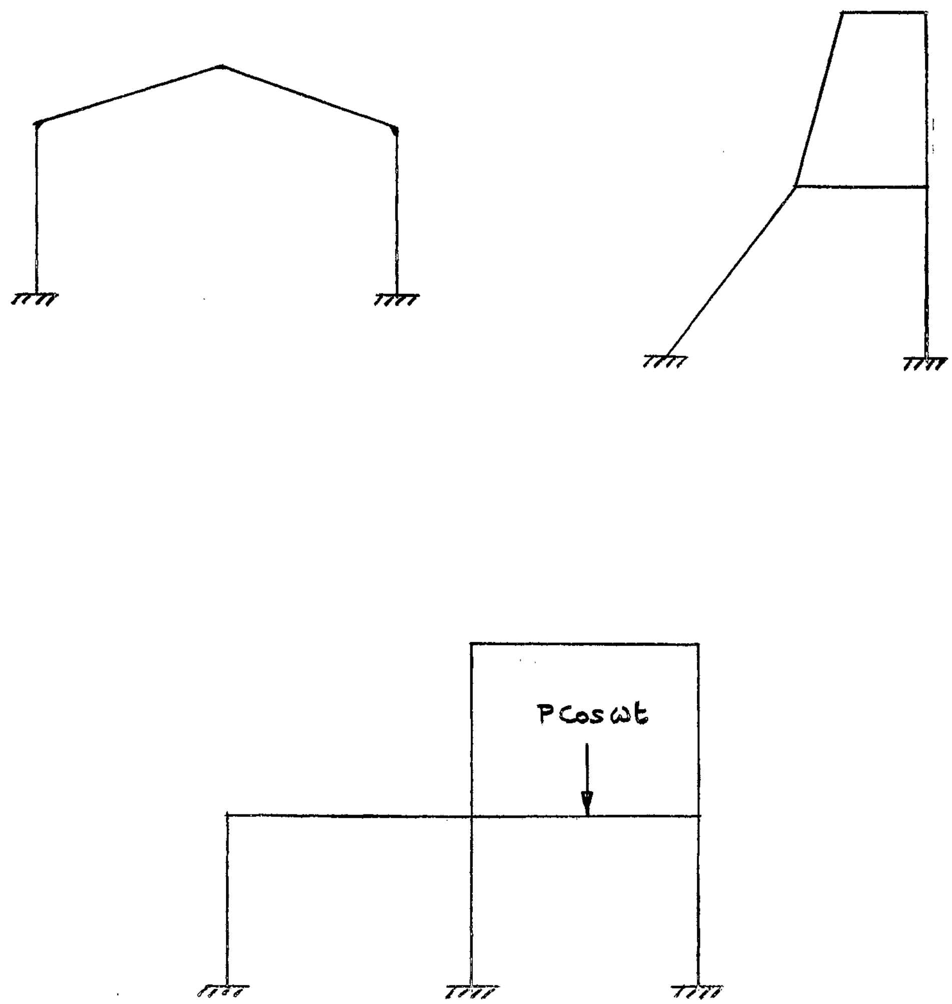


Figure 1. Planar Rigid Jointed Frames

investigation is restricted to the study of planar frames comprising of straight bars of constant sections connected rigidly at their ends. The frame supports may be pinned, fixed, guided or on rollers. Deformations are assumed to be small enough not to affect the basic geometry of the frames and positions of applied loads. Effect of axial deformations is included.

The study also includes the investigation of the most suitable choice of the primary unknowns and of the method of formulating the problem using the string polygon concepts. The application of the method is to be illustrated by numerical examples.

1.3 Assumptions and Limitations of the Problem

In addition to the commonly made assumptions in the Euler-Bernoulii small deflection theory of bending, the following assumptions and limitations apply:

1. The stresses are within the elastic limit and the stress-strain relationship is linear.
2. Each member is straight and of a constant section and has uniform properties throughout its length.
3. One principal plane of each member coincides with the plane of the frame.
4. The cross-sectional dimensions of each member are small in comparison with its length. Hence shear deformations and rotatory inertia are neglected.
5. Only small oscillations are considered. Hence transverse deformations are considered independent of axial forces.
6. Axial forces induced in the members are small compared to

their critical buckling loads.

7. Damping is considered very small and is neglected in case of free vibrations. However for forced vibrations, only steady state part is considered.

8. It is possible to express the deformations at a point as a product of a position function and a time function.

9. Response of the frame to external harmonic loads will either be in phase or out of phase by 180° with the loads.

1.4 Notations and Symbols

Notations and symbols of quantities appearing in this dissertation are defined where they first appear and are also compiled under Nomenclature. This also contains a list of circular and hyperbolic hybrid functions adopted from Bishop (16).

1.5 Historical Review

Before the classical methods in structural dynamics were developed, lumped mass approximation appears to have been widely used to obtain good results both for beams and frames. Among the first to study the dynamic analysis of beams, considering the mass distributed, seem to be Rayleigh (1) and Love (2). The application of the classical analytical methods for determining natural frequencies of beams and simple frames has been described, among others, by Darnley (3), Hohenemser and Prager (4), Timoshenko (5), Bennon (6) and Saibel (7), (8), and (9).

Gaskell (10) extended Cross's moment balancing and Grinter's angle balancing techniques to problems in structural dynamics.

The so-called stiffness analysis has been effectively used by Veletsos and Newmark (11), (12), and (13), for problems in structural dynamics. Rieger and McCallion (14) have studied the natural frequencies of single as well as multi-span, pinned and fixed base portal frames and prepared tables to aid in the design of portal frames.

An important contribution to the field of structural dynamics is made by Bishop (15), (16), and (17) in his receptance method which appears to be the first systematic approach in analyzing vibrating systems using the flexibility concept. He however gives credit to Carter (18) for introducing the dynamic flexibility concept and to Duncan (19) and Johnson (20) for extending it.

The application of matrix analysis to structural dynamics is studied by Pestel and Leckie (21) and also by Marguerre (22). Laursen, Shubinski and Clough (23) as well as Ariaratnam (24) have applied the stiffness matrix methods to vibration analysis of frames. Levien and Hartz (25) have published a paper on the dynamic flexibility matrix analysis of frames.

A good number of books on structural dynamics have been published in the last decade, e.g. Rogers (26), Biggs (27). These books explore the analysis of many types of vibrating structural systems using all generally available methods for static analysis of structures.

All the literature however seem to indicate the complexity of computation in problems in structural dynamics and indicate the use of electronic computers as imperative, particularly for complex frames. Elaborate computer programs to solve problems in this area are reported to have been developed at the University of California at Berkeley (23) and at Massachussets Institute of Technology (28).

CHAPTER II

MEMBER DYNAMIC PROPERTIES

2.1 Co-ordinate Systems

Two types of co-ordinate reference systems are used. Both are right-handed, orthogonal cartesian systems. The first referred to as a member system is a system associated with each member. It has its origin at one end of the member, its x-axis aligned along the member, its y-axis in the plane of the frame and its z-axis normal to it.

The other system, referred to as the basic system, has fixed reference axes with x-y axes in the plane of the frame and z axis normal to it. The origin of the basic reference system is located at any convenient point.

2.2 Member Dynamic Properties

The elastic properties of a straight bar in dynamic state are defined by relating its end forces and end deformations.

A typical bar ij taken out of a vibrating frame is considered (Figure 2). The figure shows the amplitudes of the end forces acting on the bar. Cross-sectional sign convention is adopted to define the orientation of the end forces which are shown in their positive sense.

The end deformations are measured in the directions of the end forces. The member reference axes are also shown.

Axial and transverse vibrations are considered separately.

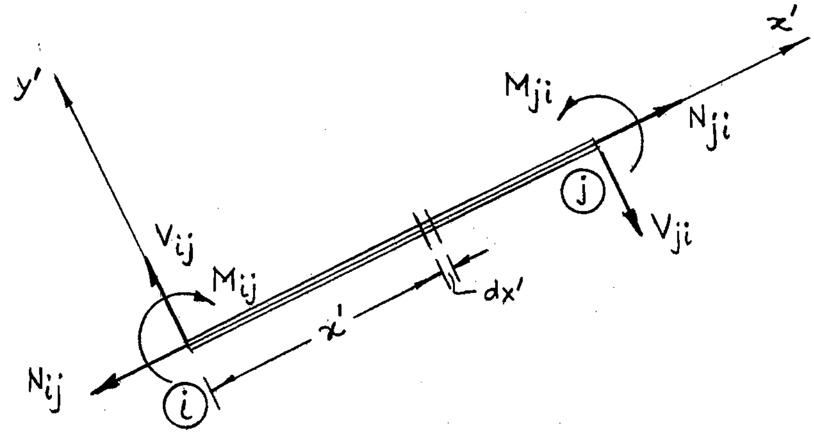


Figure 2. Typical Bar With End Froces

Axial Vibrations: The governing differential equation for axial vibrations is derived by considering the dynamic equilibrium of a small element of length dx' taken out of the member as a free body (Figure 3).

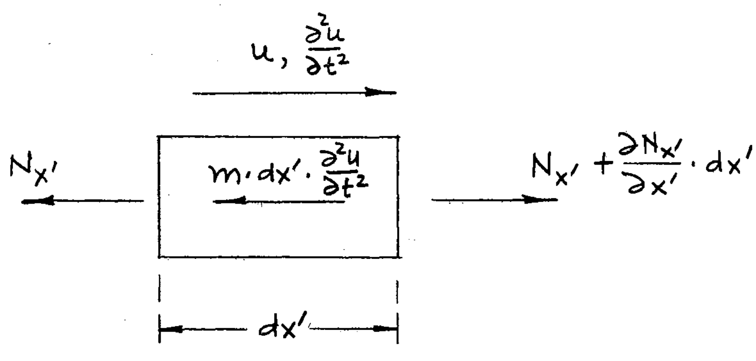


Figure 3. Differential Length of a Member in Axial Vibrations

The equation of equilibrium is

$$\frac{\partial N_{x'}}{\partial x'} = m \frac{\partial^2 u}{\partial t^2} \tag{2-1}$$

The stress-strain relation at that section gives

$$\epsilon_{x'} = \frac{\partial u}{\partial x'} = \frac{N_{x'}}{AE} \tag{2-2}$$

Combining equations (1) and (2) gives

$$\frac{\partial^2 u}{\partial x'^2} = \frac{m}{AE} \frac{\partial^2 u}{\partial t^2} \tag{2-3}$$

where

u = axial displacement

m = mass/unit length

A = area of cross section

E = modulus of elasticity

N = normal force

x' = position variable

t = time variable

ϵ = axial strain

Assuming a product solution for u of equation (2-3), $u(x',t) = X(x') \cdot T(t)$, the general solution can be shown to be

$$u(x',t) = (C'\text{Cos } kx' + D'\text{Sin } kx')\,(A'\text{Cos } pt + B'\text{Sin } pt) \tag{2-4}$$

where

p = angular frequency of vibration

$k = (mp^2/AE)^{\frac{1}{2}}$

and A', B', C' and D' are constants.

Assuming that all forces and displacements reach their amplitudes in phase (or out of phase by 180°) with each other and working with the amplitudes, the end axial forces and end axial displacements can be related by making this solution satisfy the following boundary conditions:

$$\text{at } x' = 0,\ u = -\Delta_{ijx'},\ \frac{\partial u}{\partial x'} = \frac{N_{ij}}{AE} \tag{2-5}$$

and

$$\text{at } x' = L,\ u = +\Delta_{jix'},\ \frac{\partial u}{\partial x'} = \frac{N_{ji}}{AE} \tag{2-6}$$

Applying the first two conditions gives

$$\begin{bmatrix} \Delta_{ijx'} \\ N_{ij}/AE \end{bmatrix} = \begin{bmatrix} -1.0 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} C' \\ D' \end{bmatrix} \tag{2-7}$$

Applying the remaining two conditions gives

$$\begin{bmatrix} \Delta_{jix'} \\ N_{ji}/AE \end{bmatrix} = \begin{bmatrix} \text{Cos } kL & \text{Sin } kL \\ -k \text{ Sin } kL & k \text{ Cos } kL \end{bmatrix} \begin{bmatrix} C' \\ D' \end{bmatrix} \tag{2-8}$$

Combining equations (2-7) and (2-8) and rearranging gives

$$\begin{bmatrix} N_{ji}/kAE \\ \Delta_{jix'} \end{bmatrix} = \begin{bmatrix} \text{Cos } kL & \text{Sin } kL \\ \text{Sin } kL & -\text{Cos } kL \end{bmatrix} \begin{bmatrix} N_{ij}/kAE \\ \Delta_{ijx'} \end{bmatrix} \qquad (2\text{-}9)$$

In case the axial deformations are neglected, equation (2-9) may be modified by letting $A \rightarrow \infty$, $k^2 \rightarrow 0$, and $k^2AE \rightarrow mp^2$. In that case

$$\begin{bmatrix} N_{ji} \\ \Delta_{jix'} \end{bmatrix} = \begin{bmatrix} 1.0 & mp^2L \\ 0 & -1.0 \end{bmatrix} \begin{bmatrix} N_{ij} \\ \Delta_{ijx'} \end{bmatrix} \qquad (2\text{-}9a)$$

Transverse Vibrations: The governing differential equations for transverse vibrations is derived by considering the dynamic equilibrium of the small element of length dx' under the effect of transverse forces and moments (Figure 4).

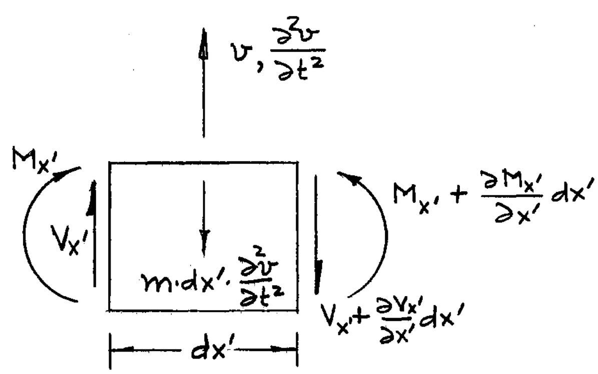


Figure 4. Differential Length of a Member in Transverse Vibrations

The equations of equilibrium are

$$\frac{\partial V_{x'}}{\partial x'} + m \frac{\partial^2 v}{\partial t^2} = 0 \qquad (2\text{-}10)$$

and

$$\frac{\partial M_{x'}}{\partial x'} = V_{x'} \qquad (2\text{-}11)$$

The moment-deformation relation is

$$EI \frac{\partial^2 v}{\partial x'^2} = M_{x'} \qquad (2\text{-}12)$$

Combining Equations (2-10), (2-11) and (2-12) gives

$$\frac{\partial^4 v}{\partial x'^4} + \frac{m}{EI} \frac{\partial^2 v}{\partial t^2} = 0 \qquad (2\text{-}13)$$

where

v = transverse displacement

I = moment of inertia of the cross section about the axis of bending

V = shear force

M = bending moment

Assuming a product solution for v of equation (2-13), $v(x',t) = X(x') \circ T(t)$, the general solution can be shown to be

$$v(x',t) = (A \cos \lambda x' + B \sin \lambda x' + C \cosh \lambda x' + D \sinh \lambda x') \cdot (A' \cos pt + B' \sin pt) \tag{2-14}$$

where

$$\lambda = (mp^2/EI)^{\frac{1}{4}}$$

and A, B, C and D are constants.

Assuming that all forces, moments and deformations reach their amplitudes in phase (or out of phase by 180^0) with each other and working with the amplitudes, the end forces and end deformations of the bar can be related by making the solution satisfy the following boundary conditions:

At $x' = 0$, $\quad v = \Delta_{ijy'}$, $\quad \dfrac{\partial v}{\partial x'} = -\theta_{ij}$

$$\frac{\partial^2 v}{\partial x'^2} = \frac{M_{ij}}{EI} , \quad \frac{\partial^3 v}{\partial x'^3} = \frac{V_{ij}}{EI}$$

At $x' = L$, $\quad v = -\Delta_{jiy'}$, $\quad \dfrac{\partial v}{\partial x'} = \theta_{ji}$

$$\frac{\partial^2 v}{\partial x'^2} = \frac{M_{ji}}{EI} , \quad \frac{\partial^3 v}{\partial x'^3} = \frac{V_{ji}}{EI}$$

Applying the conditions at $x' = 0$ gives

$$\begin{bmatrix} \Delta_{ijy'} \\ -\theta_{ij}/\lambda \\ M_{ij}/\lambda^2 EI \\ V_{ij}/\lambda^3 EI \end{bmatrix} = \begin{bmatrix} 1.0 & 0 & 1.0 & 0 \\ 0 & 1.0 & 0 & 1.0 \\ -1.0 & 0 & 1.0 & 0 \\ 0 & -1.0 & 0 & 1.0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \tag{2-15}$$

Applying the conditions at $x' = L$ gives

$$\begin{bmatrix} -\Delta_{jiy'} \\ \theta_{ji}/\lambda \\ M_{ji}/\lambda^2 EI \\ V_{ji}/\lambda^3 EI \end{bmatrix} = \begin{bmatrix} \text{Cos } \lambda L & \text{Sin } \lambda L & \text{Cosh } \lambda L & \text{Sinh } \lambda L \\ -\text{Sin } \lambda L & \text{Cos } \lambda L & \text{Sinh } \lambda L & \text{Cosh } \lambda L \\ -\text{Cos } \lambda L & -\text{Sin } \lambda L & \text{Cosh } \lambda L & \text{Sinh } \lambda L \\ \text{Sin } \lambda L & -\text{Cos } \lambda L & \text{Sinh } \lambda L & \text{Cosh } \lambda L \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \tag{2-16}$$

Combining equations (2-15) and (2-16) and rearranging gives

$$\begin{bmatrix} V_{ji}/\lambda^3 EI \\ M_{ji}/\lambda^2 EI \\ \Delta_{jiy'} \\ \theta_{ji}/\lambda \end{bmatrix} = \frac{1}{2} \begin{bmatrix} F9 & -F8 & F7 & F10 \\ F7 & F9 & -F10 & F8 \\ F8 & F10 & -F9 & F7 \\ -F10 & F7 & -F8 & -F9 \end{bmatrix} \begin{bmatrix} V_{ij}/\lambda^3 EI \\ M_{ij}/\lambda^2 EI \\ \Delta_{ijy'} \\ \theta_{ij}/\lambda \end{bmatrix} \tag{2-17}$$

where

$F7 = \text{Sin } \lambda L + \text{Sinh } \lambda L$

$F8 = \text{Sin } \lambda L - \text{Sinh } \lambda L$

$F9 = \text{Cos } \lambda L + \text{Cosh } \lambda L$

$F10 = \text{Cos } \lambda L - \text{Cosh } \lambda L$

Transport Matrix: The force-deformation relations of axial and transverse vibrations, equations (2-9) and (2-17), can be combined into a single matrix relation:

$$\begin{bmatrix} N_{ji}/kAE \\ V_{ji}/\lambda^3 EI \\ M_{ji}/\lambda^2 EI \\ \Delta_{jix'} \\ \Delta_{jiy'} \\ \theta_{ji}/\lambda \end{bmatrix} = \frac{1}{2} \begin{bmatrix} G1 & 0 & 0 & G2 & 0 & 0 \\ 0 & F9 & -F8 & 0 & F7 & F10 \\ 0 & F7 & F9 & 0 & -F10 & F8 \\ G2 & 0 & 0 & -G1 & 0 & 0 \\ 0 & F8 & F10 & 0 & -F9 & F7 \\ 0 & -F10 & F7 & 0 & -F8 & -F9 \end{bmatrix} \begin{bmatrix} N_{ij}/kAE \\ V_{ij}/\lambda^3 EI \\ M_{ij}/\lambda^2 EI \\ \Delta_{ijx'} \\ \Delta_{ijy'} \\ \theta_{ij}/\lambda \end{bmatrix} \tag{2-18}$$

where

$G1 = 2 \text{ Cos } kL$

$G2 = 2 \text{ Sin } kL$

The axial force term may be made similar to the shear force term

$$\frac{N}{kAE} = \frac{N}{\lambda^3 EI} \cdot \frac{\lambda^3 EI}{kAE} = \frac{N}{\lambda^3 EI} \cdot \frac{\lambda \cdot I \cdot \lambda^2}{A \cdot k}$$

$$= \frac{N}{\lambda^3 EI} \; \frac{\lambda \cdot I}{A} \; \frac{\left(\frac{mp^2}{EI}\right)^{\frac{1}{2}}}{\left(\frac{mp^2}{AE}\right)^{\frac{1}{2}}} = \frac{N}{\lambda^3 EI} \cdot \lambda \cdot \left(\frac{I}{A}\right)^{\frac{1}{2}} = \frac{N}{\lambda^3 EI} \cdot \frac{\lambda}{R}$$

where

$$R = \left(\frac{A}{I}\right)^{\frac{1}{2}}$$

Using this, Equation (2-18) becomes

$$\begin{bmatrix} N_{ji}/\lambda^3 EI \\ V_{ji}/\lambda^3 EI \\ M_{ji}/\lambda^2 EI \\ \Delta_{jix'} \\ \Delta_{jiy'} \\ \theta_{ji}/\lambda \end{bmatrix} = \frac{1}{2} \begin{bmatrix} G1 & 0 & 0 & \frac{R}{\lambda}G2 & 0 & 0 \\ 0 & F9 & -F8 & 0 & F7 & F10 \\ 0 & F7 & F9 & 0 & -F10 & F8 \\ \frac{\lambda}{R}G2 & 0 & 0 & -G1 & 0 & 0 \\ 0 & F8 & F10 & 0 & -F9 & F7 \\ 0 & -F10 & F7 & 0 & -F8 & -F9 \end{bmatrix} \begin{bmatrix} N_{ij}/\lambda^3 EI \\ V_{ij}/\lambda^3 EI \\ M_{ij}/\lambda^2 EI \\ \Delta_{ijx'} \\ \Delta_{ijy'} \\ \theta_{ij}/\lambda \end{bmatrix} \tag{2-19}$$

In a frame with members of many different sizes, it is convenient to work in terms of the properties of one reference member. Referring the quantities λ, I, A, m and R as λ_o, I_o, A_o, m_o and R_o for the reference member and those for any other, say ith member, as λ_i, I_i, A_i, m_i and R_i and denoting

$$\alpha_i = \frac{A_i}{A_o} = \frac{m_i}{m_o} \quad , \quad \beta_i = \frac{I_i}{I_o}$$

$$N' = \frac{N}{\lambda_o^3 EI_o} \quad , \quad V' = \frac{V}{\lambda_o^3 EI_o}$$

$$M' = \frac{M}{\lambda_o^2 EI_o} \quad , \quad \theta' = \frac{\theta}{\lambda_o} \tag{2-20}$$

Equation (2-19) may be modified to read as shown on the next page.

In case the axial deformations are neglected the following modifications should be made in the coefficient matrix of Equation (2-21).

Term (1,1) = 2.0

" (1,4) = $2\alpha_i L_i \lambda_o$

" (4,1) = 0.0

" (4,4) = -2.0

denoting

$$\{F\} = \{N' \quad V' \quad M'\}$$
$$\{\delta\} = \{\Delta_x, \Delta_y, \theta'\}$$
$$\{S\} = \{F \;\vdots\; \delta\} \tag{2-22}$$

and denoting ½ x coefficient matrix = $[T'^M]$, Equation (2-21) can be written shortly as

$$\{S^M_{ji}\} = [T'^M_{ij}]\{S^M_{ij}\} \tag{2-23}$$

This can also be written as

$$\begin{bmatrix} F^M_{ji} \\ \hline \delta^M_{ji} \end{bmatrix} = \left[\begin{array}{c|c} T(11)'^M_{ij} & T(12)'^M_{ij} \\ \hline T(21)'^M_{ij} & T(22)'^M_{ij} \end{array}\right] \begin{bmatrix} F^M_{ij} \\ \hline \delta^M_{ij} \end{bmatrix} \tag{2-24}$$

It may be noted that the relation stated above can be easily used

$$
\begin{bmatrix} N_{ji}' \\ V_{ji}' \\ M_{ji}' \\ \Delta_{jix'} \\ \Delta_{jiy'} \\ \theta_{ji}' \end{bmatrix}
= \frac{1}{2}
\begin{bmatrix}
G1 & 0 & 0 & \frac{R_o}{\lambda_o}\alpha_i \cdot G2 & 0 & 0 \\
0 & F9 & -\alpha_i^{\frac{1}{4}}\beta_i^{-\frac{1}{4}}\cdot F8 & 0 & \alpha_i^{\frac{3}{4}}\beta_i^{\frac{1}{4}}\cdot F7 & \alpha_i^{\frac{1}{2}}\beta_i^{\frac{1}{2}}\cdot F10 \\
0 & \alpha_i^{-\frac{1}{4}}\beta_i^{\frac{1}{4}}\cdot F7 & F9 & 0 & -\alpha_i^{\frac{1}{2}}\beta_i^{\frac{1}{2}}\cdot F10 & \alpha_i^{\frac{1}{4}}\beta_i^{\frac{3}{4}}\cdot F8 \\
\frac{\lambda_o}{R_o\alpha_i} G2 & 0 & 0 & -G1 & 0 & 0 \\
0 & \alpha_i^{-\frac{3}{4}}\beta_i^{-\frac{1}{4}}\cdot F8 & \alpha_i^{-\frac{1}{2}}\beta_i^{-\frac{1}{2}}\cdot F10 & 0 & -F9 & \alpha_i^{-\frac{1}{4}}\beta_i^{\frac{1}{4}}\cdot F7 \\
0 & -\alpha_i^{-\frac{1}{2}}\beta_i^{-\frac{1}{2}}\cdot F10 & \alpha_i^{-\frac{1}{4}}\beta_i^{-\frac{3}{4}}\cdot F7 & 0 & -\alpha_i^{\frac{1}{4}}\beta_i^{-\frac{1}{4}}\cdot F8 & -F9
\end{bmatrix}
\begin{bmatrix} N_{ij}' \\ V_{ij}' \\ M_{ij}' \\ \Delta_{ijx'} \\ \Delta_{ijy'} \\ \theta_{ij}' \end{bmatrix}
\tag{2-21}
$$

to derive the flexibility functions as well as the stiffness functions of the bar, if desired. The flexibility functions are obtained when, by suitable transposition, the displacement values $\{\delta\}$s are expressed in terms of the force values $\{F\}$s. On the other hand, expressing the force values in terms of the displacement values gives the stiffness functions.

2.3 Load Functions

The end deformations induced in a free-free bar due to applied harmonic loads between its ends are termed load functions, τs.

Consider a free-free bar ij subjected to a harmonic transverse load of amplitude P applied at a general section $x' = \xi$, (Figure 5).

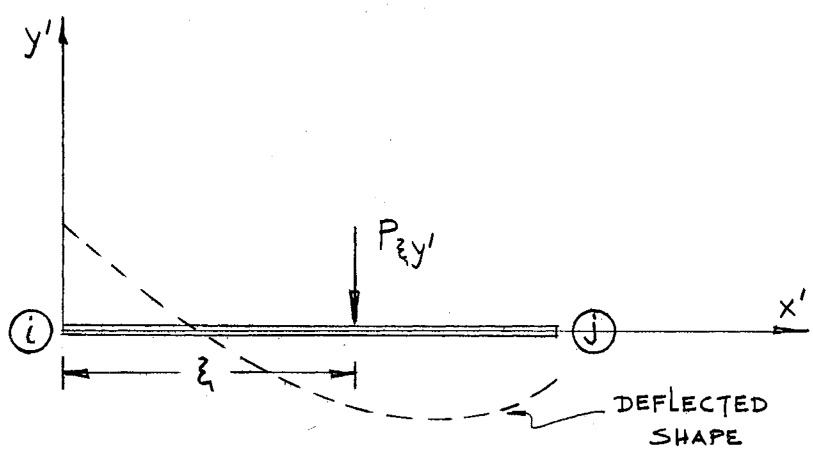


Figure 5. A Free-Free Bar With an Applied Transverse Force

The end slopes and displacements induced may be computed easily by the reciprocal deformation relations (Bishop (15)). Thus using

$$\begin{aligned}
\Delta_{ijy'}^{P_{\xi y'}=1} &= \Delta_{\xi y'}^{V_{ij}=1} \\
\Delta_{jiy'}^{P_{\xi y'}=1} &= \Delta_{\xi y'}^{V_{ji}=1} \\
\theta_{ij}^{P_{\xi y'}=1} &= \Delta_{\xi y'}^{M_{ij}=1} \\
\theta_{ji}^{P_{\xi y'}=1} &= \Delta_{\xi y'}^{M_{ji}=1} \quad ,
\end{aligned} \tag{2-25}$$

gives

$$\begin{aligned}
\Delta_{ijy'}^{P_{\xi y'}} &= P_{\xi y'} \cdot \Delta_{\xi y'}^{V_{ij}=1} \\
\Delta_{jiy'}^{P_{\xi y'}} &= P_{\xi y'} \cdot \Delta_{\xi y'}^{V_{ji}=1} \\
\theta_{ij}^{P_{\xi y'}} &= P_{\xi y'} \cdot \Delta_{\xi y'}^{M_{ij}=1} \\
\theta_{ji}^{P_{\xi y'}} &= P_{\xi y'} \cdot \Delta_{\xi y'}^{M_{ji}=1} \quad .
\end{aligned} \tag{2-26}$$

For more than one externally applied loads, the total end deformation effects may be found by superposition, and for distributed applied loads of amplitude $w_{x'}$, the total effect may be found by integrating Equations (2-26).

$$\Delta^{w}_{ijy'} = \int_{o}^{L} \Delta^{V_{ij}=1}_{\xi y'} \cdot w_{\xi y'} d\xi$$

$$\Delta^{w}_{jiy'} = \int_{o}^{L} \Delta^{V_{ji}=1}_{\xi y'} \cdot w_{\xi y'} d\xi$$

$$\theta^{w}_{ij} = \int_{o}^{L} \Delta^{M_{ij}=1}_{\xi y'} \cdot w_{\xi y'} d\xi$$

$$\theta^{w}_{ji} = \int_{o}^{L} \Delta^{M_{ji}=1}_{\xi y'} \cdot w_{\xi y'} d\xi$$

The effect of axial, applied load may be expressed similarly.

$$\Delta^{P_{\xi x'}}_{ijx'} = P_{\xi x'} \cdot \Delta^{N_{ij}=1}_{\xi x'}$$

$$\Delta^{P_{\xi x'}}_{jix'} = P_{\xi x'} \cdot \Delta^{N_{ji}=1}_{\xi x'}$$

And for a distributed axial, applied load

$$\Delta^{w}_{ijx'} = \int_{o}^{L} \Delta^{N_{ij}=1}_{\xi x'} \cdot w_{\xi x'} d\xi$$

$$\Delta^{w}_{jix'} = \int_{o}^{L} \Delta^{N_{ji}=1}_{\xi x'} \cdot w_{\xi x'} d\xi$$

The computation of the deformation functions due to unit end forces used above is shown in detail in Appendix A.

Load Effect in Transport Relation: When the effects of applied loads are to be included in a transport matrix relation such as Equation (2-24), it may be stated generally as

$$\left\{\begin{array}{c} F_{ji} \\ \hline \delta_{ji} \end{array}\right\} = [T'^M]\left\{\begin{array}{c} F_{ij} \\ \hline \delta_{ij} \end{array}\right\} + \left\{\begin{array}{c} F^w_{ji} \\ \hline \delta^w_{ji} \end{array}\right\} . \tag{2-27}$$

$\left\{\begin{array}{c} F^w_{ji} \\ \hline \delta^w_{ji} \end{array}\right\}$ represents the forces and deformations induced at j due to applied loads when the end i is constrained to some prescribed values of forces and deformations $\left\{\begin{array}{c} F_{ij} \\ \hline \delta_{ij} \end{array}\right\}$. Therefore $\left\{\begin{array}{c} F^w_{ji} \\ \hline \delta^w_{ji} \end{array}\right\}$ represents the forces and deformation needed at j in order that no forces and deformations are induced at i due to the loads on the bar ij (Figure 6). These values can be computed from the previously defined load functions of a free-free bar, as follows:

$$\left\{\begin{array}{c} F^w_{ji} \\ \hline \delta^w_{ji} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline \tau_{ji} \end{array}\right\} - [T'^M]\left\{\begin{array}{c} 0 \\ \hline \tau_{ij} \end{array}\right\} \tag{2-28}$$

where

$$\left\{\tau_{ij}\right\} = \left\{\begin{array}{c} \Delta^w_{ijx'} \\ \\ \Delta^w_{ijy'} \\ \\ \theta'^w_{ij} \end{array}\right\} , \text{ and } \left\{\tau_{ji}\right\} = \left\{\begin{array}{c} \Delta^w_{jix'} \\ \\ \Delta^w_{jiy'} \\ \\ \theta'^w_{ji} \end{array}\right\} . \tag{2-29}$$

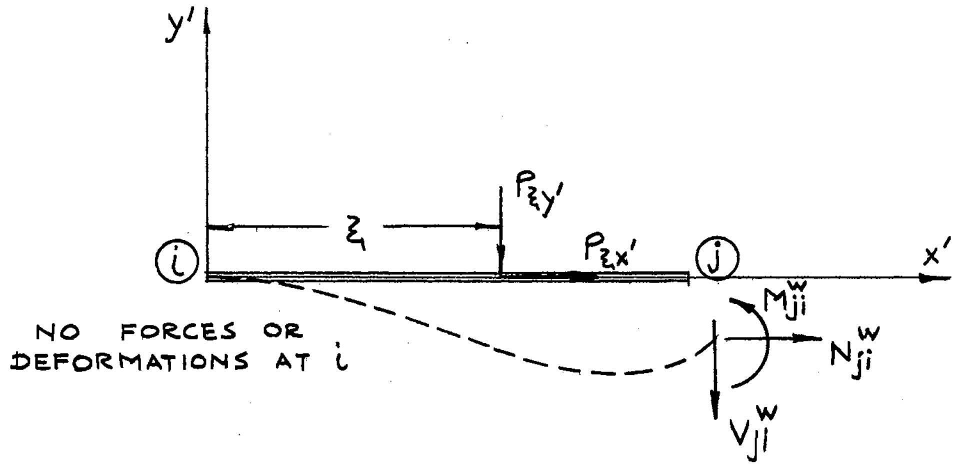


Figure 6. Load Effect in Transport Relation

2.4 Transformation to Basic Reference Axes

The dynamic properties of a member are thus far defined in the member system of co-ordinates. The inter-relation of member end forces as well as deformations at joints in which they meet can best be established if all member end values are described with respect to a set of common reference axes. The basic reference axes are used for this purpose.

A bar ij (Figure 7) inclined to the positive x axis of the basic system by an angle ω_j is considered. The figure shows the member axes and the member end forces in both the member system and the basic system in their positive sense. It may be noted that the end forces referred to in the basic system are so defined that they coincide with those referred to in the member system for $\omega_j = 0$.

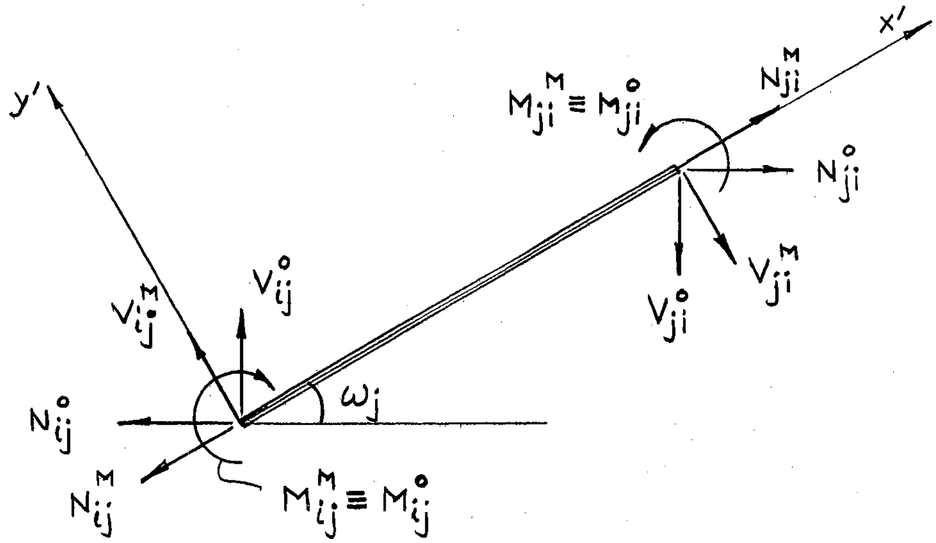


Figure 7. Inclined Member With End Forces in Member System as Well as in Basic System

The following transformation relations may now be established in terms of ω_j:

$$N_{ji}^{\circ} = N_{ji}^{M} \text{ Cos } \omega_j + V_{ji}^{M} \text{ Sin } \omega_j$$

$$V_{ji}^{\circ} = -N_{ji}^{M} \text{ Sin } \omega_j + V_{ji}^{M} \text{ Cos } \omega_j$$

$$M_{ji}^{\circ} = M_{ji}^{M} \quad . \tag{2-30}$$

Dividing the first two equations throughout by $\lambda_{\circ}^{3}EI_{\circ}$ and the last one by $\lambda_{\circ}^{2}EI_{\circ}$, these equations may be rewritten as

$$N'^{\circ}_{ji} = N'^{M}_{ji} \operatorname{Cos} \omega_j + V'^{M}_{ji} \operatorname{Sin} \omega_j$$

$$V'^{\circ}_{ji} = -N'^{M}_{ji} \operatorname{Sin} \omega_j + V'^{M}_{ji} \operatorname{Cos} \omega_j$$

$$M'^{\circ}_{ji} = M'^{M}_{ji} \tag{2-31}$$

Also

$$\Delta^{\circ}_{jix} = \Delta_{jix'} \operatorname{Cos} \omega_j + \Delta_{jiy'} \operatorname{Sin} \omega_j$$

$$\Delta^{\circ}_{jiy'} = -\Delta_{jix'} \operatorname{Sin} \omega_j + \Delta_{jiy'} \operatorname{Cos} \omega_j$$

$$\theta^{\circ}_{ji} = \theta^{M}_{ji} \tag{2-32}$$

The last equation above is divided by $\lambda_{\circ}$ to give $\theta'^{\circ}_{ji} = \theta'^{M}_{ji}$.

In matrix notation and symbolic form, these equations may be written as

$$\left\{F^{\circ}_{ji}\right\} = [\omega_j]\left\{F^{M}_{ji}\right\}$$

$$\left\{\delta^{\circ}_{ji}\right\} = [\omega_j]\left\{\delta^{M}_{ji}\right\} \tag{2-33}$$

where

$$[\omega_j] = \begin{bmatrix} \operatorname{Cos} \omega_j & \operatorname{Sin} \omega_j & 0 \\ -\operatorname{Sin} \omega_j & \operatorname{Cos} \omega_j & 0 \\ 0 & 0 & 1.0 \end{bmatrix}_{3x3} \tag{2-34}$$

or in a combined form

$$\begin{Bmatrix} F_{ji}^{\circ} \\ \hline \delta_{ji}^{\circ} \end{Bmatrix} = \left[\begin{array}{c|c} [\omega_j] & 0 \\ \hline 0 & [\omega_j] \end{array}\right] \begin{Bmatrix} F_{ji}^{M} \\ \hline \delta_{ji}^{M} \end{Bmatrix}$$

i.e.

$$\left\{S_{ji}^{\circ}\right\} = [\pi_j]\left\{S_{ji}^{M}\right\} \tag{2-35}$$

where

$$[\pi_j] = \left[\begin{array}{c|c} [\omega_j] & 0 \\ \hline 0 & [\omega_j] \end{array}\right]_{6x6} \tag{2-36}$$

It may be noted that each of the angular transformation matrices $[\omega_j]$ and $[\pi_j]$ is orthogonal, i.e.

$$[\omega_j]^T = [\omega_j]^{-1}$$

and

$$[\pi_j]^T = [\pi_j]^{-1} \tag{2-37}$$

It can similarly be shown that

$$\left\{F_{ij}^{o}\right\} = [\omega_j]\left\{F_{ij}^{M}\right\}$$

$$\left\{\delta_{ij}^{o}\right\} = [\omega_j]\left\{\delta_{ij}^{M}\right\}$$

and

$$\left\{S_{ij}^{o}\right\} = [\pi_j]\left\{S_{ij}^{M}\right\} \tag{2-38}$$

By using Equations (2-35) and (2-38), the transport matrix relation (Equation (2-23)) developed in the member system of co-ordinates may be expressed in the basic reference system.

$$\left\{ S_{ji}^{\,o} \right\} = [T_{ij}^{\prime o}] \left\{ S_{ij}^{\,o} \right\} \tag{2-39}$$

where

$$[T_{ij}^{\prime o}] = [\pi_j]\,[T_{ij}^{\prime M}]\,[\pi_j]^{-1} \tag{2-40}$$

2.5 <u>Equilibrium and Compatibility at a Joint</u>

A bar system ijkl (Figure 8) is considered. The free body diagrams of the bars ij and jk show their end forces in the basic reference system. The free body diagram of the joint j shows the effect of the forces from ends ji and jk and also shows the effect of any externally applied forces at the joint. These externally applied forces referred here may as well be the force effects of any other member meeting into the joint.

<u>Equilibrium at j</u>

$$N_{jk}^{\,o} = N_{ji}^{\,o} - N_j^o$$

$$V_{jk}^{\,o} = V_{ji}^{\,o} - V_j^o$$

$$M_{jk}^{\,o} = M_{ji}^{\,o} - M_j^o$$

Dividing the first two equations throughout by $\lambda_o^3 EI_o$ and the last one by $\lambda_o^2 EI_o$, these equations may be written in matrix notation

$$\left\{ F_{jk}^{\,o} \right\} = \left\{ F_{ji}^{\,o} \right\} - \left\{ F_j^o \right\} \tag{2-41}$$

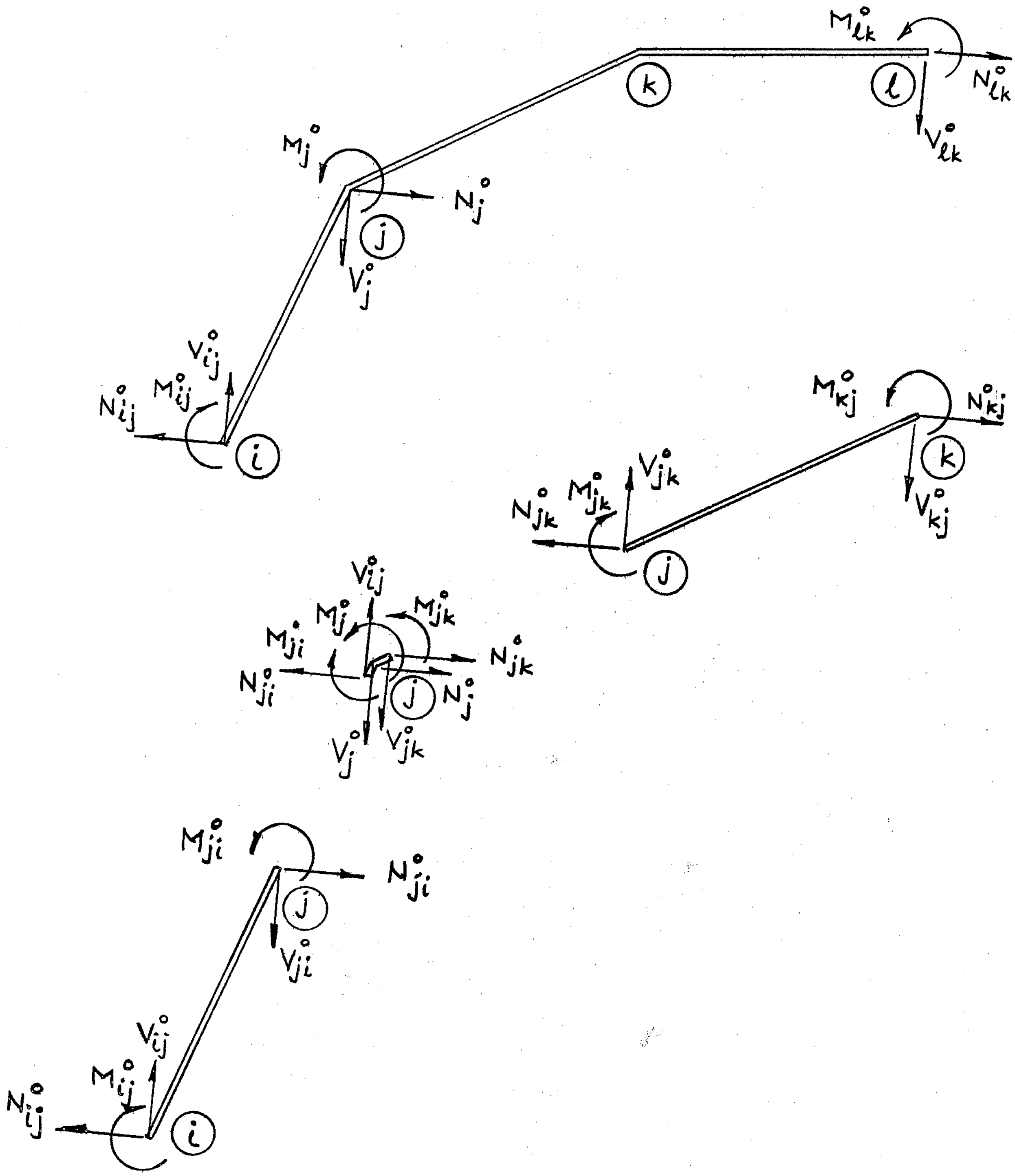


Figure 8. Equilibrium at a Joint

where

$$\left\{ F_j^o \right\} = \left\{ N_j'^o \ V_j'^o \ M_j'^o \right\}$$

and $N_j'^o = \dfrac{N_j^o}{\lambda_o^3 EI_o}$, $V_j'^o = \dfrac{V_j^o}{\lambda_o^3 EI_o}$ and $M_j'^o = \dfrac{M_j^o}{\lambda_o^2 EI_o}$

Compatibility at j

$$\Delta_{jkx}^o = -\Delta_{jix}^o$$

$$\Delta_{jky}^o = -\Delta_{jiy}^o$$

$$\theta_{jk}'^o = -\theta_{ji}'^o$$

where $\theta_{jk}'^o = \dfrac{\theta_{jk}^o}{\lambda_o}$ and $\theta_{ji}'^o = \dfrac{\theta_{ji}^o}{\lambda_o}$

In matrix notation

$$\left\{ \delta_{jk}^o \right\} = - \left\{ \delta_{ji}^o \right\} \tag{2-42}$$

Combining Equations (2-41) and (2-42) gives

$$\left\{ \begin{array}{c} F_{jk}^o \\ \hline \delta_{jk}^o \end{array} \right\} = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array} \right] \left\{ \begin{array}{c} F_{ji}^o \\ \hline \delta_{ji}^o \end{array} \right\} - \left\{ \begin{array}{c} F_j^o \\ \hline 0 \end{array} \right\} \tag{2-43}$$

where I is a unit matrix 3x3.

Symbolically

$$\left\{ S_{jk}^{o} \right\} = [J] \left\{ S_{ji}^{o} \right\} - \left\{ S_{j}^{o} \right\} \tag{2-44}$$

where
$$[J] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array} \right]$$

and
$$\left\{ S_{j}^{o} \right\} = \left\{ F_{j}^{o} \vdots 0 \right\}$$

Similarly it can be shown that

$$\left\{ S_{kl}^{o} \right\} = [J] \left\{ S_{kj}^{o} \right\} - \left\{ S_{k}^{o} \right\}$$

and so on for other joints.

In absence of any joint loads a simple chain matrix product will result for several bars. For example

$$\left\{ S_{lk}^{o} \right\} = [T_{kl}^{\prime o}] \, [J] \, [T_{jk}^{\prime o}] \, [J] \, [T_{ij}^{\prime o}] \left\{ S_{ij}^{o} \right\}$$

CHAPTER III

ELASTIC LOADS AND ELASTO-STATIC EQUATIONS

3.1 General

The String Polygon method (29), (30) and (31) is based on conjugate analogy. The real deformations are treated as conjugate loads--angular deformations as conjugate forces and linear deformations as conjugate moments. The geometric compatibility of deformations of a given frame is enforced by establishing equilibrium equations of the corresponding conjugate loads, also referred to as elastic loads. These equilibrium equations are called elasto-static equations and are neatly written if the distributed conjugate loads are replaced by statically equivalent point loads at chosen points--usually the member ends.

3.2 Member Elastic Functions

Member End Elastic Forces. Figure 9(a) shows a bar ij in its deflected form. The values shown are the amplitudes of the end forces and end deformations. The straight line joining i and j is the string line ij. The angles ϕ_{ij} and ϕ_{ji} between the string line and the end tangents to the deformation curve are taken as the end elastic forces $\bar{P}_{ij}$ and $\bar{P}_{ji}$ for the conjugate bar ij shown in Figure 9(b). In terms of the end deformations these end elastic forces can be computed as follows:

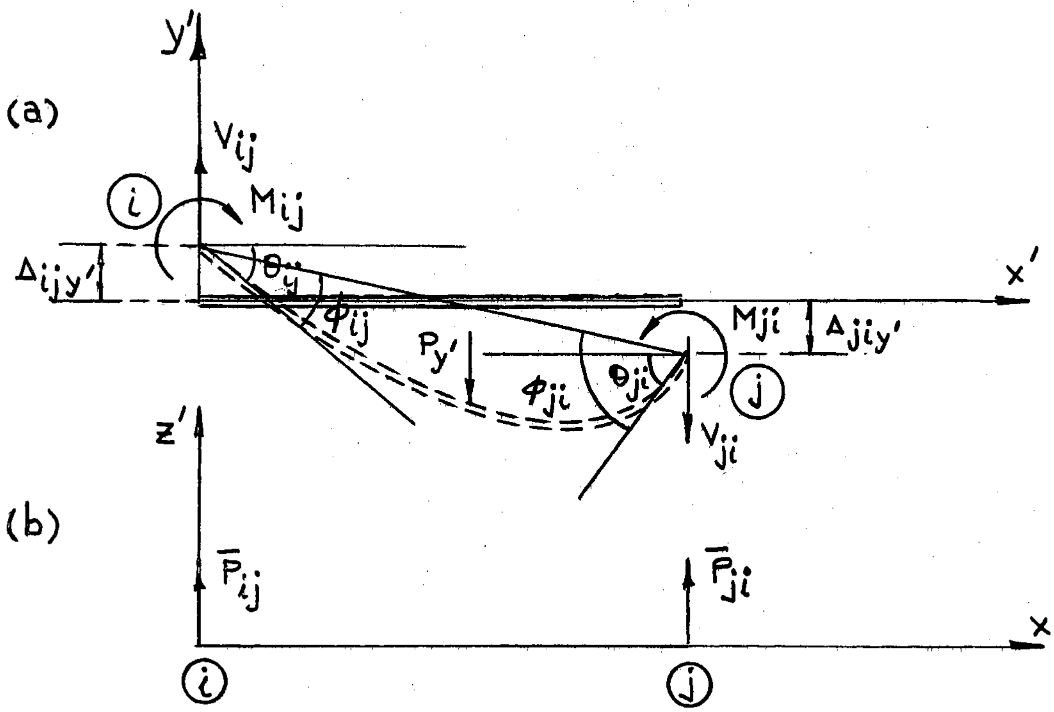


Figure 9. End Elastic Forces for Bar ij With Transverse Loads

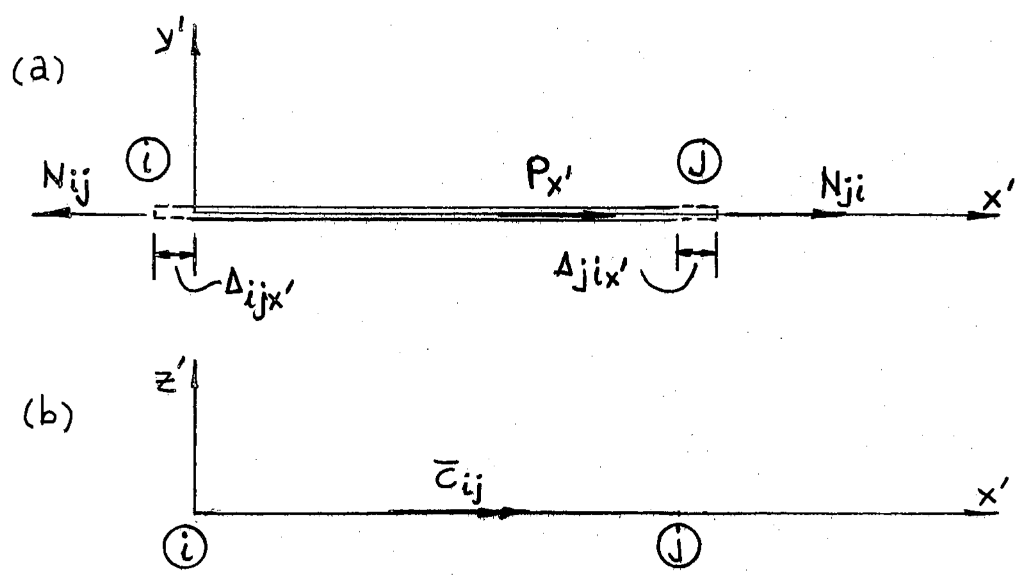


Figure 10. Member Elastic Moment for Bar ij With Axial Loads

$$\overline{P}_{ij} = \phi_{ij} = \theta_{ij} - \frac{\Delta_{ijy'} + \Delta_{jiy'}}{L_{ij}} = \lambda_o \theta'_{ij} - \frac{\Delta_{ijy'} + \Delta_{jiy'}}{L_{ij}}$$

$$\overline{P}_{ji} = \phi_{ji} = \theta_{ji} + \frac{\Delta_{ijy'} + \Delta_{jiy'}}{L_{ij}} = \lambda_o \theta'_{ji} + \frac{\Delta_{ijy'} + \Delta_{jiy'}}{L_{ij}} \qquad (3\text{-}1)$$

Member Elastic Moment. Figure 10(a) shows the axial forces and axial deformations of the bar ij. The total axial deformation is taken as the member elastic moment (30), $\overline{C}_{ij}$, shown on the conjugate bar ij in Figure 10(b). In terms of the end deformations, this gives

$$\overline{C}_{ij} = \Delta_{ijx'} + \Delta_{jix'} \qquad (3\text{-}2)$$

Equations (3-1) and (3-2) can be combined into the following matrix equation.

$$\begin{bmatrix} \overline{P}_{ij} \\ \overline{P}_{ji} \\ \overline{C}_{ij} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L_{ij}} & \lambda_o & 0 & -\frac{1}{L_{ij}} & 0 \\ 0 & +\frac{1}{L_{ij}} & 0 & 0 & +\frac{1}{L_{ij}} & \lambda_o \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{ijx'} \\ \Delta_{ijy'} \\ \theta'_{ij} \\ \Delta_{jix'} \\ \Delta_{jiy'} \\ \theta'_{ji} \end{bmatrix} \qquad (3\text{-}3)$$

This equation describes the member elastic functions in terms of its end deformations.

Written symbolically, Equation (3-3) may be stated as

$$\left\{ \bar{P}_j \right\} = [L_j] \left\{ \begin{array}{c} \delta_{ij} \\ \hdashline \delta_{ji} \end{array} \right\} \tag{3-4}$$

While writing the elasto-static equations for any given frame it is necessary to compile the elastic quantities of all members in terms of their end deformations in a single matrix. This may be done using Equation (3-4) above.

$$\begin{bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \bar{P}_j \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \bar{P}_n \end{bmatrix} = \begin{bmatrix} [L_1] & & & & \\ & [L_2] & & & \\ & & \ddots & & \\ & & & [L_j] & \\ & & & & \ddots \\ & & & & & [L_n] \end{bmatrix} \begin{bmatrix} \delta^1_{01} \\ \delta^1_{10} \\ \hdashline \delta^2_{12} \\ \delta^2_{21} \\ \hdashline \vdots \\ \hdashline \delta^j_{ij} \\ \delta^j_{ji} \\ \hdashline \vdots \\ \hdashline \delta^n_{n-1,n} \\ \delta^n_{n,n-1} \end{bmatrix} \tag{3-5}$$

In symbolic form this becomes

$$\{\overline{P}\} = [B]\{\delta\} \tag{3-6}$$

3.3 Elasto-Static Equations

Equations of equilibrium of the member elastic loads are written for necessary conjugate panels of the real structure such that all members are accounted for. For planar frame deforming in plane, the elasto-static equations are of the type $\Sigma\overline{P}_z = 0$, $\Sigma\overline{M}_x = 0$ and $\Sigma\overline{M}_y = 0$. Known as well as unknown deformations at the support are shown as externally applied elastic loads at the corresponding points on the conjugate panels. Figure 11, 12 and 13 illustrate the development of the elasto-static equations for some frames. The dotted lines indicating the bottom side of the frame members establish the member orientations. The curvilinear arrows indicate the direction in which a conjugate panel is traversed.

Elasto-static equations thus developed should be equal in number to the primary unknowns in a frame. The resulting set of equations in terms of all member elastic loads may be written symbolically as

$$[A]\{\overline{P}\} = \{\Delta\} \tag{3-7}$$

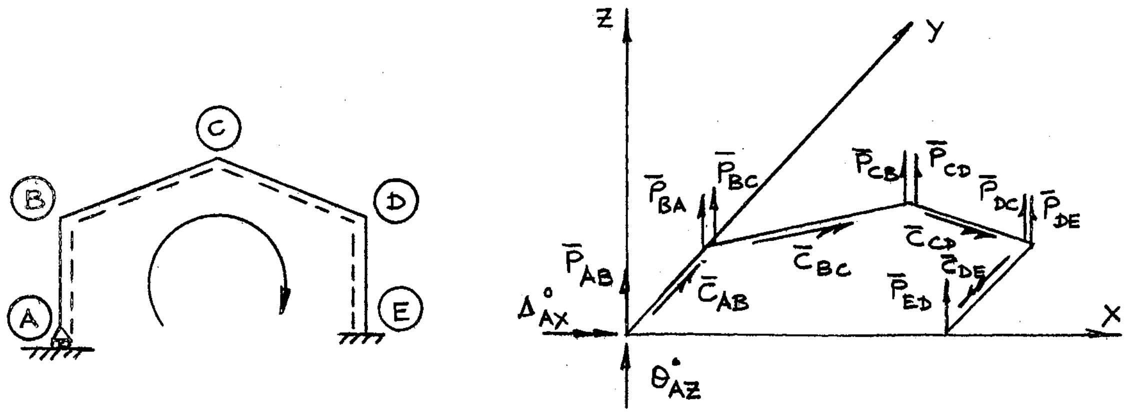


Figure 11. Single Span Gable Frame and Corresponding Conjugate Structure

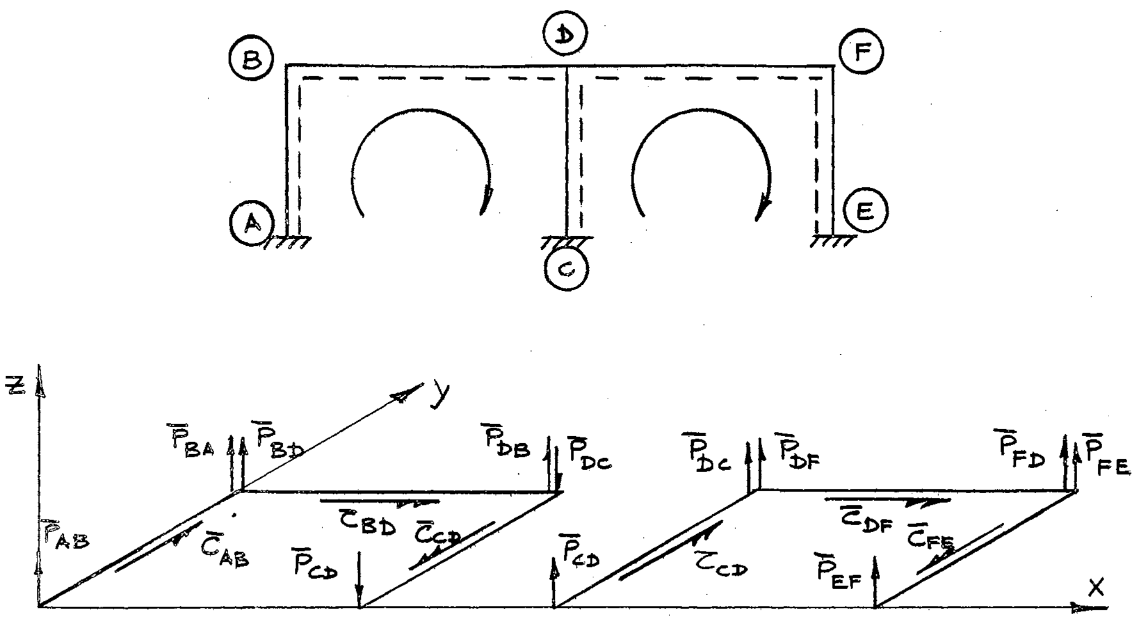


Figure 12. Two-Span Single Story Frame and Corresponding Conjugate Structures

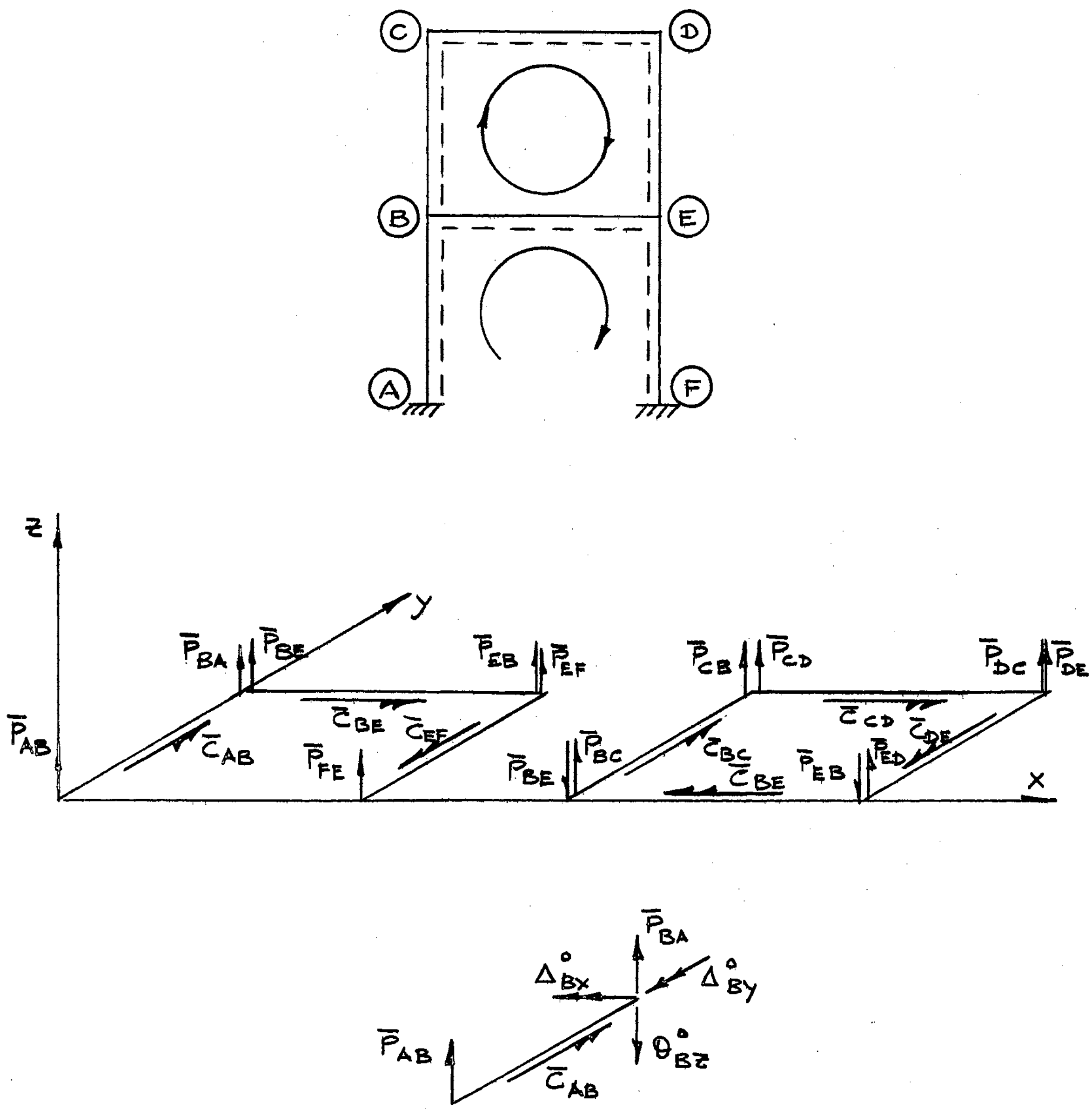


Figure 13. Single Span Two Story Frame and Corresponding Conjugate Structures

CHAPTER IV

SELECTION OF PRIMARY UNKNOWNS

4.1 Selection of Primary Unknowns

For analyzing a complex, rigid jointed frame, it is necessary to render the frame "statically determinate." This may conveniently be done by introducing cuts (Figure 14) in each closed loop (panel) preferably near a joint or a support. The ground between supports may be considered an infinitely rigid member. The idea is to convert the given frame into a "tree" wherein the end quantities of any member can be computed in terms of those at the 'free' ends.

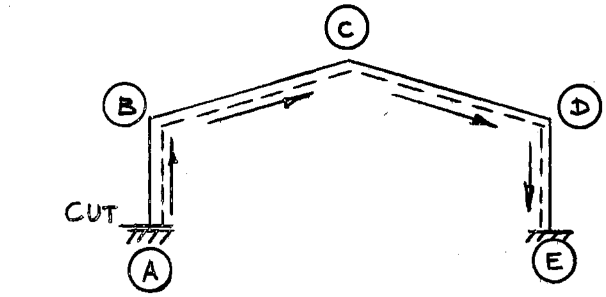
The forces and/or deformations at the cuts introduced in the frame may then be treated as externally applied functions and they constitute the set of primary unknowns. In general there are (6n-b) primary unknowns in a frame where

n = number of closed panels in the frame

and

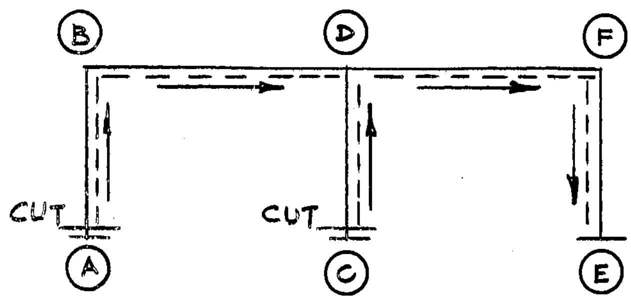
b = number of known forces and/or deformations at the supports released.

A judicious choice of the location of the cuts to be introduced in a frame may help reduce the number of primary unknowns. This is illustrated in Figure 15.



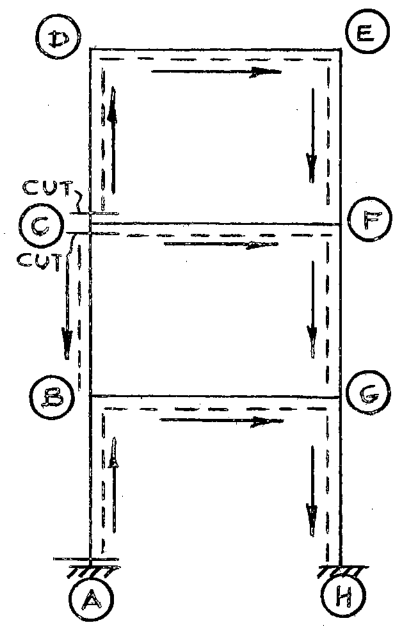
Unknowns

$\{F_{AB}\}$



Unknowns

$\{F_{AB}\}$ and $\{F_{CD}\}$

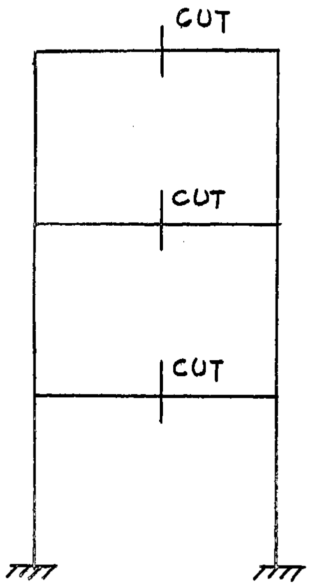


Unknowns

$\{F_{AB}\}$, $\{F_{CD}\}$, $\{F_{CB}\}$

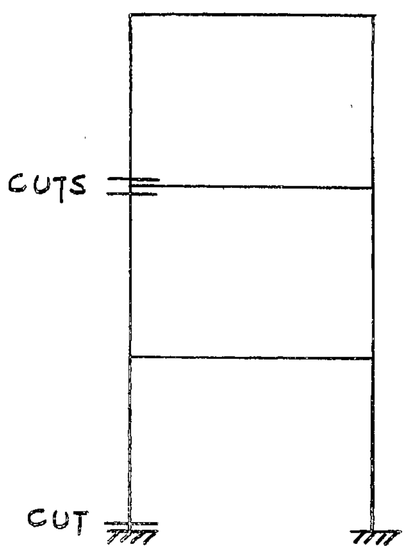
and $\{\delta^{o}_{CD}\} \equiv \{\delta^{o}_{CB}\}$

Figure 14. Selection of Primary Unknowns in Some Frames



Unknowns $\{F \vdots \delta\}$ at each cut

Total unknowns = 18



Unknowns $\{F\}$ values at all cuts and $\{\delta\}$ value at upper cuts

Total unknowns = 12

Figure 15. Effect of the Choice of Location of Cuts on the Number of Unknowns

4.2 Solution of the Primary Unknowns

The primary unknowns are solved for by setting up an equal number of elasto-static equations. Each closed panel provides three equations of elasto-static equilibrium. In addition, for every deformation chosen as a primary unknown, one equation is written using a free body of the conjugate structure involving that deformation.

The elasto-static equations however are not written in terms of the primary unknowns directly. They are actually written in terms of the member elastic quantities $\overline{P}$'s and $\overline{C}$'s (Equation 3-7) which can be expressed in terms of member end deformations as explained earlier, (Equation 3-6). To express the member end deformations in terms of the primary unknowns, transport matrices are used i.e. the primary unknowns are transported from their locations to various member ends. Figure 14 illustrates the "flow" of the unknowns through various members in some frames.

The expression of the member end deformations in terms of the primary unknowns may symbolically be written as

$$\{\delta\} = [C]\{X\} \tag{4-1}$$

CHAPTER V

APPLICATION

5.1 Procedure for Application of the Method

The various aspects of the proposed method, developed in the previous chapters, can now be fitted together in the outline of the procedure for application of the method described in the following steps:

1. The primary unknowns for the frame are identified.
2. All end values for all members are expressed in terms of the primary unknowns using the transport relations.
3. Elasto-static equations for the frame are written.
4. Elastic weights and moments are expressed in terms of member end deformations.
5. The end deformations are expressed in terms of the unknowns established for the frame.
6. The resulting final matrix equation is solved for the natural frequencies or the response of the frame.
7. The mode shapes of free vibrations or the deflection diagram for forced vibrations are computed.

The application of the method developed is now illustrated by the following three numerical examples. These examples are solved using the computer programs written for IBM 7040.

5.2 Single Span Gable Frame

A single span constant section gable frame, Figure 16, with fixed bases is analyzed for its natural frequencies. The following data are used.

$$L = 8.0 \text{ in}$$

$$a = 0.4$$

$$b = 0.2$$

$$m = 15.2174 \times 10^{-6} \text{ lb}\cdot\text{sec}^2/\text{in}^2$$

$$I = 34.2282 \times 10^{-6} \text{ in}^4$$

$$A = 20740.0 \times 10^{-6} \text{ in}^2$$

$$E = 30.6 \times 10^6 \text{ lbs/in}^2$$

$\left\{F_{AB}^{M}\right\}$ is chosen as primary unknown. The transportation of quantities and the formulation of the problem are shown in detail in Appendix B. The first four natural frequencies obtained are 236.2, 425.2, 950.7, 1482.4 cps. Figure 17 shows the corresponding mode shapes.

5.3 Two Span Single Story Frame

The frame shown in Figure 18 is analyzed for its first four natural frequencies. All members are identical in length and section. The following data are used.

$$L = 6.0 \text{ in}$$

$$E = 28.3 \times 10^6 \text{ lbs/in}^2$$

$$\text{Section} = \frac{3}{16} \text{ in} \times \frac{5}{16} \text{ in}$$

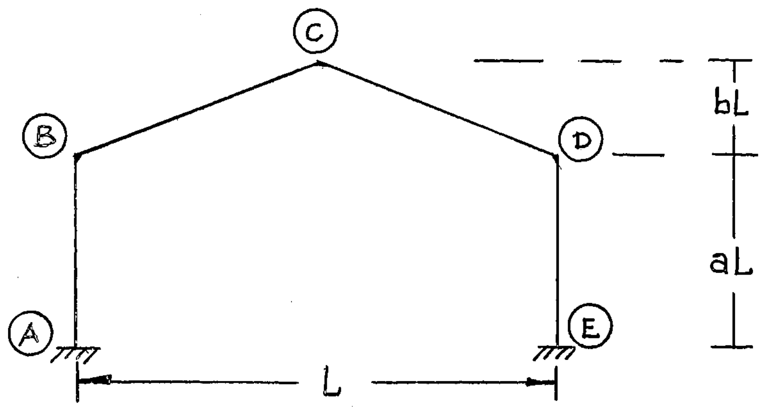


Figure 16. Single Span Gable Frame

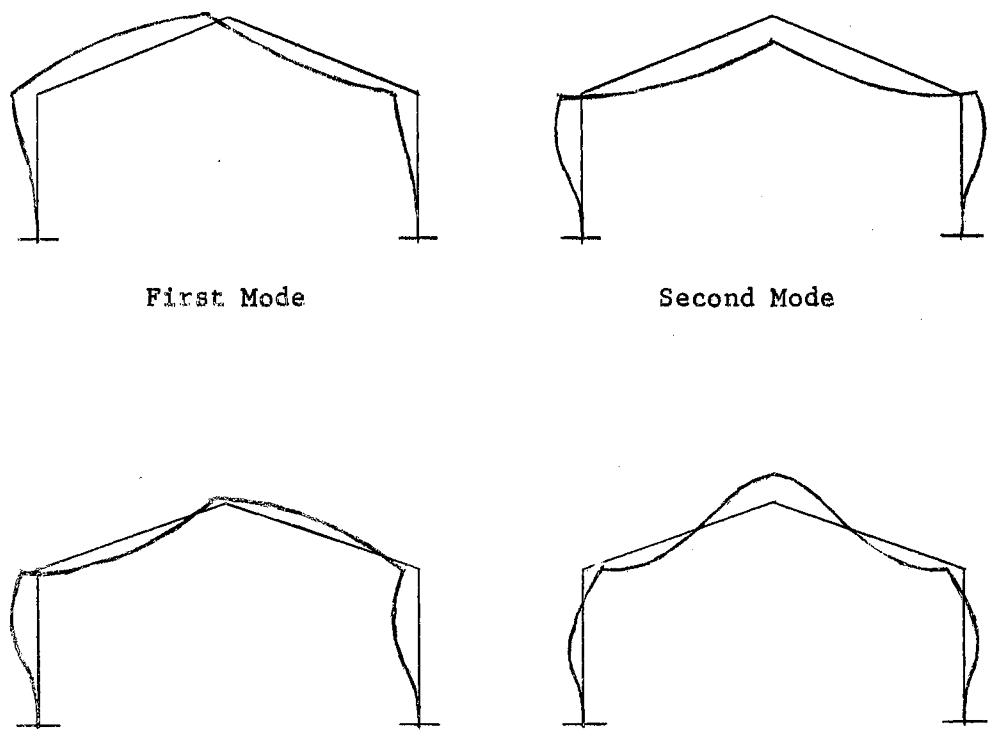


Figure 17. Mode Shapes of Free Vibrations

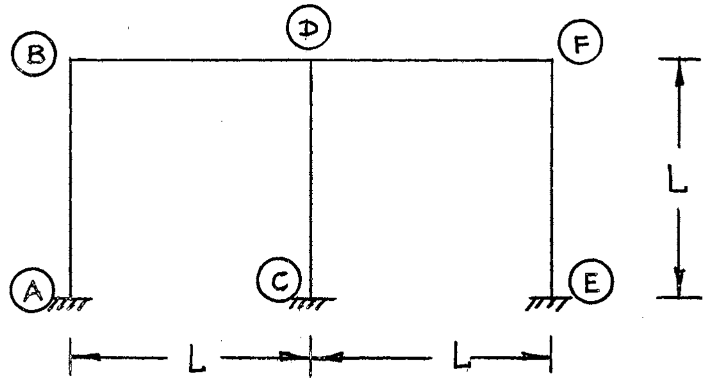


Figure 18. Two Span Single Story Frame

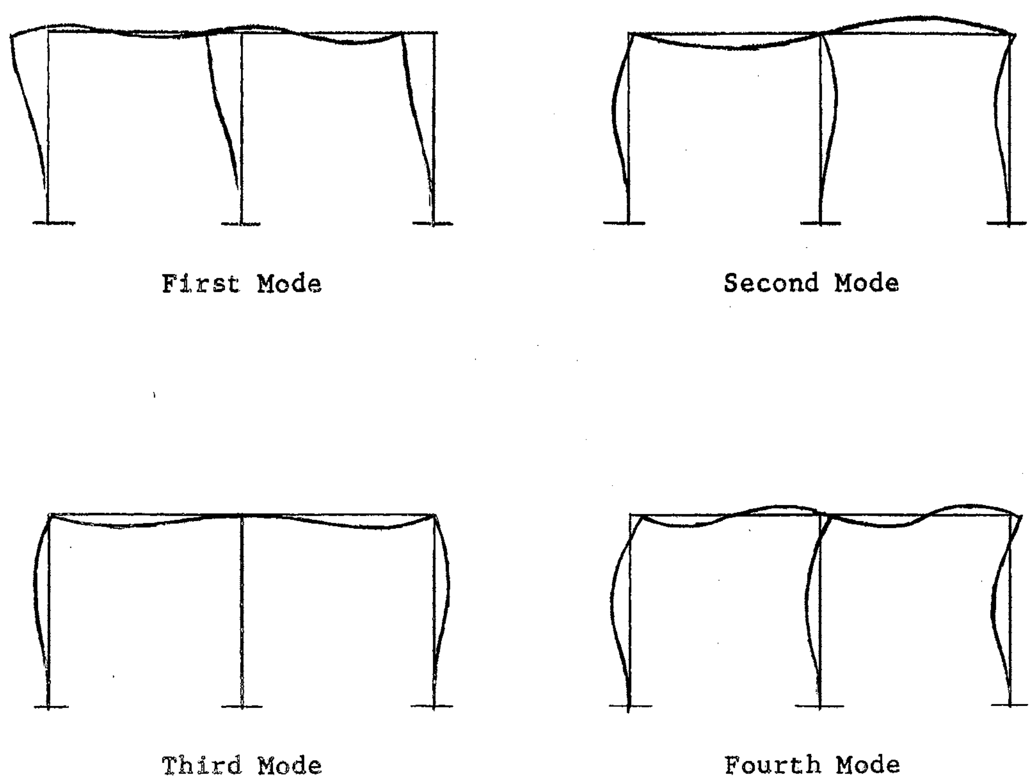


Figure 19. Mode Shapes of Free Vibrations

$\left\{F_{AB}^{M}\right\}$ and $\left\{F_{CD}^{M}\right\}$ are chosen as primary unknowns. The transportation of quantities and the formulation of the problem are shown in detail in Appendix B. The natural frequencies obtained are shown in Table I and compared with the values obtained by Rieger and McCallion (14). Figure 19 shows the corresponding mode shapes.

5.4 Single Span Three Story Frame

The frame shown in Figure 20 is analyzed for symmetrical forced vibrations due to the pulsating load shown. The following data apply:

$$L2 = 6.0 \text{ m}$$

$$L1 = 5.0 \text{ m}$$

$$I2 = 1.0 \times 10^{-4} \text{ m}^4$$

$$I1 = 2.0 \times 10^{-4} \text{ m}^4$$

$$m2 = 2.04 \times 10^{-2} \text{ Tsec}^2/\text{m}^2$$

$$m1 = 5.10 \times 10^{-2} \text{ Tsec}^2/\text{m}^2$$

$$P = 1.262 \text{ T}$$

$$\omega = 80.2 \text{ rad/sec}$$

$$E = 2.1 \times 10^7 \text{ T/m}^2$$

Because of the symmetry, the given frame is modified to an equivalent frame shown in Figure 21. The primary unknowns are N', M' and $\Delta_{y'}$ values at A, C and E. The transportation of quantities and the formulation of the problem are shown in detail in Appendix B. The primary unknowns are solved for in terms of the applied load and all the other end values are found using the same transport relations that are previously used in the problem. Table II shows the principal moment and deformation values obtained, in comparison with those

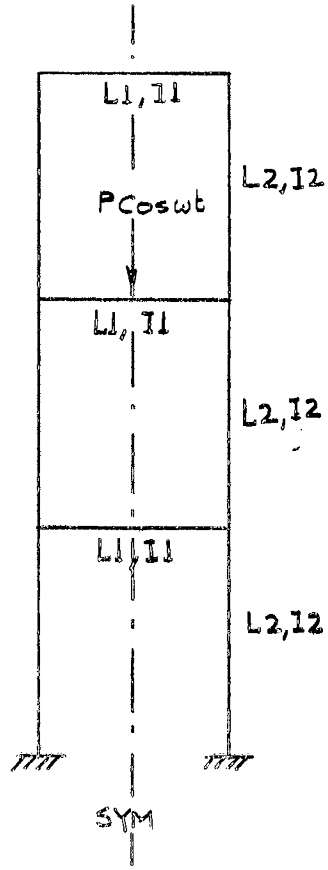


Figure 20. Single Span Three Story Frame

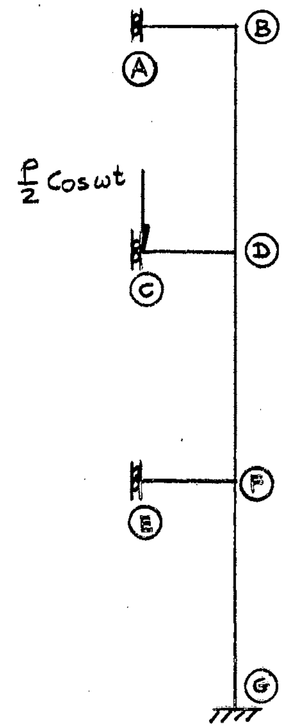


Figure 21. Equivalent Frame

obtained by Blaszkowiak and Kaczkowski (35). Figure 22 shows the computed shape of the deflected frame.

TABLE I

COMPARISON OF NATURAL FREQUENCIES (CPS) OF THE TWO SPAN SINGLE STORY FRAME

	String Polygon		Rieger and McCallion (14)
	Including Axial Deformation	Excluding Axial Deformation	Excluding Axial Deformation
1st	139.5	139.5	142.3
2nd	574.2	574.7	583.0
3rd	721.8	724.5	734.0
4th	975.8	976.0	990.0

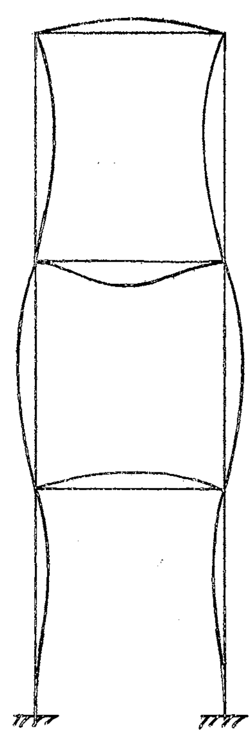


Figure 22. Shape of the Deflected Frame

TABLE II

COMPARISON OF MOMENTS AND DEFORMATION VALUES OF THE SINGLE SPAN THREE STORY FRAME

	String Polygon	Iteration (Ref. (35))
M_{AB}	- 0.350 Tm	- 0.367 Tm
M_{BA}	- 0.171 Tm	- 0.163 Tm
M_{CD}	+ 1.648 Tm	+ 1.632 Tm
M_{DC}	- 0.551 Tm	- 0.538 Tm
M_{EF}	- 0.233 Tm	- 0.241 Tm
M_{FE}	- 0.111 Tm	- 0.106 Tm
M_{BD}	- 0.171 Tm	- 0.163 Tm
M_{DB}	+ 0.249 Tm	+ 0.240 Tm
M_{DF}	- 0.302 Tm	- 0.298 Tm
M_{FD}	+ 0.233 Tm	+ 0.230 Tm
M_{FG}	+ 0.122 Tm	+ 0.124 Tm
M_{GF}	- 0.106 Tm	- 0.108 Tm
θ_B	169.5×10^{-6} rad	174.3×10^{-6} rad
θ_D	376.2×10^{-6} rad	374.3×10^{-6} rad
θ_F	112.3×10^{-6} rad	114.3×10^{-6} rad
Δ_{AY}	216.0×10^{-6} m	245.7×10^{-6} m
Δ_{CY}	766.6×10^{-6} m	740.0×10^{-6} m
Δ_{EY}	146.5×10^{-6} m	160.0×10^{-6} m

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1 Summary

The applicability of the String Polygon method to the analysis of planar rigid jointed frames for free and forced harmonic vibrations is investigated in this dissertation. Effect of axial deformations of members is included in the study.

Instead of using the general flexibility approach in formulation of the string polygon functions as in the static analysis, it was found advisible to use transport matrix relation as an effective tool to keep down the number of unknowns in a problem. A procedure to choose the primary unknowns in a given frame is explained.

The dynamic properties of a member and load functions are derived first. The formulation of a problem is simply done in the following steps:

1. Elasto-static equations are written.
2. The elastic loads are expressed in terms of member end deformations.
3. The member end deformations are expressed in terms of chosen unknowns.
4. The resulting matrix equation is solved either for natural frequencies of vibration or the response of the structure in case of forced vibrations.

Three numerical examples illustrate the method developed.

It is believed that the combination of transport matrix and string polygon used in the vibration analysis of frames is original.

6.2 Conclusions

The method proposed is quite straightforward when applied using the steps described. Several problems checked indicate that the method yields excellent results. Effect of axial deformations in the analysis of frames with members of commonly used proportions is found to be negligible. The transport matrix used can be easily modified to exclude axial deformations if desired.

The advantages of using the String Polygon method over other methods are clear. It is superior to general flexibility method because of simpler formulation and fewer unknowns involved. It also is superior to methods which involve writing shear equilibrium equations. Also, frames with sloping members do not present any special problems. The proposed method also compares very well with the general stiffness method and in some cases has fewer primary unknowns than the latter. As in all complex problems, use of electronic computer is seen to be necessary.

6.3 Extensions

The ideas presented here may readily be extended

(1) To frames with curved members

(2) To frames vibrating out-of-plane

(3) To three dimensional frames

(4) To the vibration analysis of frames due to impulsive or blast loads.

SELECTED BIBLIOGRAPHY

1. Rayleigh, Lord, The Theory of Sound, MacMillan and Co., Ltd., 1944. Also Dover Publications, 1945.

2. Love, A. E. H., The Mathematical Theory of Elasticity, McGraw-Hill Book Co., New York, 1944.

3. Darnley, E. R., The Transverse Vibrations of Beams and the Whirling of Shafts Supported at Intermediate Points, Philosophical Magazine, V. 41, pp. 81-95, 1921.

4. Hohenemser, K. and Praeger, W., Dynamic der Stabwerke, Julius Springer, Berlin, 1933.

5. Timoshenko, S., Vibration Problems in Engineering, 3rd Ed., D. Van Nostrand Co., Inc., New York, 1955.

6. Bennon, S., Natural Modes of Vibration of Simple Frames, Franklin Institute Journal, V. 243, p. 13-39, 1947.

7. Saibel, E., Vibration Frequencies of Continuous Beams, Journal--Aeronautical Sciences, V. 11, n. 1, pp. 88-90, Jan. 1944.

8. Saibel, E. and D'Appolonia, E., Forced Vibrations of Continuous Beams, ASCE Proceedings, V. 77, n. 100, Nov. 1951.

9. Lee, W. F. Z. and Saibel, E., Free Vibrations of Constrained Beams, Journal--Applied Mechanics, V. 19, pp. 471-477, 1952.

10. Gaskell, R. E., On Moment Balancing in Structural Dynamics, Quarterly of Applied Mathematics, V. 1, pp. 237-249, 1943.

11. Veletsos, A. S. and Newmark, N. M., Determination of the Natural Frequencies of Continuous Beams on Flexible Supports. Proceedings of the Second U. S. National Congress of Applied Mechanics, pp. 147-155, 1954.

12. Veletsos, A. S. and Newmark, N. M., Natural Frequencies of Continuous Flexural Members, ASCE Proceedings, V. 81, n. 735, July, 1955.

13. Veletsos, A. S., A Method for Calculating the Natural Frequencies of Continuous Beams, Frames and Certain Types of Plates, Ph.D. Thesis, 1953, University of Illinois, Urbana, Illinois.

14. Rieger, N. F. and McCallion, H., Natural Frequencies of Portal

Frames, International Journal of Mechanical Sciences, V. 7, n. 4, pp. 253-276, April, 1965.

15. Bishop, R. E. D., Analysis and Synthesis of Vibrating Systems, Royal Aeronautical Society Journal, V. 58, n. 526, pp. 703-719, Oct. 1954.

16. Bishop, R. E. D., Analysis of Vibrating Systems Which Embody Beams in Flexure, Institution of Mechanical Engineers Proceedings, V. 169, n. 51, pp. 1031-1046, 1955.

17. Bishop, R. E. D., Vibration of Frames, Institution of Mechanical Engineers Proceedings, V. 170, n. 29, pp. 955-968, 1956.

18. Carter, B. C., The Vibration of Airscrew Blades with Particular Reference to Harmonic Torque Impulses in the Drive, R & M 1758, 1936.

19. Duncan, W. J., Mechanical Admittances and Their Applications to Oscillation Problems, R & M 2000, 1947.

20. Johnson, D. C., The Application of Admittance Methods in the Classical Theory of Small Oscillations, Engineering, 171, 4453, pp. 650-652, 1951.

21. Pestel, E. C. and Leckie, F. A., Matrix Methods in Elasto Mechanics, McGraw-Hill Book Co., Inc., New York, 1963.

22. Marguerre, K., Vibration and Stability Problems of Beams Treated by Matrices, Journal of Mathematics and Physics, V. 35, n. 1, pp. 28-43, April, 1956.

23. Laursen, H. I., Shubinski, R. P. and Clough, R. W., Dynamic Matrix Analysis of Framed Structures, Proceedings Fourth U. S. National Congress of Applied Mechanics, pp. 99-105, 1962.

24. Ariaratnam, S. T., Vibration of Plane Frameworks, Applied Science Research Sec. A., V. 13, n. 4-5, pp. 249-259, 1964.

25. Levien, K. W. and Hartz, B. J., Dynamic Flexibility Matrix Analysis of Frames, ASCE Proceedings V. 89, Journal Structural Division, n. ST4, paper 3612, Aug. 1963.

26. Rogers, G. L., Dynamics of Framed Structures, John Wiley & Sons, Inc., New York, 1959.

27. Biggs, J. M., Introduction to Structural Dynamics, McGraw-Hill Book Co., Inc., New York, 1964.

28. Maldonado, D. B., Dynals--A Computer System for Structural Dynamic Analysis, D.Sc. Thesis, Department of Civil Engineering, Massachusetts Institute of Technology, 1965.

29. Tuma, J. J. and Oden, J. T., String Polygon Analysis of Frames

with Straight Members, ASCE Proceedings V. 87, Journal Structural Division, n. ST7, Paper 2956, Oct. 1961.

30. Wu, C. M., The General String Polygon, M.S. Thesis, Oklahoma State University, 1960.

31. Oden, J. T., Analysis of Fixed End Frames with Bent Members by the String Polygon Method, M.S. Thesis, Oklahoma State University, 1960.

32. Liaw, B. D., Dynamic Analysis of Beams by the Flexibility Method, M.S. Thesis, Oklahoma State University, 1963.

33. Gillespie, J. W. and Liaw, B. D., Frequency Analysis of Beams by Flexibility Method, ASCE Proceedings V. 90, Journal Engineering Mechanics Division, n. EM1, Paper 3798, Feb. 1964.

34. Gillespie, J. W. and Liaw, B. D., Dynamics of Structures--Part I--Frequency Analysis, School of Civil Engineering Reserach Publication, No. 24, Oklahoma State University, 1964.

35. Blaszkowiak, S. and Kaczkowski, Z., Iteration Methods in Structural Analysis, Pergamon Press, New York, 1966.

APPENDICES

APPENDIX A

DEFORMATION FUNCTIONS OF A BAR DUE TO UNIT END FORCES

Expressions of deformation curves of a straight free-free bar ij of constant section loaded by end forces of unit amplitude are derived here. The expressions derived may also be recognized as expressions of influence functions for various end deformations due to an applied load of unit amplitude on the bar at a general section, because of the reciprocal deformation relationships.

Transverse Deformations: The amplitude of transverse deformations may be recalled from Equation (2-14)

$$v_{x'} = A \text{ Cos } \lambda x' + B \text{ Sin } \lambda x' + C \text{ Cosh } \lambda x' + D \text{ Sinh } \lambda x' \qquad \text{(A-1)}$$

The constants A, B, C and D are to be obtained from Equation (2-15) first.

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & -0.5 & 0 \\ 0 & -0.5 & 0 & -0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & -0.5 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} \Delta_{ijy'} \\ \theta_{ij}/\lambda \\ M_{ij}/\lambda^2 EI \\ V_{ij}/\lambda^3 EI \end{bmatrix} \qquad \text{(A-2)}$$

For any end force of unit amplitude applied, the deformation values on the right hand side of the equation above can be obtained from the flexibility relation given below.

$$\begin{bmatrix} \Delta_{ijy'} \\ \theta_{ij}/\lambda \\ \Delta_{jiy'} \\ \theta_{ji}/\lambda \end{bmatrix} = \frac{1}{F3} \begin{bmatrix} -F5 & F1 & -F8 & F10 \\ F1 & F6 & -F10 & -F7 \\ -F8 & -F10 & -F5 & -F1 \\ F10 & -F7 & -F1 & F6 \end{bmatrix} \begin{bmatrix} V_{ij}/\lambda^3 EI \\ M_{ij}/\lambda^2 EI \\ V_{ji}/\lambda^3 EI \\ M_{ji}/\lambda^2 EI \end{bmatrix} \qquad \text{(A-3)}$$

where $F1 = \text{Sin } \lambda L \text{ Sinh } \lambda L$

$F3 = \text{Cos } \lambda L \text{ Cosh } \lambda L - 1$

$F5 = \text{Cos } \lambda L \text{ Sinh } \lambda L - \text{Sin } \lambda L \text{ Cosh } \lambda L$

$F6 = \text{Cos } \lambda L \text{ Sinh } \lambda L + \text{Sin } \lambda L \text{ Cosh } \lambda L$

Equation (A-3) is derived from Equation (2-17) by suitable transposition.

The deformation equation for the bar due to any unit end force may now be found by solving for the end deformations induced by that force and solving for the constants A, B, C and D from Equation (A-2) and then substituting in Equation (A-1).

For example, the deflection equation of the bar due to M_{ij} of unit amplitude can be found as follows.

Substituting $M_{ij} = 1$, $V_{ij} = 0 = V_{ji} = M_{ji}$ in Equation (A-3) gives

$$\Delta_{ijy'} = F1/(F3 \cdot \lambda^2 EI)$$

$$\theta_{ij}/\lambda = F6/(F3 \cdot \lambda^2 EI)$$

Substituting these values in Equation (A-2) gives

$$A = \frac{1}{2\ \lambda^2\ EI\ F3} (F1 - F3)$$

$$B = \frac{1}{2\ \lambda^2\ EI\ F3} \cdot (-F6)$$

$$C = \frac{1}{2\ \lambda^2\ EI\ F3} (F1 + F3)$$

$$D = \frac{1}{2\ \lambda^2\ EI\ F3} (-F6) \quad .$$

Substituting these in Equation (A-1) gives

$$v_{x'}^{M_{ij}=1} = \frac{1}{2\ \lambda^2\ EI\ F3} ((F1-F3)\ Cos\ \lambda x' - F6\ Sin\ \lambda x' + (F1+F3) \cdot$$

$$Cosh\ \lambda x' - F6\ Sinh\ \lambda x') \qquad \text{(A-4a)}$$

Remaining values calculated similarly are

$$v_{x'}^{V_{ij}=1} = \frac{1}{2\ \lambda^3\ EI\ F3} (-F5\ Cos\ \lambda x' - (F1+F3)\ Sin\ \lambda x'$$

$$-F5\ Cosh\ \lambda x' - (F1-F3)\ Sinh\ \lambda x') \qquad \text{(A-4b)}$$

$$v_{x'}^{V_{ji}=1} = \frac{1}{2\ \lambda^3\ EI\ F3} (-F8\ Cos\ \lambda x' + F10\ Sin\ \lambda x'$$

$$-F8\ Cosh\ \lambda x' + F10\ Sinh\ \lambda x') \qquad \text{(A-4c)}$$

$$v_{x'}^{M_{ji}=1} = \frac{1}{2\lambda^2 EI\ F3} (F10 \text{ Cos } \lambda x' + F7 \text{ Sin } \lambda x' + F10 \text{ Cosh } \lambda x'$$

$$+ F7 \text{ Sinh } \lambda x') \qquad \text{(A-4d)}$$

The four values computed above are respectively the same as $\Delta_{\xi y'}^{M_{ij}=1}$, $\Delta_{\xi y'}^{V_{ij}=1}$, $\Delta_{\xi y'}^{V_{ji}=1}$, and $\Delta_{\xi y'}^{M_{ji}=1}$, referred to in Chapter II.

Axial Deformations: The amplitude of axial deformation of the bar ij may be recalled from Equation (2-4).

$$u_{x'} = C' \text{ Cos } kx' + D' \text{ Sin } kx' \qquad \text{(A-5)}$$

The constants C' and D' are to be computed from Equation (2-7).

$$\begin{bmatrix} C' \\ D' \end{bmatrix} = \begin{bmatrix} -1.0 & 0 \\ 0 & 1.0 \end{bmatrix} \begin{bmatrix} \Delta_{ijx'} \\ N_{ij}/kAE \end{bmatrix} \qquad \text{(A-6)}$$

The flexibility relation given below may be used to compute $\Delta_{ijx'}$ on the right hand side of the equation.

$$\begin{bmatrix} \Delta_{ijx'} \\ \Delta_{jix'} \end{bmatrix} = \begin{bmatrix} -\text{ Cot } kL & \text{Cosec } kL \\ \text{Cosec } kL & -\text{ Cot } kL \end{bmatrix} \begin{bmatrix} N_{ij}/kAE \\ N_{ji}/kAE \end{bmatrix} \qquad \text{(A-7)}$$

The above equation is obtained by suitable transposition from Equation (2-9).

The deformation equation of the bar due to either end axial force of unit amplitude can now be computed by first computing $\Delta_{ijx'}$ from Equation (A-7) and substituting in Equation (A-6) to solve for C' and

D'. Substitution in the expression for $u_{x'}$ gives the desired result. The deformation equations thus obtained are

$$u_{x'}^{N_{ij}=1} = \frac{1}{kAE} \left(\mathrm{Cot}\ kL \cdot \mathrm{Cos}\ kx' + \mathrm{Sin}\ kx'\right) \qquad \text{(A-8a)}$$

$$u_{x'}^{N_{ji}=1} = \frac{-1}{kAE} \cdot \mathrm{Cosec}\ kL \cdot \cos kx' \qquad \text{(A-8b)}$$

These two values are the same as $\Delta_{\xi x'}^{N_{ij}=1}$ and $\Delta_{\xi x'}^{N_{ji}=1}$ referred to in Chapter II.

APPENDIX B

COMPUTATIONAL DETAILS OF THE NUMERICAL EXAMPLES

B-1. Single Span Gable Frame (Figure 23)

$$\left\{ F^{M}_{AB} \right\} \text{ is chosen as primary unknown.}$$

The members AB, BC, CD and DE are referred to as member Nos. 1, 2, 3 and 4 respectively and their corresponding transport matrices and the angular transformation matrices are referred to as $[T1]$, $[T2]$, $[T3]$, $[T4]$, and $[\pi 1]$, $[\pi 2]$, $[\pi 3]$, $[\pi 4]$.

$$[\pi B] = [\pi 2]^T \, [J] \, [\pi 1]$$

$$[\pi C] = [\pi 3]^T \, [J] \, [\pi 2]$$

$$[\pi D] = [\pi 4]^T \, [J] \, [\pi 3]$$

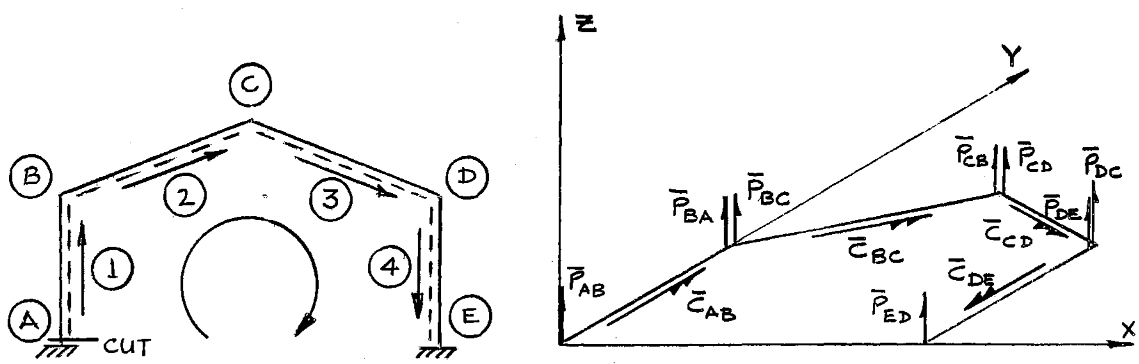


Figure 23. Single Span Gable Frame and its Conjugate Structure

All member end values are expressed in terms of the primary unknown as follows, using $\{S1\} = \{F^M_{AB} \vdots 0\}$

$$\{S^1_{AB}\} = \{S1\}$$

$$\{S^1_{BA}\} = [T1]\{S^1_{AB}\} = [T1]\{S1\}$$

$$\{S^2_{BC}\} = [\pi B]\{S^1_{BA}\} = [\pi B][T1]\{S1\} = [TR1]\{S1\}$$

$$\{S^2_{CB}\} = [T2]\{S^2_{BC}\} = [T2][TR1]\{S1\} = [TR2]\{S1\}$$

$$\{S^3_{CD}\} = [\pi C]\{S^2_{CB}\} = [\pi C][TR2]\{S1\} = [TR3]\{S1\}$$

$$\{S^3_{DC}\} = [T3]\{S^3_{CD}\} = [T3][TR3]\{S1\} = [TR4]\{S1\}$$

$$\{S^4_{DE}\} = [\pi D]\{S^3_{DC}\} = [\pi D][TR4]\{S1\} = [TR5]\{S1\}$$

$$\{S^4_{ED}\} = [T4]\{S^4_{DE}\} = [T4][TR5]\{S1\} = [TR5]\{S1\} \qquad \text{(B-1)}$$

<u>Elasto-Static Equations</u>. Three elasto-static equations, $\Sigma \overline{P}_z = 0$, $\Sigma \overline{M}_{xAE} = 0$, $\Sigma \overline{M}_{yDE} = 0$, are written using the twelve elastic quantities shown in Figure 23. These are written in a matrix form

$$\underset{3x12}{[A]} \; \underset{12x1}{\{\overline{P}\}} = 0 \qquad \text{(B-2)}$$

The elastic loads are expressed in terms of member end deformations using Equation (3-6).

$$\underset{12x1}{\{\overline{P}\}} = \underset{12x24}{[B]} \; \underset{24x1}{\{\delta\}} \qquad \text{(B-3)}$$

The member end deformations are expressed in terms of primary unknowns using Equations (4-1)

$$\underset{24x1}{\{\delta\}} = \underset{24x6}{[C]} \; \underset{6x1}{\{S1\}} \tag{B-4}$$

which is modified to

$$\underset{24x1}{\{\delta\}} = \underset{24x3}{[C1]} \underset{3x1}{\{F_{AB}^{1}\}} \tag{B-5}$$

by dropping the columns corresponding to zero elements in $\{S1\}$.

Equations (B-2), (B-3) and (B-5) are combined to give

$$[A]\,[B]\,[C1]\left\{F_{AB}^{1}\right\} = 0$$

$$= [FINAL]\left\{F_{AB}^{1}\right\} = 0 \tag{B-6}$$

For a non-trivial solution for $\left\{F_{AB}^{1}\right\}$, the determinant of $[FINAL]$ is made zero. This gives the values of the frequency parameter λ from which the frequencies are obtained in cycles per second.

The mode shapes are obtained by substituting the solution for λ in Equation (B-6) to solve for $\left\{F_{AB}^{1}\right\}$. Substitution of $\{S1\}$ in Equation (B-4) then gives all end deformations.

B-2. Two Span Single Story Frame (Figure 24)

$\left\{F_{AB}^{M}\right\}$ and $\left\{F_{CD}^{M}\right\}$ are chosen as primary unknowns. The members AB, CD, FE, BD and DF are respectively referred to as members Nos. 1, 2, 3, 4 and 5 and their corresponding angular transformation matrices are referred to as $[\pi 1]$, $[\pi 2]$, $[\pi 3]$, $[\pi 4]$ and $[\pi 5]$. All members being identical in length and cross-section, they have the same transport matrix referred to as $[T]$.

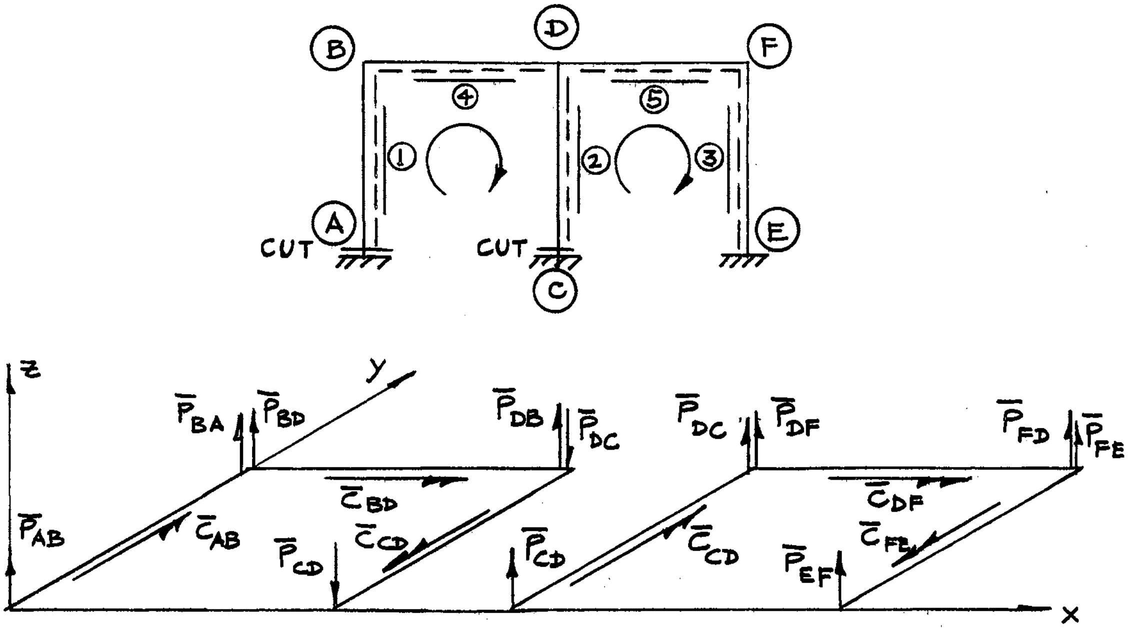


Figure 24. Two Span Single Story Frame and Corresponding Conjugate Structures

$$[\pi B] = [\pi 4]^T \; [J] \; [\pi 1]$$

$$[\pi F] = [\pi 3]^T \; [J] \; [\pi 5]$$

All member end values are expressed in terms of the primary unknowns as follows, using $\{S1S2\} = \{S1 \vdots S2\} = \{F_{AB}^{\;1} \vdots 0 \vdots F_{CD}^{\;2} \vdots 0\}$

$$\{S_{AB}^{\;1}\} = \{S1\}$$

$$\{S_{BA}^{\;1}\} = [T]\{S_{AB}^{\;1}\} = [T]\{S1\}$$

$$\{S_{BD}^{\;4}\} = [\pi B]\{S_{BA}^{\;1}\} = [\pi B]\;[T]\{S1\} = [TR1]\{S1\}$$

$$\{S_{DB}^{\;4}\} = [T]\{S_{BD}^{\;4}\} = [T]\;[TR1]\{S1\} = [TR2]\{S1\}$$

$$\left\{S_{CD}^{2}\right\} = \left\{S2\right\}$$

$$\left\{S_{DC}^{2}\right\} = [T]\left\{S_{CD}^{2}\right\} = [T]\left\{S2\right\}$$

$$\left\{S_{DC}^{0}\right\} = [\pi 2]\left\{S_{DC}^{2}\right\} = [\pi 2]\,[T]\left\{S2\right\} = [TR3]\left\{S2\right\}$$

$$\left\{S_{DB}^{0}\right\} = [\pi 4]\left\{S_{DB}^{4}\right\} = [\pi 4]\,[TR2]\left\{S1\right\} = [TR4]\left\{S1\right\}$$

$$\begin{aligned}\left\{S_{DF}^{0}\right\} &= [J]\left\{S_{DC}^{0}\right\} + \left\{S_{DB}^{0}\right\}\\ &= [J]\,[TR3]\left\{S2\right\} + [TR4]\left\{S1\right\} \qquad \text{F part only}\\ &= [TR5]\left\{S2\right\} + [TR4]\left\{S1\right\}\\ &= [D1]\left\{S1S2\right\}\end{aligned}$$

where

$$[D1] = \left[\begin{array}{c|c|c|c} TR4(11) & TR4(12) & TR5(11) & TR5(12) \\ \hline 0 & 0 & TR5(21) & TR5(22) \end{array}\right]_{6x12}$$

$$\left\{S_{DF}^{5}\right\} = [\pi 5]^{T}\left\{S_{DF}^{0}\right\} = [\pi 5]^{T}\,[D1]\left\{S1S2\right\} = [D2]\left\{S1S2\right\}$$

$$\left\{S_{FD}^{5}\right\} = [T]\left\{S_{DF}^{5}\right\} = [T]\,[D2]\left\{S1S2\right\} = [D3]\left\{S1S2\right\}$$

$$\left\{S_{FE}^{3}\right\} = [\pi F]\left\{S_{FD}^{5}\right\} = [\pi F]\,[D3]\left\{S1S2\right\} = [D4]\left\{S1S2\right\}$$

$$\left\{S_{EF}^{3}\right\} = [T]\left\{S_{FE}^{3}\right\} = [T]\,[D4]\left\{S1S2\right\} = [D5]\left\{S1S2\right\} \qquad (B\text{-}7)$$

Elasto-static equations: Three elasto-static equations are written for each panel shown in Figure (24). These equations are

$$\Sigma \overline{P}_z = 0 \qquad\qquad \Sigma \overline{P}_z = 0$$

$$\Sigma \overline{M}_{xBD} = 0 \qquad \text{and} \qquad \Sigma \overline{M}_{xDF} = 0$$

$$\Sigma \overline{M}_{yCD} = 0 \qquad\qquad \Sigma \overline{M}_{yEF} = 0$$

These equations involve fifteen elastic quantities shown and are arranged in a matrix form

$$\underset{6x15}{[A]} \underset{15x1}{\{ \overline{P} \}} = 0 \tag{B-8}$$

The elastic loads are expressed in terms of member end deformations using Equation (3-6),

$$\underset{15x1}{\{ \overline{P} \}} = \underset{15x30}{[B]} \underset{30x1}{\{ \delta \}} \tag{B-9}$$

The member end deformations are expressed in terms of primary unknowns using Equations (B-7)

$$\underset{30x1}{\{ \delta \}} = \underset{30x12}{[C]} \underset{12x1}{\{ S1S2 \}} \tag{B-10}$$

which is reduced to

$$\underset{30x1}{\{ \delta \}} = \underset{30x6}{[C1]} \underset{6x1}{\{ F_{AB}^{1} \vdots F_{CD}^{2} \}} \tag{B-11}$$

by dropping the terms corresponding to zero elements in $\{S1S2\}$.

Equations (B-8), (B-9) and (B-11) are combined to give

$$[A]\,[B]\,[C1]\left\{F_{AB}^{\,1} \vdots F_{CD}^{\,2}\right\} = 0$$

$$= \underset{6x6}{[FINAL]}\underset{6x1}{\left\{F_{AB}^{\,1} \vdots F_{CD}^{\,2}\right\}} = 0 \qquad \text{(B-12)}$$

By making the determinant of [FINAL] equal to zero, the frequency parameter λ is solved for. The mode shapes are obtained by finding the end deformations from Equation (B-11), using the values of $\{S1\}$ and $\{S2\}$ obtained by solving Equation (B-12) for $\{F_{AB}^{\,1}\}$ and $\{F_{CD}^{\,2}\}$.

B-3. Single Span Three Story Frame (Figure 25)

N', M' and Δ_y values at A, C and E are taken as primary unknowns. V' at C is a known value equal to $-\frac{P}{2}\cos\omega t$. The remaining values of V', Δ_x and θ' at A, C and E are zero.

Members AB, CD and EF are identical and identically oriented. Their transport matrix and their angular transformation matrix are referred to as [T1] and [π1]. By the same reason, similar quantities for members BD, DF and FG are referred to as [T2] and [π2].

$$[\pi B] = [\pi 2]^T\,[J]\,[\pi 1]$$

All member end values are first computed in terms of $\{S1S2S3\} = \{S1 \vdots S2 \vdots S3\}$ where $\{S1\} = \left\{F_{AB}^{\,1} \vdots \delta_{AB}^{\,1}\right\}$, $\{S2\} = \left\{F_{CD}^{\,1} \vdots \delta_{CD}^{\,1}\right\}$ and $\{S3\} = \left\{F_{EF}^{\,1} \vdots \delta_{EF}^{\,1}\right\}$.

$$\left\{S_{AB}^{\,1}\right\} = \{S1\}$$

$$\left\{S_{BA}^{\,1}\right\} = [T1]\left\{S_{AB}^{\,1}\right\} = [T1]\{S1\}$$

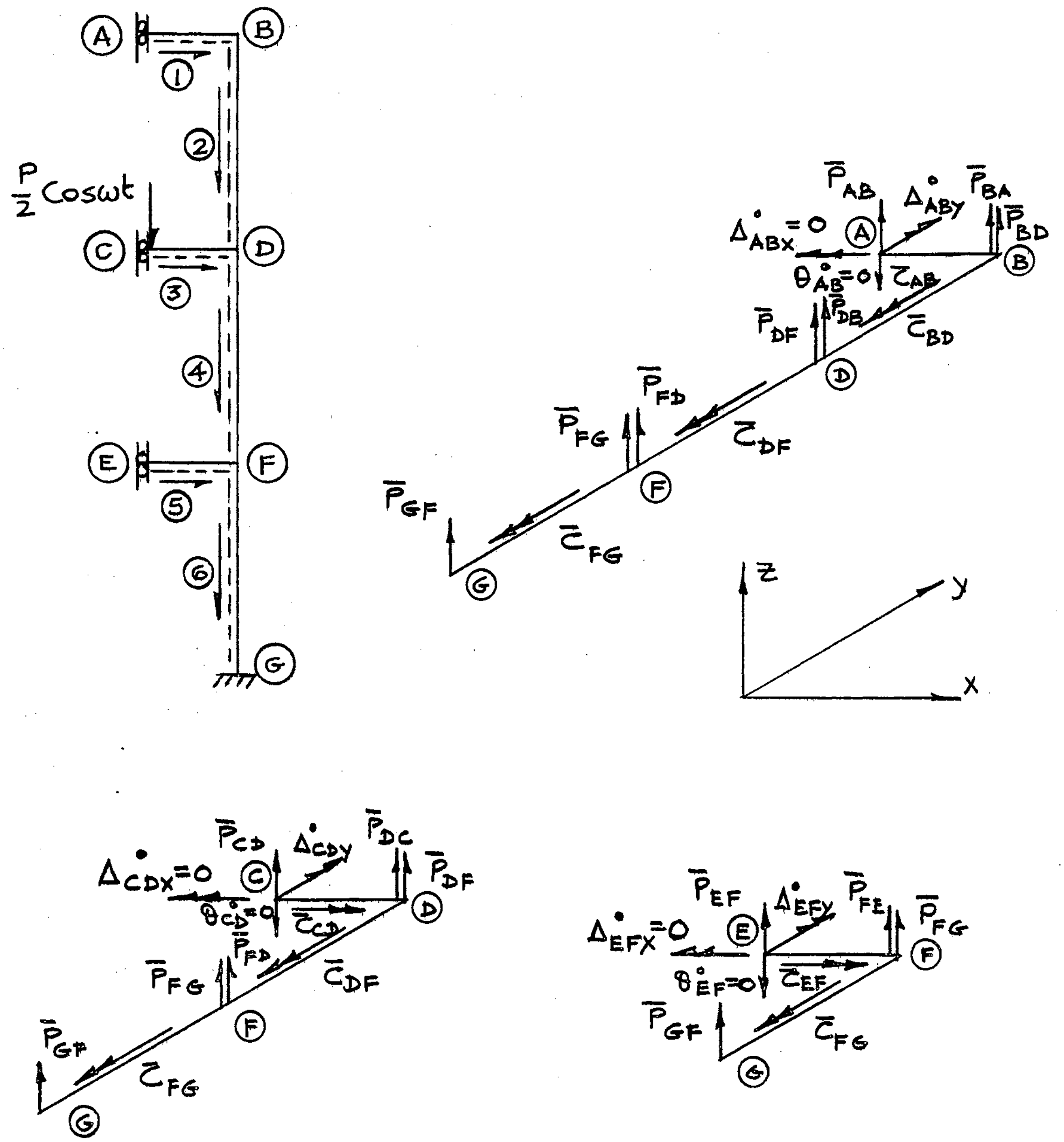


Figure 25. Single Span Three Story Modified Frame and Corresponding Conjugate Structures

$$\left\{S_{BD}^{2}\right\} = [\pi B]\left\{S_{BA}^{1}\right\} = [\pi B]\,[T1]\{S1\} = [TR1]\{S1\}$$

$$\left\{S_{DB}^{2}\right\} = [T2]\left\{S_{BD}^{2}\right\} = [T2]\,[TR1]\{S1\} = [TR2]\{S1\}$$

$$\left\{S_{DB}^{0}\right\} = [\pi 2]\left\{S_{DB}^{2}\right\} = [\pi 2]\,[TR2]\{S1\} = [TR3]\{S1\}$$

$$\left\{S_{CD}^{1}\right\} = \{S2\}$$

$$\left\{S_{DC}^{1}\right\} = [T1]\left\{S_{CD}^{1}\right\} = [T1]\{S2\}$$

$$\left\{S_{DC}^{0}\right\} = [\pi 1]\left\{S_{DC}^{1}\right\} = [\pi 1]\,[T1]\{S2\} = [TR4]\{S2\}$$

$$\left\{S_{DF}^{0}\right\} = [J]\left\{S_{DC}^{0}\right\} + \left\{S_{DB}^{0}\right\}$$

$$= [J]\,[TR4]\{S2\} + [TR3]\{S1\} \qquad \text{F part only}$$

$$= [TR5]\{S2\} + [TR3]\{S1\}$$

$$= [D1]\{S1S2\}$$

where

$$[D1] = \left[\begin{array}{c|c|c|c} TR3(11) & TR3(12) & TR5(11) & TR5(12) \\ \hline 0 & 0 & TR5(21) & TR5(22) \end{array}\right]_{6x12}$$

and $\{S1S2\} = \{S1 \,\vdots\, S2\}$

$$\left\{S_{DF}^{2}\right\} = [\pi 2]^{T}\left\{S_{DF}^{0}\right\} = [\pi 2]^{T}\,[D1]\{S1S2\} = [D2]\{S1S2\}$$

$$\left\{S_{FD}^{2}\right\} = [T2]\left\{S_{DF}^{2}\right\} = [T2]\,[D2]\{S1S2\} = [D3]\{S1S2\}$$

$$\left\{S_{FD}^{0}\right\} = [\pi 2]\left\{S_{FD}^{2}\right\} = [\pi 2]\,[D3]\{S1S2\} = [D4]\{S1S2\}$$

$$\left\{S_{EF}^{1}\right\} = \left\{S3\right\}$$

$$\left\{S_{FE}^{1}\right\} = [T1]\left\{S_{EF}^{1}\right\} = [T1]\left\{S3\right\}$$

$$\left\{S_{FE}^{0}\right\} = [\pi 1]\left\{S_{FE}^{1}\right\} = [\pi 1]\,[T1]\left\{S3\right\} = [TR4]\left\{S3\right\}$$

$$\left\{S_{FG}^{0}\right\} = [J]\left\{S_{FE}^{0}\right\} + \left\{S_{FD}^{0}\right\}$$

$$= [J]\,[TR4]\left\{S3\right\} + \underline{[D4]\left\{S1S2\right\}}$$

F part only

$$= [TR5]\left\{S3\right\} + \underline{[D4]\left\{S1S2\right\}}$$

$$= [E1]\left\{S1S2S3\right\}$$

where

$$[E1] = \left[\begin{array}{c|c|c|c|c|c} D4(11) & D4(12) & D4(13) & D4(14) & TR5(11) & TR5(12) \\ \hline 0 & 0 & 0 & 0 & TR5(21) & TR5(22) \end{array}\right]_{6x18}$$

$$\left\{S_{FG}^{2}\right\} = [\pi 2]^{T}\left\{S_{FG}^{0}\right\} = [\pi 2]^{T}\,[E1]\left\{S1S2S3\right\} = [E2]\left\{S1S2S3\right\}$$

$$\left\{S_{GF}^{2}\right\} = [T2]\left\{S_{FG}^{2}\right\} = [T2]\,[E2]\left\{S1S2S3\right\} = [E3]\left\{S1S2S3\right\} \qquad \text{(B-13)}$$

<u>Elasto-Static Equations</u>: Three elasto-static equations are written for each conjugate panel shown in Figure 25. These equations, written in terms of eighteen member elastic quantities, are

$$\Sigma\,\overline{P}_{z} = 0 \qquad \Sigma\,\overline{P}_{z} = 0 \qquad \Sigma\,\overline{P}_{z} = 0$$

$$\Sigma\,\overline{M}_{xAB} = 0 \qquad \Sigma\,\overline{M}_{xCD} = 0 \qquad \Sigma\,\overline{M}_{xEF} = 0$$

$$\Sigma\,\overline{M}_{yGB} = 0 \qquad \Sigma\,\overline{M}_{yGD} = 0 \qquad \Sigma\,\overline{M}_{yGF} = 0$$

$$
\begin{bmatrix}
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & -1 & 0 & L2 & 0 & 0 & 0 & 0 & L2 & 2L2 & 0 & 0 & 0 & 0 & 2L2 & 3L2 & 0 \\
L1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & L2 & 0 & 0 & 0 & 0 & L2 & 2L2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & L1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & L2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L1 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\overline{P}_1^1 \\
\\
\overline{P}_2^2 \\
\overline{P}_3^3 \\
\\
\overline{P}_4^4 \\
\overline{P}_5^5 \\
\\
\overline{P}_6^6
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
-\Delta^{\circ}_{ABy} \\
0 \\
0 \\
-\Delta^{\circ}_{CDy} \\
0 \\
0 \\
-\Delta^{\circ}_{EFy}
\end{bmatrix}
\qquad (B-14)
$$

$$
\begin{bmatrix} \bar{P}_1^1 \\ \hline \bar{P}_2^2 \\ \hline \bar{P}_3^3 \\ \hline \bar{P}_4^4 \\ \hline \bar{P}_5^5 \\ \hline \bar{P}_6^6 \end{bmatrix}
=
\left[\begin{array}{c|c|c|c|c|c}
[X] & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & [Y] & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & [X] & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & [Y] & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & [X] & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & [Y]
\end{array}\right]
\begin{bmatrix} \delta_{AB}^1 \\ \delta_{BA}^1 \\ \delta_{BD}^2 \\ \delta_{DB}^2 \\ \delta_{CD}^3 \\ \delta_{DC}^3 \\ \delta_{DF}^4 \\ \delta_{FD}^4 \\ \delta_{EF}^5 \\ \delta_{FE}^5 \\ \delta_{FG}^6 \\ \delta_{GF}^6 \end{bmatrix}
\tag{B-15}
$$

$$
[X] = \begin{bmatrix} 0 & {}^{-1}/L1 & \lambda_o & 0 & {}^{-1}/L1 & 0 \\ 0 & 1/L1 & 0 & 0 & 1/L1 & \lambda_o \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\qquad
[Y] = \begin{bmatrix} 0 & {}^{-1}/L2 & \lambda_o & 0 & {}^{-1}/L2 & 0 \\ 0 & 1/L2 & 0 & 0 & 1/L2 & \lambda_o \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
$$

These arranged in a matrix form are shown in Equation (B-14).

The $\overline{P}$ values are expressed in terms of the member end deformations as shown in Equation (B-15). The vector on the right hand side of Equation (B-14) is written in terms of the primary unknowns (Equation (B-14b)). The member end deformations are expressed in terms of $\{S1\}$,

$$\begin{bmatrix} 0 \\ 0 \\ -\Delta^{0}_{ABy} \\ 0 \\ 0 \\ -\Delta^{0}_{CDy} \\ 0 \\ 0 \\ -\Delta^{0}_{EFy} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} N'^{1}_{AB} \\ M'^{1}_{AB} \\ \Delta^{1}_{ABy} \\ N'^{1}_{CD} \\ M'^{1}_{CD} \\ \Delta^{1}_{CDy} \\ N'^{1}_{EF} \\ M'^{1}_{EF} \\ \Delta^{1}_{EFy} \end{bmatrix} \tag{B-14b}$$

$\{S2\}$ and $\{S3\}$ as shown in Equation (B-16). Symbolically these relations may be written as

$$[A]\{\overline{P}\} = [R]\{FD\} \tag{B-14a}$$

$$\{\overline{P}\} = [B]\{\delta\} \tag{B-15a}$$

$$
\begin{bmatrix} \delta^{1}_{AB} \\ \delta^{1}_{BA} \\ \delta^{2}_{BD} \\ \delta^{2}_{DB} \\ \delta^{3}_{CD} \\ \delta^{3}_{DC} \\ \delta^{4}_{DF} \\ \delta^{4}_{FD} \\ \delta^{5}_{EF} \\ \delta^{5}_{FE} \\ \delta^{6}_{FG} \\ \delta^{6}_{GF} \end{bmatrix}
=
\begin{bmatrix}
0 & I & 0 & 0 & 0 & 0 \\
T1(21) & T1(22) & 0 & 0 & 0 & 0 \\
TR1(21) & TR1(22) & 0 & 0 & 0 & 0 \\
TR2(21) & TR2(22) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & T1(21) & T1(22) & 0 & 0 \\
D2(21) & D2(22) & D2(23) & D2(24) & 0 & 0 \\
D3(21) & D3(22) & D3(23) & D3(24) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & T1(21) & T1(22) \\
E2(21) & E2(22) & E2(23) & E2(24) & E2(25) & E2(26) \\
E3(21) & E3(22) & E3(23) & E3(24) & E3(25) & E3(26)
\end{bmatrix}
\begin{bmatrix} F^{1}_{AB} \\ \delta^{1}_{AB} \\ F^{1}_{CD} \\ \delta^{1}_{CD} \\ F^{1}_{EF} \\ \delta^{1}_{EF} \end{bmatrix}
\qquad \text{(B-16)}
$$

$$
\{ \delta \} = [C] \{ S1S2S3 \} \qquad \text{(B-16a)}
$$

where $\{FD\}$ is the vector of primary unknowns, $\left\{ N'^{1}_{AB},\ M'^{1}_{AB},\ \Delta^{1}_{ABy},\ N'^{1}_{CD},\ M'^{1}_{CD},\ \Delta^{1}_{CDy},\ N'^{1}_{EF},\ M'^{1}_{EF},\ \Delta^{1}_{EFy} \right\}$.

Combining these gives

$$
\underset{9\times18}{[A]}\ \underset{18\times36}{[B]}\ \underset{36\times18}{[C]}\ \underset{18\times1}{\{S1S2S3\}} = \underset{9\times9}{[R]}\ \underset{9\times1}{\{FD\}} \qquad \text{(B-17)}
$$

$[C]\{S1S2S3\}$ is now modified by deleting zero elements in $\{S1S2S3\}$ and also deleting the corresponding columns from $[C]$. This leaves only the primary unknowns and the effect of the applied load in $V_{CD}^{'1}$. Rearranging the terms makes it possible to separate the primary unknowns and the applied load.

$$[A]\ [B]\ [\underset{36x9}{C1} \vdots \underset{36x1}{C2}]\{\underset{9x1}{FD} \vdots \underset{1x1}{P}\} = [R]\{FD\}$$

$$\Big[[A]\ [B]\ [C1] - [R]\Big]\{FD\} = -[A]\ [B]\{C2\}\cdot P \qquad \text{(B-18)}$$

$\{FD\}$ is now solved for in terms of P. Using the known values of $\{FD\}$ and $V_{CD}^{'1}$, $\{S1S2S3\}$ is reconstructed and used in Equations (B-13) to obtain all member end forces and deformations.

VITA

Rameshchandra Kapilram Munshi

Candidate for the Degree of

Doctor of Philosophy

Thesis: VIBRATION ANALYSIS OF PLANAR FRAMES BY THE STRING POLYGON METHOD

Major Field: Civil Engineering

Biographical:

Personal Data: Born August 6, 1930, in Bombay, India, the son of Kapilram and Chatura Munshi.

Education: Graduated from G. T. High School, Bombay, India, in June, 1947. Joined St. Xavier's College, Bombay, and passed the Intermediate Science Examination in May, 1949. Studied for one year at L. D. College of Engineering, Ahmedabad, India, and transferred to V. J. Technical Institute, Bombay. Completed the requirements and received the Degree of Bachelor of Engineering (Civil) from the University of Bombay, Bombay, India, in October, 1953. Attended the Utah State University in the Fall quarter, 1959, then transferred to the Oklahoma State University. Completed requirements for the Master of Science degree in August, 1961. Completed the requirements for the degree of Doctor of Philosophy in July, 1968. Member of Chi Epsilon and Associate Member of Sigma Xi.

Professional Experience: Worked as a Technical Assistant and Assistant Engineer with M/S Garlick and Company (P) Ltd., Bombay, India, from Nov. 1953 to September 1959; as a graduate assistant in the School of Civil Engineering, Oklahoma State University from September 1960 to January 1962; as a part time instructor in the School of Civil Engineering from February 1962 to August 1963; as an instructor in the School of Civil Engineering, Oklahoma State University from September 1963 to August 1968, engaged in teaching courses in Structural Mechanics, Analysis and Design. Associate Member of A.S.C.E. and member of A.C.I.