MULTIPLICATIVE NUMBER-THEORETIC FUNCTIONS AND THEIR GENERATING FUNCTIONS

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TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

The study of functions is an important part of modern mathematics. Today's student is introduced early to the mathematical concept of function. By the time he has completed an elementary course in calculus, the student is familiar with many functions.

There are functions peculiar to almost every branch of mathematics. There are others which, even though useful in many different areas of mathematics, have their origin in a particular branch. These are usually identified in some way with the branch of their origin.

The functions most closely identified with number theory are usually called "number-theoretic", "arithmetic" or "arithmetical". They are different from most functions found in algebra or analysis in that they are usually defined on the integers or a subset of the integers. Sometimes an extension to the real numbers is made and questions of analytic behavior are studied. However, they are usually more useful in the study of natural numbers or the integers.

Stewart (27) indicates that a number-theoretic function is one such that the function values depend upon the standard form of n. This seems unnecessarily restrictive and, although most of those used in this paper meet that requirement, the adjectives "number-theoretic" and "arithmetic" will be used more loosely. For example, occasional reference will be made to arithmetical functions associated with the additive theory of numbers. One such function, which does not meet Stewart's requirement, is the function given by $\pi(n)$ = the number of primes less than or equal to n.

Number theory is one of the oldest branches of mathematics. It is, however, not a dead issue. There are still many unsolved problems. Also, interrelations often exist between number theory and other branches of mathematics. For example, geometry has close ties with number theory. Number theory provides useful examples for abstract algebra and topology. Although primary interest will be centered on the number theoretic ideas involved, some of the relationships between analysis and number theory will be seen in this dissertation.

One area of number theory which lends itself to study by undergraduates and to undergraduate research is the study of the multiplicative functions. Multiplicative functions, as defined in Definition 2.1, are not restricted to number theory. However, some of the most interesting are peculiar to the subject. These form the principal subject matter of this paper.

The reader should have a minimum background of a complete sequence of elementary calculus which includes a study of series. Although a course in elementary number theory is presumed, the reader could possibly acquire the background needed by a careful study of the topics covered in Chapter II. The series used are usually considered as series of complex numbers. However, so as not to confuse the reader who has not been exposed to complex series, they are stated as if they were real valued. This makes no difference as far as the major points of the paper are concerned.

The material is presented in what is hoped to be a logical sequence. Thus, the reader should presume nothing concerning the level of difficulty from the location of a topic. For example, the material in Chapter V is relatively easy, but, it will take a mature reader to understand most of Chapter IV. No comprehensive historical development is attempted. However, historical facts are mentioned at various points in the development.

Chapter II provides the background from number theory and analysis which will be needed in the later chapters. This is for the purpose of making the paper self-contained thus making it unnecessary that the reader be an expert on the subject and eliminating the need for several references. Chapters III and IV develop the theory of generating functions which are useful tools in the study of number-theoretic functions. Chapter V includes material which evolved as a result of the study. A search of the literature has failed to uncover these results elsewhere. Thus, they are presumed to be new. The reader will recognize that not everything which could be developed has been proved here and can possibly make some additional conjectures and prove them.

CHAPTER II

BACKGROUND FROM NUMBER THEORY AND ANALYSIS

Multiplicative Functions

There are several different types of functions associated with number theory. The usual way of classifying them seems to be to classify them as "multiplicative", "those associated with the additive theory of numbers", and "other". This is somewhat misleading because there is more than one type of function that could be called additive and there are functions associated with the additive theory of numbers which could not be called additive functions. The class of multiplicative functions is well defined by a precise mathematical property.

Definition 2.1. A function f, defined on the natural numbers, is called multiplicative if and only if

$$f(mn) = f(m)f(n)$$

whenever $(m,n) = 1$. If

$$f(mn) = f(m)f(n)$$

for all natural numbers m and n, then f is called completely (totally, unconditionally) multiplicative.

It should be observed that some functions from algebra and calculus satisfy this definition when their domain is restricted to the natural numbers. For example, the identity function, defined by $f(n) = 1$ for every n, and the power functions, defined by $f(n) = n^x x\neq 0$, are completely multiplicative. It is also possible to define multiplicative functions in the same way for the set of all integers. However, the natural numbers are consistently used here as the domain.

Throughout the paper some notation is used which should be explained. First $d|n$ will be used, as is usual, to mean d divides n. Hence, if

$$\sum_{d|n} f(d)$$

is written $d|n$ specifies the index set for the sum as those natural numbers which divide n. By the Fundamental Theorem of Arithmetic it is known that if $n > 1$, then n can be written uniquely (except for order) as the product of primes. Thus, a natural number $n > 1$ is often expressed in standard form (or canonical form) as,

$$n = p_i^{a_1} p_2^{a_2} \ldots p_r^{a_r} \quad \text{or} \quad n = \prod_{i=1}^{r} p_i^{a_i}$$

where the subscripts are used to indicate that the primes are different. However, the same subscript being used for the exponent means only that the ith exponent belongs to the ith primes of the chosen order. The order is often chosen as the "natural order", that is,

$$p_1 < p_2 < \ldots < p_r.$$

The notation $(m,n) = 1$ used in Definition 2.1 will be used often and, although it may never be referred to directly, it is appropriate to remind the reader of its precise meaning.

<u>Definition 2.2</u>. Natural numbers m and n are relatively prime, written $(m,n) = 1$, if and only if when $p|n$ then $p\nmid m$ and when $p|m$ then $p\nmid n$ for p a prime.

Definitions 2.1 and 2.2 would thus lead one to suspect that primes play an important part in the theory of multiplicative functions. That this suspicion is a fact is seen in the important role of primes in the discussions and theorems of this paper.

The following theorem shows a very useful characteristic of multiplicative functions.

Theorem 2.1. If f is multiplicative and not identically zero, then $f(1) = 1$.

Proof: If f is not identically zero, then there exists an integer n such that $f(n) \neq 0$. But, $(1,n) = 1$, thus

$$f(n) = f(1 \cdot n) = f(1)f(n).$$

Hence, $f(n) \neq 0$ implies $f(1) = 1$.

It was indicated earlier that primes play an important role in multiplicative functions. In fact, if a function is given to be multiplicative and is defined for powers of each prime, it is determined for all natural numbers. For example, let g be a multiplicative function such that $g(p^a) = 3^a$ for every prime p. Then

$$\begin{aligned} g(n) &= g(p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}) \\ &= g(p_1^{a_1}) g(p_2^{a_2}) \cdots g(p_r^{a_r}) \\ &= 3^{a_1} 3^{a_2} \cdots 3^{a_r} = 3^{a_1 + a_2 + \cdots + a_r}. \end{aligned}$$

In this case, g was specified as multiplicative, then an expression for g(n) is calculated. Usually, a function arises in answer to a question about n, or a defining property is found for a function which can then be shown to be multiplicative. As an example, if prime p is given, let $M_p(n) = a + 1$ where a is the highest power of p that divides n. The function M_p then answers the question: Does p^b divide n? Observe that if $M_p(n) = 1$, then $p^b \nmid n$ and if $M_p(n) = k$, $k > 1$, then $p^b | n$ for $b < k$. Also, M_p is multiplicative. For if $(m,n) = 1$ and $p^a | m$, then $p \nmid n$. Thus,

$$M_p(m)M_p(n) = M_p(m) \cdot 1 = M_p(m) = M_p(mn).$$

Later, when the formula for the function τ is given, the reader should note that:

$$\tau(n) = \prod_{p|n} M_p(n).$$

Later in this chapter an operation on multiplicative functions, called the convolution product, will be discussed. However, a special case, given by the following theorem is needed earlier.

Theorem 2.2. If f is a multiplicative function then the function g defined by

$$g(n) = \sum_{d|n} f(d)$$

is also multiplicative.

Proof: If $(m,n) = 1$, then $d|mn$ means that d can be written $d = st$ where $s|m$, $t|n$ and $(s,t) = 1$. Then,

$$g(mn) = \sum_{d|mn} f(d) = \sum_{s|m,t|n} f(st).$$

Hence, since f is multiplicative, $f(st) = f(s)f(t)$.

Thus,

$$g(mn) = \sum_{s|m,t|n} f(s)f(t) = \sum_{s|m} f(s) \cdot \sum_{t|n} f(t)$$

$$= g(m)g(n).$$

It is no surprise that the set of multiplicative functions is not closed under some operations. It is obvious that f + g is not always multiplicative when f and g are. For example,

$$f(n) = n^2,\ g(n) = 1$$

define multiplicative functions but

$$(f + g)n = n^2 + 1$$

does not.

Because of the obvious close relation between multiplicative functions and the operation of ordinary multiplication of functions, the following is not an unexpected result.

Theorem 2.3. If f and g are multiplicative functions, then f·g is multiplicative and, if $g(n) \neq 0$ for every n, f/g is multiplicative.

Corollary 2.4. If g is a multiplicative function such that $g(n) \neq 0$ for every n, then 1/g is a multiplicative function.

Corollary 2.4 follows easily from Theorem 2.3 if one recalls that f(n) = 1 for every n defines a multiplicative function. The corollary is stated so that it may be observed that not all multiplicative functions have an inverse under the operation of ordinary multiplication of functions. For example, note that f(n) = 1 for every n is the identity and that

$$g(1) = 1,\ g(n) = 0 \text{ for } n > 1,$$

defines a multiplicative function which, by Corollary 2.4 has no inverse.

Another theorem, which was used earlier but needs to be formally stated since it will be assumed many times, is the following.

Theorem 2.5. If f is multiplicative and n is written in standard form, then

$$f(n) = \prod_{i=1}^{r} f(p_i^{a_i}).$$

Proof: The proof is by induction on r.

The arithmetical functions discussed in this paper are defined on P, the set of positive integers. The notation f(a,n) and similar notations are used but this is intended to mean that f is a function of n which involves an arbitrary constant a.

Functions defined on P x P or P x P x ... x P may be studied in very much the same manner as is used here. In fact, the definition of multiplicative can be easily extended by requiring that a function be multiplicative in each component of $(n_1,n_2,\ldots,n_k)$. Unfortunately, the functions themselves do not generalize as easily. Cohen (5) and Vaidyanathaswamy (30) are two of the many mathematicians to generalize number theoretic functions. There seems to be a number of ways to generalize most of them as functions of several variables. For this reason, and because it would lead too far afield, it is not deemed desirable to include functions of several variables in the present discussion.

Some Special Multiplicative Arithmetic Functions

There are several multiplicative functions which have arisen naturally in the investigations of number theory. There are others, which because of their unusual nature or because they are representative of a larger class, have become a part of the "folklore" of the subject. In order to have examples upon which to base later discussion, they are presented here. Their definition, elementary formula (if it is not a part of the definition), and some useful properties are given. Some proofs are omitted in order to avoid boring the reader. For proofs, he may refer to one of several elementary number theory texts.

Probably the function of number theory most often encountered is the Euler totient function or phi function. It occurs in counting problems and in the additive theory of numbers as well as in the multiplicative theory.

Definition 2.3. The Euler totient function, denoted by ϕ, is defined as follows: $\phi(n)$ is the number of positive integers less than or equal to n and relatively prime to n.

To see that the behavior of ϕ is quite irregular, one should note that $\phi(15) = 8$, $\phi(16) = 8$, $\phi(17) = 16$, $\phi(18) = 6$, $\phi(19) = 18$ and $\phi(20) = 8$.

The following theorems, which are easily proved by several different methods, give the formula for $\phi(n)$ and for the sum of $\phi(n)$ over the divisors of n.

Theorem 2.6. If n is written in standard form then,

$$\phi(n) = n\left(\frac{p_1-1}{p_1}\right)\left(\frac{p_2-1}{p_2}\right)\cdots\left(\frac{p_r-1}{p_r}\right) = n\prod_{p\|n}\left(\frac{p-1}{p}\right).$$

Theorem 2.7. For any positive integer n,

$$n = \sum_{d|n}\phi(d).$$

The function ϕ is multiplicative. This may be proved by independent methods and the above theorems will follow by using Theorems 2.2 and 2.5.

As an alternate procedure, Theorem 2.6 can be proved directly from the definition of ϕ. Then, it follows from the formula in Theorem 2.6 that ϕ is multiplicative.

There are many useful formulas involving ϕ, but some other functions must be defined first. One of these is the Mobius μ function which has a unique role in the theory of multiplicative functions.

<u>Definition 2.4</u>. Let μ be the function defined for positive integers such that,

$$\text{(i) } \mu(1) = 1; \quad \text{(ii) } \mu(n) = 0$$

if $n > 1$ and n has a perfect square as a factor; $\mu(n) = (-1)^r$ if $n > 1$ and n is the product of r different prime factors.

The fact that μ is multiplicative follows from the definition. The next theorem, which provides some useful equalities, follows easily from Theorem 2.6 and Definition 2.4.

<u>Theorem 2.8</u>. If ϕ is the Euler totient function, then $\phi(n)$ is equal to any of the following equivalent sums.

$$n \sum_{d|n} \mu(d)/d = \sum_{d|n} n\mu(d)/d$$

$$= \sum_{d|n} d\mu(n/d) = \sum_{cd=n} c\mu(d).$$

<u>Theorem 2.9</u>.

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$$

and

$$\sum_{d|n} |\mu(d)| = 2^r$$

where r is the number of distinct prime factors of n.

The function defined by the first sum is usually designated by ε. Thus, hereafter, $\varepsilon(n) = \sum_{d|n} \mu(d)$. The function defined by the second sum is a member of a class of functions which is discussed later.

The unique role of μ in the theory of multiplicative functions is expressed in a relationship called the Mobius inversion formula. It is given by the next theorem.

Theorem 2.10. The functions g and f are related by

$$g(n) = \sum_{d|n} f(d)$$

if and only if

$$f(n) = \sum_{d|n} \mu(n/d)g(d) = \sum_{d|n} \mu(d)g(n/d).$$

It is not intended by the previous statements to indicate that Theorem 2.10 applies only to multiplicative functions. Not only does it apply to all functions on the positive integers but there is an extension of this inversion which applies for functions f and g on positive real numbers.

There are other types of inversion involving other functions. However, it has been shown by Satyanarayana (24) that this particular inversion is unique to the Mobius function. This was accomplished by proving independently some of the consequences of Theorem 2.10 and showing that if Theorem 2.10 defined a function μ^*, then $\mu^* = \mu$.

There are some well-known functions of positive integers whose function values at n depend on the divisors of n.

Definition 2.5. For any real number k, the function σ_k is the function such that $\sigma_k(n)$ is the sum of the kth powers of the divisors of n.

That is,

$$\sigma_k(n) = \sum_{d|n} d^k.$$

Since n^k is multiplicative it is easy to see that, by Theorem 2.2, σ_k is also multiplicative.

Two special cases are particularly useful. If $k = 1$, note that $\sigma_k(n)$ gives the sum of the divisors of n. This function is denoted by σ. If

$$k = 0,\ \sigma_k(n) = \sum_{d|n} 1,$$

and it is seen that σ_0 simply counts the number of divisors of n. The function σ_0 is usually designated by τ and is often called the tau-function.

Theorem 2.11 gives formulas for $\tau(n)$, $\sigma(n)$ and $\sigma_k(n)$ based on the standard form of n. It may be observed that, in this form, the formula for $\tau(n)$ cannot be derived by taking $k = 0$ in the formula for σ_k. However, it may be done taking the limit as k goes to zero.

<u>Theorem 2.11</u>. If $n = \prod_{i=1}^{r} p_i^{a_i}$, then:

$$\text{(i)}\quad \tau(n) = \prod_{i=1}^{r} (a_i+1)$$

$$\text{(ii)}\quad \sigma(n) = \prod_{i=1}^{r} \left(\frac{p_i^{a_i+1}-1}{p_i-1}\right)$$

$$\text{(iii)}\quad \sigma_k(n) = \prod_{i=1}^{r} \left(\frac{p_i^{(a_i+1)k}-1}{p_i^k - 1}\right).$$

The functions defined by

$$f_k(n) = \sum_{d|n} \sigma_k(d)$$

are not usually considered in elementary number theory. Since functions created in this manner are an important part of the theory of generating functions, the formula for $f_0(n) = \sum_{d|n} \tau(n)$ is derived here.

Let

$$f_0(n) = \sum_{d|n} \tau(n).$$

Then,

$$\begin{aligned} f_0(p^a) &= \tau(1) + \tau(p) + \tau(p^2) + \ldots + \tau(p^a) \\ &= 1 + 2 + 3 + \ldots + (a+1) \\ &= \frac{(a+1)(a+2)}{2}. \end{aligned}$$

Thus, since f_0 is multiplicative by Theorem 2.2,

$$\begin{aligned} f_0(n) &= f_0\left(p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}\right) \\ &= f_0\left(p_1^{a_1}\right) f_0\left(p_2^{a_2}\right) \ldots f_0\left(p_r^{a_r}\right) \\ &= (a_1+1)(a_1+2)(a_2+1)(a_2+2)\ldots(a_r+1)(a_r+2)2^{-r}. \end{aligned}$$

Another function which occurs frequently in the study of the theory of numbers is the lambda function (sometimes called Liouville's function).

<u>Definition 2.6.</u> If

$$n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r},$$

let

$$\lambda(n) = (-1)^q, \text{ where } q = a_1 + a_2 + \ldots + a_r.$$

From this definition, it is apparent that q is the total number of primes, distinct or not, in the standard form of n and that the sign of $\lambda(n)$ tells whether this is even or odd.

It is a fact that λ, as given by Definition 2.6, is a member of a special class of multiplicative functions which is discussed in the following theorem.

Theorem 2.12. If k is any non-zero real number and ν is a function on the positive integers such that $\nu(mn) = \nu(m) + \nu(n)$ whenever $(m,n) = 1$, then the function defined by $h_k(n) = k^{\nu(n)}$ is a multiplicative function of n. If $\nu(mn) = \nu(m) + \nu(n)$ for all m and n, then h_k is completely multiplicative.

Proof:

$$\begin{aligned} h_k(mn) &= k^{\nu(mn)} \\ &= k^{\nu(m) + \nu(n)} \\ &= k^{\nu(m)} k^{\nu(n)} \\ &= h_k(m)h_k(n). \end{aligned}$$

The function λ is completely multiplicative and is thus an example of the last statement in the theorem. An example of the first type may be seen by letting $k = 2$ and $\nu(n) = r$ where

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}.$$

Thus $h_2(n) = 2^r$ and $h_2(1) = 1$ since $\nu(1) = 0$. This function could be considered as one measure of the "compositeness" of n. However, $g(n) = 2^q$ where $q = a_1 + a_2 + \ldots + a_r$ would probably be a better measure.

The next, and last, function to be considered is a multiplicative function which has application in the study of prime numbers. It is really one of a class of multiplicative function called group characters. This function, designated here by χ, is one of the simpler non-trivial group characters. The study of group characters, as such, has no place in this discussion. However, since they are multiplicative functions, it is appropriate to include a representative. For a complete but concise development of group characters, Lectures on Elementary Number Theory, by Hans Radamacher, is recommended.

Definition 2.7. For natural numbers n, let χ be the function defined by

$$\chi(n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases}$$

It is important to notice that if n is in the sequence S such that $s_k = 4k + 1$, then $\chi(n) = 1$ and if n is in the sequence T such that $t_k = 4k + 3$, then $\chi(n) = -1$. In fact, this statement is an alternate form of the definition and is used in Chapter III. It may also be seen that

$$\sum_{d|n} \chi(d) = d_1(n) - d_3(n)$$

where $d_1(n)$ is the number of divisors of n that are in S and $d_3(n)$ is the number of divisors of n that are in T.

The purpose of this section was to give a representative collection of number-theoretic functions. Others which have not been given here are defined as the need arises. Properties of most functions used in the paper are stated briefly in the Appendix.

The Convolution Product of Multiplicative Functions

Earlier the ordinary products of arithmetic functions were discussed. There is a different kind of product which is more naturally related to the study of generating functions. It is called the convolution product (sometimes the Dirichlet product) of two arithmetic functions. Convolution products have been the subject of much study. Lehmer (15) and Carlitz (2) published papers in 1931 and 1964, respectively, which deal with convolution products.

The treatment given here is largely due to Shockly (26). His recent text is the only elementary text that this writer has seen that covers convolution products. In his seventh chapter, he shows that the set of all arithmetic functions is a ring, with identity, under the operations of ordinary addition of functions and convolution multiplication. He shows also that the multiplicative functions are a subset which is closed under convolution multiplication and is, in fact, a group under that operation.

This last statement is the fact that makes convolution products important in this study. It is really the basis for one method of finding generating functions (Theorems 3.18 and 3.19).

Definition 2.8. Let α and β be arithmetic functions. The function $\alpha \otimes \beta$ called the convolution product of α and β, is defined by

$$(\alpha \otimes \beta)(n) = \sum_{d|n} \alpha(d)\beta(n/d).$$

Two examples of convolution multiplication have already been seen. If $\alpha(n) = 1$ for every n, then

$$(\alpha \otimes \beta)(n) = \sum_{d|n} \beta(n/d) = \sum_{d|n} \beta(d)$$

which was the sum considered in Theorem 2.2. The other example was seen in connection with the Möbius inversion formula. The reader will have ample opportunity to see other examples where products are actually computed.

It is almost obvious that convolution multiplication is commutative and associative. While the definition, and these properties as well, hold for all arithmetic functions, the most important fact for this study is that the multiplicative functions form a subset of the arithmetic functions which is closed under convolution multiplication.

Theorem 2.13. If α and β are multiplicative functions then so is $\alpha \otimes \beta$.

Proof: If $(m,n) = 1$ and $d|mn$ it is easily seen that d can be written as $d = st$ where $s|m$ and $t|n$. Also, $(s,t) = (m/s,n/t) = 1$. Thus,

$$(\alpha \otimes \beta)(mn) = \sum_{d|mn} \alpha(d)\beta(mn/d) = \sum_{\substack{s|m \\ t|n}} \alpha(st)\beta(m/s\cdot n/t).$$

Since α and β are multiplicative, this sum is equal to $\sum_{s|m} \alpha(s)\beta(m/s)\alpha(t)\beta(n/t)$ which is equal to the product of

$$\sum_{s|m} \alpha(s)\beta(m/s) \text{ and } \sum_{t|n} \alpha(t)\beta(n/t).$$

However, these are $(\alpha \otimes \beta)(m)$ $(\alpha \otimes \beta)(n)$, respectively. Thus

$$(\alpha \otimes \beta)(mn) = (\alpha \otimes \beta)(m)\cdot(\alpha \otimes \beta)(n)$$

for $(m,n) = 1$, and $\alpha \otimes \beta$ is multiplicative.

It has been observed already that μ has a unique place in the theory of multiplicative functions because of its use in the Möbius inversion formula. Recall that

$$\sum_{d|n} \mu(d) = \varepsilon(n) = \begin{cases} 1, & \text{if } n = 1. \\ 0, & \text{if } n > 1. \end{cases}$$

Theorem 2.14. The function ε is the identity for convolution multiplication. That is

$$\alpha \otimes \varepsilon = \varepsilon \otimes \alpha = \alpha$$

for any arithmetic function α.

Since the identity for convolution multiplication is not the same as the identity for ordinary multiplication, it should be expected that, if an inverse exists for α, it is not $1/\alpha$. The following theorem shows that the convolution inverse exists if $\alpha(1) \neq 0$. Note that an explicit expression for α^{-1} is given in the proof.

Theorem 2.15. If $\alpha(1) \neq 0$, then the arithmetic function α has a convolution inverse.

Proof: Let $\alpha(1) \neq 0$. Define the function β by,

$$\beta(1) = 1/\alpha(1),$$

$$\beta(n) = -1/\alpha(1) \sum_{\substack{d|n \\ d<n}} \beta(d)\alpha(n/d), \text{ if } n > 1.$$

The following shows that $\beta = \alpha^{-1}$ where α^{-1} means, in this discussion, the convolution inverse of α.

If $n = 1$, $(\beta \otimes \alpha)(1) = \beta(1)\alpha(1) = 1 = \varepsilon(1)$.

If $n > 1$, then

$$(\beta \otimes \alpha)(n) = \sum_{d|n} \beta(d)\alpha(n/d) = \sum_{\substack{d|n \\ d<n}} \beta(d)\alpha(n/d) + \beta(n)\alpha(1)$$

$$= \sum_{\substack{d|n \\ d<n}} \beta(d)\alpha(n/d) + \{ \frac{-1}{\alpha(1)} \sum_{\substack{d|n \\ d|n}} \beta(d)\alpha(n/d) \}\alpha(1)$$

$$= 0 = \varepsilon(n).$$

Hence,

$$\alpha \otimes \beta = \beta \otimes \alpha = \varepsilon$$

and $\beta = \alpha^{-1}$.

Theorem 2.16. If α is a multiplicative function, $\alpha \neq 0$ then α^{-1} exists and is multiplicative.

Proof: Since $\alpha(1) = 1$, by Theorem 2.1, the inverse exists, by Theorem 2.15, and is defined by $\alpha^{-1}(1) = 1$, $\alpha^{-1}(n) = -\sum_{\substack{d|n \\ d<n}} \alpha^{-1}(d)\alpha(n/d)$ if $n > 1$.

The proof that α^{-1} is multiplicative is relatively simple but, because of the computations involved, is very long. A short outline of the proof follows. The computations which are omitted are very similar to those in the proofs of Theorems 2.13 and 2.15.

It can be verified for a finite number of cases that if $(a,b) = 1$,

$$\alpha^{-1}(ab) = \alpha^{-1}(a)\alpha^{-1}(b).$$

If this is not true for all positive integers a and b, there is a pair $(m,n) = 1$ such that mn is the smallest product with relatively prime factors, m and n, and

$$\alpha^{-1}(mn) \neq \alpha^{-1}(m)\alpha^{-1}(n)$$

That is, if $cd < mn$ and $(c,d) = 1$, then the equality holds. Using this fact and the fact that if $(m,n) = 1$ and $d|mn$, then d can be written as $d = st$ where $s|m$ and $t|n$, it is possible to show that

$$\alpha^{-1}(m)\alpha^{-1}(n) - \alpha^{-1}(mn) = 0.$$

The computation depends on the definition of α^{-1} and the fact $st < mn$ implies

$$\alpha^{-1}(st) = \alpha^{-1}(s)\alpha^{-1}(t)$$

where s and t are given above. Thus,

$$\alpha^{1}(m)\alpha^{-1}(n) = \alpha^{-1}(mn),$$

contrary to the assumption, and it is impossible to choose a first product mn as was done. Hence, α^{-1} is multiplicative.

Using Theorem 2.16 and the construction in the proof of Theorem 2.15, it is possible to find $\alpha^{-1}(p^a)$ and, using Theorem 2.5, to find

$\alpha^{-1}(n)$. It is interesting that, since

$$\sum_{d|n} \mu(d) = \varepsilon(n),\ \mu^{-1}(n) = 1$$

for every natural number n. That is, the identity function for ordinary multiplication is the inverse of the Möbius function under convolution multiplication.

Although not all of them are referred to directly, these last four theorems are the basis for the conclusions drawn in the last section of Chapter III.

The following theorem shows that the identity is the only function which is its own inverse.

<u>Theorem 2.17.</u> If α is its own inverse under convolution multiplication, then $\alpha = \varepsilon$.

Proof: Suppose $\alpha \otimes \alpha = \varepsilon$. Then

$$\varepsilon(n) = \sum_{d|n} \alpha(d)\alpha(n/d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1. \end{cases}$$

Since α is multiplicative, $\alpha(1) = 1$. Thus,

$$\varepsilon(p) = \alpha(1)\alpha(p) + \alpha(p)\alpha(1) = 0$$

implies

$$2\alpha(p) = 0 \text{ or } \alpha(p) = 0.$$

Suppose

$$\alpha(p^i) = 0 \text{ when } i < a.$$

Then

$$\begin{aligned} \varepsilon(p^a) &= \alpha(1)\alpha(p^a) + \alpha(p)\alpha(p^{a-1}) + \dots + \alpha(p^a)\alpha(1) \\ &= 2\alpha(p^a) = 0. \end{aligned}$$

Or,

$$\alpha(p^a) = 0 \text{ for all } a > 0.$$

Hence, also,

$$\alpha(n) = 0 \text{ if } n > 1.$$

Therefore $\alpha = \varepsilon$.

Composition of Multiplicative Functions

It would be very useful if the ordinary composition of two multiplicative functions were again a multiplicative function. This, however, is not the case. The purpose of this section is to examine some types of functions whose compositions are multiplicative.

First, to verify that not all compositions of multiplicative functions are multiplicative, consider the following example.

Note that $\phi(21) = \phi(3)\phi(7) = 2 \cdot 6 = 12$. Then,

$$(\tau \circ \phi)(21) = \tau(12) = 6.$$

But,

$$(\tau \circ \phi)(3) \; (\tau \circ \phi)(7) = \tau(2)\tau(6) = 2 \cdot 4 = 8.$$

Thus,

$$(\tau \circ \phi)(21) \neq (\tau \circ \phi)(3) \; (\tau \circ \phi)(7).$$

Hence, τ and ϕ are multiplicative but $\tau \circ \phi$ is not.

There are examples of compositions of functions which are multiplicative. It will be recalled from elementary algebra that

$$|ab| = |a||b|$$

for all integers a and b. Thus,

$$|\mu(mn)| = |\mu(m)\mu(n)| = |\mu(m)||\mu(n)|.$$

Hence $|\mu|$ is multiplicative. This is an example of the following theorem.

Theorem 2.18. If α is completely multiplicative and β is a multiplicative function such that the range of β is a subset of the domain of α, then their composit $\alpha \circ \beta$ is also a multiplicative function. If β is completely multiplicative then so is $\alpha \circ \beta$.

Proof: With the conditions as stated for α and β, and $(m,n) = 1$,

$$(\alpha \circ \beta)(mn) = \alpha[\beta(mn)] = \alpha[\beta(m)\beta(n)]$$
$$= \alpha[\beta(m)]\alpha[\beta(n)] = (\alpha \circ \beta)(m)\cdot(\alpha \circ \beta)(n).$$

If β is completely multiplicative the same proof holds for all m and n, thus $\alpha \circ \beta$ is completely multiplicative.

There are other functions whose composites are multiplicative. If $f(n) = n^2$ for all n, then $\tau \circ f$ is multiplicative. Since

$$(\tau \circ f)n = \tau(n^2),$$
$$(\tau \circ f)(mn) = \tau[(mn)^2] = \tau(m^2n^2).$$

But, if $(m,n) = 1$, then $(m^2,n^2) = 1$ and

$$\tau(m^2n^2) = \tau(m^2)\tau(n^2).$$

Hence,

$$(\tau \circ f)(mn) = (\tau \circ f)m\cdot(\tau \circ f)(n)$$

when $(m,n) = 1$ and $\tau \circ f$ is multiplicative.

It is obvious that this example does not satisfy the hypothesis of Theorem 2.18. However, it is included in the following theorem.

Theorem 2.19. Let α be a multiplicative function. If β is a multiplicative function such that

$$(\beta(m),\beta(n)) = 1$$

whenever $(m,n) = 1$, and the range of β is a subset of the domain of α, then $\alpha \circ \beta$ is a multiplicative function.

Proof: With the given conditions,

$$\begin{aligned}(\alpha \circ \beta)(mn) &= \alpha[\beta(mn)] \\ &= \alpha[\beta(m)\beta(n)] \\ &= \alpha[\beta(m)]\alpha[\beta(n)] \\ &= (\alpha \circ \beta)(m)\cdot(\alpha \circ \beta)(n).\end{aligned}$$

Hence, $\alpha \circ \beta$ is a multiplicative function.

It is possible that these two theorems do not cover all available cases. However, for the purposes of this study, they are sufficient. It is only necessary that one know that there is a sufficient number of examples so that general discussion is merited. The reader should be able to see other examples which satisfy the hypotheses of these theorems.

Dirichlet Series

The discussion of generating functions in the chapters that follow uses certain elements of analysis. Although the formal properties of series are essentially all that is needed it seems advisable to state for the reader the theorems from analysis which are applicable and which sometimes make a proof easier.

Definition 2.9. A Dirichlet series is a series of the form

$$\sum_{n=1}^{\infty} a(n)n^{-s}.$$

The variable s is usually considered complex and, when the series converges, it converges in some half plane to a complex valued function of s. In order to make the discussion appear more elementary, attention will be restricted to those complex s which lie on the real line. Thus, in

$$\sum_{n=1}^{\infty} a(n)n^{-s},$$

the reader may consider s a real number and the series will converge to a real number for $s > s_0$.

Hardy and Wright (12) explain the role of Dirichlet series as generating functions in the following manner.

> The theory of Dirichlet series, when studied seriously for its own sake, involves many delicate questions of confergence. These are mostly irrelevant here, since we are concerned primarily with the formal side of the theory; and most of our results could be proved . . . without the use of any theorem of analysis or even the notion of the sum of an infinite series. There are however, some theorems which must be considered as theorems of analysis; and, even when this is not so, the reader will probably find it easier to think of the series which occur as sums in the ordinary analytical sense.

One of the tools needed is the operation called the formal product of Dirichlet series. The formal product of two Dirichlet series is formed by taking all possible products of the terms of one series with the terms of the other and combining powers of n^{-s}. Thus, if

$$\left(\sum_{u=1}^{\infty} a(u)u^{-s}\right) \cdot \left(\sum_{n=1}^{\infty} b(v)v^{-s}\right) = \sum_{n=1}^{\infty} c(n)n^{-s},$$

by collecting coefficients on the left side

$$c(n) = \sum_{uv=n} a(u)b(v) = \sum_{d|n} a(d)b(n/d) = \sum_{d|n} a(n/d)b(d).$$

The following theorem shows the uniqueness of coefficients in Dirichlet series.

<u>Theorem 2.20</u>. If

$$\sum_{n=1}^{\infty} a(n)n^{-s} \text{ and } \sum_{n=1}^{\infty} b(n)n^{-s}$$

converge to the same function in some region, then

$$a(n) = b(n).$$

The convergence is necessary for an analytic proof of the theorem. However, in the formal sense this would be taken as the definition of equality of the series and convergence is not a factor.

Formal products may be extended to a finite number of series. The formal product of the series

$$\sum_{u=1}^{\infty} a(u)u^{-s}, \quad \sum_{v=1}^{\infty} b(v)v^{-s}, \quad \sum_{w=1}^{\infty} c(w)w^{-s}, \ \ldots$$

is, then,

$$\sum_{n=1}^{\infty} y(n)n^{-s}$$

where

$$y(n) = \sum_{uvw\ldots=n} a(u)b(v)c(w)\ldots .$$

It is possible under certain circumstances to extend the definition of formal product to an infinite set of series. For this purpose, suppose that $a(1) = b(1) = c(1) = \ldots = 1$ so that the term $a(u)b(v)c(w)\ldots$ contains only a finite number of factors which are not 1. Then, $y(n)$ is the same as given in the finite case. This holds if the series are absolutely convergent or, in the formal sense, if the order of multiplication has been specified.

A most important theorem, as far as generating functions are concerned, is derived by using a formal product of series. First, let f be a multiplicative function and recall that $f(1) = 1$. Take the collection of all series of the form

$$1 + f(p)p^{-s} + f(p^2)p^{-2s} + \ldots + f(p^a)p^{-as} + \ldots$$

for p a prime. For example, if $p = 2$, then $a(u) = f(2^a)$ when $u = 2^a$ and is zero otherwise. If the series are multiplied, in the natural order of the primes, then, by the Fundamental Theorem of Arithmetic, each n occurs just once as a product $n = uvw \ldots$, and

$$y(n) = f(p_1^{a_1})f(p_2^{a_2}) \dots f(p_r^{a_r}) = f(n)$$

when

$$n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}.$$

Since the order of multiplication is specified and $y(n)$ reduces to a single term, no question of convergence arises. This proves the following theorem.

Theorem 2.21. If $f(n)$ is multiplicative then

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p \{1+f(p)p^{-s}+f(p^2)p^{-2s} + \dots + f(p^a)p^{-as} + \dots\}$$

where the product is taken over primes in the natural order.

A similar theorem which depends on the absolute convergence of the series can be proved. However, since it is desirable that questions of convergence be avoided, Theorem 2.21 is sufficient here.

The simplest of the Dirichlet series is

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

It is convergent for $s > 1$, and its sum $\zeta(s)$ is called the Riemann zeta function. Volumes have been written involving this function and it is not the purpose here to delve deeply into theory concerning it. It is a fact that one of the outstanding unsolved problems of mathematics is the location of its zeros. In this paper, it is mostly a useful tool. The next theorem could be considered a corollary to Theorem 2.21.

Theorem 2.22. If $s > 1$ then $\zeta(s) = \prod_p (1-p^{-s})^{-1}$.

Of course, by Theorem. 2.19

$$\sum_{n=1}^{\infty} n^{-s} = \prod_p \{1 + p^{-s} + p^{-2s} + \dots\}$$

without considering convergence.

Another Dirichlet series which is often encountered is given by

$$L(s) = 1^{-s} - 3^{-s} + 5^{-s} - \dots .$$

It can be seen that

$$L(s) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

It is also possible to write L in product form as;

$$L(s) = \prod_{p} \frac{1}{1-\chi(p)p^{-s}}$$

or;

$$L(s) = \prod_{q} \frac{1}{1-q^{-s}} \prod_{r} \frac{1}{1-r-s}$$

where

$$q \equiv 1 \pmod 4 \text{ and } r \equiv 3 \pmod 4.$$

Before closing this chapter it is necessary that some comments be made concerning notation and nonemclature.

When

$$F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$$

is written it usually means that the series converges to a function of s. In this paper, both the symbol and the name function are used somewhat loosely. The symbol, and the name generating function, will be intended to convey the idea that there are some values of s for which the series converges to F(s). However, it should later be obvious to the reader that when the proper order of terms is chosen the series generates f even if it does not converge. Sometimes the series is referred to as the generating series. The two designations are almost interchangeable, however, generating series is often used to indicate that F is in series rather than product form. The symbolism also provides a convenient name, such as ζ, L, or F by which to refer to the series.

CHAPTER III

GENERATING FUNCTIONS FOR MULTIPLICATIVE ARITHMETIC FUNCTIONS: ZETA FUNCTIONS AND RELATED FUNCTIONS

Introduction to Generating Functions

From the discussion in Chapter II, it may be observed that some properties of a multiplicative arithemtic function are attainable from its definition or elementary formla. There are questions concerning such functions which are not easily answered from the elementary theory. The purpose of the present chapter is not to raise and answer such questions but to examine one tool used by mathematicians in doing so. This tool is the generating function.

The first study of generating functions is attributed to Euler. According to Dickson (7), the study of partitions led Euler to discover the first generating functions. The greatest usefulness of such functions has probably been in the additive theory of numbers. However, the impetus provided by their study has led to developments in the theory of multiplicative functions as well. Their usefulness extends also to the theory of complex variables.

Writers in the field have found difficulty in giving a precise definition of "generating function". Some have attempted a definition; others give a discussion only; still others define it to be whatever type function is useful at the moment; and the remainder assume that the reader is fully acquainted with them so that no definition is necessary.

Vaidyanathaswamy (30) gives the following somewhat confusing definition:

The "generator" of a multiplicative function f(N) is a function f(x,z) of two arguments, such that $f(p,z) = f(p)z + f(p^2)z^2 + \ldots$. The "generating function" F(x,z) or f(N) is defined by $F(x,z) = 1 + f(x,z)$.

Hardy and Wright (12) handle the idea of a generating function in the following manner. Their first approach is thus:

A Dirichlet Series is a series of the form

$$F\,(s) = \sum_{n=1}^{\infty} \alpha(n)/n^s$$

The variable s may be real or complex, but here we shall be concerned with real values only. F (s), the sum of the series, is called the generating function of α_n.

This would lead one to believe that they intended that all generating functions should be Dirichlet series. However, this is not the case for later in the same chapter (12) the following discussion is added.

The generating functions discussed in this chapter have been defined by Dirichlet series; but any function

$$F\,(s) = \sum \alpha_n\, \mu_n\,(s)$$

may be regarded as a generating function of α_n. The most usual form of $\mu_n(s)$ is

$$\mu_n(s) = e^{-\lambda_n s}$$

Where λ_n is a sequence of positive numbers which increases steadily to infinity. The most important cases are the cases $\lambda_n = \log n$ and $\lambda_n = n$. When $\lambda_n = \log n$ $\mu_n(s) = n^{-s}$ and the series is a Dirichlet series. When $\lambda_n = n$, it is a power series in $x = e^{-s}$.

Though something may be lacking in the way of mathematical precision, this approach at least yields a general form for generating functions. This lack may even be good. For, a too restrictive definition might so delimit the concept as to make it useless.

The power series is of little use in the theory of multiplicative functions. Its usefulness is restricted chiefly to the additive theory and particularly to the theory of partitions. Power series do arise in the multiplicative theory, but, as will be seen in Chapter IV, their role is secondary.

The present chapter will be concerned with generating functions which involve Dirichlet series and series (or infinite products) which are related to them. In Chapter IV another type of generating function will be considered.

Generating Functions of Some Well-Known Arithmetic Functions

The generating functions included in this section are well known. They are found in standard works such as those by Hardy and Wright (12), LeVeque (16) or Titchmarsh (29). No attempt is made here to assign them to their orignators. The purpose of including them is threefold: (1) to acquaint the reader with the generating functions of the arithmetic functions which are usually included in an elementary number theory course; (2) to illustrate techniques which will be used later; and (3) to provide the reader with basic formulas from which others may be developed by processes to be shown in this chapter.

Most of the functions considered here have generating functions which are combinations (products and quotients) of zeta functions. In fact, it seems natural that this should be the case for multiplicative functions, since the terms of the zeta function are themselves factorable. That is, if $n = ab$, then $n^{-s} = (ab)^{-s} = a^{-s}b^{-s}$. Hardy observed in an address to the London Mathematical Society (12) that it was natural to use the zeta function in connection with the theory of primes because it was more natural to multiply primes than to add them.

Proofs of some theorems are included to illustrate the techniques. However, to avoid repetition, some are stated without proof.

Theorem 3.1. $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n)n^{-s}$ $(s > 1)$.

Proof: Since $\zeta(s) = \prod_{p} (1-p^{-s})^{-1}$ and $\mu(p^a) = 0$, for $a > 1$,

$$1/\zeta(s) = \prod_{p} (1-p^{-s}) = \prod_{p} \{1+\mu(p)p^{-s}+\mu(p^2)p^{-2s}+ \ldots\}.$$

But by Theorem 2.21 this is

$$\sum_{n=1}^{\infty} \mu(n)n^{-s}.$$

At this point, it is desirable to digress from the stated purpose of this section to consider an idea which is inherent, though not obviourly so, in the theory of generating functions. The equation of Theorem 3.1 may be used as a definition for μ.

To see that this statement is a fact, suppose h is a multiplicative arithmetic function, defined on P, such that

$$1/\zeta(s) = \sum_{n=1}^{\infty} h(n)n^{-s}.$$

Then, by Theorem 2.20, $h(n) = \mu(n)$ for every $n \in P$.

Another property of μ may be seen from the following:

$$\zeta(s) \quad 1/\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \cdot \sum_{n=1}^{\infty} \mu(n)n^{-s}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{d|n} \mu(d) \right) n^{-s} = 1$$

Hence,

$$\sum_{d|n} \mu(d) = 1, \text{ if } n = 1, \text{ and } 0, \text{ if } n > 1.$$

But this formula for this sum is the same as that derived in Theorem 2.9.

The other generating functions mentioned in this paper lend themselves to the same treatment. However, the elementary formulas are not always easily derived. This method of defining the multiplicative arithmetic functions will therefore probably not replace the usual methods. It should be mentioned, though, that for the purpose of applications this type of definition is sometimes used.

<u>Theorem 3.2.</u>

$$\zeta(s - 1)/\zeta(s) = \sum_{n=1}^{\infty} \phi(n)n^{-s} \quad (s > 2).$$

Proof: Use Theorem 3.1 and multiply the Dirichlet series to get:

$$\zeta(s-1)/\zeta(s) = \sum_{n=1}^{\infty} n^{1-s} \sum_{n=1}^{\infty} \mu(n)n^{-s} = \sum_{n=1}^{\infty} n \cdot n^{-s} \sum_{n=1}^{\infty} \mu(n)n^{-s}$$

$$= \sum_{n=1}^{\infty} h(n)n^{-s}, \text{ where } h(n) = \sum_{d|n} d\mu(n/d).$$

Thus, $h(n) = \phi(n)$ and

$$\zeta(s-1)/\zeta(s) = \sum_{n=1}^{\infty} \phi(n)n^{-s}.$$

It will be seen in the next chapter that $\phi(n)$ also has a generating function of another type. Thus, by the uniqueness theorem (Theorem 2.20) the arithmetic function generated by a given generating function is unique but the generating function for a given arithmetic function may not be unique. In fact, since for an arithmetic function f, f(n) is determined by the nth term of the series, this uniqueness must be present before a series can be a generating function.

<u>Theorem 3.3.</u> $\zeta(s)\zeta(s-k) = \sum_{n=1}^{\infty} \sigma_k(n)n^{-s}$ $(s > k + 1)$.

Proof: As in the proof of Theorem 3.2, $\zeta(s)\zeta(s-k)$

$$= \sum_{n=1}^{\infty} n^{-s} \sum_{n=1}^{\infty} n^{k} \cdot n^{-s} = \sum_{n=1}^{\infty} n^{-s} \sum_{d|n} d^{k} = \sum_{n=1}^{\infty} \sigma_k(n)n^{-s}.$$

<u>Corollary 3.4.</u> $\zeta^2(s) = \sum_{n=1}^{\infty} \tau(n)n^{-s}$ $(s > 1)$.

Proof: Let $k = 0$ in Theorem 3.3.

<u>Corollary 3.5.</u> $\zeta(s)\zeta(s-1) = \sum_{n=1}^{\infty} \sigma(n)n^{-s}$ $(s > 2)$.

Proof: Let $k = 1$ in Theorem 3.3.

Corollary 3.4 can also be arrived at from another direction. If $d_k(n)$ is the number of ways of expressing n as the product of k factors where the same factors in a different order are counted as different, then note that $d_2(n) = \tau(n)$. Then Corollary 3.4 is also a corollary of the following theorem.

<u>Theorem 3.6.</u> $\zeta^k(s) = \sum_{n=1}^{\infty} d_k(n)n^{-s}$.

Proof: $\zeta^k(s) = \left\{ \sum_{n=1}^{\infty} n^{-s} \right\}^k = \sum_{n=1}^{\infty} n^{-s} \sum_{d_1 \cdots d_k = n} 1 = \sum_{n=1}^{\infty} d_k(n)n^{-s}.$

The following theorems are given here because of their relation to those already given. They are related to later work as well because they are simple examples of methods of "discovering" functions through the use of generating functions. However, reference will be made to this section at the appropriate time in order to remind the reader of their importance.

Theorem 3.7. $$\zeta(s)/\zeta(2s) = \sum_{n=1}^{\infty} |\mu(n)| n^{-s}.$$

Proof:
$$\zeta(s)/\zeta(2s) = \prod_p \left(\frac{1-p^{-2s}}{1-p^{-s}}\right) = \prod_p (1 + p^{-s}).$$

Recall the proof of Theorem 3.1 and notice that p^{-s} has the same coefficient except that it is positive. Thus the function generated here is the same except that all signs are positive. Therefore,
$$\zeta(s)/\zeta(2s) = \sum_{n=1}^{\infty} |\mu(n)| n^{-s}.$$

It is easily seen that $\{\mu(n)\}^2 = |\mu(n)|$ and, in fact, $\{\mu(n)\}^{2k} = |\mu(n)|$ when k is a natural number. Also, $\{\mu(n)\}^{2k+1} = \mu(n)$ for k any natural number. Thus, by Theorems 3.1 and 3.7, all positive integral powers of μ are generated by these same series.

Theorems 3.8 and 3.9 were given by Titchmarsh (29) who proved them in a manner similar to the proof for Theorem 3.7. The proofs used here will be referred to later as examples of a more general case.

Theorem 3.8.
$$\zeta^3(s)/\zeta(2s) = \sum_{n=1}^{\infty} \tau(n^2) n^{-s} \quad (s > 1).$$

Proof: By Corollary 3.4, $\zeta^2(s) = \sum_{n=1}^{\infty} \tau(n) n^{-s}$ $(s > 1)$

and, by Theorem 3.7, $\zeta(s)/\zeta(2s) = \sum_{n=1}^{\infty} |\mu(n)| n^{-s}$ $(s > 1)$.

Thus,

$$\begin{aligned}
\zeta^3(s)/\zeta(2s) &= \zeta^2(s) \cdot \zeta(s)/\zeta(2s) \\
&= \sum_{n=1}^{\infty} \tau(n)n^{-s} \cdot \sum_{n=1}^{\infty} |\mu(n)\ n^{-s} \\
&= \sum_{n=1}^{\infty} h(n)n^{-s}
\end{aligned}$$

where

$$h(n) = \sum_{d|n} |\mu(d)|\tau(n/d).$$

Recall that $h(n)$ is multiplicative and that $h(1) = 1$. Then, since

$$\begin{aligned}
|\mu(p^a)| &= 0 \text{ for } a > 1,\ h(p^a) \\
&= \sum_{d|p^a} |\mu(d)|\tau(p^a/d) \\
&= |\mu(1)|\tau(p^a) + |\mu(p)|\tau(p^{a-1}) \\
&= \tau(p^a) + \tau(p^{a-1}) \\
&= (a + 1) + a = 2a + 1 = \tau(p^{2a}).
\end{aligned}$$

Thus, if $n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$, then

$$\begin{aligned}
h(n) &= h\left(p_1^{a_1}\right) h\left(p_2^{a_2}\right) \ldots h\left(p_r^{a_r}\right) \\
&= \left(p_1^{2a_1}\right) \left(p_2^{2a_2}\right) \ldots \tau\left(p_r^{2a_r}\right) \\
&= \tau\left(p_1^{2a_1}\right) \tau\left(p_2^{2a_2}\right) \ldots \tau\left(p_r^{2a_r}\right) = \tau(n^2).
\end{aligned}$$

Hence,

$$\zeta^3(s)/\zeta(2s) = \sum_{n=1}^{\infty} \tau(n^2)n^{-s}.$$

<u>Theorem 3.9.</u>

$$\zeta^4(s)/\zeta(2s) = \sum_{n=1}^{\infty} \{\tau(n)\}^2\ n^{-s} \quad (s > 1).$$

Proof: Multiply $\zeta(s) \cdot \zeta^3(s)/\zeta(2s)$ in their series form.

Other Generating Functions Involving Zeta Functions

There are many other less well-known functions of number theory which have generating functions defined by Dirichlet series. Although these are not as often used in elementary number theory, some of them warrant consideration from a historical point of view while others are of a practical nature and relate handily to discussions in this chapter.

The multiplicative functions of the form $k^{\nu(n)}$, where $k \neq 0$ and $\nu(n)$ is an arithmetic function such that

$$\nu(mn) = \nu(m) + \nu(n),$$

were discussed in Chapter II. Two special functions mentioned where $\lambda(n)$ and one such that $k = 2$, $\nu(n) = r$ where

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}.$$

Theorem 3.10.

$$\zeta(2s)/\zeta(s) = \sum_{n=1}^{\infty} \lambda(n)n^{-s} \quad (s > 1).$$

Proof:

$$\zeta(2s)/\zeta(s) = \prod_p \left(\frac{1-p^{-s}}{1-p^{-2s}} \right) \quad \text{by Theorem 2.21.}$$

Thus,

$$\zeta(2s)/\zeta(s) = \prod_p (1 + \frac{1}{p})^{-1} = \prod_p (1 - \frac{1}{p^s} + \frac{1}{p^{2s}} \cdots)$$

$$= \sum_{n=1}^{\infty} (-1)^q n^{-s} = \sum_{n=1}^{\infty} \lambda(n)n^{-s}$$

where q is the total number of prime factors of n.

Theorem 3.11.

$$\zeta(s)/\zeta(2s) = \sum_{n=1}^{\infty} 2^r n^{-s} \quad (s > 1)$$

where $n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$.

Proof. By Theorem 3.7,

$$\zeta^2(s)/\zeta(2s) = \zeta(s) \cdot \zeta(s)/\zeta(2s)$$
$$= \sum_{n=1}^{\infty} n^{-s} \sum_{n=1}^{\infty} |\mu(n)| n^{-s}.$$

Thus,

$$\zeta^2(s)/\zeta(2s) = \sum_{n=1}^{\infty} n^{-s} \sum_{d|n} |\mu(n)| .$$

If $g(n) = \sum_{d|n} |\mu(n)|$, then $g(1) = 1$ and

$$g(p^a) = |\mu(1)| + |\mu(p)| + |\mu(p^2)|$$
$$+ \ldots + |\mu(p^a)| = 1 + 1 + 0 + \ldots + 0 = 2.$$

Then,

$$g(n) = g(p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r})$$

and since g is multiplicative,

$$g(n) = g(p_1^{a_1}) g(p_2^{a_2}) \ldots g(p_r^{a_r}) = 2^r.$$

Thus,

$$\zeta^2(s)/\zeta(2s) = \sum_{n=1}^{\infty} 2^r n^{-s}.$$

Theorem 3.12 is a more general theorem of the same nature as Theorem 3.7. In fact, Theorem 3.12 reduces to Theorem 3.7 for $k = 2$.

Theorem 3.12. If $q_k(n) = 0$ when n has a kth power, $k > 1$, as a factor and $q_k(n) = 1$ otherwise, then,

$$\zeta(s)/\zeta(ks) = \sum_{n=1}^{\infty} q_k(n) n^{-s}. \quad (s > 1)$$

Proof: The proof is essentially the same as the proof of Theorem 3.7.

It is implicit that $k > 1$ for if $k = 1$, then q_k is the identity function. This gives the obvious fallacy $\zeta(s)/\zeta(s) = \zeta(s)$. This would mean $\zeta(s) = 1$ for every s such that $\zeta(s) \neq 0$. However, it is known that $\zeta(2) = \pi^2/6$.

The great Indian, Ramanujan, discovered several generating functions (23). It is interesting that he found them "incidentally in the course of other investigations". Also interesting is that he generally handled series with a casual disregard for convergence. In fact, many of his earlier proofs were wrong for that reason, but G. H. Hardy, to whom he communicated them, was surprised to find many of his calculations correct. As was usual in his day, his original proofs were not published with the results. However, his methods were employed by others who followed him and by his teacher, G. H. Hardy, as well.

The first theorem from Ramanujan is an example of a generating function for the ordinary product of two arithmetic functions. It should be recalled that the generating functions for $\sigma_a(n)$ and $\sigma_b(n)$ are given by Theorem 3.3. The proof used here was given by Titchmarsh (29), who also specified the region of convergence.

<u>Theorem 3.13.</u>

$$\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)/\zeta(2s-a-b)$$
$$= \sum_{n=1}^{\infty} \sigma_a(n)\sigma_b(n)\ n^{-s}$$
$$(s > \max\{1,\ a+1,\ b+1,\ a+b+1\}).$$

Proof: By Theorem 2.21 the left-hand side becomes

$$\prod_p \frac{1-p^{-2s+a+b}}{(1-p^{-s})(1-p^{-s+a})(1-p^{-s+b})(1-p^{-s+a+b})}$$

and with $z = p^{-s}$ this is then

$$\prod_p \frac{1-p^{a+b}z^2}{(1-z)(1-p^a z)(1-p^b z)(1-p^{a+b}z)} \ .$$

The fraction can then be broken down by the method of partial fractions to give:

$$\frac{1}{(1-p^a)(1-p^b)} \left\{ \frac{1}{1-z} - \frac{p^a}{1-p^a z} - \frac{p^b}{1-p^b z} + \frac{p^{a+b}}{1-p^{a+b}z} \right\} .$$

Then, write each fraction in the braces in series form and combine to get:

$$\frac{1}{(1-p^a)(1-p^b)} \sum_{m=0}^{\infty} (1-p^{(m+1)a}-p^{(m+1)b}+p^{(m+1)(a+b)})z^m$$

$$= \frac{1}{(1-p^a)(1=p^b)} \sum_{m=0}^{\infty} (1-p^{(m+1)a})(1-p^{(m+1)b})z^m.$$

Hence:

$$\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)/\zeta(2s-a-b)$$

$$= \prod_p \sum_{m=0}^{\infty} \frac{1-p^{(m+1)a}}{1-p^a} \cdot \frac{1-p^{(m+1)b}}{1-p^b} \cdot \frac{1}{p^{ms}}$$

$$= \prod_p \sum_{m=0}^{\infty} \sigma_a(p^m)\sigma_b(p^m)p^{-ms}$$

$$= \sum_{n=1}^{\infty} \sigma_a(n)\sigma_b(n)n^{-s}$$

by applying the formula for σ_k and Theorem 2.21 in that order.

It is reasonable to expect that a theorem as this would have some interesting special cases. If $a = b = 0$ then Theorem 3.9 is a special case. Some other special cases are included in the following corollaries.

Corollary 3.14.

$$\zeta^2(s)\zeta^2(s-1)/\zeta(2s-1)$$
$$= \sum_{n=1}^{\infty} \tau(n)\sigma(n)n^{-s} \qquad (s > 2).$$

Proof: Let a = 0, b = 1 in the theorem.

Corollary 3.15.

$$\zeta(s)\zeta^2(s-1)\zeta(s-2)/\zeta(2s-2)$$
$$= \sum_{n=1}^{\infty} \sigma^2(n)n^{-s} \qquad (s > 3).$$

Proof: Let a = b = 1 in the theorem.

The next theorem was also "found" by Ramanujan (23). It is included here without proof. A method by which it and others like it could be discovered, will be included in the final section of this chapter.

Theorem 3.16.

$$L(s)L(s-a)L(s-b)L(s-a-b)/(1-2^{-2s+a+b})\zeta(2s-a-b)$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1}\sigma_a(2n-1)\sigma_b(2n-1)(2n-1)^{-s}.$$

Corollary 3.17.

$$L^4(s)/(1-2^{-2s})\zeta(2s)$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1}\tau^2(2n-1)(2n-1)^{-s}.$$

It is a fact that the coefficients of the terms of the series for L(s) give the multiplicative function $\chi(n)$ defined in Chapter II.

That is,

$$L(s) = \sum_{n=1}^{\infty} \chi(n)n^{-s}.$$

Hence, products formed from L(s) and $\zeta(s)$ generate multiplicative functions. Both L(s) and $\zeta(s)$ are also involved in the study of the additive theory of numbers. In fact, L(s) is the basis for the study of the distribution of primes of the forms 4m + 1 and 4M + 3.

It will be observed that $\zeta(s)$ L(s) generates the multiplicative arithmetic function $\sum_{d|n} \chi(d)$. However, $4\zeta(s)L(s)$ generates the function r(n) from additive theory where r(n) is defined to be the number of representations of n as the sum of two squares. These representations include squares of all integers and different orders are counted as different.

The proof of the above statement requires theory which has not been developed. For a discussion of r(n) and a proof, the reader is referred to Chapter XVI and Chapter XVII of The Theory of Numbers by G. H. Hardy and E. M. Wright.

Some Classes of Multiplicative Arithmetic Functions and Methods of Finding Their Generating Functions

The foregoing sections of this chapter would almost lead one to believe that generating series were discovered by accident. The reader might even get the feeling that one just takes some combination of ζ or L, multiplies or divides them, and waits to see what happens. It is a fact that this would produce a multiplicative arithmetic function and give its generating function. However, it is desirable to have a more systematic approach.

Unfortunately, there is such a wide range of arithmetic functions that it is impossible to classify them all. Vaidyanathaswamy (29) attempted to classify the elementary functions. However, his classifications seem to be of a little use in this study. The purpose here is to discuss some methods of finding generating functions and to extend these methods.

Indications of some possibilities have already been seen. For example, it is evident that Theorem 3.3 gives a generating function for an infinite class of arithmetic functions. However, Theorem 3.13 would indicate that this is merely a subclass of a much larger class of functions which are generated by the series in Theorem 3.13. By taking a different direction, it can be seen that

$$A_k(n) = \sum_{d|n} \sigma_k(d)$$

is also a multiplicative function of n and that if F(s) generates σ_k then $\zeta(s)F(s)$ generates A_k. In fact, the general case is given in the following theorem.

<u>Theorem 3.18.</u> If

$$F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$$

is the generating series for the multiplicative function f, then $\zeta(s)F(s)$ gives the generating series for the multiplicative function A defined by $A(n) = \sum_{d|n} f(d)$.

In Theorem 2.13 it was found that the convolution product of two multiplicative arithmetic functions is multiplicative. That is, if B is the arithmetic function defined by

$$B(n) = \sum_{d|n} f(d)g(n/d)$$

where f and g are multiplicative, then B is multiplicative. If f and g are functions whose generating series are known, it is possible to find the generating series for B. In fact, an example of this was seen in the proof of Theorem 3.8 where a method of determining the function values of B was also used. The next theorem covers the general case.

If g(n) = 1 for all n, Theorem 3.19 is a special case of Theorem 3.19.

Theorem 3.19. If

$$F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$$

and

$$G(s) = \sum_{n=1}^{\infty} g(n)n^{-s},$$

then

$$F(s)G(s) = \sum_{n=1}^{\infty} B(n)n^{-s},$$

where

$$B(n) = \sum_{d|n} f(d)g(n/d).$$

Proof: The proof follows by the Dirichlet multiplication of the two series.

The generating function found by using Theorem 3.19 depends only on those chosen for f and g. If $\varepsilon(n)$ is 1, if $n = 1$, and 0, if $n > 1$, ε is the identity of convolution multiplication and no new generating functions are found by Theorem 3.19, if $g = \varepsilon$ or $f = \varepsilon$.

In Theorems 3.2 and 3.3 the functions $\zeta(s-1)$ and $\zeta(s-k)$ were used. It should be noted that $h(n) = n^k$ defines a completely multiplicative function and that

$$\zeta(s-k) = \sum_{n=1}^{\infty} n^{k-s} = \sum_{n=1}^{\infty} n^k n^{-s}$$

is its generating function. If $g(n) = n^k$ and $f(n) \neq \varepsilon(n)$, then for a fixed $f(n)$ an infinite class of functions may be created by using Theorem 3.19.

From the foregoing discussion it is evident that many generating functions of multiplicative functions may be found by using the relatively few elementary functions for which the generating functions are known. There are still many questions unanswered.

Two such questions stem from Theorems 3.8, 3.9 and 3.13. In Theorem 3.8, a generating function for $\tau(n^2)$ is given. In Theorems 3.9 and 3.13 generating functions are given for ordinary products of some elementary multiplicative functions whose generating functions are known. The function $\tau(n^2)$ is a special case of the composition of multiplicative functions as described in Chapter II.

Is there a general method of arriving at a generating function for composites and ordinary products? Unfortunately, the answer is not available. Theorems 3.8 and 3.9 were established by essentially the same process. While there is a similarity in the generating function of σ_k (Theorem 3.3) and $\sigma_a \sigma_b$ (Theorem 3.13), there is no obvious way of obtaining the second from the first.

The most promising method seems to be the method used by Nadler (19)(20). It is difficult to determine if this method was originated by Nadler but in the research for this paper it has not been found in the writings of his predecessors. This includes those mentioned in his bibliography. It could actually have been the method used by Ramanujan to find the formulas in Theorems 3.13 and 3.16.

By this method, it is possible to find generating functions for some, if not all, ordinary products and compositions. It should also be applicable to other functions. Essentially it might be described as an inverse for the method used in some of the proofs of this chapter (see Theorem 3.13). Some examples will serve as illustrations.

Example 3.1. Let

$$f(n) = \lambda(n)\tau(n) = (-1)^q\tau(n)$$

where q is the total number of prime factors of n. Suppose that

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p \sum_{t=0}^{\infty} f(p^t)p^{-ts}.$$

Then,

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p \sum_{t=0}^{\infty} (-1)^t(t+1)p^{-ts}$$

$$= \prod_p \{1 - 2p^{-s} + 3p^{-2s} - 4p^{-3s} + 5p^{-4s} - \ldots\}$$

$$= \prod_p \frac{1}{(1+p^{-s})^2} = \prod_p \left(\frac{1-p^{-s}}{1-p^{-2s}}\right)^2 = \frac{\zeta^2(2s)}{\zeta^2(s)} .$$

The formula found in Example 3.1 may be proved by the method used to prove Theorem 3.8 using the result of Theorem 3.10. An interesting side result here is that $\lambda \otimes \lambda = \lambda \cdot \tau$ where $\otimes$ is convolution multiplication and $\cdot$ is ordinary multiplication.

Consider the more complicated example which gives the generating function of $\lambda \cdot h_2$ where λ and h_2 are given in Definition 2.6 and Theorem 2.12 respectively.

Example 3.2. Let $g(n) = (-1)^q 2^r$ where q is the total number of prime factors of n and r is the number of distinct prime factors of n. Then

$$\sum_{n=1}^{\infty} g(n)n^{-s} = \prod_p \sum_{t=0}^{\infty} g(p^t)p^{-ts}$$

gives

$$\sum_{n=1}^{\infty} (-1)^q 2^r n^{-s} = \prod_p \{1-2p^{-s} + 2p^{-2s} - 2p^{-3s} + \ldots\}$$

$$= \prod_p \{1 + \sum_{t=1}^{\infty} (-1)^t 2p^{-ts}$$

$$= \prod_p \{1 + 2\sum_{t=1}^{\infty} \left(\frac{-1}{p^s}\right)^t$$

$$= \prod_p \{1 + 2(1-p^s)/(p^{2s}-1)$$

$$= \prod_p \{1 - 2(1-p^s)/(1-p^{2s})$$

$$= \prod_p (-1 + 2p^s - p^{2s})/(1-p^{2s})$$

$$= \prod_p (1-2p^{-s} + p^{-2s})/(1-p^{-2s})$$

$$= \prod_p (1-p^{-s})^2/(1-p^{-2s}) = \zeta(2s)/\zeta^2(s).$$

The following example due to Nadler (19) not only involves an ordinary product of multiplicative functions but also a composition.

Example 3.3. Define the function ρ_a by

$$\rho_a(n) = n^{-a/2}\sigma_a(n)$$

where a is a real number, and, as usual, n is a natural number. If $a = 0$ then $\rho_a(n) = \tau(n)$. Suppose $a \neq 0$ and $n = \prod_{i=1}^{r} p_i^{t_i}$. From its elementary formula,

$$\sigma_a(n) = \prod_{i=1}^{r} \left(p_i^{(t_i+1)a} - 1\right) / (p_i^a - 1).$$

Also,

$$n^{-a/2} = \prod_{i=1}^{r} p_i^{t_i(-n/2)}.$$

Then

$$\rho_a(n) = n^{-a/2}\sigma_a(n)$$

$$= \prod_{i=1}^{r} \left(p_i^{-t_i a/2} - p_i^{t_i a/2 + a} \right) \Big/ \left(1 - p_i^a \right). \quad \text{(A)}$$

Now, consider the function $\rho_a(n^x)$, where x is a natural number greater than unity. Form the expression

$$\sum_{n=1}^{\infty} \rho_a(n^x) n^{-s} = \prod_p \sum_{t=0}^{\infty} \rho_a(p^{tx}) p^{-ts}$$

and consider the series in the right-hand side. By (A),

$$\sum_{t=0}^{\infty} \rho_a(p^{tx}) p^{-ts} = 1/(1-p^a) \sum_{t=0}^{\infty} (p^{-axt/2} - p^{axt/2+a}) p^{-ts}$$

This latter series may be summed as the difference of two geometric series to get

$$1/(1-p^a)\{1/(1-p^{-ax/2-s}) - p^a/(1-p^{ax/2-s})$$

$$= \frac{(1-p^{ax/2-s})/(1-p^a) - p^a(1-p^{-ax/2-s})/(1-p^a)}{(1-p^{-ax/2-s})(1-p^{ax/2-s})}$$

$$= \frac{(1-p^a)/(1-p^a) + (p^{-ax/2+a-s} - p^{ax/2-s})/(1-p^a)}{(1-p^{-ax/2-s})(1-p^{ax/2-s})}$$

$$= \frac{1 + (p^{-ax/2+a} - p^{ax/2})p^{-s}/(1-p^a)}{(1-p^{-ax/2-s})(1-p^{ax/2-s})}$$

$$= \frac{1 + \rho_a(p^{x-2})p^{-s}}{(1-p^{-ax/2-s})(1-p^{ax/2-s})} \quad . \qquad (B)$$

Thus, using (B),

$$\sum_{n=1}^{\infty} \rho_a(n^x)n^{-s} = \prod_p \sum_{t=0}^{\infty} \rho_a(p^{tx})p^{-ts}$$

$$= \prod_p \frac{1 + \rho_a(p^{x-2})p^{-s}}{(1-p^{-(s+ax/2)})(1-p^{-(s-ax/2)})}$$

$$= \zeta(s+ax/2)\zeta(s-ax/2)\prod_p\{1+\rho_a(p^{x-2})p^{-s}. \qquad (C)$$

The special case of $x = 2$ gives

$$\sum_{n=1}^{\infty} \rho_a(n^2)n^{-s} = \zeta(s-a)\zeta(s+a)\zeta(s)/\zeta(2s).$$

The product remaining in (C) is not always a zeta function but it may be observed that

$$\prod_p\{1 + \rho_a(p^{x-2})p^{-s}\} = \sum_{n=1}^{\infty} |\mu(n)|\rho(n^{x-2})n^{-s}.$$

Thus (C) may then be written as a product of Dirichlet series.

Some further results are easily obtained. One of these is that (C) remains the same if $(-a)$ replaces a. This is due to the fact that

$$\rho_{-a}(n) = \rho_a(n).$$

If

$$\overline{\rho}_a(n) = \rho_{2a}(n)/\rho_a(n)$$

then a formula similar to (C) exists for $\overline{\rho}_a$, of which a special case is,

$$\sum_{n=1}^{\infty} \overline{\rho}_a(n^2)n^{-s} = \frac{\zeta(s+a)\zeta(s-a)}{\zeta(s)} \quad .$$

Nadler (19), (20) also showed that the same technique, with a slight alteration, could be applied to yield functions involving L(s). Essentially, the only change necessary is to write

$$L(s) = \prod_{q}(1-q^{-s})^{-1} \prod_{r}(1-r^{-s})^{-1}$$

where q and r are primes such that $q \equiv 1 \pmod 4$ and $r \equiv 3 \pmod 4$.

Using this method Nadler (19) developed a formula for $\rho_a\ (2n+1)^x$ which is similar to (C) above. A special case which can be proved by the methods used earlier in this chapter is given by the formla,

$$\sum_{n=0}^{\infty} (-1)^n \rho_a\{(2n+1)^2\}(2n-1)^{-s} = L(s)L(s+a)1(s-a)/(1-2^{-2s})\zeta(2s).$$

Two other special cases of more complicated formulas found by Nadler (20) are of interest. They may also be proved by methods used earlier.

They are:

$$\sum_{n=1}^{\infty} \sigma_a(n^2)n^{-s} = \zeta(s)\zeta(s-a)\zeta(s-2a)/\zeta(2s-2a)$$

and

$$\sum_{n=1}^{\infty} \chi(n)\sigma_a(n^2) = L(s)L(s-a)L(s-2a)/(1-2^{2a-2})\zeta(2s-2a).$$

It is evident that by using this method, a product representation of the generating function can be obtained. If the series in the product representation can be summed as in the Examples 3.1, 3.2, and 3.3 the generating function can be written as a product of Dirichlet series. However, as pointed out earlier, these Dirichlet series are not always zeta series or L series.

CHAPTER IV

GENERATING FUNCTIONS INVOLVING LAMBERT SERIES

Some Lambert Series and the Associated Power Series

A problem of recent interest concerns generating functions of a type other than Dirichlet series.

It will be recalled that Hardy and Wright (12) were quoted, in Chapter III, as proposing that any function

$$F(s) = \sum \alpha_n u_n(s)$$

might be regarded as a generating function for α_n. If

$$u_n(s) = \frac{e^{-ns}}{1 - e^{-ns}}$$

and

$$x = e^{-s},$$

then

$$u_n(s) = \frac{x^n}{1 - x^n} .$$

Thus, a function F defined by

$$F(x) = \sum_{n=1}^{\infty} \frac{a(n)x^n}{1 - x^n}$$

may be regarded as the generating function of a(n). A series of the form

$$\sum_{n=1}^{\infty} \frac{a(n)x^n}{1 - x^n}$$

is called a Lambert series.

The following theorem gives a useful property of Lambert series.

Theorem 4.1. If

$$F(x) = \sum_{n=1}^{\infty} \frac{a(n)x^n}{1 - x^n} \quad \text{and} \quad b(N) = \sum_{n|N} a(n),$$

then

$$F(x) = \sum_{n=1}^{\infty} b(N)x^n.$$

Proof:

$$\frac{x^n}{1 - x^n} = (x^n + x^{2n} + x^{3n} + \ldots),$$

thus

$$F(x) = \sum_{n=1}^{\infty} a(n) \sum_{m=1}^{\infty} x^{mn} = \sum_{n=1}^{\infty} b(N)x^n$$

where

$$b(N) = \sum_{n|N} a(n).$$

This relation between a(n) and b(N) is the same as that considered earlier in connection with the Dirichlet series. In fact, the following theorem shows that the entire relationship expressed in Theorem 4.1 is equivalent to

$$\zeta(s)f(s) = g(s)$$

where f(s) and g(s) are the Dirichlet series associated with a(n) and b(N), respectively. Dickson (7) attributes this theorem to Cesaro.

Theorem 4.2. If

$$f(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$$

and

$$g(s) = \sum_{n=1}^{\infty} b(n)n^{-s}$$

then

$$F(x) = \sum_{n=1}^{\infty} \frac{a(n)x^n}{1 - x^n} = \sum_{n=1}^{\infty} b(n)x^n$$

if and only if

$$\zeta(s)f(s) = g(s).$$

Proof: If

$$\zeta(s)f(s) = g(s)$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \sum_{n=1}^{\infty} \frac{b(n)}{n^s} .$$

If the series are multiplied as usual,

$$\sum_{n=1}^{\infty} \frac{\gamma(n)}{n^s} = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

where

$$\gamma(n) = \sum_{d|n} a(d).$$

Thus, by uniqueness,

$$b(n) = \gamma(n) = \sum_{d|n} a(d).$$

If

$$F(x) = \sum_{n=1}^{\infty} \frac{a(n)x^n}{1 - x^n}$$

and

$$b(n) = \sum_{d|n} a(d)$$

the conditions in the hypothesis of Theorem 4.1 are met, thus

$$F(x) = \sum_{n=1}^{\infty} b(n)x^n.$$

If the conditions in the hypothesis of Theorem 4.1 are met then

$$b(n) = \sum_{d|n} a(d).$$

The steps of the proof may then be reversed to prove that

$$\zeta(s)f(s) = g(s).$$

It has been shown that if a(n) is a multiplicative arithmetic function then

$$b(n) = \sum_{d|n} a(d)$$

is also multiplicative. Not all such functions have been explicitly defined in the theory. However, there are many examples where both a(n) and $\sum_{d|n} a(d)$ define well-known arithmetic functions.

The following examples show the relationship implied by Theorems 4.1 and 4.2 in the case of a few well-known arithmetic functions.

Example 4.1. If

$$a(n) = \mu(n),$$

then

$$b(n) = \sum_{d|n} \mu(d).$$

Since b(n) = 1 if n = 1 and b(n) = 0 if n > 1,

$$\sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1 - x^n} = \sum_{n=1}^{\infty} [\sum_{d|n} \mu(d)]x^n = x.$$

Example 4.2. Since

$$n = \sum_{d|n} \phi(d),$$

$$\sum_{n=1}^{\infty} \frac{\phi(n)x^n}{1 - x^n} = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1 - x)^2}.$$

then

$$b(n) = \sum_{d|n} d^k = \sigma_k(n).$$

Thus,

$$\sum_{n=1}^{\infty} \frac{n^k x^n}{1 = x^n} = \sum_{n=1}^{\infty} \sigma_k(n)x^n.$$

Two special cases of this example were reported by Dickson (7). If k = 1, then

$$\sum_{n=1}^{\infty} \frac{nx^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma(n)x^n$$

and if k = 0, then

$$\sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} \tau(n)x^n.$$

Since a(n) is multiplicative if and only if b(n) is multiplicative, it is apparent that for each Lambert series which generates a multiplicative function there exists a corresponding power series. One notes also that if a(n) is given, then

$$b(n) = \sum_{d|n} a(n)$$

and by the Möbius inversion formula if b(n) is given then

$$a(n) = \sum_{d|n} \mu(d)b(n/d).$$

Suppose that

$$\sum_{n=1}^{\infty} \frac{h(n)x^n}{1 - x^n} = \sum_{n=1}^{\infty} \lambda(n)x^n$$

where $\lambda(n) = (-1)^q$ and q represents the total number of prime factors of n.

Then, $\lambda(n) = \sum_{d|n} h(d)$ and h may be determined from this relationship.

Since

$$h(n) = \sum_{d|n} \mu(d)\lambda(n/d), \; h(p^a) = \sum_{d|p^a} \mu(d)\lambda(n/d)$$

$$= \mu(1)\lambda(p^a) + \mu(p)\lambda(p^{a-1}) + \mu(p^2)\lambda(p^{a-2}) + \ldots + \mu(p^a)\lambda(1)$$

$$= (-1)^a + (-1)(-1)^{a-1}$$

$$= (-1)^a 2.$$

If

$$n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r},$$

then

$$h(n) = h\left(p_1^{a_1}\right) h\left(p_2^{a_2}\right) \ldots h\left(p_r^{a_r}\right)$$

since h is multiplicative. Thus,

$$h(n) = \{(-1)^{a_1 2}\}\{(-1)^{a_2 2}\} \ldots \{(-1)^{a_r 2}\}$$

$$= (-1)^{a_1 + a_2 + \ldots + a_r 2^r}.$$

But,

$$a_1 + a_2 + \ldots + a_r = q$$

and r is the number of distinct prime factors of n. Hence

$$h(n) = (-1)^q 2^r$$

where q is the total number of distinct prime factors of n.

Classes of Arithmetic Functions from Designated Power Series

Generalization of some of the theorems of the last section produces classes of functions which are of some interest in themselves. The first theorem of this section derives a class of arithmetic functions by choosing $\sigma_k(an)$ as the coefficients of the associated power series. The second theorem shows how this method may be used to apply to other functions. Other theorems of the section characterize the functions found and develop theory needed for a further generalization.

Harris and Warren (13) proved the following theorem concerning $\sigma_k(an)$. In this theorem and in all theorems and discussions in the remainder of the present chapter, r and s are used as follows: r is the largest factor of a for which $(r,n) = 1$ and $a = rs$.

Theorem 4.3. If $f_k(a,n)$ is an arithmetic function,

$$\sum_{n=1}^{\infty} \frac{f_k(a,n)x^n}{(1 - x^n)} = \sum_{n=1}^{\infty} \sigma_k(an)x^n$$

if and only if

$$f_k(a,n) = s^k\sigma_k(r)n^k.$$

The proof is not included since this theorem becomes a special case of the next theorem. It was given here as a theorem for two reasons. First the proof given by Subba Rao (28) for the more general theorem was patterned after that given for this one. Second, further reference to it will be made in examples which occur later in the chapter.

It should be observed that Example 4.3 is really a special case of Theorem 4.3. If $a = 1$, then

$$f_k(a,n) = 1^k\sigma_k(1)n^k = n^k$$

and the result is exactly Example 4.3.

The following theorem is a generalization of Theorem 4.3.

Theorem 4.4. Let g and h be multiplicative functions defined on P such that

$$h(n) = \sum_{d|n} g(d)$$

and let r and s be as given for Theorem 4.3. Let $f(a,n)$ be the arithmetic function defined by the relation

$$\sum_{n=1}^{\infty} \frac{f(a,n)x^n}{(1 - x^n)} = \sum_{n=1}^{\infty} h(an)x^n$$

then

$$f(a,n) = h(r)g(sn).$$

Proof. Since, by Theorem 4.1,

$$h(an) = \sum_{d|n} f(a,n),$$

$$f(a,n) = \sum_{d|n} h(an/d)\ \ (d),$$

by the Möbius inversion formula, with r and s defined as stated, s has no factors except those which are already factors of n. Hence any divisor of sn which is not a divisor of n has a square factor. If d is such a divisor then $\mu(d) = 0$. Thus,

$$\sum_{d|n} h(an/d)\mu(d) = \sum_{d|sn} h(an/d)\mu(d).$$

Since $(r,n) = 1$ and h is multiplicative,

$$\begin{aligned}\sum_{d|sn} h(an/d)\mu(d) &= \sum_{d|sn} h(r\cdot sn/d)\mu(d)\\ &= \sum_{d|sn} h(r)h(sn/d)\mu(d)\\ &= h(r) \sum_{d|sn} h(sn/d)\mu(d).\end{aligned}$$

But, by the Möbius inversion formula the latter sum is just $g(sn)$. Thus,

$$f(a,n) = h(r)g(sn)$$

as claimed.

The converse is also true. It may be established by reversing the steps of the proof.

Further insight into the nature of $f(a,n)$ and $f_k(a,n)$ of Theorems 4.4 and 4.3 may be gained from the following theorems.

Theorem 4.5. Let

$$F_k(a,n) = f_k(a,n)/n^k$$

where $f_k(a,n)$ is defined by

$$\sum_{n=1}^{\infty} \frac{f_k(a,n)x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_k(an)x^n.$$

Then, $F_k(a,n)$ is periodic in n with least period P where P is the product of the distinct prime factors of a.

Proof: The proof is given for the more general case.

Theorem 4.6. Let g(n) be a positive valued and unconditionally multiplicative function of n, so that

$$g(m,n) = g(m)g(n)$$

for all positive integers m and n. Then, the function

$$k(a,n) = f(a,n)/g(n)$$

is periodic in n with least period P, where f(a,n) is defined in Theorem 4.4 and P is the product of the distinct prime factors of a.

Proof: First, it must be shown that if b is any factor of a such that $(b,n) = 1$, then $(b,n + P) = 1$, and conversely.

If $(b,n + P) \neq 1$, then there exists a prime p_1 such that $p_1 | b$ and $p_1 | n + P$, But $p_1 | n + P$ means that

$$n + P = cp_1$$

and

$$n = cp_1 - P.$$

However, b is a factor of a, thus $p_1 | b$ implies that $p_1 | P$. Hence,

$$n = cp_1 - P'p_1 = p_1(c-P') \text{ and } p_1 | n.$$

Since $p_1 | b$ and $p_1 | n$, then $(b,n) \neq 1$ contrary to assumption. Therefore, if $(b,n) = 1$ then $(b,n + P) = 1$.

If $(b,n + P) = 1$, then, since every prime divisor of b also divides P, no prime divisor of b divides n. Thus, $(b,n) = 1$. By the fact just proved, r and s are the same if n is replaced by n + P.

$$\frac{f(a,n)}{g(n)} = \frac{h(r)g(sn)}{g(n)} = \frac{h(r)g(s)g(n)}{g(n)} = h(r)g(s).$$

This follows from Theorem 4.4 and the fact that g is completely multiplicative. Therefore,

$$\frac{f(a,n+P)}{g(n+P)} = h(r)g(s) = \frac{f(a,n)}{g(n)} \quad \text{and } k(a,n)$$

has least period less than or equal to P.

If R is any period then

$$k(a,n) = k(a,n+R)$$

for all n. If n = a, then

$$k(a,a) = h(1)g(a).$$

Let

$$k(a,a + r) = h(t)g(u)$$

where t is the largest factor of a such that $(t,a + R) = 1$ and $a = tu$. If R is a period, then

$$h(1)g(a) = h(t)g(u).$$

But, h is multiplicative so that $h(1) = 1$; therefore,

$$\begin{aligned} h(t)g(u) &= g(a) \\ &= g(tu) = g(t)g(u). \end{aligned}$$

Hence, $h(t) = g(t)$; or

$$g(t) = h(t) = \sum_{d|t} g(d).$$

This holds only if $t = 1$. If $t = 1$, then $(a, a + R) = a$ and every prime factor of a is a factor of R. Thus $P|R$ and $k(a,n)$ has least period P.

Since $g(n) = n^k$ is a positive valued completely multiplicative function it may be seen that Theorem 4.5 is a special case of Theorem 4.6.

It will be seen later, in the examples, that some values of the function F_k of Theorem 4.5 occur several times in each period. One such value is $\sigma_k(a)$. The following proposition which was conjectured and proved by this writer, shows that the number of times that $\sigma_k(a)$ occurs depends on the period.

Theorem 4.7. Let F_k be defined as in Theorem 4.5. Then F_k assumes the value $\sigma_k(a)$ exactly $\phi(P)$ times per period.

Proof: If $(a,m) = 1$, then $r = a$ and $s = 1$. By Theorem 4.3

$$f_k(a,m)/m^k = s^k\sigma_k(r) = \sigma_k(a).$$

If $m \leq P$, where p is the product of the distinct primes that divide a, then $(a,m) = 1$ if and only if $(P,m) = 1$. But, $(P,m) = 1$ exactly $\phi(P)$ times as $m\varepsilon\{1, 2, \ldots P\}$. Thus

$$F_k(a,m) = \sigma_k(a)$$

at least $\phi(P)$ times in the first period, hence, in every period.

To show that

$$F_k(a,m) = \sigma_k(a)$$

no more than $\phi(P)$ times per period, it will suffice to show that if $s \neq 1$ then

$$s^k\sigma_k(r) \neq \sigma_k(a).$$

In fact, it is shown that when $a = rs$ and $s \neq 1$,

$$s^k\sigma_k(r) < \sigma(rs).$$

If $r = 1$, then

$$s^k < 1 + s^k \leq \sigma_k(s).$$

If $r \neq 1$, let $1, d_1, d_2, \ldots, d_k, r$ be the divisors of r. Then,

$s, d_1 s, \ldots, d_k s$ and rs are all distinct divisors of rs. However, there are other divisors of rs for $r \neq 1$ and $s \neq 1$ implies that 1 is not among those listed and neither is r. Then,

$$s^k \sigma_k(r) = s^k \sum_{d|n} d^k = s^k(1 + d_1^k + d_2^k + \ldots + d_t^k + r^k)$$

$$= s^k + s^k d_1^k + s^k d_2^k + \ldots + s^k d_t^k + s^k r^k.$$

And,

$$\sigma_k(rs) = s^k + s^k d_1^k + s^k d_1^k + s^k d_2^k + \ldots + s^k d_t^k + s^k r^k + 1 + \sum D^k$$

where $\sum D^k$ is the sum of the kth powers of all other divisors of rs and hence a positive quantity. Then, comparing the two values,

$$s^k \sigma_k(r) < \sigma_k(rs) = \sigma_k(a).$$

Thus, F_k assumes the value $\sigma_k(a)$ no more than $\phi(P)$ times per period.

The next theorem and its corollary are based on Theorems 4.3 and 4.5. They initially appear to add little to the theory. However, they result indirectly in the generalization of Theorem 4.5 to numbers of the form an + b. Also, they lead to a novel identity which proves useful in computation of one of the derived arithmetic functions.

Before the theorem is proved, a definition and a lemma are needed.

Definition 4.1. If $\mu(n)$ is the Möbius function, let

$$\overline{\mu}(m/a) = \begin{cases} \mu(m/a), & \text{if } a | m \\ 0, & \text{if } a \nmid m. \end{cases}$$

Lemma 4.8. (i) If a divides m and μ is the Mobius function, then

$$\sum_{d|(m/a)} \mu(d) = \begin{cases} 1, & \text{if } m = a \\ 0, & \text{if } m > a. \end{cases}$$

Proof: Since m/a = 1 for m = a and m/a > a, this is equivalent to summing the Mobius function over the divisors of n = m/a.

(ii) If $\overline{\mu}$ is as defined, then

$$\sum_{m=1}^{\infty} \overline{\mu}(m/a)x^m/(1-x^m) = x^a.$$

Proof: By Theorem 4.1,

$$\sum_{m=1}^{\infty} \mu(m/a)x^m/(1-x^m) = \sum_{m=1}^{\infty} \{\sum_{d|m} \overline{\mu}(d/a)\}x^m$$

which, by Definition 4.1, becomes

$$\sum_{\substack{m=1 \\ a|m}}^{\infty} \{\sum_{d|m/a} \mu(d)\}x^m.$$

By Lemma 4.8 (i),

$$\sum_{\substack{m=1 \\ a|m}}^{\infty} \{\sum_{d|(m/a)} \mu(d)\}x^m = x^a.$$

The conclusion follows by combining these results.

Theorem 4.9. Let

$$g_k(t;a,m) = f_k(a,m) - \sum_{j=1}^{t} \sigma_k(ja)\overline{\mu}(m/j)$$

with the usual restriction that the sum is zero if t = 0.

Then,

$$\sum_{m=1}^{\infty} \frac{g_k(t;a,m)x^m}{1-x^m} = \sum_{m=t+1}^{\infty} \sigma_k(am)x^m.$$

Proof: Since

$$\sum_{m=t+1}^{\infty} \sigma_k(am)x^m = \sum_{m=1}^{\infty} \sigma_k(am)x^m - \sum_{j=1}^{t} \sigma_k(ja)x^j$$

then by Theorem 4.3

$$\sum_{m=1}^{\infty} \sigma_k(am)x^m = \sum_{m=1}^{\infty} \frac{f_k(a,m)x^m}{(1-x^m)}$$

and, (1)

$$\sum_{m=t+1}^{\infty} \sigma_k(am)x^m = \sum_{m=1}^{\infty} \frac{f_k(a,m)x^m}{(1-x^m)} - \sum_{j=1}^{t} \sigma_k(aj)x^j .$$

By Lemma 4.8 (ii),

$$x^j = \sum_{m=1}^{\infty} \frac{\overline{\mu}(m/j)x^m}{(1-x^m)} .$$

Thus, substitute in the second term of (1) to get,

$$\sum_{m=t+1}^{\infty} \sigma_k(am)x^m = \sum_{m=1}^{\infty} \frac{f_k(a,m)x^m}{1-x^m} - \sum_{j=1}^{t} \sigma_k(aj) \sum_{m=1}^{\infty} \frac{\overline{\mu}(m/j)x^m}{1-x^m}$$

$$= \sum_{m=1}^{\infty} \frac{f_k(a,m)x^m}{1-x^m} - \sum_{j=1}^{t} \sum_{m=1}^{\infty} \frac{\sigma_k(aj)\overline{\mu}(m/j)x^m}{1-x^m}$$

$$= \sum_{m=1}^{\infty} \frac{f_k(a,m)x^m}{1-x^m} - \sum_{m=1}^{\infty} \frac{\sum_{j=1}^{t} \sigma_k(aj)\overline{\mu}(m/j)x^m}{1-x^m}$$

$$= \sum_{m=1}^{\infty} \frac{g_k(t;a,m)x^m}{1-x^m} .$$

<u>Corollary 4.10</u>. If $m \leq t$ then $g_k(t;a,m) = 0$. Hence,

$$\sum_{m=t+1}^{\infty} \frac{g_k(t;a,m)x^m}{1-x^m} = \sum_{m=t+1}^{\infty} \sigma_k(am)x^m .$$

Proof: If $m \leq t$ then, by Definition 4.1,

$$\sum_{j=1}^{t} \sigma_k(ja)\overline{\mu}(m/j) = \sum_{d|m} \sigma_k(ad)\mu(m/d)$$

$$= \sum_{d|n} \sigma_k\left(\frac{am}{d}\right) \mu(d)$$

$$= f_k(a,m) .$$

Thus

$$g_k(t;a,m) = f_k(a,m) - f_k(a,m) = 0.$$

The final statement of the corollary is then immediate since the first t terms of

$$\sum_{m=1}^{\infty} \frac{g_k(t;a,m)x^m}{1-x^m}$$

are then identically zero.

The following theorem establishes the identity which was mentioned earlier.

Theorem 4.11. Let

$$\delta_k(a,m) = \begin{cases} 0 \text{ if } m \leq t \\ \sigma_k(am) \text{ if } m \geq t. \end{cases}$$

Then

$$\sum_{d|m} g_k(t;a,d) = \delta_k(a,m).$$

Proof: By Theorem 4.9 and the definition of δ_k,

$$\sum_{m=1}^{\infty} \frac{g_k(t;a,m)x^m}{1-x^m} = \sum_{m=1}^{\infty} \delta_k(a,m)x^m$$

But, by Theorem 4.1,

$$\delta_k(a,m) = \sum_{d|m} g_k(t;a,d).$$

The novelty of this identity lies in the fact that $g_k(t;a,m)$ depends heavily on t, whereas, its sum over the divisors of m depends almost entirely on a and m.

Some Computational Examples

Some examples will serve to illustrate the theorems of this chapter.

The first two examples illustrate Theorem 4.11. It should first be noted that Theorem 4.11 reduces considerably the amount of computation involved in finding the sum of $g_k(t;a,m)$ over the divisors of m. Second, the two examples together emphasize the fact that, once m is taken greater than t, the sum no longer depends on t.

Example 4.4. Let

$$a = 9,\ k = 1,\ t = 5 \text{ and } m = 15,$$

then by Theorem 4.11,

$$\sum_{d|15} g_1(t;9,d) = \sigma(135) = \sigma(5)\sigma(27) = 6(40) = 240.$$

If the sum is computed directly the computation is as follows. As a result of Corollary 4.8,

$$\sum_{d|15} g_1(5,9,d) = \sum_{\substack{d|15 \\ d\geq 6}} g_1(t;9,d) = g_1(5,9,15).$$

The definition of g_1 then gives

$$\begin{aligned} g_1(5;9,15) &= f_1(9,15) - \sum_{j=1}^{5} \sigma(9j)\overline{\mu}(15/j) \\ &= 9(15) - [\sigma(9) - \sigma(27) - \sigma(45)] \\ &= 9(15) - \sigma(9) + \sigma(27) + \sigma(45) \\ &= 135 - 13 + 40 + 78 \\ &= 240 = \sigma(135). \end{aligned}$$

Example 4.5. If t = 4 and a, k, and m remain the same as in Example 4.4, then

$$\sum_{d|15} g_1(4,9,d) = \sigma(135)$$

by Theorem 4.11.

If the direct method is used, the computation is as follows.

$$\sum_{d|15} g_1(4;9,15) = \sum_{\substack{d|15 \\ d \geq 5}} g_1(4;9,d)$$

$$= g_1(4;9,5) + g_1(4;9,15)$$

$$= [f(9,5) - \sum_{j=1}^{4} \sigma(9j)\overline{\mu}(5/j)]$$

$$+ [f(9,15) - \sum_{j=1}^{4} \sigma(9j)\overline{\mu}(15/j)]$$

$$= 5\sigma(9) - [-\sigma(9)] + 9(15) - [\sigma(9) - \sigma(27)]$$

$$= 5\sigma(9) + 9(15) + \sigma(27)$$

$$= 5(13) + 9(15) + 40$$

$$= 65 + 135 + 40 = 240$$

$$= \sigma(135).$$

The next example will illustrate the periodicity of $f_k(a,m)/m^k$ which was proved in Theorem 4.5. The results will then be used to show an example of Theorem 4.3. An example where the period is much longer could obviously have been chosen. However, little is to be gained by looking at a longer example.

Example 4.6. Let k = 2 and a = 24, then by Theorem 4.5 the period is p = 6.

If m = 1, then

$$\frac{f_2(24,1)}{1^2} = \sigma_2(24) = 850.$$

$$\text{If } m = 2, \text{ then } \frac{f_2(24,2)}{2^2} = 8^2\ \sigma_2(3) = 64(10) = 640.$$

$$\text{If } m = 3, \text{ then } \frac{f_2(24,3)}{3^2} = 3^2\sigma_2(8) = 9(85) = 765$$

$$\text{If } m = 4, \text{ then } \frac{f_2(24,4)}{4^2} = 8^2\sigma_2(3) = 64(10) = 640.$$

$$\text{If } m = 5, \text{ then } \frac{f_2(24,5)}{5^2} = \sigma_2(24) = 850$$

$$\text{If } m = 6, \text{ then } \frac{f_2(24,6)}{6^2} = (24)^2\ \sigma_2(1) = 24^2 = 576.$$

It was shown in the proof of Theorem 4.5 that r and s are unchanged by adding P (and thus any multiple of P) to m. Thus,

$$\frac{f_2(24,6b+1)}{(6b+1)^2} = \sigma_2(24) = \frac{f_2(24,1)}{1^2}.$$

Or, if $m \equiv 0, 1, 2, 3, 4, 5$

$$\frac{f_2(24,6b+m)}{(6b+m)^2} = s^2\ \sigma_2(r)$$

where r and s are the same as for m in the first period which is computed above.

<u>Example 4.7.</u> If $k = 2$ and $a = 24$, then by Example 4.6,

$$\gamma(n) = \frac{f_2\ (24,n)}{n^2} = \begin{cases} 850, & \text{if } n\equiv 1 \pmod 6 \\ 640, & \text{if } n\equiv 2 \pmod 6 \\ 765, & \text{if } n\equiv 3 \pmod 6 \\ 640, & \text{if } n\equiv 4 \pmod 6 \\ 850, & \text{if } n\equiv 5 \pmod 6 \\ 576, & \text{if } n\equiv 0 \pmod 6). \end{cases}$$

By Theorem 4.3, then

$$\sum_{n=1}^{\infty} \frac{\gamma(n)n^2x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_2(24n)x^n$$

where $\gamma(n)$ is given above.

Harris and Warren (13) uncovered an interesting computational scheme which is based on the theorems of this chapter.

If

$$\sum_{n=1}^{\infty} \frac{f_k(a,n)x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_k(an)x^n,$$

then, by Theorem 4.1,

$$\sigma_k(an) = \sum_{d|n} f_k(a,d).$$

The function $f_k(a,n)$ has now been characterized for all values of a. If a is taken to be a prime p, then by Theorem 4.5, $f_k(p,n)/n^k$ is periodic with least period p. Since

$$f_k(p,n) = s^k\sigma_k(r)n^k,$$

where rs = p, there are just two cases to consider. First, if p divides n, then r = 1 and s = p yields $f_k(p,n) = p^k n^k$. If p does not divide n, then r = p and s = 1 yields

$$f_k(p,n) = \sigma_k(p)n^k = (p^k + 1)n^k.$$

In view of the preceding it is then easy to compute $f_k(p,n)$ and thus to compute $\sum_{d|n} f_k(a,d)$. If $\sigma_k(p)$, $\sigma_k(2p)$, $\sigma_k(3p)$ . . . are computed by this method for each prime p then $\sigma_k(n)$ will be computed at least twice with the exception of n = 1 and n a power of a prime. But $\sigma_k(1) = 1$ and

$$\sigma_k(p^i) = \sigma_k(p\cdot p^{i-1})$$

will be computed exactly once.

The method of computation is illustrated in Table I. If k = 2 the numbers in each column are just $f_2(a,d)$ where d is a divisor of n. In the table a = 3, n = 1, 2, 3, . . ., 12.

TABLE I

an	3	6	9	12	15	18
	10	10	10	10	10	10
		40		40		
			81			
				160		
					250	
						324
σ_2(an)	10	50	91	210	260	455

an	21	24	27	30	33	36
	10	10	10	10	10	10
		40		40		40
			81			81
		160				160
				250		
	440					324
		640				
			729			
				1000		
					1210	
						1296
σ_2(an)	500	850	820	1300	1220	1911

A Generalization to Power Series Where the Powers Are an Arbitrary Arithmetic Progression

It has been seen in previous theorems of this chapter that for $a > 0$ an arithmetical function $f_k(a,n)$ exists such that

$$\sum_{n=1}^{\infty} \frac{f_k(a,n)x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_k(an)x^n.$$

Harris and Warren (13) investigated the question of the existence of a function $f_k(an+b)$ with $a > 0$ and $b > 0$ such that

$$\sum_{n=1}^{\infty} \frac{f_k(an+b)x^{an+b}}{1-x^{an+b}} = \sum_{n=1}^{\infty} \sigma_k(an+b)x^{an+b}.$$

The final two theorems of this chapter show under what conditions the function exists and characterize it when it exists.

Theorem 4.12. If there is an arithmetic function $f_k(an+b)$ with $b > 0$ such that

$$\sum_{n=1}^{\infty} \frac{f_k(an+b)x^{an+b}}{1-x^{an+b}} = \sum_{n=1}^{\infty} \sigma_k(an+b)x^{an+b},$$

then $a = (a,b)$.

Proof: Let $g = (a,b)$ such that $1 = (a/g,\ b/g)$. By Dirichlet's theorem there exists an N_0 and a prime p such that

$$(a/g)N_0 + (b/g) = p.$$

Thus, x^{gp} appears on the right side (in the power series) with a non-zero coefficient.

Write the left side as the double series:

$$\sum_{n=1}^{\infty} f_k(an+b)\ [x^{an+b} + x^{2(an+b)} + x^{3(an+b)} + \ldots] .$$

Note that the coefficient of x^{aN+b} is $\sum f_k(an+b)$ summed all n for which $an+b \mid aN+b$. Since $aN_o + b = gp$, the coefficient of x^{gp} in this form of the left side is $\sum f_k(an+b)$ where the sum is over all n such that $an+b \mid aN_o+b$. Suppose $an+b \mid gp$, then $g = (a,b)$ implies that $(a/g)n + (b/g) \mid p$. But $b > 0$ means that

$$(a/g)n + b/g = p$$

or

$$an + b = gp = aN_o + b.$$

Thus the coefficient of x^{gp} is just $f_k(aN_o+b)$.

Now the coefficient of x^{2gp} in the double series form of the left side is $\sum f_k(an+b)$ summed over all n such that $an+b \mid 2gp$. This includes $n = N_o$ but may include other values of n.

Consider the case where there is an $n \neq N_o$. Then $(a/g)n + (b/g) \mid 2p$ and hence

$$(a/g)n + (b/g) = 2$$

or

$$(a/g)n + (b/g) = p.$$

The second case implies $n = N_o$, so consider the first only. From the first case $n = \frac{2g-b}{a}$, hence $a \mid 2g-b$. But $b = kg$ so $a \mid 2g-kg$ or $a \mid (2-k)g$. Now, $(2-k)g = an$ and $g > 0$ means $2-k > 0$. Hence $k = 1$ and $b = g$ giving also $a = g$.

If N_o is the only n for which $an + b \mid 2gp$, then the coefficient of x^{2gp} in the left side is just $f_k(aN_o+b)$ which is not zero. Hence x^{2gp} must occur on the right side so there is n_1 such that

$$an_1 + b = 2gp.$$

Combine

$$an_1 + b = 2gp$$

and

$$aN_\circ + b = gp$$

to get $a/g\ (n_1 - N_\circ) = p$. Thus $a/g|p$ and $a/g = 1$ or $a/g = p$. The possibility that $a/g = p$ is ruled out by $b/g > 0$. Hence $a = g$.

For all possible cases $a = g$, thus $a = (a,b)$.

Theorem 4.13. If $b = at$ and

$$h_k(t;a,n) = g_k(t;a,n+t)$$

where g_k in the function defined in Theorem 4.9, then

$$\sum_{n=1}^{\infty} \frac{h_k(t;a,n)x^{an+b}}{1-x^{an+b}} = \sum_{n=1}^{\infty} \sigma_k(an+b)x^{an+b}.$$

Proof:

$$\sum_{n=1}^{\infty} \frac{h_k(t;a,n)x^{an+b}}{1-x^{an+b}}$$

$$= \sum_{n=1}^{\infty} \frac{g_k(t;a,n+t)x^{a(n+t)}}{1-x^{a(n+t)}}$$

$$= \sum_{n=t+1}^{\infty} \frac{g_k(t;a,m)y^{m}}{1-y^{m}} \qquad \text{where } y = x^a.$$

By Corollary 4.10,

$$\sum_{m=t+1}^{\infty} \frac{g_k(t;a,m)y^{m}}{1-y^{m}} = \sum_{m=t+1}^{\infty} \sigma_k(am)y^{m}.$$

And, it follows that

$$\sum_{m=t+1}^{\infty} \sigma_k(am)y^{m} = \sum_{n=1}^{\infty} \sigma_k(a(n+t))y^{n+t}$$

$$= \sum_{n=1}^{\infty} \sigma_k(an+b)x^{an+b}.$$

Theorems 4.12 and 4.13 thus say that $f_k(a,m)$ exists and satisfies the conditions given in Theorem 4.13 if and only if a = (a,b). It may also be seen that h_k is a function which meets the requirements for f_k in the hypothesis of Theorem 4.12. Hence a specific function has been exhibited which meets all requirements.

CHAPTER V

CONTRACTIONS OF MULTIPLICATIVE FUNCTIONS

The search for methods of finding multiplicative functions and their generating functions motivated the definition and theorems of this chapter. The ideas involved are simple but lead to a method of deriving infinitely many different functions from a single know function.

A function, called the characteristic function of a set, is used extensively in many other areas of mathematics. The reader, however, need know nothing more than the definition. This function is usually designated by the Greek letter chi which has already been used to designate a number-theoretic function. For this reason, and in the interest of simplicity, a notation is adopted so that, after the initial definitions, chi need not be used.

<u>Definition 5.1</u>. Let S be an arbitrary subset of a universal set U. Then the function defined by

$$\chi_S(m) = 1, \text{ if } m \in S, \text{ and } \chi_S(m) = 0, \text{ if } m \in U - S,$$

is called the characteristic function of S.

In the following definition and in the remainder of the chapter the universal set is always P, the set of positive integers.

<u>Definition 5.2</u>. Let f be a multiplicative function defined on P. Define f_S, the contraction of f to S, as follows:

$$f_S(n) = f(n)\chi_S(n).$$

It should be observed that the domain of f_S is still P. Thus, the definition does not restrict the domain of f to S but merely makes all values zero on P - S. Also it is seen that the set S greatly influences the character of f_S. For example, if S is finite, the range of f_S contains at most a finite set of non-zero elements.

It is desirable, for reasons which are later obvious, to separate the multiplicative functions into two classes. One class will include those functions which have a zero. The other will contain those which have no zeros. For instance, the Mobius function is of the first type and the phi and sigma functions are of the second type. Some theorems relate to one type but not the other and some to either of the two.

The first theorem is a collection of observations which are immediate consequences of the definition. They are stated without proof.

Theorem 5.1. Let f be a multiplicative function having no zero. Then:

(i) if $n \varepsilon S \cap R$, $f_S(n) = f_R(n) \neq 0$;

(ii) if $n \notin S$ and $n \notin R$, $f_S(n) = f_R(n) = 0$;

(iii) if $n \varepsilon S - R$ or $n \varepsilon R - S$, $f_S(n) \neq f_R(n)$;

(iv) if $Q \subset P$ and $R \subset P$, $f_S(Q \cup R) = f_S(Q) \cup f_S(R)$;

(v) if $R = S$ and $Q \subset P$, $f_S(Q) = f_R(Q)$.

If f has a zero the inequalities in Theorem 5.1 (i) and (iii) may or may not hold. Thus if f has a zero those parts of the theorem are invalid.

The next theorem shows the relationship of the ranges when contractions are to two sets and to their union. It holds for the contraction of any multiplicative function.

Theorem 5.2. Let f be a multiplicative function. If $S \subset P$, $Q \subset P$ and $S \cup Q \neq P$, then

$$f_{S \cup Q}(P) = f_S(P) \cup f_Q(P).$$

Proof: If $x \neq 0$ and $x \in f_{S \cup Q}(P)$ then there exists $m \in S \cup Q$ such that $f(m) = x$. But $m \in S \cup Q$ implies $m \in S$ or $m \in Q$. Then, either

$$f_S(m) = f(m) = x \quad \text{or} \quad f_Q(m) = f(m) = x.$$

In either case, $x \in f_S(P) \cup f_Q(P)$. Conversely, if $x \in f_S(P) \cup f_Q(P)$ then $x \in f_S(P)$ or $x \in f_Q(P)$. Thus there exists m such that $f(m) = x$ and $m \in S$ or $m \in Q$. Thus $m \in S \cup Q$ and

$$f_{S \cup Q}(m) = f(m) = x.$$

Hence, $x \in f_{S \cup Q}(P)$.

If $x = 0$, $S \cup Q \neq P$ makes zero an element in both members. When f is a function which has a zero, this condition is unnecessary. If $S \cup Q = P$, the right side may include zero when the left side does not.

A result similar to Theorem 5.2 can be proved for intersections.

Theorem 5.3. Let f be a multiplicative function. If S and Q are proper subsets of P, then $f_{S \cap Q}(P) \subset f_S(P) \cap f_Q(P)$.

Proof: If $x = 0$ then S, Q and $S \cap Q$ proper subsets of P implies $x \in f_{S \cap Q}(P)$ and $x \in f_S(P) \cap f_Q(P)$. Thus, zero is in both sides of the equation by hypothesis.

Suppose $x \neq 0$ and $x \in f_{S \cap Q}(P)$. Then, there exists $m \in S \cap Q$ such that $f(m) = x$. But $m \in S$ and $m \in Q$; hence $f_S(m) = f(m) = x$ and $f_Q(m) = f(m) = x$. Thus, $x \in f_S(P)$ and $x \in f_Q(P)$; therefore $x \in f_S(P) \cap f_Q(P)$ and, $f_{S \cap Q}(P) \subset f_S(P) \cap f_Q(P)$.

The containment cannot be shown to be reversed unless f is a special type of function. For example, let f be the tau-function and let S and Q be given by $S = \{1, 2, 4, 6, 8, 9, 10\}$ and $Q = \{1, 3, 4, 8\}$. Since $\tau(2) = \tau(3) = 2$, $2 \epsilon \tau_S(P) \cap \tau_Q(P)$ but $2 \notin \tau_{S \cap Q}(P) = \{0, 1, 3, 4\}$. When f is a one to one function defined on P, $m \neq n$ implies $f(m) \neq f(n)$. Thus the difficulty in the above example is avoided and

$$f_{S \cap Q}(P) = f_S(P) \cap f_Q(P)$$

under the hypotheses of Theorem 5.3.

There are other properties of contractions which could be considered here. Those in the first three theorems are given so that the reader may see some of the possibilities. However, the principal purpose of this paper is the discussion of multiplicative functions.

It is apparent that not all contractions of a multiplicative function on P are themselves multiplicative functions on P. For, suppose S is any finite set such that there are $a \epsilon S$ and $b \epsilon S$ with $(a,b) = 1$ and $ab \notin S$. Then

$$\tau_S(a)\tau_S(b) = \tau(a)\tau(b) \neq 0 = \tau_S(ab).$$

This would lead one to conjecture that properties of the set S determine whether or not f_S is multiplicative. Also, one would surmise that S should be a closed set under multiplication. These assumptions are reasonable but not entirely correct. In fact, it will be seen that whether or not f has zeros is also a factor.

Before proceeding further, it seems desirable to consider some examples of contractions which are multiplicative.

Example 5.1. Let $S = \{1, 2, 3, 4, 6\}$. Then, it is easily shown that μ_S is a multiplicative function on P. But τ_S is not a multiplicative function.

Example 5.2. Let $S = \{n | n = 2^i 3^j$ where $i = 0, 1, 2, \ldots,$ and $j = 0, 1, 2, \ldots,\}$. Let $Q = \{n | n = 2^{2i} 3^{2j}$ where $i = 0, 1, 2, \ldots,$ and $j = 0, 1, 2, \ldots\}$. It can be verified that if f is multiplicative then f_S and f_Q are also multiplicative.

From Example 5.1 it can be concluded that it is not generally necessary that S be infinite, nor that S be closed under multiplication, in order that f_S be multiplicative for some f. Some of the difficulty involved in characterizing sets S for which f_S is multiplicative is due to the fact that f may have zeros. The location of the zeros of f seems to influence considerably the type of set, and in fact the specific sets, for which the contraction of f is multiplicative.

If multiplicative functions without zeros are considered, it is possible to find conditions on S which are both necessary and sufficient to make f_S multiplicative. Essentially, the problem reduces to one of finding sets whose characteristic function is multiplicative on P. Since the ordinary product of multiplicative functions is multiplicative, and

$$f_S(n) = f(n)\chi_S(n),$$

it is seen that χ_S being multiplicative implies f_S is also.

For practical reasons, the final result is not given as stated in the preceeding paragraph. The next theorem gives the conditions for an arbitrary function. Since $\chi_P(n) = 1$, for every $n \varepsilon P$, satisfies the hypothesis of the theorem it is obvious that χ_S is a special case.

Theorem 5.4. Let f be a multiplicative function, defined on P, such that $f(n) \neq 0$ for every $n \varepsilon P$. Then f_S is multiplicative on P if and only if S meets the following conditions:

(i) if $(a,b) = 1$ and $a \varepsilon S$, $b \varepsilon S$ then $ab \varepsilon S$;

(ii) for every factorization $n = ab$ where $(a,b) = 1$, $n \varepsilon S$ implies $a \varepsilon S$ and $b \varepsilon S$.

Proof: If $(a,b) = 1$ and $a, b \varepsilon S$, then by (i) $ab \varepsilon S$. Thus

$$f_S(ab) = f(ab) = f(a)f(b) = f_S(a)f_S(b).$$

If $(a,b) = 1$ and either $a \notin S$ or $b \notin S$, then (i) and (ii) imply $ab \notin S$. Thus,

$$f_S(ab) = 0 = f_S(a)f_S(b).$$

Hence (i) and (ii) imply that f_S is multiplicative on P. (Note that (ii) implies that $1 \varepsilon S$, hence $f_S(1) = f(1) = 1$.)

Suppose S does not satisfy condition (i). Then, there exists $a \varepsilon S$ and $b \varepsilon S$ with $(a,b) = 1$ and $ab \notin S$. Hence

$$f_S(a) = f(a) \neq 0 \text{ and } f_S(b) = f(b) \neq 0$$
$$\text{and } f_S(a)f_S(b) \neq 0.$$

However, $f_S(ab) = 0$. Thus

$$f_S(ab) \neq f_S(a)f_S(b)$$

and f_S is not multiplicative.

Suppose S does not satisfy condition (ii). Then $n \varepsilon S$ with a factorization $n = ab$ where $(a,b) = 1$ and either $a \notin S$ or $b \notin S$. Hence

$$f_S(ab) = f(ab) \neq 0$$

and either $f_S(a) = 0$ or $f_S(b) = 0$. Thus, $f_S(ab) \neq 0$ and $f_S(a)f_S(b) = 0$ gives

$$f_S(ab) \neq f_S(a)f_S(b).$$

Hence, f_S is not multiplicative.

Thus, f_S multiplicative on P implies (i) and (ii).

The proof of the sufficiency of (i) and (ii) in Theorem 5.4 did not employ the fact that f had no zero. The implication thus goes that direction for any multiplicative function f. Most of the discussion in the remainder of the chapter requires only that sets exist for which f_S is multiplicative. This theorem gives some, if not all, sets such that f_S is multiplicative whenever f is multiplicative. Two sets which satisfy (i) and (ii) are seen in Example 5.2. One might observe that infinitely many such sets are possible by using any pair of primes as bases. In fact, it is possible to create sets such as these by using any finite number of primes and taking all possible products.

The following corollary gives the case for functions which are completely multiplicative.

Corollary 5.5. If f has no zeros and is completely multiplicative, then f_S is completely multiplicative, if and only if S satisfies the following conditions:

(i)* S is closed under multiplication;

(ii)* $n \in S$ and $d|n$ implies $d \in S$.

Proof: The proof is the same as for Theorem 5.5 with the conditions $(a,b) = 1$ dropped.

Conditions (i)* and (ii)* are sufficient for any multiplicative f. However, they are stronger than is necessary. In fact, the set Q of Example 5.2 fails to satisfy (ii)*.

To summarize the results so far, one might say that new multiplicative functions may be found by taking a known function and judiciously defining a set of zeros. The next step is to show that still others may follow from these by the usual processes.

The next theorem is basic to the discussion which follows.

Theorem 5.6. Let

$$h(n) = \sum_{d|n} f_S(d) \quad \text{and} \quad k(n) = \sum_{d|n} f(d).$$

Then $h = k_S$ if and only if, for every $n \in P$,

$$\sum_{\substack{d|n \\ d \in S}} f(d) = \begin{cases} \sum_{d|n} f(d), & \text{if } n \in S \\ 0, & \text{if } n \notin S. \end{cases}$$

Proof: By Definition 5.2, $h(n)$ is the left member of the equation and $k_S(n)$ is the right member.

That h and k_S are equal is unusual. First, consider the following example.

Let $S = \{n | n = 2^{2i}3^{2j}$ where $i = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots\}$

$= \{1, 4, 9, 16, 36, 81, \ldots\}$. Let f be the Euler ϕ-function so that $n = \sum_{d|n} \phi(d)$.

Thus, $k(36) = 36$ and $k_S(36) = 36$. But,

$$\begin{aligned} h(36) &= \phi_S(1) + \phi_S(2) + \phi_S(3) + \phi_S(4) + \phi_S(6) + \phi_S(9) + \phi_S(12) \\ &\quad + \phi_S(18) + \phi_S(36) \\ &= \phi(1) + \phi(4) + \phi(9) + \phi(36) \\ &= 1 + 2 + 6 + 12 = 21. \end{aligned}$$

Also, since $2 \notin S$, $k_S(2) = 0$ whereas

$$h(2) = \phi_S(1) + \phi_S(2) = 1.$$

The following theorem shows that the equality in Theorem 5.6 never holds if f is a function such as ϕ, τ or σ_k.

Theorem 5.7. Let h and k be as given in Theorem 5.6. If f is a positive valued multiplicative function and S is a proper subset of P for which f_S is multiplicative, then $F \neq G_S$.

Proof: If S is a proper subset then there exists $m \in P - S$ for which $f_S(m) = 0$ and $k_S(m) = 0$. By Theorem 5.4, it can be concluded that $1 \in S$. Thus, since $1|m$,

$$h(m) = \sum_{\substack{d|m \\ d\in S}} f(d) \geq f(1) = 1,$$

Hence, $k_S(m) = 0$ and $h(m) \geq 1$ implies $h \neq k_S$.

Where χ_S is the characteristic function of S and S is a proper subset of P, this theorem indicates that $\tau_S \neq h$ when

$$h(n) = \sum_{d|n} \chi_S(n).$$

When the condition in Theorem 5.6 is not met, two different functions are created by summing on divisors of n of a contraction to a set S and by reversing this order. Thus, by using Theorem 3.18 and contraction to a given set, two different functions and their generating functions may be found.

Earlier it was mentioned that summing on the divisors of n was a special case of convolution. In fact, it is the case if one function is the identity for ordinary multiplication. If 1_S represents the identity function restricted to S, it can be seen by example, that $\sum_{d|n} f_S(d)$ is not, in general, equal to $\sum_{d|n} f_S(n)1_S(n/d)$.

To see this, let S be the set Q of Example 5.2. Then,

$$Q = \{1, 4, 9, 16, 36, \ldots\}$$

and

$$\sum_{d|12} f_S(d)1_S(n/d) = f_S(1)1_S(12) + f_S(2)1_S(6)$$

$$+ \quad f_S(3)1_S(4) + f_S(4)1_S(3) + f_S(6)1_S(2) + f_S(12)1_S(1)$$

$$= \quad 0.$$

But,

$$\sum_{d|12} f_S(d) = f_S(1) + f_S(2) + f_S(3) + f_S(4) + f_S(6)$$

$$+ \quad f_S(12) = f_S(1) + f_S(4)$$

$$= f(1) + f(4).$$

If $f(4) \neq -1$, these sums are not the same. Thus, the equality holds on the given set for a very restricted type of function.

The general case for convolution products is given in the next two theorems.

Theorem 5.8. If $$H(n) = \sum_{d|n} f_S(d)g_S(n/d)$$

and

$$K(n) = \sum_{d|n} f(d)g(n/d),$$

then $H = K_S$ if and only if

$$K_S(n) = \begin{cases} \sum_{d|n} f(d)g(n/d), d \varepsilon S, \text{ if } n \varepsilon S \\ 0, \text{ if } n \notin S. \end{cases}$$

Proof: The theorem follows from the definition of f_S, g_S and K_S.

Theorem 5.9. If H and K are defined as in Theorem 5.8 and S satisfies the conditions (i)* and (ii)* of Corollary 5.5, then $H = K_S$.

Proof: If S is closed under multiplication then $d \varepsilon S$ and $n/d \varepsilon S$ makes $n \varepsilon S$. By (ii)*, $n \varepsilon S$ implies $d \varepsilon S$ and $n/d \varepsilon S$ for every divisor d of n. Hence,

$$K_S(n) = \sum_{d|n} f(d)g(n/d) = \sum_{\substack{d|n \\ d\varepsilon S}} f(d)g(n/d) \text{ when } n \varepsilon S.$$

If $n \notin S$, then either $d \notin S$ or $n/d \notin S$ for every divisor d of n. (If both are in S then $n \varepsilon S$.) Hence, $K_S(n) = 0$ and

$$H(n) = \sum_{d|n} f_S(d)g_S(n/d) = 0.$$

Therefore, $K_S(n) = H(n)$ for all n.

By Theorem 2.13, if S is a set for which f_S and g_S are each multiplicative and if

$$h(n) = \sum_{d|n} f_S(n)g_S(n/d),$$

then h is also multiplicative. In particular, when f, g and S satisfy the conditions in Theorem 5.4, it is true that h is a multiplicative function.

A few comments can be made concerning the generating functions of contractions of multiplicative functions.

First, regardless of whether f_S is multiplicative or not, whenever

$$F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$$

is the generating function of f, then

$$F^*(s) = \sum_{n=1}^{\infty} f_S(n)n^{-s} = \sum_{n\varepsilon S} f(n)n^{-s}$$

is the generating function of f_S. It might also be pointed out, in case the question of convergence arises, that $|f_S(n)| \leq |f(n)|$ for every n. Thus, by comparison the latter series converges absolutely whenever the previous one does.

Second, whenever the contractions of multiplicative functions are multiplicative, the methods discussed in Chapters III and IV still hold for finding other functions based on them. However, by Theorems 5.8 and 5.9 it follows that Theorem 3.19 can yield two different functions which depend on whether the contraction to S is done before the convolution or after it. That is, if

$$F(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \text{ and } G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$$

it is recalled that F(s)G(s) generates a function h defined by

$$h(n) = \sum_{d|n} f(d)g(n/d).$$

Then, also, if F* and G* are as stated earlier, then F*(s)G*(s) generates the function defined by

$$k(n) = \sum_{d|n} f_S(d)g_S(n/d).$$

But, h_S and k are not necessarily the same function.

Since there is an infinite number of sets which satisfy the conditions in Theorem 5.4, from a given function and its generating function it is possible to derive an infinite number of multiplicative functions and their generating functions. As described above, it is also possible to find others from these.

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APPENDIX

MULTIPLICATIVE FUNCTIONS OF ELEMENTARY NUMBER THEORY

The following list includes several of the most common multiplicative functions of number theory. Listed here are (1) their usual designation; (2) the definition; (3) related formulas; (4) generating function or functions.

A. (1) The Euler totient function, $\phi(n)$;

(2) $\phi(n)$ is the number of positive integers m such that $m \leq n$ and $(m,n) = 1$;

(3) $\phi(n) = n \prod_{P|n} (1 - 1/P)$;

$$n = \sum_{d|n} \phi(d);$$

(4) $\zeta(s-1)/\zeta(s) = \sum_{n=1}^{\infty} \phi(n)n^{-s}$;

$$\sum_{n=1}^{\infty} \frac{\phi(n)x^n}{1-x^n} = \frac{x}{(1-x)^2} .$$

B. (1) The Möbius function, $\mu(n)$;

(2) $\mu(n) = 1$, if $n = 1$, $\mu(n) = 0$, if n has the square of a prime as a factor, and $\mu(n) = (-1)^r$, if n is the product of r distinct primes;

(3) $\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$;

$|\mu(n)| = 0$ if n has the square of a prime as factor and $|\mu(n)| = 1$, otherwise.

(4) $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n)n^{-s}$;

$$\sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1-x^n} = x.$$

C. (1) The convolution identity, $\varepsilon(n)$;

(2) $\varepsilon(n) = 1$, if $n = 1$ and $\varepsilon(n) = 0$ if $n > 1$;

(3) $\varepsilon(n) = \sum_{d|n} \mu(d)$;

(4) $\sum_{n=1}^{\infty} \varepsilon(n)n^{-s} = 1$.

D. (1) $\tau(n)$;

(2) $\tau(n)$ is the number of positive divisors of n;

(3) If $n = \prod_{i=1}^{r} p_i^{a_i}$, then $\tau(n) = \prod_{i=1}^{r} (a_i + 1)$;

$\tau(n) = \sum_{d|n} 1$;

(4) $\zeta^2(s) = \sum_{n=1}^{\infty} \tau(n)n^{-s}$.

E. (1) $\sigma(n)$;

(2) $\sigma(n)$ is the sum of the positive divisors of n;

(3) $\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+1}-1}{p_i-1} = \sum_{d|n} d$;

(4) $\zeta(s)\zeta(s-1) = \sum_{n=1}^{\infty} \sigma(n)n^{-s}$.

F. (1) $\sigma_k(n)$;

(2) $\sigma_k(n)$ is the sum of the kth powers of the positive divisors of n;

(3) $\sigma_k(n) = \prod_{i=1}^{r} \left\{\frac{p_i^{(a_i+1)k}-1}{p_i^k - 1}\right\} = \sum_{d|n} d^k$;

Note: $\sigma_1 = \sigma$ and $\sigma_0 = \tau$.

(4) $\zeta(s)\zeta(s-k) = \sum_{n=1}^{\infty} \sigma_k(n)n^{-s}$;

$$\sum_{n=1}^{\infty} \frac{n^k x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_k(n)x^n.$$

G. (1) Liouville's function, $\lambda(n)$;

(2) $\lambda(n) = (-1)^q$, where q is the total number (distinct or not) of prime factors of n;

(4) $\zeta(2s)/\zeta(s) = \sum_{n=1}^{\infty} \lambda(n)n^{-s}$.

H. (1) $\chi(n)$;

(2) $\chi(n) = 0$, if $2|n$, and $\chi(n) = (-1)^{(n-1)/2}$, if n is odd;

(4) $L(s) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$.

I. (1) $q_k(n)$;

(2) $q_k(n) = 0$, if n has the kth power of a prime as a factor and $q_k(n) = 1$, otherwise;

(3) $q_1(n) = |\mu(n)|$;

(4) $\zeta(s)/\zeta(ks) = \sum_{n=1}^{\infty} q_k(n)n^{-s}$.

VITA

William Wayne Malone

Candidate for the Degree of

Doctor of Education

Thesis: MULTIPLICATIVE NUMBER-THEORETIC FUNCTIONS AND THEIR GENERATING FUNCTIONS

Major Field: Higher Education

Biographical:

Personal Data: Born near Paris, Mississippi, October 22, 1930, son of Irving and Mary Malone.

Education: Attended public schools in Lafayette County, Mississippi; graduated from Taylor High School, Taylor, Mississippi, in 1948; attended Delta State College in the summer and fall of 1948; received the Bachelor of Arts in Education degree from the University of Mississippi in August, 1954, with a major in mathematics; received the Master of Education degree from the University of Mississippi in August, 1958, with a major in secondary education; received the Master of Arts degree from Louisiana State University with a major in mathematics in August, 1960; attended the University of Mississippi in the summer of 1961; completed requirements for the Doctor of Education degree at Oklahoma State University in July, 1968.

Professional Experience: Taught elementary grades at Pine Flat Elementary School, Lafayette County, Mississippi, from 1949 to 1951; taught high school mathematics at Pinedale High School, Etta, Mississippi, the year 1951-1952; taught high school mathematics at Taylor High School, Taylor Mississippi, from 1952 to 1956; served as principal of Black Jack High School, Batesville, Mississippi, for the year 1956-57; was principal of Cardwell High School, Carwell, Missouri, from 1957 to 1959; taught mathematics at Louisiana State University at Alexandria, Alexandria, Louisiana, since September, 1960; was on leave for 1965-66 and the spring semester 1967-68.

Organizations: Member of Mathematical Association of America, National Council of Teachers of Mathematics, and Phi Delta Kappa.