HOMOGENEOUS BOUNDED

PLANE CONTINUA

By

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PREFACE

Any list of people to whom gratitude should be expressed for aid given me in my quest for the degree for which this paper has been prepared, would almost certainly be inadequate. However, several people have given assistance of such significance that I must delineate some of their specific contributions.

Professor John Jobe guided me through all the preliminary study of topology which was necessary to begin a thesis on that subject. He made the original suggestion which eventually led to the formation of the theme of this thesis. Finally, he read the original drafts in their worst forms and continued to offer suggestions and give assistance until the thesis was completed.

Professors H. S. Mendenhall, Ware Marsden, Vernon Troxel and John Jewett all served on my advisory committee, and each offered helpful suggestions at various stages of the development of the thesis. I particularly wish to thank Dr. Jewett for his willingness to accept the chairmanship of my committee at a very late date in my program, following the retirement of my original chairman, Dr. Mendenhall.

The typing of this paper was the result of the combined efforts of my wife, Karen, who typed the original draft, and Judy Roach who typed the last two drafts and gave much assistance in developing the correct format. Without their patience and assistance the task of preparing this paper would have been immeasurably more difficult.

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CHAPTER I

THE HISTORY OF THE PROBLEM

Introduction

This chapter will present a chronological account of the development of the properties and examples of homogeneous bounded plane continua.

In order not to destroy the continuity of the story, no effort will be made to define topological concepts or terminology within this chapter. The topological terms which are less well known are defined in Chapter II. Other terms can be found in the references listed in the introduction to the second chapter.

It is necessary at this point to explain the referencing style used in this paper. When only one number appears in the parentheses following some item, i.e., (3), then that number refers to the number that has been assigned to the source being cited in the bibliography of this paper. When a sequence of three entries in parentheses follow some item, i.e., (3, 48, Theorem 16), then the first number gives the number of the source as given in the bibliography of this paper, the second number gives the page number within that source, and the third entry identifies the specific item that is being utilized.

The formal presentation of examples of homogeneous bounded plane continua and the proofs of the theorems giving their basic properties will constitute Chapters III and IV of this paper. Hence, no attempt will be made at this time to verify how the results given in this chapter are achieved.

As stated above, the results presented in this chapter are given approximately in the chronological order of their discovery. When those same results are presented again in later chapters, they will be given in the order that most efficiently facilitates their proof.

Fact and Fantasy Before 1948

The story of homogeneous bounded plane continua began in 1920 when a topological definition for the word "homogeneous" was first given by Waclaw Sierpinski (26). The definition, as given by Sierpinski, stated simply that a set M is homogeneous if and only if for every pair of points x and y belonging to M, there exists a homeomorphism mapping M to M and x to y. Examples of such sets are easy to construct (for instance, any line in the plane is homogeneous). However, when certain other restrictions are required of the set, examples become less numerous.

The simple closed curve is an example of a homogeneous bounded plane continuum (see Theorem 3.2). In 1920, B. Knaster and C. Kuratowski (19) stated a problem on homogeneous continua which took twenty-six years to resolve. The problem which they presented can be stated as follows: Is every nondegenereate homogeneous bounded plane continuum a simple closed curve?

Even though he could not verify his suspicions, in 1922 Knaster (18) himself gave a description of a hereditarily indecomposable continuum which he suspected of being homogeneous. This continuum was indeed homogeneous, but this fact was not proved until 1951 (3).

In a brilliant paper in 1924, Stefan Mazurkiewicz (20) proved a result which gave support to the idea that simple closed curves are the only homogeneous bounded plane continua. Mazurkiewicz proved that the only locally connected nondegenerate homogeneous bounded plane continuum is the simple closed curve.

Mazurkiewicz's paper was the last published on the problem until 1937. In that year, Zenon Waraszkiewicz (27) announced that he had proved that the only homogeneous bounded plane continuum is the simple closed curve. However, the following statement can be found in Waraszkiewicz's paper:

> It [the proof] is composed of two parts, of which the first, profiting only from the local homogeneity, permits the restriction of the class of continua under consideration to the one of irreducible sections of the plane such that every subcontinua is a simple arc or a proper indecomposable continuum. Now that second hypothesis is impossible since each automorphic transformation of a curve of which every part is indecomposable, reduces to the identity, so that one can consider only the irreducible plane curves, each part of which is a simple arc.

Of course, it is not immediately evident that it is incorrect to assume that ". . . each automorphic transformation of a curve of which every part is indecomposable, reduces to the identity, . . .". However, as will later be proved (Theorems 3.5 and 3.8), such is the case.

In 1944, Gustav Choquet (8) stated the following theorem without proof: "Any compact homogeneous plane set is either (1) finite, or (2) perfect and totally disconnected, or (3) homeomorphic to a union of concentric circles of positive radius which cuts any diameter in a set of type (1) or (2)." This result is also false and was probably based on Waraszkiewicz's earlier paper.

Hence, by 1945 the principle results that had appeared in print seemed to leave no doubt that the only nondegenerate homogeneous bounded plane continua were the simple closed curves.

The First Example

In 1948, Edwin E. Moise (22) published an example of a continuum which he proved to be indecomposable and homeomorphic to each of its nondegenerate subcontinua. The methods used by Moise to describe his continuum suggested immediately to R. H. Bing that Moise's continuum might be homogeneous. Later in the same year, Bing (1) presented a proof that the pseudo-arc (as Moise had called his continuum) was indeed homogeneous. Shortly thereafter, Moise (23) also published a proof of the homogeneity of the pseudo-arc.

It is interesting to note that Moise suggested in his original paper on the pseudo-arc that it might be homeomorphic to the continuum described by Knaster (18) in 1922. R. H. Bing (3) established in 1951 that any pair of linearly chainable compact nondegenerate hereditarily indecomposable plane continua are homeomorphic. That result is sufficient to show that the continuum described by Knaster was a pseudo-arc.

In 1949 and 1951, two papers appeared which would have added support to the notion that a simple closed curve is the only homogeneous bounded continuum, if Moise and Bing had not already published their results. In the first of these papers, F. B. Jones (11) showed that every compact plane continuum, that is both homogeneous and aposyndetic, is a simple closed curve. The second paper, by H. J. Cohen (9), used the result of the first paper to prove that the only homogeneous bounded plane continuum that contains a simple closed curve is a simple closed curve. Another theorem in Cohen's paper extended the theorem which had earlier been proved by Mazurkiewicz (20). This new theorem stated that the only locally connected, locally homogeneous, bounded plane continuum is the simple closed curve.

Since several papers had been published indicating that any one of several additional restrictions can force a homogeneous bounded plane continuum to be a simple closed curve, it may not be surprising that two papers by Issac Kapuano (15 and 16), which challenged the homogeneity of the pseudo-arc, were published in 1953. Kapuano's proofs (as they appeared in print) were only vague outlines. However, his error seemed to lie in some sort of assumption that the points of the pseudo-arc had a kind of natural linear order.

A More General Problem

Once Bing had published his proof of the existence of a homogeneous bounded plane continuum, other than the simple closed curve, a more general problem immediately arose. The newer problem asked the question: How many distinct examples of homogeneous bounded plane continua exist, and how can they be classified?

One of the most important results relating to the new problem appeared in 1951. In that year, F. B. Jones (12) proved that a homogeneous bounded plane continuum that does not separate the plane is indecomposable. Since the pseudo-arc does not separate the plane and is indecomposable, and since Bing had shown that all linearly chainable, compact, nondegenerate, hereditarily indecomposable plane continua are homeomorphic, it appeared that all bounded homogeneous continua, not separating the plane, might be pseudo-arcs. By adding the restriction that the continua under consideration be linearly chainable, Bing (4) was able to establish that all homogeneous bounded plane continua, that do not separate the plane and are not degenerate, are hereditarily indecomposable. Hence, it was established that the only homogeneous, linearly chainable, bounded plane continuum is the pseudo-arc.

In 1954, R. H. Bing and F. B. Jones announced simultaneously, but separately, that each had discovered a (circularly) chainable homogeneous bounded plane continuum that was neither a simple closed curve nor a pseudo-arc. It was discovered that the two examples were essentially the same, and hence, the result was published jointly in 1959 (7). This example was called a "circle of pseudo-arcs". It was shown to separate the plane and to be the union of an upper semi-continuous collection of pseudo-arcs.

Between the announcement of the discovery of the circle of pseudo-arcs and the actual publication of its description, F. B. Jones (14) published a proof that every decomposable, homogeneous bounded plane continuum that separates the plane, but which is not a simple closed curve, is a union of an upper semi-continuous collection of pseudo-arcs. In their joint paper, Bing and Jones established that all homogeneous bounded plane continua, that separate the plane and are a union of an upper semi-continuous collection of pseudo-arcs, are homeomorphic.

Thus, the following is an exhaustive classification system for chainable homogeneous bounded plane continua:

Type 1: Pseudo-arcs;

Type 2: Simple closed curves;

Type 3: Circle of pseudo-arcs;

Type 4: Indecomposable continua that separate the plane.

It is not yet known whether continua of Type 4, which are homogeneous, actually exist. At least one example which may belong to that class has been defined (3; 48, Example 2). That continuum is shown to be an indecomposable continuum that separates the plane, but no proof of its homogeneity has been published.

The last significant paper to be published on homogeneous bounded plane continua appeared in 1960, and was another paper showing that added restrictions almost always cause such continua to be simple closed curves. In that paper, R. H. Bing (2) proved that the only homogeneous bounded plane continuum which contains an arc is the simple closed curve.

CHAPTER II

FUNDAMENTAL TOPOLOGICAL CONCEPTS

Introduction

In this chapter, the basic topological concepts necessary to read this paper are presented. It will be generally assumed that the reader is familiar with the basic definitions and theorems that occur in a first course in elementary point set topology. In particular, any topological term appearing in Elementary Topology by D. W. Hall and G. L. Spencer (10) is not defined in this chapter.

In order to preserve space, many theorems that can be found in the literature which are used to prove the theorems in this paper have not been stated. In each such case, a reference which includes the proof of the theorem is given. Of course, the hypotheses of all theorems utilized in this paper have been carefully checked to assure the applicability of the conclusions. The majority of the theorems that are used, but not explicitly stated, may be found in one of the three books, Elementary Topology by D. W. Hall and G. L. Spencer (10), Foundations of Point Set Theory by R. L. Moore (24), or Analytic Topology by G. F. Whyburn (28).

Certain of the definitions in this chapter are of such a nature that examples are necessary to clarify their statement. In such a case, either an example will be given following the definition, or a reference will be given where an appropriate example can be found.

A clear understanding of the definitions and theorems associated with the concept of "crooked chains" is necessary for reading many of the proofs of this paper. Hence, nearly all such definitions are illustrated by example and all such theorems are followed by reasonably complete proofs.

The Topological Setting

The basic topological space assumed in all theorems and examples of this paper is the ordinary Cartesian plane with the usual metric topology. Care has been taken to assure that all results from other sources, that are used in this paper, are valid in this topological setting. Examples are presented in such a manner that their existence in the plane is clear.

Some confusion could arise by the frequent use of the term "domain" throughout this paper since "open set" is more commonly used in discussion of the Cartesian plane. The following definition should clarify the relationship between the two terms.

Definition 2.1: Let S be a topological space and D be a subset of S. Then D is a *domain* if and only if D is an open set of S. If a set D is open relative to a set M in S, then D is said to be a *domain relative to M* or just a *domain in M*.

The Concept of Homogeneity

The fundamental topological property of point sets that is studied in particular in this paper is the property of being homogeneous.

Definition 2.2: A point set M is said to be homogeneous if and only if for every pair of points x and y of M there exists a homeomorphism mapping M to M and x to y.

Example 2.3: Any simple closed curve is a homogeneous point set (see Theorem 3.2).

A somewhat weaker property than homogeneity is the property of being locally homogeneous. In some theorems of this paper the hypotheses only require that the set under consideration be locally homogeneous.

Definition 2.4: The set M is locally homogeneous if, for each pair of its points x and y of M there exists a homeomorphism between two domains in M, one containing x, the other containing y, such that x is mapped to y.

Of course, any homogeneous set is locally homogeneous. The converse of this statement is not necessarily true. The following example illustrates that fact.

Example 2.5: Let H be the open arc in the Cartesian plane given by $\{(x,y): 2<x<3,\ y=0\}$. Let K be the unit circle. If M is the union of H and K then M is locally homogeneous but not homogeneous.

Certain Types of Connected Sets

The only class of point sets that will be studied in this paper relative to the concept of homogeneity is the class containing those sets that are both closed and connected. A special name is given to the members of this class and certain members of the class are further classified by additional properties. The following sequence of definitions is concerned with the naming of special classes of connected sets.

Definition 2.6: A closed and connected set is called a continuum.

Definition 2.7: A connected subset C of a set M is called a component of M if and only if C is not properly contained in any connected subset of M.

Notation: An arc with end points x and y will usually be denoted by xy. Occasionally, an arc xy will be denoted by xzy to emphasize that xy passes through the point z, where $z \neq x$ and $z \neq y$. On other occasions, the notations (xy) and (xzy) are useful to indicate the open arc xy; that is, the arc xy except for its end points. When (xy) and (xzy) are used in a discussion, [xy] and [xzy] may also be used to give added emphasis to the fact that the end points are to be included.

Definition 2.8: An arc component of a set M is a subset C of M such that each pair of points of C belongs to an arc in M but C is not properly contained in any subset of M with that same property.

Example 2.9: Let $H = \{(x,y): x=0, -1 \le y \le 1\}$ and let $K = \{(x,y): y = \sin 1/x, 0 < x \le 1\}$. Let M be the continuum $H \cup K$. Then each of the sets H and K is an arc component of M.

Definition 2.10: If p and q are two points of the same arc component of the set M then the union of all arcs in M that have p as an end point and contain q is called a ray starting at p.

Example 2.11: Let K be the arc component of M in Example 2.9 and p be any point of K. If q is a point of K whose x coordinate is less than the x coordinate of p, then the ray starting at p and containing q is the set of points belonging to K with x coordinate less than the x coordinate of p. Similarly, if the x coordinate of q is greater than the x coordinate of p, then the ray starting at p and containing q is the set of points belonging to K with x coordinate greater than the x coordinate of p.

Definition 2.12: If M is a continuum, a composant of M is a point set K such that, for some point p of M, the point x belongs to K if and only if there is a proper subcontinuum of M containing both p and x.

Definition 2.13: A set of points M is said to be cyclicly connected provided every pair of points of M lie together on some simple closed curve in M.

Example 2.14: Let H be the set of points in the Cartesian plane and on the circles centered at the origin and having radii one and two respectively. Let $K = \{(x,y): -2 \leq x \leq 2, y=0\}$. If M is the union of H and K then M is cyclicly connected. Obviously, any simple closed curve is also cyclicly connected.

Definition 2.15: The point set M is said to be connected im kleinen at the point p if and only if p belongs to M and, for every domain D relative to M that contains p there exists a domain relative to M which contains p and is a subset of a component of D.

Example 2.16: Let M be the continuum of Example 2.9. Then each of the sets H and K is connected and connected im kleinen, but M is not connected im kleinen at any point of H.

Definition 2.17: A continuum which is locally connected and which contains no simple closed curve is called a dendrite.

Examples of dendrites are easy to construct. Of course, an arc is one such example.

Definition 2.18: A continuum M is said to be unicoherent if and only if for every pair of continua H and K such that M is the union of H and K, the intersection of H and K is a continuum. A continuum is said to be hereditarily unicoherent if every subcontinuum is unicoherent.

The pseudo-arc presented in Chapter III is shown in Chapter IV (Theorem 4.7) to be hereditarily unicoherent.

Definition 2.19: The continuum M is aposyndetic at the point z of M with respect to the point x of M provided that M contains a continuum K and a set V which is open relative to M, such that M - {x} contains K, V contains z, and V is a subset of K.

Example 2.20: Let M be the continuum of Example 2.9. Then M is not aposyndetic at any point of H with respect to any other point of H. However, M is aposyndetic at any point of K with respect to any other point of M.

Definition 2.21: The continuum M is said to be indecomposable if and only if it is not the union of two subcontinua distinct from M. If every subcontinuum of M is indecomposable then M is said to be hereditarily indecomposable.

Examples of indecomposable continua are not easy to describe. Several such examples can be found in "Concerning Hereditarily Indecomposable Continua," by R. H. Bing (3).

Definition 2.22: A continuum is decomposable if and only if it is not indecomposable.

Definition 2.23: If H and K are disjoint closed point sets, the continuum M is said to be an irreducible continuum from H to K if M intersects both H and K but no proper subcontinuum of M intersects both H and K.

Definition 2.24: Suppose a_0b_0, a_1b_1, . . ., is a sequence of arcs converging to an arc xy. The sequence is called a folded sequence

<u>of arcs</u> converging to xy if a_o, b_o, a_1, b_1, . . ., converges to x.

<u>Example 2.25</u>: Let the coordinates of the point a_i be $((1/2)^{2i},0)$, $i = 0, 1, 2, \ldots$; let the coordinates of the point b_i be $((1/2)^{2i+1},0)$, $i = 0, 1, 2, \ldots$; and let the coordinates of the point c_i be $((1/2)^{2i+1},1)$, $i = 0, 1, 2, \ldots$. Let a_ib_i, $i = 0, 1, 2, \ldots$, denote the arc formed by the union of two line segments joining a_i to c_i and b_i to c_i, respectively. Let xy be the line segment joining $x = (0,0)$ to $y = (0,1)$. Then a_ob_o, a_1b_1, . . . is a folded sequence of arcs converging to xy.

<u>Definition 2.26</u>: A <u>simple triod</u> is the union of three arcs such that the intersection of any two of them is the same point p.

<u>Definition 2.27</u>: If S is the Cartesian plane and M is a closed proper subset of S, then every component of S - M is called a <u>complementary domain</u> of M.

<u>Definition 2.28</u>: The set T is said to <u>separate</u> the connected point set M if and only if M - T is the union of two separated point sets.

Properties of Sets Associated with Special Points

Certain properties possessed by points, by virtue of their being members of homogeneous sets, are preserved under homeomorphisms of the set to itself. Consequently, one method of determining whether a set is homogeneous is to examine particular points under a homeomorphism of the set to itself. Thus, it is convenient to have special names for points having properties that are sometimes preserved under a homeomorphism.

Definition 2.29: A point p is called a boundary point of a point set M if and only if every open set containing p contains a point of M and a point not belonging to M. The union of all boundary points of a set is called the boundary of the set.

The next point property that will be identified is one that is always preserved by a homeomorphism of a continuum to itself. That fact will be proved after an example is given illustrating the definition.

Definition 2.30: If k is a positive integer, the point p of the continuum M is said to be of Menger order k with respect to M if and only if it is true that (1) every domain with respect to M that contains p contains a domain with respect to M which contains p and whose boundary with respect to M contains only k points, (2) if n is a positive integer less than k, there exists a domain D with respect to M, containing p, such that if U is any domain with respect to M which contains p and which is a subset of D, then the boundary of U with respect to M contains more than n points.

Example 2.31: Let M be the continuum of Example 2.14. Then the points having coordinates (-2,0) and (2,0) have Menger order three; the points having coordinates (-1,0) and (1,0) have Menger order four; and all other points of M have Menger order two.

Theorem 2.32: Let M be a continuum and p_1 and p_2 be distinct points of M. Suppose there exists two open sets of M, say E and F, such that p_1 and p_2 belong to E and F respectively, and a homeomorphism from E to F that maps p_1 to p_2. Then p_1 and p_2 have the same Menger order.

Proof: Suppose the Menger order of p_1 is k_1 and the Menger order of p_2 is $k_2 \neq k_1$. Without loss of generality, let $k_1 > k_2$. Then there exists an open set D_1 of M such that D_1 is a subset of E and whose boundary with respect to M contains more than k_2 points. Let D_2 be the subset of F that is the image of D_1 under the homeomorphism. Then D_2 is open in M and the boundary of D_2 with respect to M contains at most k_2 points. Therefore, there exists some point p_3 of the boundary of D_1 with respect to M which maps to some point p_4 of D_2 that is not on the boundary of D_2 with respect to M. Let D_4 be an open set of M such that p_4 is in D_4 and D_4 is a subset of D_2. If D_3 is the inverse image of D_4, then D_3 is an open subset of D_2 and contains p_3. But this is impossible because p_2 is a boundary point of D_2 with respect to M and hence no open subset of D_2 contains p_3.

Definition 2.33: A point p is called an end point of a continuum M if p has Menger order one with respect to M.

Definition 2.34: The point p is called a cut point of the connected point set M if and only if M - $\{p\}$ is not connected.

Definition 2.35: A point p will be called a separating point of a set M provided there exist two points a and b of some component C of M such that $M - \{p\} = M_a \cup M_b$, where M_a and M_b are mutually separated and contain a and b respectively.

Definition 2.36: A point p of a continuum M will be called a local separating point of M provided that there exists a compact neighborhood R of p such that if C is the component of the intersection of M with the closure of R that contains p, then $M \cap (\bar{R} - \{p\}) = M_1 \cup M_2$ where M_1 and M_2 are mutually separated sets and neither $M_1 \cap C$ nor $M_2 \cap C$ is empty.

Example 2.37: Let M be the continuum of Example 2.14. Then every point of M is a local separating point, but no point is either a cut point or a separating point. In connected sets cut points and separating points are equivalent concepts.

Definition 2.38: A point x cuts a point w from a point z in a continuum M if and only if there exists no subcontinuum of M lying in $M - \{x\}$ that contains both w and z.

Example 2.39: It should be clear that if a continuum M is cyclicly connected (as in Example 2.14) then no point x cuts a point w from a point z in M. But, let M be the continuum of Example 2.9, then any point of H other than (0,1) or (0,-1) cuts (0,1) from (0,-1) in M.

Definition 2.40: The point p is said to be accessible from the point set M if and only if for every point x of M there exists an arc xp lying, except for p, wholly in M.

Example 2.41: Let M be any open set in the plane. Then every point of $\bar{\bar{M}}$ is accessible from M.

Sequences

Several of the continua used as examples in this paper occur as limit sets of sequences. Most of the terminology associated with sequences and generalized sequences (well-ordered sets) that is used in this paper is standard. However, two terms, not so commonly used, are defined here so that their meaning will be clear.

Definition 2.42: If, for each positive integer n, M_n is a point set, then the limiting set of the sequence $M_1, M_2, M_3, \ldots$ is a point set M such that p belongs to M if and only if for every open set R containing p there exist infinitely many integers n such that M_n contains a point of R. If L is the limiting set of every subsequence of $M_1, M_2, M_3, \ldots$, then $M_1, M_2, M_3, \ldots$, is said to converge to L.

Definition 2.43: Let α be any sequence (finite, countable or uncountable). The subsequence β of the sequence α is said to be an initial segment of α if and only if every term of α that precedes any term of β belongs to β.

Upper Semi-Continuous and Continuous Collections

The proofs of several theorems in this paper are completed by showing that certain sets can be decomposed into disjoint collections of subsets which can then be considered to be a topological space with the subsets as points. The terminology introduced in this section will provide the foundation for such considerations.

Definition 2.44: A collection G of mutually exclusive closed point sets is said to be upper semi-continuous if and only if it is true that if g is a point set of the collection G, and g_1, g_2, g_3, . . . is a sequence of point sets from G, and for every n, x_n and y_n are points in g_n such that x_1, x_2, x_3, . . ., converges to a point in g, then every infinite subsequence of y_1, y_2, y_3, . . . has a subsequence converging to a point that lies in g.

Definition 2.45: A collection G of subsets of a metric space M is said to give an upper semi-continuous decomposition of M if and only if (1) the sets of G are compact, (2) G fills up M (every point of M belongs to a set of G), and (3) G is upper semi-continuous.

Example 2.46: Let M be the subspace of the Cartesian plane whose points are the points of $A \cup B$ where $A = \{(x,y): 0\leq x<1, 0\leq y\leq 1\}$ and $B = \{(x,y): 1\leq x\leq 2, 0\leq y\leq 2\}$. For each x_o such that $0\leq x_o\leq 2$ define $g_{x_o} = \{(x_o,y): (x_o,y)$ is an element of $M\}$. If $G = \{g_x\}$, $0\leq x\leq 2$, then G is an upper semi-continuous collection of sets that gives an upper semi-continuous decomposition of M.

Definition 2.47: A collection G of closed point sets is said to be continuous if and only if it is true that if g is a point set of the collection G and g_1, g_2, g_3, . . . is a sequence of point sets of this collection and, for every n, x_n and y_n are points of g_n, and the sequence x_1, x_2, x_3, . . . converges to a point in g, then every infinite subsequence of y_1, y_2, y_3, . . . has a subsequence converging to a point that lies in g and, furthermore, $g_1, g_2, g_3, \ldots$ converges to g.

Definition 2.48: A collection G of subsets of a metric space M is said to give a continuous decomposition of M if and only if (1) the sets of G are compact, (2) G fills up M, (3) G is continuous.

Example 2.49: Let M be the subspace of the Cartesian plane whose points are the points of A where $A = \{(x,y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$. For each x such that $0 \leq x \leq 1$, define $g_x = \{(x,y): 0 \leq y \leq 1\}$. Then if $G = \{g_x\}$, $0 \leq x \leq 1$, G is a continuous collection that gives a continuous decomposition of M.

Crooked Chains

The definitions and theorems contained in this section are the least well known of any in the chapter. However, they are probably the most important since they are ideas from which the pseudo-arc and the circle of pseudo-arcs are developed.

The principle definitions will be given first, along with illustrative examples. A sequence of theorems that give the important properties of crooked chains will then be proved.

Definition 2.50: A collection of domains $D = (d_1, d_2, \ldots, d_n)$ is called a linear chain if and only if $d_i \cap d_j \neq \emptyset$ if and only if $|i - j| \leq 1$, $i = 1, 2, \ldots, n$. If p and q are points belonging only to d_1 and d_n respectively then D is called a linear chain from p to q. If $D = (d_1, d_2, \ldots, d_n)$ is a linear chain then d_1 and d_n

are called end links; all other links are called interior links. The link d_i is called the i-th link. If $d_i \cap d_j \neq \emptyset$ then d_i and d_j are called adjacent links.

Definition 2.51: A linear chain such that no link has diameter greater than the positive number ε is called an ε-chain.

Definition 2.52: A continuum M such that for every positive number ε there is an ε-chain covering M is called linearly chainable or ε-chainable.

Example 2.53: An arc is linearly chainable but a simple triod is not.

Definition 2.54: A collection of domains $D = (d_1, d_2, \ldots, d_n)$ $(n > 2)$ is called a circular chain if and only if $d_i \cap d_j \neq \emptyset$ if and only if $|i - j| \leq 1$, $i = 1, 2, 3, \ldots, n$, except that $d_1 \cap d_n \neq \emptyset$.

Definition 2.55: A continuum M is said to be circularly chainable if for every positive number ε there is a circular chain covering M such that no link has diameter greater then ε.

Example 2.56: A simple closed curve is circularly chainable whereas an arc is not.

Definition 2.57: If D and E are either both linear chains or both circular chains then D contains E if and only if every link of E is a subset of some link of D.

Example 2.58: In Figure 1, the chain D contains the chain E.

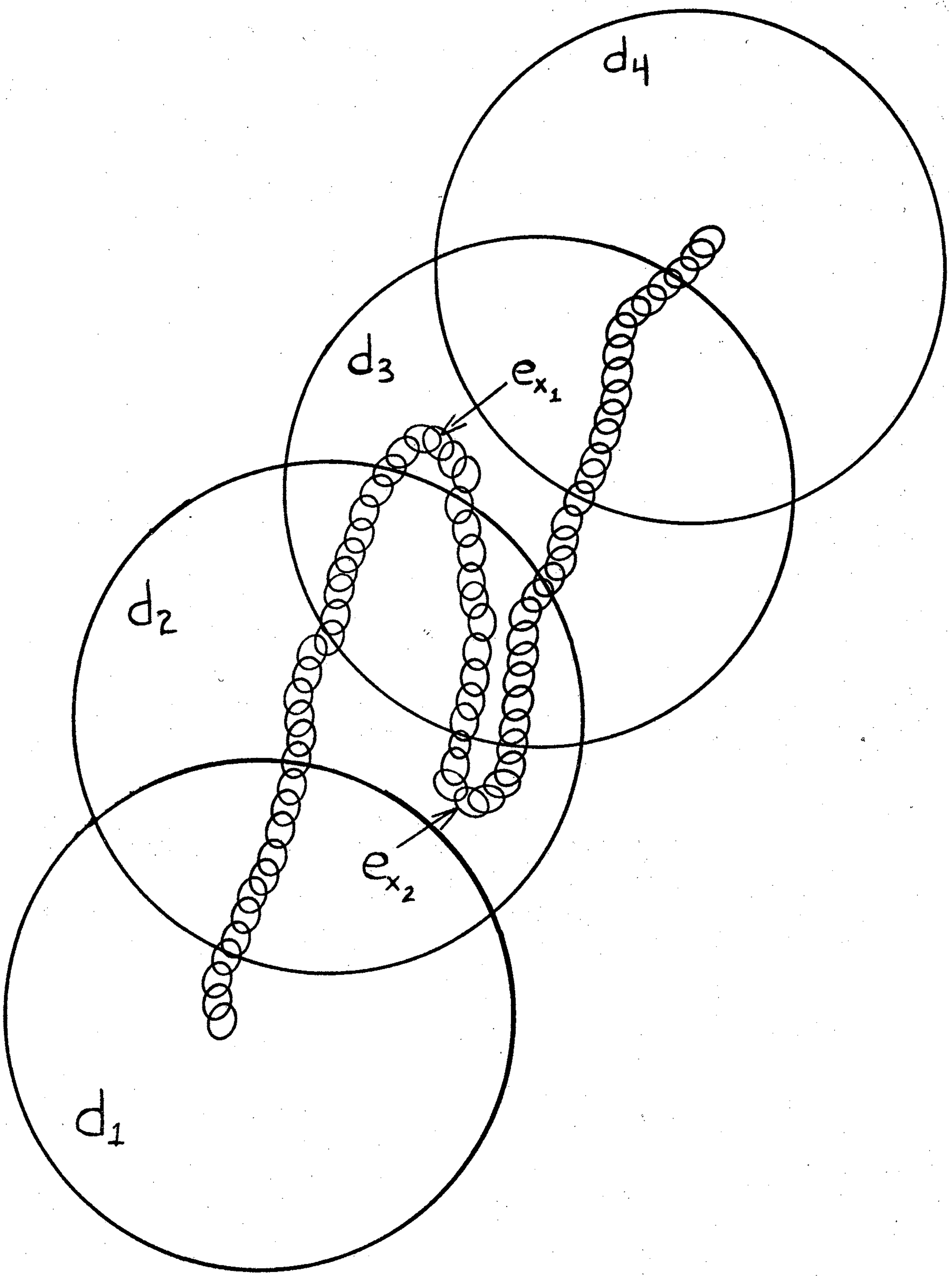

Figure 1. A Chain Crooked in a Chain
with Four Links.

Definition 2.59: The word chain refers to either a linear or circular chain. When the word chain is used alone, it will generally be clear from the context whether reference is being made to a circular or to a linear chain. If it is not clear, then it may be assumed that the statement is applicable to either type of chain.

Definition 2.60: If E and D are chains (linear or circular) then E is a subchain of D if and only if each link of E is a link of D. If E is a linear chain then E will be denoted by $D_{(i,j)}$ if the i-th and j-th links of D are the end links of E.

The following definition is the key idea in the description of the pseudo-arc. Special attention should be given to parts (b) and (c) of the definition. One is tempted to read the subscript on d in part (c) as a k instead of an h. It is the arrangement of these subscripts which essentially achieves the desired "crookedness" of the chains.

Definition 2.61: The linear chain $E = (e_1, e_2, \ldots, e_n)$ is crooked in the linear chain $D = (d_1, d_2, \ldots, d_m)$ if and only if:

(1) D contains E.

(2) For every subchain $E_{(i,j)}$ of E such that $e_i \cap d_h \neq \emptyset$, $e_j \cap d_k \neq \emptyset$, where $|h - k| > 2$, the following conditions hold:

(a) $E_{(i,j)}$ is the union of three chains $E_{(i,r)}$, $E_{(r,s)}$, and $E_{(s,j)}$ such that $(s-r)(j-i) > 0$,

(b) e_r is a subset of a link of $D_{(h,k)}$ adjacent to d_k, and

(c) e_s is a subset of a link $D_{(h,k)}$ adjacent to d_h.

Examples will be given illustrating Definition 2.61 after additional notation is introduced in Definition 2.62.

Definition 2.62: Let (x_1,y_1), (x_2,y_2), . . ., (x_n,y_n) be a collection of ordered pairs of integers. Then the chain E follows the pattern (x_1,y_1), (x_2,y_2), . . ., (x_n,y_n) in the chain D if and only if the x_i-th link of E is a subset of the y_i-th link of D, $i = 1, 2, \ldots, n$.

Example 2.63: This example is given to illustrate the pattern which must exist if a linear chain D has a specified number of links, the linear chain E is crooked in D, the first link of E is a subset of the first link of D, and the last link of E is a subset of the last link of D.

Case 1: The chain D has exactly four links. Definition 2.61 implies that there must exist links numbered 1, x_1, x_2, and x_3 such that $1<x_1<x_2<x_3$ and such that E follows the pattern (1,1), $(x_1,3)$, $(x_2,2)$, $(x_3,4)$ in the chain D. In this case the only possible links of d_h and d_k of D where $|h - k| > 2$ occur when h=1, k=4, or h=4, k=1. One subchain $E_{(i,j)}$ of E such that $e_i \cap d_1 \neq \emptyset$ and $e_j \cap d_4 \neq \emptyset$ is the subchain $E_{(1,x_3)}$. If one lets $r = x_2$ and $s = x_3$ then it can be seen that Definition 2.61 is satisfied. Figure 1 illustrates case 1.

Case 2: The chain D has exactly five links. Definition 2.61 implies that there must exist links numbered 1, x_1, x_2, x_3, x_4, x_5, x_6, and x_7 of E such that $1<x_1<x_2<x_3<x_4<x_5<x_6<x_7$ and such that E

follows the pattern $(1,1)$, $(x_1,3)$, $(x_2,2)$, $(x_3,4)$, $(x_4,2)$, $(x_5,4)$, $(x_6,3)$, $(x_7,5)$ in D. In this example one could select links d_h and d_k where $|h - k| > 2$ in any of the following ways, h=1, k=5; h=5, k=1; h=1, k=4; h=4, k=1; h=2, k=5; h=5, k=2. If h=1 and k=5 then an example of a subchain $E_{(i,j)}$ of E where $e_i \cap d_h \neq \emptyset$ and $e_j \cap d_k \neq \emptyset$ is the subchain $E_{(1,x_7)}$. In this case one can let $r = x_3$ and $s = x_4$ and then conditions (a), (b), and (c) of Definition 2.61 are satisfied. If h=5 and k=2 then an example of a subchain $E_{(i,j)}$ of E where $e_i \cap d_h \neq \emptyset$ and $e_j \cap d_k \neq \emptyset$ is the subchain $E_{(x_7,x_4)}$. In this case one can let $r = x_5$ and $s = x_6$ and again conditions (a), (b), and (c) of Definition 2.61 are satisfied. Similar selections can be made for the other possible choices of values for h and k. Figure 2 illustrates case 2.

Case 3: The chain D has exactly 6 links. Figure 3 illustrates case 3.

Even Figure 3 is not adequate to illustrate the complexities involved in a sequence of crooked chains. For example, suppose it is desired to draw a chain F crooked in E from a point p in the first link of E in Figure 3 to a point q in the last link of E. Suppose F has n links. Now obviously, as F traverses any six links of E, the pattern that E follows in D in Figure 3 must be followed by F in E. But notice also that just <u>one</u> of the <u>many</u> other patterns that must be followed by F in E is that there must be a subchain F_1 of F whose first link intersects e_1 and whose last link intersects e_{n-1}; there must be another subchain F_2 of F (distinct from F_1 except for its first link which is the last link of F_1) whose <u>first</u> link

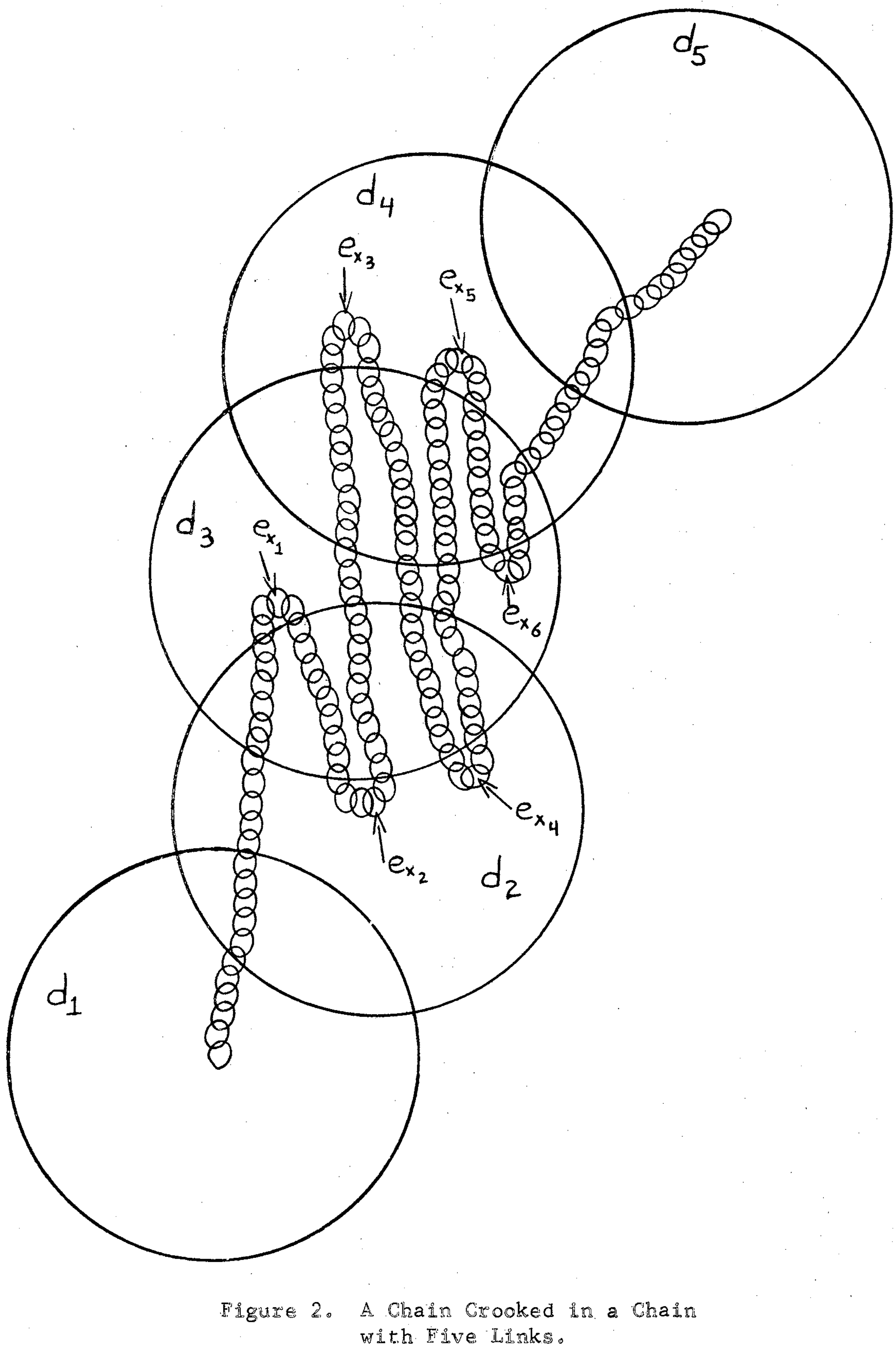

Figure 2. A Chain Crooked in a Chain with Five Links.

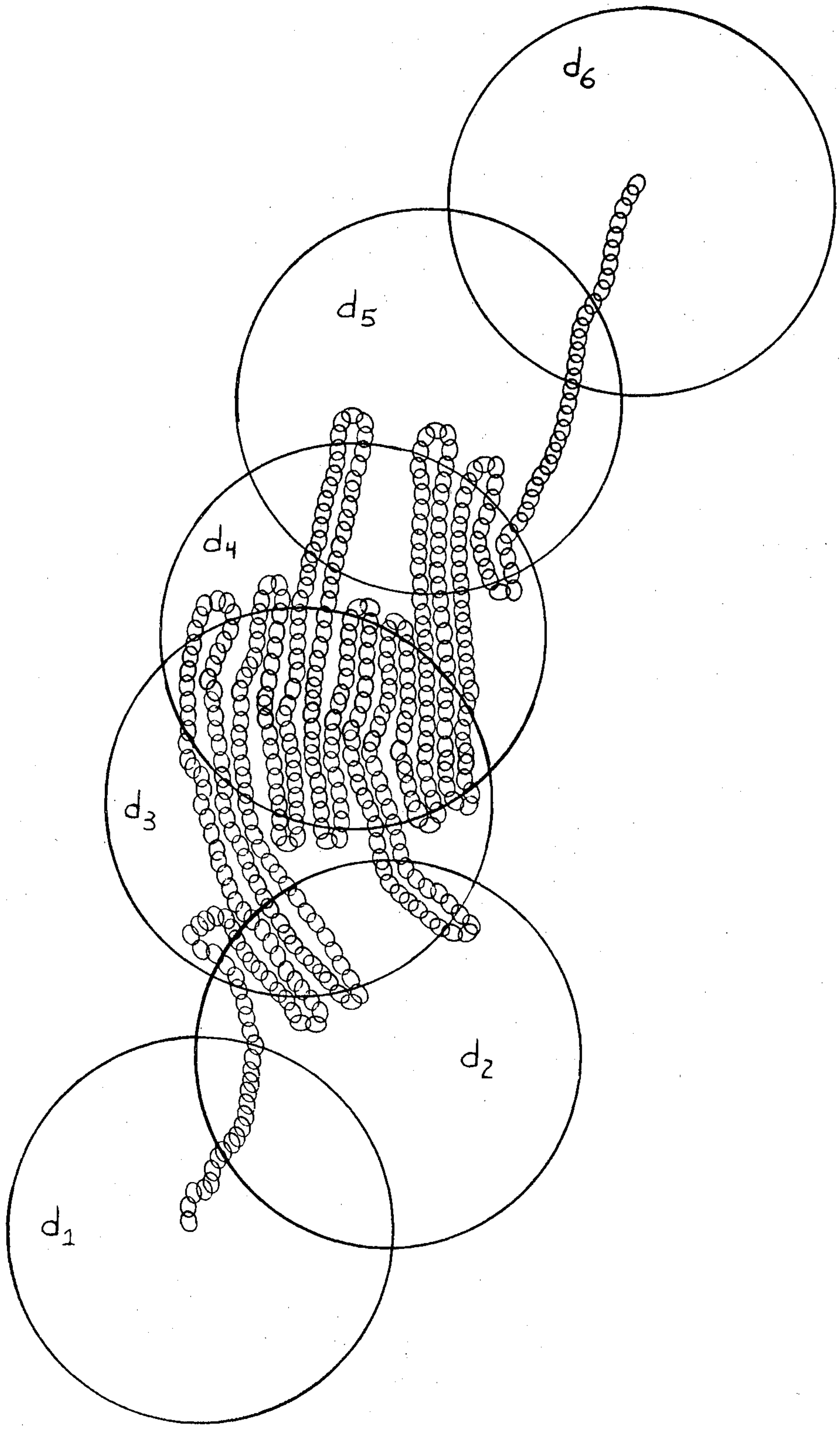

Figure 3. A Chain Crooked in a Chain with Six Links.

intersects e_{n-1} and whose last link intersects e_2; and there must be still another subchain F_3 of F (distinct from F_1 and F_2 except that the first link of F_3 is the last link of F_2 such that the first link of F_3 intersects e_2 and the last link intersects e_n. Now each of the chains F_1, F_2, and F_3 must be the union of three distinct chains following patterns similar to the one described above; each of those nine chains must be the union of three distinct chains following that pattern; and etc., until the point is finally reached that none of the subchains under consideration traverse more than three links of E.

Definition 2.64: The chain E is a consolidation of the chain D if and only if:

(1) Each link of E is the union of a subcollection of links of D, and

(2) D is contained in E.

It is now possible to establish some properties of crooked chains. The first two theorems in this section should seem reasonable even without considering their proofs. The last four theorems are almost impossible to visualize; however, the techniques employed in their proofs help to make the theorems understandable.

Theorem 2.65: If D, E, and F are chains such that D is a consolidation of E and F is crooked in E, then F is crooked in D.

Proof: It must be shown that if $F_{(h,k)}$ is a subchain of F such that the end links f_h and f_k of $F_{(h,k)}$ intersect links d_r and d_s

of D with $|r - s| > 2$ then $F_{(h,k)}$ is the union of three chains $F_{(h,u)}$, $F_{(u,v)}$, and $F_{(v,k)}$ such that $(k-h)(v-u)$ is positive and f_u and f_v are subsets of links of $D_{(r,s)}$ adjacent to d_s and d_r respectively.

The special case where no interior link of $F_{(h,k)}$ intersects either d_r or d_s will be considered first. Let e_m and e_n be links of E contained in d_r and d_s such that e_m and e_n intersect f_h and f_k respectively. Now D is a consolidation of E, hence $E_{(m,n)}$ is contained in $D_{(r,s)}$. The link $e_{m'}$ of $E_{(m,n)}$ adjacent to e_m intersects an interior link of $F_{(h,k)}$ and hence is not contained in d_r. Therefore, the link $e_{m'}$ of $E_{(m,n)}$ is contained in the link of $D_{(r,s)}$ adjacent to d_r. Similarly, the link $e_{n'}$ of $E_{(m,n)}$ adjacent to e_n is in the link of $D_{(r,s)}$ adjacent to d_s.

Now F is crooked in E so $F_{(h,k)}$ is the union of three chains $F_{(h,u)}$, $F_{(u,v)}$, and $F_{(v,k)}$ such that $(k-h)(v-u)$ is positive and f_u and f_v lie in the links of $E_{(m,n)}$ adjacent to e_n and e_m respectively. But from the previous argument we then have f_u and f_v contained in links adjacent to d_s and d_r respectively.

Now that the special case is proved, the remainder of the proof is easy. For suppose that $F_{(w,z)}$ is any subchain of F such that the end links f_w and f_z of $F_{(w,z)}$ intersect links d_r and d_s of D with $|r - s| > 2$. There exists a subchain $F_{(h,k)}$ of $F_{(w,z)}$ such that f_h and f_k intersect links of d_r and d_s of D but no interior links of $F_{(h,k)}$ intersect either d_r or d_s. By the special case, $F_{(h,k)}$ is the union of three chains $F_{(h,u)}$, $F_{(u,v)}$, and $F_{(v,k)}$ such that $(k-h)(v-u)$ is positive and f_u and f_v are contained in links adjacent to d_s and d_r respectively. But this means $F_{(w,z)}$ is the union of the three chains $F_{(w,u)}$, $F_{(u,v)}$, and $F_{(v,z)}$ where $(z-w)(v-u)$ is positive and f_u and f_v are contained in links adjacent to d_s and d_r respectively.

Therefore F is crooked in D.

<u>Theorem 2.66</u>: If D, E, and F are chains such that E contains F and is crooked in D, then F is crooked in D.

Proof: Let f_h and f_u be links of F intersecting links d_r and d_s respectively of D and $|r - s| > 2$. It must be shown that $F_{(h,k)}$ is the union of three chains $F_{(h,u)}$, $F_{(u,v)}$, and $F_{(v,k)}$ such that $(k-h)(v-u)$ is positive and f_u and f_v are subsets of links of $D_{(r,s)}$ adjacent to d_s and d_r respectively. Let e_m and e_n be links of E containing links f_h and f_k respectively. Now $E_{(m,n)}$ is the union of three chains $E_{(m,x)}$, $E_{(x,y)}$, and $E_{(y,n)}$ where $(n-m)(y-x) > 0$ and e_x and e_y are contained in links of $D_{(r,s)}$ adjacent to d_s and d_r respectively.

Let f_u be a link of $F_{(h,k)}$ contained in e_x and let f_v be a link of $F_{(u,k)}$ contained in e_y.

It is clear that $F_{(h,k)}$ is the union of three chains $F_{(h,u)}$, $F_{(u,v)}$, and $F_{(v,k)}$ with $(k-h)(v-u) > 0$, and that f_u and f_v are subsets of links of $D_{(r,s)}$ adjacent to d_s and d_r respectively.

The next two theorems are useful because they show how to create a chain following a desired pattern from some chain in an existing sequence of crooked chains. The previous two theorems show that this new chain will retain certain desirable properties.

Theorem 2.67: If the chain D is crooked in the chain $E = (e_1, e_2, \ldots, e_m)$ and d_j is a particular link of D, then there is a chain F such that F is a consolidation of D, d_j is contained in only the first link of F, each link of F is a subset of the union of two adjacent links of E, and any link of F containing an end link of D which intersects e_1 or e_m is a subset of $e_1 \cup e_2$ or $e_{m-1} \cup e_m$.

Proof: In case m = 1, 2, 3, or 4, let one link of F be the union of the links of D contained in $e_1 \cup e_2$ and the other link of F be the union of the links of D contained in $e_3 \cup e_4$. Note that it may be true that F has only one link, but whether F has one link or two, the links of F can still be numbered so that the conclusion of the theorem is satisfied.

The proof is completed by induction on m. Suppose the theorem is true for $m = 1, \ldots, r-1$, where $r-1 > 4$.

The special case in which no interior link of D intersects either e_1 or e_r will be proved first. There is no loss of generality in the following argument if it is assumed that the end links of d_1 and d_n of D intersect e_1 and e_r respectively.

Let $D = D_{(1,h)} \cup D_{(h,k)} \cup D_{(k,n)}$ where $1<h<k<n$, d_h is a subset of e_{r-1} and d_k is a subset of e_2. The link d_j may be a link in any of the three subchains of D in the above union. The case where d_j is a link of $D_{(1,h)}$ will be argued. If d_j is a link of $D_{(h,k)}$ or $D_{(k,n)}$ the theorem may be proved by techniques similar to the ones used below.

The chain $D_{(1,h)}$ does not intersect e_r because no interior link of D intersects e_r. Hence $E_{(1,r-1)}$ contains $D_{(1,h)}$. Since $E_{(1,r-1)}$

has less than r links, the induction hypothesis implies the existence of a chain H such that H is a consolidation of $D_{(1,h)}$, only the first link of H contains d_j, each link of H is a subset of the union of two adjacent links of $E_{(1,r-1)}$, and any link of H containing d_1 or d_h is a subset of $e_1 \cup e_2$ or $e_{r-2} \cup e_{r-1}$.

Let h_u be the first link of H that contains d_h. Note the h_u is a subset of $e_{r-2} \cup e_{r-1}$ because d_h is a subset of e_{r-1}, $r > 4$, and hence d_h is not a subset of $e_1 \cup e_2$.

The possibility exists that u=1. If u=1 then d_j is a subset of $e_{r-2} \cup e_{r-1}$ because d_j is a subset of $h_1 = h_u$. Let s_1 be the union of the links of D in $e_1 \cup e_2$ but not in $D_{(k,n)}$; let s_2 be the union of the links of D in e_3 but not in e_2 nor in $D_{(k,n)}$; . . .; let s_{r-4} be the union of the links of D in e_{r-3} but not in e_{r-4} nor in $D_{(k,n)}$; and let s_{r-3} be the union of the links of D in $e_{r-2} \cup e_{r-1}$ but not in e_{r-3} and not in $D_{(k,n)}$. Now let $F = (s_{r-3}, s_{r-4}, \ldots, s_1, d_k, d_{k+1}, \ldots, d_n)$. Then F is a chain satisfying the conclusions of the theorem. Virtually the same proof as that constructed in the next case ($u \geq 2$) can be used to show that F actually does satisfy all conditions of the theorem.

Now suppose $u \geq 2$. Let s_1 be the union of the links of D which are contained in $e_1 \cup e_2$ but not in e_3 and not in $H_{(1,u-1)}$ nor $D_{(k+1,n)}$; let s_2 be the union of the links of D which are in e_3 but not in e_2 and not in $H_{(1,u-1)}$ nor $D_{(k+1,n)}$; . . .; let s_{r-4} be the union of the links of D which are in e_{r-3} but not in e_{r-4} and not in $H_{(1,u-1)}$ nor $D_{(k+1,n)}$; and let s_{r-3} be the union of the links of D which are in $e_{r-2} \cup e_{r-1}$ but not in $H_{(1,u-1)}$ nor $D_{(k+1,n)}$.

Define F as follows: $F = (h_1, \ldots, h_{u-1}, s_{r-3}, \ldots, s_1, d_{k+1}, \ldots, d_n)$. It will now be shown that F has the properties asserted in the conclusions of the theorem.

It is not difficult to see that F actually is a chain. To show that F is a consolidation of D just note first that each link of F is a union of a subcollection of links of D, by definition of the links of F. Then note that F clearly contains D because, by definition, F contains all of D that is in $E_{(1,r)}$, and since no interior link of D intersects e_r, then all of D must be contained in $E_{(1,r)}$. Now if the links of F are numbered so that the first link of F is h_1, then d_j is in only the first link of F. Each link of F is a subset of two adjacent links of E because of the corresponding property of H, the definition of $s_1, s_2, \ldots, s_{r-3}$, and the fact that each link of $D_{(k+1,n)}$ is a subset of one link of E. Since h_u is a subset of $e_{r-2} \cup e_{r-1}$ and $r > 4$, and since d_1 intersects e_1, then d_1 is not a subset of h_u. Hence, d_1 is contained in $H_{(1,u-1)}$. Therefore, any link of F containing d_1 which intersects e_1 is a subset of $e_1 \cup e_2$ because H has that property. Now d_n, the other end link of D, is also an end link of F. Hence, it is clear that any link of F containing d_n which intersects e_1 or e_m is a subset of $e_1 \cup e_2$ or $e_{m-1} \cup e_m$.

Thus, the special case of the theorem in which D has no interior links intersecting either e_1 or e_r has been proved.

The more general case in which D may have interior links intersecting e_1 or e_r can now be established.

Let $D_{(h,k)}$ be the maximal subchain of D with the property that $D_{(h,k)}$ contains d_j, and no interior link of $D_{(h,k)}$ intersects either e_1 or e_r. If no link of $D_{(h,k)}$ intersects e_1, or if no link of $D_{(h,k)}$ intersects e_r, then by the induction hypothesis, there exists a chain H such that H is a consolidation of $D_{(h,k)}$, only the first link of H contains d_j, each link of H is a subset of the union of two adjacent links of E, and any link of H containing an end link of $D_{(h,k)}$ which intersects e_1 or e_r is a subset of $e_1 \cup e_2$ or $e_{r-1} \cup e_r$. If one link of $D_{(h,k)}$ intersects e_1 and one link intersects e_r, then H exists by the special case which was previously proved.

Let h_u be the first link of H that intersects either e_1 or e_r. The case where h_u intersects e_1 will be argued. The other case can be proved in a similar fashion. If h_u intersects e_1 then h_u is a subset of $e_1 \cup e_2$ by the properties of H.

Let s_1 be the union of the links of D that are in $e_1 \cup e_2$ but not in $H_{(1,u-1)}$; let s_2 be the union of the links of D that are in e_3 but not in e_2 and not in $H_{(1,u-1)}$; . . .; let s_{r-3} be the union of the links of D that are in e_{r-2} but not in e_{r-3} and not in $H_{(1,u-1)}$; and let s_{r-2} be the union of the links in D that are in $e_{r-1} \cup e_r$ but not in $H_{(1,u-1)}$.

Define F as follows: $F = (h_1, \ldots, h_{u-1}, s_1, \ldots, s_{r-2})$. Virtually the same argument as the one used to prove the special case can be applied to show that F satisfies the conclusions of the theorem.

Theorem 2.68: If $D = (d_1, d_2, \ldots, d_n)$ is a chain crooked in the chain $E = (e_1, e_2, \ldots, e_m)$ and $D_{(r,s)}$ is a subchain of D such that a link of $D_{(r,s)}$ intersects e_1 and a link of $D_{(r,s)}$ intersects

e_m, then there is a chain F such that F is a consolidation of D, each element of F is a subset of the union of two adjacent links of E, d_r is contained in only the first link of F and d_s is contained in only the last.

Proof: If $m \leq 4$, let F be the chain whose links belong to the set $\{f_x, f_y\}$ where f_x is the union of the links of D that are in $e_1 \cup e_2$ and f_y is the union of the links of D that are in $e_3 \cup e_4$. It is clear that F satisfies the conclusions of the theorem.

Now let $m > 4$. A subchain $D_{(h,k)}$ of $D_{(r,s)}$ may be chosen so that (1) $h < k$, (2) d_h intersects e_1, d_k intersects e_m, and (3) d_r and d_s are links of $D_{(1,h)}$ and $D_{(k,n)}$ respectively.

By applying Theorem 2.67 to $D_{(1,h)}$ and to $D_{(k,n)}$ both of which are crooked in E, two chains H and G can be found with the following properties: (1) The chain H is a consolidation of $D_{(1,h)}$, each link of H is a subset of the union of two adjacent links of E, d_r is contained in only the first link of H, and any link of H containing d_h is a subset of either $e_1 \cup e_2$ or $e_{m-1} \cup e_m$, and (2) the chain G is a consolidation of $D_{(k,n)}$, each link of G is a subset of the union of two adjacent links of E, d_s is contained in only the first link of G, and any link of G containing d_k is a subset of $e_1 \cup e_2$ or $e_{m-1} \cup e_m$.

Since $m > 4$, if h_u and g_v are the first links of H and G containing d_h and d_k respectively then h_u is a subset of $e_1 \cup e_2$ and g_v is a subset of $e_{m-1} \cup e_m$.

Let s_1 be the union of the links of D that are in $e_1 \cup e_2$ but not in $H_{(1,u-1)}$ nor $G_{(v-1,1)}$; let s_2 be the union of the links of D that are in e_3 but not in e_2 and not in $H_{(1,u-1)}$ nor $G_{(v-1,1)}$; . . .; let s_{m-3} be the union of the links of D that are in e_{m-2} but not in e_{m-3} and not in $H_{(1,u-1)}$ nor $G_{(v-1,1)}$; and let s_{m-2} be the union of the links of D that are in $e_{m-1} \cup e_m$ but not in $H_{(1,u-1)}$ nor $G_{(v-1,1)}$.

Now define the chain F as follows: (1) the first u-1 links of F are the links of $H_{(1,u-1)}$, (2) the last v-1 links of F are the links of $G_{(v-1,1)}$, and (3) the other links of F are s_1, s_2, . . ., s_{m-2}.

It is not difficult to see that F is a chain and is a consolidation of D. Each link of F is a subset of the union of two adjacent links of E because of the corresponding property of H and G, and because the links s_1, s_2, . . ., s_{m-2} were defined in such a way that they were subsets of the union of adjacent links of E. The link d_r is contained in only the first link of F because it is contained in only the first link of H. The link d_s is contained in only the last link of F because it is contained in only the first link of G and the first link of G is the last link of F.

As in the case of the previous two theorems, the next theorem shows how to create chains following desired patterns from existing chains. This particular theorem will be utilized only to establish the more important theorem, Theorem 2.70.

<u>Theorem 2.69</u>: Suppose $(1,x_1)$, $(2,x_2)$, . . ., (n,x_n) is a collection of ordered pairs of positive integers such that $h = x_1 \leq x_i \leq x_n = k$ and $|x_i - x_{i+1}| \leq 1$ for i = 1, 2, . . ., n-1. Suppose D_1, D_2, . . ., D_m, . . . is a sequence of chains from P to Q such that for each

positive integer i, D_{i+1} is crooked in D_i, and no link of D_i has a diameter of more than 1/i. Denote the r-th link of D_i by d_{ir}. Suppose that the subchain $D_{2(u,v)}$ of D_2 is contained in the subchain of $D_{1(h,k)}$ of D_1 and the closures of d_{2u} and d_{2v} are mutually exclusive subsets of d_{1h} and d_{1k} respectively. Then for each integer w, there is an integer j greater than w and a chain $E = (e_1, e_2, \ldots, e_n)$ following the pattern $(1,x_1), (2,x_2), \ldots, (n,x_n)$ in D_1 such that E is a consolidation of the links of D_j contained in $D_{2(u,v)}$ and no interior link of E intersects $d_{2u} \cup d_{2v}$.

Proof: Since $x_1 = h$, $x_n = k$ and $|x_i - x_{i+1}| \leq 1$, then $n \geq k-h+1$. The theorem will first be proven for $n = k-h+1$ and then completed by induction on n.

Since the closure of d_{2u} is a subset of d_{1h}, the closure of d_{2v} is a subset of d_{1k} and the diameter of any link of D_i is less than or equal to 1/i, then there exists an integer m greater than w such that any link of D_m that intersects d_{2u} is a subset of d_{1h} and any link of D_m that intersects d_{2v} is a subset of d_{1k}.

Let $n = k-h+1$ and let j be any integer greater than m. Let e_1 be the union of the links of D_j contained in $d_{1h} \cap D^*_{2(u,v)}$; let e_2 be the union of the links of D_j contained in $d_{1(h+1)} \cap D^*_{2(u,v)}$ but not d_{1h}; . . .; and let e_n be the union of the links of D_j contained in $d_{1k} \cap D^*_{2(u,v)}$. Now certainly e_1 is a subset of d_{1h}, e_2 is a subset of $d_{1(h+1)}$, . . ., and e_n is a subset of d_{1k}. Hence, E follows the pattern $(1,h), (2,h+1), \ldots, (n,k)$ in D_1. But in this case, this is the pattern $(1,x_1), (2,x_2), \ldots, (n,x_n)$. Obviously, E is a consolidation of the links of D_j contained in E. Since $j > m$ then from the corresponding property of D_m, it can be seen that any link of D_j intersecting d_{2u} is contained in d_{1h} and any link intersecting d_{2v} is contained in d_{1k}. Thus, the only links of E which intersect either d_{2u} or d_{2v} are e_1 and e_m respectively. This establishes the theorem for $n = k-h+1$.

Now suppose the theorem is true for all integers less than n.

The special case where $x_1 = x_2$ will be considered first. By the induction hypothesis there exists an integer $s > w$ and a chain $F = (f_1, f_2, \ldots, f_{n-1})$ following the pattern $(1,x_2), (2,x_3), \ldots, (n-1,x_n)$ in D_1 such that F is a consolidation of the links of D_s in $D_{2(u,v)}$ and such that only the first link of F intersects d_{2u} and only the last intersects d_{2v}.

The same reasoning utilized to establish the existence of D_m in the case $n = k-h+1$ can now be used to assert the existence of an integer $j > s$ such that any three linked suchain of D_j which intersects d_{2u} is a subset of d_{1h}.

Let e_1 be the union of the links of D_j which are contained in f_1 and which intersect d_{2u}. Since f_1 is a subset of $x_1 = x_2$ then e_1 is a subset of x_1. Let e_2 be the union of all links of D_j which are contained in f_1 but do not intersect d_{2u}. The property of D_j described in the preceding paragraph shows that e_2 actually exists. Also, e_2 is a subset of f_1 which is a subset of x_2, and thus, e_2 is a subset of x_2. Let e_3 be the union of the links of D_j contained in f_2 but not f_1; . . .; let e_n be the union of the links of D_j contained in f_{n-1} but not f_{n-2}. This construction of $E = (e_1, e_2, \ldots, e_n)$ shows that E is a consolidation of the links of D_j contained in $D_{2(u,v)}$, and that E follows the pattern $(1,x_1)$, $(2,x_2)$, . . ., (n,x_n) in D_1. The corresponding property of F together with the construction process for the links e_1 and e_2 show that no interior links of E intersect d_{2u} or d_{2v}.

The next special case which should be considered is the case where an integer r exists such that $2<r<n$ and $x_1 = x_r$. Techniques similar to those which have been employed to this point in the proof can be used to construct a proof for this case. However, the proof for this special case is extremely long and tedious, and thus has been omitted.

The final case that must be considered is the one where for every integer i such that $1<i<n$, $x_i \neq x_1$.

It must now be noted that the fact that D_2 is crooked in D_1 has not been used in this proof. Indeed, the only case where any part of the hypothesis that D_{i+1} is crooked in D_i is ever used is in the case which was omitted. And in that case, it is not necessary to have D_2 crooked in D_1. These facts are pointed out because a chain W such that W has all the necessary properties of $D_{2(u,v)}$ will now be constructed. It will then be asserted that the induction hypothesis applies to W since W will have the essential properties of $D_{2(u,v)}$ and since $D_{2(u,v)}$ is an arbitrary chain contained in an arbitrary subchain of D_1. That is, W will be contained in a subchain of D_1, the closures of the first and last links of W will be subsets of the first and last links respectively of that subchain of D_1, and by virtue of Theorem 2.65, subchains of D_j, $j > 2$, contained in W will be crooked in W.

Let $W = (w_1, w_2, \ldots, w_t)$ be defined as follows: The link w_1 is the consolidation of the links of $D_{2(u,v)}$ that are not contained in d_{1h} but intersect d_{1h}; w_2 is the consolidation of the links of $D_{2(u,v)}$ that are contained in $d_{1(h+1)}$ but do not intersect d_{1h}; w_3 is the consolidation of the links of $D_{2(u,v)}$ that are in $d_{1(h+2)}$ but not in $d_{1(h+1)}$; . . .; w_{t-1} is the consolidation of all links of $D_{2(u,v)}$ except d_{2v} that are in d_{1k} but not in $d_{1(k-1)}$; and $w_t = d_{2v}$. Hence, W is a chain contained in $D_{1(h+1,k)}$ such that the closure of w_1 is a subset of $d_{1(h+1)}$ and the closure of w_t is a subset of d_{1k}. Thus, by the induction hypothesis, there exists an integer j greater than w and a chain $F = (f_1, f_2, \ldots, f_{n-1})$ such that F is a consolidation of the links of D_j in W, F follows the pattern $(1,x_2)$, $(2,x_3)$, . . ., $(n-1,x_n)$ in D_1, f_1 is the only element of F intersecting w_1, and f_{n-1} is the only link of F intersecting $w_t = d_{2v}$.

Define e_1 to be the union of all elements of D_j in $D^*_{2(u,v)} \cap d_{1h}$, $e_2 = f_1, \ldots, e_n = f_{n-1}$. Then $E = (e_1, e_2, \ldots, e_n)$ satisfies all conclusions of the theorem.

The next theorem is the most important one of this section. This theorem furnishes the result that will eventually provide the key to proving the homogeneity of the pseudo-arc.

<u>Theorem 2.70</u>: Suppose $(1,x_1)$, $(2,x_2)$, . . ., (n,x_n) is a collection of ordered pairs of positive integers such that $1 = x_1 \le x_i \le x_n$ and $|x_i - x_{i+1}| \le 1$ for $i = 1, 2, \ldots, n-1$. Suppose $D_1, D_2, \ldots$ is a sequence of chains from P to Q such that D_1 has x_n links and for each positive integer i, D_{i+1} is crooked in D_i, the closure of each

link of D_{i+1} is a compact subset of a link of D_i, and no link of D_i has a diameter of more than $1/i$. Then there is an integer j and a chain E from P to Q such that E is a consolidation of D_j and E follows the pattern $(1,x_1), (2,x_2), \ldots, (n,x_n)$ in D_1.

Proof: Let $h = 1$, $k = x_n$, $u = 1$, and $v = m$ where D_2 has m links, then the conclusion of this theorem is immediate from Theorem 2.69.

CHAPTER III

EXAMPLES OF HOMOGENEOUS BOUNDED PLANE CONTINUA

Introduction

The three known examples of nondegenerate homogeneous bounded plane continua which were briefly discussed in Chapter I will be presented in more detail in this chapter. The first two examples, the simple closed curve and the pseudo-arc, will be thoroughly discussed and the proofs of their homogeneity will be presented. The third example, the circle of pseudo-arcs, will be briefly described, but the proof of its homogeneity will be omitted.

It will be shown in Chapter IV that any chainable nondegenerate homogeneous bounded plane continuum that is not homeomorphic to a simple closed curve, a pseudo-arc, or a circle of pseudo-arcs must be an indecomposable continuum separating the plane. It is not known whether such a homogeneous continuum actually exists. The final example of the chapter will be an example of a chainable hereditarily indecomposable continuum which separates the plane. This continuum is suspected of being homogeneous.

All examples given in this chapter will be described in such a way that their existence in the plane is clear. Hence, any topological properties of the plane which are needed in proofs will be used without hesitation.

The Simple Closed Curve

The fact that a simple closed curve is a homogeneous bounded continuum is almost immediate from its definition (10; 170, Definition 3.2). Since a simple closed curve is defined to be a homeomorph of the unit circle, then by the compactness of the unit circle any simple closed curve is bounded (10; 170, Theorem 3.3 and 10; 75, Theorem 4.16). The crux of the remainder of the proof is contained in the following lemma.

<u>Lemma</u> <u>3.1</u>: The unit circle U is homogeneous.

Proof: Let x_1 and x_2 be any two points on the unit circle U. A function F from U to U will be defined such that $F(x_1) = x_2$ and F is a homeomorphism.

Let the coordinates of x_1 be given by $(\cos \theta_1, \sin \theta_1)$, $0 \leq \theta_1 < 2\pi$, and the coordinates of x_2 be given by $(\cos \theta_2, \sin \theta_2)$, $0 \leq \theta_2 < 2\pi$.

Either $\theta_1 \leq \theta_2$ or $\theta_2 \leq \theta_1$. It will be supposed that $\theta_1 \leq \theta_2$. The argument for the case $\theta_2 \leq \theta_2$ is similar to the one which follows.

Let $\theta_2 - \theta_1 = \emptyset$. It is clear that $0 \leq \emptyset < 2\pi$.

Let $x \in U$ and the coordinates of x be $(\cos \theta, \sin \theta)$. Either $0 \leq \theta + \emptyset < 2\pi$ or $0 \leq \theta + \emptyset - 2\pi < 2\pi$. If $0 \leq \theta + \emptyset < 2\pi$, define $F(x) = y$ where the coordinates of y are $(\cos (\theta + \emptyset), \sin (\theta + \emptyset))$. If $0 \leq \theta + \emptyset - 2\pi < 2\pi$, define $F(x) = y$ where the coordinates of y are $(\cos (\theta + \emptyset - 2\pi), \sin (\theta + \emptyset - 2\pi))$.

Since F is just a function which rotates the unit circle through the angle $\emptyset$, then it is not hard to show that F is a homeomorphism.

Also, $F(x_1) = x_2$ because $(\cos(\theta_1 + \emptyset), \sin(\theta_1 + \emptyset)) = (\cos \theta_2, \sin \theta_2)$.

Theorem 3.2: A simple closed curve is a homogeneous bounded continuum.

Proof: It was argued above that a simple closed curve is bounded. Since a simple closed curve is homeomorphic to the unit circle then it is immediate from the corresponding property of the unit circle that it is a continuum (10: 170, Theorem 3.3). Lemma 3.1 implies that it is homogeneous.

The Pseudo-arc

The simplicity of the description of the first example of a homogeneous bounded plane continuum, and of the proof that it is indeed homogeneous, gives no indication of the difficulties which are involved in presenting the second example. This second example, the pseudo-arc, is defined in terms of sequences of crooked chains. In addition to the theorems on crooked chains, some very delicate proofs of preliminary theorems are necessary in order to establish the homogeneity of the pseudo-arc.

Definition 3.3: Let S be a compact metric space and let P and Q be distinct points of S. Let $D_1, D_2, \ldots$ be a sequence such that:

(1) D_i is a chain from P to Q, $i = 1, 2, \ldots,$

(2) D_{i+1} is crooked in D_i, $i = 1, 2, \ldots,$

(3) if $D_i = (d_{i1}, d_{i2}, \ldots, d_{in_i})$, then the diameter of d_{ij} is less than or equal to $1/i$, $j = 1, 2, \ldots, n_i$.

(4) if d is a link of D_{i+1} then there exists a link d' of D_i such that $\bar{d} \subset d'$.

Let $M = \bigcap_{i=1}^{\infty} D_i^*$. Then M is called a pseudo-arc.

It is virtually impossible to visualize a pseudo-arc. One can conceive of some of the difficulties involved in trying to describe the pseudo-arc with pictures if Figures 1, 2, and 3 are studied. In any one of these figures one could think of chain D as being D_1 and chain E as being D_2 in the sequence given in Definition 3.3. Suppose it is desired to draw the chain D_3 contained in the chain D_2 in Figure 3. The virtual impossibility of the task of drawing the chain D_3 was discussed in the paragraph immediately following case 3 of Example 2.63. Of course, it should be noted that in these figures the difficulties are somewhat exaggerated since condition (3) of Definition 3.3 has been more than amply satisfied. However, even if all conditions are satisfied in such a way that the minimum number of links exist in each of the chains D_2, D_3, . . ., it is difficult to draw any chain after the second one of the sequence.

It should also be noted that it has not been assumed that the links of the chains in Definition 3.3 are connected. It may therefore be surprising that M is a continuum. However, that such is the case is the main result of the following theorem.

Theorem 3.4: The pseudo-arc M is a compact continuum.

Proof: It will first be proved that M is closed. Since each D_i has only a finite number of links then (4) of Definition 3.3 implies that

$\overline{D^*_{i+1}} \subset D^*_i$, $i = 1, 2, \ldots$. Now suppose M is not closed. Let P' be a limit point of M such that P' does not belong to M. If P' does not belong to M then there exists an integer j such that P' does not belong to D^*_j. Now if P' does not belong to D^*_j then P' does not belong to $\overline{D^*_{j+1}}$ since $\overline{D^*_{j+1}} \subset D^*_j$. But $\overline{D^*_{j+1}}$ is a closed set containing M and hence all limit points of M. This contradicts P' is a limit point of M.

It is immediate that M is compact since M is a closed subset of a compact space.

Suppose M is not connected. Then M is the union of two separated sets H and K. Let the distance between H and K be k. The number k is positive (10; 91, Theorem 1.13). There exists an integer i such that $3/i < k$. Since every link of D_i has diameter less than or equal to $1/i$, there exists an interior link d_i of D_i such that $d_i \cap H = \emptyset$ and $d_i \cap K = \emptyset$. Now by the definition of M, for every $j > i$ there exists a link of D_j whose closure is a subset of d_i. Since the intersection of a monotonic collection of closed and compact sets is not empty, it is clear that $\bigcap_{n=i}^{\infty} d_n \neq \emptyset$ (10; 69, Theorem 3.30). Let P_1 belong to $\bigcap_{n=i}^{\infty} d_n$, then P_1 belongs to M. Hence P_1 belongs to H or K. This is a contradiction.

The fact that the pseudo-arc is indecomposable will be used in establishing its homogeneity. The proof that the pseudo-arc is indecomposable could be deduced from the fact that the pseudo-arc is a homogeneous bounded plane continuum that does not separate the plane, if this latter statement could somehow be proven first. That every homogeneous bounded plane continuum that does not separate the plane, is indecomposable, is one of the important results which has been achieved since the pseudo-arc was first defined (see Theorem 4.8 in Chapter IV). However, no proof that the pseudo-arc is homogeneous which does not make use of the fact that it is indecomposable has yet been published.

Theorem 3.5: Each subcontinuum of the pseudo-arc M is indecomposable.

Proof: Suppose there exists a subcontinuum M' of M which is decomposable. Then $M' = H \cup K$ where H and K are proper subcontinua of M'. Let P_1 be a point of K not belonging to H and P_2 be a point of H not belonging to K. The distance between P_1 and H is greater than zero (10; 91, Theorem 1.13). Similarly, the distance between P_2 and K is greater than zero. Hence, there exists an integer j such that the distance from P_1 to H is greater than 2/j and the distance from P_2 to K is greater than 2/j. Let $D_{j(h,k)}$ $(h < k)$ and $D_{(j+1)(u,v)}$ be subchains of D_j and D_{j+1} respectively such that P_1 and P_2 belong to end links of each of these subchains. Without loss of generality, let P_1 belong to d_{jh} (the link numbered h in the chain D_j), Suppose there exists a link d_{jm} $(h < m < k)$ of $D_{j(h,k)}$ which contains no point of M'. Since M' is a subset of M and M is a subset of D_j then M' is a subset of the union of $D_{j(1,m-1)}$ and $D_{j(m+1,n)}$ where d_{jn} is the last link of D_j.

Now P_1 belongs to $D^*_{j(1,m-1)}$ and P_2 belongs to $D^*_{j(m+1,n)}$ and $M' = (M' \cap D^*_{j(1,m-1)}) \cup (M' \cap D^*_{j(m+1,n)})$. But by definiton of chain, $D^*_{j(1,m-1)}$ and $D^*_{j(m+1,n)}$ are separated. This contradicts the fact that M' is a continuum. Therefore, every link of $D_{j(h,k)}$ contains a point of M'. Similarly, every link of $D_{(j+1)(u,v)}$ contains a point of M'. Therefore, since the distance from P_1 to H and the distance from P_2 to K are both greater than 2/j, it can be seen that $d_{j(h+1)}$ contains a point of K but none of H and $d_{j(k-1)}$ contains a point of H but none of K. It follows that $d_{j(h+1)} \neq d_{j(k-1)}$ and so $|h - k| > 2$. Hence, D_{j+1} is crooked in D_j implies that $D_{(j+1)(u,v)}$ is the union of three chains, $D_{(j+1)(u,r)}$, $D_{(j+1)(r,s)}$, and $D_{(j+1)(s,v)}$ $(r < s)$ such that $d_{(j+1)r}$ and $d_{(j+1)s}$ are subsets of $d_{j(k-1)}$ and $d_{j(h+1)}$ respectively. Since the definition of "crooked" only requires that $d_{(j+1)r}$ and $d_{(j+1)s}$ be subsets of links of D_j adjacent to d_{jk} and d_{jh} respectively, then it may not be clear that these links can be specified to be subsets of $d_{j(k-1)}$ and $d_{j(h+1)}$. However, if it is recalled that the requirement that $h < k$ was also imposed on $D_{j(h,k)}$, then it is not difficult to see that no generality is lost in specifying which links of $D_{j(h,k)}$ contain the end links of $D_{(j+1)(r,s)}$. Now since $d_{(j+1)s}$ is a subset of $d_{j(h+1)}$, $d_{(j+1)v} \cap d_{jk} \neq \emptyset$, and $k-(h+1) \geq 2$ then there must be at least one link $d_{(j+1)t}$, $(s < t \leq v)$ such that $d_{(j+1)t}$ is a subset of $d_{j(k-1)}$. Now every link of $D_{(j+1)(u,v)}$ contains points of M', $d_{(j+1)r}$ and $d_{(j+1)t}$ are subsets of $d_{j(k-1)}$, which contains points of H but not of K, so $d_{(j+1)r}$ and $d_{(j+1)t}$ both contain points of H. But $d_{(j+1)s}$ is a subset of $d_{j(h+1)}$ which contains no points of H. The definition of chain and the fact that

$r < s < t$ shows that H is not a continuum. Therefore, the assumption that M' could be written as the union of two proper subcontinua is false.

In order to prove the existence of the type of homeomorphism which shows that the pseudo-arc is homogeneous, two existence theorems will first be established. The first of these theorems (Theorem 3.6) guarantees the existence of a homeomorphism between certain pairs of compact closed sets. The unusual and very restrictive set of hypotheses for this theorem may make the theorem seem to be so limited in applicability that it would be of little use. However, if both of the sets, M_n $(n = 1, 2)$, mentioned in the hypotheses are the pseudo-arc M and both sequences of domains are the sequence of crooked chains used to define M, then it can be seen that the theorem produces a homeomorphism from M to M. Of course, the existence of a homeomorphism from M to M is obvious (the identity, for instance); however, a special kind of homeomorphism from M to M can be deduced with the aid of the second theorem.

The second theorem (Theorem 3.7) makes use of Theorem 3.6 to prove that for certain pairs of continua there exists a homeomorphism that will map any arbitrary fixed pair of points of the first continuum to any arbitrary fixed pair of the second continuum. Obviously, if it is possible to allow both members of such a pair of continua to be the continuum M, then the homogeneity of M will be established.

<u>Theorem 3.6</u>: Suppose M_1 and M_2 are compact closed sets; $\varepsilon_1, \varepsilon_2, \ldots$ is a sequence of positive numbers with a finite sum; and $X_{(1,1)}$

$X_{(1,2)}, \ldots$ and $X_{(2,1)}, X_{(2,2)}, \ldots$ are sequences of well-ordered collections of domains such that for each n (n = 1, 2) and for each positive integer i, (1) $X_{(n,i)}$ covers M_n, (2) each element of $X_{(n,i)}$ intersects M_n, (3) no element of $X_{(n,i)}$ has a diameter of more than ϵ_i, and (4) if the j-th element of $X_{(n,i+1)}$ intersects the k-th element of $X_{(n,i)}$, then the distance between the j-th element of $X_{(m,i+1)}$ (m = 1, 2) and the k-th element of $X_{(m,i)}$ is less than ϵ_i. Then there is a homeomorphism T carrying M_1 into M_2.

Proof: The first step will be to define T(P) for any point P belonging to M_1.

Let the k-th element of $X_{(n,i)}$ be $x_{(n,i)k}$. Let $Y_{(n,i)}$ denote the well-ordered collection whose k-th element is $y_{(n,i)k}$ where $y_{(n,i)k}$ denotes the set of all points Q such that the distance from Q to $x_{(n,i)k}$ is less than $\epsilon_i + 2(\epsilon_{i+1} + \epsilon_{i+2} + \ldots)$.

If $x_{(n,i)r}$ intersects $x_{(n,i+1)s}$ then by hypothesis (4) the distance between $x_{(m,i)r}$ and $x_{(m,i+1)s}$ is less than ϵ_i. Also, by hypothesis (2), the diameter of $x_{(m,i+1)s}$ is no more than ϵ_{i+1}. Now suppose Q belongs to $y_{(m,i+1)s}$ closure so that the distance from Q to $y_{(m,i+1)s}$ is zero. Then the distance from Q to $x_{(m,i+1)s}$ is less than or equal to $\epsilon_{i+1} + 2(\epsilon_{i+2} + \epsilon_{i+3} + \ldots)$, so the distance from Q to $x_{(n,i)r}$ is less than or equal to $\epsilon_i + 2(\epsilon_{i+1} + \epsilon_{i+2} + \ldots)$. Therefore, the closure of $y_{(m,i+1)s}$ is a subset of the closure of $y_{(m,i)r}$. In general, if $i < j$ and $x_{(n,i)r}$ intersects $x_{(n,j)s}$ the closure of $y_{(m,j)s}$ is a subset of $y_{(m,i)r}$.

Now let P be a point of M_1. By hypothesis, there is a sequence of domains $x_{(1,1)a}, x_{(1,2)b}, \ldots$ containing P. Define T(P) to be the common part of $y_{(2,1)a}$, $y_{(2,2)b}$, ... Since the diameters of the elements of the sequence $y_{(2,1)a}$, $y_{(2,2)b}$, . . . are approaching zero and since the closure of $y_{(2,i+1)u}$ is a subset of $y_{(2,i)v}$ if $x_{(1,i+1)u}$ intersects $x_{(1,i)v}$, then it is clear that T(P) is a point and does not depend on which sequence $x_{(1,1)p}$, $x_{(1,2)q}$, . . . of domains containing P is selected.

Now let D be any domain containing T(P). Since T(P) belongs to every element of $y_{(2,1)a}$, $y_{(2,2)b}$, . . . and since the diameters of the elements of this sequence approach zero, then there exists some term $y_{(2,k)z}$ such that T(P) belongs to $y_{(2,k)z}$ and $y_{(2,k)z}$ is a subset of D. Now since $x_{(2,k)z}$ is a subset of $y_{(2,k)z}$ which is a subset of D and since $x_{(2,k)z}$ contains a point of M_2, then D contains some point of M_2. Thus T(P) is a limit point of M_2. But M_2 is closed and compact. Therefore T(P) belongs to M_2.

To show that T is continuous, let T(P) be a point of M_2 and D be a domain containing T(P). There is an integer j such that any element of $Y_{(2,j)}$ containing T(P) is a subset of D. By definition of T, if $x_{(1,j)r}$ is an element of $X_{(1,j)}$ containing P, $T(M_1 \cap x_{(1,j)r})$ is a subset $y_{(2,j)r}$. Now T(P) belongs to $y_{(2,j)r}$ which is a subset of D. Therefore T is continuous.

Suppose T is not one to one. Then there exist distinct points P_1 and P_2 of M_1 such that $T(P_1) = T(P_2)$. Let the distance from P_1 to P_2 be d. There exists an integer k such that the diameter of every element of $Y_{(1,k)}$ is less than d. Hence no element of $Y_{(1,k)}$ contains both P_1 and P_2. Since $X_{(2,k)}$ covers M_2, some element of $X_{(2,k)}$, say $x_{(2,k)r}$, contains $T(P_1) = T(P_2)$. Now there exists an integer j greater than k such that every element of $Y_{(2,j)}$ containing $T(P_1) = T(P_2)$ is a subset of $x_{(2,k)r}$. Let $x_{(1,j)u}$ and $x_{(1,j)v}$ be elements of $X_{(1,j)}$ containing P_1 and P_2 respectively. This means that both $y_{(2,j)u}$ and $y_{(2,j)v}$ contain $T(P_1) = T(P_2)$ and so both $y_{(2,j)u}$ and $y_{(2,j)v}$ are subsets of $x_{(2,k)r}$. Now $x_{(2,j)u}$ is a subset of $y_{(2,j)u}$ and $x_{(2,j)v}$ is a subset of $y_{(2,j)v}$ and so both $x_{(2,j)u}$ and $y_{(2,j)v}$ are subsets of $x_{(2,k)r}$. It has been established in general that if $i < j$ and $x_{(n,i)r}$ intersects $x_{(n,j)s}$, then the closure of $y_{(m,j)s}$ is a subset of $y_{(m,i)r}$. Now in this case we have $k < j$, $x_{(2,k)r}$ intersects both $x_{(2,j)u}$ and $x_{(2,j)v}$ and so the closures of both $y_{(1,j)u}$ and $y_{(1,j)v}$ are subsets of $y_{(1,k)r}$. But $x_{(1,j)u}$ is a subset of $y_{(1,j)u}$ and $x_{(1,j)v}$ is a subset of $y_{(1,j)v}$; hence, $x_{(1,j)u}$ is a subset of $y_{(1,k)r}$ and $x_{(1,j)v}$ is a subset of $y_{(1,k)r}$. This is a contradiction because P_1 belongs to $x_{(1,j)u}$ and P_2 belongs to $x_{(2,j)v}$ but no element of $Y_{(1,k)}$ contains P_1 and P_2.

If T can be shown to be a closed map, then the proof that T is a homeomorphism will be complete. Now M_1 and M_2 are closed and compact and so T is closed (10: 75, Theorem 4.16 and 10; 66, Theorem 3.19).

Theorem 3.7: Suppose M_1 and M_2 are compact continua; P_1 and Q_1 are points of M_1; P_2 and Q_2 are points of M_2; the sequence of positive numbers $\epsilon_1, \epsilon_2, \ldots$ has limit zero; and the sequences $D_{(1,1)}$, $D_{(1,2)}, \ldots$ and $D_{(2,1)}, D_{(2,2)}, \ldots$ are sequences of chains from P_1 to Q_1 and from P_2 to Q_2 respectively. Suppose also that for each n (n = 1, 2), and for each positive integer i, (1) $D_{(n,i+1)}$ is crooked in $D_{(n,i)}$, (2) the closure of each link of $D_{(n,i+1)}$ is a subset of a link of $D_{(n,i)}$, (3) no link of $D_{(n,i)}$ has a diameter of more than $1/i$, and (4) $M_n = \bigcap_{i=1}^{\infty} D^*_{(n,i)}$. Then there is a homeomorphism carrying M_1 into M_2 that carries P_1 to P_2 and Q_1 to Q_2.

Proof: By hypothesis, there exists an integer t such that no link of $D_{(1,t)}$ has a diameter of more than 1/2. Define $X_{(1,1)}$ to be $D_{(1,t)}$. Define $X_{(2,1)}$ to be a chain from P_2 to Q_2 which has the same number of links as $X_{(1,1)}$ and which is a consolidation of some $D_{(2,i)}$.

By hypothesis, there exists an integer k such that $D_{(2,k)}$ is contained in $D_{(2,i)}$ and no element has a diameter of more than 1/2. Define $X_{(2,2)}$ to be $D_{(2,k)}$. Since $X_{(2,1)}$ is a consolidation of $D_{(2,i)}$ then $X_{(2,2)}$ is contained in $X_{(2,1)}$.

Let $(1,x_1)$, $(2,x_2)$, . . ., (r,x_r) be a pattern followed by $X_{(2,2)}$ in $X_{(2,1)}$ where x_r is the number of links in $X_{(2,1)}$. Since $X_{(2,1)}$ and $X_{(1,1)}$ have the same number of links, then $X_{(1,1)}$ is a chain from P_1 to Q_1 which has x_r links.

Since no link of $D_{(1,t)}$ has a diameter greater than 1/2 and for every i no link of $D_{(1,i)}$ has a diameter greater than ϵ_i (where the limit of the ϵ_i's is zero), then it is possible to define a subsequence $D_{(1,t_1)}$, $D_{(1,t_2)}$, . . . of $D_{(1,1)}$, $D_{(1,2)}$, . . . such that (1) $D_{(1,t_1)} = D_{(1,t)}$, and (2) no link of $D_{(1,t_i)}$ has a diameter greater than $1/i$. By Theorem 2.66, $D_{(1,t_{i+1})}$ is crooked in $D_{(1,t_i)}$, $i = 1, 2, \ldots$.

The sequence $D_{(1,t_i)}$, $i = 1, 2, \ldots$, satisfies the hypotheses of Theorem 2.70. Therefore, there exists an integer j and a chain

E from P_1 to Q_1 such that E is a consolidation of $D_{(1,t_j)}$ and follows the pattern $(1,x_1)$, $(2,x_2)$, . . ., (r,x_r) in $D_{(1,t_1)}$. That is, if $X_{(1,2)}$ is defined to be E, then: (1) $X_{(1,2)}$ is a chain from P_1 to Q_1, (2) $X_{(1,2)}$ is a consolidation of a term of the sequence $D_{(1,i)}$, $i = 1, 2, \ldots$, and (3) $X_{(1,2)}$ follows a pattern in $X_{(1,1)}$ that $X_{(2,2)}$ follows in $X_{(2,1)}$.

Now since $X_{(1,2)}$ is a consolidation of some term of $D_{(1,i)}$, $i = 1, 2, \ldots$, then by hypotheses (1) and (3) there exists an integer j such that $D_{(1,j)}$ is contained in $X_{(1,2)}$ and no link of $D_{(1,j)}$ has diameter greater than 1/4. Define $X_{(1,3)} = D_{(1,j)}$. Using Theorem 2.70 and the same technique as above, it can be shown that there is an integer j and a chain $X_{(2,3)}$ from P_2 to Q_2 such that $X_{(2,3)}$ is a consolidation of $D_{(2,j)}$ and follows a pattern in $X_{(2,2)}$ that $X_{(1,3)}$ follows in $X_{(1,2)}$

The process used to define $X_{(1,2)}$, $X_{(2,2)}$, $X_{(1,3)}$ and $X_{(2,3)}$ can be continued to define the sequences $X_{(1,1)}$, $X_{(1,2)}$, $X_{(1,3)}$, . . . and $X_{(2,1)}$, $X_{(2,2)}$, $X_{(2,3)}$, ...

The following properties of $X_{(n,i)}$, $n = 1, 2$ are immediate: (1) $X_{(n,1)}$, $X_{(n,2)}$, . . . are collections of domains covering M_n, (2) each link of $X_{(n,i)}$, $i = 1, 2, \ldots$, intersects M_n, (3) no link of $X_{(n,2i-1)}$ nor $X_{(n,2i)}$ has a diameter of more than 1/2i and (4) $X_{(n,i+1)}$ is a chain from P_n to Q_n that follows a pattern in $X_{(n,i)}$ that $X_{(m,i+1)}$ follows in $X_{(m,i)}$.

Properties (1), (2), and (3) above show that $X_{(n,1)}$, $X_{(n,2)}$, . . . is a sequence satisfying hypotheses (1), (2), and (3) of Theorem 3.6. It will now be shown that hypothesis (4) of Theorem

3.6 is also satisfied.

Let $(1,x_1)$, $(2,x_2)$, . . ., (s,x_s) be the pattern which $X_{(n,i+1)}$ follows in $X_{(n,i)}$ that $X_{(m,i+1)}$ follows in $X_{(m,i)}$. Suppose the j-th link of $X_{(n,i+1)}$ intersects the k-th link of $X_{(n,i)}$. Since every link of $X_{(n,i+1)}$ is a subset of some link of $X_{(n,i)}$ then the j-th link of $X_{(n,i+1)}$ must be a subset of the (k-1)-th, the k-th or the (k+1)-th link of $X_{(n,i)}$. Therefore, one of the ordered pairs (j,k-1), (j,k), or (j,k+1) must belong to the collection $(1,x_1)$, $(2,x_2)$, . . ., (s,x_s). Suppose it is (j,k). Then it is also true that the j-th link of $X_{(m,i+1)}$ is a subset of the k-th link of $X_{(m,i)}$, and hence the distance between the j-th link of $X_{(m,i+1)}$ and the k-th link of $X_{(m,i)}$ is zero. Now suppose the ordered pair which belongs to the pattern is (j,k-1). Then the j-th link of $X_{(m,i+1)}$ is a subset of the (k-1)-th link of $X_{(m,i)}$. The (k-1)-th link of $X_{(m,i)}$ intersects the k-th link of $X_{(m,i)}$. Thus, the distance between the j-th link of $X_{(m,i+1)}$ and the k-th link of $X_{(m,i)}$ must be less than the diameter of the (k-1)-th link of $X_{(m,i)}$. A similar argument shows that if the ordered pair (j,k+1) belongs to the pattern then the distance between the j-th link of $X_{(m,i+1)}$ and the k-th link of $X_{(m,i)}$ must be less than the diameter of the (k+1)-th link of $X_{(m,i)}$. Regardless of which of the above three cases is true, it is clear that hypothesis (4) of Theorem 3.6 is satisfied.

Therefore, by Theorem 3.6, there exists a homeomorphism carrying M_1 into M_2 which carries P_1 and Q_1 into P_2 and Q_2 respectively.

The next theorem is the final one in the current sequence of theorems. This theorem furnishes the primary result of this chapter.

The homogeneity of the pseudo-arc is proven by the technique which was suggested in the discussion preceding Theorem 3.6. Hence, the majority of the proof is concerned with satisfying the hypotheses of Theorem 3.7 in an appropriate fashion. The collection of theorems in Chapter II on the properties of crooked chains will be employed frequently in order to create a sequence of chains necessary for the utilization of Theorem 3.7.

Theorem 3.8: The pseudo-arc M is homogeneous.

Proof: Since M is indecomposable, there exist two points R and S of M which belong to different composants of M (24: 59, Theorems 138 and 139).

Let D_j be any term of the sequence $D_1, D_2, \ldots$ which was used to define M. It will be shown that there exists a term D_k $(k > j)$ of the sequence such that if $D_k(R,S)$ is the subchain of D_k from R to S then $D_k(R,S)$ has a link that intersects the first link of D_j and has a link that intersects the last link of D_j.

Consider the limiting set L of the sequence $D^*_{j+1}(R,S)$, $D^*_{j+2}(R,S)$, $\ldots$. It is clear that L is a subset of M. Suppose L is not a continuum. Since L is closed but not a continuum then L is not connected. Let L be the union of H and K where H and K are closed separated point sets. Let U and V be domains such that H is a subset of U, K is a subset of V, and the distance from U to V is h. Now suppose R and S belong to the same component C of L. Now C is a subset of H or C is a subset of K, and thus $\bar{C}$ is a proper subcontinuum of M containing R and S. This contradicts the assumption that R and S belong to different composants of M. Hence R and S belong to different components of L and so it can be assumed that R belongs to H and S belongs to K. Let $D_{j+t}(R,S)$ be an element of the sequence $D_{j+1}(R,S)$, $D_{j+2}(R,S)$, . . . such that every link of $D_{j+t}(R,S)$ has diameter less than h/3. The first link of $D_{j+t}(R,S)$ intersects U, the last link intersects V, no link has diameter as large as h/3, and the distance from U to V is h. Therefore, there exists a link of $D_{j+t}(R,S)$ which intersects neither U nor V. Let T be a set formed by selecting a link from each subchain $D_{j+s}(R,S)$, $s \geq t$, which intersects neither U nor V. There are two possibilities: (1) there exist an infinite number of elements of T which contain a common point P_1, or (2) an infinite sequence Z of distinct points can be selected from distinct elements of T. In case (1) the point P_1 would also have to belong to L. But this contradicts the assumption that L is a subset of the union of U with V. In case (2) there exists a point P_2 which is a limit point of Z because it was assumed that M was defined in a compact space. By definition of L, P_2 belongs to L. But P_2 does not belong to U union V because no element of T intersects the domain U union V. This contradicts that L is a subset of the union of U with V. Therefore, neither case (1) nor case (2) is possible. Hence the assumption that L is not a continuum leads to a contradiction.

Since R and S belong to different composants of M and L is a subcontinuum of M containing R and S, then L must be M. Therefore, the limiting set of $D^*_{j+1}(R,S)$, $D^*_{j+2}(R,S)$, . . . is M. Thus, it can be seen that there exists an infinite subsequence of the sequence $D_{j+1}(R,S)$, $D_{j+2}(R,S)$, . . . such that each term has a link which intersects the first link of D_j. By an argument similar to the one above, it can be shown that the limiting set of this subsequence is a subcontinuum of M containing R and S. Since R and S belong to different composants of M, then this subcontinuum is M. But if the subcontinuum is M, then some term of the subsequence must have a link which intersects the last link of D_j. Hence, there is an integer k greater than j such that $D_k(R,S)$ intersects both end links of D_j.

Let j be an integer such that the union of any two adjacent links of D_j is a domain with diameter no more than 1. By the above argument, there exists an integer h greater than j such that the sub-chain $D_h(R,S)$ has a link which intersects the first link of D_j and has a link which intersects the last link of D_j. By Theorem 2.68 there is a chain E_1 such that E_1 is a consolidation of D_h, each element of E_1 is a subset of two adjacent links of D_j, the first link of $D_h(R,S)$ is contained in only the first link of E_1, and the last link of $D_h(R,S)$ is contained in only the last link of E_1. So E_1 is a chain from R to S such that E_1 is a consolidation of D_h and no element of E_1 has a diameter of more than 1.

Let k be an integer greater than h such that no element of D_k is of diameter more than 1/2. By Theorem 2.65, D_k is crooked in E_1. A simple argument that makes use of properties (3) and (4) in the definition of the pseudo-arc, together with the assumed compactness, will show that there is an integer t such that the closure of the union of each pair of intersecting links of D_t is a subset of a link of D_k. As previously demonstrated, there exists an integer m greater than t such that the subchain $D_m(R,S)$ intersects the first and last links of D_t. Therefore, by Theorem 2.68, there is a chain E_2 from R to S such that E_2 is a consolidation of D_m and each link of E_2 is a subset of two adjacent links of D_t. By Theorem 2.66, E_2 is crooked in E_1. Also, since each link of E_2 is a subset of the union of two adjacent links of D_t, the closure of each pair of intersecting links of D_t is a subset of a link of D_k, and D_k is crooked in E_1, then the closure of each link of E_2 is a subset of a link of E_1. It is also clear that since no element of D_k has diameter of more than 1/2 then no element of E_2 has diameter of more than 1/2.

If the above process is continued, a sequence E_1, E_2, . . . of chains from R to S is defined such that for each integer i, (1) E_{i+1} is crooked in E_i, (2) the closure of each element of E_{i+1} is a subset of an element of E_i, (3) no element of E_i has a diameter of more than 1/i, and (4) E_i is a consolidation of some D_j.

Now let P_1 and P_2 be any two points of M. Since M is indecomposable, there exist points Q_n (n = 1, 2) such that Q_n belongs to M and Q_n and P_n belong to different composants of M.

Using the results established above (letting $P_n = R$, $Q_n = S$, and $Y_{(n,i)} = E_i$), it follows that there exists a sequence $Y_{(n,1)}$, $Y_{(n,2)}$, . . . of chains from P_n to Q_n such that for each positive integer i, (1) $Y_{(n,i+1)}$ is crooked in $Y_{(n,i)}$, (2) the closure of each link of $Y_{(n,i+1)}$ is a subset of a link of $Y_{(n,i)}$, (3) no link of $Y_{(n,i)}$ has a diameter of more than 1/i, and (4) $Y_{(n,i)}$ is a consolidation of some D_j.

Theorem 3.7 gives immediately that there is a homeomorphism carrying M into itself and P_1 into P_2. Therefore, M is homogeneous.

The Circle of Pseudo-arcs

The final example of a bounded homogeneous plane continuum to be presented is the circle of pseudo-arcs. A formal presentation of this example would require the development of topological properties which are not presented in this paper. Hence, no proof that the circle of pseudo-arcs is homogeneous will be given. A proof by R. H. Bing and F. B. Jones of the homogeneity of the circle of pseudo-arcs can be found in the literature (7). The proof that the circle of pseudo-arcs, M, is homogeneous also points out that there is a continuous decomposition of M into pseudo-arcs such that the decomposition space is a simple closed curve. This fact, together with some well-known theorems on upper semi-continuous decompositions (28), can be used to prove that the circle of pseudo-arcs is decomposable.

The particular approach used to present the example will be analogous to the process created by F. B. Jones (7). This process is not presented as a definition for a circle of pseudo-arcs, but is described in such a way that it is reasonable to believe that the example has the critical asserted properties. A weakness in the presentation which will be obvious is that no justification will be offered that the steps can be repeated a countably infinite number of times, as will be asserted. However, several illustrations will be given together with a careful description of the critical phases of the first three steps of the process. It is hoped that since the process is, in a sense, cyclic with cycle length three, then sufficient information will be present to make all assertions at least seem plausible.

In order to make it easier to visualize the positions of various parts of chains in the sequence of circular chains necessary for describing the circle of pseudo-arcs, two preliminary figures are given. Figure 4 shows a set of arcs called "the first layer of V's." Figure 4 was drawn in three stages as follows:

(1) Construct two concentric circles W_1 and W_2 centered at the origin and having radii one and two respectively.

(2) Define twelve points lying on the two circles in terms of their polar coordinates. Let $a_1 = (2, -\pi/12)$, $b_1 = (1,0)$, $c_1 = (2, \pi/12)$, $a_2 = (1, -\pi/2 - \pi/12)$, $b_2 = (2, -\pi/2)$, $c_2 = (1, -\pi/2 + \pi/12)$, $a_3 = (2, \pi - \pi/12)$, $b_3 = (1, \pi)$, $c_3 = (2, \pi + \pi/12)$, $a_4 = (1, \pi/2 - \pi/12)$, $b_4 = (2, \pi/2)$, and $c_4 = (1, \pi/2 + \pi/12)$. Now b_i is connected by line segments to a_i and c_i, $i = 1, 2, 3, 4$. Note that four "V's" are thus formed. The point b_i will be said to be the vertex of V_i and a_i and c_i will be called the end points of V_i. Note that if two V's are adjacent then their vertices are on different circles.

(3) Now locate three points on the circular arc between a_i and b_j and three points on the circular arc between b_i and c_j, $j \equiv i+1 \bmod 4$, in such a way that the circular arcs are divided into four congruent sub-arcs. As in step (2), connect the points lying between V_i and V_j, $j \equiv i+1 \bmod 4$, in such a way that two new V's are formed with vertices on different circles. When all points are connected between each pair V_i and V_j, a total of twelve V's will have been formed such that no two adjacent V's have vertices on the same circle. Number

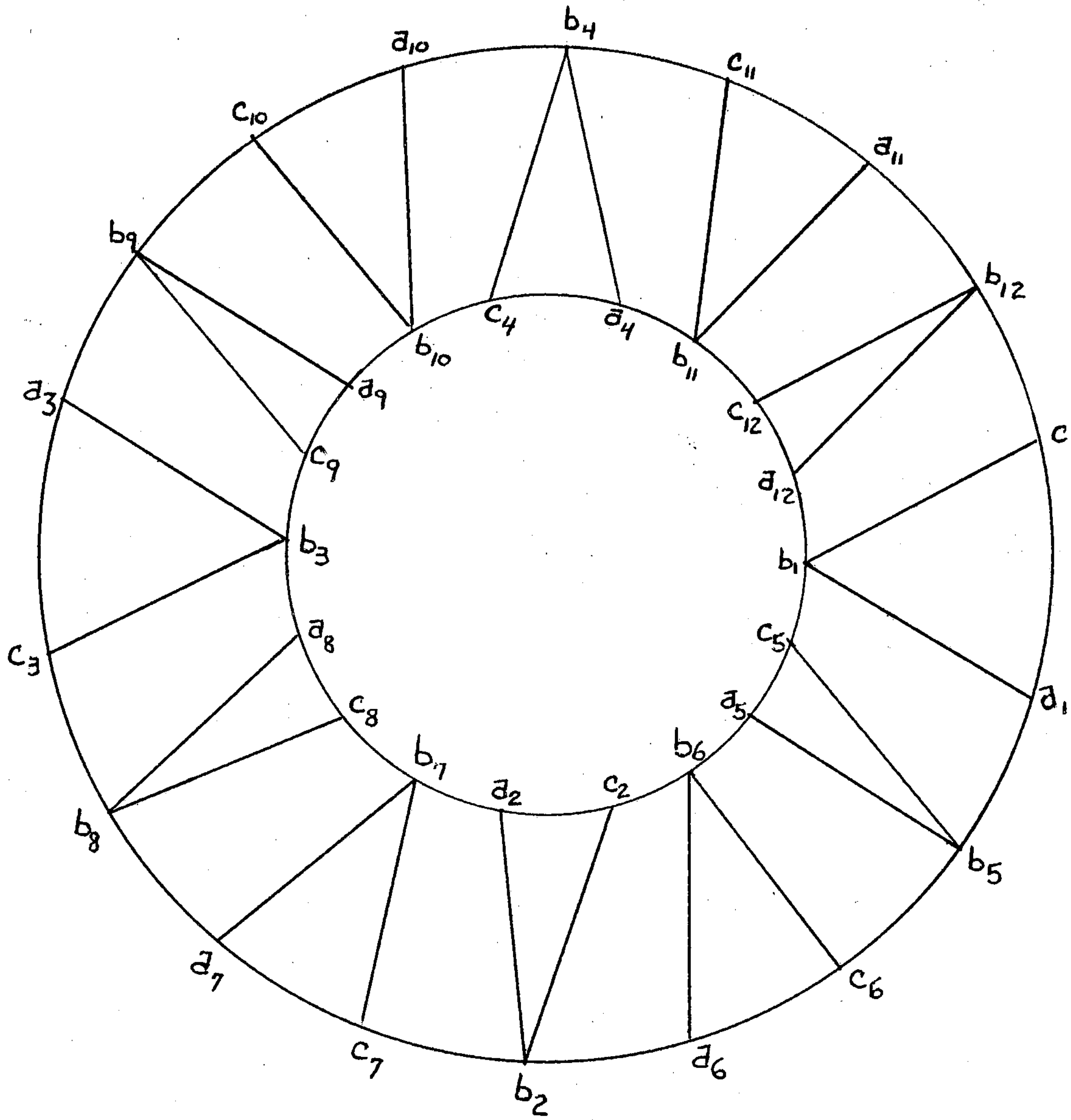

Figure 4. The First Layer of V's.

the last eight V's in a clockwise fashion beginning with V_5 where V_5 is the V adjacent to V_1 and in the clockwise direction from V_1. The end points of the V's should be numbered so that a_i is located in a clockwise direction from c_i.

This completes the construction of the first layer of V's. The result of steps (1), (2), and (3) can be seen in Figure 4.

Figure 5 shows the second layer of V's. Note that the second layer includes the first layer. The additional V's in layer two were constructed by subdividing the circular arcs between the adjacent V's of layer one and proceeding as in step (3) above. Note that the pattern of having the vertices of adjacent V's on different circles and the clockwise numbering pattern have been maintained.

The process of constructing layers of V's is now continued a countably infinite number of times by subdividing the circular arcs between adjacent V's belonging to the preceding layer. The pattern for alternating vertices and the numbering pattern are maintained.

It will not be proved, but it should be clear that the closure of the union of the infinite collection of V's is a continuum which separates the plane.

The goal in describing the circle of pseudo-arcs is to show how each member of the infinite collection of V's can be replaced with a pseudo-arc.

It might be thought that if L is the continuum formed by the closure of the union of the V's then L is homogeneous and perhaps homeomorphic to a simple closed curve. However, consider any small neighborhood R of b_i. Let C be the component of $L \cap \bar{\bar{R}}$ containing b_i.

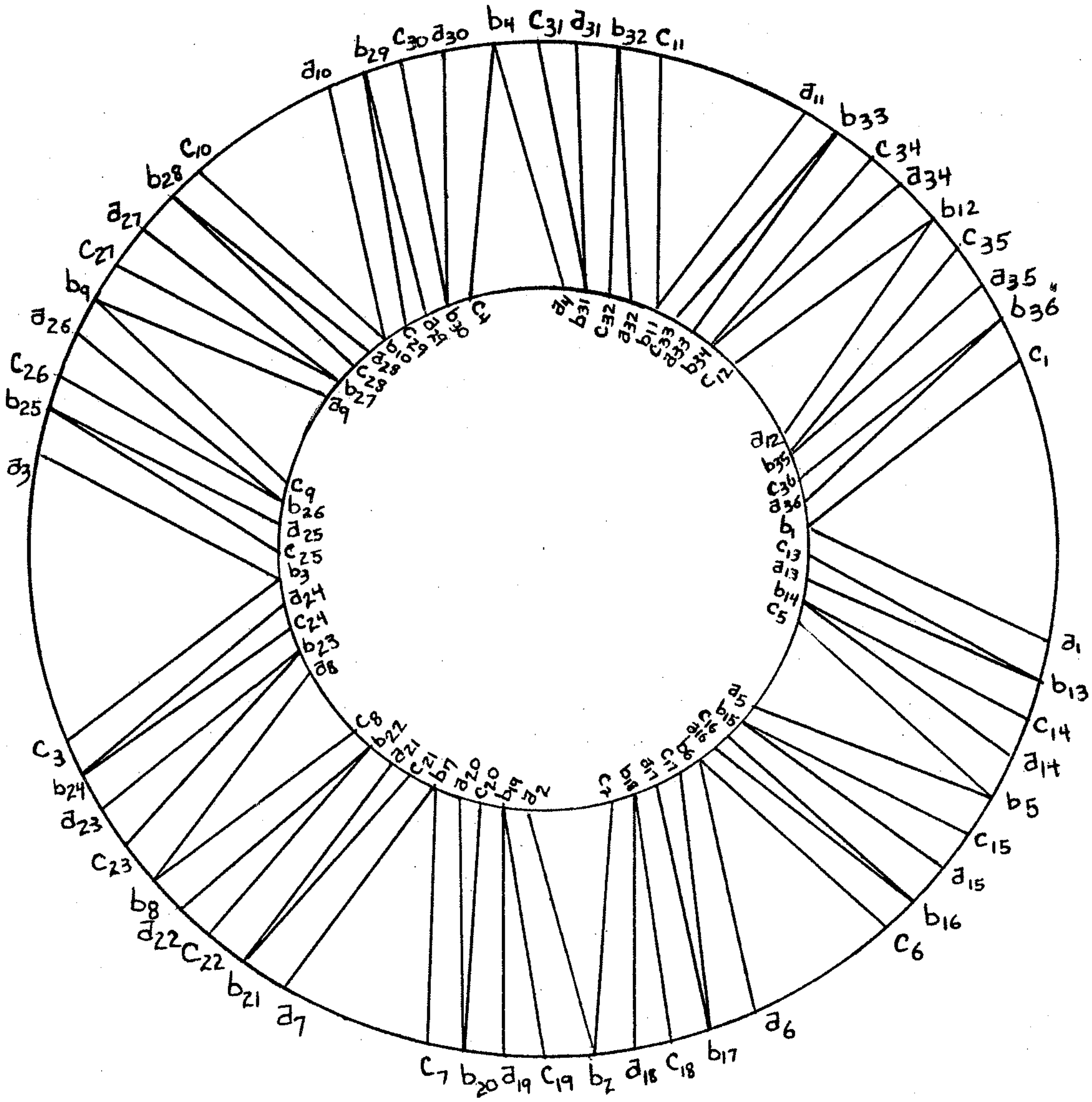

Figure 5. The Second Layer of V's.

Then C is seen to be $V_i \cap \bar{R}$. Moreover, $L \cap (\bar{R} - \{b_i\})$ is the union of two separated point sets each of which intersect C. Thus b_i is a local separating point (Definition 2.36) of L. Now consider any small neighborhood R_1 of c_i. Let C_1 be the component of $L \cap \bar{R}_1$ containing c_i. Then C_1 is $V_i \cap \bar{R}_1$. But in this case $C_1 - \{c_i\}$ is connected. Thus if $L \cap (\bar{R}_1 - \{b_i\})$ is the union of two separated point sets M_1 and M_2, then $C_1 - \{c_i\}$ is a subset of one of the two sets M_1 or M_2 and hence does not intersect the other. Therefore c_i is not a local separating point of L. Thus, it should seem reasonable that there does not exist a homeomorphism mapping L to L and c_i to b_i.

The existence of points of L that are local separating points and points that are not, make it seem unlikely that a homogeneous continuum can be constructed by simply substituting pseudo-arcs for the V's of the continuum L. The process will be called "replacing the V's"; however, one of the essential ingredients of the process (described in property (6) below) is necessary in order to overcome difficulties caused by the existence of local separating points in L.

The general plan for replacing the V's by pseudo-arcs is to describe a sequence of circular chains in such a way that the i-th chain covers the i-th layer of V's and has subchains crooked in sub-chains of the preceding chain of the sequence.

The most important properties of the sequence of chains are listed below. In describing the construction of the chains no reference will be made to some of the properties since they are natural results of the processes necessary for guaranteeing other properties. For instance, (8) and (9) are naturally satisfied by the process used to satisfy (1) through (7). However, (8) and (9) must be included in order to characterize the necessary "crookedness" with property (10).

The ten properties of the sequence $D_1, D_2, D_3, \ldots$ are:

(1) The sequence $D_1, D_2, D_3, \ldots$ is a sequence of circular chains of connected domains.

(2) For each positive integer i, the closure of each element of D_{i+1} is a subset of some element of D_i.

(3) For each i, each element of D_i intersects the annulus between W_1 and W_2, and not both of two intersecting links of D_i intersect $W_1 \cup W_2$.

(4) If for each i, δ_i is the maximum diameter of a link of D_i then δ_i approaches 0 as i approaches infinity.

(5) The subscripts of the elements of D_i which intersect W_1 preserve the clockwise order on W_1 and the subscripts of those intersecting W_2 preserve the clockwise order on W_2.

(6) If a_i, b_i, and c_i are the end points and vertex of V_i, there is a natural number m_i such that the shortest subchain of D_{m_i} irreducible from a_i to c_i contains b_i, the subchain of $D_{m_{i+1}}$, irreducible from a_i to b_i contains c_i, the subchain of $D_{m_{i+2}}$, irreducible from b_i to c_i contains a_i, the subchain of $D_{m_{i+3}}$, irreducible from a_i to c_i contains b_i and so on.

(7) $(W_1 \cup W_2) \cap (\bigcap_{i=1}^{\infty} D_i^*) = \bigcup_{i=1}^{\infty} \{a_i, b_i, c_i\}$.

(8) For each i, D_i is the union of finitely many subchains $T_{i1}, T_{i2}, \ldots, T_{in_i}$ such that (a) $T^*_{i1}, T^*_{i2}, \ldots, T^*_{in_i}$ is a circular chain, and (b) for each j, $1 \leq j \leq n_i$, T_{ij} is either irreducible from W_1 to W_2 or (for some k) irreducible about $\{a_k, b_k, c_k\}$.

(9) If $h < i$, each element of $\{T^*_{i1}, T^*_{i2}, \ldots, T^*_{in_i}\}$ is a subset of two intersecting links of $\{T^*_{h1}, T^*_{h2}, \ldots, T^*_{hn_h}\}$.

(10) If $h < i$ and T_{ij} is contained in $T_{hk} \cup T_{ht}$, then T_{ij} is crooked in $T_{hk} \cup T_{ht}$ where $t \equiv (k+1) \bmod n_h$.

A procedure for constructing D_1 and portions of D_2 and D_3 will now be given. Attention will be centered on constructing the chains so that properties (1) through (6) are satisfied. It is not difficult to see that properties (7) through (9) occur as a natural result of the procedure. Property (10) will be omitted because of the physical limitations imposed by the width of a pencil lead. However, it will be clear that property (10) could be satisfied without destroying the other properties. The omission of property (10) is not meant to detract from its significance, since property (10) is actually the main item which justifies naming the continuum circle of "pseudo-arcs".

To construct D_1 proceed as follows:

(1) Group the vertices and end points of the V's in layer one in sets S_k, ($k = 1, 2, \ldots, 12$) of three each such that (a) each S_k is a subset only of W_1 or only of W_2, (b) each S_k is of the form $\{a_i, b_j, c_k\}$ $i \neq j$, $j \neq k$, $i \neq k$, (c) no vertex or end point in layer one is between any pair of points in any particular S_k unless it belongs to S_k.

(2) Enclose each set S_k constructed in step (1) in a domain such that (a) the domain intersects W_i if and only if S_k intersects W_i, and (b) no two such domains intersect.

(3) For each domain D that intersects W_2 construct two distinct non-intersecting chains such that D is an end-link of each chain, the other two end links are the two domains on W_1 containing the end points of the V of layer one whose vertex is in D, and the only links of either chain that intersect $W_1 \cup W_2$ are the end links.

The chain D_1 is pictured in Figure 6. The only properties in the list of ten that necessarily apply to D_1 are (1), (3), (5), and (8). Those four properties are satisfied. However, it can be seen that if $m_i = 1$, then property (6) is also satisfied.

A portion of chain D_2 will now be constructed. In fact only the part of D_2 in the vicinity of V_1 will be discussed. However, it should be clear that the process is general and could have been done for any V of layer one. The subscript "1" is specified so that reference can be made to specific points and V's in layer two.

Since at some stage property (6) must be satisfied, D_2 will be constructed in such a way that it is possible to let $m_i = 1$ (and hence $m_{i+1} = 2$). It should be noted that if no attempt were being made to satisfy property (6) until some later stage, then D_2 could be constructed in exactly the same manner as D_1, if the additional restriction imposed by property (2) were appropriately satisfied. Accordingly, attention will be centered on satisfying property (6).

To construct D_2 proceed as follows:

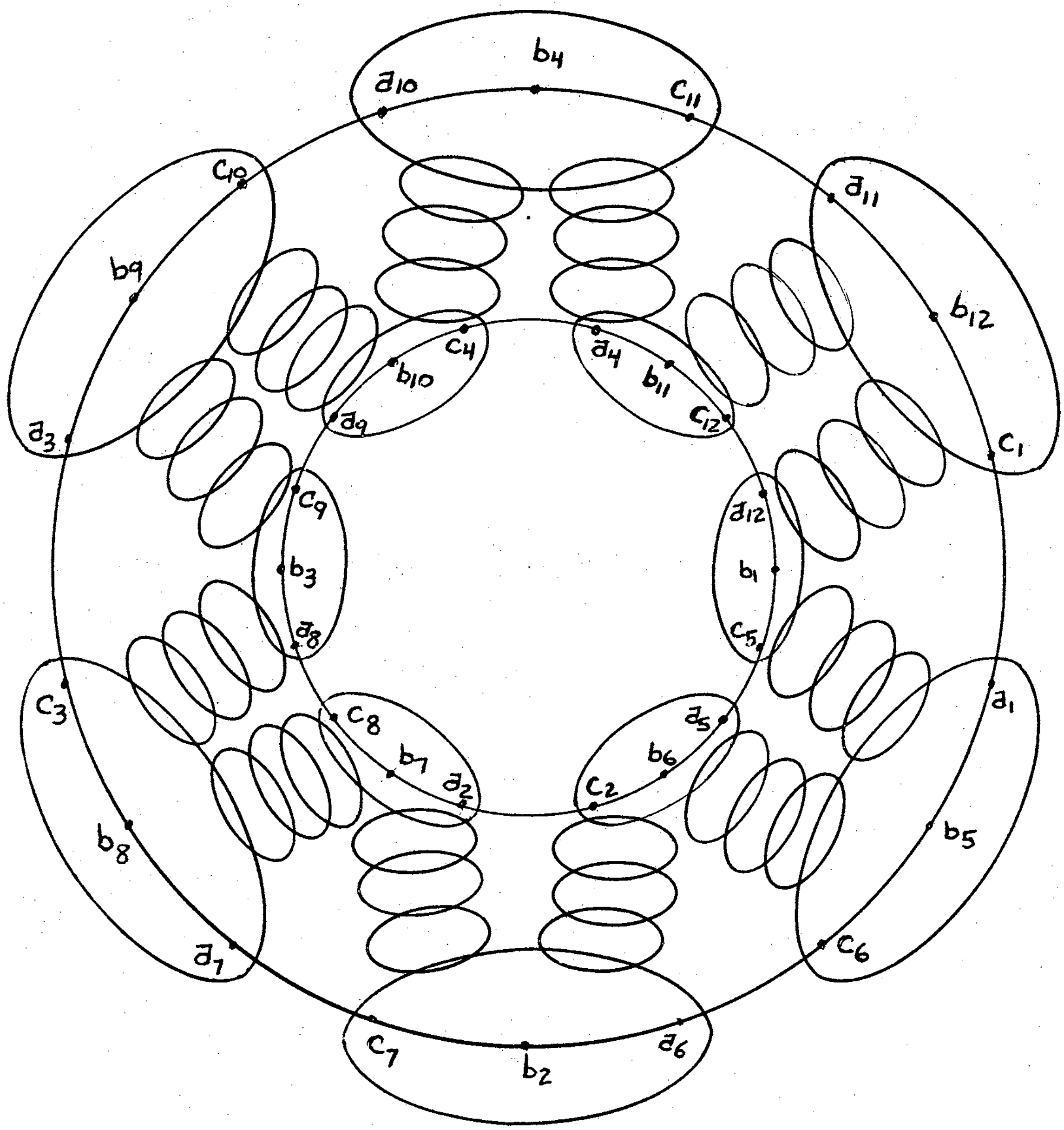

Figure 6. The Chain D_1.

(1) Using the points of layer two instead of layer one repeat the process described in steps (1) and (2) of the construction of D_1, but construct the domains so that property (2) is also satisfied. For convenience, the domains constructed in this step will be referred to by the subscript on the vertex of the V of layer two included in the domain.

(2) Let T_{11} be the subchain of D_1 that is irreducible about a_1, b_1, c_1. It can be seen that each end link of T_{11} contains three of the domains which intersect W_2 constructed in step (1) and that the interior link of T_{11} that intersects W_1 contains three such domains that also intersect W_1. (See Figure 7). Being careful to preserve property (2) construct eight mutually disjoint chains $M_1, M_2, \ldots, M_8$ such that (a) the end links of M_1 are domains 33 and 12, (b) the end links of M_2 are domains 12 and 35, (c) the end links of M_3 are domains 35 and 1, (d) the end links of M_4 are domains 1 and 36, (e) the end links of M_5 are domains 36 and 13, (f) the end links of M_6 are domains 13 and 14, (g) the end links of M_7 are domains 14 and 5, (h) the end links of M_8 are domains 5 and 10. Now the second part of property (6) is satisfied if $i = 1$ and $m_i = 1$. But also note that for sets of points such as $\{a_{35}, b_{35}, c_{35}\}$ and $\{a_{14}, b_{14}, c_{14}\}$, the first part of property (6) is satisfied. If Figure 7 is examined carefully, it can also be seen that properties (8) and (9) are satisfied. Repetition of the above process within each element of $\{T_{11}, T_{12}, \ldots, T_{1n_1}\}$ will satisfy all appropriate properties.

Details of constructing the chain D_3 are omitted; however, if Figure 8 is studied it can be seen that the third part of property

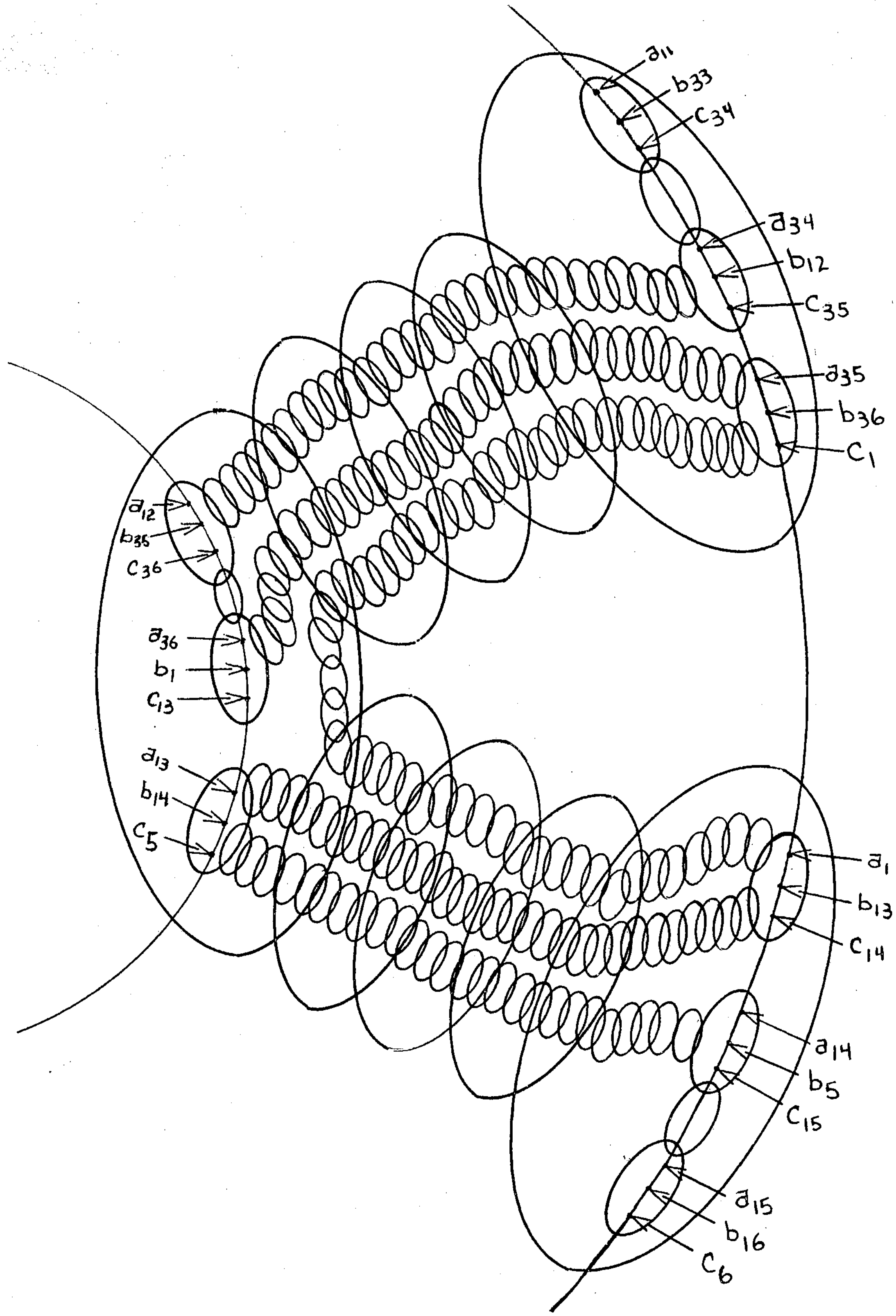

Figure 7. The Chain D_2 Near V_1.

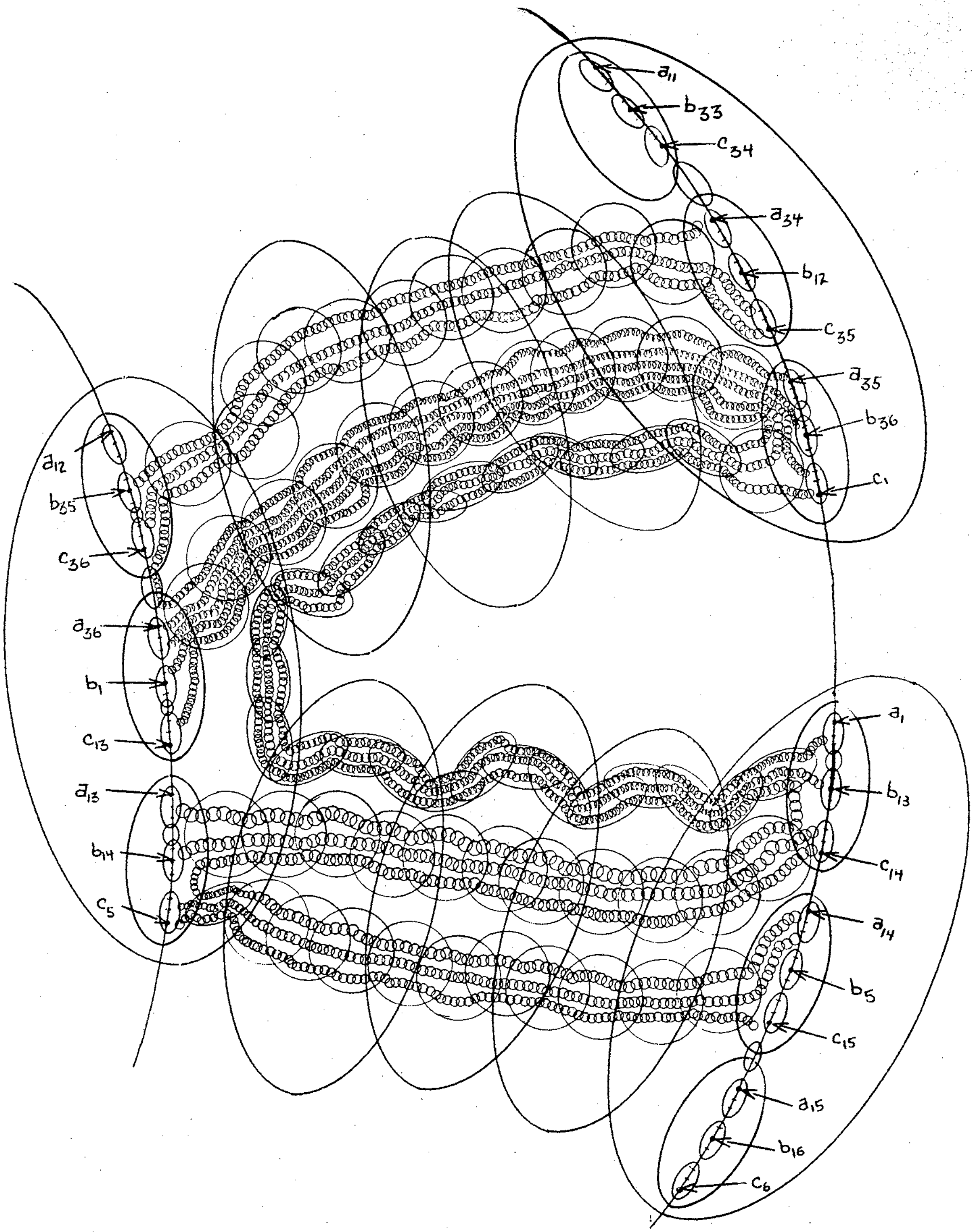

Figure 8. The Chain D_3 Near V_1.

(6) has been satisfied for the set of points $\{a_1,b_1,c_1\}$, that the second part of property (6) has been satisfied for the sets $\{a_{14},b_{14},c_{14}\}$ and $\{a_{35},b_{35},c_{35}\}$, and that the first part has been satisfied for the new points (marked but not named) from layer three that appear in the figure. All other properties, except property (10), are also seen to be satisfied.

Of course, the circle of pseudo-arcs is just $\bigcap_{i=1}^{\infty} D_i^*$. It can be shown that $\bigcap_{i=1}^{\infty} D_i^*$ separates the plane and therefore is not homeomorphic to the pseudo-arc. Moreover, $\bigcap_{i=1}^{\infty} D_i^*$ contains no local separating points and hence is not homeomorphic to a simple closed curve.

An Indecomposable Continuum Separating the Plane

This chapter will be concluded with the presentation of an example of a continuum which is strongly suspected of being homogeneous, but which has not been shown to be so. It is known that this continuum is hereditarily indecomposable and separates the plane (3).

Let D_1, D_2, . . . be a sequence of circular chains such that (a) each link of D_i is the interior of a circle with diameter less than $1/i$, (b) the closure of each link of D_{i+1} is contained in a link of D_i, (c) each complementary domain of D_{i+1}^* contains a complementary domain of D_i^*, and (d) if E_i is a proper subchain of D_i and E_{i+1} is a subchain of D_{i+1} contained in E_i, then E_{i+1} is crooked in E_i.

It is by no means obvious that a sequence of chains with the asserted properties actually exists in the plane. The proof that the continuum is indecomposable and separates the plane includes suggestions that help to establish patterns useful in constructing the sequence D_1, D_2, . . . (3). However, if D_1 has less than four links then no restriction on D_2 is imposed by property (d). If D_1 has four or more links then property (d) forces D_2 to have so many links that it is virtually impossible to show both D_1 and D_2 in one illustration. Hence no illustration for this example has been included.

CHAPTER IV

THE CLASSIFICATION OF CHAINABLE HOMOGENEOUS PLANE CONTINUA

INTRODUCTION

The goal of Chapter IV is to show that the list of examples presented in Chapter III sufficiently illustrates all types of chainable homogeneous bounded plane continua known to exist. This will be accomplished by presenting a classification system for chainable homogeneous bounded plane continua, and then proving that such continua always belong to one of the classes in that system.

The results appearing in this chapter indicate that every chainable homogeneous bounded plane continuum must belong to one of the following four classes:

Type 1: Pseudo-arcs.

Type 2: The simple closed curves.

Type 3: Circles of pseudo-arcs.

Type 4: Indecomposable continua that separate the plane.

The question of whether continua of type four actually exist was discussed in the last section of Chapter III. Examples contained in Chapter III show that continua of type one, type two, and type three can be constructed in the plane.

Many of the proofs necessary for showing that the four classifications given above exhaust the set of homogeneous bounded plane continua are exceedingly long. Some require an extensive development of topological concepts not considered in this paper. However, several of the fundamental theorems can be presented in a manner that will require very little additional work to be done by the reader. The proofs of most of the other theorems can be outlined with sufficient details so that the more knowledgeable reader can supply the remainder of the proof with the assistance of the references listed. Only one major theorem will be severely neglected. The conclusion of that theorem is that all homogeneous decomposable bounded plane continua, that are not simple closed curves, are circles of pseudo-arcs. This particular theorem is one of those that is dependent on the presentation of several additional topological concepts. But more importantly, any sort of proof of the theorem would require that a proof of the homogeneity of the circle of pseudo-arcs be given. The proof that any two circles of pseudo-arcs are homeomorphic is merely one of the side results of the proof of the homogeneity of such continua. The proof of the homogeneity of the circle of pseudo-arcs is prohibitively long, even in the condensed version in which it was originally published (7). Hence, this result will be given but the proof will be indicated by reference.

The results contained in the second section of this chapter do not at first appear to contribute to the problem of classifying homogeneous bounded plane continua. However, the fact that the only locally connected homogeneous bounded plane continua are the simple closed curves is useful in the fourth section in the proofs of some obviously important results. Theorem 4.3 of the second section of this chapter could have been listed as a lemma in the fourth section. However, it was felt that the conclusion of Theorem 4.3 was sufficiently interesting to be presented in a separate section.

The main result of the third section of this chapter, that a simple closed curve is the only homogeneous bounded plane continuum containing an arc, does not contribute to the main purpose of the chapter. However, since the result is closely related to the problem and since the second section of the chapter sets the stage for its proof, the theorem is given along with a brief resumé of its proof.

The last three sections of the chapter contain the remainder of the theorems necessary to assure that the types one, two, three, and four, given previously, are actually sufficient to exhaustively classify homogeneous bounded plane continua.

Locally Connected Homogeneous Continua

The purpose of this section will be to prove that the only locally connected homogeneous bounded plane continua are the simple closed curves. The theorem that is actually proved is slightly stronger in the sense that the hypothesis of homogeneity is not used, but is replaced by local homogeneity in the proof.

Two lemmas are required to establish the main result. Since the first of the two lemmas contains a result that is related to the topic of this chapter, its proof is included. The second lemma is stated without proof; however, the proof may be found in the reference cited.

Lemma 4.1: If M is a locally connected, locally homogeneous, nondegenerate, bounded plane continuum, then M contains a simple closed curve.

Proof: Suppose the theorem is false. Then it is clear from the definition of dendrite (Definition 2.17) that M is a dendrite. It is known that every point of a dendrite is either a cut point or an end point and that every pair of points of a dendrite are separated by a third point (28; 88, Theorem 1.1). Since M is nondegenerate, it follows that there exist points of M which are not end points. That is, M must contain cut points. Now every nondegenerate compact continuum has at least two non-cut points and so M also has end points (24; 38, Theorem 93). An end point of M is defined to be a point of M with Menger order one with respect to M (Definition 2.33). So if an end point is a non-cut point, then cut points must have Menger order greater than one.

Let x be an end point of M and y be a cut point of M. Since x and y have different Menger orders with respect to M, then Theorem 2.32 implies that there do not exist open sets E and F with respect to M containing x and y respectively and a homeomorphism mapping E to F and x to y. This contradicts the hypothesis that M is locally homogeneous.

Therefore, it must be true that M contains a simple closed curve.

Since the result of the next lemma is of no particular interest other than as a tool in the proof of Theorem 4.3, and since the lemma's conclusion is rather easy to visualize, no proof for the lemma is given. The proof can be found in the reference cited.

Lemma 4.2: Suppose the simple closed curve J is the boundary of a complementary domain of the locally connected plane continuum K. Let W be a connected open subset of K containing the open arc (ab) of J, but neither a nor b. Then, if the open arc (ab) contains no local separating point of K, it does not separate W (9).

The main result of this section can now be proved:

Theorem 4.3: If M is a locally connected, locally homogeneous, nondegenerate bounded plane continuum, then M is a simple closed curve.

Proof: Lemma 4.1 shows that M must contain a simple closed curve C.

Suppose M is not a simple closed curve, then there exists a point p_1 of M that does not belong to C. Let p_2 be any point of C. It is known that every pair of points of a locally connected continuum may be joined by an arc lying in the continuum (28; 36, Theorem 5.1). Designate some arc from p_1 to p_2 that lies in M by p_1p_2. It is not difficult to see that some point on the arc p_1p_2 has Menger order of at least three with respect to M. In particular, the first point on p_1p_2 in the order from p_1 to p_2 that belongs to C has Menger order greater than 2 with respect to M. Hence, by Theorem 2.32, every point of M has Menger order of at least three.

Now M has no local separating point because if there exists a local separating point of M, then by the local homogeneity every point of M is a local separating point. But this is impossible since all save a countable number of the local separating points of M must be of Menger order two (28; 61, Theorem 9.2). Therefore, M has no separating point.

Since all locally connected continua that do not have separating points are cyclicly connected then M is cyclicly connected (28; 79, Theorem 9.3).

Now the boundary of each complementary domain of a cyclicly connected, locally connected, locally compact continuum is a simple closed curve (28; 107, Theorem 2.5). Thus, the boundary of each complementary domain of M is a simple closed curve.

It will now be shown that there exists a point of M which is not on the boundary of any complementary domain of M. Suppose that this is not the case. Then M is the union of simple closed curves, each of which is the boundary of a complementary domain of M. Since any two complementary domains of a continuum are disjoint, and since the complementary domains of M are bounded by simple closed curves, then M has at most a countable number of complementary domains. Thus, M is the union of a finite or, at most, a countable number of simple closed curves. If M is the union of a finite number of simple closed curves, then certainly one of those curves contains an open subset of M. Since no locally compact closed point set M is the union of countably many point sets such that if K is any one of them, every point of K is a limit point of M - K, then even if M is the union of a countable number of simple closed curves, one of them must still contain an open subset of M (24; 21, Theorem 53). It should be clear than an open subset of M which is also an open subset of a simple closed curve can contain no points of Menger order higher than two. But this is impossible because every point of M has Menger order three or more. Therefore, there exists some point q of M which is not on the boundary of any complementary domain of M.

Let p be any point on the boundary J of a complementary domain of M. Because M is locally homogeneous, there exists a homeomorphism between two open subsets E and F of M containing p and q respectively, such that p is mapped to q.

Consider any arc containing p, say [cpd], lying in E. Let the image of [cpd] be [c'qd']. There exists a circle G such that q is the center of G, the interior of G intersects M only at points of F, and neither c' nor d' belong to the interior of G. Now the interior of G will contain an open subarc (x'qy') of [c'qd'] which separates the interior of G into two domains D_1 and D_2.

Let H be the component of the common part of M and the interior of G that contains (x'qy'). With reference to the homeomorphism between E and F, let V be the inverse image of H. The inverse image of (x'qy') will be denoted by (xpy). By Lemma 4.2, V - (xpy) is connected. This means that H - (x'qy') is a subset of D_1 or D_2. Without loss of generality, suppose H - (x'qy') lies in D_2. Now H being the component of the common part of M and the interior of G, and (x'qy') being part of the boundaries of both D_1 and D_2 imply that q is on the boundary of a complementary domain of M. This is a contradiction.

Homogeneous Continua Containing an Arc

Theorem 4.3 is the point from which one starts in order to prove that the simple closed curve is the only homogeneous bounded plane continuum that contains an arc. A very brief outline of the remainder of the proof is given below. It should not be supposed that enough of the proof is given that the details would be easy to supply. Each statement should be viewed as a lemma requiring a lengthy proof to verify.

Theorem 4.4: The only homogeneous bounded plane continuum that contains an arc is a simple closed curve.

Indication of Proof: Suppose there exists a homogeneous bounded plane continuum M that contains an arc but is not a simple closed curve. The proof that no such continuum can exist is accomplished by investigating the properties which such a continuum would have to possess. A list of twenty properties can be obtained. It can be shown that the twentieth property leads to a contradiction. In order to illustrate the relationship of Theorem 4.4 to Theorem 4.3, compact descriptions of the proofs of the first five properties are given. The remaining fifteen properties can be established in the order given but generally require several rather lengthy lemmas for their complete demonstration.

Property 1. The set M is not locally connected. This property is immediate from Theorem 4.3.

Property 2. The set M is not connected im kleinen (Definition 2.15) at any point. If M were connected im kleinen at some point then by the homogeneity of M, it would be connected im kleinen at every point. But a continuum connected im kleinen at every point is locally connected (24; 90, Theorem 10). This contradicts Property 1.

Property 3. The set M contains an open set U with uncountably many components. Property 2 provides the key for proving this property.

Property 4. The set M contains no simple triod (Definition 2.26). The homogeneity of M implies that if M contains a simple triod then every component in U contains a simple triod. It can be shown that the plane contains at most a countable number of triods. This contradicts Property 3.

Property 5. The set M contains no simple closed curve. It is possible to prove from Theorem 4.3 that the simple closed curve is the only homogeneous bounded plane continuum containing a simple closed curve.

Property 6. Each ray (Definition 2.10) in M is the union of a countable number of arcs.

Property 7. For each point p of an arc component A of M (Definition 2.8), A is the union of two rays R_1 and R_2 starting at p such that the intersection of R_1 and R_2 is p.

Property 8. The set M has uncountably many arc components.

Property 9. If R is a ray of M and p is a point of $\bar{R}$, one of the rays starting at p lies in $\bar{R}$.

Property 10. If $\bar{R}_1$ is the closure of a ray of M, it contains a continuum $\bar{R}$ that is irreducible with respect to being the closure of a ray.

Property 11. If R is a ray in an arc component A of M, $\bar{R} = \bar{A}$.

Property 12. If the closures of two arc components of M intersect, the closures are equal.

Property 13. The closure of each arc component A of M is homogeneous.

By making use of the thirteen properties listed thus far, it is now possible to prove that the existence of M implies the existence of another continuum M' which is the closure of one of its arc components. That is, properties one through thirteen imply the existence of a homogeneous bounded plane continuum M' one of whose arc components is dense in M' but which is not a simple closed curve. The remaining seven properties are properties which can be shown to be possessed by M'.

Property 14. If C is a non-degenerate subcontinuum of M' that is not an arc, then C intersects uncountably many arc components of M'.

Property 15. Each non-degenerate proper subcontinuum of M' is an arc.

Property 16. The set M' is indecomposable.

Property 17. For each positive number ϵ and each arc xy in M' there is an ϵ-chain $d_1, d_2, \ldots, d_n$ covering xy such that x belongs to d_1, y belongs to d_n, and the common part of M' and the union of the boundaries of d_1, d_2, . . ., d_n is a subset of the union of $\bar{d}_1$ and $\bar{d}_n$.

Property 18. For each positive number ϵ there is a positive number δ such that if ab is an arc in M' with the distance between a and b less than δ, then either the diameter of the set containing the points of ab is less than ϵ or each point of M' is within a distance of ϵ of some point of ab.

Property 19. If a point p of M' is accessible (Definition 2.40) from a component T of the complement of M' in the plane, each point of any arc in M' is accessible from T.

Property 20. The set M' contains a folded sequence of arcs (Definition 2.24) converging to an arc.

The proof of Theorem 4.4 can now be completed by proving that it is impossible for the compact continuum M' to contain a folded sequence of arcs converging to an arc (2).

Homogeneous Continua That Do Not Separate the Plane

The main result of this section is contained in Theorem 4.8. This result will show that every homogeneous bounded plane continuum which does not separate the plane must have one of the prominent features of the pseudo-arc. That is, such continua are always indecomposable.

It may seem strange to include details of the proof of the following lemma when it is noted that many details of the proof of the main theorem (Theorem 4.6) resulting from the leamma have been omitted. However, more than just the conclusions of the lemm- are utilized in the proof of Theorem 4.6. Certain facts that are noted in the proof of the lemma are used in proving Theorem 4.6, as well as certain techniques that occur in the proof. In particular, the technique which shows how a certain uncountable sequence of points can be created, is a useful tool in filling in details that have been omitted from the proof of Theorem 4.6. Hence, the inclusion of the details of the proof of Lemma 4.5 make it possible to omit many details from the proof of Theorem 4.6.

<u>Lemma</u> <u>4.5</u>: Let M be a homogeneous bounded plane continuum. Let x and y be distinct points of M. For every point t of M denote by U_t the set of all points z of M such that M is aposyndetic (Definition 2.19) at z with respect to t. Then U_y is not a proper subset of U_x.

Proof: Notice first that the definition of aposyndetic shows that U_t is open in M for every point t of M.

Suppose that U_y is a proper subset of U_x. Since M is homogeneous, there exists a homeomorphism T such that T(M) = M and T(x) = y.

Let p belong to U_x. Then M contains a continuum K_p and a subset V_p open in M such that p belongs to V_p, V_p is a subset of K_p, and K_p is a subset of $M - \{x\}$. Now $T(K_p)$ is a continuum in M, $T(V_p)$ is a subset of $T(K_p)$ which is open in M, and $T(V_p)$ contains T(p). Also, $T(K_p)$ is a subset of $T(M - \{x\}) = T(M) - \{T(x)\} = M - \{y\}$. Hence, T(p) belongs to U_y, and so $T(U_x)$ is a subset of U_y. Now let p belong to U_y. Since T is a homeomorphism, there exists a point z of M such that $T(z) = p$. Because p belongs to U_y there exists a continuum K_p that is contained in $M - \{y\}$ and a set V_p open in M which is a subset of K_p and contains p. The open set $T^{-1}(V_p)$ contains the point $T^{-1}(p) = z$ and is contained in the continuum $T^{-1}(K_p)$. Now $T^{-1}(K_p)$ is contained in $T^{-1}(M - \{y\}) = T^{-1}(M) - \{T^{-1}(y)\} = M - \{x\}$. That is, z belongs to U_x. But if z belongs to U_x then $T(z) = p$ belongs to $T(U_x)$. Therefore, U_y is a subset of $T(U_x)$ and it follows that $U_y = T(U_x)$.

It will now be shown that $U_{T(y)} = T(U_y)$. Let p belong to U_y. By definition of aposyndetic there exists a continuum K_p and a set V_p open in M such that p belongs to V_p, V_p is a subset of K_p, and K_p is a subset of $M - \{y\}$. The set $T(K_p)$ is a continuum in M, $T(V_p)$ is open in M and is a subset of $T(K_p)$. Also, $T(K_p)$ is a subset of $T(M - \{y\}) = M - \{T(y)\}$. This shows that $T(U_y)$ is a subset of $U_{T(y)}$. Let p belong to $U_{T(y)}$. There exists a continuum K_p and a set V_p open in M such that p belongs to V_p, V_p is a subset of K_p, and K_p is a subset of $M - \{T(y)\}$. Now the continuum $T^{-1}(K_p)$ is a subset of $M - \{y\}$. Also, $T^{-1}(K_p)$ contains the set $T^{-1}(V_p)$ which is open in M and contains $T^{-1}(p) = z$. Hence, z belongs to U_y. This means $T(z) = p$ belongs to $T(U_y)$. Thus, $U_{T(y)}$ is a subset of $T(U_y)$. It follows that $U_{T(y)} = T(U_y)$.

Since U_y is a proper subset of U_x and $T(U_x)$ is equal to U_y, then $T(U_y)$ is a proper subset of U_y. Thus, $U_{T(y)}$ is a proper subset of U_y.

Now $y \neq T(y)$ because $U_{T(y)}$ is a proper subset of U_y. The hypotheses of the theorem are now satisfied by y and $T(y)$. That is, y and T(y) are distinct points of M such that $U_{T(y)}$ is a proper subset of U_y. Therefore, the same reasoning as that used in the preceding three paragraphs can be applied to show that T(y) and T(T(y)) are distinct points, and that $U_{T(T(y))}$ is a proper subset of $U_{T(y)}$. The process thus far described can be repeated a countably infinite number of times to produce the sequence, $x_o, x_1, x_2, \ldots$, where $x_o = x$, $x_1 = y$, $x_2 = T(y)$, . . ., $x_n = T^n(y)$, . . ., and for each positive integer n, U_{x_n} is a proper subset of $U_{x_{n-1}}$. Also, if $i \neq j$ then $x_i \neq x_j$ because if $x_i = x_j$ then $U_{x_i} = U_{x_j}$.

The continuum M is compact and so the sequence $x_o, x_1, x_2, \ldots$, has a limit point x_w in M. Let p be a point of U_{x_w}. There exists a continuum K_p in M and a set V_p open in M such that V_p is a subset of K_p containing p and K_p is a subset of $M - \{x_w\}$. Since K_p is closed and does not contain x_w then there must be an infinite number of points of the sequence $x_o, x_1, x_2, \ldots$ that do not belong to K_p. Hence, for infinitely many positive integers n, $M - \{x_n\}$ contains K_p. That is, for infinitely many integers n, M is aposyndetic at p with respect to x_n. This means that for infinitely many integers n, p belongs to U_{x_n}. Now U_{x_n} is a subset of $U_{x_{n-1}}$ for every n. Thus, p belongs to all U_{x_n}. It follows that U_{x_w} is a proper subset of every U_{x_n}. It also follows that $x_w \neq x_n$ for any n because if $x_w = x_n$ then $U_{x_w} = U_{x_n}$ which would mean that U_{x_n} is also a subset of $U_{x_{n+1}}$. But this is not possible because $U_{x_{n+1}}$ is a proper subset of U_{x_n}.

Since M is homogeneous, there exists a homeomorphism T_1 such that $T_1(M) = M$ and $T_1(x) = x_w$.

Consider the map $T_1TT_1^{-1}$. Certainly $T_1TT_1^{-1}$ is a homeomorphism of M onto itself. Let $T_1TT_1^{-1}(x_w) = x_{w_1}$. Then, as argued in previous cases, $T_1TT_1^{-1}(U_{x_w}) = U_{x_{w_1}}$. Notice also that $T_1TT_1^{-1}(U_{x_w}) = T_1T(U_x) = T_1(U_y)$. But U_y is a proper subset of U_x and $T_1(U_x) = U_{x_w}$. Thus, $T_1(U_y)$ is a proper subset of U_{x_w}. That is, $U_{x_{w_1}}$ is a proper subset of U_{x_w}. The argument used to show that the points of the sequence, $x_o, x_1, x_2, \ldots$, are distinct can be applied to show that x_{w_1} does not equal x_w nor any x_n that precedes x_w.

It will now be shown that the process thus far described can be carried out in such a way that an uncountable sequence of sets, $U_{x_o}, U_{x_1}, U_{x_2}, \ldots$, is produced. It should be clear that the process described produces sequences that may be dependent on the particular homeomorphisms T and T_1 that are selected. Since it is not necessarily true that T is the only homeomorphism which maps M to M and x to y, then the sequence produced by the process may not be the only sequence with the ascribed properties. It will be shown that some such sequence must be uncountable.

Let S be the class which contains every sequence of sets that can be produced by repeating the process, and suppose that each element of S is a countable sequence. Now if S_a is an arbitrary member of S then (1) $S_a = U_{a_1}, U_{a_2}, U_{a_3}, \ldots$, (2) S_a is countable, (3) $U_{a_1} = U_x$, (4) $U_{a_2} = U_y$, (5) U_{a_n} is a proper subset of every U_{a_k}

that precedes it, and (6) U_{a_n} is open for all n.

Let S_a and S_b be elements of S. Define the relation ($\leq$) by $S_a \leq S_b$ if and only if $U_{a_1}, U_{a_2}, U_{a_3}, \ldots, U_{a_t}$ is an initial segment of S_a implies that $U_{a_1} = U_{b_1}, U_{a_2} = U_{b_2}, U_{a_3} = U_{b_3}, \ldots,$ $U_{a_t} = U_{b_t}$. The notation chosen for initial segments is intended to indicate that they may be either finite for infinite. Indeed, the process used to create $U_{x_0}, U_{x_1}, U_{x_2}, \ldots$ shows that initial segments may be infinite and still not include the whole sequence. Note also that $S_a \leq S_b$ simply means that S_a is an initial segment of S_b. It follows that if $S_a \leq S_b$ then $S_b \leq S_a$ if and only if $S_a = S_b$, and that if $S_a \leq S_b$ and $S_b \leq S_c$ then $S_a \leq S_c$. Hence, the relation ($\leq$) produces a partial order on S.

A sequence of elements of S, say $S_{a_1}, S_{a_2}, S_{a_3}, \ldots,$ is called a chain if and only if $S_{a_1} \leq S_{a_2} \leq S_{a_3} \leq \ldots$. Let B = $S_{a_1}, S_{a_2}, S_{a_3}, \ldots$ be a chain in S and consider the union of the elements of this chain, B*. Now B* will be a sequence, say $U_{c_1}, U_{c_2}, U_{c_3}, \ldots,$ such that every initial segment of B* is an initial segment of some element of B. Since B* is a countable union of countable sequences then B* is a countable sequence. Also, by the definition of the elements of B, (1) $U_{a_1} = U_{c_1}$, (2) $U_{a_2} = U_{c_2}$, (3) U_{c_n} is a proper subset of every U_{c_k} that precedes it, and (4) U_{c_n} is open for every n. The fact that every initial segment of B* is an initial segment of some element of B shows that B* can be produced by the process that produced the elements of S. Hence, B* belongs to S. If S_{a_k} is an element of B, then S_{a_k} is an initial segment of every element of B that follows S_{a_k} and so S_{a_k} is also an initial segment of B*. That is, B* is an upper bound of the chain B. Since B was an arbitrary chain in S then every chain of S has an upper bound. Therefore, by Zorn's Lemma there exists an element S_b of S such that if S_p belongs to S and $S_b \leq S_p$, then $S_b = S_p$ (17; 33, Theorem 25).

Let x_o, x_1, x_{b_2}, x_{b_3}, . . . be the sequence of points that corresponds to $S_b = U_{x_o}$, U_{x_1}, $U_{x_{b_2}}$, $U_{x_{b_3}}$, . . ., where $x_o = x$ and $x_1 = y$. Now x_o, x_1, x_{b_2}, . . . is a sequence of distinct points of the compact continuum M. Therefore, there exists a limit point x_{b_w} of the sequence that belongs to M. The same argument used to extend the sequence, x_o, x_1, x_2, . . ., to include its limit point x_w can now be used to extend the sequence, x_o, x_1, x_{b_2}, . . ., to include its limit point x_{b_w}. The same argument shows that $U_{x_{b_w}}$ is a proper subset of every element of S_b. Thus, if $S_{b'} = U_{x_o}$, U_{x_1}, $U_{x_{b_2}}$, . . ., $U_{x_{b_w}}$, then $S_{b'}$ belongs to S. But $S_b \leq S_{b'}$ and $S_b \neq S_{b'}$. This contradicts the definition of S_b.

Therefore, the assumption that every element of S is countable is false.

Let the sequence, U_{x_o}, U_{x_1}, U_{x_2}, . . ., U_{x_w}, . . ., be some uncountable sequence in S. Then this sequence is well-ordered, uncountable, monotonically decreasing, and each member of the sequence is an open set. Each member of the sequence is a proper subset of all members of the sequence that precede it, so it is possible to select the sequence Y of distinct points $Y = y_1, y_2, y_3, \ldots$ in such a way that y_w belongs to U_{x_w} but to no member of the sequence that follows U_{x_w}. Since M is compact, then every uncountable subset of Y has a limit point. Therefore, there exists a point y_v of Y which is a limit point of the set of all points of Y that precede y_v in Y and a limit point of the set of all points of Y that follow y_v in Y, (24; 3, Theorem 6). But U_{x_v} is an open set containing y_v but no point of Y that precedes y_v in Y. This is a contradiction.

Therefore, U_y is not a proper subset of U_x.

A complete exposition of the proof of the following theorem would require the development of several concepts which are not considered in this paper. An outline of the proof has been provided.

<u>Theorem</u> <u>4.6</u>: A homogeneous, hereditarily unicoherent, bounded plane continuum M is indecomposable.

Indication of Proof: Assume M is not indecomposable. It is known that a compact continuum M is indecomposable if and only if there do not exist two distinct points x and y of M such that M is aposyndetic at x with respect to y (13; 407, Theorem 9). Therefore, the assumption that M is not indecomposable is equivalent to the assumption that there exist two points x and y of M such that M is aposyndetic at y with respect to x. That is, for some point x of M the set U_x is non-empty. As was noted in the proof of Lemma 4.5, since U_x is non-empty for some point x of M, and since M is homogeneous, then U_z is non-empty for every point z of M.

Let x be an arbitrary point of M and let H be a set such that y belongs to H if and only if $U_x = U_y$. Define $U = U_x$ for all x in H. As in the proof of Lemma 4.5, it can be seen that U is open. Also, it is clear that H is a subset of M - U. Lemma 4.5 can be used to establish that the set H is closed.

It is impossible for every point of a compact continuum to cut every point of a domain relative to the continuum from every point of another domain relative to the continuum (6; 501, Corollary 2). If it is assumed that some point x of H cuts a point w of M from a point z of U but that x does not cut w from some other point of U, then the homogeneity of M leads to a contradiction of the preceding statement. Thus, if x cuts a point w of M from a point of U, then x cuts w from all points of U. It can be shown that U is a subset of U_w. Lemma 4.5 will then imply that $U = U_w$ and hence that w also belongs to H.

If o is a point of H, let N_o be the set of all points x of H such that x cuts o from U. For every point o of H the set N_o is closed and o cuts all points of N_o from every point of U.

The set H does not contain a domain with respect to M. For suppose H contains a domain D. Let o be any point of H and consider the set N_o. Suppose D is also a subset of N_o. If this were true, then since every point of N_o cuts every other point of N_o

from each point of U, it follows that any point x of D would cut every point of the domain $D - \{x\}$ from each point of U. But the homogeneity of M would then imply that every point of M would cut each point of some open subset of M from each point of some other open subset of M. As noted earlier in this proof, such a situation cannot occur in a compact continuum (6; 501, Corollary 2). Hence, D is not a subset of N_o. Thus $D - D \cap N_o$ is non-empty. Since N_o is closed, then $D - D \cap N_o$ is a domain. Suppose M is aposyndetic at some point x of $D - D \cap N_o$ with respect to some point y of N_o. By definition of U_y, x belongs to U_y. Since y belongs to N_o which is a subset of H, then $U_y = U$. But x belongs to D which is a subset of H and so $U_x = U$. Therefore $U_x = U_y$. This is impossible because it would imply that x belongs to U_x. Hence, M is aposyndetic at no point of $D - D \cap N_o$ with respect to a point of N_o. The following conditions are now clearly satisfied: (1) M is a compact continuum, (2) $D - D \cap N_o$ is an open subset of M, (3) N_o is a closed subset of M such that $(D - D \cap N_o) \cap N_o$ is empty. (4) M is not aposyndetic at any point of $D - D \cap N_o$ with respect to a point of N_o. Hence, if z belongs to U, $D - D \cap N_o$ contains a point x and N_o contains a point y such that y cuts x from z (13; 405, Theorem 6). As pointed out at the end of the preceding paragraph, this means y cuts x from every point of U. Therefore, x belongs to N_o. This is clearly impossible, because x belongs to $D - D \cap N_o$. It follows that H contains no domain.

Since U is open in M, then $M - U$ is closed. It can be shown that $M - U$ is connected and hence that $M - U$ is a continuum.

Suppose the domain U is not dense in M. Then $M - \bar{\bar{U}}$ is not empty. The set $M - \bar{\bar{U}}$ is not a subset of H because $M - \bar{\bar{U}}$ is a domain with respect to M and H contains no domain with respect to M. Thus, $M - (\bar{\bar{U}} \cap H)$ is non-empty. Since $\bar{\bar{U}}$ and H are closed, then $M - (\bar{\bar{U}} \cap H)$ is open with respect to M. Let y be a point of H. By definition of U, M is not aposyndetic at any point of $M - (\bar{\bar{U}} \cap H)$ with respect to y. Let z be any point of U. As in the preceding paragraph, sufficient conditions have been satisfied to guarantee the existence of a point x in $M - (\bar{\bar{U}} \cap H)$ such that y cuts x from z in M. The argument in paragraph four of this proof shows that x belongs to H. Obviously, this is a contradiction because x belongs to $M - (\bar{\bar{U}} \cap H)$. Therefore, U is dense in M.

The facts that M is homogeneous and hereditarily unicoherent, U is dense in M, and $M - U$ is a continuum can be utilized to show that if o is an arbitrary point of H, then $N_o = M - U$.

Now by definition of N_o, N_o is a subset of H. By definition of H, H is a subset of $M - U$. Since $M - U = N_o$, then $N_o = H$.

It has now been shown that H is a continuum, and that the union of H and U is M. Also, H is the boundary of U and every point of H cuts every point of H from every point of U.

If G is defined to be the collection of all images of H under homeomorphisms of M to itself, it can be shown that G is an upper semi-continuous collection of point sets (Definition 2.44) filling up M. With respect to its elements as points, G can be shown to be a continuum M' which is compact, aposyndetic, homogeneous, and hereditarily unicoherent. Under these conditions M' must contain a nonseparating point (24; 38, Theorem 93; 13; 404, Theorem 0; and 29; 737, Theorem 6.6). Since M' is homogeneous, it follows that every point of M' is a nonseparating point.

Let a and b be distinct points of M' and T be an irreducible subcontinuum of M from a to b. Let x be any point of T distinct from a and b. Since x is a nonseparating point of M', there exists a continuum T_1 in M - $\{x\}$ that contains both a and b. But M' is hereditarily unicoherent. So the common part of T and T_1 is a subcontinuum containing a and b but not x. This contradicts that T was irreducible from a to b.

This contradiction is sufficient to imply that the original assumption of the existence of the sets U and H was invalid.

Therefore, M must be indecomposable.

Some additional details that were omitted from the preceding proof can be found in the paper, "Homogeneous Unicoherent Indecomposable Continua," by F. B. Jones, which is listed in the bibliography of this paper.

<u>Theorem 4.7</u>: If M is a homogeneous bounded plane continuum that does not separate the plane, M is hereditarily unicoherent.

Proof: Suppose M is not hereditarily unicoherent. Then there exist two points x and y of M such that there exist at least two distinct irreducible subcontinua C_1 and C_2 of M from x to y (21; 179, Theorem 1.1). The common part of C_1 and C_2 is not connected because if the common part were connected then it would contain a subcontinuum from x to y. Since C_1 and C_2 are distinct, that subcontinuum would have to be a proper subcontinuum of C_1 or C_2. But this is not possible because C_1 and C_2 are irreducible. Hence, there exist two complementary domains H and K of $C_1 \cup C_2$ (24; 175, Theorem 22). Therefore $C_1 \cup C_2$ separates the plane. Let S be the plane and $S - (C_1 \cup C_2) = H_1 \cup K_1$ where H_1 and K_1 are open sets with no points in common.

Now $S - M = (H_1 - M) \cup (K_1 - M)$ and neither $H_1 - M$ nor $K_1 - M$ is empty. For suppose either $H_1 - M$ or $K_1 - M$ is empty, say $H_1 - M$. Then H_1 is an open set that is a subset of M. But Theorem 4.3 implies that M is not locally connected. Since M is homogeneous then M cannot be locally connected at any point. Therefore, M cannot contain H_1. It follows from $S - M = (H_1 - M) \cup (K_1 - M)$ that M separates the plane. This is a contradiction and so M must be hereditarily unicoherent.

<u>Theorem 4.8</u>: If M is a homogeneous bounded plane continuum which does not separate the plane then M is indecomposable.

Proof: The theorem is an immediate result of Theorems 4.6 and 4.7.

Homogeneous Linearly Chainable Continua

In the preceding section it was shown that all homogeneous bounded plane continua that do not separate the plane are indecomposable. Theorem 4.9 in this section will show that all compact, hereditarily indecomposable, linearly chainable continua are homeomorphic. The definition of a pseudo-arc given in Chapter III together with Theorem 3.5 show that a pseudo-arc is a nondegenerate, hereditarily indecomposable, linearly chainable, compact continuum. Theorem 4.10 will prove that every homogeneous, nondegenerate, linearly chainable, compact continuum is a pseudo-arc. Thus, the results of this section together with those of the preceding section, are sufficient to prove that all homogeneous bounded plane continua that do not separate the plane are homeomorphic and are pseudo-arcs.

Theorem 4.9: If M_1 and M_2 are compact, nondegenerate, hereditarily indecomposable, linearly chainable, continua, then M_1 and M_2 are homeomorphic.

Proof: Since M_1 is linearly chainable, there exists a sequence of chains, $C_1, C_2, C_3, \ldots$, such that no link of C_i has diameter greater than $1/i$, each element of C_i intersects M, and the closure of every link of C_{i+1} is contained in a link of C_i.

It will be shown that the fact that M_1 is hereditarily indecomposable implies that for some integer n_2, C_{n_2} is crooked in C_1.

Let the links of C_1 be $c_{11}, c_{12}, c_{13}, \ldots, c_{1n_1}$. Suppose no chain of the sequence, $C_1, C_2, C_3, \ldots$, is crooked in C_1. Then there exist links c_{1h} and c_{1k} of C_1 such that $k-h > 2$ and for infinitely many integers m, $C_m = (c_{m1}, c_{m2}, \ldots, c_{mt_n})$ has two links c_{mi} and c_{mj} in c_{1h} and c_{1k} respectively such that if c_{mr} is in $c_{1(k-1)}$ and between c_{mi} and c_{mj}, then there is not a link of C_m in $c_{1(n+1)}$ which is between c_{mr} and c_{mj}. The preceding statement is less confusing when it is noted that all the assertions of the sentence are justified by the existence of infinitely many chains in the sequence, C_1, C_2, C_3, . . ., that are <u>not</u> crooked in C_1. It can be supposed that the link c_{mr} identified above is such that no link of C_m is contained in $c_{1(k-1)}$ and is between c_{mi} and c_{mr}. Let W_m be the union of c_{mi}, c_{mr}, and the links of C_m between them. Let V_m be the union of c_{mr}, c_{mj}, and the links of C_m between them.

A sequence, a_1, a_2, a_3, . . ., of integers can be selected in such a way that the sequences, W_{a_1}, W_{a_2}, W_{a_3}, . . . and V_{a_1}, V_{a_2}, V_{a_3}, . . ., converge (24; 24, Theorem 59).

Let W be the limiting set of W_{a_1}, W_{a_2}, W_{a_3}, . . . and let V be the limiting set of V_{a_1}, V_{a_2}, V_{a_3}, ... Both W and V are continua (28; 14, Theorem 9.1). Now W intersects the closure of c_{1h} but not the closure of c_{1k} and V intersects the closure of c_{1k} but not the closure of c_{1h}. Thus W and V are distinct. But W and V are not disjoint because for every m both W_m and V_m contain the link c_{mr}.

A contradiction has been reached since it is now possible to conclude that the hereditarily indecomposable continuum M_1 has a decomposable subcontinuum $V \cup W$.

Therefore, there exists a subsequence, C_{n_1}, C_{n_2}, C_{n_3}, . . ., of C_1, C_2, C_3, . . . such that $C_{n_{i+1}}$ is crooked in C_{n_i}.

The continuum M_1 has uncountably many distinct composants (24; 59, Theorem 139). Therefore, there exist two distinct points p and q belonging to different composants of M_1. For every i, let W_i be the union of the links of the subchain of C_{n_i} from p to q. The argument contained in the third paragraph of the proof of Theorem

3.8 shows that the limiting set of W_1, W_2, . . . is a continuum containing p and q. Since p and q belong to different composants of M_1 and the limiting set of W_1, W_2, W_3, . . . is a subcontinuum of M_1 containing p and q, then that limiting set must be M_1. It follows that for every integer j, some W_k $(k > j)$ intersects both the first and last links of C_{nj}, and hence the subchain C_{nk} intersects the first and last links of C_{nj}.

The hypotheses of Theorem 2.68 have now been satisfied. Therefore, there is a chain E_j such that the first link of E_j contains p, the last link contains q, E_j is a consolidation of C_{nj} and each link of E_j lies in the union of two adjacent links of C_{nj}. It is clear that the diameter of every link of E_j is less than $2/j$.

A short induction argument that makes use of Theorems 2.65 and 2.66 will show that for every j, E_{j+1} is crooked in E_j.

Therefore, from the sequence, E_1, E_2, E_3, . . ., a sequence, D_1, D_2, D_3, . . ., can be selected such that for every positive integer i, (1) D_i is a chain from p to q, (2) D_{i+1} is crooked in D_i, (3) the closure of each link of D_{i+1} is a subset of a link of D_i, (4) no link of D_i has diameter greater than $1/i$, and (5)

$$M_1 = \bigcap_{i=1}^{\infty} D_i^*.$$

Let p' and q' be points of M_2 belonging to different composants of M_2. The process employed to create the sequence D_1, D_2, D_3, . . . can be repeated to create a sequence G_1, G_2, G_3, . . . such that for every i, (1) G_i is a chain from p' to q', (2) G_{i+1} is crooked in G_i, (3) the closure of each link of G_{i+1} is a subset of a link of G_i,

(4) no link of G_i has diameter greater than $1/i$, and (5) $M_2 = \bigcap_{i=1}^{\infty} G_i^*$.

The hypotheses of Theorem 3.7 have been satisfied. Therefore, there is a homeomorphism mapping M_1 to M_2.

An end point of a continuum has been defined in general to be a point with Menger order one with respect to that continuum. In the case of a linearly chainable continuum M, a point p will be called an end point of M if and only if for each positive number ϵ there is an ϵ-chain covering M such that the first link contains p. In this case the two definitions of end point are equivalent but that fact is unimportant in the discussion that follows, since no theorems that were proved using the first definition will be used here.

<u>Theorem</u> <u>4.10</u>: Each homogeneous, nondegenerate, linearly chainable, bounded plane continuum is a pseudo-arc.

Proof: It will be shown that M has an end point p. Let $T_1, T_2, T_3, \ldots$ be a sequence of $1/n$-chains covering M. Let $q_1, q_2, q_3, \ldots$ be points of M such that q_n belongs to the first link of T_n for every n. Since M is compact, some subsequence of $q_1, q_2, q_3, \ldots$ converges to a point q.

For each neighborhood N of q and each positive number ϵ there is an ϵ-chain covering M one of whose end links intersects M and lies in N. Call this property, "the property of q".

It will be shown that every point of M has the property of q. Let x be an arbitrary point of M and let F be a homeomorphism mapping

M to M and q to x. Now F is uniformly continuous (10; 135, Theorem 8.16). Thus, given an $\epsilon > 0$, there exists a $\delta > 0$ such that if p_1 and p_2 belong to M and the distance between p_1 and p_2 is less than δ then the distance between $F(p_1)$ and $F(p_2)$ is less than ϵ. That is, for every n there is some point q_t of $q_1, q_2, q_3, \ldots$ such that $F(q_t)$ is within a distance of 1/n of $F(q) = x$. Now let $E_{n1}, E_{n2}, E_{n3}, \ldots, E_{nm}$ be the links of T_n. Consider the sets $F(E_{n1} \cap M), F(E_{n2} \cap M), \ldots, F(E_{nm} \cap M)$. It is clear that $(F(E_{ni} \cap M)) \cap (F(E_{nj} \cap M))$ is empty or non-empty according as $(E_{ni} \cap M) \cap (E_{nj} \cap M)$ is empty or non-empty. It is also clear that $F(E_{ni} \cap M)$ is an open subset of M for every i. It follows that for each set $F(E_{ni} \cap M)$ there exists and open subset of the plane G_{ni}, such that $G_{ni} \cap M = F(E_{ni} \cap M)$ and $G_{ni} \cap G_{nj}$ is empty or non-empty according as $(F(E_{ni} \cap M)) \cap (F(E_{nj} \cap M))$ is empty or nonempty. This means that $G_{n1}, G_{n2}, \ldots, G_{nm}$ are links of a chain covering M. Let the chain whose links are $G_{n1}, G_{n2}, \ldots, G_{nm}$ be denoted by S_n. It follows from the uniform continuity of F that for every n, there exists an S_t whose links have diameter less than 1/n. Therefore, for each neighborhood N of F(q) and each positive number ϵ there is an ϵ-chain covering M, one of whose end links intersects M and lies in N. Thus, x has the property of q. Since x was arbitrary, every point of M has the property of q.

Let d_1 be an end link of a 1-chain covering M and let p_1 be any point of M belonging to d_1. Since p_1 has the property of q, there is an end link d_2 of a 1/2-chain covering M such that d_1 contains $\bar{d}_2$ and d_2 contains a point p_2 of M. Similarly, there is an end link of a 1/3-chain covering M such that d_2 contains $\bar{d}_3$, and d_3 contains a point p_3 of M. This process may be continued to define the sequences, d_1, d_2, d_3, . . ., and p_1, p_2, p_3, ... The intersection of the sets d_1, d_2, d_3, . . . is non-empty since $\bigcap_{i=2}^{\infty} \bar{d}_i$ is a subset of $\bigcap_{i=1}^{\infty} d_i$ and $\bigcap_{i=2}^{\infty} \bar{d}_i$ is non-empty (10; 69, Theorem 3.30). Let p belong to $\bigcap_{i=1}^{\infty} d_i$. The point p belongs to M because p is a limit point of p_1, p_2, p_3, . . . and M is compact. Now for every $\epsilon > 0$ there is some set in the sequence, d_1, d_2, d_3, . . ., whose diameter is less than ϵ. Therefore, for every $\epsilon > 0$ there is an ϵ-chain covering M whose first link contains p. That is, p is an end point of M.

It can now be shown that M is hereditarily indecomposable. Assume that M is not hereditarily indecomposable. This implies that M contains a continuum H which is the union of two proper subcontinua H' and H''. Certainly, the intersection of H' and H'' is non-empty. Let p belong to both H' and H''. An argument similar to the one used to show that every point of M has the property of q will show that every point of M is an end point. Hence, p is an end point of M. But it is known that a necessary and sufficient conditions that a point p be an end point of a linearly chainable continuum M is that for every pair of subcontinua H' and H'' containing p, either H' contains H'' or H'' contains H' (5; 66, Theorem 13). This is impossible since both H' and H'' are proper subcontinua of their union. Therefore, M is hereditarily indecomposable.

Since by hypothesis M was nondegenerate and linearly chainable, and all nondegenerate, hereditarily indecomposable, linearly chainable continua are homeomorphic (Theorem 4.9) then M must be a pseudo-arc (Definition 3.3 and Theorem 3.5).

Homogeneous Continua That Separate the Plane

The theorem presented in this section will complete the list of theorems necessary to justify the classification system presented in the introduction to this chapter. The theorem will not be proved for the reasons cited in the introduction.

Theorem 4.11: Every homogeneous bounded plane continuum that separates the plane and is decomposable, but is not a simple closed curve, is a circle of pseudo-arcs (14; 732, Theorem 2, and 7; 181, Theorem 10).

The above theorem is proved by showing that every homogeneous bounded plane continuum that separates the plane and is decomposable, but is not a simple closed curve, can be decomposed into an upper semi-continuous collection of pseudo-arcs that fill up the continuum (14; 732, Theorem 2). This result would be sufficient to justify the name "circle of pseudo-arcs". However, as in the case of the pseudo-arc, it is also shown that any two such continua are homeomorphic (7; 181, Theorem 10). Thus, the example presented in Chapter III is representative of all members of the class.

CHAPTER V

SUMMARY

The historical development of the examples and theorems on homogeneous bounded plane continua is given in Chapter I of this paper. This chapter will provide a review of the development of those same examples and theorems as they are found within this paper.

Chapter II delineates the topological concepts necessary for the later presentation of specific examples and major theorems. In particular, a detailed presentation of the properties of crooked chains is given in Chapter II.

In Chapter III, the three distinct examples of homogeneous bounded plane continua, which have been discovered to this date, are given. The simple closed curve and the pseudo-arc are shown to be homogeneous. The circle of pseudo-arcs is described in enough detail that its homogeneity should at least seem probable. A fourth example, distinct from the first three, but which has neither been shown to be homogeneous nor non-homogeneous, is also presented in Chapter III.

A classification system, which places all chainable homogeneous bounded plane continua in four distinct classes, is given in Chapter IV. All such continua are classified according to whether they are (1) pseudo-arcs, (2) simple closed curves, (3) circles of pseudo-arcs, or (4) indecomposable continua that separate the plane.

Theorems that show that the classification system has the asserted properties are given in the remainder of the chapter.

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