ANALYSIS OF MULTI-LAYERED, ANISOTROPIC SHELLS OF REVOLUTION SUBJECTED TO AXISYMMETRIC THERMAL GRADIENTS

By

TUNG DJU LIN

Bachelor of Science
Taiwan Christian College
Taiwan
June, 1959

Master of Science
University of Tennessee
Knoxville, Tennessee
August, 1962

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College of the Oklahoma State University
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Thesis Approved:

Donald E. Boyd
Thesis Adviser

Ahmed Salama

Thomas Scott Dean

Jim Allen

N. Durham
Dean of the Graduate College

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Tung Dju Lin

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NOMENCLATURE

A_1, A_2	Undetermined coefficients;
$A^{\prime}, A^{*}$	Defined by equations (3.2.5.a) and (3.2.10.a);
a_1, a_2	Undetermined coefficients;
$\bar{\bar{a}}$	Undetermined factor;
a^{k}_{ij}	Elastic constants;
B_1, B_2	Undetermined coefficients;
$B^{\prime}, B^{*}, B^{**}$	Defined by equations (3.2.5.a), (3.2.10.a) and (3.4.6);
B^{k}_{ij}	Constant coefficients;
b_1, b_2	Undetermined constants;
C^{*}	Defined by equation (3.2.10.a);
C_{ij}	Extensional stiffness;
D_{11}, D_{12}, D_{22}	Defined by equation (2.4.8.a);
D^{*}, D^{**}	Defined by equations (3.2.10.a) and (3.4.7);
D_{ij}	Bending stiffness;
E	Modulus of elasticity of steel;
E^{k}_{1}, E^{k}_{2}	Elastic moduli of k^{th} layer of shell in the meridional and circumferential directions respectively;
$\bar{\bar{E}}_1, \bar{\bar{E}}_2$	Undetermined constants;
E_L	Modulus of elasticity of fiber glass;
E_r	Modulus of elasticity of resin;
E_{xm}	Generalized transverse modulus of elasticity;

e_i^k	Total strain at k^{th} layer;
$\bar{\bar{F}}_1$, $\bar{\bar{F}}_2$	Undetermined constants;
G^*, G^{**}	Defined by equation (3.4.20);
H°, H^*, H^{**}	Defined by equations (A.2.52.a) and (A.2.53.a);
h	Thickness of shell;
i	Indice;
j	Indice;
K	Thermal conductivity;
K_{ij}	Stiffness of interaction of tension and bending;
k	Layer number;
$\bar{K}$	Coefficient of intensity function defined by equation (3.1.3);
L	Longitudinal length of shell;
$\bar{\bar{L}}$	Linear operator;
M_1, M_2	Bending moments in the meridional and circumferential directions respectively;
M_{1t}, M_{2t}	Bending moments, due to temperature, in the meridional and circumferential directions respectively;
m	Number of layers above the middle surface of shell;
N	Transverse shear force;
n	Number of layers below the middle surface of shell;
n'	Total number of layers of shell;
P_1, P_2, P_3	Defined by equation (2.4.9.a);
P	Coefficient of the longitudinal temperature variation equation;
R_1, R_2	Radii of curvature in the meridional and circumferential directions

	respectively;
r	Distance from points of middle surface to the axis of revolution;
s	Variable along the meridional direction;
T_1, T_2	Resultant forces in the meridional and circumferential directions respectively;
T_{1t}, T_{2t}	Resultant forces in the meridional and circumferential directions respectively;
T^k	Temperature of the k^{th} layer;
T_m	Temperature of the middle layer;
u	Displacement in meridional direction;
V	Auxiliary function;
V^*, V^{**}	Defined by equation (3.4.25);
W	Meissener's function;
w	Displacement in radial direction;
x	Defined by equation (A.2.54.a);
z	Coordinate axis of shell element normal to shell surface;
α_o	Defined by equation (3.3.6.a);
α_i^k	Thermal coefficients for k^{th} layer;
β	Variability function, defined by equation (3.1.2);
ΔT^k	Temperature difference between the k^{th} layer and the middle layer;
γ	Distance between layer and middle surface;
$\emptyset$	Defined in figure 2;
η	Component of shear deformation;
τ	Torsional deformation;

ν	Poisson's ratio of steel;
ν_f	Poisson's ratio of fiber glass;
ν_r	Poisson's ratio of resin;
ν_{LT}	Poisson's ratio, defined by equation (A.3.3);
ν_{TL}	Poisson's ratio, defined by equation (A.3.4);
ε_1, ε_2	Tangential strains of middle surface of shell in the meridional and circumferential directions respectively;
δ_{m+n}	Defined in figure 1;
χ_1, χ_2	Curvature changes of the middle surface of shell in the meridional and circumferential directions respectively;
λ	Arbitrary constant;
θ	Geometrical variable;
Δ	Defined in figure 1;
$\bar{\bar{\Delta}}$	Nondimensional height of an element;
$\bar{\bar{\sigma}}$	Defined by equation (2.5.10);
σ_1^k, σ_2^k	Layer stresses in the meridional and circumferential direction respectively;
π_k	Defined by equation (2.1.5.a);
Ω	Defined by equation (2.4.6.a);
$\theta(\beta)$, $\mathcal{S}(\beta)$, $\emptyset(\beta)$, $\psi(\beta)$	Defined by equation (3.3.8.a);
φ	Percentage of fiberglass by volume;
$\Phi_1(s)$	Defined by equation (2.4.9.b);
$\Phi_2(s)$	Defined by equation (2.4.10.a);

CHAPTER I

INTRODUCTION

1.1 Statement of the Problem

Shells are widely used as structural elements in modern buildings, aircraft, ships, rockets, etc. A careful study of shells used in engineering shows that they are frequently laminated from anisotropic materials. A very important application is found in spacecraft and supersonic missile construction in which aerodynamic heating around the forward portion of such craft often results in large temperature gradients over the surface and through the shell thickness. Nonuniform temperatures induce stresses within the structure and may have a pronounced effect on its design. Thus, it is important to be able to predict stresses arising from temperature effects.

Many scientists today in the technically advanced countries engage in the research and development of the fastest possible ballistic missiles to meet the need of military defensive and offensive system. It is believed that the present analysis will contribute to the structural design of nose cones of missiles flying at zero angle of attack.

A common type of shell is one which possesses symmetry with respect to an axis of revolution. An isothermal theory for such shells has been developed previously and applied to conical shells subjected to certain special cases of surface loading (6), (7). This thesis extends the theory of anisotropic shells by including thermal effects and applies the results to the analysis of the fiberglass wound conical shell subjected to axisymmetric thermal gradients.

1.2 Historical Background

The development of the theory of laminated anisotropic shells of revolution subjected to isothermal conditions provides the background for the content of this thesis. Historically, analytical studies of structural members of multi-layered construction have been of interest to technical people in the fields of aircraft construction, rocket construction and ship building. It is becoming increasingly important with the rapid development of modern aerospace technology and the introduction of new, efficient, anisotropic material to expand further the technology.

There have been exhaustive studies dealing with the theory of homogeneous, isotropic shells, but there have been few which investigated the general theory of anisotropic laminated shells, until Ambartsumyan (1) completed his book, Theory of Anisotropic shells in 1959. His effort shed light on many fundamental problems in the theory of anisotropic shells. Although this book is considered as a good reference in the theory of anisotropic shell analysis and its application, many important problems are not covered, such as the theories of stability and vibration, nonlinear theory and temperature problems of anisotropic laminated shells. It is the purpose of this thesis to fill one of these gaps; that of axisymmetric thermal stresses in laminated, anisotropic shells of revolution.

Temperature problems in the theory of isotropic shells have been studied by some prominent investigators. In 1952 Huth (7) presented a paper analyzing thermal stresses in conical shells, considering aerodynamic heating of a missile nose cone. By using Meriam's analysis of the rotating conical shell (12) and a standard procedure developed by Meissner (11), Huth (7) obtained a fourth-order ordinary differential equation. Thompson's function was used in the solution of the homogeneous part and a polynomial series expansion was applied to obtain the particular solution.

In 1962, S. B. Dong, K. S. Pister and R. L. Taylor presented a paper on the theory of laminated anisotropic shells and plates (4). Following classical isotropic shell procedure, but incorporating specialized elasticity relations for orthotropic laminations, governing equations for small displacements were presented for shells of revolution. The specialization of equations to the case of the cylindrical shell was made using the well-known Donnell approximations. Incorporating the Airy stress function and the transverse displacement, the system of equations were reduced to two fourth-order differential equations. Following a procedure suggested by Vlasov (19), the equations were simplified to a single equation in terms of the transverse displacement. The general solution was obtained by a trial function method for the homogeneous solution and by polynomial series for the particular solution.

In 1963 Radkowski presented a paper on stress analysis of orthotropic thin multi-layered shells of revolution (13). From strain-displacement relations, stress-strain relations and equilibrium equations, two, coupled, second-order differential equations were obtained in terms of two unknown variables; i. e. horizontal force and reference surface meridional curvature change. A computer program was written to solve the matrix difference equations. An analytical method was employed for solving cylindrical shell equations.

In 1966, papers were presented by Dong (3) and Grinchenko (6) concerning the temperature problems in laminated shells of revolution. Dong (3) used the finite element method to carry out the solution of the problem. He established the stiffness for each finite element and used conventional structural techniques to enforce equilibrium and continuity of displacements at all joints. Grinchenko

(6) used an analytical method to solve the problem of the shell consisting of isotropic layers. He set up the equations of equilibrium and compatibility of deformation of the shell elements in association with conventional relationships between forces, moments and deformations by applying the Kirchoff-Love hypothesis and combined these through the use of Meissner's functions. He obtained two second-order governing differential equations. By differentiation, he reduced the two simultaneous differential equations to one third order hypergeometric equation and used the sum of three partial solutions to represent the sought solution of the governing differential equation. Both Dong's and Grinchenko's works were limited to the conical shell. This thesis, however, covers general shells of revolution.

1.3 Basic Concepts

The theory of shells is a part of the theory of elasticity of elastic bodies. In the theory of elasticity, the term shell is applied to bodies bounded by two curved surfaces, the distance between the surfaces being small in comparison with the other dimensions. The locus points which lie at equal distances from these two surfaces define the reference surface of the shell. The distance between two curved surfaces of the shell determines its thickness and will be designated by h. The investigation was made with an infinitely small anisotropic element defined at different points of the body by three orthogonal coordinate lines. In the general case of a uniform curvilinearly anisotropic body, the elastic body obeys the generalized Hooke's Law.

1.4 Assumptions

1) The shell was considered to be thin. (The ratio of its thickness to the radius of curvature of the reference surface being very small compared to unity).

2) The shell is of uniform thickness h consisting of an arbitrary number of homogeneous anisotropic layers, each having uniform thickness, t^k.

3) It was assumed that at each point of each layer there is only one plane of axial symmetry parallel to the reference surface of the shells.

4) The curvilinear coordinates were selected to coincide with the lines of principal curvature of the shell surface.

5) All layers of the shell obey the generalized Hooke's Law and function simultaneously without slipping.

6) After deformation, a rectilinear element normal to the undeformed coordinate surface of the shell remains rectilinearly normal to the deformed coordinate surface of the shell with its length preserved. Thus, the normal stresses on an area parallel to the reference surface of the shell were neglected in comparison with other stresses.

CHAPTER II

DERIVATION OF THE GOVERNING EQUATIONS

The following development proceeds along lines similar to those established by Ambartsumyan (1). For completeness of this presentation some of his work will be briefly redeveloped here. However, major emphasis will be placed in this chapter on rederiving the governing equations to include the effects of nonuniform, axisymmetric temperature distributions.

2.1 Basic Elastic Constants

Consider a shell consisting of a number of curvilinearly anisotropic layers as shown in figure (1). The shell is undergoing small deformations while obeying the generalized Hooke's Law. Using the tensor notation, the stress-strain relation can be expressed as

$$e_i^k - \alpha_i^k T^k = a_{ij}^k \sigma_j^k \tag{2.1.1}$$

where i, j are indices and k is layer designation; e_i^k is the total strain and $\alpha_i^k T^k$ are the strains due to the local temperature with α_i^k being coefficients of linear thermal expansion and T^k is the local layer temperature. The constants a_{ij}^k are called the elastic constants and there is generally a total of 21 independent elastic constants. In the case of three orthogonal planes of elastic symmetry, the number of a^k_{ij} reduces to 9. If, at a point in a body, there are three mutually perpendicular planes of elastic symmetry the body is known as an orthogonally anisotropic or an orthotropic body.

The geometric hypothesis of nondeformable normals (after deformation a rectilinear element normal to a reference surface of a shell remains rectilinear, normal to the deformed reference surface of the shell preserves its length) gives:

$$\begin{aligned} e^k_3 &= 0 \\ e^k_4 \ (= e^k_{23}) &= 0 \\ e^k_5 \ (= e^k_{13}) &= 0 \end{aligned} \tag{2.1.2}$$

where 1, 2, 3, are the three principal coordinate lines corresponding to the meridians, parallels and normals to the surface, respectively. Furthermore, the unit elongation of a fiber at a distance γ from the reference surface undergoes the unit elongation of a fiber at the reference surface plus the elongation of fiber due to the curvature change at the corresponding point: i. e.

$$\begin{aligned} e^k_1 &= \mathcal{E}_1 + \gamma\chi_1 \\ e^k_2 &= \mathcal{E}_2 + \gamma\chi_2 \\ e^k_6 = e^k_{12} &= \eta + \gamma\tau \end{aligned} \tag{2.1.3}$$

where ε_1, ε_2 and χ_1, χ_2 are tangential strains and curvature changes of the reference surface of the shell. For a shell of revolution, the component of shear deformation (η) on the coordinate surface and the torsional deformation (τ) are zero for the case of axially symmetric loading. Then, inverting equations (2.1.1), incorporating equations (2.1.2) and (2.1.3), and setting both σ_3^k and e_{12}^k to zero, one obtains

$$\sigma_1^k = \frac{1}{\pi_k}\Big[a_{22}^k\varepsilon_1 - a_{12}^k\varepsilon_2 + a_{22}^k\gamma\chi_1 - a_{12}^k\gamma\chi_2 - a_{22}^k\alpha_1^k T^k + a_{12}^k\alpha_2^k T^k\Big] \qquad (2.1.4)$$

$$\sigma_2^k = \frac{1}{\pi_k}\Big[a_{11}^k\varepsilon_2 - a_{12}^k\varepsilon_1 + a_{11}^k\gamma\chi_2 - a_{12}^k\gamma\chi_1 - a_{11}^k\alpha_2^k T^k + a_{12}^k\alpha_1^k T^k\Big] \qquad (2.1.5)$$

where

$$\pi_k = a_{11}^k a_{22}^k - (a_{12}^k)^2 \qquad (2.1.5.a)$$

Equations (2.1.4) and (2.1.5) relate the stresses in each layer to the reference surface strains and curvature changes and local temperature of the layer. For a shell consisting of orthotropic layers with its principal directions of elasticity at each layer coinciding with the directions of coordinate lines 1, 2 and 3, the desired elastic constants of the k^{th} layer are (1);

$$a_{11}^{k} = \frac{1}{E_1^k}$$

$$a_{22}^{k} = \frac{1}{E_2^k} \tag{2.1.6}$$

$$a_{12}^{k} = -\frac{\nu_2^k}{E_2^k} = -\frac{\nu_1^k}{E_1^k}$$

The stress resultants are defined as

$$T_1 = \sum_{k=1}^{n'} \int_{(\delta_{k-1} - \Delta)}^{(\delta_k - \Delta)} \sigma_1^k d\gamma \tag{2.1.7}$$

$$T_2 = \sum_{k=1}^{n'} \int_{(\delta_{k-1} - \Delta)}^{(\delta_k - \Delta)} \sigma_2^k d\gamma \tag{2.1.8}$$

$$M_1 = \sum_{k=1}^{n'} \int_{(\delta_{k-1} - \Delta)}^{(\delta_k - \Delta)} \sigma_1^k \gamma d\gamma \tag{2.1.9}$$

$$M_2 = \sum_{k=1}^{n'} \int_{(\delta_{k-1} - \Delta)}^{(\delta_k - \Delta)} \sigma_2^k \gamma d\gamma \tag{2.1.10}$$

where $n' = n+m$, total number of layers in the shell, δ_k is the distance from inner surface of shell to the k^{th} layer and Δ is the distance between inner surface and reference surface of the shell (see figure 1). By introducing equations (2.1.4) and (2.1.5) into equations (2.1.7) through (2.1.10), one obtains

$$T_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + K_{11}\chi_1 + K_{12}\chi_2 - T_{1t} \qquad (2.1.11)$$

$$T_2 = C_{22}\varepsilon_2 + C_{12}\varepsilon_1 + K_{12}\chi_1 + K_{22}\chi_2 - T_{2t} \qquad (2.1.12)$$

$$M_1 = D_{11}\chi_1 + D_{12}\chi_2 + K_{11}\varepsilon_1 + K_{12}\varepsilon_2 - M_{1t} \qquad (2.1.13)$$

$$M_2 = D_{22}\chi_2 + D_{12}\chi_1 + K_{12}\varepsilon_1 + K_{22}\varepsilon_2 - M_{2t} \qquad (2.1.14)$$

where

$$T_{1t} = \sum_{k=1}^{n'} \int_{(\delta_{k-1} - \Delta)}^{(\delta_k - \Delta)} \frac{(a_{22}^k \alpha_1^k - a_{12}^k \alpha_2^k)}{\pi_k} (T^k) d\gamma$$

$$T_{2t} = \sum_{k=1}^{n'} \int_{(\delta_{k-1} - \Delta)}^{(\delta_k - \Delta)} \frac{(a_{11}^k \alpha_2^k - a_{12}^k \alpha_1^k)}{\pi_k} (T^k) d\gamma \qquad (2.1.15)$$

$$M_{1t} = \sum_{k=1}^{n'} \int_{(\delta_{k-1} - \Delta)}^{(\delta_k - \Delta)} \frac{(a_{22}^k \alpha_1^k - a_{12}^k \alpha_2^k)}{\pi_k} (T^k) \gamma d\gamma \qquad (2.1.16)$$

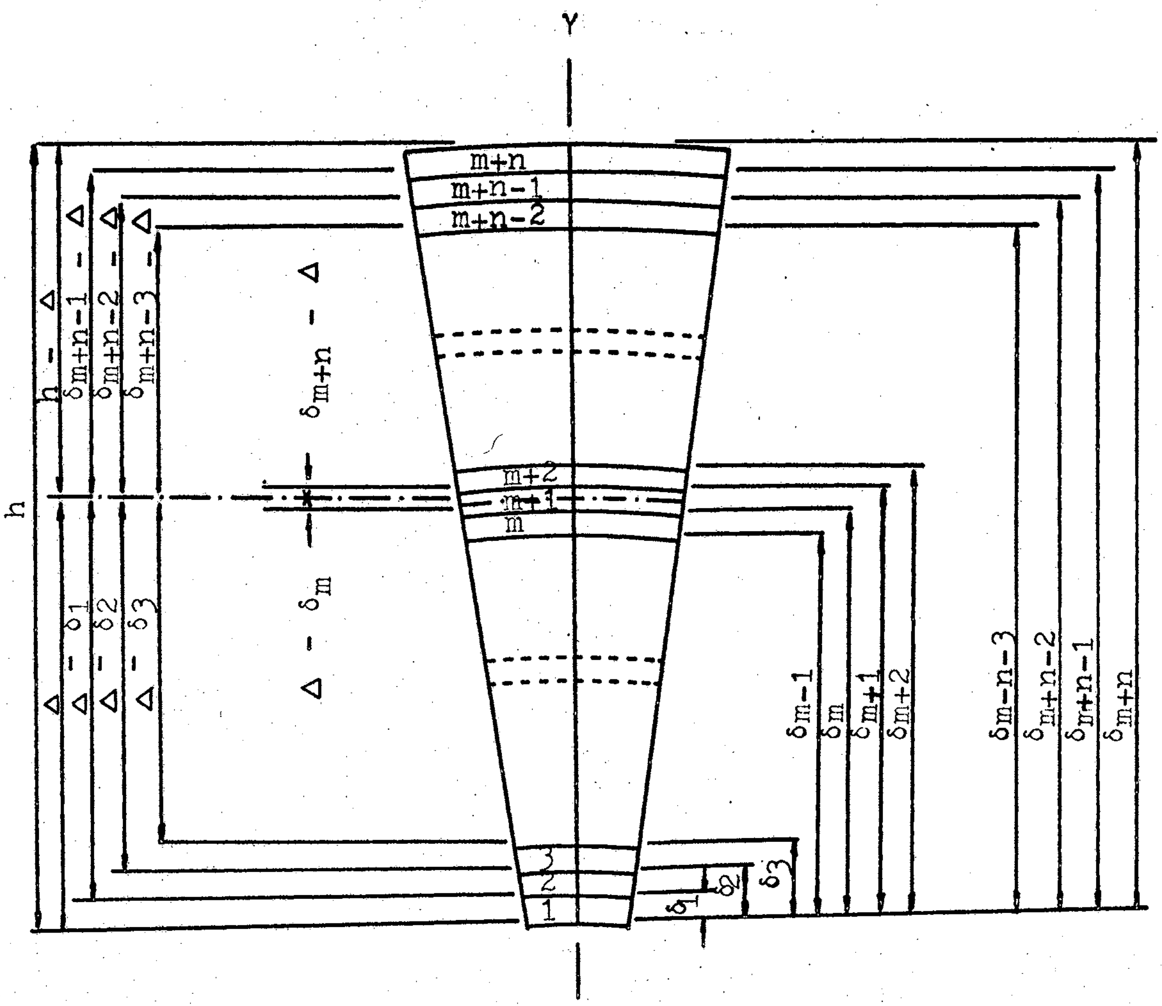

Figure 1. Geometry of A Shell Element

$$M_{2t} = \sum_{k=1}^{n'} \int_{(\delta_{k-1} - \Delta)}^{(\delta_k - \Delta)} \frac{(a_{11}^k \alpha_2^k - a_{12}^k \alpha_1^k)}{\pi_k} (T^k) \gamma d\gamma$$

$$C_{ij} = \sum_{k=1}^{n'} B_{ij}^k (\delta_k - \delta_{k-1})$$

$$K_{ij} = \frac{1}{2} \sum_{k=1}^{n'} B_{ij}^k \left[(\delta_k^2 - \delta_{k-1}^2) - 2\Delta(\delta_k - \delta_{k-1}) \right] \qquad (2.1.17)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n'} B_{ij}^k \left[(\delta_k^3 - \delta_{k-1}^3) - 3\Delta(\delta_k^2 - \delta_{k-1}^2) + 3\Delta^2 (\delta_k - \delta_{k-1}) \right]$$

wherein

$$B_{11}^k = \frac{a_{22}^k}{\pi_k}$$

$$B_{22}^k = \frac{a_{11}^k}{\pi_k} \qquad (2.1.17.a)$$

$$B_{12}^k = \frac{-a_{12}^k}{\pi_k}$$

C_{ij} characterizes the influence of the elongation along the coordinate lines, D_{ij} represents the bending stiffness and torsional stiffnesses about the coordinate lines and K_{ij} represents the stiffness of interaction of tension and bending.

For a shell consisting of an odd number of anisotropic layers symmetrically arranged relative to the middle surface of the shell, all interaction stiffnesses K_{ij} become zero. Equations (2.1.11) to (2.1.14) can therefore be simplified to

$$T_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 - T_{1t} \tag{2.1.18}$$

$$T_2 = C_{22}\varepsilon_2 + C_{12}\varepsilon_1 - T_{2t} \tag{2.1.19}$$

$$M_1 = D_{11}\chi_1 + D_{12}\chi_2 - M_{1t} \tag{2.1.20}$$

$$M_2 = D_{22}\chi_2 + D_{12}\chi_1 - M_{2t} \tag{2.1.21}$$

2.2 Equations of Equilibrium

The conditions for equilibrium involve consideration of stresses acting on an infinitesimal element whether these stresses are caused by temperature or other effects. Thus, the equations remain identical to those derived for the isotropic shell of uniform temperature. Consider the notation of figures 2 & 3. The equations of static equilibrium, in the absence of surface forces, written for orthogonal curvilinear coordinates are,

$$\frac{d(rT_1)}{ds} + T_2 \sin\theta + \frac{r}{R_1} N = 0 \tag{2.2.1}$$

$$\frac{d(rN)}{ds} - r\left(\frac{T_1}{R_1} + \frac{T_2}{R_2}\right) = 0 \tag{2.2.2}$$

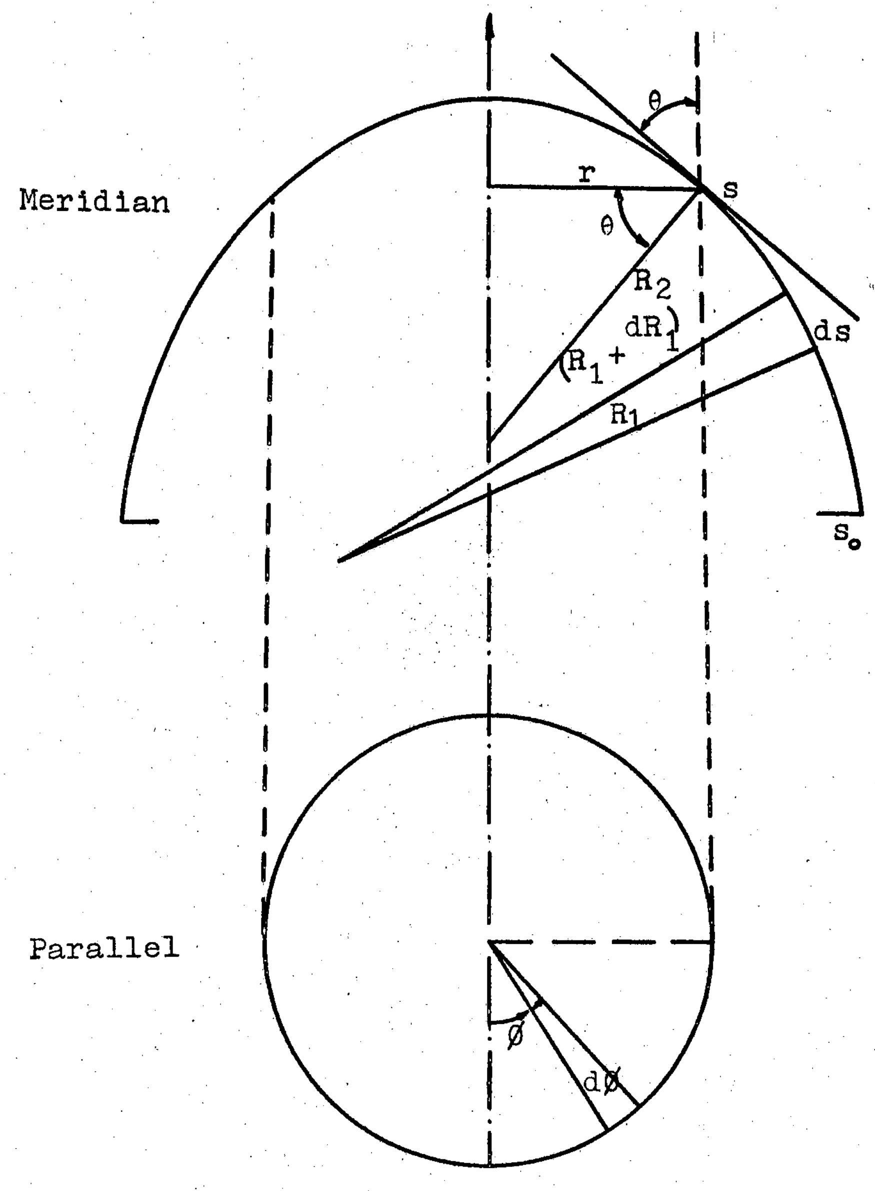

Figure 2. Shell of Rotation

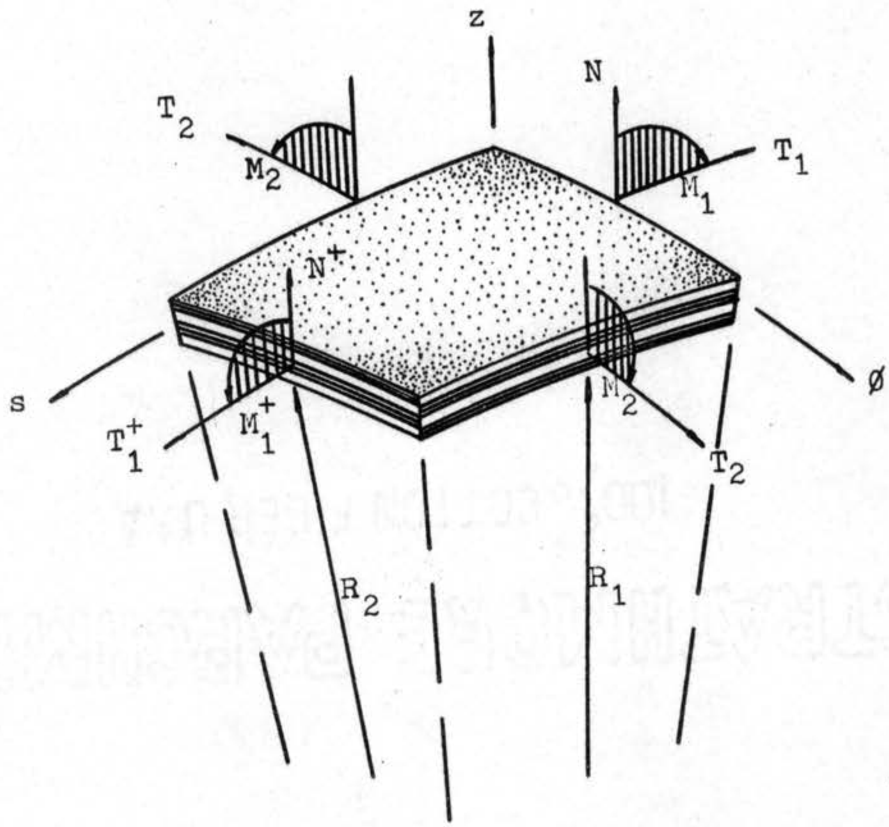

Figure 3. Forces and Bending Moments Acting on An Element of Shell

$$\frac{d(rM_1)}{ds} + M_2 \sin \theta - rN = 0 \qquad (2.2.3)$$

where, in addition to terms defined previously,

s = the variable along the meridional direction

N = transverse shear force

r = the distance from points to the axis of revolution

R_1 = the radius of curvature of the meridian

R_2 = the second principal radius of curvature of the surface

2.3 Kinematic Equations

One may relate middle surface strains and curvatures to displacements at a point on the middle surface. Again, these are unaffected by thermal effects and remain the same as those from classical shell theory (1). They are, using the notation and geometry from figure 4,

$$\varepsilon_1 = \frac{du}{ds} + \frac{w}{R_1} \qquad (2.3.1)$$

$$\varepsilon_2 = \frac{1}{r}(w \cos \theta - u \sin \theta) \qquad (2.3.2)$$

$$\chi_1 = -\frac{dW}{ds} \qquad (2.3.3)$$

$$\chi_2 = W \frac{\sin \theta}{r} \qquad (2.3.4)$$

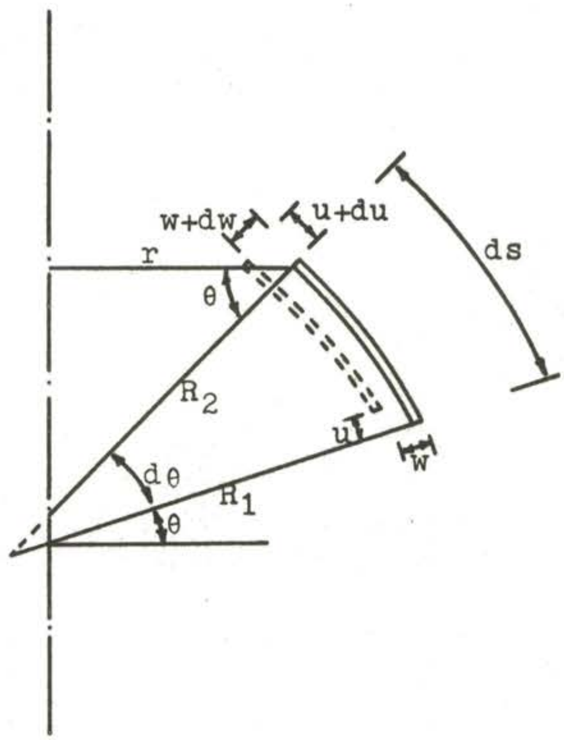

Figure4. Displacement of An Element of Shell in Plane of Meridian

where

$$W = \frac{dw}{ds} - \frac{u}{R_1}$$

u = displacement in meridional direction

w = displacement in radial direction

The equation of compatibility is (1)

$$r\frac{d\chi_2}{ds} - (\mathcal{E}_2 - \mathcal{E}_1)\sin\theta - W\cos\theta = 0 \qquad (2.3.5)$$

2.4 Combination of Equations

Following the procedure of Meissner (11), an auxiliary function $V = V(s)$ may be introduced to reduce the number of equations involved in the solution. Let the stress resultants be defined as follows:

$$T_2 = \frac{dV}{ds} \qquad (2.4.1)$$

$$T_1 = -\frac{\sin\theta}{r} V \qquad (2.4.2)$$

$$N = \frac{\cos\theta}{r} V \qquad (2.4.3)$$

where $V = V(s)$ is a function to be determined. The form of these definitions is such as to satisfy inherently the two force equilibrium equations. The moment equilibrium equation becomes

$$\frac{d(rM_1)}{ds} + M_2\sin\theta - V\cos\theta = 0 \qquad (2.4.4)$$

Again, substituting the values of T_1, T_2 and N in terms of Meissner's function into equations (2.1.11) and (2.1.12) and solving for the strains $\mathcal{E}_1$ and $\mathcal{E}_2$, one obtains

$$\mathcal{E}_1 = -\frac{1}{\Omega}\Big[C_{22}\frac{\sin\theta}{r}V + C_{12}\frac{dV}{ds} + (K_{12}C_{22} - K_{22}C_{12})\frac{\sin\theta}{r}W - (K_{11}C_{22} - K_{12}C_{12})\frac{dW}{ds} + C_{12}T_{2t} - C_{22}T_{1t}\Big] \tag{2.4.5}$$

$$\mathcal{E}_2 = \frac{1}{\Omega}\Big[C_{12}\frac{\sin\theta}{r}V + C_{11}\frac{dV}{ds} - (K_{22}C_{11} - K_{12}C_{12})\frac{\sin\theta}{r}W + (K_{12}C_{11} - K_{11}C_{12})\frac{dW}{ds} + C_{11}T_{2t} - C_{12}T_{1t}\Big] \tag{2.4.6}$$

where

$$\Omega = C_{11}C_{22} - C_{12}^2 \tag{2.4.6.a}$$

Likewise, substitution of these functions into equations (2.1.13), (2.1.14) and solving for M_1 and M_2, leads to

$$M_1 = (-D_{11} + D^{\circ}_{11})\frac{dW}{ds} + (D_{12} - D^{\circ}_{12})\frac{\sin\theta}{r}W + \left(\frac{-K_{11}C_{12} + K_{12}C_{11}}{\Omega}\right)\frac{dV}{ds} + \left(\frac{-K_{11}C_{22} + K_{12}C_{12}}{\Omega}\right)\frac{\sin\theta}{r}V + \left(\frac{K_{11}C_{12} - K_{12}C_{12}}{\Omega}\right)T_{1t} + \left(\frac{-K_{11}C_{12} + K_{12}C_{12}}{\Omega}\right)T_{2t}$$

$$- M_{1t} \tag{2.4.7}$$

and

$$M_2 = (-D_{12} + D^{\circ}_{12})\frac{dW}{ds} + (D_{22} - D^{\circ}_{22})\frac{\sin\theta}{r}W$$

$$+ \left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega}\right)\frac{dV}{ds} + \left(\frac{K_{22}C_{11} - K_{12}C_{22}}{\Omega}\right)\frac{\sin\theta}{r}V$$

$$+ \left(\frac{-K_{22}C_{12} + K_{12}C_{22}}{\Omega}\right)T_{1t} + \left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega}\right)T_{2t}$$

$$- M_{2t} \tag{2.4.8}$$

where

$$D^{\circ}_{11} = \frac{K_{11}^2C_{22} - 2K_{11}K_{12}C_{12} + K_{12}^2C_{11}}{\Omega}$$

$$D^{\circ}_{22} = \frac{K_{22}^2C_{11} - 2K_{22}K_{12}C_{12} + K_{12}^2C_{22}}{\Omega} \tag{2.4.8.a}$$

$$D^{\circ}_{12} = \frac{K_{11}K_{12}C_{22} - (K_{11}K_{22} + K_{12}^2)C_{12} + K_{22}K_{12}C_{11}}{\Omega}$$

These auxiliary functions are introduced primarily to satisfy the force equilibrium equations. The compatibility condition and moment equilibrium provide two simultaneous equations for the two unknown functions V and W. With this in mind, the values of $\mathcal{E}_1$ and $\mathcal{E}_2$ from equations (2.4.5), (2.4.6) may be substituted into the compatibility equation (2.3.5). The compatibility equation is then expressed in terms of auxiliary functions as

$$\frac{d^2V}{ds^2} - \frac{\sin\theta}{r}\frac{dV}{ds} + \left(\frac{C_{12}}{C_{11}}\frac{1}{R_1R_2} - \frac{C_{22}}{C_{11}}\frac{\sin^2\theta}{r^2}\right) V$$

$$= \frac{P_1}{C_{11}}\frac{d^2W}{ds^2} + \frac{P_2 - P_1}{C_{11}}\frac{\sin\theta}{r}\frac{dW}{ds} + \left(\frac{\Omega}{C_{11}}\frac{1}{R_2}\right.$$

$$\left. + \frac{K_{22}C_{11} - K_{12}C_{12}}{C_{11}}\frac{1}{R_1R_2} - \frac{P_3}{C_{11}}\frac{\sin^2\theta}{r^2}\right) W$$

$$+ \Phi_1(s) \qquad (2.4.9)$$

where

$$P_1 = K_{11}C_{12} - K_{12}C_{11}$$

$$P_2 = K_{22}C_{11} - K_{11}C_{22} \qquad (2.4.9.a)$$

$$P_3 = K_{22}C_{12} - K_{12}C_{22}$$

and

$$\Phi_1(s) = -\frac{dT_{2t}}{ds} + \frac{C_{12}}{C_{11}}\frac{dT_{1t}}{ds} + \frac{C_{11} + C_{12}}{C_{11}}\frac{\sin\theta}{r}T_{2t}$$

$$- \frac{C_{22} + C_{12}}{C_{11}}\frac{\sin\theta}{r}T_{1t} \qquad (2.4.9.b)$$

The moment equilibrium equation (2.4.4) may also be expressed in terms of the auxiliary function as

$$\frac{d^2W}{ds^2} - \frac{\sin\theta}{r}\frac{dW}{ds} - \left(\frac{D_{12} - D^\circ_{12}}{D_{11} - D^\circ_{11}}\frac{1}{R_1R_2} + \frac{D_{22} - D^\circ_{22}}{D_{11} - D^\circ_{11}}\frac{\sin^2\theta}{r^2}\right) W$$

$$= \frac{P_1}{\Omega(D_{11} - D^{\circ}_{11})} \frac{d^2V}{ds^2} + \frac{P_2 + P_1}{\Omega(D_{11} - D^{\circ}_{11})} \frac{\sin\theta}{r} \frac{dV}{ds}$$

$$- \left(\frac{1}{D_{11} - D^{\circ}_{11}} \frac{1}{R_2} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega(D_{11} - D^{\circ}_{11})} \frac{1}{R_1R_2}\right.$$

$$\left.- \frac{P_3}{\Omega(D_{11} - D^{\circ}_{11})} \frac{\sin^2\theta}{r^2}\right) V + \Phi_2(s) \qquad (2.4.10)$$

where

$$\Phi_2(s) = \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega(D_{11} - D^{\circ}_{11})} \frac{dT_{1t}}{ds} - \frac{P_3 + K_{11}C_{22} - K_{12}C_{12}}{\Omega(D_{11} - D^{\circ}_{11})} T_{1t}$$

$$- \frac{K_{11}C_{12} - K_{12}C_{11}}{\Omega(D_{11} - D^{\circ}_{11})} \frac{dT_{2t}}{ds} + \frac{K_{22}C_{11} - K_{12}C_{12} + P_1}{\Omega(D_{11} - D^{\circ}_{11})} T_{2t}$$

$$- \frac{1}{D_{11} - D^{\circ}_{11}} \frac{dM_{1t}}{ds} + \frac{1}{D_{11} - D^{\circ}_{11}} \frac{1}{r} M_{1t}$$

$$- \frac{1}{D_{11} + D^{\circ}_{11}} \frac{1}{r} M_{2t} \qquad (2.4.10.a)$$

Equations (2.4.9) and (2.4.10) comprise a complete system of differential equations in terms of the two required functions V and W, by means of which one may determine all the design force of the problem.

2.5 The Governing Equation for the Orthotropic, Odd-Number-of Layeres Shell

The elastic constants for shells of revolution consisting of an odd number of orthotropic layers symmetrically arranged relative to the middle surface of the shell have been obtained in section 1. The material of the shell at each point has only one plane of elastic symmetry parallel to the middle surface of the shell. If the coordinate surface of the shell coincides with the middle surface, one has the advantage of all interactional stiffness K_{ij} being zero. By virtue of this, one obtains

$$P_1 = P_2 = P_3 = D^{\circ}_{11} = D^{\circ}_{22} = D^{\circ}_{12} = 0$$

The moment resultants M_1 and M_2 become

$$M_1 = -D_{11}\frac{dW}{ds} + D_{12}\frac{\sin\theta}{r}W - M_{1t} \qquad (2.5.1)$$

$$M_2 = -D_{12}\frac{dW}{ds} + D_{22}\frac{\sin\theta}{r}W - M_{2t} \qquad (2.5.2)$$

Then, equations (2.4.9) and (2.4.10) reduce to

$$\frac{d^2V}{ds^2} - \frac{\sin\theta}{r}\,\frac{dV}{ds} + \left(\frac{C_{12}}{C_{11}}\,\frac{1}{R_1R_2} - \frac{C_{22}}{C_{11}}\,\frac{\sin^2\theta}{r^2}\right)V$$

$$= \frac{\Omega}{C_{11}}\,\frac{1}{R_2}W + \Phi_1(s) \qquad (2.5.3)$$

$$\frac{d^2W}{ds^2} - \frac{\sin\theta}{r}\,\frac{dW}{ds} - \left(\frac{D_{12}}{D_{11}}\,\frac{1}{R_1R_2} + \frac{D_{22}}{D_{11}}\,\frac{\sin^2\theta}{r^2}\right)W$$

$$= -\frac{1}{D_{11}}\,\frac{1}{R_2}\,V + \Phi_2(s) \qquad (2.5.4)$$

where $\Phi_1(s)$ remains as defined in equation (2.4.9.b) but $\Phi_2(s)$ becomes;

$$\Phi_2(s) = -\frac{1}{D_{11}}\,\frac{\sin\theta}{r}(M_{2t} - M_{1t}) + \frac{dM_{1t}}{ds}$$

As discussed in Appendix 1 and reference (1), the values of $\frac{C_{22}}{C_{11}}$ and $\frac{D_{22}}{D_{11}}$ may reasonably be assumed to equal some constant λ.i.e.

$$\frac{C_{22}}{C_{11}} = \frac{D_{22}}{D_{11}} = \lambda$$

The system may then be written in a compact form as

$$\bar{L}(V) + \frac{C_{12}}{C_{11}}\,\frac{1}{R_1R_2}\,V = \frac{\Omega}{C_{11}}\,\frac{1}{R_2}\,W + \Phi_1(s) \qquad (2.5.5)$$

$$\bar{L}(W) - \frac{D_{12}}{D_{11}}\,\frac{1}{R_1R_2}\,W = -\frac{1}{D_{11}}\,\frac{1}{R_2}V + \Phi_2(s) \qquad (2.5.6)$$

where

$$\bar{L} = \frac{d^2}{ds^2} - \frac{\sin\theta}{r}\,\frac{d}{ds} - \lambda\frac{\sin^2\theta}{r^2}$$

Multiplying equation (2.5.5) by an undetermined factor, $\bar{a}$, and combining with equation (2.5.6) leads to

$$\bar{L}(W) - \left(\frac{D_{12}}{D_{11}}\frac{1}{R_1} + \frac{\Omega}{C_{11}}\bar{a}\right)\frac{W}{R_2} + \bar{a}\left[L(V) + \left(\frac{C_{12}}{C_{11}}\frac{1}{R_1}\right.\right.$$

$$\left.\left. + \frac{1}{\bar{a}D_{11}}\right)\frac{V}{R_2}\right] = \Phi_2(s) + \bar{a}\Phi_1(s) \qquad (2.5.7)$$

To determine the value of $\bar{a}$, let

$$\frac{D_{12}}{D_{11}}\frac{1}{R_1} + \frac{\Omega}{C_{11}}\bar{a} = -ik^2 \qquad (2.5.8)$$

$$\frac{C_{12}}{C_{11}}\frac{1}{R_1} + \frac{1}{\bar{a}D_{11}} = ik^2 \qquad (2.5.9)$$

By means of this substitution it is possible to find the complex function $\bar{\sigma}$ in terms of V and W

$$\bar{\sigma} = W + \bar{a}V \qquad (2.5.10)$$

which satisfies equation (2.5.7). Addition of equations (2.5.8) and (2.5.9) results in

$$\bar{a}^2 + \frac{C_{11}}{\Omega}\left(\frac{D_{12}}{D_{11}} + \frac{C_{12}}{C_{11}}\right)\frac{\bar{a}}{R_1} + \frac{C_{11}}{D_{11}\Omega} = 0 \qquad (2.5.11)$$

The solution to this quadratic equation is

$$\bar{a} = -\frac{C_{11}}{2\Omega R_1}\left(\frac{D_{12}}{D_{11}} + \frac{C_{12}}{C_{11}}\right) \pm \frac{1}{2}\sqrt{\frac{C_{11}^2}{\Omega^2}\left(\frac{D_{12}}{D_{11}} + \frac{C_{12}}{C_{11}}\right)^2\frac{1}{R_1^2} - \frac{4C_{11}}{D_{11}\Omega}}$$

From Appendix 4 it is seen that

$$\frac{C_{11}}{\Omega}\left(\frac{D_{12}}{D_{11}}+\frac{C_{12}}{C_{11}}\right) \ll 1$$

The value of $\frac{1}{R_1}$ is practically always very small. Therefore, the terms involving $\frac{1}{R_1}$ can be discarded and $\bar{\bar{a}} \doteq -i\sqrt{\frac{C_{11}}{\Omega D_{11}}}$ will identically satisfy equation (2.5.7). Inserting the value of $\bar{\bar{a}}$ in equation (2.5.7) one obtains

$$\frac{d^2W}{ds^2}-\frac{\sin\theta}{r}\frac{dW}{ds}-\lambda\frac{\sin^2\theta}{r^2}W+i\sqrt{\frac{\Omega}{C_{11}D_{11}}}\frac{W}{R_2}$$

$$-i\sqrt{\frac{C_{11}}{D_{11}\Omega}}\left[\frac{d^2V}{ds^2}-\frac{\sin\theta}{r}\frac{dV}{ds}-\lambda\frac{\sin^2\theta}{r^2}V\right]+\frac{1}{D_{11}}\frac{1}{R_2}V$$

$$=\Phi_2(s)-i\sqrt{\frac{C_{11}}{\Omega D_{11}}}\Phi_1(s) \qquad (2.5.12)$$

Because

$$\bar{\bar{\sigma}}=W-i\sqrt{\frac{C_{11}}{D_{11}\Omega}}V \qquad (2.5.13)$$

equation (2.5.12) can be written as

$$\frac{d^2\bar{\bar{\sigma}}}{ds^2}-\frac{\sin\theta}{r}\frac{d\bar{\bar{\sigma}}}{ds}-\lambda\frac{\sin^2\theta}{r^2}\bar{\bar{\sigma}}+i\sqrt{\frac{\Omega}{C_{11}D_{11}}}\frac{\bar{\bar{\sigma}}}{R_2}$$

$$=\Phi_2(s)-i\sqrt{\frac{C_{11}}{D_{11}\Omega}}\Phi_1(s)$$

Or, in linear operator form,

$$L(\bar{\bar{\sigma}}) + i\sqrt{\frac{\Omega}{C_{11}D_{11}}}\,\frac{\bar{\bar{\sigma}}}{R_2} = \bar{\Phi}_2(s) - i\sqrt{\frac{C_{11}}{D_{11}\Omega}}\,\bar{\Phi}_1(s) \qquad (2.5.14)$$

This is the governing differential equation for a shell of revolution consisting of an odd number of orthotropic layers subjected to axisymmetric thermal gradients. As expected, the form of this equation is identical to that derived by Ambartsumyan (1) for the isothermal shell except for an additional term, reflecting thermal effects, on the right side. The solution of equation (2.5.14) consists of the sum of the homogeneous and particular solutions and only the particular solution can be affected by this new term. Therefore, the homogeneous solution obtained by Ambartsumyan through asymptotic integration is perfectly applicable to the present problem.

CHAPTER III

SOLUTION OF THE GOVERNING EQUATION

3.1 Solution of the Homogeneous Equation

As explained in section 2.5, the homogeneous solution obtained by Ambartsumyan (1) is applicable to this problem. It is (1)

$$\bar{\sigma} = (E_1 \cos\beta - F_1 \sin\beta)e^{-\beta} + (E_2 \cos\beta + F_2 \sin\beta)e^{\beta}$$
$$+ i\Big[(E_1 \sin\beta + F_1 \cos\beta)e^{-\beta} - (E_2 \sin\beta$$
$$- F_2 \cos\beta)e^{\beta}\Big] \tag{3.1.1}$$

where E_1, E_2, F_1 and F_2 are unknown constants, and β can be expressed as

$$\beta = \frac{\bar{k}}{\sqrt{2}} \int_{s_o}^{s} \frac{ds}{\sqrt{R_2}} \tag{3.1.2}$$

in which

$$\bar{k} = \sqrt{\frac{\Omega}{C_{11} D_{11}}} \tag{3.1.3}$$

From equations (3.1.2) and (3.1.3), it is seen that β changes with respect to the shape, thickness and elastic properties of shell.

3.2 Solution of Particular Equation

A particular solution of equation (2.5.14) would apply to the general shell of revolution. Such a particular solution was not obtained in this study, however, particular solutions were obtained for two important shell shapes; the circular cylindrical shell and the conical shell with special material properties. These solutions will be discussed in the following sections.

3.2.a Particular Solution for the Circular, Cylindrical Shell

The governing equation for the general shell of revolution becomes the governing equation for a circular cylindrical shell, if the geometric variable θ is set to zero. The governing equation then becomes

$$\frac{d^2\bar{\sigma}}{dx^2} + i\sqrt{\frac{\Omega}{C_{11}D_{11}}}\,\frac{\bar{\sigma}}{F} = \left[\frac{H^\circ}{D_{11}} + i\sqrt{\frac{C_{11}}{D_{11}}}\left(H^* - \frac{C_{12}}{C_{11}}H^{**}\right)\right](x-L) \qquad (3.2.1)$$

The derivation of right hand side of above equation can be found in Appendix 2.

Let

$$\bar{\sigma}_p = (a_1 + a_2 i)x + (b_1 + b_2 i) \qquad (3.2.2)$$

Its derivative with respect to x is

$$\frac{d\bar{\bar{\sigma}}_p}{dx} = a_1 + a_2 i \tag{3.2.3}$$

also the second derivative is

$$\frac{d^2\bar{\bar{\sigma}}_p}{dx^2} = 0 \tag{3.2.4}$$

Substituting the assumed polynomial and its derivatives into equation(3.2.1) and equating the corresponding terms, one obtains a particular solution for circular cylindrical shell in the form

$$\bar{\bar{\sigma}}_p = (A' + B'i)(x-L) \tag{3.2.5}$$

where

$$\begin{aligned} A' &= -\frac{\bar{\bar{r}}C_{11}}{\Omega}\left(H^* - \frac{C_{12}}{C_{11}}H^{**}\right) \\ B' &= \bar{\bar{r}}\sqrt{\frac{C_{11}}{D_{11}\Omega}}\,H^{\circ} \end{aligned} \tag{3.2.5.a}$$

3.2.b Particular Solution for the Conical Shell

The governing equation for the general shell of revolution (2.5.4) becomes the governing equation for a conical shell, if the geometric variable θ is a certain constant α. As shown in Appendix 1, λ rapidly approaches as the number of layers increases to infinity if the material properties alternate. Equation (2.5.14) can be written in this case as

$$x^2\frac{d^2\bar{\bar{\sigma}}}{dx^2} + x\frac{d\bar{\bar{\sigma}}}{dx} - \bar{\bar{\sigma}} + i\sqrt{\frac{\Omega}{C_{11}D_{11}}}\,\frac{x\bar{\bar{\sigma}}}{\tan\alpha} = \left[\frac{H^*}{D_{11}}\right.$$

$$+ i\sqrt{\frac{C_{11}}{\Omega D_{11}}}\left(H^* - \frac{C_{12}}{C_{11}}H^{**}\right)\Big] (x^3 - Lx^2) \qquad (3.2.6)$$

The derivation of the right hand side of the above equation can be found in Appendix 2.

Let

$$\bar{\sigma}_p = (a_1 + a_2 i)x^2 + (b_1 + b_2 i)x + (c_1 + c_2 i) \qquad (3.2.7)$$

Its derivative with respect to x is

$$\frac{d\bar{\sigma}_p}{dx} = 2(a_1 + a_2 i)x + (b_1 + b_2 i) \qquad (3.2.8)$$

Also the second derivative is

$$\frac{d^2\bar{\sigma}_p}{dx^2} = 2(a_1 + a_2 i) \qquad (3.2.9)$$

Substituting the assumed polynomial and its derivatives into equation (3.2.6) and equating the corresponding terms provides enough conditions to solve for the unknown constants. The obtained particular solution then can be written

$$\bar{\sigma}_p = (A^* + B^* i)x^2 + (C^* + D^* i)x \qquad (3.2.10)$$

where

$$A^* = \frac{C_{11}\tan\alpha}{\Omega}\left(H^* - \frac{C_{12}}{C_{11}}H^{**}\right)$$

$$B^* = -\sqrt{\frac{C_{11}}{\Omega D_{11}}}\tan\alpha H^{\circ} \qquad (3.2.10.a)$$

$$C^* = -\frac{C_{11}\tan\alpha}{\Omega}\left(H^* - \frac{C_{12}}{C_{11}}H^{**}\right)L + \frac{3C_{11}\tan^2\alpha}{\Omega}H^{\circ}$$

$$D^* = \sqrt{\frac{C_{11}}{\Omega D_{11}}}\ L\tan\alpha\ H^* + \frac{3\tan^2\alpha\ C_{11}\sqrt{C_{11}D_{11}}}{\Omega\sqrt{\Omega}}(H^* - \frac{C_{12}}{C_{11}} H^{**})$$

3.3 The General Solution for the Circular Cylindrical Shell

In section 3.1 the homogeneous solution was obtained for the general shell of revolution. If the geometric variable θ is zero, equation (3.1.1) yield the homogeneous solution for the circular cylindrical shell

$$\begin{aligned}\bar{\sigma}_h &= (E_1\cos\beta - F_1\sin\beta)e^{-\beta} + (E_2\cos\beta + F_2\sin\beta)e^{\beta} \\ &+ i\left[(E_1\sin\beta + F_1\cos\beta)e^{-\beta} - (E_2\sin\beta\right. \\ &\left. - F_2\cos\beta)e^{\beta}\right]\end{aligned} \tag{3.3.1}$$

where

$$\beta = \sqrt{\frac{\bar{k}^2}{2r}}\ s \tag{3.3.2}$$

The general solution of equation (3.2.1) is the summation of equations (3.3.1) and (3.2.5). On the basis of equation (2.5.13) the general solution is

$$\begin{aligned}W - i\sqrt{\frac{C_{11}}{\Omega D_{11}}}\,V &= (E_1\cos\beta - F_1\sin\beta)e^{-\beta} + (E_2\cos\beta \\ &+ F_2\sin\beta)e^{\beta} + A's + i\left[(E_1\sin\beta + F_1\cos\beta)e^{-\beta}\right. \\ &\left. - (E_2\sin\beta - F_2\cos\beta)e^{\beta} + B's\right]\end{aligned} \tag{3.3.3}$$

Separating the imaginary and real parts, one obtains

$$W = (E_1\cos\beta - F_1\sin\beta)e^{-\beta} + (E_2\cos\beta + F_2\sin\beta)e^{\beta}$$

$$+ A's \qquad (3.3.4)$$

$$V = -\frac{\Omega}{\bar{k}^2 C_{11}}(E_1\sin\beta + F_1\cos\beta)e^{-\beta} - (E_2\sin\beta$$

$$- F_2\cos\beta)e^{\beta} - \sqrt{\frac{D_{11}}{C_{11}}}B's \qquad (3.3.5)$$

Let

$$\beta_1 = \alpha_o - \sqrt{\frac{\bar{k}^2}{2r}}\, s \qquad (3.3.6)$$

where

$$\alpha_o = \sqrt{\frac{\bar{k}^2}{2r}}L \qquad (3.3.6.a)$$

By introducing the new constants

$$A_1 = E_1$$

$$A_2 = (E_2\cos\alpha_o + F_2\sin\alpha_o)e^{\alpha}$$

$$B_1 = -F_1$$

$$B_2 = (E_2\sin\alpha_o - F_2\cos\alpha_o)e^{\alpha}$$

The equations (3.3.4) and (3.3.5) can be written in the following forms;

$$W = A_1\theta(\beta) + B_1\mathcal{S}(\beta) + A_2\theta(\beta_1) + B_2\mathcal{S}(\beta_1) + A's \qquad (3.3.7)$$

$$V = \frac{\Omega}{\bar{k}^2 C_{11}}\Big[-A_1\mathcal{S}(\beta) + B_1\theta(\beta) - A_2\mathcal{S}(\beta_1)$$

$$+ B_2\theta(\beta_1)\Big] - \sqrt{\frac{D_{11}\Omega}{C_{11}}}B's \qquad (3.3.8)$$

where

$$\theta(\beta) = e^{-\beta}\cos\beta$$

$$\mathcal{S}(\beta) = e^{-\beta}\sin\beta$$

$$\phi(\beta) = \theta(\beta) + \mathcal{S}(\beta)$$

$$\psi(\beta) = \theta(\beta) - \mathcal{S}(\beta) \qquad (3.3.8.a)$$

The sought functions W and V are thereof obtained, and their derivatives are

$$\frac{dW}{ds} = -\bar{\bar{k}}\sqrt{\frac{1}{2\bar{\bar{r}}}}\Big[A_1\phi(\beta) - B_1\psi(\beta) - A_2\phi(\beta_1) + B_2\psi(\beta_1)\Big] + A' \qquad (3.3.9)$$

$$\frac{dV}{ds} = D_{11}\,\bar{\bar{k}}^3\sqrt{\frac{1}{2\bar{\bar{r}}}}\Big[-A_1\psi(\beta) - B_1\phi(\beta) + A_2\psi(\beta_1) + B_2\phi(\beta_1)\Big] - \sqrt{\frac{D_{11}\Omega}{C_{11}}}B' \qquad (3.3.10)$$

With these obtained functions and their derivatives, the design stress resultants are determined

$$T_1 = -\frac{\sin 0^\circ}{\bar{\bar{r}}}V = 0 \qquad (3.3.11)$$

$$T_2 = D_{11}\frac{\bar{\bar{k}}^3}{\sqrt{2\bar{\bar{r}}}}\Big[-A_1\psi(\beta) - B_1\phi(\beta) + A_2\psi(\beta_1) + B_2\phi(\beta_1)\Big] - \sqrt{\frac{\Omega D_{11}}{C_{11}}}B' \qquad (3.3.12)$$

$$M_1 = D_{11}\frac{\bar{\bar{k}}^2}{\sqrt{2\bar{\bar{r}}}}\Big[A_1\phi(\beta) - B_1\psi(\beta) - A_2\phi(\beta_1)$$

$$+ B_2\psi(\beta_1)\Big] - D_{11}\ A's - M_{1t} \tag{3.3.13}$$

$$M_2 = D_{12}\frac{\bar{\bar{k}}}{\sqrt{2\bar{\bar{r}}}}\Big[A_1\phi(\beta) - B_1\psi(\beta) - A_2\phi(\beta_1)$$

$$+ B_2\psi(\beta_1)\Big] - M_{2t} \tag{3.3.14}$$

$$N = \frac{1}{\bar{\bar{r}}}\bar{\bar{k}}^2 D_{11}\Big[- A_1\varphi(\beta) + B_1\theta(\beta) - A_2\varphi(\beta_1)$$

$$+ B_2\theta(\beta_1)\Big] - \sqrt{\frac{D_{11}\Omega}{C_{11}}}\frac{B'}{\bar{\bar{r}}}s \tag{3.3.15}$$

On the basis of the property of long circular cylindrical shell (1), β_1 shall be discarded in calculating the quantities at edge where $s = 0$. Also β shall be discarded at the other end where $s = L$. With this in mind, the constants A_1, B_1, A_2, B_2 can be determined from the boundary conditions.

3.4 The General Solution for the Conical Shell

Again, in section 3.1 the homogeneous solution was obtained for the general shell of revolution. Equation (3.1.1) is the homogeneous solution for the conical shell if the geometric variable θ is constant α.

$$\bar{\bar{\sigma}}_h = (E_1\cos\beta - F_1\sin\beta)e^{-\beta} + (E_2\cos\beta + F_2\sin\beta)e^{\beta}$$

$$+ i\Big[(E_1\sin\beta + F_1\cos\beta)e^{-\beta} - (E_2\sin\beta$$

$$- F_2\cos\beta)e^{\beta}\Big] \tag{3.4.1}$$

where, in the case of the conical shell

$$\beta = \frac{\bar{k}}{\sqrt{2\tan\alpha}}\int_{s_0}^{s} \frac{ds}{\sqrt{L-s}} \tag{3.4.2}$$

On basis of equation (2.5.13), the general solution is

$$\begin{aligned} W - i\sqrt{\frac{C_{11}}{D_{11}\Omega}}\,V = {} & (E_1\cos\beta - F_1\sin\beta)e^{-\beta} + (E_2\cos\beta \\ & + F_2\sin\beta)e^{\beta} + A^*(L-s)^2 + C^*(L-s) + i\Big[(E_1\sin\beta \\ & + F_1\cos\beta)e^{-\beta} - (E_2\sin\beta - F_2\cos\beta)e^{\beta} + B^*(L-s)^2 \\ & + D^*(L-s)\Big] \end{aligned} \tag{3.4.3}$$

Following the same procedure of section 3.3.a, the functions W, V and their derivatives are obtained as

$$\begin{aligned} W = {} & A_1\,\theta(\beta) + B_1\varphi(\beta) + A_2\,\theta(\beta_1) + B_2\varphi(\beta_1) \\ & + A^*(L-s)^2 + C^*(L-s) \end{aligned} \tag{3.4.4}$$

$$\begin{aligned} V = {} & \frac{\Omega}{\bar{k}^2 C_{11}}\Big[-A_1\varphi(\beta) + B_1\,\theta(\beta) - A_2\varphi(\beta_1) \\ & + B_2\,\theta(\beta_1)\Big] - B^{**}(L-s)^2 + D^{**}(L-s) \end{aligned} \tag{3.4.5}$$

where

$$B^{**} = -\tan\alpha\, H^{\circ} \tag{3.4.6}$$

$$D^{**} = L\tan\alpha\, H^{\circ} + \frac{3\tan^2\alpha\; C_{11}}{\Omega}\left(H^* - \frac{C_{12}}{C_{11}}H^{**}\right) \tag{3.4.7}$$

$$\frac{dW}{ds} = -\bar{k}\sqrt{\frac{1}{2(L-s)\tan\alpha}}\Big[A_1\phi(\beta) - B_1\psi(\beta) - A_2\phi(\beta\ )$$

$$+ B_2\psi(\beta\ )\Big] - 2A^*(L-s) - C^* \qquad (3.4.8)$$

$$\frac{dV}{ds} = D_{11}\bar{k}^3\sqrt{\frac{1}{2(L-s)\tan\alpha}}\Big[-A_1\psi(\beta) - B_1\phi(\beta) + A_2\psi(\beta_1)$$

$$+ B_2\phi(\beta_1)\Big] - 2B^{**}(L-s) + D^{**} \qquad (3.4.9)$$

Substituting equations (3.4.4) through (3.4.9) into equations (2.4.1) through (2.4.3) and (2.5.1) through (2.5.2), the design quantities then are

$$T_1 = \frac{\bar{k}^2 D_{11}}{L-s}\Big[A_1\mathcal{Y}(\beta) - B_1\theta(\beta) + A_2\mathcal{Y}(\beta_1)$$

$$- B_2\theta(\beta_1)\Big] + B^{**}(L-s) + D^{**} \qquad (3.4.10)$$

$$T_2 = \bar{k}^3 D_{11}\sqrt{\frac{1}{2(L-s)\tan\alpha}}\Big[-A_1\psi(\beta) - B_1\phi(\beta) + A_2\psi(\beta_1)$$

$$+ B_2\phi(\beta_1)\Big] - 2B^{**}(L-s) + D^{**} \qquad (3.4.11)$$

$$N_1 = \frac{\cot\alpha}{L-s}\bar{k}^2 D_{11}\Big[-A_1\mathcal{Y}(\beta) + B_1\theta(\beta) + B_2\theta(\beta_1)$$

$$- A_2\mathcal{Y}(\beta_1)\Big] - \cot\alpha B^{**}(L-s) - \cot\alpha D^{**} \qquad (3.4.12)$$

$$M_1 = D_{11}\bar{k}\sqrt{\frac{1}{2(L-s)\tan\alpha}}\Big[A_1\phi(\beta) - B_1\psi(\beta) - A_2\phi(\beta_1)$$

$$+ B_2\psi(\beta_1)\Big] + \frac{D_{12}}{L-s}\Big[A_1\theta(\beta) + B_1\mathcal{Y}(\beta) + A_2\theta(\beta_1)$$

$$+ B_2\mathcal{Y}(\beta_1)\Big] + D_{11}\Big[2A^*(l-s) + C^*\Big]$$

$$+ D_{12}\left[A^*(L-s) + C^*\right] - M_{1t} \tag{3.4.13}$$

$$M_2 = -D_{12}\sqrt{\frac{\bar{k}^2}{2(L-s)\tan}}\left[A_1\phi(\beta) - B_1\psi(\beta) + A_2\psi(\beta_1)\right.$$

$$\left. + B_2\phi(\beta_1)\right] + \frac{D_{22}}{L-s}\left[A_1\theta(\beta) + B_1\varphi(\beta) + A_2\theta(\beta_1)\right.$$

$$\left. + B_2\varphi(\beta_1)\right] + D_{12}\left[2A^*(L-s) + C^*\right]$$

$$+ D_{22}\left[A^*(L-s) + C^*\right] - M_{2t} \tag{3.4.14}$$

where A_1, A_2, B_1, and B_2 are four unknown constants which shall be determined by boundary conditions. For a shell containing a conical vertex, to assure the continuity of slope at the vertex point, it is necessary to assume

$$A_2 = B_2 = 0 \quad \text{(see equation (3.4.4))}$$

A closed conical shell with a clamped base was used to illustrate the application of this theory. The unknown constants were determined on basis of the clamped edge condition which require

$$R_2^{\circ}N^{\circ} + \sqrt{2R_2^{\circ}}\, kM_1^{\circ} = 0 \tag{3.4.15}$$

at the edge where $s = 0$; $(\beta = 0)$

From equations (3.4.12) and (3.4.13) it was found that

$$N^{\circ} = -\frac{1}{L\tan\alpha}\bar{k}^2 D_{11}B_1 - \frac{L}{\tan\alpha}B^{**} - \frac{1}{\tan\alpha}D^{**} \tag{3.4.16}$$

$$M_1^{\circ} = \frac{D_{11}\bar{k}}{\sqrt{2L\tan\alpha}}\left[A_1 - B_1\right] + \frac{D_{12}}{L}A_1 + D_{11}\left[2A^{*}L + C^{*}\right]$$
$$+ D_{12}\left[A^{*}L + C^{*}\right] - M_{1t} \qquad (3.4.17)$$

$$R_2^{\circ} = L\tan\alpha \qquad (3.4.18)$$

Inserting equations (3.4.16) - (3.4.18) into equation (3.4.15), A_1 was found to be

$$A_1 = \frac{G^{*}}{G^{**}} \qquad (3.4.19)$$

where

$$G^{*} = D_{11}\bar{k}^2 + \sqrt{\frac{2\tan\alpha}{L}}\,\bar{k}\,D_{12}$$
$$(3.4.20)$$
$$G^{**} = L(B^{**} + D^{**}) - \sqrt{2L\tan\alpha}\,\bar{k}\left[A^{*}L\,(2D_{11} + D_{12})\right.$$
$$\left. + C^{*}(D_{11} + D_{12})\right] + 2\sqrt{2L\tan\alpha}\,\bar{k}\,M_{1t}^{\circ}$$

The normal displacement w of the conical shell was derived (1) as

$$w = \frac{C_{11}T_2 - C_{12}T_1}{\Omega}(L-s)\,\sin\alpha\cos\alpha + \left[\,e_z^{\circ} + \int_s^s (W\sin\alpha\right.$$
$$\left. + \frac{C_{22}T_1 - C_{12}T_2}{\Omega}\cos\alpha)\,ds\right]\sin\alpha \qquad (3.4.21)$$

Again, the boundary condition for the edge s = 0 ; $\beta = 0$ is w = 0

On basis of equation (3.4.4), (3.4.10), (3.4.11) at s = 0 ; $\beta = 0$ e_z° is obtained

$$e_z = -\left[C_{11}\left(\frac{\bar{\bar{k}}^3 D_{11}}{\sqrt{2L\tan\alpha}}(-A_1 - B_1) - 2B^{**}L + D^{**}\right)\right.$$

$$\left. - C_{12}\left(\frac{\bar{\bar{k}}^2 D_{11}}{L}(-B_1) + B^{**}L + D^{**}\right)\right]\frac{L\cos\alpha}{\Omega} \qquad (3.4.22)$$

Also, the boundary condition for $s = L$; $\beta_1 = 0$ is

$w = 0$

Similarly, equations (3.4.4), (3.4.10), (3.4.11) at $s = L$, give

$$0 = \mathring{e}_z + (C_{22} - C_{12})D^{**}\frac{\cos\alpha\ L}{\Omega} \qquad (3.4.23)$$

Solving equations (3.4.22) and (3.4.23) simultaneously, one obtains

$$B_1 = \frac{V^*}{V^{**}} \qquad (3.4.24)$$

where

$$V^* = (2C_{11} + C_{12})B^{**}L + (C_{22} - C_{11})D^{**} - \frac{C_{11}\bar{\bar{k}}^3 D_{11}}{\sqrt{2L\tan\alpha}}A_1$$

$$V^{**} = -\frac{C_{11}\bar{\bar{k}}^3 D_{11}}{\sqrt{2L\tan\alpha}} + \frac{C_{12}\bar{\bar{k}}^2 D_{11}}{L} \qquad (3.4.25)$$

Let $\bar{\bar{A}}_1$ and $\bar{\bar{B}}_1$ be the obtained constants A_1 and B_1. Then, the design quantities can be expressed as

$$T_1 = \frac{\bar{\bar{k}}^2 D_{11}}{L-s}\left[\bar{\bar{A}}_1\mathcal{Y}(\beta) - \bar{\bar{B}}_1\theta(\beta)\right] + B^{**}(L-s) + D^{**} \qquad (3.4.26)$$

$$T_2 = \bar{\bar{k}}^3 D_{11} \sqrt{\frac{1}{2(L-s)\tan\alpha}} \left[- \bar{A}_1 \psi(\beta) - \bar{\bar{B}}_1 \phi(\beta) \right]$$

$$- 2B^{**}(L-s) + D^{**} \qquad (3.4.27)$$

$$M_1 = D_{11} \bar{\bar{k}} \sqrt{\frac{1}{(2(L-s)\tan\alpha}} \left[\bar{\bar{A}}_1 \phi(\beta) - \bar{\bar{B}}_1 \psi(\beta) \right]$$

$$+ \frac{D_{12}}{L-s} \left[\bar{\bar{A}}_1 \theta(\beta) + \bar{\bar{B}}_1 \varphi(\beta) \right] + D_{11} \left[2A^*(L-s) + C^* \right]$$

$$+ D_{12} \left[A^*(L-s) + C^* \right] - M_{1t} \qquad (3.4.28)$$

$$M_2 = D_{12} \sqrt{\frac{\bar{\bar{k}}^2}{2(L-s)\tan\alpha}} \left[\bar{A}_1 \phi(\beta) - \bar{\bar{B}}_1 \psi(\beta) \right] + \frac{D_{22}}{L-s} \left[\bar{\bar{A}}_1 \theta(\beta) \right.$$

$$+ B_1 \varphi(\beta) + D_{12} \left[2A^*(L-s)+C^* \right] + D_{22} \left[A^*(L-s) \right.$$

$$\left. + C^* \right] - M_{2t} \qquad (3.4.29)$$

$$N = \frac{\cot\alpha}{L-s} \bar{\bar{k}}^2 D_{11} \left[-\bar{A}_1 \varphi(\beta) + \bar{\bar{B}}_1 \theta(\beta) \right]$$

$$- \cot\alpha \left[B^{**}(L-s) + D^{**} \right] \qquad (3.4.30)$$

Inserting equation (2.1.5) in equations (2.1.10) and (2.1.11) the layer stresses were found to be

$$\overset{k}{\sigma}_1 = \frac{\overset{k}{E}_1}{(1-\nu_1\nu_2)} \left[\varepsilon_1 + \overset{k}{\nu}_2 \varepsilon_2 + \gamma(\chi_1 + \overset{k}{\nu}_2 \chi_2) \right.$$

$$\left. - (\overset{k}{\alpha}_1 + \overset{k}{\nu}_2 \overset{k}{\alpha}_2)(\overset{k}{T}) \right] \qquad (3.4.31)$$

$$\sigma_2^k = \frac{E_2^k}{(1-\nu_1\nu_2)}\Big[\varepsilon_2 + \nu_1^k\varepsilon_1 + \gamma(\chi_2 + \nu_1^k\chi_1)$$

$$- (\alpha_2^k + \nu_1^k\alpha_1^k)(T^k)\Big] \qquad (3.4.32)$$

Alternately, these can be expressed in terms of Meissner's functions. If equations (2.3.3), (2.3.4), (2.4.5) and (2.4.6) are substituted in the above equation;

$$\sigma_1^k = \frac{E_1^k}{(1-\nu_1\nu_2)}\Big[\frac{(-C_{22}+\nu_2^k C_{12})}{(L-s)\,\Omega}V + \frac{(-C_{12}+\nu_2^k C_{11})}{\Omega}\frac{dv}{ds}$$

$$+ \frac{(-C_{12}+\nu_2^k C_{11})}{\Omega}T_{2t} + \frac{(C_{22}-\nu_2^k C_{12})}{\Omega}T_{1t}$$

$$+ \gamma(\nu_2^k\frac{W}{L-s} - \frac{dW}{ds}) - (\alpha_1^k + \nu_2^k\alpha_2^k)(\Delta T^k + T_m)\Big] \qquad (3.4.33)$$

$$\sigma_2^k = \frac{E_2^k}{(1-\nu_1\nu_2)}\Big[\frac{(C_{12}-\nu_1^k C_{22})}{(L-s)\,\Omega}V + \frac{(C_{11}-\nu_1^k C_{12})}{\Omega}\frac{dW}{ds}$$

$$+ \frac{(C_{11}-\nu_1^k C_{12})}{\Omega}T_{2t} + \frac{(-C_{12}+\nu_1^k C_{22})}{\Omega}T_{1t}$$

$$+ \gamma(\frac{W}{L-s} - \nu_1^k\frac{dW}{ds}) - (\alpha_2^k+\nu_1^k\alpha_1^k)(\Delta T^k + T_m)\Big] \qquad (3.4.34)$$

In Appendix 2. it was proved that

$$T_{1t} = \sum_{k=1}^{n'} \frac{E_1^k(\alpha_1^k+\nu_2^k\alpha_2^k)t^k}{1-\nu_1\nu_2}T_m$$

$$T_{2t} = \sum_{k=1}^{n'} \frac{E_2^k(\alpha_2^k + \nu_1^k\alpha_1^k)t^k}{1 - \nu_1\nu_2} T_m$$

Thus,

$$\frac{(-C_{12} + \nu_2^k C_{11})}{\Omega} T_{2t} + \frac{(C_{22} - \nu_2^k C_{12})}{\Omega} T_{1t}$$

$$= \frac{(-C_{12} + \nu_2^k C_{11})}{\Omega} \sum_{k=1}^{n'} \frac{E_2^k(\alpha_2^k + \nu_1^k\alpha_1^k)t^k}{1 - \nu_1\nu_2} T_m$$

$$+ \frac{(C_{22} - \nu_2^k C_{12})}{\Omega} \sum_{k=1}^{n'} \frac{E_1^k(\alpha_1^k + \nu_2^k\alpha_2^k)t^k}{1 - \nu_1\nu_2} T_m$$

Based on equation (2.1.17)

$$\frac{(-C_{12} + \nu_2^k C_{11})}{\Omega} T_{2t} + \frac{(C_{22} - \nu_2^k C_{12})}{\Omega} T_{1t}$$

$$= \frac{(-C_{12} + \nu_2^k C_{11})}{\Omega}(C_{22}\alpha_2^k + C_{12}\alpha_1^k)T_m$$

$$+ (C_{22} - \nu_2^k C_{12})(C_{11}\alpha_1^k + C_{12}\alpha_2^k)T_m$$

$$= \frac{1}{\Omega}\Big[-C_{12}C_{22}\alpha_2^k + C_{11}C_{22}\nu_2^k\alpha_2^k - C_{12}\alpha_1^k + C_{11}C_{12}\alpha_1^k\nu_2^k$$

$$+ C_{11}C_{22}\alpha_1^k - C_{11}C_{12}\nu_2^k\alpha_1^k + C_{12}C_{22}\alpha_2^k - C_{12}^2\nu_2^k\alpha_2^k\Big] T_m$$

$$= \frac{1}{\Omega}(C_{11}C_{22} - C_{12}^2)(\alpha_1^k + \nu_2^k\alpha_2^k) T_m$$

Equation (3.4.33) can be reduced to

$$\sigma_1^k = \frac{E_1^k}{(1 - \nu_1\nu_2)}\left[\frac{(-C_{22} + \nu_2^k C_{12})}{(L - s)\Omega} V + \frac{(-C_{12} + \nu_2^k C_{11})}{\Omega}\frac{dV}{ds}\right.$$

$$\left. + \gamma\left(\nu_2^k\frac{W}{L - s} - \frac{dW}{ds}\right) - (\alpha_1^k + \nu_2^k\alpha_2^k)\,\Delta T^k\right] \qquad (3.4.35)$$

Likewise, equation (3.4.34) also reduces to

$$\sigma_2^k = \frac{E_2^k}{(1 - \nu_1\nu_2)}\left[\frac{(C_{12} - \nu_1^k C_{22})}{(L - s)} V + \frac{(C_{11} - \nu_1^k C_{12})}{\Omega}\frac{dV}{ds}\right.$$

$$\left. + \gamma\left(\frac{W}{L - s} - \nu_1^k\frac{dW}{ds}\right) - (\alpha_2^k + \nu_1^k\alpha_1^k)\,\Delta T^k\right] \qquad (3.4.36)$$

3.5 Numerical Examples

As an illustration of the application of this theory to typical problems, a typical numerical solution of the conical shell will be discussed.

To establish partially the validity of the solution, the equations were first specialized to the case of the isotropic shell and results compared with those of an existing solution for this special problem by Huth (7).

Example 1. The isotropic conical shell.

The elasticity properties were for an isotropic material and all data and thermal gradients were taken from reference (7). The thermal variations are shown in figure (5) and the ν, E and α are

$$\nu = \frac{1}{4}$$

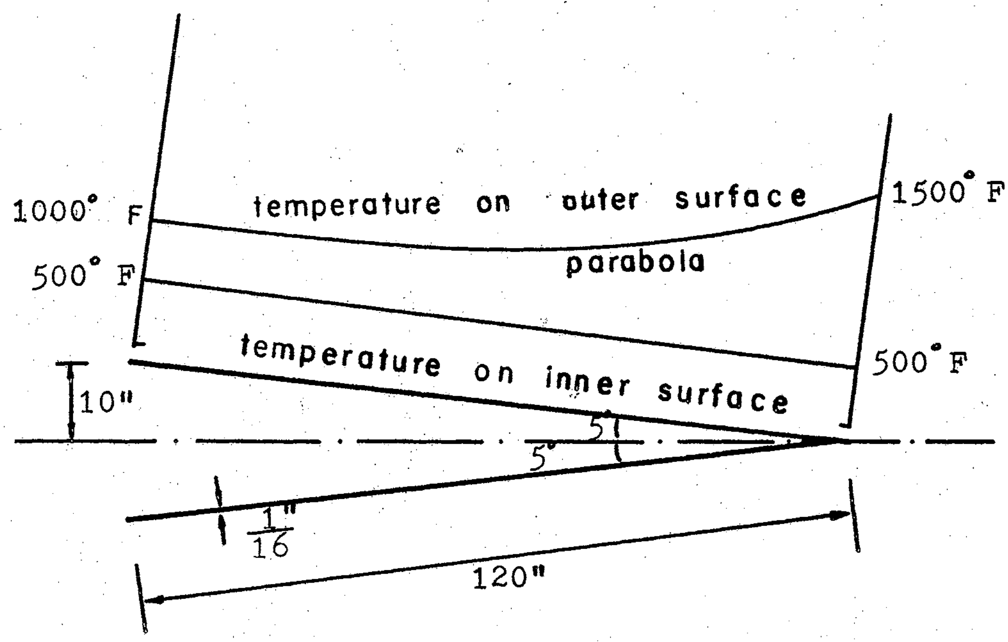

Figure 5. Temperature Variations Along The Inner and Outer Shell Surface

$$E = 30 \times 10^6 \text{ lb/in}^2$$

$$\alpha = 7 \times 10^{-6} \text{ in/in/deg F}$$

The elasticity properties become

$$C_{11} = C_{22} = 2 \times 10^6 \text{ lb/in}^2 \qquad D_{11} = D_{22} = 650 \text{ lb/in}^2$$

$$C_{12} = \frac{1}{2} \times 10^6 \text{ lb/in}^2 \qquad D_{12} = 159 \text{ lb/in}^2$$

$$\Omega = 4.25 \times 10^{12} \text{ lb}^2\text{/in}$$

$$\bar{k}^2 = 57.2 \text{ 1/in}$$

$$M_{1t} = - \frac{E}{1 - \nu} \cdot \frac{1}{3(32)^2} \cdot \left(500 + \frac{5}{144} s^2\right)$$

$$P = \frac{5}{144} \text{ deg F/in}^2$$

From Appendix 2 one obtains

$$H^* = H^{**} = 1.22 \quad \text{lb/in}^3 \qquad H^\circ = 0.0064 \quad \text{lb/in}^2$$

Also from the definitions of A^* and C^* following equation (3.2.10), it was found that

$$A^* = 0.376 \times 10^{-7} \qquad \text{1/in}^3$$

$$C^* = -0.451 \times 10^{-4} \qquad \text{1/in}^2$$

and were used to determine M_1, M_2 which are contributed by the particular solutions ;

$$D_{11} \left[2A^*(L - s) + C^* \right] + D_{12} \left[A^*(L - s) + C^* \right]$$

and

$$D_{12}\left[A^{*}(L - s) + C^{*}\right] + D_{22}\left[A^{*}(L - s) + C^{*}\right]$$

These are very small compared to that contributed by M_{1t}. The values of T_1, T_2, and N contributed by the particular solution are

$$B^{**}(L - s) + D^{**} = H^{\circ} s \tan\alpha$$

and

$$-2B^{**}(L - s) + D^{**} = (-L + 2s)\, H^{\circ} \tan\alpha$$

Because

$$\frac{3 \tan^2\alpha\, C_{11}}{\Omega}\left(H^{*} - \frac{C_{12}}{C_{11}} H^{**}\right) = 0.98 \times 10^{-8}\ 1/\text{in}$$

is very small, it can be neglected.

A computer program was written to solve the formulated equations (3.4.26) to (3.4.28). The results are plotted as figures 6, 7 and 8.

Example 2. The orthotropic conical shell

A conical shell consisting of nine orthotropic layers with the same shell dimensions and thermal variations as those of example 1 was also investigated.

From Appendices 1-4

$$\nu_1 = 0.083$$

$$\nu_2 = 0.253$$

$$C_{11} = 0.305 \times 10^{6}\ \text{lb/in}^2$$

$$C_{12} = 0.375 \times 10^{5}\ \text{lb/in}^2$$

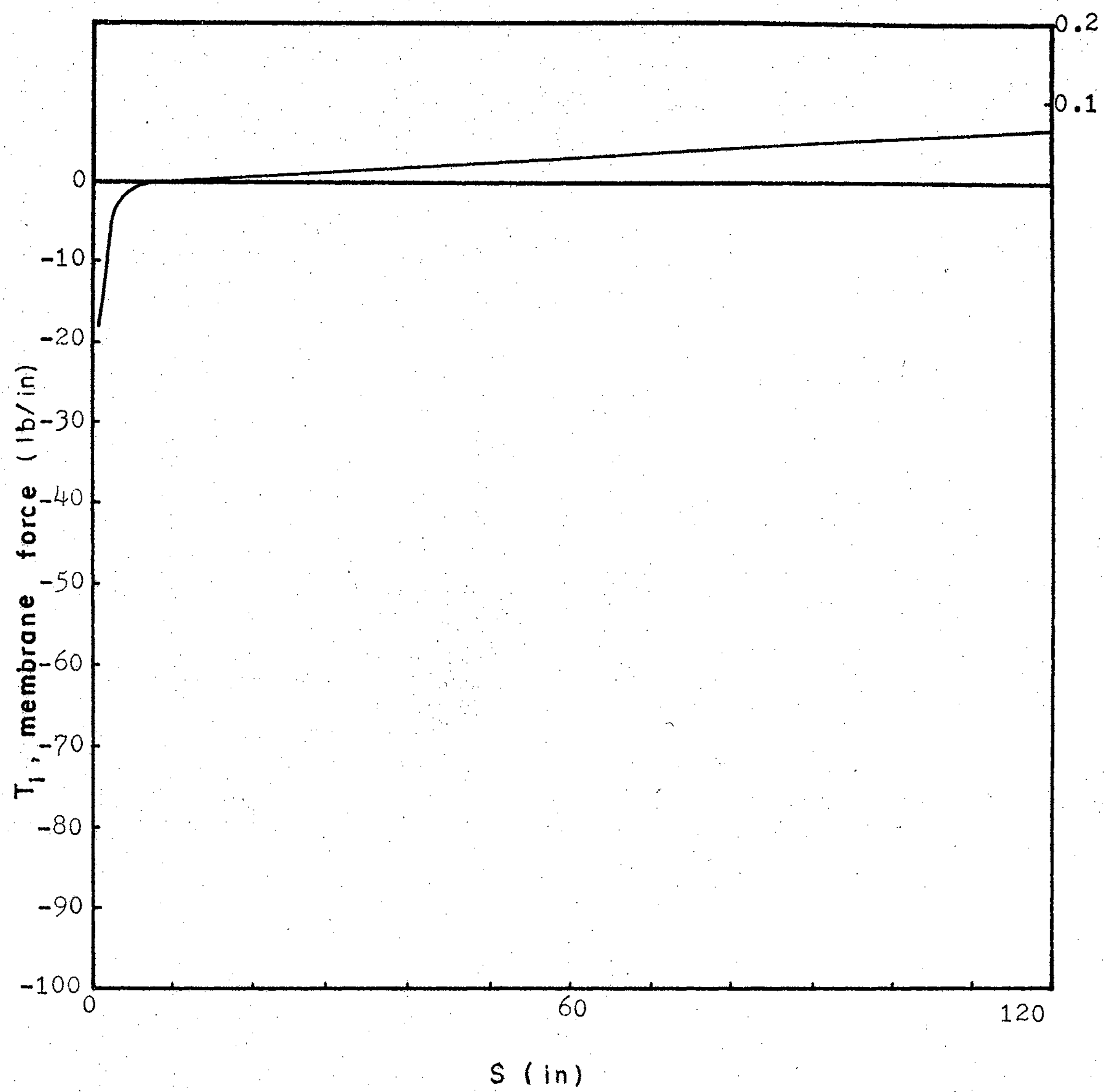

Figure 6. Membrane Force, T_1, as A Function of s in A Heated Conical Shell

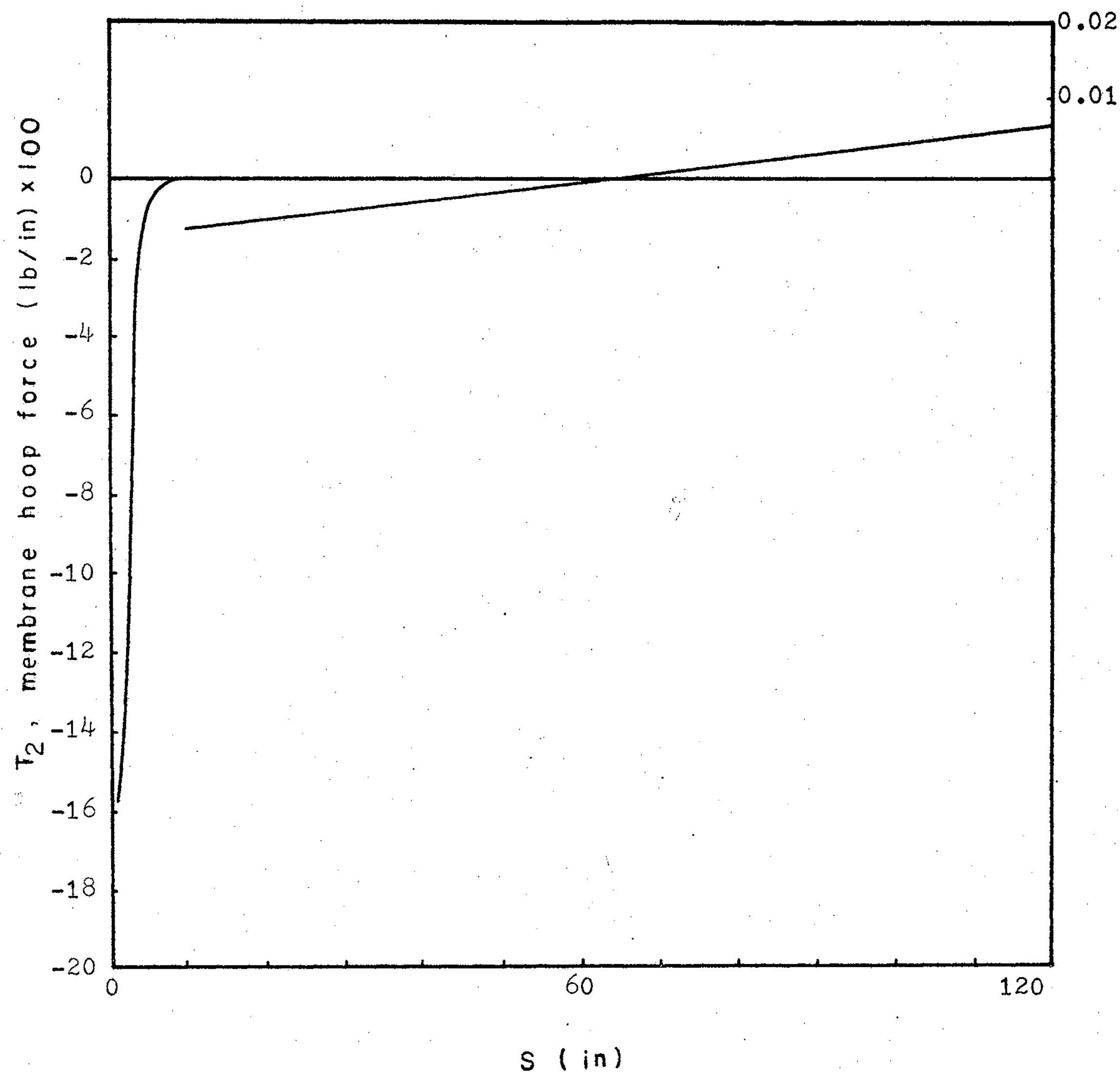

Figure 7. Membrane Hoop Force, T_2, as A Function of s in A Heated Conical Shell

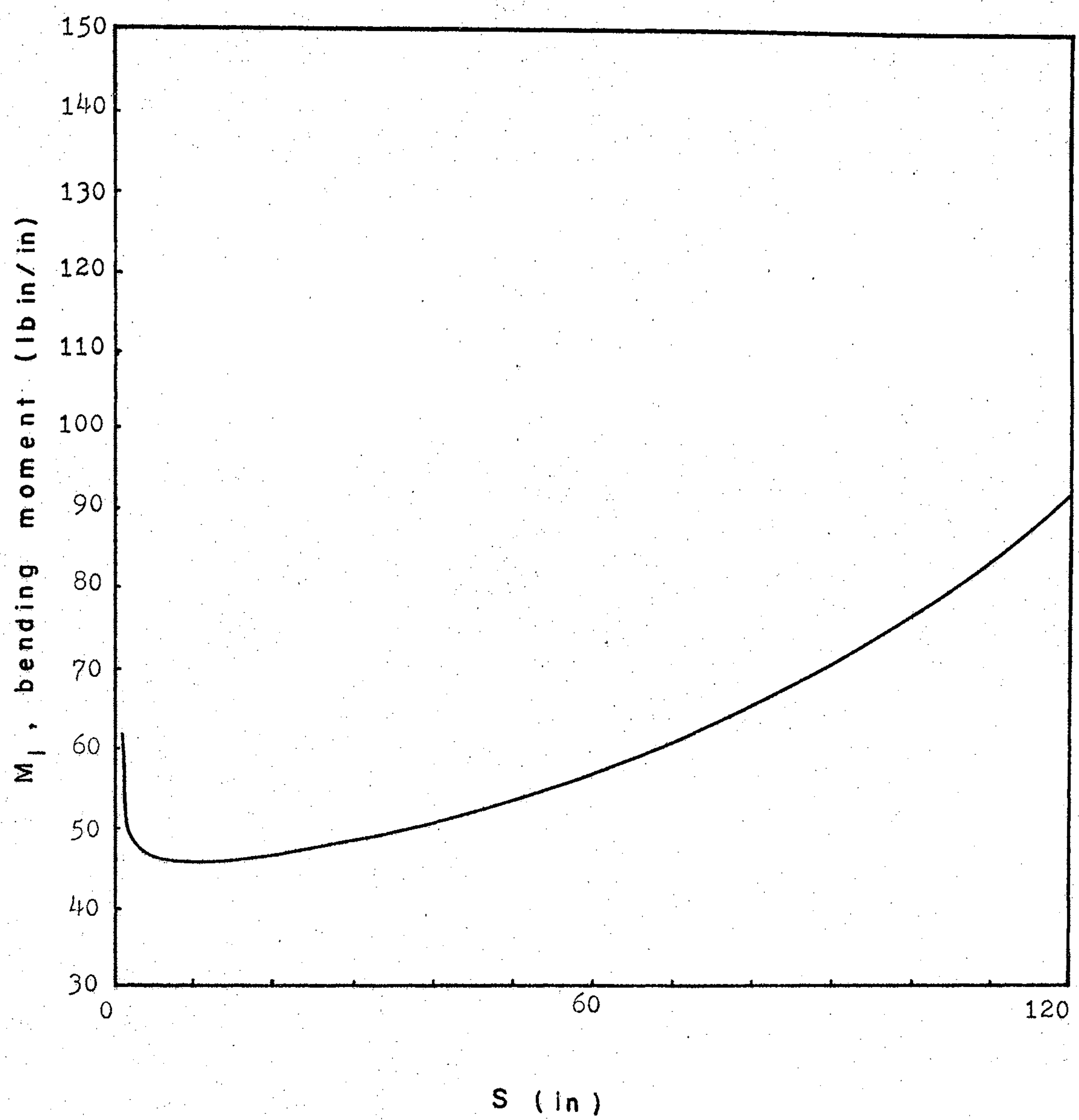

Figure 8. Bending Moment, M_1, as A Function of s in A Heated Conical Shell

$C_{22} = 0.275 \times 10^6 \quad lb/in^2$

$D_{11} = 0.111 \times 10^3 \quad lb/in^2$

$D_{12} = 12.15 \quad lb/in^2$

$D_{22} = 80 \quad lb\ in^2$

$\bar{k}^2 = 49.2\ 1/in$

$K_1 = 1$

$K_2 = 5$

$E_1 = 6.87 \times 10^6 \quad lb/in^2$

$E_2 = 2.31 \times 10^6 \quad lb/in^2$

$t^k = 0.00684 \quad in$

$\alpha_1 = 6.67 \times 10^{-6} \quad in/in/deg\ F$

$\alpha_2 = 6.08 \times 10^6 \quad in/in\ deg\ F$

$\Omega = 8.21 \times 10^{10} \quad lb^2/in^4$

Following the procedure stated in Example 1, one obtains

$H^* = 0.167 \quad lb/in^3$

$H^{**} = 0.146 \quad lb/in^3$

$H' = 0.00093 \quad lb/in^2$

$A^* = 0.486 \times 10^{-7} \quad 1/in^3$

$$C^{*} = -0.582 \times 10^{-5} \quad 1/in^2$$

A computer program was written to solve the formulated equations (3.4.26) - (3.4.30). The results were plotted as in figures 9- 15.

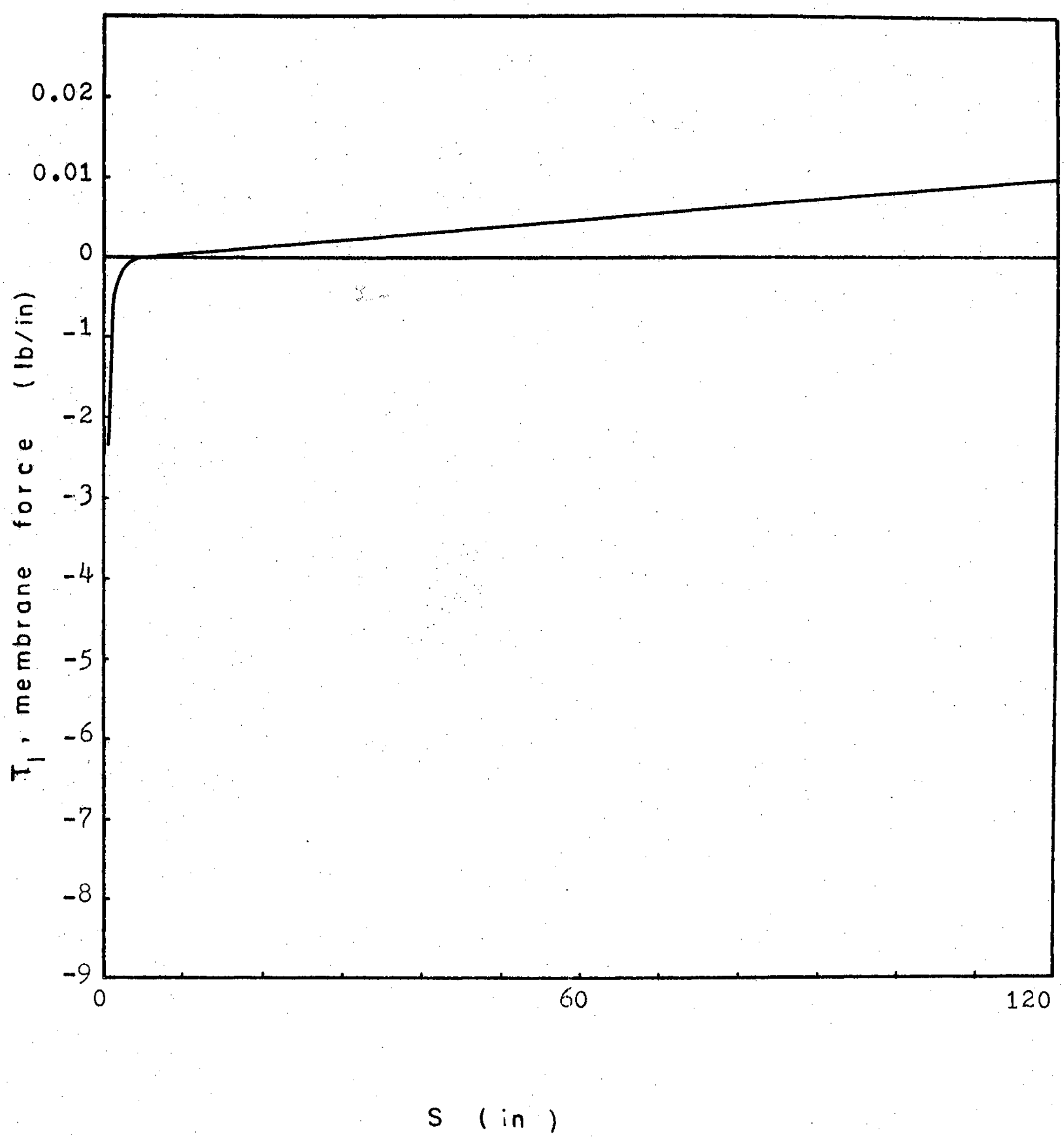

Figure 9. Membrane Force, T_1 as A Function of s in A Heated Multi-Layered Conical Shell

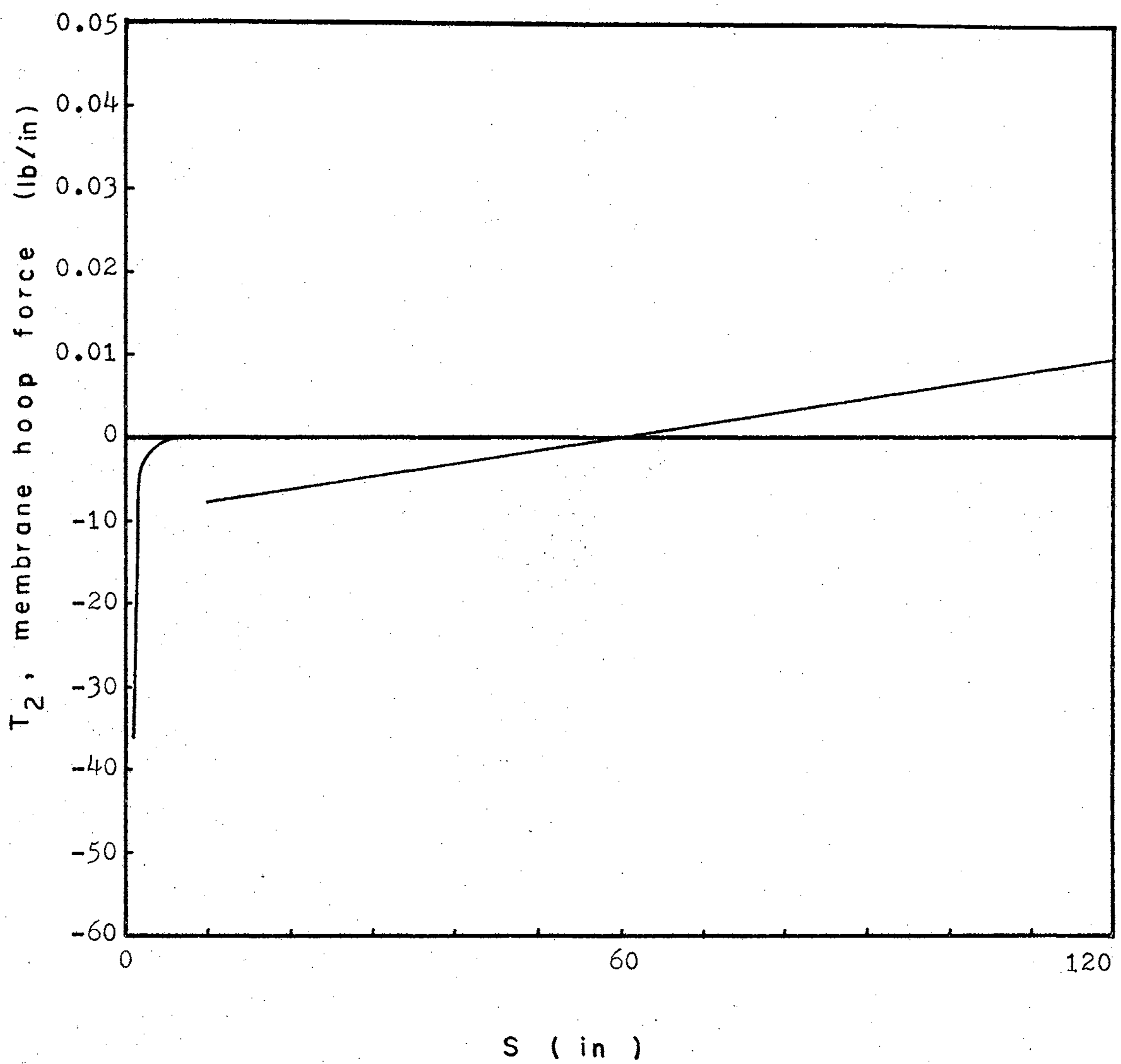

Figure 10. Membrane Hoop Force, T_2, as A Function of s in A Heated Multi-Layered Conical Shell

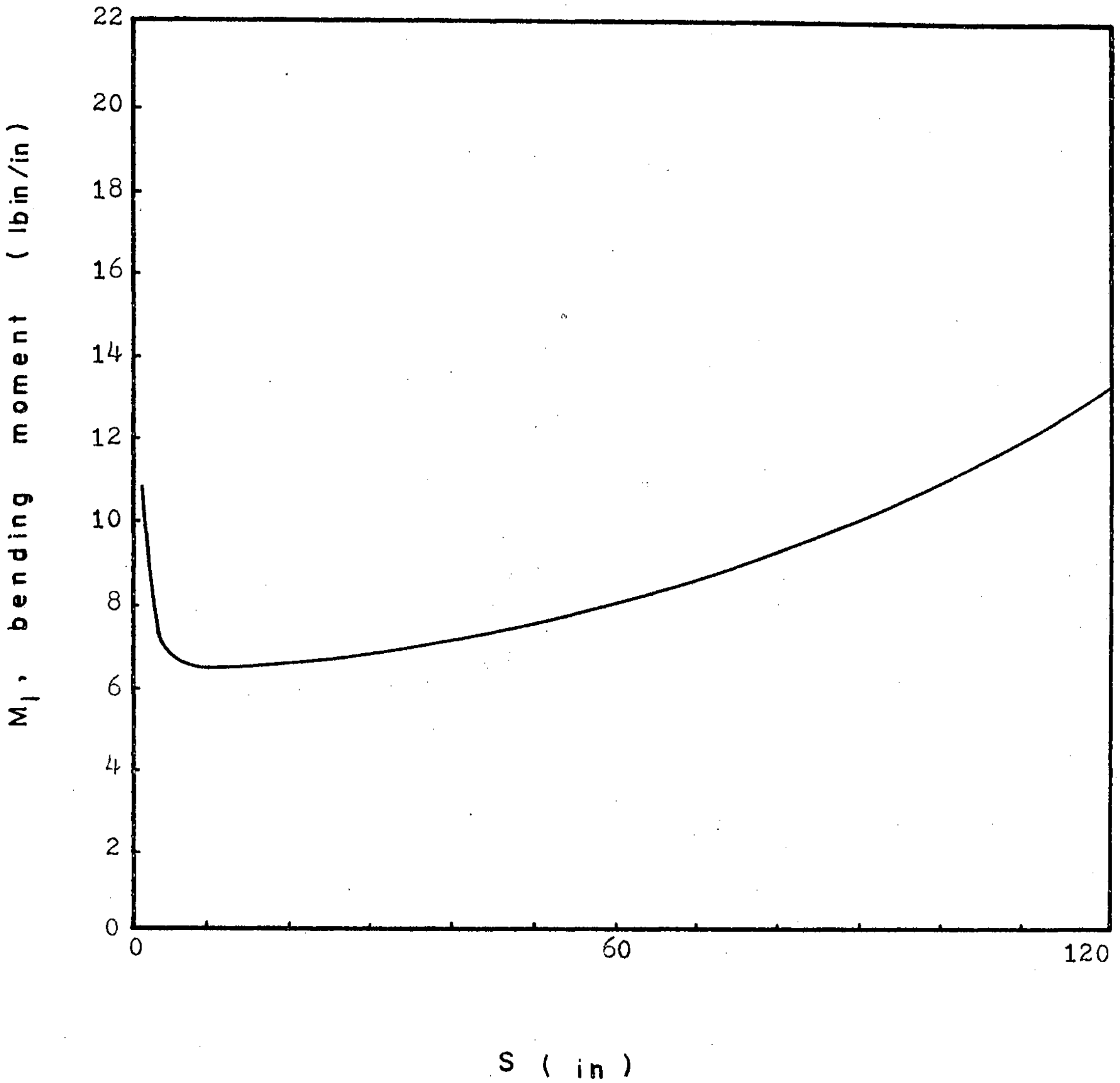

Figure 11. Bending Moment, M_1 as A Function of s in A Multi-Layered Conical Shell

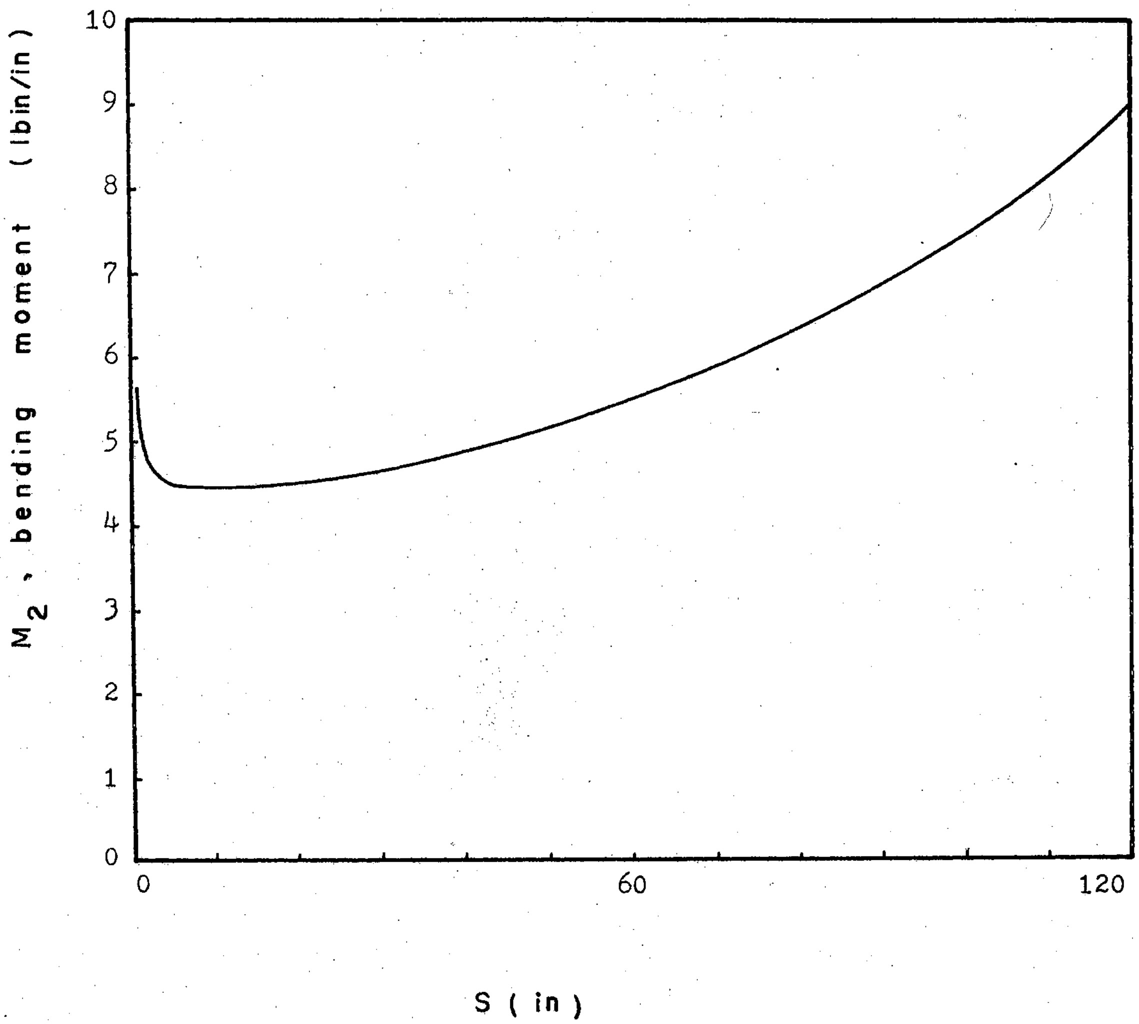

Figure 12. Bending Moment, M_2, as A Function of s in A Multi-Layered Conical Shell

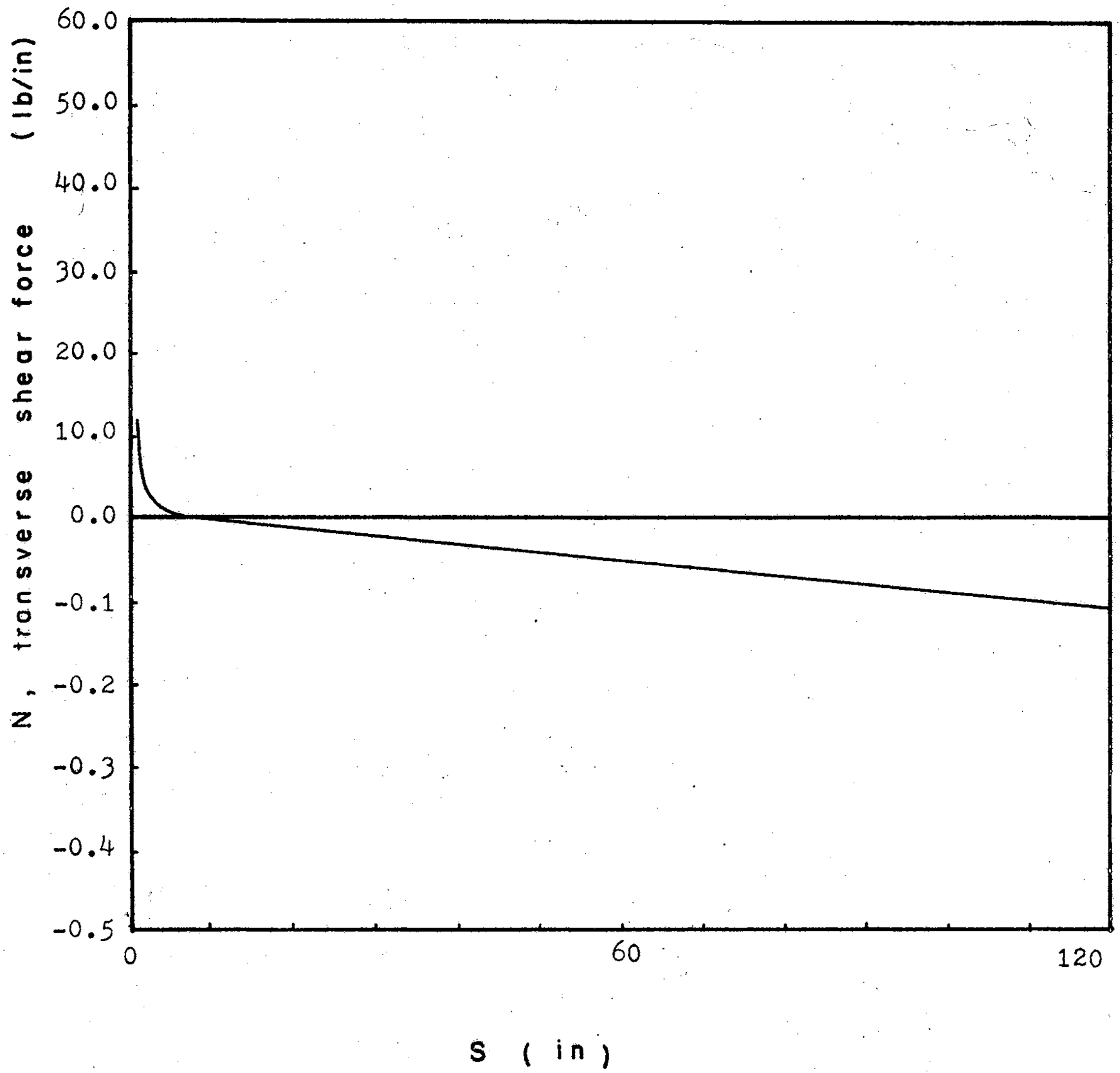

Figure 13. Transverse Shear Force, N, as A Function of s in A Heated Multi-Layered Conical Shell

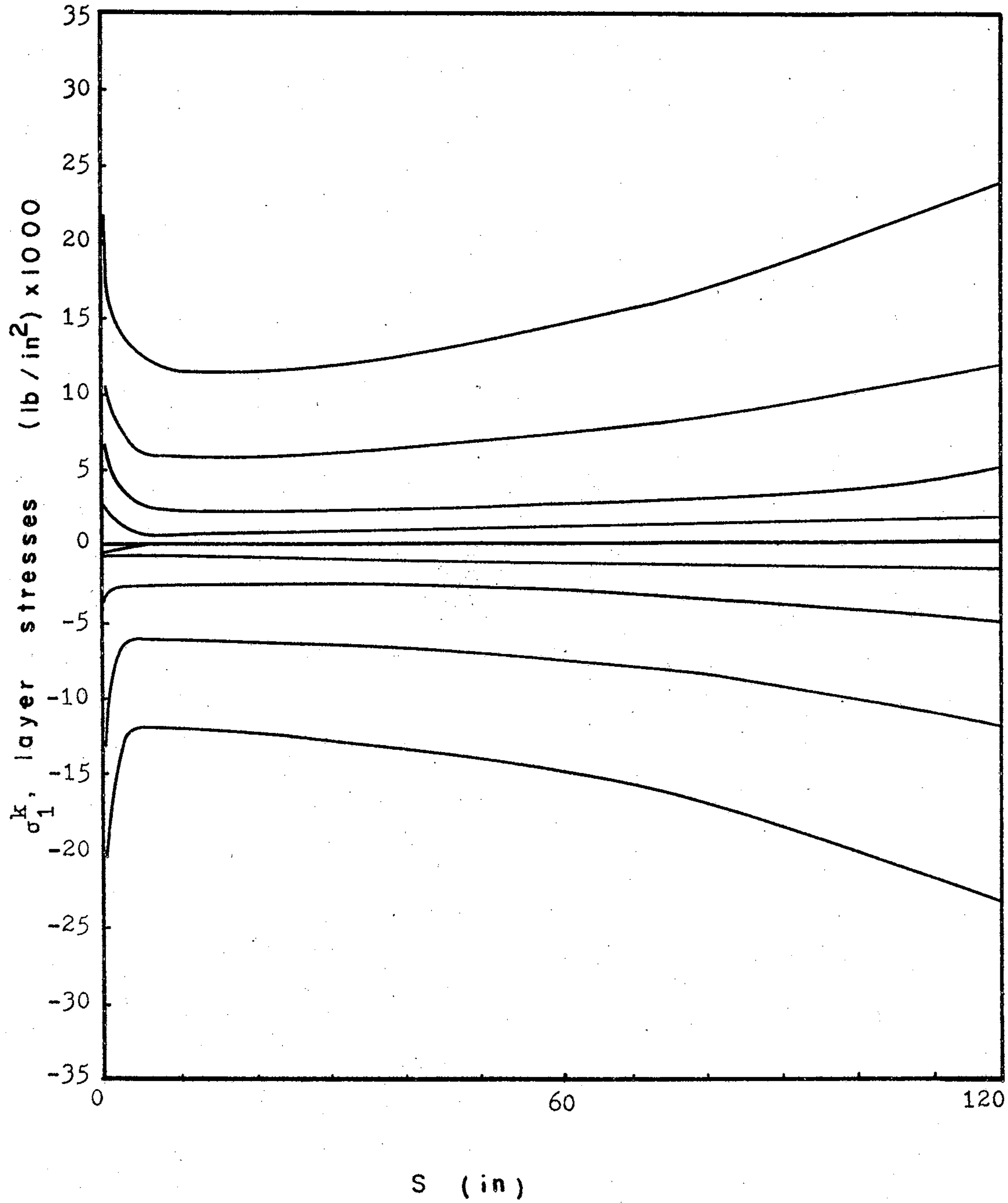

Figure 14. Layer Stresses in Direction 1, as A Function of s in A Heated Multi-Layered Conical Shell

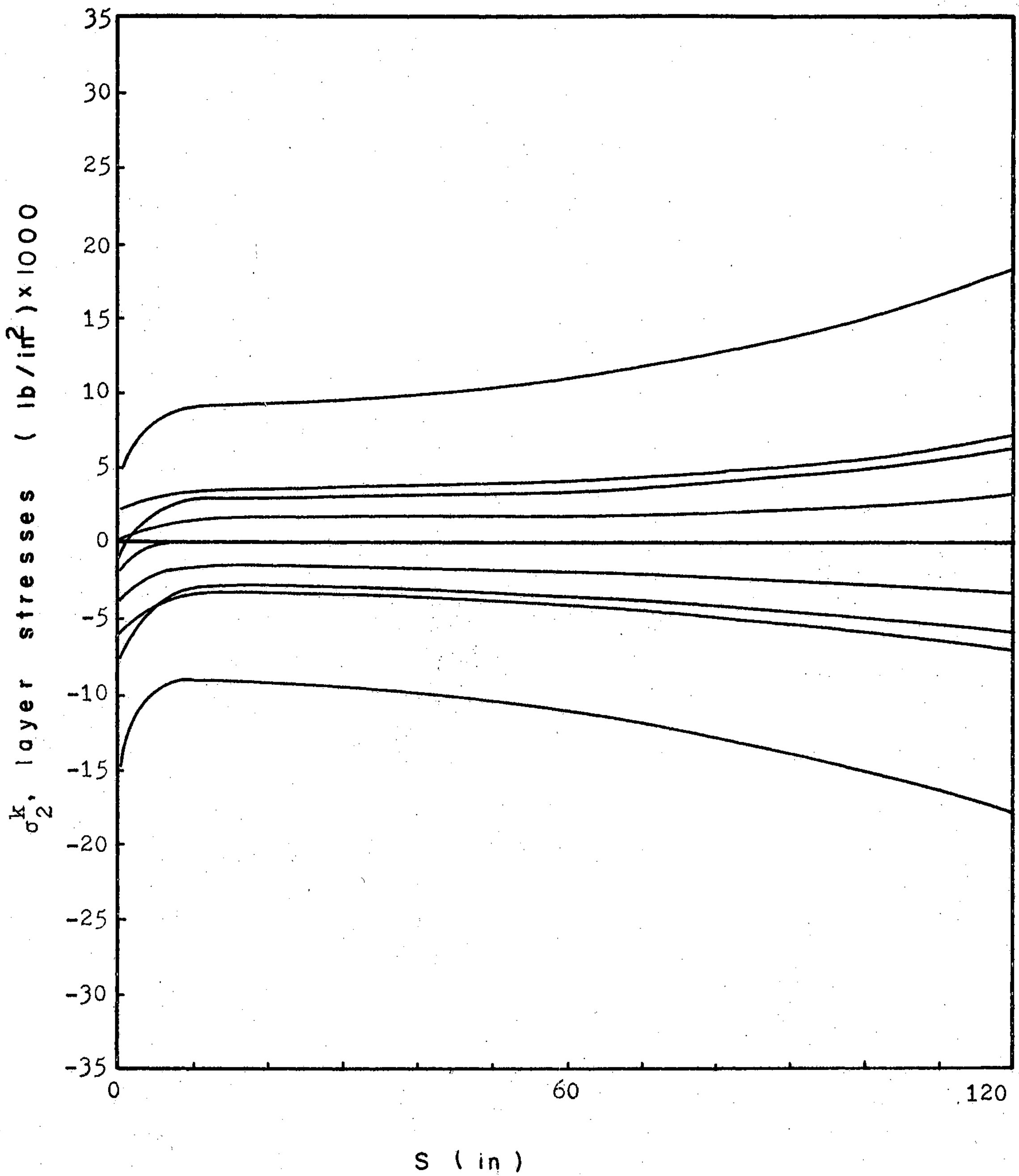

Figure 15. Layer Stresses in Direction 2, as A Function of s in A Heated Multi-Layered Conical Shell

CHAPTER IV

DISCUSSION AND CONCLUSION

In the analysis of this problem, several significant assumptions have been made. For example, it was assumed the ratios of elastic constants ; C_{22} to C_{11}, D_{22} to D_{11} were considered to be equal to an arbitrary constant λ in order to reduce the system of differential equations to a single governing equation. Furthermore, this arbitrary constant λ was taken to be unity in order to make the particular solution possible. In addition to these assumptions, the first approximation of asymptotic integration was taken in the solution to the homogeneous part ; those equations which are obtained when the coefficients of $1/k^2$ and $1/k$ vanish. By elementary reasoning, it was shown in Reference 1 that the first approximation has an error of the order of h/R in comparision with unity. Therefore, it leads one to conclude that the results obtained by this analytical method are only approximate, but adequate for most engineering purposes.

It is seen from the design quantities that the homogeneous solution is in the form of a damping function which converges so rapidly that the edge effect zone is very small compared to the corresponding dimension of the shell.

From the plotted figures, one sees the abrupt changes in the design quantities occurring somewhere between 5 in. and 10 in. from the rigidly clamped end. As pointed out by Huth (7), in an actual missile configuration, rigid clamping certainly will not be achieved, so the considered case represents a limiting situation.

In the analysis of Example 2 fiberglass was used as the construction material. Of course, the numerical values for typical elastic properties are low compared to those of steel which was considered in Example 1. Correspondingly, the design quantities become proportionally less.

It should be noted also that there are deviations existing in the comparisons of the summation of the layer forces against the corresponding tangential resultant forces. The deviations amount to 9.6 % for the meridional direction and 5.6 % for the circumferential direction. Both are less than 10 % and are considered to be good for engineering purposes. The accumulated error is belived to be the result of the approximations discussed previously.

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APPENDIX 1

THE ARBITRARY CONSTANT

For convenience, the layer thickness is considered to be unity in the following calculations. The definitions of C_{ij}, Ω, D_{ij} and $\bar{k}^2$ show that they are functions of the layer thickness t^k, $(t^k)^2$, $(t^k)^3$ and $\frac{1}{t^k}$ respectively. Therefore, these elastic constants shall be modified by the layer thickness before applying to the problems. The ratios $\frac{C_{22}}{C_{11}}$ and $\frac{D_{22}}{D_{11}}$ have been plotted against the total number of layers in the shell as shown in Figure 16. It is seen that the curve behaves in an oscillating damped fashion and converges relatively rapidly to unity. Alternately, the ratios can be shown mathematically to approach unity in the limit by using equations (2.1.17), (2.2.17.a) and (2.1.6): i. e. for the case of alternating layers of equal thicknesses,

$$\frac{C_{22}}{C_{11}} = \frac{\sum_{k=1}^{n'} B_{22}(\delta_k - \delta_{k-1})}{\sum_{k=1}^{n'} B_{11}(\delta_k - \delta_{k-1})} \tag{A.1.1}$$

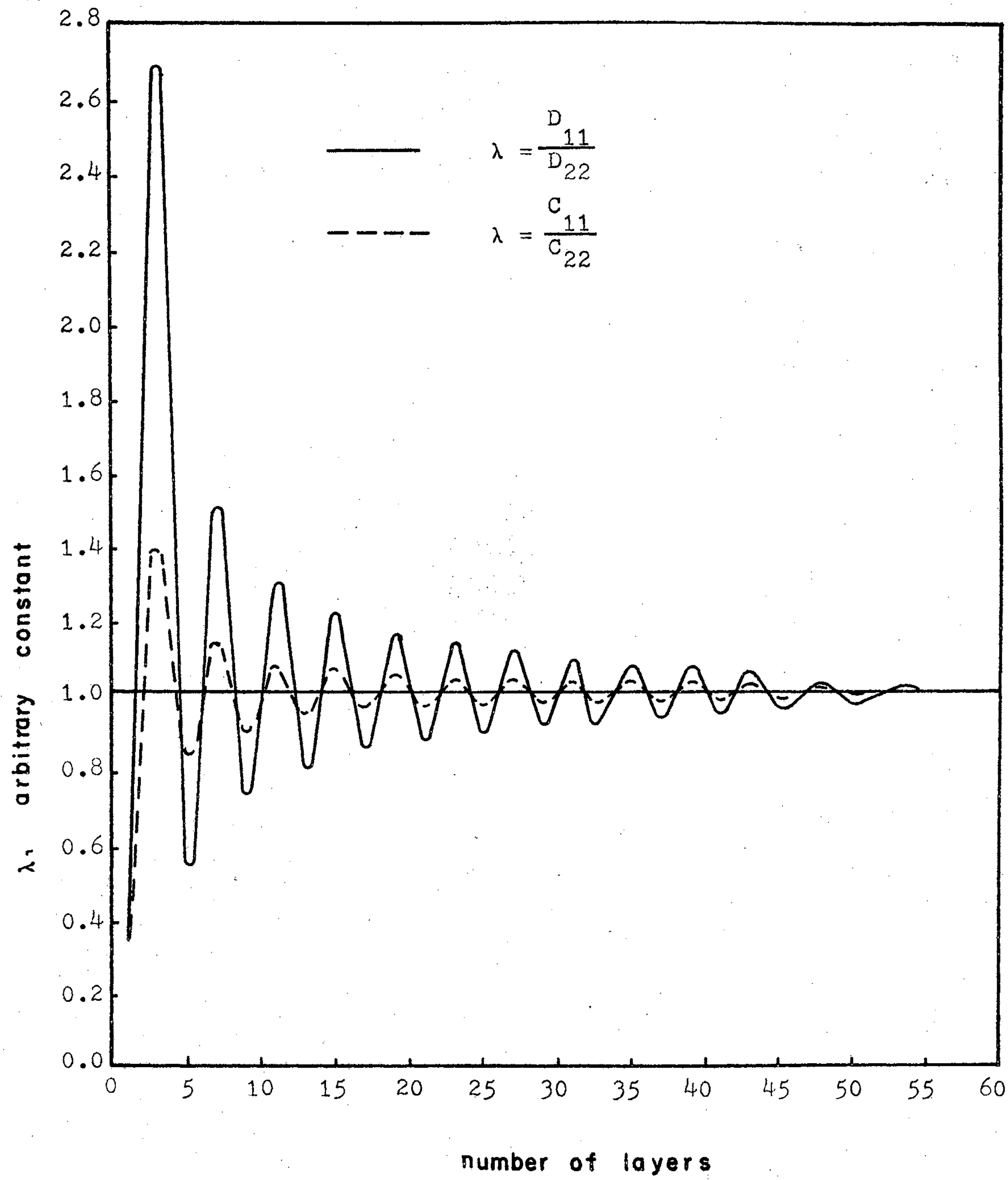

Figure 16. Arbitrary Constant λ, as A Function of Number of Layers in A Multi-Layered Conical Shell

$$\lambda = \frac{C_{22}}{C_{11}} = \frac{\sum_{k=1}^{n'} \frac{E_2^k}{1 - \nu_1\nu_2} t^k}{\sum_{k=1}^{n'} \frac{E_1^k}{1 - \nu_1\nu_2} t^k}$$

$$= \lim_{n' \to \infty} \frac{(E_2 + F_1 + E_2 + E_1 + \cdots\cdots n' \text{term}) \frac{t}{1 - \nu_1\nu_2}}{(E_1 + E_2 + E_1 + E_2 + \cdots\cdots n' \text{term}) \frac{t}{1 - \nu_1\nu_2}}$$

$$= 1 \qquad \text{(A.1.2)}$$

APPENDIX 2

THE THERMAL GRADIENTS

In this section is discussed the calculation of the temperature distribution through the thickness of the shell. With the assistance of some insulating material on the inner surface of the shell, the temperature along the inner surface can be held uniform. To determine the temperature at each point of each layer, a steady-state heat conduction condition was assumed to exist through the shell thickness. Taking a ninelayered shell, for example, the heat-balance equations are

$$K_1(T_1 - T_{in}) = K_2(T_2 - T_1) \tag{A.2.1}$$

$$K_2(T_2 - T_1) = K_1(T_3 - T_2) \tag{A.2.2}$$

$$K_1(T_3 - T_2) = K_2(T_4 - T_3) \tag{A.2.3}$$

$$K_2(T_4 - T_3) = K_1(T_5 - T_4) \tag{A.2.4}$$

$$K_1(T_5 - T_4) = K_2(T_6 - T_5) \tag{A.2.5}$$

$$K_2(T_6 - T_5) = K_1(T_7 - T_6) \tag{A.2.6}$$

$$K_1(T_7 - T_6) = K_2(T_8 - T_7) \tag{A.2.7}$$

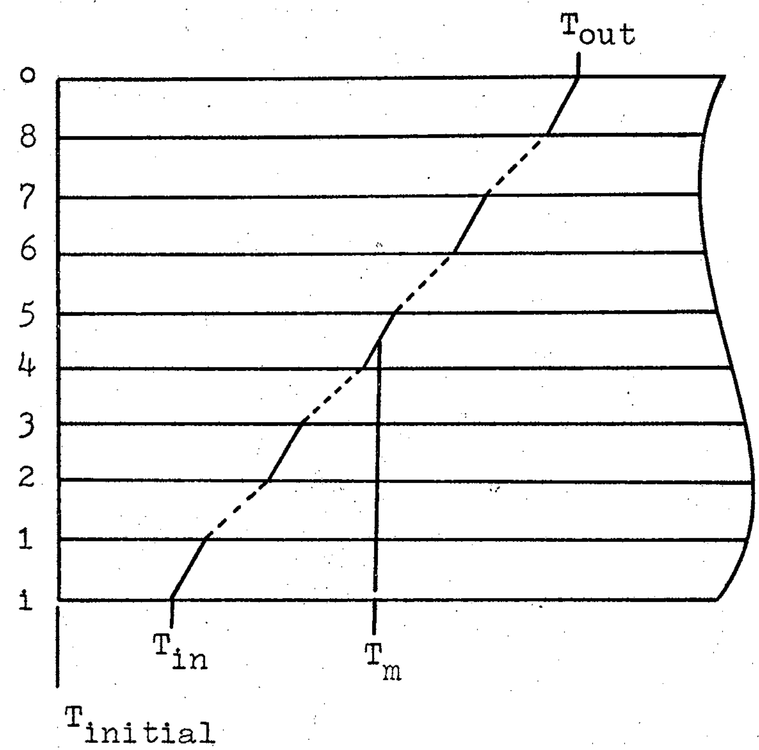

Figure 17. Temperature Distribution Across the Thickness

$$K_2(T_8 - T_7) = K_1(T_{out} - T_8) \tag{A.2.8}$$

where K_1, K_2 are the alternating thermal conductivies. The solutions to these eight simutaneous equations are

$$T_1 = \frac{1}{K_1 + K_2}\left[K_1 T_{in} + K_2 T_2\right] \tag{A.2.9}$$

$$T_2 = \frac{1}{K_1 + 2K_2}\left[K_2 T_{in} + (K_1 + K_2)T_3\right] \tag{A.2.10}$$

$$T_3 = \frac{1}{2K_1 + 2K_2}\left[K_1 T_{in} + (K_1 + 2K_2)T_4\right] \tag{A.2.11}$$

$$T_4 = \frac{1}{2K_1 + 3K_2}\left[K_2 T_{in} + (2K_1 + 2K_2)T_5\right] \tag{A.2.12}$$

$$T_5 = \frac{1}{3K_1 + 3K_2}\left[K_1 T_{in} + (2K_1 + 3K_2)T_6\right] \tag{A.2.13}$$

$$T_6 = \frac{1}{3K_1 + 4K_2}\left[K_2 T_{in} + (3K_1 + 3K_2)T_7\right] \tag{A.2.14}$$

$$T_7 = \frac{1}{4K_1 + 4K_2}\left[K_1 T_{in} + (3K_1 + 4K_2)T_8\right] \tag{A.2.15}$$

$$T_8 = \frac{1}{4K_1 + 5K_2}\left[K_2 T_{in} + (4K_1 + 4K_2)T_{out}\right] \tag{A.2.16}$$

where T_{in} and T_{out} are given temperatures on inner surface and outer surface respectively. T_8, T_7, $T_6 \cdots\cdots T_1$ are found in reverse order and are obtained as

$$T_1 = \frac{1}{4K_1 + 5K_2}\left[(4K_1 + 4K_2)T_{in} + K_2 T_{out}\right] \tag{A.2.17}$$

$$T_2 = \frac{1}{4K_1 + 5K_2}\Big[(3K_1 + 4K_2)T_{in}$$

$$+ (K_1 + K_2)T_{out}\Big] \tag{A.2.18}$$

$$T_3 = \frac{1}{4K_1 + 5K_2}\Big[(3K_1 + 3K_2)T_{in}$$

$$+ (K_1 + K_2)T_{out}\Big] \tag{A.2.19}$$

$$T_4 = \frac{1}{4k_1 + 5K_2}\Big[(2K_1 + 3K_2)T_{in}$$

$$+ (2K_1 + 2K_2)T_{out}\Big] \tag{A.2.20}$$

$$T_5 = \frac{1}{4K_1 + 5K_2}\Big[(2K_1 + 2K_2)T_{in}$$

$$+ (2K_1 + 3K_2)T_{out}\Big] \tag{A.2.21}$$

$$T_6 = \frac{1}{4K_1 + 5K_2}\Big[(K_1 + 2K_2)T_{in}$$

$$+ (3K_1 + 3K_2)T_{out}\Big] \tag{A.2.22}$$

$$T_7 = \frac{1}{4K_1 + 5K_2}\Big[(K_1 + K_2)T_{in}$$

$$+ (3K_1 + 4K_2)T_{out}\Big] \tag{A.2.23}$$

$$T_8 = \frac{1}{4K_1 + 5K_2}\Big[K_2T_{in} + (4K_1 + 4K_2)T_{out}\Big] \tag{A.2.24}$$

Because heat flows linearly from face to face of each layer, the half of the summation of temperatures at each face of the layer gives the average temperature for each corresponding layer as:

$$T_{1-in} = \frac{1}{2(4K_1 + 5K_2)}\Big[(8K_1 + 9K_2)T_{in} + K_2 T_{out}\Big] \tag{A.2.25}$$

$$T_{2-1} = \frac{1}{2(4K_1 + 5K_2)}\Big[(7K_1 + 8K_2)T_{in} + (K_1 + 2K_2)T_{out}\Big] \tag{A.2.26}$$

$$T_{3-2} = \frac{1}{2(4K_1 + 5K_2)}\Big[(6K_1 + 7K_2)T_{in} + (2K_1 + 3K_2)T_{out}\Big] \tag{A.2.27}$$

$$T_{4-3} = \frac{1}{2(4K_1 + 5K_2)}\Big[(5K_1 + 6K_2)T_{in} + (3K_1 + 4K_2)T_{out}\Big] \tag{A.2.28}$$

$$T_{5-4} = \frac{1}{2(4K_1 + 5K_2)}\Big[(4K_1 + 5K_2)T_{in} + (4K_1 + 5K_2)T_{out}\Big] \tag{A.2.29}$$

$$T_{6-5} = \frac{1}{2(4K_1 + 5K_2)}\Big[(3K_1 + 4K_2)T_{in} + (5K_1 + 6K_2)T_{out}\Big] \tag{A.2.30}$$

$$T_{7-6} = \frac{1}{2(4K_1 + 5K_2)} \Big[(2K_1 + 3K_2)T_{in}$$

$$+ (6K_1 + 7K_2)T_{out}\Big] \qquad (A.2.31)$$

$$T_{8-7} = \frac{1}{2(4K_1 + 5K_2)} \Big[(K_1 + 2K_2)T_{in}$$

$$+ (7K_1 + 8K_2)T_{out}\Big] \qquad (A.2.32)$$

$$T_{out-8} = \frac{1}{2(4K_1 + 5K_2)} \Big[K_2T_{in}$$

$$+ (8K_1 + 9K_2)T_{out}\Big] \qquad (A.2.33)$$

From these expressions, by subtracting the temperature of the middle layer from that of each layer, one obtains

$$\Delta T^K = T^K - T_m \qquad (A.2.34)$$

$$T^K = \Delta T^K + T_m \qquad (A.2.35)$$

where

T^K = Temperature of the K^{th} layer

T_m = Temperature of the middle layer

ΔT^K = Temperature difference between the K^{th} layer and the middle layer

For instance, the temperature difference between the first layer (T_{1-in}) and the middle layer (T_{5-4}) can be written as

$$T_{1-in} - T_{5-4} = \frac{1}{2(4K_1 + 5K_2)} \left[(8K_1 + 9K_2) + K_2 T_{out} \right]$$

$$- \frac{1}{2(4K_1 + 5K_2)} \left[(4K_1 + 5K_2) T_{in} \right.$$

$$\left. + (4K_1 + 5K_2) T_{out} \right]$$

$$= \frac{1}{2(4K_1 + 5K_2)} \left[(4K_1 + 4K_2) T_{in} \right.$$

$$\left. - (4K_1 + 4K_2) T_{out} \right]$$

$$= \frac{4(K_1 + K_2)}{2(4K_1 + 5K_2)} (T_{in} - T_{out}) \qquad \text{(A.2.36)}$$

In general, the temperature difference for any layer, ΔT^K, in the presence of any arbitrary number of layers, n', may be expressed as

$$\Delta T^k = \frac{\left[k - \left(\frac{n'+1}{2}\right) \right] (K_1 + K_2)}{2\left[\left(\frac{n'-1}{2}\right) K_1 + \left(\frac{n'+1}{2}\right) K_2 \right]} (T_{out} - T_{in}) \qquad \text{(A.2.37)}$$

where n' = total number of layers

k = k^{th} layer

Then, the resulant force due to temperature are, using equations (2.1.15)

$$T_{1t} = \sum_{k=1}^{n'} \int_{(\delta_{k-1} - \Delta)}^{(\delta_k - \Delta)} \sigma_{1t}^{k} \, d\gamma$$

$$= \sum_{k=1}^{n'} \frac{E_1^k(\alpha_1^k + \nu_2^k \alpha_2^k)}{1-\nu_1\nu_2} \left\{ \frac{\left[k-(\frac{n'+1}{2})\right](K_1+K_2)}{2\left[(\frac{n'-1}{2})K_1+(\frac{n'+1}{2})K_2\right]} (T_{out}-T_{in}) \right.$$

$$\left. + T_m \right\} t^k \qquad (A.2.38)$$

$$T_{2t} = \sum_{k=1}^{n'} \int_{(\delta_{k-1}-\Delta)}^{(\delta_k-\Delta)} \sigma_{2t}^k \, d\gamma$$

$$= \sum_{k=1}^{n'} \frac{E_2^k(\alpha_2^k+\nu_1^k\alpha_1^k)}{1-\nu_1\nu_2} \left\{ \frac{\left[K-(\frac{n'+1}{2})\right](K_1+K_2)}{2\left[(\frac{n'-1}{2})K_1+(\frac{n'+1}{2})K_2\right]} (T_{out}-T_{in}) \right.$$

$$\left. + T_m \right\} t^k \qquad (A.2.39)$$

It is seen that k is the variable in the series, if expansion is made only regariding to term $\left[k - (\frac{n'+1}{2})\right]$

$$\sum_{k=1}^{n'} \left[k-(\frac{n'+1}{2})\right] = -(\frac{n'-1}{2}) - (\frac{n'-3}{2}) - (\frac{n'-5}{2}) \ldots\ldots\ldots\ldots$$

$$+ (\frac{n'-5}{2}) + (\frac{n'-3}{2}) + (\frac{n'+1}{2}) = 0 \qquad (A.2.40)$$

Therefore,

$$T_{1t} = \sum_{k=1}^{n'} \frac{E_1^k(\alpha_1^k + \nu_2^k\alpha_2^k)t^k}{1-\nu_1\nu_2} T_m \qquad (A.2.41)$$

$$T_{2t} = \sum_{k=1}^{n'} \frac{E_2^k(\alpha_2^k + \nu_1^k\alpha_1^k)t^k}{1-\nu_1\nu_2} T_m \qquad (A.2.42)$$

As n' approaches infinity the values for T_{1t} and T_{2t} are

$$T_{1t} = T_{2t} = T_t \tag{A.2.43}$$

Also, from Appendix 1, it is seen that

$$C_{22} = C_{11}$$

Thus $\bar{\Phi}_1(s)$ is reduced to

$$\bar{\Phi}_1(s) = -(1 - \frac{C_{12}}{C_{11}}) \frac{dT_t}{ds} \tag{A.2.44}$$

The moments due to temperature are

$$M_{1t} = -\sum_{k=1}^{n'} \int_{(\delta_{k-1} - \Delta)}^{(\delta_k - \Delta)} \sigma_{1t}^k \, \gamma d\gamma$$

$$= -\sum_{k=1}^{n'} \frac{E_1^k(\alpha_1^k + \nu_2^k \alpha_2^k)(t^k)^2}{1-\nu_1\nu_2} \left\{ \frac{\left[k - (\frac{n'+1}{2})\right]^2 (K_1+K_2)}{2\left[(\frac{n'-1}{2})K + (\frac{n'+1}{2})K_2\right]} (T_{out} - T_{in}) + \left[k-(\frac{n'+1}{2})\right] T_m \right\} \tag{A.2.45}$$

and

$$M_{2t} = -\sum_{k=1}^{n'} \int_{(\delta_{k-1} - \Delta)}^{(\delta_k - \Delta)} \sigma_{2t}^k \, \gamma d\gamma$$

$$= -\sum_{k=1}^{n'} \frac{E_2^k(\alpha_2^k + \nu_1^k \alpha_1^k)(t^k)^2}{1-\nu_1\nu_2} \left\{ \frac{\left[k - (\frac{n'+1}{2})\right]^2 (K_1+K_2)}{2\left[(\frac{n'-1}{2})K_1 + (\frac{n'+1}{2})K_2\right]} T_{out}\right.$$

$$- T_{in}) + \left[k-(\frac{n'+1}{2}) \right] T_m \Big\} \qquad (A.2.46)$$

In these series $\left[k-(\frac{n'+1}{2})\right]$ and $\left[k-(\frac{n'+1}{2})\right]^2$ are the variable terms. Likewise, as n' approaches infinity the values for M_{1t} and M_{2t} are

$$M_{1t} = M_{2t} = M_t \qquad (A.2.47)$$

Thus, $\Phi_2(s)$ is reduced to

$$\Phi_2(s) = -\frac{1}{D_{11}} \frac{dM_t}{ds} \qquad (A.2.48)$$

If the longitudinal temperature variation is given by

$$T_{out} = ps^2 + R \qquad (A.2.49)$$

$$T_{in} = I \qquad (A.2.50)$$

then

$$-\frac{1}{D_{11}} \frac{dM_t}{ds} = + \frac{H'}{2D_{11}} 2S \qquad (A.2.51)$$

or

$$-\frac{1}{D_{11}} \frac{dM_t}{ds} = + \frac{H'}{D_{11}} (L-x) \qquad (A.2.52)$$

where

$$x = L - s$$

$$H' = p\sum_{k=1}^{n'} \frac{\left[k-(\frac{n'+1}{2})\right]^2 (K_1+K_2)E_1^k(\alpha_1^k+\nu_2^k\alpha_2^k)(t^k)^2}{\left[(\frac{n'-1}{2})K + (\frac{n'+1}{2})k\right](1-\nu_1\nu_2)} \qquad (A.2.52.a)$$

Also $\Phi_1(s)$ can be written

$$\Phi_1(s) = -(H^* - \frac{C_{12}}{C_{11}} H^{**})(x - L) \tag{A.2.53}$$

where

$$H^* = 2p\sum_{k=1}^{n'} \frac{E_1^k(\alpha_1^k + \nu_2^k \alpha_2^k)t^k}{1 - \nu_1 \nu_2}$$

$$H^{**} = 2p\sum_{k=1}^{n'} \frac{E_2^k(\alpha_2^k + \nu_1^k \alpha_1^k)t^k}{1 - \nu_1 \nu_2} \tag{A.2.53.a}$$

therefore,

$$\Phi_2(s) - i\sqrt{\frac{C_{11}}{D_{11}\Omega}}\Phi_1(s) = \frac{H^{\circ}}{D_{11}}(x - L)$$

$$+ i\sqrt{\frac{C_{11}}{\Omega D_{11}}}(H^* - \frac{C_{12}}{C_{11}} H^{**})(x - L) \tag{A.2.54}$$

APPENDIX 3

THE MATERIAL PROPERTIES OF RESIN-GLASS

For an orthotropic material the elastic constants are functions of resin-glass proportion, the filament orientation with respect to the load direction, and the material properties in the principal directions. To determine the equivalent elastic constants and Poisson's ratio, the known solution was applied and only the formulated expressions shall be introduced

i.e.

$$E_L = \rho E_f + (1 - \rho)E_r \tag{A.3.1}$$

$$E_t = \sqrt{\frac{4}{\pi}} \sum_{m=1}^{m=\frac{1}{\Delta}} E_{xm}\Delta + E_r\left[1 - \sqrt{\frac{4\rho}{\pi}}\right] \tag{A.3.2}$$

$$\nu_{LT} = \rho\nu_f + (1 - \rho)\nu_r \tag{A.3.3}$$

$$\nu_{TL} = \frac{E_T}{E_L}\nu_{LT} \tag{A.3.4}$$

where

E_f = modulus of elasticity of fiber glass

E_r = modulus of elasticity of resin

ρ = percentage of fiberglass by volune

E_{xm} = generalized transverse modulus elasticity

$\bar{\Delta}$ = non-dimensional height of an element

ν_f = Poisson's ratio of fiber glass

ν_r = Poisson's ratio of resin

The data of elastic constants and Poisson's ratio used in Example 2 are;

$E_f = 10 \times 10^6 \quad lb/in^2$

$E_r = .5 \times 10^6 \quad lb/in^2$

$\nu_f = .2$

$\nu_r = .36$

$\rho = .67$

Using equations (A.3.1, 2, 3, 4) one obtains

$E_L = 6.87 \times 10^6 \ lb/in^2$

$E_T = 2.313 \times 10^6 \ lb/in^2$

$\nu_{LT} = .253$

$\nu_{TL} = 0.083$

Determination of thermal coefficients for the combined material, taking filament structure for analysis, it is to determine the equivalent thermal coefficients in the fiber direction and transverse direction. Examing an element as shown in figure 18.

The thermal coefficient in the (T) direction simply is

$$\alpha_T = f\alpha_f + (1 - f)\alpha_r$$

The thermal coefficient in (L) direction is obtained by solving the compatibility equation

$$\alpha_f - \frac{\sigma f}{E_f} = \alpha_r + \frac{\sigma r}{E_r} = \alpha_L \qquad (A.3.5)$$

and the equilibrium equation

$$f\sigma_f = (1 - f)\sigma_r \qquad (A.3.6)$$

From equation (1)

$$\frac{\sigma f}{E_f} + \frac{\sigma r}{E_r} = \alpha_f - \alpha_r \qquad (A.3.7)$$

from equation (2)

$$\sigma_f = \left(\frac{1 - f}{f}\right)\sigma_r \qquad (A.3.8)$$

substituting equation (A.3.8) in equation (A.3.7), one obtains

$$\sigma_r \quad \frac{\alpha_f - \alpha_r}{\frac{1}{E_r} + \frac{1-f}{f E_f}} \qquad (A.3.9)$$

inserting σ_r value in equation (A.3.5)

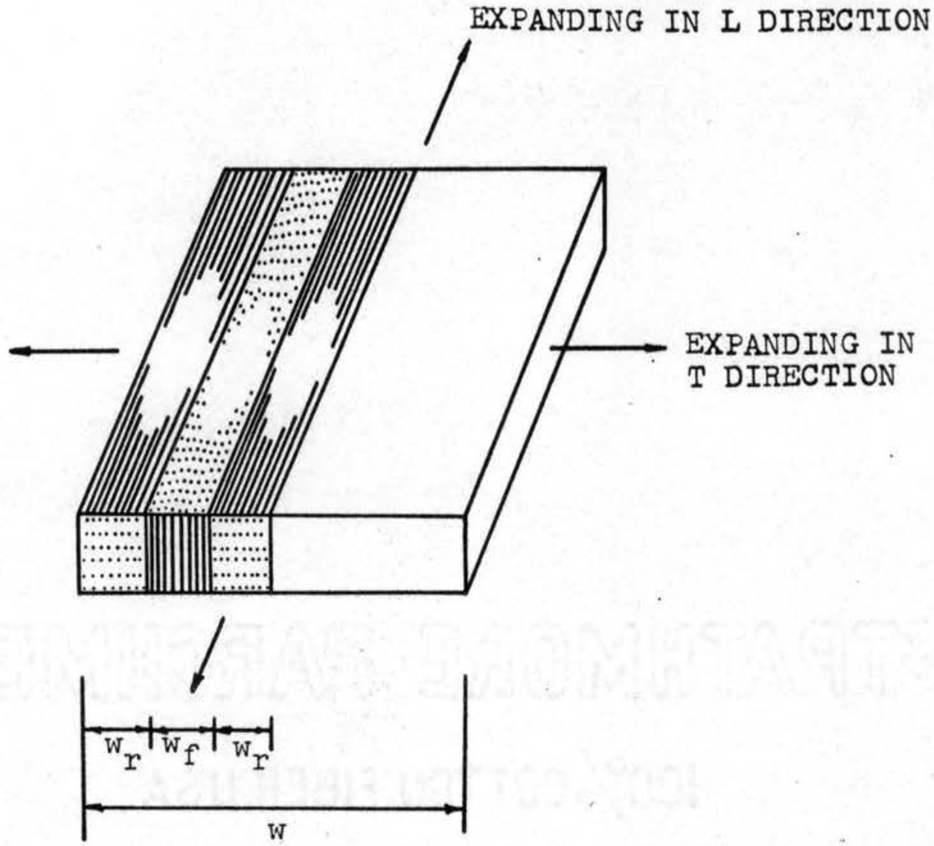

Figure 18. Element of Filament Structure

$$\alpha_L = \alpha_r + \frac{\alpha_f - \alpha_r}{1 + \frac{1-\varphi}{\varphi}\frac{E_r}{E_f}}$$

$$\alpha_L = \frac{\alpha_f + \left(\frac{1-\varphi}{\varphi}\frac{E_r}{E_f}\right)\alpha_r}{1 + \frac{1-\varphi}{\varphi}\frac{E_r}{E_f}} \qquad \text{(A.3.10)}$$

if one uses

$$E_f = 10 \times 10^6 \quad \text{lb/in}^2$$

$$E_r = 0.5 \times 10^6 \quad \text{lb/in}^2$$

APPENDIX 4

THE SIMPLIFICATION OF VALUE $\bar{a}$

The value $\bar{a}$ was obtained through the solution of the quadratic equation (2.5.11) as

$$\bar{a} = -\frac{C_{11}}{2\Omega R_1}\left(\frac{D_{12}}{D_{11}} + \frac{C_{12}}{C_{11}}\right)$$

$$\pm\sqrt{\left[\frac{1}{2\Omega R_1}\left(\frac{D_{12}}{D_{11}} + \frac{C_{12}}{C_{11}}\right)\right]^2 - \frac{C_{11}}{D_{11}\Omega}} \qquad \text{(A.4.1)}$$

By virtue of equations (2.1.17), the following terms were found

$$\frac{C_{11}}{\Omega} = \sum_{k=1}^{n'} \frac{1-\nu_1\nu_2}{t^k E_1^k (1-\nu_2^2)}$$

$$\frac{D_{12}}{D_{11}} = -\nu_2$$

$$\frac{C_{12}}{C_{11}} = -\nu_2$$

Then, it is easily seen that

$$\frac{C_{11}}{2R_1\Omega}\left(\frac{D_{12}}{D_{11}} + \frac{C_{12}}{C_{11}}\right) = \frac{-1}{R_1}\sum_{k=1}^{n'} \frac{\nu_2(1-\nu_1\nu_2)}{t^k E_1^k (1-\nu_2^2)}$$

where $\frac{1}{R_1}$ and the summation of series is very small.

Equation (A.4.1) is, therefore, simplified to $\bar{a} = -i\sqrt{\frac{C_{11}}{D_{11}\Omega}}$.

VITA

TUNG DJU LIN

Candidate for the Degree of

Doctor of Philosophy

Thesis: ANALYSIS OF MULTI-LAYERED, ANISOTROPIC SHELLS OF REVOLUTION SUBJECTED TO AXISYMMETRIC THERMAL GRADIENTS

Major Field: Civil Engineering

Biographical:

Personal Data: Born in Taiwan, Free China, May 10, 1933, the son of H. and A. T. Lin.

Education: Graduated from I-Lan High School, Taiwan, June 1954; received the Bachelor of Science degree from Taiwan Christian College, Taiwan, June 1959; received the Master of Science degree from University of Tennessee, Knoxville, Tennessee, August 1962 with a major in Civil Engineering; completed requirements for the Doctor of Philosophy degree in May 1968.

Professional Experience: Graduate Assistant at Taiwan Christian College, Taiwan, 1959-1960. Engineer at Tennessee Highway Department, Knoxville, Tennessee, 1962-1964. Graduate Assistant at Oklahoma State University, Stillwater, Oklahoma, 1965-1967. Research Engineer at the Prescon Corporation, Corpus Christi, Texas, 1967. Associate member of ASCE.