THE DISTRIBUTION OF CERTAIN FUNCTIONS OF PARAMETERS: PRIOR AND POSTERIOR

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CHAPTER I

INTRODUCTION

In recent years considerable emphasis has been placed on studies relating to subjective probability and Bayesian statistics. We will not discuss the foundations of subjective probability and Bayesian statistics as set forth by Ramsey [8], de Finetti [2], Savage [10], and others, since the objectives of this paper are methodological rather than philosophical. That is, we are concerned with developing methods for actually picking a specific prior distribution from a family of prior distributions which can represent the available prior knowledge. We desire to obtain workable methods and tools for applying the concept of critical betting odds or critical probabilities as described by Lindley [5] and developed by Ramsey [8] and de Finetti [2].

From the applications viewpoint, the Bayesian procedure involves the choosing of a prior distribution $\Pi(\theta)$ to reflect the prior information available about the parameter. Then an experiment is performed involving $f(x|\theta)$ to obtain further information concerning the parameter θ. All information contained in the sample $x_{1,n}$ about θ is contained in the likelihood function $\ell(x_{1,n}|\theta)$. The prior information and sample information are then combined via Bayes theorem to obtain the posterior distribution $\Pi''(\theta|x_{1,n})$ as

$$\Pi''(\theta|x_{1,n}) \propto \ell(x_{1,n}|\theta)\ \Pi(\theta).$$

Bayesian inference is then based upon the posterior distribution of θ. That is, if one desires an interval estimate or a point estimate for θ, it is derived from the posterior distribution of θ. The posterior distribution for θ is regarded as possessing all the available information concerning θ after experimentation. Interval estimates are taken as percentage points of the posterior distribution. To obtain a Bayesian point estimate for θ one introduces the concept of a loss function.

To briefly discuss loss functions, one considers the space A of all possible acts based upon the possible results from available experiments which might be performed. That is, A is thought of as the set of all image points of statistics on the sample space or outcome space for these experiments. Then a function is defined on the cartesian product of the parameter space with the action space A to the set of real numbers. This function, called the loss function, should reflect the penalty or loss incurred by the experimenter from taking action $a(x_{1,n})$ when the value of the parameter is θ.

Assuming that a loss function can be determined, the Bayes point estimate is that $a(x_{1,n})$ in A which minimizes the expected loss. The expectation is taken with respect to the posterior distribution of θ. More precisely, the Bayes point estimate for θ is that $a^*(x_{1,n})$ in A such that

$$E''[L(\theta, a^*(x_{1,n}))] = \min_{A} \int_{\Omega} L(\theta, a(x_{1,n}))\, \Pi''(\theta)\, d\theta .$$

Returning to consideration of prior distributions, there are two classes of prior distributions which are of primary importance: the locally diffuse uniform and the conjugate prior distributions. The locally diffuse uniform prior distribution is a uniform distribution on some unspecified set of real numbers of finite measure. It is, of course, an improper distribution since the range of positive probability for the random variable is not specified. This class of distributions is discussed by Jeffreys [4] and put forth as representing the absence of prior information.

The conjugate prior distributions are discussed by Raiffa and Schlaifer [7]. The distributions in this class are proper distributions and possess a functional form closely related to that of the sampling distribution. To define the class of conjugate prior distributions we need the concept of a kernel of the likelihood function.

<u>Definition 1</u>. Let $\ell(x_{1,n}|\theta)$ be the likelihood of the sample $x_{1,n}$ for given θ and $k(x_{1,n}|\theta)$ be a function such that

$$\ell(x_{1,n}|\theta = r(x_{1,n})\, k(x_{1,n}|\theta),$$

then $k(x_{1,n}|\theta)$ is a <u>kernel</u> of the likelihood and $r(x_{1,n})$ is the residue.

It should be noted that all information in the likelihood concerning θ must be contained in any kernel of that likelihood. That is, a residue must be a function of the sample alone and cannot involve θ.

Conjugate prior distributions, in addition to having a functional form closely related to the sampling distribution $f(x|\theta)$, possess the property of invariance. That is, the conjugate prior distribution and its resulting posterior distribution must belong to the same distributional family. Thus, if one's prior information can be represented by a conjugate prior distribution, then no amount of sample information can change the distributional family for the parameter. We define the conjugate distribution below.

Definition 2. Let $f(y|\theta)$ be a density for y with parameter θ. If $k(y|\theta)$ is a kernel of f, then

$$g(\theta) = N(y)\, k(y|\theta)$$

where

$$[N(y)]^{-1} = \int_{\Omega} k(y|\theta)\, d\theta ,$$

is a conjugate prior distribution for θ.

In Appendix B we give some of the specific conjugate prior distributions for the particular distributions considered in this paper. These are included since in all cases we consider at least the conjugate prior distribution on the parameter.

This paper is a study of functions of parameters and makes an effort to determine the distributional properties of the ordinates of these functions for various prior and posterior distributions on the parameters. In many instances it is a function of the parameters which is of primary interest rather than the explicit parameters. Thus, this is an effort to make a more direct approach to the problem as it relates to Bayesian inference and the assignment of prior distributions.

This study is not without precedent. Lindley [5] shows by taking locally diffuse uniform prior distributions on the parameters in the simple linear regression model, with normally distributed errors, that certain resulting functions of the parameters and the sample have posterior distributions which are identical with the same functions under frequentist theory. Mood and Graybill [6] speak of determining the distribution of the ordinate of chi-square density function in their chapter on hypothesis testing.

The motivation for studying the particular functions included in this paper is the desire to obtain workable intuitive methods for actually assessing the consistency of one's prior information with a given prior distribution. That is, if one has assigned a prior distribution without considering the prior probability content of certain intervals, explicitly in the determination of the specifications for the prior density, then certainly such deliberations should be consistent with those criteria which the experimenter used to determine his prior distribution. Thus, if one has chosen a prior distribution which is completely specified by its mean and variance, using intuitive devices other than interval content, then, if interval content notions are available, their consistency should be evaluated subjectively by the distributional properties of these functions of parameters.

In particular, we study the distributional properties of certain density functions and probability mass functions $f(x_o|\theta)$, as well as their corresponding distribution functions

$$F(x_o|\theta) = \int_{-\infty}^{x_o} f(t|\theta)\, dt,$$

for various prior and posterior distributions on θ. Throughout the study x_o is regarded as a fixed, though arbitrary, value. It can be seen that the function $u = F(x_o|\theta)$ is the content of the interval $(-\infty, x_o]$ for each value of θ. Given a prior or posterior distribution on θ one can determine the prior or posterior distribution for the content of this interval.

If $u = f(x_o|\theta)$ is a probability mass function, then u represents the probability for each θ that $x = x_o$. We therefore study the distribution of this probability for various prior and posterior distributions on θ. If u is a probability density function, then we are determining the prior and posterior distribution of a likelihood.

In general, the plan for presentation of these functions in Chapters I and II is to (1) identify the particular function being considered, (2) give relevant properties possessed by the function and its inverse if the inverse can be determined, (3) identify the density of the measure assigned to the parameter space Ω or the ordinate space S, (4) find the density function for the induced measure on the ordinate space S or the parameter space Ω if possible, both prior and posterior, (5) determine the induced cumulative distribution function if the induced density cannot be explicitly determined, (6) determine the moment generating function and/or exhibit the mean and variance of the ordinate, and (7) determine that member of the conjugate family for the parameter which will induce a uniform measure on the ordinate if one exists.

Bayesian q tolerance limits for p coverage and Bayes estimators for the fraction of acceptable product are considered briefly in Chapter IV as applications for the results of the previous chapters.

Notation

Let X be a random variable which has a chi-square distribution with n degrees of freedom. Then we shall denote its distribution function by $F_{\chi^2}(x_o|n)$ evaluated at x_o. The solution for t in $F_{\chi^2}(t|n) = q$ will be denoted by $\chi_q^2(n)$. We will in general denote the prior distribution on the parameter by Π and the posterior by Π''. A standard normal deviate will be denoted by z, its distribution function at $z = t_o$ by $F_z(t_o)$, and the solution for t in $F_z(t) = q$ is represented as z_q. If a deviate U has a Student's t-distribution with r degrees of freedom, the distribution function for U evaluated at $U = u_o$ is given as $F_t(u_o|r)$, the upper q probability point is denoted by $t_q(r)$. The expression $\Pr\{u:\text{statement}\} = q$ will be interpreted as follows: Let $S(u)$ be the values of u which satisfy the statement, then

$$q = \int_{S(u)} f(u)\, du ,$$

where $f(u)$ is the probability density function for u.

CHAPTER II

DENSITY FUNCTIONS

Let $u = f(x_o | \theta)$, where $f(x|\theta)$ is a probability density function and x_o is a specified value of the random variable x. In this case u is regarded as a function of θ for θ in some subset of the real numbers which we denote by Ω. If we further suppose that u is a unimodal differentiable function of θ such that $f(x_o | -\infty) = f(x_o | -\infty) = 0$, then there exists a point $\theta^* \epsilon \Omega$ such that u is a monotone increasing differentiable function of θ for $\theta < \theta^*$, and u is a monotone decreasing differentiable function for $\theta > \theta^*$. Then for each value of u with $0 < u < f(x_o | \theta^*)$ there are two distinct values of θ, say θ_1 and θ_2 such that $u_o = f(x_o | \theta_1) = f(x_o | \theta_2)$. So we can partition the domain Ω of θ in such a way that Ω is mapped doubly onto the domain of u. There are two functions $u = f_1(x_o | \theta)$ for $-\infty < \theta < \theta^*$ and $u = f_2(x_o | \theta)$ for $\theta^* < \theta < \infty$ which are monotone and differentiable, hence they have inverses, say $\theta = f_1^{-1}(u;x_o)$ and $\theta = f_2^{-1}(u;x_o)$.

Now the distribution function of u is given by

$$F(u_o;x_o) = \Pr \{u \leq u_o\} = \int_{-\infty}^{f_1^{-1}(u_o;x_o)} \Pi(\theta)\, d\theta + \int_{f_2^{-1}(u_o;x_o)}^{\infty} \Pi(\theta)\, d\theta \qquad (2.1)$$

where $\Pi(\theta)$ is the density of a measure, either prior or posterior, on Ω and $0 \leq u_o \leq f(x_o | \theta^*)$. Now since $f_1^{-1}(u;x_o)$ and $f_2^{-1}(u;x_o)$ are monotone differentiable functions, the density for u exists and is given by

$$f(u;x_o) = \Pi(f_1^{-1}(u;x_o)) \frac{df_1^{-1}(u;x_o)}{du} - \Pi(f_2^{-1}(u;x_o)) \frac{df_2^{-1}(u;x_o)}{du} \qquad (2.2)$$

So, for any continuously differentiable unimodal function of a single parameter, an arbitrary measure with density Π on Ω induces a measure on the space of u which has a density given by (2.2). However, in actual application for most of the cases considered, the functions f_1^{-1} and f_2^{-1} cannot be exhibited in simple explicit form. In the following paragraphs we examine several density functions belonging to this class.

Binomial Probability Ordinate

For this case we take

$$u = \binom{n}{x_o} p^{x_o} (1-p)^{n-x_o} \qquad (2.3)$$

for $x_o = 0, 1, \ldots, n$ specified. We take a uniform density on p and determine that

$$h(u) = \frac{1}{n} u^{-\frac{n-1}{n}}$$

for $x_o = 0, n$. If $x_o \neq 0, n$ the density for u has not been determined in explicit terms.

The distribution function for u relative to a uniform density on p can be determined as

$$F(u_o) = \Pr\{u \leq u_o\} = \Pr\{p \leq p_1\} + \Pr\{p \geq p_2\} \qquad (2.4)$$

where

$$u_o = \binom{n}{x_o} p_1^{x_o} (1-p_1)^{n-x_o} = \binom{n}{x_o} p_2^{x_o} (1-p_2)^{n-x_o}$$

with $0 \leq p_1 \leq \frac{x_o}{n}$ and $\frac{x_o}{n} \leq p_2 \leq 1$. But Pr $\{p \leq p_1\} = p_1$ and Pr $\{p \geq p_2\} = 1 - p_2$, hence (2.4) implies

$$F(u_o) = p_1 + 1 - p_2 . \tag{2.5}$$

This completely determines the distribution of u for a uniform density on p.

The posterior distribution for u is given by

$$F''(u_o) = \frac{\Gamma(mn+2)}{\Gamma(m\bar{x}+1)\,\Gamma(mn-m\bar{x}+1)} \left[\int_o^{p_1} p^{m\bar{x}} (1-p)^{mn-m\bar{x}} dp + \int_{p_2}^{1} p^{m\bar{x}} (1-p)^{mn-m\bar{x}} dp \right] \tag{2.6}$$

where p_1 and p_2 are as specified in (2.4) and m is the sample size. Hence, posterior percentage points for u can be determined by consulting tables of the incomplete beta distribution function.

For example, suppose we have taken a sample of size m = 5 and obtained $\bar{x} = 5$ from a binomial population with n = 10. If we are concerned about $x_o = 3$ and $u_o = 0.05$, then it is determined that $p_1 = 0.09405$ and $p_2 = 0.58656$.

If a uniform distribution is taken on p, then the value of the posterior distribution function for $u_o = .05$ is given by

$$F(.05) = \int_0^{.09405} \beta(t; 26, 25)dt + \int_{.58656}^{1} \beta(t; 26, 25)dt$$

$$= 0 + .15879524$$

$$= 0.158 .$$

For a uniform density on p the moment generating function is obtained by considering

$$M_u(t) = E(e^{tu}).$$

Then we expand the exponential in series and interchange the order of summation and integration to obtain the form

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(u^k). \tag{2.7}$$

Finally, we substitute for u and recognize the complete beta function $\beta(kx_o + 1, k(n-x_o) + 1)$ to obtain

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{(kn+1)(k!)} \frac{\binom{n}{x_o}^k}{\binom{kn}{kx_o}} . \tag{2.8}$$

We obtain the prior mean and variance of u as

$$E[u] = \frac{1}{n+1} \quad \text{and} \quad V(u) = \frac{\binom{n}{x_o}^2}{(2n+1)\binom{2n}{2x_o}} - \frac{1}{(n+1)^2} \tag{2.9}$$

for any x_o directly from (2.8). We see that E(u) is just the prior marginal distribution of x for a uniform prior on p. That is, the prior marginal distribution on x_o, for a uniform prior on p, is a uniform distribution. Regarding the variance of u as a function of x_o, we note that

$$V(u_{n-x_o}(p)) = V(u_{x_o}(p))$$

where

$$u_{x_o}(p) = \binom{n}{x_o} p^{x_o} (1-p)^{n-x_o} .$$

It can also be seen that u has maximum variance when $x_o = 0$ or $x_o = n$. We see that u has minimum variance for $x_o = \frac{n}{2}$, if n is even, or two points of minimum variance, $x_o = \frac{n-1}{2}$ and $x_o = \frac{n+1}{2}$, if n is odd.

The truth of this statement can be seen by observing that

$$V(u_{x_o}(p)) > V(u_{x_o+1}(p)) \quad \text{for} \quad x_o < \frac{n-1}{2} .$$

That is, the variance is monotone decreasing for x_o in

$$\{ x:0 \leq x < \frac{n-1}{2}, \quad x \text{ an integer}\} ,$$

and for x_o in

$$\{ x:\frac{n+1}{2} < x \leq n, \quad \text{integer } x \}$$

the variance of u is monotone increasing.

This is intuitively reasonable since the mode of u occurs at $\frac{x_o}{n}$; and, as we look across the various values of x_o, we believe the minimum range for u occurs at either $x_o = \frac{n-1}{2}$, $\frac{n+1}{2}$ for n odd or at $x_o = \frac{n}{2}$ if n is even, though we do not have a rigorous proof of this conjecture.

The posterior moment generating function can be obtained from a result we will obtain in a following paragraph, as well as the posterior mean and variance.

Now let us take the natural conjugate prior on p, the beta density, (see Appendix B)

$$\Pi(p; \nu_1, \nu_2) = \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} p^{\nu_1-1} (1-p)^{\nu_2-1}, \quad \nu_1, \nu_2 > 0, \ 0 \leq p \leq 1. \tag{2.10}$$

The induced densities on u can be obtained for $x_o = 0$, n because in these cases u is a monotone function of p. We solve for p as

$p = 1-u^{1/n}$ for $x_o = 0$, and if $x_o = n$, then $p = u^{1/n}$, which yields the densities on u:

$$h(u; x_o = 0) = \frac{\Gamma(\nu_1+\nu_2)}{n\Gamma(\nu_1)\Gamma(\nu_2)} (1-u^{1/n})^{\nu_1-1} (u^{1/n})^{\nu_2-n}$$

$$h(u; x_o = n) = \frac{\Gamma(\nu_1+\nu_2)}{n\Gamma(\nu_1)\Gamma(\nu_2)} (u^{1/n})^{\nu_1-n} (1-u^{1/n})^{\nu_2-1}$$

The explicit form of the prior densities induced on u for $x_o = 1, \ldots, (n-1)$ has not been obtained, since we cannot solve for the inverse functions of p in terms of u. Indeed, the inverse relations are not functions for these cases. Also, we will see that the prior moment generating function for u is not that of a known distribution.

The prior distribution function of u for (2.10) on p is given as

$$F(u; \nu_1, \nu_2, x_o) = \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \left[\int_0^{p_1} p^{\nu_1-1} (1-p)^{\nu_2-1} \, dp + \int_{p_2}^{1} p^{\nu_1-1} (1-p)^{\nu_2-1} \, dp \right] \tag{2.11}$$

where p_1 and p_2 are as specified in (2.4).

The prior moment generating function for u under (2.10) is obtained from (2.7). Upon substitution of u in terms of p we obtain

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \binom{n}{x_o}^k \frac{1}{\beta(\nu_1, \nu_2)} \int_0^1 p^{\nu_1+kx_o-1}(1-p)^{\nu_2+k(n-x_o)-1} \, dp.$$

Recognizing the integral as a beta function and simplifying we get the result

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\binom{n}{x_o}^k}{\binom{kn}{kx_o}} \frac{\binom{kx_o+\nu_1-1}{\nu_1-1}\binom{k(n-x_o)+\nu_2-1}{\nu_2-1}}{\binom{kn+\nu_1+\nu_2-1}{\nu_1+\nu_2-1}} \tag{2.12}$$

which implies

$$E[u] = \frac{\binom{x_o+\nu_1-1}{\nu_1-1}\binom{n-x_o+\nu_2-1}{\nu_2-1}}{\binom{n+\nu_1+\nu_2-1}{\nu_1+\nu_2-1}} .$$

If ν_1 and ν_2 are integers, we can interpret this prior mean of u relative to (2.10) as the marginal prior distribution on x. It is recognized as a hypergeometric waiting-time distribution. That is, suppose we have a population of $n+\nu_1+\nu_2-1$ elements consisting of $\nu_1+\nu_2-1$ c's and n $\bar{c}$'s. Let elements be drawn successively until exactly ν_1 c's are drawn. Then E(u) gives the probability that exactly $x_o+\nu_1$ elements will have to be drawn to obtain ν_1 c's. If ν_1 and ν_2 are not integers, then E(u) is not a hypergeometric waiting-time distribution. It is, however, the marginal prior distribution for x.

The prior variance of u can be obtained directly from (2.12); however, one observes that it is a rather complicated expression for this case.

As was previously promised, the posterior moment generating function for a uniform prior distribution on p can be obtained from the result (2.12). That is, if we replace (ν_1, ν_2) by $(m\bar{x}+1,\ m(n-\bar{x}))$ in (2.12), the resulting expression is the desired moment generating function.

One might ask if there is a member of the natural conjugate family of densities which will induce a uniform distribution on u. To answer this question, we consider the moment generating function for u with a uniform density on u, namely

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{(k+1)!} \tag{2.13}$$

and compare corresponding terms of (2.12) and (2.13). That is, we must be able to determine ν_1 and ν_2 such that

$$\frac{\binom{n}{x_o}^k}{\binom{kn}{kx_o}} \frac{\binom{kx_o+\nu_1-1}{\nu_1-1}\binom{k(n-x_o)+\nu_2-1}{\nu_1-1}}{\binom{kn+\nu_1+\nu_2-1}{\nu_1+\nu_2-1}} \equiv \frac{1}{k} \tag{2.14}$$

for every positive integer k. However, if we consider the special case of n=2, x_o=1, we see that no values exist such that (2.14) is an identity in k. Thus, there exists no beta distribution which will induce a uniform distribution on all binomial ordinates.

Poisson Probability Ordinate

If we take

$$u = \frac{\lambda^{x_o} e^{-\lambda}}{x_o!} \qquad \lambda > 0, \quad x_o = 0, 1, \ldots \quad \text{(fixed)}$$

then u is a unimodal differentiable function of λ, the mode occurring at $\lambda = x_o$. Hence, for any arbitrary density on λ, a density function exists for the measure induced on the space of u. For an arbitrary density Π on λ we have

$$h(u; x_o = 0) = \Pi(\log \frac{1}{u}) \cdot \frac{1}{u} \quad 0 \leq u < 1.$$

The explicit form of the density induced on u for an arbitrary density on λ for $x_o = 1, 2, \ldots,$ has not been determined. An inverse function for λ in terms of u does not exist, since u is a unimodal function of λ for this set of values for x_o.

Taking the natural conjugate prior density on λ for this case, we have

$$\Pi(\lambda; a, b) = \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} \qquad a, b > 0; \quad \lambda > 0 \qquad (2.15)$$

a gamma-1 density. (See Appendix B)

The prior distribution function for u induced by (2.15) is given by

$$F(u_o | x_o) = \int_0^{\lambda_1} \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} dx + \int_{\lambda_2}^{\infty} \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} dx \qquad (2.16)$$

where

$$u_o = \frac{\lambda_1^{x_o} e^{-\lambda_1}}{x_o!} = \frac{\lambda_2^{x_o} e^{-\lambda_2}}{x_o!}$$

with $0 \leq \lambda_1 \leq x_o$ and $x_o \leq \lambda_2 < \infty$ while

$$0 \leq u_o \leq \frac{x_o^{x_o} e^{-x_o}}{x_o!} .$$

Alternatively, we may write (2.16) as

$$F(u_o; x_o, a, b) = F_{\chi^2}(2\lambda_1 b | 2a) + 1 - F_{\chi^2}(2\lambda_2 b | 2a).$$

The posterior distribution function for u is obtained from (2.16) by replacing a and b with $a'' = n\bar{x} + a$ and $b'' = b + n$ respectively, where n is the sample size.

The moment generating function for u relative to (2.15) on λ is found by considering equation (2.7). Then, after substitution for u in terms of λ and interchanging the order of summation and integration, we obtain

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!\,(x_o!)^k} E(\lambda^{kx_o} e^{-k\lambda}).$$

Evaluating the resulting integral by recognizing the gamma function involved, we obtain the desired result

$$M_u(t) = \frac{b^a}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\Gamma(kx_o + a)}{[\Gamma(x_o+1)]^k (k+b)^{kx_o+a}}, \qquad (2.17)$$

from which we obtain directly the prior mean and variance as

$$E(u) = \binom{x_o + a - 1}{a-1} \left(\frac{b}{b+1}\right)^a \left(\frac{1}{b+1}\right)^{x_o},$$

and

$$V(u) = \frac{b^a}{\Gamma(a)\,(x_o!)^2} \left[\frac{\Gamma(2x_o + a)}{(2+b)^{2x_o+a}} - \frac{b^a}{\Gamma(a)} \frac{[\Gamma(x_o+a)]^2}{(1+b)^{2(x_o+a)}} \right].$$

We note that, if a is an integer, then E(u) is a negative binomial distribution. That is, for a gamma-1 prior density on λ, the prior marginal on x is a binomial waiting-time distribution. If a is not an integer, then E(u) is still the prior marginal distribution for x, but it is not a negative binomial. Also, we note that the posterior distribution function, moment generating function, mean, and variance are obtained from the prior, if (a, b) is replaced by $(n\bar{x} + a, n + b)$.

Negative Exponential Density Ordinate

Take $u = \theta e^{-x_o\theta}$, $\theta > 0$ with $x_o > 0$ specified. Then u is a unimodal differentiable function of θ with the mode occurring at $\theta = \frac{1}{x_o}$. We know a density exists for u if a density is placed on θ.

The natural conjugate prior density on θ is given by (2.15) from which it follows that the prior cumulative distribution on u is given by

$$F(u_o; a, b, x_o) = \frac{b^a}{\Gamma(a)}\left[\int_0^{\theta_1} \theta^{a-1} e^{-b\theta}\, d\theta + \int_{\theta_2}^{\infty} \theta^{a-1} e^{-b\theta}\, d\theta\right], \qquad (2.18)$$

$0 \le u_o \le \frac{1}{x_o e}$, and θ_1, θ_2 are such that $0 \le \theta_1 \le \frac{1}{x_o}$, $\frac{1}{x_o} \le \theta_2 < \infty$, where $u_o = \theta_1 e^{-x_o\theta_1} = \theta_2 e^{-x_o\theta_2}$. The posterior cumulative distribution function for u is given by (2.18) with (a, b) replaced by (a", b") where $a'' = a+n$ and $b'' = b+n\bar{x}$.

The moment generating function for u relative to the natural conjugate prior is found by considering equation (2.7) and substituting for u in terms of θ to obtain

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(\theta^k e^{-k\theta}).$$

By recognizing the gamma function involved, we evaluate the integral to obtain

$$M_u(t) = \frac{b^a}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\Gamma(k+a)}{(b+kx_o)^{a+k}}. \qquad (2.19)$$

We obtain directly from (2.19) the mean and variance of u as

$$E(u) = \frac{ab^a}{(b+x_o)^{a+1}} \qquad (2.20)$$

and

$$V(u) = ab^a \left[\frac{a+1}{(b+2x_o)^{a\,2}} - \frac{ab^a}{(b+x_o)^{2(a+1)}} \right].$$

We note that both the mean and variance are decreasing functions of x_o, as should be expected, since u is a decreasing function of x_o. The mean of u is also the prior marginal density of x. It can be seen that this distribution is related to a beta distribution. That is, if we let

$$y = \frac{b}{b+x},$$

then y is distributed as $\beta(y; a, 1)$.

We conjecture from (2.19) that no choice of (a, b) in (2.14) will induce a uniform prior on u. That is, there are no real numbers (a, b) such that

$$\frac{\Gamma(k+a)}{(b+kx_o)^{a+k}} = \frac{e^{-k}}{(k+1)\,x_o^k} \qquad (2.21)$$

for all positive integers k with $x_o > 0$. We state this as a conjecture because a rigorous proof of this has not occurred to us.

Ordinate of Normal Density

We shall consider three cases, namely σ known, μ known, and μ and σ unknown. In all cases the prior distribution $\Pi(\theta)$ will be the natural conjugate prior.

The function under consideration is

$$u = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x_o-\mu)^2}{2\sigma^2}}, \quad \sigma > 0, \quad -\infty < \mu < \infty.$$

Case 1: σ known.

We have

$$\Pi(\mu|a,b,\sigma) = \sqrt{\frac{b}{2\pi}}\ \frac{1}{\sigma}\ e^{-\frac{b(\mu-a)^2}{2\sigma^2}} \qquad b,\sigma > 0. \tag{2.22}$$

u is a unimodal differential function of μ with mode at $\mu = x_o$ and

$$0 \le u \le \frac{1}{\sqrt{2\pi}\ \sigma} .$$

We see that u is symmetric about x_o and furthermore, we can solve explicitly for the functions $\mu = f_1^{-1}(u;x_o)$ and $\mu = f_2^{-1}(u;x_o)$; namely,

$$\mu = f_1^{-1}(u;x_o) = x_o - \sigma \sqrt{\log \frac{1}{2\pi\sigma^2 u^2}} \tag{2.23}$$

and

$$\mu = f_2^{-1}(u;x_o) = x_o + \sigma \sqrt{\log \frac{1}{2\pi\sigma^2 u^2}} .$$

From (2.23) we see that

$$\frac{df_1^{-1}(u;x_o)}{du} = \frac{\sigma}{u} \left[\log \frac{1}{2\pi\sigma^2 u^2}\right]^{-\frac{1}{2}} = -\frac{df_2^{-1}(u;x_o)}{du}. \tag{2.24}$$

By setting up the distribution function for u and taking the derivative of the resulting integrals involving $f_1^{-1}(u;x_o)$ and $f_2^{-1}(u;x_o)$ with respect to u according to (2.24), factoring out common terms and recognizing the hyperbolic cosine we obtain the induced prior density on u as

$$h(u;a,b,x_o,\sigma) = \frac{2b^{\frac{1}{2}}(2\pi)^{\frac{b-1}{2}}\sigma^b u^{b-1}}{\left[\log \frac{1}{2\pi\sigma^2 u^2}\right]^{1/2}} e^{-\frac{b}{2\sigma^2}(x_o-a)^2} \cdot \tag{2.25}$$

$$\cdot \operatorname{Cosh}\left[\frac{b(x_o-a)}{\sigma}\left(\log \frac{1}{2\pi\sigma^2 u^2}\right)^{1/2}\right].$$

The posterior distribution on u is obtained by replacing (a, b) in (2.25) with (a", b") where

$$a'' = \frac{ba + n\bar{x}}{b+n} \quad \text{and} \quad b'' = b + n.$$

The moment generating function for u is obtained by considering (2.7). Substituting for u in terms of μ and taking constants from beneath the integral we obtain

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!\,(2\sigma^2)^{k/2}} E(e^{\frac{-k(x_o-\mu)^2}{2\sigma^2}}).$$

We use (2.22) to take the expectation where we combine the exponents and remove the constant

$$\frac{\sigma}{(kb+\sigma^2)^{1/2}} e^{\frac{-k(x_o-a)^2}{2(kb+\sigma^2)^{1/2}}}$$

from beneath the integral to leave the integral equal to unity and obtain

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{e^{\frac{-k(x_o-a)^2}{2(kb+\sigma^2)}}}{(2\pi)^{k/2}\sigma^{k-1}(kb+\sigma^2)^{1/2}} \tag{2.26}$$

for the density (2.22) on μ.

If we had a uniform density on u for

$$0 < u \leq \frac{1}{\sqrt{2\pi}\,\sigma} \quad ,$$

the moment generating function would be

$$M_u^*(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \quad \frac{1}{(\sqrt{2\pi}\,\sigma)^k\,(k+1)} \quad . \tag{2.27}$$

To see that there exist no real numbers a and b with $b > 0$ such that (2.22) induces a uniform density on u, we compare terms of (2.26) and (2.27). We note that if they did exist, we would require that

$$\frac{1}{k+1} = \frac{\sigma\, e^{-\frac{k(x_o-a)^2}{2(kb+\sigma^2)}}}{(kb+\sigma^2)^{1/2}} \quad ,$$

for some a and b with $b > 0$ be satisfied for every positive integer k. This can be solved for $(x_o-a)^2$ to yield the identity in k

$$(x_o-a)^2 = \frac{(kb+\sigma^2)}{k} \quad \log \frac{\sigma^2(k+1)^2}{kb+\sigma^2} \quad . \tag{2.28}$$

Thus, the right side of (2.28) must be independent of k which means that if there is a real number b satisfying (2.28) then we must have

$$\left(\frac{4\sigma^2}{b+\sigma^2}\right)^{b+\sigma^2} \left(\frac{kb+\sigma^2}{\sigma^2\,(k+1)^2}\right)^{\frac{b+\sigma^2}{k}} = 1 \tag{2.29}$$

for all k. Hence, taking the limit of the left side of (2.29) as k becomes unbounded, we see the limit is zero which contradicts the existence of a real number b satisfying (2.29). This fact means that no b will satisfy (2.28). Thus, no pair (a, b) exists which will induce a uniform density on u.

For this case in our study of the normal ordinate, if we assign a uniform density to u and agree to assign equal measures to each of the sets $\{\mu : \mu < x_o - y\}$ and $\{\mu : \mu > x_o + y\}$ for each positive real number y, the density induced on μ is

$$h(\mu|x_o, \sigma) = \frac{|x_o - \mu|}{2\sigma^2} e^{-\frac{(\mu - x_o)^2}{2\sigma^2}} \quad -\infty < \mu < \infty. \qquad (2.30)$$

One observes that h is a bimodal density function with modes occuring at $\mu = x_o \pm \sigma$. The density function (2.30), sketched below in Figure 1, is symmetric about $\mu = x_o$. Its mean is at x_o, and it has a variance of $2\sigma^2$.

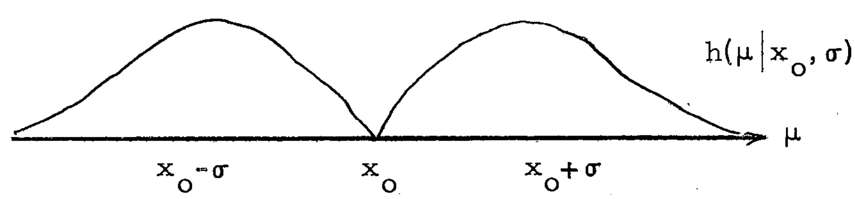

Figure 1

The moment generating function for μ is given by

$$M_\mu(t) = e^{x_o t} \left[1 - t\sigma\sqrt{\frac{\pi}{2}} + \sqrt{2}\; t\sigma \int_{\frac{t\sigma}{2}}^{\infty} e^{-z^2} dz\right] \qquad (2.31)$$

If we take a sample of size n from $N(x; \mu, \sigma^2)$ using (2.30) as the prior measure on μ, we obtain the posterior distribution on μ as

$$h''(\mu \mid x_o, \sigma) = k \frac{|\mu - x_o|}{2\sigma^2} e^{-\frac{n+1}{2\sigma^2}\left(\mu - \frac{n\bar{x}+x_o}{n+1}\right)^2} \qquad (2.32)$$

where

$$k^{-1} = \frac{1}{n+1} e^{-\frac{n^2}{n+1}\frac{(\bar{x}-x_o)^2}{2\sigma^2}} + \frac{n(\bar{x}-x_o)}{\sigma(n+1)^{3/2}}\left[\int_{\frac{n}{\sqrt{n+1}}\frac{(\bar{x}-x_o)}{\sigma}}^{\infty} e^{-\frac{z^2}{2}} dz - \sqrt{\frac{\pi}{2}}\right].$$

Using equations (2.23) and result (2.32) we obtain the posterior distribution function for u, from which we determine the density of the posterior distribution on u as

$$f''(u \mid \sigma, x_o, x_{1,n}) = k(2\pi\sigma^2 u^2)^{n/2} \sqrt{2\pi}\, \sigma\, e^{\frac{-n^2(\bar{x}-x_o)^2}{2(n+1)\sigma^2}} \cdot$$

$$\cdot \cosh\left[\frac{n(\bar{x}-x_o)}{\sigma}\left(\log \frac{1}{2\pi\sigma^2 u^2}\right)^{1/2}\right] \qquad (2.33)$$

$0 \le u \le \frac{1}{\sqrt{2\pi}\,\sigma}$ where k is given in (2.32).

We note the posterior density of u will be the same whether $\bar{x} > x_o$ or $\bar{x} < x_o$, since $f''(u \mid \sigma, x_o, x_{1,n})$ is the same if $(\bar{x}-x_o)$ is replaced by $-(\bar{x}-x_o)$.

The moment generating function for u is given by

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{1}{(\sqrt{2\pi}\,\sigma)^k} e^{\frac{-kn^2(\bar{x}-x_o)^2}{2\sigma^2}} c\ I\ (k) \qquad (2.34)$$

where c = k in (2.32) and

$$I(k) = \frac{2\sigma n}{\sqrt{b}\,(n+k+1)}\,(x_o-\bar{x})\,F_z\left(\frac{\sqrt{n+k+1}\,\,(n(x_o-\bar{x}))}{2\sigma(n+k+1)}\right) + \frac{2\sigma^2}{n+k+1}\, e^{\frac{-n^2(x_o-\bar{x})^2}{2\sigma^2(n+k+1)}}.$$

Case 2. μ known.

For this case the natural conjugate prior is the inverted-gamma-2 density, whose form is

$$\Pi(\sigma;\nu,v) = \frac{2\left(\frac{\nu v}{2\sigma^2}\right)^{\frac{\nu+1}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{\nu v}{2}\right)^{1/2}}\, e^{-\frac{\nu v}{2\sigma^2}}. \qquad (2.35)$$

For this case u is a unimodal differentiable function of σ whose mode occurs at $\sigma = |x_o-\mu|$. We know a density exists for u, but we have not obtained it.

The prior distribution function for u is given by

$$F\,(u_o|x_o,\nu,v,\mu) = \int_0^{\sigma_1} \Pi(\sigma;\nu,v)\,d\sigma + \int_{\sigma_2}^{\infty} \Pi(\sigma;\nu,v)\,d\sigma \qquad (2.36)$$

where

$$u_o = \frac{1}{\sqrt{2\pi}\,\sigma_1}\, e^{\frac{-(x_o-\mu)^2}{2\sigma_1^2}} = \frac{1}{\sqrt{2\pi}\,\sigma_2}\, e^{\frac{-(x_o-\mu)^2}{2\sigma_2^2}}$$

and Π is given by (2.35) and $0 < \sigma_1 < |x_o-\mu| < \sigma_2 < \infty$.

The moment generating function for u relative to (2.35) is obtained by considering (2.7). Removing constants from under the integral, we obtain

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!\,(2\pi)^{k/2}} \, E\left[\left(\frac{1}{\sigma^2}\right)^{k/2} e^{\frac{-k(x_o-\mu)^2}{2\sigma^2}} \right]. \qquad (2.37)$$

The inverted-gamma-2 density is recognized under the integral and evaluated to give

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{1}{\pi^{k/2}} \frac{(\nu v)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\Gamma\left(\frac{\nu+k}{2}\right)}{\left[\nu v + k(x_o-\mu)^2\right]^{(\nu+k)/2}} \qquad (2.38)$$

which yields

$$E[u] = \frac{1}{\sqrt{\pi}} \frac{(\nu v)^{\nu/2}\,\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left[\nu v + (x_o-\mu)^2\right]^{(\nu+1)/2}}$$

$$= \frac{1}{\sqrt{\nu\pi}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left[1 + \frac{\frac{(x_o-\mu)^2}{v}}{\nu}\right]^{-\frac{\nu+1}{2}}.$$

We note that $E[u]$ is a specific value of Student's t-density with ν degrees of freedom. That is, the marginal prior distribution of x for an inverted gamma-2 prior on σ is Student's t with ν degrees of freedom.

The posterior distribution function and moment generating function is obtained by replacing (ν, v) by

$$\left(\nu+n, \ \frac{1}{\nu+n}\left(\nu v + \sum_{i=1}^{n} (x_i - \mu)^2\right)\right)$$

in (2.36) and (2.38) respectively.

Case 3. μ and σ unknown.

The natural conjugate prior distribution on (μ, σ) is the normal gamma density. (See Appendix B.)

$$\Pi(\mu, \sigma \mid a, b, \nu, v) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-b(x_o-\mu)^2}{2\sigma^2}} \frac{2\left(\frac{\nu v}{2\sigma^2}\right)^{(\nu+1)/2}}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{\nu v}{2}\right)^{1/2}} e^{\frac{-\nu v}{2\sigma^2}} \tag{2.39}$$

For this case the prior cumulative distribution function for u involves the evaluation of a double integral which we choose to represent as

$$F(u_o) = 1 - \int_0^{(2\pi u_o^2)^{-1/2}} \left\{ F_z\left(\frac{\sqrt{b}}{2}\left(x_o - a + \sigma\sqrt{\log\frac{1}{2\pi u_o^2\sigma^2}}\right)\right) - F_z\left(\frac{\sqrt{b}}{\sigma}\left(x_o - a - \sigma\sqrt{\log\frac{1}{2\pi u_o^2\sigma^2}}\right)\right)\right\} f_{i\gamma 2}(\sigma; \nu, v)\, d\sigma . \tag{2.40}$$

The posterior is obtained by replacing (a, b, ν, v) by $(a'', b'', \nu'', v''$ where

$$a'' = \frac{ba + n\bar{x}}{b+n}, \quad b'' = b + n$$

$$\nu'' = \begin{cases} \nu + n, & b > 0 \\ \nu + n - 1, & b = 0 \end{cases}$$

$$v'' = \frac{1}{\nu''}\left[\nu v + ba^2 + \Sigma x_i^2 - b''a''^2\right].$$

The moment generating function for u is given by considering equation (2.37). The expectation is taken with respect to (2.39). We integrate first with respect to μ and then with respect to σ to obtain

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sqrt{\frac{b}{b+k}} \; \frac{\Gamma \frac{\nu+k}{2}}{\Gamma \frac{\nu}{2} \; (\pi\nu v)^{k/2}} \left[1 + \frac{\frac{bk(x_o-a)^2}{v(b+k)}}{\nu} \right]^{-\frac{\nu+k}{2}} \qquad (2.41)$$

which gives

$$E[u] = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left[1 + \frac{\frac{b(x_o-a)^2}{v(b+1)}}{\nu} \right]^{-\frac{\nu+1}{2}} \sqrt{\frac{b}{(b+1)v}} \; .$$

We note with

$$t = \sqrt{\frac{b}{v(b+1)}} \; (x_o-a),$$

$E[u]$ is the value of Student's t-density with ν degrees of freedom, and is also the marginal prior distribution of x_o. That is, the marginal prior distribution of x for a normal-gamma prior distribution on (μ, σ) is given by $E(u)$. The posterior moment generating function is obtained by replacing (a, b, ν, v) by (a'', b'', ν'', v'') above.

CHAPTER III

DISTRIBUTION FUNCTIONS

Introduction

We will consider functions of the form $u = F(x_o | \theta)$ where F is a distribution function for the random variable x. We regard u as a function of θ for a fixed, but arbitrary, value of x which we denote by x_o. Distributional properties of u will concern us throughout this chapter as well as physical interpretations for u. In all cases u is the probability that $x \leq x_o$, given θ. We study the distribution of this probability for various prior measures on Ω, the space of θ. In some instances x_o may be regarded as an engineering upper specification limit in which case u would be the fraction of acceptable product. If x_o is regarded as a Bayesian upper tolerance limit, then u is the coverage for the resulting interval. Thus, we see that physical interpretations for u are available for this class of functions of parameters.

It is also of interest to assign a prior measure to S, the space of u and determine the induced measure on Ω the space of θ. This is possible provided $u = F(x_o | \theta)$ is a monotone function of θ. Otherwise, a unique measure is not induced on Ω. In this line of thought, the following theorem gives one result of this approach which is useful in determining that conjugate prior distribution which induces a uniform distribution on u.

Theorem: If $f(x)$ is a continuous probability density function and a parameter θ enters the density of a random variable T for given θ in the form of $x = \theta t$ or $x = \theta + t$ such that $h(t|\theta) = f(\theta t) \cdot \theta$ or $h(t|\theta) = f(\theta + t)$, then assigning a uniform prior distribution to

$$u = \int_{-\infty}^{t_o} h(t|\theta)\, dt$$

induces a measure on θ which has a density belonging to the conjugate family for θ.

Proof: We suppose the density for u is given by

$$\Pi(u) = \frac{1}{a}, 0 \leq u \leq a$$

where

$$a = \max_{\theta \in \Omega} \int_{-\infty}^{t_o} h(t|\theta)\, dt.$$

By hypothesis

$$u = \int_{-\infty}^{t_o} f(\theta t) \cdot \theta dt \quad \text{or} \quad \int_{-\infty}^{t_o} f(\theta + t)\, dt.$$

Then making a change of variable transformation on the integral, we see that

$$u = \int_{-\infty}^{\theta t_o} f(x)\, dx \quad \text{or} \quad \int_{-\infty}^{t_o + \theta} f(x)\, dx.$$

Both of the above imply that u is a monotone function of θ, thus the density of the measure induced on Ω by Π is given by

$$g(\theta; t_o) = \frac{1}{a} \left|\frac{du}{d\theta}\right|$$

$$= \frac{1}{a} f(t_o\theta)|t_o| \quad \text{or} \quad \frac{1}{a} f(t_o + \theta)$$

$$= \left|\frac{t_o}{a}\right| \; h(t_o|\theta) \quad \text{or} \quad \frac{1}{a}\, h(t_o|\theta).$$

Hence, in either case g belongs to the conjugate family of densities for θ.

We note that the above theorem gives a method for determining that special member of the conjugate family which induces a uniform distribution on the cumulative ordinate, but does not give necessary conditions for the existence of a conjugate prior which will accomplish this task.

The remainder of this chapter is concerned with the ordinate of the distribution functions whose densities were studied in Chapter II.

Binomial Distribution Function

The objective function of this section is

$$u = \sum_{k=0}^{x_o} \binom{n}{k} p^k \; (1-p)^{n-k}, \text{ for } x_o = 0, 1, \ldots, n \text{ fixed}$$

with $p \epsilon \Omega = [0, 1]$. We note that u is a monotone differentiable decreasing function of p, which is easily seen from the well known form for u; namely,

$$u = \frac{1}{\beta(x_o+1,\ n-x_o)} \int_p^1 t^{x_o} \, (1-t)^{n-x_o-1} \, dt. \qquad (3.\ 1)$$

If we assign a uniform density to u and observe the form of u given by (3. 1), then the density for u is given as

$$h(p; x_o, n) = \Pi(u(p)) \; \left|\frac{du}{dp}\right| .$$

Noting that $\Pi(u(p)) = 1$ for all u(p), and that u is a monotone decreasing differentiable function of p, the desired density is simply the negative of $\frac{du}{dp}$ which is most easily obtained from (3. 1). Thus, we obtain

$$h(p; x_o, n) = \frac{1}{\beta(x_o+1,\ n-x_o)}\ p^{x_o}(1-p)^{n-x_o} \tag{3. 2}$$

as the induced density on p. Certainly (3. 2) belongs to the conjugate family of densities for p.

Assignment of a beta density (2. 10) to u induces

$$h(p; x_o, \nu_1, \nu_2) = \frac{p^{x_o}(1-p)^{n-x_o}}{\beta(\nu_1, \nu_2)\,[\beta(x_o+1,\ n-x_o)]^{(\nu_1+\nu_2-1)}} \left[\int_p^1 t^{x_o}(1-t)^{n-x_o}\,dt\right]^{\nu_1-1}$$
$$\left[\int_0^p t^{x_o}(1-t)^{n-x_o}\,dt\right]^{\nu_2-1}$$

as the density on p, a rather involved density function.

In view of the preceding paragraph, we see that for an arbitrary density on u the functional form for the density induced on p can always be determined. The usefulness of this distribution as a prior density on p and its interpretation are problems for further study.

Suppose we now pursue the alternative approach and assign a density to p, then study the induced distribution on u. The form of the density for an arbitrary prior density on p has not been determined; however, we do determine the prior and posterior moment generating function and the prior and posterior cumulative distribution function for uniform and beta densities on p.

Taking a uniform density on p we obtain the moment generating function for u by considering

$$M_u(t) = E(e^{tu}). \tag{3.3}$$

Expanding the exponential in a Maclaurin series and interchanging the order of integration and summation we obtain

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E\left[\sum_{r=0}^{x_o} \binom{n}{x_o} p^{x_o} (1-p)^{n-x_o}\right]^k .$$

Now applying the multinomial theorem under the integral sign and again interchanging the order of summation and integration we obtain the form

$$M_u(t) = \sum_{k=0}^{\infty} \sum_{\Sigma r_j = k} \frac{t^k}{k!} k! \prod_{j=0}^{x_o} \left[\frac{\binom{n}{j}^{r_j}}{r_j!}\right] E\left[p^{\Sigma j r_j} (1-p)^{kn-\Sigma j r_j}\right], \tag{3.4}$$

where the second summation is over all integer partitions of k.

Finally, evaluation of the beta function under the integral symbol yields

$$M_u(t) = \sum_{k=0}^{\infty} \sum_{\Sigma r_j = k} t^k \frac{\left(\prod_{j=0}^{x_o} \frac{\binom{n}{j}^{r_j}}{r_j!}\right)}{(kn+1)\binom{kn}{\Sigma j r_j}} \tag{3.5}$$

where the first summation is over all combinations of r_j's whose sum is k. From $M_u(t)$ we see that

$$E[u] = \frac{x_o + 1}{n+1} .$$

This prior mean on u could have been anticipated from (2.9) since the expected value of a sum is the sum of the expected values. That is, this prior mean is the marginal distribution function for x_o relative to a uniform prior on p.

Alternatively, if we take a beta density on p and return to equation (3.4), and evaluate the expected value relative to (2.10), the resulting moment generating function is given by

$$M_u(t) = \sum_{k=0}^{\infty} \sum_{\Sigma r_j = k} t^k \frac{\left[\prod_{j=0}^{x_o} \frac{\binom{n}{j}^{r_j}}{r_j!}\right]}{\binom{kn}{\Sigma j r_j}} \frac{\binom{\Sigma j r_j + \nu_1 - 1}{\nu_1 - 1}\binom{kn - \Sigma j r_j + \nu_2 - 1}{\nu_2 - 1}}{\binom{kn + \nu_1 + \nu_2 - 1}{\nu_1 + \nu_2 - 1}}, \quad (3.6)$$

which yields the mean of u as

$$E[u] = \sum_{j=0}^{x_o} \frac{\binom{j + \nu_1 - 1}{\nu_1 - 1}\binom{n - j + \nu_2 - 1}{\nu_2 - 1}}{\binom{n + \nu_1 + \nu_2 - 1}{\nu_1 + \nu_2 - 1}} .$$

We observe that the mean of u is the cumulative hypergeometric waiting-time distribution function with $N = n + \nu_1 + \nu_2 - 1$, $Np = \nu_1 + \nu_2 - 1$, $k = \nu_1$, and $x = j + \nu_1$ from Wilks [11] on page 142. This result could have been anticipated from (2.12), because E(u) is the marginal prior distribution function for x. We also note that E(u) is an increasing function of x_o just as u is an increasing function of x_o for fixed p. That is, the monotone increasing property is not affected by the taking of expected values.

We note that the posterior moment generating function for u with respect to the uniform prior on p is given by (3.6) with (ν_1, ν_2) replaced by $(m\bar{x}+1,\ mn-m\bar{x}+1)$, and with respect to a beta prior on p by replacing (ν_1, ν_2) by $(\nu_1 + m\bar{x},\ \nu_2 + mn - m\bar{x})$.

The induced distribution function for u is given by

$$F(u_o) \doteq Pr\{u \leq u_o\} = Pr\{p : p \geq p_o\}$$

where

$$u_o = \frac{1}{\beta(x_o+1,\ n-x_o)} \int_{p_o}^{1} t^{x_o}\ (1-t)^{n-x_o-1}\, dt.$$

Hence, for a uniform prior on p we obtain $F(u_o) = 1-p_o$ while for the beta prior on u we have

$$F(u_o) = \frac{1}{\beta(\nu_1, \nu_2)} \int_{p_o}^{1} t^{\nu_1 -1}\, (1-t)^{\nu_2 -1}\, dt \qquad (3.7)$$

as the prior distribution function for u. The posterior distribution function for u relative to a uniform prior on p is given by (3.7) with (ν_1, ν_2) replaced by $(m\bar{x}+1,\ mn-n\bar{x}+1)$ and relative to a beta prior by replacing (ν_1, ν_2) by $(\nu_1+m\bar{x},\ \nu_2+mn-m\bar{x})$.

The two measures used above indicate that for any measure on p, for which we know the posterior distribution, we can determine the posterior distribution function for the ordinate of a binomial distribution evaluated at a specified x_o. Suppose we are taking random samples of size n from a production process which produces a good item with probability p and a reject with probability 1-p. We decide to accept or reject a lot on the basis of each sample accordingly as the number of good items is greater than or equal to x_o. Then given p, 1-u is the fraction of acceptable lots. Hence, for a squared error loss function, 1-E(u) is the Bayes estimator of fraction of acceptable lots. The expectation is taken with respect to the posterior distribution of u.

Poisson Distribution Function

We consider the function

$$u = \sum_{k=0}^{x_o} \frac{\lambda^k e^{-\lambda}}{k!} \qquad \lambda \geq 0 \quad x_o = 0, 1, 2, \ldots$$

with x_o fixed. That is, u is a function of λ for which a more informative representation may be

$$u = \int_{\lambda}^{\infty} \frac{t^{x_o} e^{-t}}{\Gamma(x_o+1)} \, dt \qquad \lambda \geq 0. \tag{3.8}$$

Observing (3.8), it is easy to see that u is a monotone decreasing differentiable function of λ. Hence, we are able to determine several distributional properties for u as well as use both approaches for assigning prior distributions to the spaces $\dot{S}$ and Ω. Also, the range of u does not depend upon the specification limit x_o. Thus, assigning a uniform prior density to u induces a conjugate prior density on λ; namely,

$$h(\lambda; x_o) = \frac{\lambda^{x_o} e^{-\lambda}}{\Gamma(x_o+1)} \qquad \lambda \geq 0,$$

a gamma distribution according to the classifications of Raiffa and Schlaifer [7]. On the other hand, if we assign a beta density (2.10) to u, the induced density on λ is given by

$$g(\lambda; x_o, \nu_1, \nu_2) = \frac{1}{\beta(\nu_1, \nu_2)} \left[\int_{\lambda}^{\infty} \frac{t^{x_o} e^{-t}}{\Gamma(x_o+1)}\right]^{\nu_1 - 1} \left[\int_0^{\lambda} \frac{t^{x_o} e^{-\lambda}}{\Gamma(x_o+1)}\right]^{\nu_2 - 1} \frac{\lambda^{x_o} e^{-\lambda}}{\Gamma(x_o+1)}$$

$$= \frac{\lambda^{x_o} e^{-\lambda}}{\beta(\nu_1, \nu_2)\,\Gamma(x_o+1)} \quad [1 - F_{\gamma 1}(\lambda; x_o)]^{\nu_1 - 1} [F_{\gamma 1}(\lambda; x_o)]^{\nu_2 - 1}.$$

Due to the monotone property of u, if we use g as a prior density on λ, then the beta density (2.10) is induced on u.

Now, if we take the alternative point of view and assign a measure to Ω by means of a density on λ, the things we can do are somewhat limited because we cannot give an explicit form for λ as a function of u. That is, we cannot solve equation (3.8) for λ in terms of u. We know a density exists for the distribution induced on u, provided the measure on Ω has a density; but we cannot determine its algebraic form explicitly. We can give a rule for determining points on the cumulative distribution function of u for an arbitrary density on λ. However, we will do this for an arbitrary conjugate prior distribution.

Suppose we take a gamma-1 prior density (2.15) on λ, then

$$F(u_o; x_o, a, b) = \Pr\{u \leq u_o\} = \Pr(\lambda \geq \lambda_o)$$

where

$$u_o = \int_{\lambda_o}^{\infty} \frac{t^{x_o} e^{-t}}{\Gamma(x_o+1)} \, dt.$$

Thus, in more explicit terms

$$F(u_o; x_o, a, b) = \int_{\lambda_o}^{\infty} \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} \, d\lambda \qquad (3.9)$$

$$= 1 - F_{\gamma 1}(\lambda_o; a, b)$$

The moment generating function for u relative to this density on λ is found by considering (3.3) and expanding the exponential in a Maclaurin series. Then we interchange the order of integration and summation to obtain the form

$$M_u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(u^k).$$

Next, we expand the finite polynomial u by the multinomial theorem, interchange integration and summation, and remove the constants from under the integral to obtain

$$M_u(t) = \sum_{k=0}^{\infty} \sum_{\Sigma r_j = k} \frac{t^k}{k!} \prod_{j=0}^{x_o} \frac{\Gamma(a+\Sigma j r_j)\, k!}{(j!)^{r_j}\, r_j!} E(\lambda^{\Sigma j r_j} e^{-k\lambda}).$$

Finally, by recognizing the integral as a gamma function we obtain the moment generating function of u as

$$M_u(t) = \frac{b^a}{\Gamma(a)} \sum_{k=0}^{\infty} \sum_{\Sigma r_j = k} t^k \frac{\Gamma\left[\sum_{j=1}^{x_o} j r_j + a\right]}{(k+b)^{a+\Sigma j r_j}} \prod_{j=0}^{x_o} \frac{1}{(j!)^{r_j}\, r_j!} \tag{3.10}$$

from which we obtain the prior mean of u relative to (2.15) as

$$E[u] = \sum_{k=0}^{x_o} \binom{k+a-1}{a-1} \left(\frac{b}{b+1}\right)^a \left(\frac{1}{b+1}\right)^k \tag{3.11}$$

which is a cumulative negative binomial with $p = \frac{b}{b+1}$, $q = \frac{1}{b+1}$, and $k = a$, provided, of course, that a is an integer. If a is not an integer, the form of (3.11) would involve gamma functions; however, its interpretation as the prior marginal distribution function for x would be valid. We should note that (3.11) is not a negative binomial distribution function in this case.

The invariance property for conjugate prior distributions is very nice at this state since all we must do to obtain the posterior distribution function, moment generating function, and mean is to substitute the posterior specifications (a", b") for the prior (a, b) into (3.9), (3.10), and (3.11) respectively where (a", b") are given on page 16.

We envision properties (3.9) and (3.11) as intuitive properties of u which can be used to assess one's consistency in the assigning of a conjugate prior distribution to λ. More precisely, the assigning of specific values to the specifications a and b leaves something to be desired from an intuitive point of view. We regard the prior mean and variance (m, v) as specifications which have an intuitive appeal and by being given these, we can solve for a and b as $a = \frac{m^2}{v}$ and $b = \frac{m}{v}$. A further assessment of these values should be made by observing (3.9) and (3.11), based on information available prior to experimentation.

Exponential Distribution Function

In this section we are considering the function

$$u = \int_0^{x_o} \theta e^{-\theta t}\, du \qquad \theta > 0, \quad x_o > 0 \tag{3.12}$$

where x_o is regarded as fixed and $\Omega = (0, \infty)$. From (3.12) we see that $u = 1-e^{-\theta x_o}$, which is a differentiable monotone increasing function of θ. This function has been widely used in life testing, though from a slightly different point of view. Equation (3.12) is solvable for θ in terms of u as

$$\theta = \frac{1}{x_o} \log \frac{1}{1-u}, \tag{3.13}$$

which is a monotone increasing function of u. Thus, for an arbitrary density Π on θ, the induced density for u is

$$g(u;x_o) = \Pi\left(\frac{1}{x_o}\log\frac{1}{1-u}\right)\frac{1}{x_o(1-u)} . \tag{3.14}$$

In view of (3.14), if we assign a conjugate prior to θ, the gamma-1, the prior density on u is given by

$$f(u;x_o,a,b) = \frac{\left(\frac{b}{x_o}\right)^a}{\Gamma(a)}\left[\log\frac{1}{1-u}\right]^{a-1}(1-u)^{\frac{b}{x_o}-1} . \tag{3.15}$$

For this case we can determine an intuitive property, from some points of view, known as the entropy. The entropy ξ for a random variable X with density g is given by

$$\xi = E\left[-\log g(x)\right] .$$

Hence, for our particular case the entropy for u is

$$\begin{aligned}\xi &= E\left[-\log f(u;x_o,a,b)\right]\\ &= -\log\frac{\left(\frac{b}{x_o}\right)^a}{\Gamma(a)} - (a-1)\left[\log x_o + \frac{\Gamma(a+1)}{b^{a+1}}\left(\sum_{r=1}^{a}\frac{1}{r} + \log b - \gamma\right)\right]\\ &\quad - \left(\frac{b}{x_o}-1\right)\frac{ax_o}{b} ,\end{aligned} \tag{3.16}$$

where γ is Euler's constant, and we require a to be an integer. Otherwise, some of the integrals evaluated in determining ξ could not be evaluated explicitly. We note that for $a = 1$, $b = x_o$, the entropy for u is maximized, namely $\xi = 0$. Furthermore, $a = 1$, $b = x_o$ induces a uniform prior on u.

By taking expectations relative to (2.10), we can determine a series form for the moment generating function of u as

$$M_u(t) = e^t \sum_{k=0}^{\infty} \frac{t^k(-1)^k}{k!} \left(1 + k\frac{x_o}{b}\right)^{-a}, \qquad (3.17)$$

which yields

$$E[u] = 1 - \left(1 + \frac{x_o}{b}\right)^{-a}. \qquad (3.18)$$

As in the preceding sections where conjugate priors were used, the posterior equivalents of (3.14) to (3.18) are obtained by replacing (a, b) by (a", b") given on page 18. We note that (3.18) is the distribution function for the density function (2.20).

Normal Distribution Function

The objective function of this section is given by

$$u = \int_{-\infty}^{x_o} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(t-u)^2}{2\sigma^2}} dt \qquad (3.19)$$

for x_o a fixed real number and $\Omega = \{(u, \sigma): -\infty < \mu < \infty, 0 < \sigma < \infty\}$. The function u is regarded as a function of the two variables μ and σ. As in Chapter II, we will be concerned with three cases and will use only conjugate prior densities on the parameters in each case.

Case 1. σ known.

For this case $\Omega = \{\mu: -\infty < \mu < \infty\}$, and we will assume a normal prior measure on Ω whose density is given by (2.21). This particular function is somewhat more restrictive in what we are able to determine relative to prior measures on Ω, in the sense that the moment generating function for u is very complicated; however, we can determine the mean as

$$E[u] = \int_{-\infty}^{\infty} \int_{-\infty}^{x_o} \frac{\sqrt{b}}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[(t-u)^2 + b(u-a)^2]} dt\, d\mu$$

$$= F_z\left[\sqrt{\frac{b}{b+1}}\ \frac{(x_o-a)}{\sigma}\right]. \tag{3.20}$$

We can also determine points on the prior cumulative distribution function for u by considering

$$F(u_o) = \Pr(u \leq u_o) = \Pr\left\{\mu: \frac{x_o-\mu}{\sigma} \leq z_o\right\} \tag{3.21}$$

where $u_o = F_z(z_o)$. Then considering (3.21), we can determine points on F by

$$F(u_o; x_o, a, b, \sigma) = 1 - F_z\left[\frac{b^{1/2}(x_o-a-\sigma z_o)}{\sigma}\right] \tag{3.22}$$

for $0 \leq u_o \leq 1$. We note the range of u_o is independent of (x_o, a, b, σ).

Again, the corresponding posterior properties of u for (3.20) and (3.22) are obtained by replacing (a, b) by (a", b") of page 21. Let us further note that u is a monotone decreasing function of μ, which will be of use to us in Chapter IV.

Case 2. μ known.

For this case, as in Chapter II, the conjugate prior distribution is the inverted-gamma-2 given by equation (2.34). We observe that for this case the range of u depends upon the location of x_o relative to μ. Without loss of generality let us take $\mu = 0$, which will enable us to conserve space in discussing u. That is, with u = 0, we have u as a monotone decreasing function of σ with [0, 1/2] as its range for $x_o < 0$. We note that u is also a monotone decreasing function with range [1/2, 1] for $x_o > 0$. Thus, we will have probability 1 assigned to one or the other of these two intervals by (2.35).

The prior distribution function is completely determined by considering

$$F(u_o; x_o \mu=0, \nu, v) = \Pr\{\sigma : \frac{x_o}{\sigma} \leq z_o\} \text{where } u_o = F_z(z_o).$$

Essentially this leads to two distinct cumulative distribution functions for u. That is, if $x_o < 0$, we have

$$F(u_o; x_o < 0, \mu=0, \nu, v) = \begin{cases} 0, & \text{if } u_o \leq 0 \\ F_{i\gamma 2}\left(\frac{x_o}{z_o}, \nu, v\right), & 0 < u_o \leq 1/2 \\ 1, & u_o > 1/2 \end{cases} \tag{3.23}$$

while if $x_o > 0$, we obtain

$$F(u_o; x_o > 0, \mu=0, \nu, v) = \begin{cases} 0, & \text{if } u_o \leq 1/2 \\ 1 - F_{i\gamma 2}\left(\frac{x_o}{z_o}; \nu, v\right) & 1/2 < u_o \leq 1 \\ 1, & u_o > 1. \end{cases} \tag{3.24}$$

Perhaps a more readily usable form for (3.23) and (3.24) is obtained by recognizing that

$$F_{i\gamma 2}(\sigma | \nu, v) = 1 - F_{\chi^2}\left(\frac{\nu v}{\sigma} \middle| \nu\right). \tag{3.25}$$

Then using (3.25) in (3.23) and (3.24) we obtain

$$F(u_o; x_o < 0, \mu=0, \nu, v) = 1 - F_{\chi^2}\left(\frac{\nu v z_o}{x_o} \middle| \nu\right) \quad 0 < u_o \leq 1/2$$

and

$$F(u_o; x_o > 0, \mu=0, \nu, v) = F_{\chi^2}\left(\frac{\nu v z_o}{x_o} \middle| \nu\right) \quad 1/2 < u_o \leq 1$$

We note that given the distribution function F for $x_o < 0$, the distribution function F for $x_o > 0$ can be obtained as

$$F(1-u_o;\ x_o < 0,\ \mu=0,\ \nu, v) = 1 - F(u_o; x_o > 0,\ \mu=0,\ \nu, v),$$

for $1/2 < u_o < 1$. This follows because x_o and z_o are of the same sign and hence do not affect the sign of the chi-square variate.

As in the case of σ known, the moment generating function is complicated. We can, however, determine the first moment as

$$E[u] = \int_0^\infty \int_{-\infty}^{x_o} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(\mu-t)^2}{2\sigma^2}} \Pi(\sigma; \nu, v)\, dt\, d\sigma$$

$$= F_t\left[\frac{x_o-\mu}{\sqrt{v}} \,\middle|\, \nu\right]$$

where for this case we are not concerned with the sign of $(x_o - \mu)$ in determining the form of $E[u]$.

Points on the posterior cumulative distribution function and the posterior mean of u are determined by substituting (ν, v) by (ν'', v'') where

$$\nu'' = \nu + n, \quad v'' = \frac{1}{\nu''}\left[\nu v + \sum_{i=1}^{n} (x_i - \mu)^2\right].$$

Hence, tables are available to obtain certain percentage points on the distribution of u.

Case 3. μ and σ unknown.

The conjugate prior for this case is the normal gamma of equation (2.38) while u is a rather involved function of (μ, σ). We are unable to exhibit either the moment generating function or the density for u; we can determine the prior mean of u as

$$E[u] = F_t\left[\sqrt{\frac{b}{(b+1)v}}\ (x_o-a)\,\Big|\,\nu\right]. \tag{3.26}$$

We also can determine the distribution function for u by considering first the functions

$$\eta = \frac{b^{1/2}(\mu-a)}{\sigma} \quad \text{and} \quad \xi = \frac{v}{\sigma^2}$$

then relative to (2.38), we see that η, ξ are independently distributed while η is normally distributed with zero mean and unit variance and ξ has a chi-square divided by ν with ν degrees of freedom. Now the distribution function for u is given by

$$F(u_o) = \Pr\left\{(\mu,\sigma) : \frac{x_o-\mu}{\sigma} \le z_o\right\} \quad \text{where } u_o = F_z(z_o)$$

$$= \Pr\left\{(\mu,\sigma) : \frac{b^{1/2}(x_o-a)}{v^{1/2}} \le \frac{\eta + b^{1/2} z_o}{\sqrt{\xi}}\right\},$$

and the variable

$$t = \frac{\eta + b^{1/2} z_o}{\sqrt{\xi}}$$

has a noncentral t distribution with ν degrees of freedom and noncentrality

$$\delta = b^{1/2} z_o.$$

Therefore,

$$F(u_o) = 1 - F_{t'}\left(\frac{b^{1/2}(x_o-a)}{v^{1/2}}\,\Big|\,\nu, \delta = b^{1/2} z_o\right)$$

where $F_{t'}(\cdot \mid \nu, \delta)$ is the distribution function for the noncentral Student's t distribution with ν degrees of freedom and noncentrality δ.

Most tables of the noncentral t-distribution involve considerable computation to determine percentage points of F. Therefore, we have constructed tables of percentage points for F from tables of the noncentral t-distribution constructed by Resnikoff [9]. These tables involve a minimum of computation to determine specified percentage points for u. These tables may be used to determine one-sided q tolerance limits for p content, as well as to aid in assessing the consistency of one's subjective prior distribution with other intuitive information which may be available. We shall discuss uses of these tables further in Chapter IV.

CHAPTER IV

APPLICATIONS

The work of the preceding chapter has many potential applications, some of which were mentioned by way of introduction. In this chapter we wish to give particular consideration in some detail to a few of these areas.

Bayesian q Tolerance Intervals for p Content

In this section we give the classical formulation of q tolerance intervals for p coverage followed by a formulation of Bayesian tolerance intervals as given by Aitchison [1], then we propose an alternative formulation for Bayesian tolerance intervals which employs the results of Chapter III.

We desire to obtain an interval $[r(x_{1,n}), \infty)$ for which we have q confidence that the coverage of this interval will be at least p. That is, if we have the sampling distribution $f(x|\theta)$ we desire a lower limit L such that we have q confidence that at least 100p% of the distribution of f exceeds L for all θ. From the frequentist point of view this is accomplished by solving the following equation

$$\int_{d}^{\infty} f(x|\theta)\, dx = p \tag{4.1}$$

for a function $d_{1-p}(\theta)$. If d is a monotone increasing function of θ, a lower q confidence limit δ_q is substituted into $d_{1-p}(\theta)$ for θ. If d is a monotone decreasing function of θ, an upper q confidence limit δ_{1-q} is substituted for θ, since we desire q confidence that $d_{1-p}(\theta)$ is less than L.

The interpretation of the resulting interval is that, if samples of size n are repeatedly taken and a tolerance interval is computed for each sample, then 100q% of these intervals will cover the interval $[d_{1-p}(\theta), \infty)$ regardless of what value θ has.

The Bayesian formulation of q tolerance intervals for p coverage also solves equation (4.1) for the function $d_{1-p}(\theta)$, but instead of using a confidence limit for θ, a probability point from the posterior distribution of θ is substituted into $d_{1-p}(\theta)$ for θ. That is, if $d_{1-p}(\theta)$ is a monotone increasing function of θ, we substitute an upper q probability point θ_q into $d_{1-p}(\theta)$ for θ while if $d_{1-p}(\theta)$ is a monotone decreasing function of θ, a lower q probability point θ_{1-q} is used for θ. More precisely, we suppose (4.1) can be solved for $d = d_{1-p}(\theta)$ as an increasing function of θ. Then we solve the equation

$$\int_{-\infty}^{\delta_q} \Pi''(\theta | x_{1,n}) d\theta = q \tag{4.2}$$

for δ_q, and we take $r(x_{1,n}) = d_{1-p}(\delta_q)$ as the lower q tolerance limit for at least p coverage. The interpretation of the resulting interval is that if we consider the set of values of θ for which the coverage of our interval is at least p, then this set has probability measure, given $x_{1,n}$, of q. In other words, the probability is q that the value of the θ we drew was one of those for which the coverage of the interval was at least p.

Another approach to obtaining Bayesian q tolerance intervals for p coverage can be formulated by considering directly the distribution of

$u = F(x_o | \theta)$ to determine an upper limit while a lower limit can be obtained similarly by use of the distribution of $u = R(x_o | \theta) = 1 - F(x_o | \theta)$, provided the range of u is not affected by the value of x_o. We will see that if a Bayesian q tolerance interval for p coverage can be obtained by Aitchison's approach, then the same interval is produced by this method. We will discuss only the lower q tolerance limit for p coverage, the derivation being identical for upper limits if the distribution function replaces the reliability function.

The posterior distribution of $u = R(x_o | \theta)$ is determined from the posterior distribution of θ as in Chapter III. Then we solve the equation

$$\int_p^1 \Pi''(u | x_o, x_{1,n})\, du = q \tag{4.3}$$

for x_o in terms of p, q, and the sample $x_{1,n}$; x_o is then a Bayesian lower q tolerance limit for p coverage, while $[x_o, \infty)$ is an upper q tolerance interval for at least p coverage. The truth of this statement is perhaps easier to see if we write (4.3) as

$$q = \Pr\{u : u \geq p\} = \Pr\{\theta : \int_{x_o}^{\infty} f(t | \theta)\, dt \geq p\}. \tag{4.4}$$

The interpretation of the interval obtained from this procedure is precisely the same as that obtained by Aitchison. The following theorem is useful in showing that our limit coincides with Aitchison's.

<u>Theorem</u>: If $u = R(d | \theta)$ can be solved for $d = d_u(\theta)$ as a continuous monotone function of θ for each fixed but arbitrary $u \in (0, 1)$ and the range of u is independent of d, then u is a monotone function of θ.

Proof: By definition of $R(d|\theta)$ we have

$$d_1 \leq d_2 \text{ implies that } R(d_1|\theta) \geq R(d_2|\theta) \text{ for all } \theta. \tag{4.5}$$

Now let us assume $d_u(\theta)$ is monotone decreasing, then

$$\theta_1 \leq \theta_2 \text{ implies that } d_u(\theta_1) \geq d_u(\theta_2) \text{ for all } u. \tag{4.6}$$

Then (4.5) and (4.6) imply that $R(d_u(\theta_1)|\theta) \leq R(d_u(\theta_2)|\theta)$ for all $\theta \epsilon \Omega$, so in particular we have

$$R(d_u(\theta_1)|\theta_1) \leq R(d_u(\theta_2)|\theta_1) \text{ and } R(d_u(\theta_1)|\theta_2) \leq R(d_u(\theta_2)|\theta_2). \tag{4.7}$$

By hypothesis $u \equiv R(d_u(\theta)|\theta)$, for all θ, means that $R(d_u(\theta_1)|\theta_1) = R(d_u(\theta_2)|\theta_2)$. Hence, from (4.7) we see that $R(d_u(\theta_1)|\theta_2) \leq R(d_u(\theta_1)|\theta_1)$. That is, $0_1 \leq \theta_2$ implies that $R(d|\theta_1) \geq R(d|\theta_2)$. Hence, R is a monotone decreasing function of θ. If we change (4.6) so that $d_u(\theta)$ is monotone increasing, then $R(d|\theta)$ is also monotone increasing.

Suppose the approach due to Aitchison for obtaining a Bayesian q tolerance interval for p coverage is possible. That is, suppose one can solve the equation (4,1) for $d = d_{1-p}(\theta)$ as a monotone function of θ where p is fixed but arbitrary. By the theorem above, u is also a monotone function of θ. Thus, there exists a monotone function $\theta = r_d(u)$ for each d such that $u \equiv R(d|r_d(u))$, provided R is a continuous function of d. Hence, the density of the posterior distribution for u is given by

$$\Pi''(u|d, x_{1,n}) = f''(r_d(u)|x_{1,n}) \left|\frac{dr}{du}\right|,$$

where f'' denotes the posterior density for θ, provided, of course, that $r_d(u)$ is differentiable. Restricting our consideration to the case where $r_d(u)$ is monotone decreasing, we have

$$\int_{p}^{1} \Pi''(u|d, x_{1,n})\, du = \int_{-\infty}^{r_d(p)} f''(\theta)|x_{1,n})\, d\theta = q.$$

Thus, $r_d(p)$ is an upper q probability point of the posterior distribution for θ, $\delta_q \equiv r_d(p)$, but

$$p \equiv R(d)\,|r_d(p)) \equiv R(d_{1-p}(\delta_q)\,|\,\delta_q)$$

implies that $d = d_{1-p}(\delta_q)$ since u is a monotone function of d. This proves the two intervals are identical provided Aitchison's method is possible.

We have been discussing lower q tolerance limits for at least p coverage. It should be noted that if all statements involving p and q are replaced by 1-p and 1-q, the resulting x_o is a Bayesian upper q tolerance limit for at least p coverage. That is, a lower 1-q tolerance limit for 1-p coverage is an upper q tolerance limit for p coverage. To see this we make the substitution into (4.4) to obtain

$$1-q = Pr\,\{u{:}u \geq 1-p\} = Pr\,\{\theta: \int_{x_o}^{\infty} f(t|\theta)\, dt \geq 1-p\}. \tag{4.8}$$

We know $Pr\{\Omega\} = 1$, and

$$\{\theta: \int_{x_o}^{\infty} f(t|\theta)\, dt \leq 1-p\} = \Omega - \{\theta: \int_{x_o}^{\infty} f(t|\theta)\, dt \geq 1-p\},$$

implies from (4.8 that

$$q = Pr\,\{\theta: \int_{x_o}^{\infty} f(t|\theta)\, dt \leq 1-p\}. \tag{4.9}$$

But we see that

$$\{\theta: \int_{x_o}^{\infty} f(t|\theta)\, dt \leq 1-p\} = \{\theta: \int_{-\infty}^{x_o} f(t|\theta)dt \geq p\},$$

thus (4.9) becomes

$$q = \Pr \{\theta : \int_{-\infty}^{x_o} f(t|\theta)\, dt \geq p\}. \qquad (4.10)$$

Now (4.8) makes x_o a lower 1-q tolerance limit for 1-p coverage while (4.10) shows x_o is an upper q tolerance limit for p coverage, which completes our proof.

Perhaps more simply stated our procedure is: Determine a number x_o such that the posterior probability is q that the coverage of the interval $[x_o, \infty)$ is at least p, by determining the posterior distribution of the coverage induced from the posterior distribution of θ. Using the arbitrariness of the specification limit x_o, we are able to explicitly give some of these intervals.

In the form of an example let us consider a uniform sampling distribution

$$f(x|\theta) = \frac{1}{\theta} \qquad 0 \leq x \leq \theta,$$

with the prior distribution

$$\Pi(\theta; a, b) = \frac{(a-1)b^{a-1}}{\theta^a} \qquad a > 1, \ \theta > b > 0.$$

Then, taking a sample of size n and considering

$$y = x_{max} = \max\{x_i, \ i=1, \ldots, n\}.$$

The posterior distribution of θ is given by

$$\Pi''(\theta; x_{1,n}, a, b) = \begin{cases} \dfrac{(a+n-1)b^{a+n-1}}{\theta^{a+n}}; & \theta > b \text{ if } y < b \\[2ex] \dfrac{(a+n-1)y^{a+n-1}}{\theta^{a+n}}; & \theta \geq y \text{ if } y \geq b. \end{cases}$$

Alternatively, if we let $\omega = \max(b, y)$, the posterior distribution of θ is

$$\Pi''(\theta; y, a, \omega) = \frac{(a+n-1)\omega^{a+n-1}}{\theta^{a+n}} \qquad \theta \geq \omega .$$

The function $u = F(x_o | \theta) = \frac{x_o}{\theta}$ is a monotone decreasing function of θ with range $(0, \frac{x_o}{\omega}]$. To determine a q tolerance limit for p coverage we find the posterior distribution of u, namely

$$h''(u; x_o, y, q, \omega) = (a+n-1)\left(\frac{\omega}{x_o}\right)^{a+n-1} u^{a+n-2} \;;\; 0 < u < \frac{x_o}{\omega} .$$

Solving the equation

$$\Pr\{u : u \geq p\} = q \tag{4.11}$$

or

$$(a+n-1)\left(\frac{\omega}{x_o}\right)^{a+n-1} \int_p^{\frac{x_o}{\omega}} u^{a+n-2}\, du = q$$

for x_o, we obtain

$$x_o = \omega p\, (1-q)^{-\frac{1}{a+n-1}} \tag{4.12}$$

as the upper q tolerance limit for at least p coverage.

If we desire a lower q tolerance limit for p coverage we obtain x_o such that

$$x_o = \omega(1-p)\, q^{-\frac{1}{a+n-1}} . \tag{4.13}$$

It can be seen that the limits (4.12) and (4.13) coincide precisely with those intervals obtained by Aitchison's procedure; however, the range of u is dependent upon the value taken for x_o.

Tolerance intervals for the normal distribution are well known; however, we wish to exhibit the ease of obtaining these limits in view of the results of the studies in Chapter III and the tables included in the Appendix A to this paper. In particular, we refer to the three cases studied relating to the distribution of the ordinate for the normal distribution function.

Case 1. σ known.

Referring to equation (4.11) as it applies to equation (3.21), it can be transformed to

$$1 - q = \Pr\{u : u \leq p\},$$

which in view of (3.22) becomes

$$q = F_z\left(\frac{b^{1/2}(x_o-(a+\sigma z_p))}{\sigma}\right).$$

Then we solve

$$z_q = \frac{b^{1/2}(x_o-(a+\sigma z_p))}{\sigma}$$

for x_o to obtain

$$x_o = a + \sigma z_p + \frac{\sigma z_q}{b^{1/2}} \tag{4.14}$$

as an upper q tolerance limit for at least p coverage where the posterior specifications are used for a and b. Referring to (4.14) we see that a lower q tolerance limit for at least p coverage is given by

$$x_o = a + \sigma z_{1-p} + \frac{\sigma z_{1-q}}{b^{1/2}} . \tag{4.15}$$

We note that (4.14) coincides with Aitchison's result if (a, b) is replaced by (a'', b'').

Case 2. μ unknown.

For this case, in view of the nature of the distribution function for u (3.23) to determine an explicit limit, we specify $p \geq 1/2$. This implies from equation (3.16) that we must solve the equation

$$q = \Pr\{u : u \geq p\} = \Pr\{\sigma : \sigma \leq \frac{x_o - \mu}{z_p}\} = F_{\chi^2}\left(\frac{\nu v z_p^2}{(x_o-\mu)^2} \,\middle|\, \nu \right)$$

for x_o. But this requires determining that x_o for which

$$\frac{\nu v z_p^2}{(x_o-\mu)^2} = \chi_q^2 \quad \text{or} \quad x_o = \mu + \left(\frac{\nu v z_p^2}{\chi_q^2} \right), \tag{4.16}$$

and yields x_o, an upper q tolerance limit for at least p coverage. The desired tolerance limit is obtained by replacing (ν, v) in (4.16) by the posterior specifications (ν'', v'') from this case in Chapter II.

Case 3. μ and σ unknown.

This is perhaps the most interesting of the three cases considered for the normal distribution since it is the most general and hence has the widest application. We observed in Chapter III that points on the distribution function for u at u_o can be determined from points on the distribution of a noncentral t with ν degrees of freedom and noncentrality $\delta = z_{u_o} b^{1/2}$. We can, therefore, determine an upper q tolerance limit for p coverage by finding x_o such that

$$q = \Pr\left\{ t' : t'(\nu, z_p b^{1/2}) \leq \frac{b^{1/2}(x_o - a)}{v^{1/2}} \right\} \tag{4.17}$$

That is, we determine x_o so that

$$\frac{b^{1/2}(x_o - a)}{v^{1/2}} = t'_q\,(\nu, z_p b^{1/2}).$$

Then solving for x_o we obtain

$$x_o = a + \sqrt{\frac{v}{b}}\; t'_q\,(\nu, z_p b^{1/2}). \tag{4.18}$$

Now (4.18) corresponds to Aitchison's result, provided we take (a'', b'', ν'', v'') as in Chapter II. We have previously mentioned the tables in Appendix A which lead to an easy calculation of (4.18). These tables have a set of values for q as .005, .010, .025, .050, .950, .975, .990, and .995 by using complement probabilities. The values for u_o = .05 (.05) .95, and the posterior parameters b'' = 1 (1) 10 (5) (30), and ν'' = 4 (1) 9, 16, 36, 144. Tabled in the body are values for $T(u_o, q, \nu'', b'')$ where

$$q = \Pr\{\, t'(\nu, z_{u_o} b^{1/2}) \geq T(u_o, q, \nu'', b'')\} . \tag{4.19}$$

Thus, to determine an upper q tolerance limit for p coverage, we observe (4.18) and note that

$$1\text{-}q = \Pr\left\{ t' : t'\,(\nu, z_p b^{1/2}) \geq \frac{b^{1/2}(x_o - a)}{v^{1/2}} \right\}$$

which in view of (4.19) implies we should solve for x_o in

$$\frac{b''^{1/2}(x_o - a'')}{v''^{1/2}} = T\,(p, 1\text{-}q, \nu'', b'').$$

The solution for x_o is given as

$$x_o = a'' + \sqrt{\frac{v''}{b''}}\ T(p, 1-q, \nu'', b''),$$

where T is taken directly from Table I.

For example, suppose we desire an upper 99.5% tolerance limit for 95% coverage, given the posterior specifications $a'' = 5$, $b'' = 30$, $\nu'' = 36$ and $v'' = 1.44$. Then $T(.95, .005, 36, 30) = 13.8980$ which yields the desired limit as

$$x_o = 5 + \sqrt{\frac{1.44}{30}}\ 13.8980$$

$$= 8.0449.$$

Bayesian p-Expected Coverage Tolerance Intervals

A Bayesian p-expected coverage tolerance interval is an interval $(-\infty, r(x_1, n)]$ such that the expected coverage of the interval is p. Certainly, the functions considered in Chapter III are coverages for intervals $(-\infty, x_o]$ for each $\theta \epsilon \Omega$. Thus, their posterior expectations set equal to p and solved for x_o will permit x_o to be interpreted as the upper limit of such an interval. Alternatively, if an interval of the form $[r(x_1, n), \infty)$ is desired which will have p expected coverage, we solve the equation

$$E''(u) = 1 - p$$

for x_o in terms of p and the sample $x_{1,n}$. For example, if we consider the normal distribution function ordinate as in Case 3, Chapter III, we noted by equation (3.26) that

$$E''(u) = F_t\left(\sqrt{\frac{b''}{(b''+1)v''}}\quad (x_o - a'')\,\Big|\,\nu''\right).$$

Then setting this equation to 1-p, we obtain the equation

$$\sqrt{\frac{b''}{(b''+1)v''}}\ (x_o - a'') = t_{1-p}\,(\nu''),$$

which yields x_o in the form

$$x_o = a'' + \sqrt{\frac{(b''+1)\ v''}{b''}}\quad t_{1-p}\,(\nu'')$$

$$= a'' - \sqrt{\frac{(b''+1)\ v''}{b''}}\quad t_p\,(\nu'')$$

as the desired limit

Fraction of the Acceptable Product

We imagine a production process from which items are produced which must meet an engineering specification limit to be acceptable. That is, if a measurement x is to be made on the item and the engineering specification limit is x_o, then for any item with measurement x such that $x \leq x_o$, the item would be acceptable. Suppose we are concerned with the problem of estimating that fraction of the population of items produced which is acceptable. Folks, Pierce, and Stewart [3] give the UMV estimates for some of the more common distributions $f(x|\theta)$ for the random variable x. The Bayes estimate of the fraction of acceptable product is $E\,[u]$ where $u = F(x_o|\theta)$, and the expectation is taken with respect to the posterior distribution of u.

For comparison purposes consider our Case 1 for the normal distribution function. The UMV estimator of $g(\theta) = F(x_o|\theta)$ is given by [3] as

$$\hat{g}(\theta) = F_z\left(\sqrt{\frac{n}{n-1}}\ \frac{(x_o-\bar{x})}{\sigma}\right).$$

From (3.20) the Bayes estimator is given by

$$E''[u] = F_z\left[\sqrt{\frac{b+n}{b+n+1}}\ \frac{\left(x_o - \frac{ba+n\bar{x}}{b+n}\right)}{\sigma}\right].$$

Note that for no values of a and b does the Bayes estimator coincide with the UMV estimator, but for large n with b = 0, they are approximately equal.

Assessing A Conjugate Prior Distribution

As a final application of the results of this paper, we consider the usefulness of the idea of interval content as an intuitive concept to aid in determining a conjugate prior distribution which will adequately reflect the available prior information. For many conjugate prior distributions the explicit specifications which appear as parameters in the prior distribution are not easily determined directly, but could be solved for in terms of the first few moments or in terms of betting odds as suggested by Lindley [5]. We maintain that betting odds and critical probabilities can be more directly determined by using the notion of content of specific intervals whose distribution can be based on these criteria.

Suppose a conjugate prior distribution can be completely determined by its mean and variance. At first glance an experimenter may feel he has adequately approximated his prior information by specifying these two moments, but when confronted with the content of some specific intervals based on these moments, may decide his choice can be improved. Perhaps critical probabilities are more amenable to the idea of content than betting odds. Although both could be used since they are not independent concepts, indeed given critical betting odds, critical probabilities are uniquely determined. Diligent pursuit of this approach could lead the experimenter to the conclusion that no conjugate prior distribution for θ can adequately represent his available prior information.

We believe the prior tolerance intervals determined earlier in this chapter are usable devices for completely or partially determining an acceptable prior distribution as well as determining that a proposed prior is either inadequate or totally unacceptable. That is, use of these devices should lend credence to a proposed prior distribution, suggest an alteration which would improve its status, or render it wholly unacceptable.

CHAPTER V

SUMMARY AND CONCLUSIONS

In this thesis we have examined some of the distributional properties exhibited by the ordinate of the binomial, Poisson, negative exponential, and normal probability laws, and the ordinate of their corresponding distribution functions. We have examined these ordinates as functions of their parameters for a fixed value of the variable x. The study involved the use of various prior distributions on the parameter. In all cases we were able to determine percentage points for the prior and posterior distribution function for these ordinates. In some cases, we were able to determine the moment generating function for these ordinates in a series form.

We have applied the notion of regarding the distribution function $u = F(x_o | \theta)$ as a function of its parameter to obtain an alternative method for arriving at an upper Bayesian q-tolerance limit for at least p-coverage. We have shown that, if such a limit can be obtained by Aitchison's procedure, then our technique renders the same limit.

We have considered the problem of determining the density on the parameter which will induce a given density upon the ordinate. This density is most easily obtained by assigning the desired density to the ordinate and determining the density induced upon the parameter. In the presence of a monotone function with certain regularity conditions, this approach gives good results. For the normal ordinate with

σ known, a uniform density on the ordinate induces a bimodal curve as the density on μ if we agree to assign equal probability to equal tail areas.

We have suggested a method for assessing the consistency of a given prior distribution by using the notion of interval content. That is, by using the prior equivalent to Bayesian q-tolerance intervals for p-coverage, we obtain an interval such that the prior probability is q that it contains at least 100p% of the distribution of $f(x|\theta)$. By restricting ourselves to the conjugate class of prior distributions we could use these limits to obtain either the specifications for a particular member of the family or assess the consistency of interval content notions for a given member of this family. Thus, we could either assign a particular prior distribution or we could subjectively evaluate a given prior distribution by this method.

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APPENDIX A

TABLES FOR PERCENTAGE POINTS OF THE ORDINATE OF THE NORMAL DISTRIBUTION μ AND σ UNKNOWN

We consider the function

$$u = F_z\left(\frac{x_o - \mu}{\sigma}\right), \qquad \sigma \geq 0, \quad -\infty < \mu < \infty$$

for the normal-inverted-gamma-2 prior distribution on the parameter (μ, σ). That is,

$$\Pi(\mu, \sigma \mid a, b, \nu, v) = \sqrt{\frac{b}{2\pi\sigma^2}}\, e^{-\frac{b(\mu-a)^2}{2\sigma^2}} \frac{2\left(\frac{\nu v}{2\sigma^2}\right)^{\frac{\nu+1}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{\nu v}{2}\right)^{1/2}}\, e^{-\frac{\nu v}{2\sigma^\sigma}}$$

$$b, v > 0, \; -\infty < \mu < \infty, \; \sigma > 0.$$

It is easily seen that the expressions

$$\eta = \frac{b^{1/2}(\mu - a)}{\sigma} \quad \text{and} \quad \xi = \frac{v}{\sigma^2}$$

are independently distributed. The variable η has a standard normal distribution while ξ is distributed as $\frac{\chi^2(\nu)}{\nu}$. It then follows that

$$t' = \frac{\eta + z_u b^{1/2}}{\xi^{1/2}} \tag{A. 1}$$

has a noncentral t-distribution with ν degrees of freedom and non-centrality parameter $\delta = z_u b^{1/2}$.

The distribution function for u is given by

$$F(u|x_o, a, b, \nu, v) = \Pr \{(\mu, \sigma): F_z \frac{x_o - \mu}{\sigma} \leq u\}$$

$$= \Pr \{(\mu, \sigma): \frac{x_o - \mu}{\sigma} \leq z_u\}$$

where $u = F_z(z_u)$. Operating on the inequalities in the last expression we obtain

$$F(u|x_o, a, b, \nu, v) = \Pr \left\{(\mu, \sigma) : \frac{\eta + z_u b^{1/2}}{(\xi)^{1/2}} \geq \frac{b^{1/2}(x_o - a)}{v^{1/2}}\right\}.$$

Thus, in view of the distribution of t' indicated in (A. 1), we may write

$$F(u|x_o, a, b, \nu, v) = \Pr \left\{t' : t' \geq \frac{b^{1/2}(x_o - a)}{v^{1/2}}\right\}$$

where t' has the noncentral t-distribution with ν degrees of freedom and non-centrality parameter $z_u b^{1/2}$.

Certain percentage points for the noncentral t-distribution have been tabulated by Resnikoff [9] . These tables are particularly well adapted to computation of percentage points for u. These tables make it possible to find the point t_ϵ which is exceeded with probability ϵ in the noncentral t-distribution, with f degrees of freedom and non-centrality parameter δ, for 17 values of ϵ. The computation may be carried out as follows. First compute

$$r = \delta\ (2f)^{-1/2} \text{ and } \eta = r(1+r^2)^{-1/2}.$$

The table corresponding to the desired ϵ is entered with η and f as arguments to obtain the tabular value λ. Then the percentage point is obtained by computing

$$t_\epsilon = \frac{\delta + \lambda\left(1 + r^2 - \frac{\lambda^2}{2f}\right)^{1/2}}{1 - \frac{\lambda^2}{2f}} . \qquad (A.2)$$

For values of ϵ greater than 1/2 the relation

$$t_{1-\epsilon}(\delta) = -t(-\delta) \qquad (A.3)$$

may be used.

The values of λ are tabled for $\eta = -1.0\,(.1)\,1$ and $r = 4(1)\,9$, 16, 36, 144, ∞. To obtain values of λ for η values not tabled, linear interpolation on η is recommended. For values of f not tabled, it is recommended to interpolate linearly on $\frac{12}{f^{1/2}}$.

The following tables were constructed from [9] where we have chosen to write q and ν in place of ε and f respectively. Linear interpolation was performed in the above recommended fashion to obtain the tabular value $T(u, q, \nu, b)$ from (A.2), for arguments u and b in the table indexed by q and ν. The values of $T(u, q, \nu, b)$ are tabled for $u = 0.05(0.05)\,0.95$, $b = 1(1)\,10\,(5)\,30$, $q = 0.005, 0.010, 0.025, 0.050$, and the same values for ν that were used for f in [9], except $\nu = \infty$.

The relation (A.3) implies the following in our notation

$$T(u, 1-q, \nu, b) = -T(1-u, q, \nu, b) \text{ and } T(1-u, 1-q, \nu, b) = -T(u, q, \nu, b).$$

This relation can be used to obtain percentage points for q greater than 0.50. The procedure for interpolation within these tables and between tables for tabular $T(u, q, \nu, b)$ has not been established. It has been noted that linear interpolation yields unsatisfactory results.

TABLE I

PERCENTAGE POINTS FOR THE DISTRIBUTION OF THE NORMAL ORDINATE

$q = 0.005$ $\nu = 4$

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	1.1842	0.2753	-0.2722	-0.6773	-1.0012	-1.2678	-1.4999	-1.7086	-1.8995	-2.0760	-2.8033	-3.3788	-3.8740	-4.3148
0.10	1.7824	0.9304	0.4001	0.0145	-0.2892	-0.5449	-0.7648	-0.9590	-1.1318	-1.2859	-1.9196	-2.4180	-2.8264	-3.1851
0.15	2.2201	1.4630	0.9547	0.5812	0.2840	0.0397	-0.1680	-0.3531	-0.5185	-0.6680	-1.2582	-1.6983	-2.0649	-2.3841
0.20	2.6024	1.9479	1.4792	1.1260	0.8353	0.5987	0.3946	0.2157	0.0578	-0.0838	-0.6487	-1.0672	-1.3951	-1.6768
0.25	2.9687	2.3777	1.9860	1.6640	1.3974	1.1748	0.9735	0.7999	0.6471	0.5100	-0.0337	-0.4295	-0.7462	-1.0119
0.30	3.3117	2.8184	2.4658	2.1999	1.9785	1.7796	1.5973	1.4377	1.2957	1.1652	0.6375	0.2558	-0.0436	-0.2925
0.35	3.6284	3.2659	2.9850	2.7548	2.5606	2.3972	2.2542	2.1253	2.0077	1.8997	1.4231	1.0650	0.7746	0.5381
0.40	3.9424	3.6914	3.5062	3.3522	3.2186	3.0935	2.9792	2.8750	2.7789	2.6897	2.3270	2.0492	1.8137	1.5984
0.45	4.2712	4.1351	4.0316	3.9450	3.8692	3.8058	3.7481	3.6946	3.6446	3.5974	3.3929	3.2235	3.0697	2.9322
0.50	4.6042	4.6042	4.6042	4.6042	4.6042	4.6042	4.6042	4.6042	4.6042	4.6042	4.6042	4.6042	4.6042	4.6042
0.55	4.9418	5.0825	5.1908	5.2822	5.3628	5.4428	5.5173	5.5869	5.6523	5.7144	5.9874	6.2191	6.4242	6.6102
0.60	5.2849	5.5911	5.8354	6.0426	6.2260	6.3924	6.5459	6.6891	6.8239	6.9515	7.5127	7.9881	8.4196	8.8456
0.65	5.6735	6.1607	6.5381	6.8580	7.1410	7.3976	7.6340	7.8546	8.0621	8.2588	9.2199	10.0468	10.7832	11.4550
0.70	6.0978	6.7681	7.2868	7.7261	8.1147	8.4847	8.8478	9.1878	9.5087	9.8137	11.1644	12.3950	13.5210	14.5498
0.75	6.5602	7.4292	8.1011	8.7121	9.2769	9.7919	10.2687	10.7153	11.1371	11.5414	13.4817	15.1469	16.6319	17.9866
0.80	7.0790	8.1702	9.0973	9.9035	10.6214	11.2763	11.9175	12.5380	13.1246	13.6828	16.1594	18.2803	20.0732	21.7078
0.85	7.6917	9.1324	10.3154	11.3275	12.2955	13.1945	14.0285	14.8105	15.5493	16.2518	19.3096	21.8357	24.0868	26.1388
0.90	8.4793	10.3765	11.8991	13.2920	14.5356	15.6712	16.7236	17.7095	18.6236	19.4621	23.1802	26.3610	29.2580	31.9013
0.95	9.7705	12.3261	14.4637	16.2967	17.9303	19.3626	20.6714	21.8977	23.0560	24.1568	29.0901	33.3522	37.1253	40.5475

$q = 0.005$ $\nu = 5$

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	1.1253	0.2749	-0.2772	-0.6870	-1.0179	-1.3003	-1.5454	-1.7618	-1.9602	-2.1441	-2.9168	-3.5285	-4.0527	-4.5190
0.10	1.6560	0.9008	0.3947	0.0141	-0.2942	-0.5530	-0.7759	-0.9745	-1.1545	-1.3197	-1.9812	-2.5011	-2.9418	-3.3234
0.15	2.0353	1.3817	0.9227	0.5679	0.2833	0.0396	-0.1720	-0.3587	-0.5263	-0.6775	-1.2900	-1.7511	-2.1326	-2.4657
0.20	2.3643	1.7954	1.3965	1.0746	0.8118	0.5849	0.3894	0.2169	0.0580	-0.0860	-0.6579	-1.0857	-1.4368	-1.7288
0.25	2.6631	2.1732	1.8292	1.5595	1.3217	1.1171	0.9395	0.7781	0.6317	0.4990	-0.0349	-0.4360	-0.7570	-1.0288
0.30	2.9463	2.5410	2.2499	2.0176	1.8226	1.6537	1.5037	1.3586	1.2278	1.1087	0.6225	0.2560	-0.0449	-0.2976
0.35	3.2210	2.9068	2.6763	2.4892	2.3298	2.1901	2.0652	1.9520	1.8484	1.7526	1.3453	1.0209	0.7541	0.5260
0.40	3.4924	3.2761	3.1145	2.9812	2.8661	2.7640	2.6716	2.5871	2.5088	2.4359	2.1288	1.8850	1.6812	1.5046
0.45	3.7628	3.6506	3.5655	3.4945	3.4324	3.3768	3.3260	3.2790	3.2351	3.1939	3.0163	2.8704	2.7447	2.6335
0.50	4.0394	4.0394	4.0394	4.0394	4.0394	4.0394	4.0394	4.0394	4.0394	4.0394	4.0394	4.0394	4.0394	4.0394
0.55	4.3054	4.4169	4.5030	4.5758	4.6403	4.6987	4.7550	4.8090	4.8599	4.9081	5.1211	5.3027	5.4640	5.6109
0.60	4.5780	4.8123	5.0024	5.1643	5.3081	5.4390	5.5601	5.6734	5.7804	5.8820	6.3314	6.7428	7.1132	7.4522
0.65	4.8764	5.2568	5.5539	5.8075	6.0332	6.2388	6.4338	6.6259	6.8077	6.9808	7.7510	8.4138	9.0067	9.5492
0.70	5.2075	5.7361	6.1499	6.5139	6.8539	7.1650	7.4540	7.7254	7.9821	8.2266	9.3144	10.2503	11.0874	11.8531
0.75	5.5714	6.2642	6.8420	7.3459	7.7966	8.2091	8.5923	8.9520	9.2924	9.6165	11.0581	12.2981	13.4067	14.4202
0.80	5.9836	6.9028	7.6530	8.2987	8.8763	9.4048	9.8957	10.3565	10.7926	11.2077	13.0537	14.6402	16.0571	17.3983
0.85	6.4839	7.6811	8.6298	9.4462	10.1764	10.8445	11.4650	12.0473	12.5983	13.1227	15.4528	17.5033	19.3503	21.0322
0.90	7.1607	8.6790	9.8820	10.9170	11.8426	12.6892	13.4752	14.2127	14.9100	15.5735	18.6067	21.2143	23.5301	25.6015
0.95	8.1919	10.1992	11.7890	13.1562	14.3780	15.4947	16.5471	17.5542	18.5048	19.4077	23.3965	26.7353	29.6882	32.3709

TABLE I (Continued)

q = 0.005 ν = 6

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	1.0876	0.2700	-0.2719	-0.6869	-1.0249	-1.3145	-1.5705	-1.8010	-2.0084	-2.2007	-3.0093	-3.6570	-4.2016	-4.6864
0.10	1.5812	0.8718	0.3878	0.0170	-0.2890	-0.5497	-0.7782	-0.9806	-1.1648	-1.3343	-2.0303	-2.5742	-3.0355	-3.4417
0.15	1.9399	1.3274	0.8930	0.5566	0.2783	0.0418	-0.1669	-0.3536	-0.5225	-0.6772	-1.3039	-1.7898	-2.1886	-2.5372
0.20	2.2399	1.7144	1.3407	1.0390	0.7883	0.5731	0.3826	0.2130	0.0596	-0.0815	-0.6571	-1.0943	-1.4548	-1.7663
0.25	2.5097	2.0661	1.7465	1.4888	1.2736	1.0797	0.9092	0.7568	0.6182	0.4896	-0.0308	-0.4314	-0.7587	-1.0361
0.30	2.7635	2.3998	2.1360	1.9236	1.7402	1.5791	1.4359	1.3067	1.1851	1.0717	0.6094	0.2513	-0.0408	-0.2923
0.35	3.0084	2.7282	2.5216	2.3530	2.2087	2.0815	1.9673	1.8633	1.7648	1.6735	1.2948	0.9875	0.7342	0.5159
0.40	3.2422	3.0574	2.9135	2.7947	2.6918	2.6003	2.5174	2.4413	2.3707	2.3049	2.0255	1.7997	1.6054	1.4368
0.45	3.4709	3.3762	3.3042	3.2440	3.1913	3.1439	3.1007	3.0600	3.0210	2.9843	2.8260	2.6956	2.5830	2.4831
0.50	3.7038	3.7038	3.7038	3.7038	3.7038	3.7038	3.7038	3.7038	3.7038	3.7038	3.7038	3.7038	3.7038	3.7038
0.55	3.9424	4.0429	4.1205	4.1864	4.2448	4.2978	4.3467	4.3924	4.4355	4.4763	4.6568	4.8106	4.9475	5.0721
0.60	4.1884	4.3952	4.5562	4.6933	4.8152	4.9262	5.0290	5.1262	5.2223	5.3138	5.7204	6.0649	6.3603	6.6301
0.65	4.4494	4.7717	5.0237	5.2468	5.4502	5.6363	5.8092	5.9716	6.1168	6.2548	6.8677	7.4230	7.9232	8.3817
0.70	4.7299	5.1825	5.5558	5.8769	6.1536	6.4015	6.6315	6.8473	7.0594	7.2653	8.1831	8.9752	9.6849	10.3351
0.75	5.0386	5.6594	6.1441	6.5455	6.9040	7.2505	7.5735	7.8770	8.1645	8.4386	9.6601	10.7133	11.6562	12.5193
0.80	5.4055	6.1926	6.7898	7.3260	7.8131	8.2596	8.6749	9.0652	9.4348	9.7870	11.3558	12.7067	13.9144	15.0182
0.85	5.8516	6.8121	7.6051	8.2946	8.9126	9.4789	10.0054	10.5001	10.9685	11.4146	13.3992	15.1046	16.6265	18.0151
0.90	6.3981	7.6466	8.6633	9.5404	10.3261	11.0458	11.7146	12.3425	12.9366	13.5021	16.0134	18.1655	20.0817	21.8269
0.95	7.2361	8.9318	10.2806	11.4431	12.4833	13.4349	14.3182	15.1465	15.9294	16.6738	19.9710	22.7863	25.2862	27.5582

q = 0.005 ν = 7

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	1.0564	0.2645	-0.2741	-0.6945	-1.0418	-1.3359	-1.5969	-1.8331	-2.0498	-2.2478	-3.0786	-3.7492	-4.3184	-4.8205
0.10	1.5259	0.8494	0.3808	0.0130	-0.2913	-0.5551	-0.7875	-0.9963	-1.1837	-1.3561	-2.0727	-2.6312	-3.1055	-3.5238
0.15	1.8695	1.2818	0.8698	0.5449	0.2727	0.0377	-0.1693	-0.3565	-0.5275	-0.6846	-1.3251	-1.8214	-2.2353	-2.5931
0.20	2.1617	1.6531	1.2946	1.0100	0.7688	0.5607	0.3757	0.2080	0.0555	-0.0847	-0.6642	-1.1122	-1.4789	-1.7970
0.25	2.4152	1.9925	1.6837	1.4372	1.2297	1.0489	0.8855	0.7383	0.6040	0.4804	-0.0347	-0.4352	-0.7677	-1.0532
0.30	2.6450	2.3150	2.0606	1.8535	1.6777	1.5238	1.3864	1.2618	1.1475	1.0413	0.5955	0.2460	-0.0445	-0.2947
0.35	2.8651	2.6132	2.4260	2.2717	2.1313	2.0075	1.8962	1.7947	1.7011	1.6141	1.2502	0.9607	0.7164	0.5057
0.40	3.0730	2.9079	2.7801	2.6731	2.5803	2.4975	2.4222	2.3529	2.2885	2.2249	1.9530	1.7343	1.5490	1.3872
0.45	3.2773	3.1927	3.1284	3.0745	3.0274	2.9849	2.9461	2.9101	2.8764	2.8435	2.7014	2.5838	2.4818	2.3910
0.50	3.4845	3.4845	3.4845	3.4845	3.4845	3.4845	3.4845	3.4845	3.4845	3.4845	3.4845	3.4845	3.4845	3.4845
0.55	3.7053	3.7983	3.8702	3.9312	3.9853	4.0345	4.0798	4.1223	4.1621	4.1984	4.3587	4.4953	4.6167	4.7274
0.60	3.9331	4.1249	4.2693	4.3911	4.4994	4.5979	4.6891	4.7745	4.8551	4.9349	5.2915	5.5983	5.8730	6.1247
0.65	4.1745	4.4608	4.6844	4.8761	5.0547	5.2179	5.3694	5.5117	5.6465	5.7748	6.3467	6.8316	7.2532	7.6394
0.70	4.4237	4.8217	5.1472	5.4288	5.6807	5.9114	6.1260	6.3277	6.5187	6.6985	7.4722	8.1567	8.7819	9.3551
0.75	4.6976	5.2381	5.6719	6.0457	6.3807	6.6860	6.9585	7.2143	7.4565	7.6873	8.7600	9.6887	10.5211	11.2853
0.80	5.0154	5.7169	6.2739	6.7497	7.1605	7.5366	7.8924	8.2360	8.5616	8.8719	10.2558	11.4564	12.5592	13.5675
0.85	5.4065	6.2948	6.9852	7.5660	8.1016	8.6004	9.0644	9.5007	9.9139	10.3077	12.0887	13.6464	15.0132	16.2233
0.90	5.9082	7.0202	7.8822	8.6546	9.3472	9.9822	10.5726	11.1273	11.6663	12.1826	14.4766	16.3545	18.0273	19.5531
0.95	6.6738	8.1186	9.3071	10.3329	11.2526	12.1213	12.9280	13.6847	14.4000	15.0544	17.9306	20.3929	22.6163	24.6591

TABLE I (Continued)

q = 0.005 ν = 8

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	1.0417	0.2686	-0.2771	-0.6984	-1.0478	-1.3497	-1.6166	-1.8584	-2.0806	-2.2868	-3.1377	-3.8240	-4.4133	-4.9309
0.10	1.4933	0.8427	0.3828	0.0141	-0.2945	-0.5589	-0.7917	-1.0018	-1.1938	-1.3704	-2.1041	-2.6805	-3.1652	-3.5931
0.15	1.8193	1.2593	0.8627	0.5438	0.2769	0.0392	-0.1710	-0.3601	-0.5313	-0.6885	-1.3387	-1.8464	-2.2738	-2.6416
0.20	2.0936	1.6144	1.2716	0.9976	0.7639	0.5594	0.3778	0.2116	0.0572	-0.0850	-0.6681	-1.1202	-1.4958	-1.8215
0.25	2.3315	1.9351	1.6435	1.4085	1.2091	1.0346	0.8779	0.7340	0.6021	0.4802	-0.0343	-0.4392	-0.7718	-1.0595
0.30	2.5517	2.2354	1.9990	1.8043	1.6378	1.4913	1.3597	1.2400	1.1297	1.0274	0.5937	0.2500	-0.0442	-0.2979
0.35	2.7611	2.5212	2.3419	2.1943	2.0652	1.9492	1.8445	1.7486	1.6600	1.5774	1.2288	0.9506	0.7125	0.5052
0.40	2.9598	2.8017	2.6807	2.5785	2.4897	2.4104	2.3382	2.2717	2.2098	2.1519	1.8980	1.6915	1.5153	1.3605
0.45	3.1554	3.0745	3.0129	2.9613	2.9161	2.8754	2.8382	2.8038	2.7715	2.7411	2.6055	2.4930	2.3954	2.3083
0.50	3.3538	3.3538	3.3538	3.3538	3.3538	3.3538	3.3538	3.3538	3.3538	3.3538	3.3538	3.3538	3.3538	3.3538
0.55	3.5565	3.6417	3.7076	3.7634	3.8129	3.8577	3.8992	3.9379	3.9744	4.0090	4.1618	4.2922	4.4082	4.5139
0.60	3.7651	3.9403	4.0766	4.1928	4.2961	4.3902	4.4773	4.5589	4.6360	4.7094	5.0229	5.2906	5.5302	5.7581
0.65	3.9862	4.2592	4.4728	4.6557	4.8155	4.9585	5.0910	5.2152	5.3326	5.4443	5.9591	6.4059	6.8065	7.1741
0.70	4.2238	4.6041	4.8966	5.1428	5.3624	5.5650	5.7593	5.9418	6.1148	6.2796	7.0149	7.6502	8.2204	8.7433
0.75	4.4855	4.9762	5.3547	5.6866	5.9898	6.2678	6.5264	6.7695	7.0000	7.2197	8.2004	9.0478	9.8076	10.5040
0.80	4.7811	5.3939	5.8932	6.3282	6.7183	7.0762	7.4092	7.7225	8.0194	8.3025	9.5654	10.6553	11.6314	12.5246
0.85	5.1233	5.9121	6.5517	7.1042	7.6000	8.0548	8.4781	8.8761	9.2533	9.6128	11.2148	12.5945	13.8277	14.9542
0.90	5.5622	6.5850	7.3999	8.1042	8.7361	9.3156	9.8547	10.3613	10.8410	11.2980	13.3307	15.0764	16.6329	18.0520
0.95	6.2562	7.6155	8.6995	9.6358	10.4750	11.2437	11.9580	12.6285	13.2627	13.8661	16.5429	18.8328	20.8686	22.7205

q = 0.005 ν = 9

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	1.0266	0.2604	-0.2757	-0.6979	-1.0511	-1.3582	-1.6298	-1.8752	-2.1010	-2.3108	-3.1876	-3.8925	-4.4978	-5.0332
0.10	1.4707	0.8307	0.3740	0.0117	-0.2929	-0.5573	-0.7921	-1.0045	-1.1991	-1.3793	-2.1249	-2.7181	-3.2159	-3.6555
0.15	1.7839	1.2416	0.8502	0.5346	0.2684	0.0364	-0.1707	-0.3580	-0.5295	-0.6880	-1.3469	-1.8630	-2.2976	-2.6782
0.20	2.0447	1.5874	1.2537	0.9830	0.7532	0.5501	0.3690	0.2049	0.0541	-0.0859	-0.6673	-1.1245	-1.5075	-1.8377
0.25	2.2712	1.8945	1.6154	1.3886	1.1921	1.0196	0.8650	0.7236	0.5926	0.4712	-0.0359	-0.4367	-0.7720	-1.0630
0.30	2.4806	2.1794	1.9554	1.7695	1.6099	1.4688	1.3405	1.2226	1.1137	1.0124	0.5843	0.2423	-0.0457	-0.2963
0.35	2.6792	2.4517	2.2811	2.1401	2.0182	1.9079	1.8080	1.7163	1.6312	1.5518	1.2116	0.9364	0.7023	0.4961
0.40	2.8707	2.7183	2.6030	2.5062	2.4218	2.3463	2.2776	2.2141	2.1550	2.0996	1.8591	1.6614	1.4920	1.3414
0.45	3.0591	2.9812	2.9219	2.8722	2.8287	2.7895	2.7536	2.7204	2.6892	2.6599	2.5318	2.4249	2.3320	2.2490
0.50	3.2500	3.2500	3.2500	3.2500	3.2500	3.2500	3.2500	3.2500	3.2500	3.2500	3.2500	3.2500	3.2500	3.2500
0.55	3.4416	3.5221	3.5843	3.6369	3.6836	3.7259	3.7649	3.8014	3.8358	3.8684	4.0116	4.1333	4.2416	4.3402
0.60	3.6385	3.8037	3.9319	4.0405	4.1370	4.2248	4.3061	4.3822	4.4541	4.5224	4.8241	5.0823	5.3131	5.5247
0.65	3.8469	4.1025	4.3019	4.4724	4.6242	4.7620	4.8897	5.0095	5.1228	5.2307	5.7116	6.1268	6.4986	6.8387
0.70	4.0695	4.4243	4.7024	4.9397	5.1516	5.3454	5.5258	5.6956	5.8563	6.0095	6.6914	7.2793	7.8067	8.2905
0.75	4.3137	4.7791	5.1442	5.4582	5.7402	5.9985	6.2388	6.4643	6.6776	6.8809	7.7882	8.5722	9.2753	9.9198
0.80	4.5909	5.1820	5.6503	6.0547	6.4169	6.7481	7.0563	7.3461	7.6208	7.8827	9.0512	10.0600	10.9640	11.7935
0.85	4.9210	5.6679	6.2623	6.7741	7.2328	7.6535	8.0451	8.4134	8.7623	9.0950	10.5780	11.8584	13.0042	14.0512
0.90	5.3427	6.2932	7.0477	7.6992	8.2838	8.8200	9.3188	9.7877	10.2319	10.6550	12.5423	14.1646	15.6096	16.9277
0.95	5.9877	7.2471	8.2499	9.1163	9.8930	10.6047	11.2673	11.8900	12.4791	13.0399	15.5260	17.6532	19.5456	21.2689

TABLE I (Continued)

q = 0.005 ν = 16

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.9822	0.2560	-0.2747	-0.7054	-1.0727	-1.3965	-1.6876	-1.9531	-2.1986	-2.4277	-3.3963	-4.1822	-4.8543	-5.4510
0.10	1.3874	0.7992	0.3660	0.0123	-0.2920	-0.5608	-0.8027	-1.0238	-1.2282	-1.4189	-2.2246	-2.8745	-3.4278	-3.9176
0.15	1.6680	1.1796	0.8175	0.5199	0.2639	0.0367	-0.1692	-0.3578	-0.5323	-0.6952	-1.3845	-1.9398	-2.4132	-2.8303
0.20	1.8966	1.4931	1.1907	0.9417	0.7263	0.5347	0.3612	0.2021	0.0542	-0.0845	-0.6739	-1.1497	-1.5558	-1.9123
0.25	2.0948	1.7652	1.5182	1.3126	1.1344	0.9756	0.8315	0.6985	0.5750	0.4594	-0.0348	-0.4377	-0.7819	-1.0851
0.30	2.2750	2.0148	1.8184	1.6553	1.5133	1.3856	1.2694	1.1623	1.0625	0.9690	0.5671	0.2384	-0.0446	-0.2954
0.35	2.4432	2.2504	2.1033	1.9804	1.8732	1.7770	1.6893	1.6082	1.5325	1.4609	1.1522	0.8983	0.6785	0.4832
0.40	2.6049	2.4763	2.3785	2.2966	2.2249	2.1599	2.1003	2.0451	1.9935	1.9449	1.7342	1.5594	1.4067	1.2701
0.45	2.7624	2.6974	2.6478	2.6061	2.5695	2.5365	2.5062	2.4781	2.4517	2.4268	2.3183	2.2276	2.1475	2.0755
0.50	2.9204	2.9204	2.9204	2.9204	2.9204	2.9204	2.9204	2.9204	2.9204	2.9204	2.9204	2.9204	2.9204	2.9204
0.55	3.0790	3.1451	3.1961	3.2393	3.2774	3.3119	3.3437	3.3734	3.4013	3.4277	3.5442	3.6430	3.7302	3.8094
0.60	3.2405	3.3752	3.4793	3.5677	3.6459	3.7167	3.7820	3.8431	3.9006	3.9552	4.1964	4.4023	4.5846	4.7507
0.65	3.4103	3.6180	3.7787	3.9152	4.0365	4.1468	4.2488	4.3444	4.4345	4.5196	4.8968	5.2201	5.5080	5.7709
0.70	3.5912	3.8768	4.0991	4.2887	4.4572	4.6100	4.7516	4.8843	5.0097	5.1289	5.6571	6.1107	6.5157	6.8858
0.75	3.7881	4.1604	4.4514	4.6987	4.9191	5.1204	5.3071	5.4815	5.6465	5.8035	6.5015	7.1011	7.6376	8.1276
0.80	4.0099	4.4813	4.8489	5.1641	5.4448	5.7010	5.9388	6.1622	6.3733	6.5738	7.4670	8.2342	8.9212	9.5495
0.85	4.2738	4.8627	5.3253	5.7210	6.0749	6.3984	6.6982	6.9798	7.2464	7.5004	8.6280	9.5987	10.4671	11.2617
0.90	4.6079	5.3492	5.9322	6.4334	6.8807	7.2905	7.6707	8.0272	8.3648	8.6865	10.1170	11.3479	12.4479	13.4530
0.95	5.1120	6.0859	6.8548	7.5166	8.1072	8.6483	9.1510	9.6226	10.0691	10.4942	12.3843	14.0067	15.4518	16.7690

q = 0.005 ν = 36

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.9527	0.2519	-0.2742	-0.7103	-1.0891	-1.4271	-1.7346	-2.0181	-2.2821	-2.5295	-3.5932	-4.4654	-5.2205	-5.8922
0.10	1.3332	0.7783	0.3596	0.0116	-0.2915	-0.5629	-0.8101	-1.0384	-1.2510	-1.4507	-2.3101	-3.0167	-3.6280	-4.1710
0.15	1.5931	1.1391	0.7958	0.5094	0.2596	0.0357	-0.1687	-0.3577	-0.5340	-0.6998	-1.4145	-2.0039	-2.5138	-2.9681
0.20	1.8028	1.4312	1.1495	0.9142	0.7085	0.5237	0.3550	0.1988	0.0529	-0.0843	-0.6781	-1.1691	-1.5947	-1.9745
0.25	1.9825	1.6826	1.4545	1.2636	1.0966	0.9465	0.8091	0.6818	0.5628	0.4506	-0.0350	-0.4382	-0.7888	-1.1020
0.30	2.1449	1.9102	1.7314	1.5814	1.4499	1.3316	1.2233	1.1228	1.0288	0.9402	0.5551	0.2346	-0.0447	-0.2949
0.35	2.2963	2.1227	1.9902	1.8790	1.7814	1.6934	1.6127	1.5378	1.4676	1.4014	1.1134	0.8729	0.6626	0.4738
0.40	2.4408	2.3260	2.2382	2.1645	2.0996	2.0412	1.9875	1.9376	1.8909	1.8467	1.6541	1.4926	1.3512	1.2239
0.45	2.5805	2.5230	2.4790	2.4419	2.4093	2.3798	2.3527	2.3276	2.3039	2.2816	2.1841	2.1021	2.0300	1.9651
0.50	2.7198	2.7198	2.7198	2.7198	2.7198	2.7198	2.7198	2.7198	2.7198	2.7198	2.7198	2.7198	2.7198	2.7198
0.55	2.8592	2.9172	2.9618	2.9994	3.0325	3.0626	3.0902	3.1159	3.1401	3.1630	3.2636	3.3486	3.4237	3.4917
0.60	3.0005	3.1175	3.2076	3.2838	3.3511	3.4121	3.4682	3.5206	3.5698	3.6165	3.8211	3.9945	4.1480	4.2873
0.65	3.1480	3.3271	3.4653	3.5823	3.6857	3.7792	3.8654	3.9458	4.0215	4.0933	4.4094	4.6782	4.9162	5.1326
0.70	3.3041	3.5495	3.7388	3.8989	4.0407	4.1693	4.2881	4.3990	4.5034	4.6026	5.0390	5.4108	5.7414	6.0419
0.75	3.4735	3.7907	4.0357	4.2437	4.4280	4.5955	4.7500	4.8943	5.0303	5.1594	5.7299	6.2161	6.6488	7.0435
0.80	3.6631	4.0609	4.3694	4.6317	4.8640	5.0751	5.2703	5.4529	5.6252	5.7888	6.5111	7.1290	7.6792	8.1814
0.85	3.8864	4.3809	4.7651	5.0916	5.3816	5.6456	5.8898	6.1180	6.3334	6.5380	7.4446	8.2207	8.9118	9.5419
0.90	4.1675	4.7849	5.2649	5.6742	6.0378	6.3689	6.6755	6.9627	7.2338	7.4914	8.6338	9.6102	10.4812	11.2758
0.95	4.5885	5.3906	6.0168	6.5511	7.0271	7.4609	7.8629	8.2398	8.5957	8.9333	10.4309	11.7135	12.8570	13.8980

TABLE I (Continued)

q = 0.005 ν = 144

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.9370	0.2505	-0.2735	-0.7133	-1.0993	-1.4470	-1.7658	-2.0618	-2.3391	-2.6007	-3.7394	-4.6913	-5.5241	-6.2726
0.10	1.3041	0.7674	0.3569	0.0120	-0.2909	-0.5641	-0.8147	-1.0474	-1.2655	-1.4714	-2.3687	-3.1196	-3.7772	-4.3687
0.15	1.5525	1.1173	0.7844	0.5043	0.2581	0.0360	-0.1679	-0.3573	-0.5349	-0.7027	-1.4340	-2.0469	-2.5841	-3.0676
0.20	1.7517	1.3980	1.1273	0.8996	0.6992	0.5184	0.3523	0.1979	0.0531	-0.0836	-0.6806	-1.1813	-1.6205	-2.0161
0.25	1.9217	1.6377	1.4202	1.2373	1.0763	0.9310	0.7974	0.6732	0.5567	0.4465	-0.0345	-0.4384	-0.7930	-1.1125
0.30	2.0746	1.8534	1.6840	1.5414	1.4159	1.3025	1.1984	1.1016	1.0107	0.9248	0.5492	0.2334	-0.0441	-0.2943
0.35	2.2165	2.0537	1.9290	1.8239	1.7315	1.6480	1.5712	1.4998	1.4328	1.3695	1.0925	0.8595	0.6544	0.4693
0.40	2.3513	2.2442	2.1621	2.0929	2.0320	1.9770	1.9264	1.8793	1.8351	1.7934	1.6106	1.4567	1.3213	1.1991
0.45	2.4813	2.4279	2.3869	2.3524	2.3220	2.2945	2.2692	2.2457	2.2236	2.2027	2.1113	2.0343	1.9665	1.9053
0.50	2.6103	2.6103	2.6103	2.6103	2.6103	2.6103	2.6103	2.6103	2.6103	2.6103	2.6103	2.6103	2.6103	2.6103
0.55	2.7396	2.7931	2.8342	2.8689	2.8995	2.9271	2.9525	2.9762	2.9984	3.0194	3.1116	3.1893	3.2579	3.3199
0.60	2.8699	2.9776	3.0603	3.1301	3.1916	3.2473	3.2985	3.3461	3.3909	3.4333	3.6191	3.7759	3.9143	4.0395
0.65	3.0056	3.1697	3.2958	3.4023	3.4961	3.5810	3.6591	3.7319	3.8003	3.8650	4.1489	4.3887	4.6003	4.7919
0.70	3.1487	3.3724	3.5444	3.6895	3.8176	3.9335	4.0402	4.1395	4.2329	4.3214	4.7092	5.0371	5.3268	5.5892
0.75	3.3032	3.5915	3.8131	4.0003	4.1655	4.3150	4.4527	4.5809	4.7014	4.8156	5.3167	5.7407	6.1155	6.4553
0.80	3.4756	3.8359	4.1131	4.3473	4.5540	4.7411	4.9134	5.0741	5.2251	5.3681	5.9966	6.5288	6.9995	7.4265
0.85	3.6782	4.1234	4.4661	4.7557	5.0114	5.2430	5.4564	5.6554	5.8426	6.0199	6.7992	7.4598	8.0445	8.5748
0.90	3.9319	4.4837	4.9086	5.2681	5.5856	5.8734	6.1386	6.3860	6.6187	6.8393	7.8094	8.6321	9.3604	10.0218
0.95	4.3089	5.0193	5.5673	6.0312	6.4413	6.8131	7.1560	7.4760	7.7772	8.0627	9.3184	10.3849	11.3299	12.1885

q = 0.010 ν = 4

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.8091	-0.0005	-0.5153	-0.9007	-1.2154	-1.4791	-1.7109	-1.9209	-2.1137	-2.2927	-3.0406	-3.6404	-4.1581	-4.6197
0.10	1.3236	0.5896	0.1144	-0.2441	-0.5312	-0.7736	-0.9851	-1.1741	-1.3439	-1.4970	-2.1341	-2.6412	-3.0646	-3.4383
0.15	1.7175	1.0512	0.6109	0.2791	0.0076	-0.2204	-0.4172	-0.5914	-0.7484	-0.8917	-1.4695	-1.9104	-2.2814	-2.6066
0.20	2.0613	1.4683	1.0651	0.7592	0.5060	0.2949	0.1093	-0.0557	-0.2033	-0.3372	-0.8731	-1.2801	-1.6061	-1.8888
0.25	2.3599	1.8632	1.5030	1.2233	0.9945	0.8010	0.6273	0.4747	0.3384	0.2146	-0.2896	-0.6637	-0.9671	-1.2258
0.30	2.6451	2.2376	1.9452	1.6988	1.4962	1.3213	1.1664	1.0293	0.9062	0.7928	0.3298	-0.0185	-0.2990	-0.5344
0.35	2.9234	2.6052	2.3732	2.1858	2.0270	1.8812	1.7489	1.6303	1.5228	1.4244	1.0168	0.7067	0.4523	0.2401
0.40	3.1997	2.9794	2.8153	2.6804	2.5642	2.4613	2.3685	2.2837	2.2054	2.1326	1.8161	1.5607	1.3501	1.1674
0.45	3.4757	3.3611	3.2743	3.2018	3.1386	3.0819	3.0302	2.9824	2.9378	2.8959	2.7159	2.5685	2.4419	2.3302
0.50	3.7586	3.7586	3.7586	3.7586	3.7586	3.7586	3.7586	3.7586	3.7586	3.7586	3.7586	3.7586	3.7586	3.7586
0.55	4.0494	4.1720	4.2668	4.3473	4.4185	4.4832	4.5429	4.5986	4.6512	4.7010	4.9206	5.1076	5.2735	5.4244
0.60	4.3497	4.6021	4.7982	4.9651	5.1132	5.2478	5.3723	5.4886	5.5983	5.7023	6.1835	6.6139	6.9984	7.3497
0.65	4.6682	5.0604	5.3659	5.6261	5.8572	6.0800	6.2929	6.4926	6.6814	6.8610	7.6590	8.3445	8.9568	9.5167
0.70	5.0096	5.5529	5.9807	6.3762	6.7293	7.0521	7.3516	7.6325	7.8982	8.1510	9.2744	10.2903	11.2179	12.0663
0.75	5.3839	6.1085	6.7170	7.2395	7.7063	8.1329	8.5289	8.9003	9.2517	9.5883	11.1854	12.5592	13.7859	14.9058
0.80	5.8065	6.7800	7.5576	8.2255	8.8222	9.3677	9.8975	10.4080	10.8912	11.3512	13.3954	15.1487	16.7110	18.1347
0.85	6.3450	7.5867	8.5676	9.4104	10.2085	10.9487	11.6363	12.2814	12.8915	13.4718	16.0451	18.2460	20.2028	21.9832
0.90	7.0476	8.6185	9.8823	11.0291	12.0546	12.9921	13.8616	14.6766	15.4465	16.1781	19.4152	22.1758	24.5768	26.7527
0.95	8.1152	10.2337	11.9953	13.5088	14.8593	16.0913	17.2322	18.3000	19.3073	20.2636	24.4387	27.9481	31.0603	33.8863

TABLE I (Continued)

q = 0.010 ν = 5

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.7793	0.0051	-0.5165	-0.9130	-1.2379	-1.5181	-1.7635	-1.9823	-2.1835	-2.3706	-3.1617	-3.7938	-4.3377	-4.8226
0.10	1.2578	0.5759	0.1162	-0.2393	-0.5328	-0.7826	-1.0000	-1.1952	-1.3731	-1.5373	-2.2048	-2.7352	-3.1874	-3.5815
0.15	1.6119	1.0084	0.5958	0.2752	0.0130	-0.2153	-0.4159	-0.5948	-0.7567	-0.9038	-1.5078	-1.9713	-2.3588	-2.6990
0.20	1.9117	1.3880	1.0215	0.7335	0.4959	0.2907	0.1113	-0.0489	-0.1979	-0.3341	-0.8847	-1.3050	-1.6542	-1.9488
0.25	2.1823	1.7387	1.4194	1.1678	0.9550	0.7719	0.6111	0.4657	0.3332	0.2121	-0.2856	-0.6694	-0.9814	-1.2487
0.30	2.4309	2.0717	1.8082	1.5953	1.4133	1.2557	1.1162	0.9878	0.8713	0.7643	0.3249	-0.0125	-0.2951	-0.5361
0.35	2.6698	2.3965	2.1943	2.0248	1.8805	1.7541	1.6398	1.5341	1.4373	1.3480	0.9760	0.6850	0.4440	0.2369
0.40	2.9004	2.7177	2.5772	2.4613	2.3611	2.2721	2.1900	2.1134	2.0426	1.9766	1.6986	1.4716	1.2814	1.1170
0.45	3.1237	3.0312	2.9609	2.9021	2.8507	2.8045	2.7611	2.7202	2.6821	2.6463	2.4918	2.3648	2.2553	2.1555
0.50	3.3513	3.3513	3.3513	3.3513	3.3513	3.3513	3.3513	3.3513	3.3513	3.3513	3.3513	3.3513	3.3513	3.3513
0.55	3.5979	3.7021	3.7828	3.8515	3.9124	3.9677	4.0171	4.0622	4.1047	4.1451	4.3232	4.4751	4.6101	4.7390
0.60	3.8535	4.0650	4.2239	4.3593	4.4796	4.5892	4.6937	4.7948	4.8903	4.9813	5.3856	5.7337	6.0453	6.3306
0.65	4.1185	4.4367	4.6882	4.9146	5.1170	5.3021	5.4740	5.6355	5.7883	5.9339	6.5824	7.1415	7.6424	8.1014
0.70	4.3955	4.8507	5.2220	5.5413	5.8271	6.0888	6.3322	6.5608	6.7772	6.9835	7.9026	8.6952	9.4051	10.0553
0.75	4.7038	5.3250	5.8171	6.2411	6.6208	6.9687	7.2922	7.5962	7.8840	8.1583	9.3803	10.4333	11.4072	12.3069
0.80	5.0725	5.8682	6.4998	7.0443	7.5322	7.9792	8.3948	8.7853	9.1550	9.5073	11.0938	12.5022	13.7596	14.9071
0.85	5.5161	6.5235	7.3239	8.0142	8.6326	9.1991	9.7257	10.2202	10.6897	11.1551	13.2234	14.9969	16.5762	18.0147
0.90	6.0852	7.3655	8.3832	9.2606	10.0463	10.7704	11.4680	12.1227	12.7417	13.3305	15.9403	18.1704	20.1516	21.9280
0.95	6.9542	8.6519	10.0008	11.1848	12.2695	13.2606	14.1795	15.0404	15.8532	16.6253	20.0372	22.9009	25.4347	27.7364

q = 0.010 ν = 6

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.7661	-0.0036	-0.5204	-0.9185	-1.2498	-1.5369	-1.7924	-2.0236	-2.2323	-2.4265	-3.2484	-3.9133	-4.4800	-4.9852
0.10	1.2201	0.5653	0.1088	-0.2466	-0.5367	-0.7862	-1.0070	-1.2061	-1.3882	-1.5567	-2.2544	-2.8052	-3.2751	-3.6907
0.15	1.5479	0.9875	0.5851	0.2689	0.0043	-0.2227	-0.4209	-0.5983	-0.7600	-0.9091	-1.5264	-2.0123	-2.4143	-2.7675
0.20	1.8272	1.3414	0.9996	0.7211	0.4871	0.2845	0.1038	-0.0582	-0.2055	-0.3402	-0.8898	-1.3184	-1.6768	-1.9887
0.25	2.0748	1.6654	1.3706	1.1356	0.9379	0.7588	0.6002	0.4576	0.3272	0.2055	-0.2925	-0.6726	-0.9881	-1.2609
0.30	2.2947	1.9759	1.7305	1.5327	1.3649	1.2181	1.0871	0.9684	0.8563	0.7514	0.3188	-0.0214	-0.3019	-0.5399
0.35	2.5055	2.2643	2.0852	1.9323	1.7981	1.6798	1.5734	1.4765	1.3872	1.3043	0.9574	0.6732	0.4364	0.2304
0.40	2.7178	2.5475	2.4240	2.3217	2.2328	2.1535	2.0815	2.0144	1.9488	1.8876	1.6277	1.4188	1.2421	1.0879
0.45	2.9308	2.8424	2.7754	2.7194	2.6705	2.6265	2.5864	2.5497	2.5163	2.4848	2.3487	2.2361	2.1385	2.0517
0.50	3.1487	3.1487	3.1487	3.1487	3.1487	3.1487	3.1487	3.1487	3.1487	3.1487	3.1487	3.1487	3.1487	3.1487
0.55	3.3635	3.4540	3.5241	3.5835	3.6362	3.6841	3.7283	3.7696	3.8086	3.8456	4.0091	4.1487	4.2731	4.3867
0.60	3.5853	3.7722	3.9179	4.0422	4.1529	4.2538	4.3474	4.4350	4.5179	4.5968	4.9471	5.2486	5.5185	5.7657
0.65	3.8212	4.1134	4.3426	4.5390	4.7144	4.8747	5.0237	5.1635	5.2959	5.4220	5.9839	6.4688	6.9037	7.3024
0.70	4.0755	4.4836	4.8053	5.0820	5.3295	5.5562	5.7670	5.9652	6.1528	6.3317	7.1297	7.8188	8.4368	9.0032
0.75	4.3561	4.8946	5.3208	5.6881	6.0172	6.3189	6.5995	6.8635	7.1136	7.3519	8.4151	9.3327	10.1547	10.9316
0.80	4.6758	5.3651	5.9123	6.3845	6.8079	7.1962	7.5575	7.8972	8.2190	8.5257	9.8928	11.1008	12.1909	13.1869
0.85	5.0601	5.9328	6.6271	7.2267	7.7644	8.2574	8.7160	9.1470	9.5552	9.9440	11.7259	13.2636	14.6019	15.8232
0.90	5.5531	6.6632	7.5474	8.3109	8.9954	9.6226	10.2056	10.7721	11.3083	11.8187	14.0627	15.9555	17.6408	19.1757
0.95	6.3063	7.7811	8.9557	9.9689	10.8992	11.7582	12.5553	13.3005	13.9889	14.6436	17.5434	20.0268	22.2547	24.2780

TABLE I (Continued)

q = 0.010 ν = 7

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.7514	-0.0020	-0.5183	-0.9232	-1.2617	-1.5534	-1.8138	-2.0505	-2.2684	-2.4700	-3.3221	-4.0132	-4.6014	-5.1213
0.10	1.1924	0.5554	0.1084	-0.2422	-0.5348	-0.7883	-1.0135	-1.2171	-1.4023	-1.5736	-2.2914	-2.8626	-3.3498	-3.7806
0.15	1.5073	0.9638	0.5748	0.2649	0.0058	-0.2185	-0.4176	-0.5973	-0.7617	-0.9136	-1.5427	-2.0387	-2.4573	-2.8235
0.20	1.7696	1.3110	0.9758	0.7076	0.4787	0.2800	0.1036	-0.0557	-0.2015	-0.3360	-0.8938	-1.3314	-1.6959	-2.0143
0.25	1.9992	1.6180	1.3392	1.1095	0.9148	0.7443	0.5896	0.4497	0.3215	0.2031	-0.2880	-0.6727	-0.9942	-1.2730
0.30	2.2097	1.9072	1.6792	1.4929	1.3338	1.1905	1.0619	0.9450	0.8374	0.7371	0.3134	-0.0195	-0.2974	-0.5380
0.35	2.4108	2.1806	2.0091	1.8677	1.7425	1.6316	1.5314	1.4397	1.3550	1.2747	0.9341	0.6609	0.4288	0.2274
0.40	2.6073	2.4509	2.3332	2.2354	2.1505	2.0746	2.0056	1.9420	1.8828	1.8260	1.5825	1.3851	1.2140	1.0627
0.45	2.8014	2.7210	2.6599	2.6088	2.5641	2.5239	2.4871	2.4530	2.4211	2.3911	2.2612	2.1536	2.0602	1.9769
0.50	2.9988	2.9988	2.9988	2.9988	2.9988	2.9988	2.9988	2.9988	2.9988	2.9988	2.9988	2.9988	2.9988	2.9988
0.55	3.1936	3.2755	3.3388	3.3925	3.4400	3.4832	3.5230	3.5602	3.5954	3.6300	3.7830	3.9136	4.0301	4.1364
0.60	3.3941	3.5625	3.6977	3.8140	3.9175	4.0120	4.0996	4.1817	4.2593	4.3306	4.6454	4.9160	5.1581	5.3798
0.65	3.6073	3.8806	4.0951	4.2786	4.4363	4.5804	4.7142	4.8397	4.9585	5.0716	5.5754	6.0100	6.3997	6.7571
0.70	3.8451	4.2271	4.5181	4.7665	4.9886	5.1920	5.3810	5.5586	5.7268	5.8872	6.6023	7.2201	7.7743	8.2871
0.75	4.1077	4.5983	4.9809	5.3102	5.6053	5.8757	6.1272	6.3637	6.5878	6.8015	7.7549	8.5934	9.3579	10.0568
0.80	4.4017	5.0206	5.5112	5.9345	6.3139	6.6619	6.9858	7.2904	7.5790	7.8541	9.1142	10.2047	11.1579	12.0299
0.85	4.7469	5.5296	6.1519	6.6892	7.1713	7.6134	8.0248	8.4207	8.8002	9.1619	10.7511	12.0981	13.3106	14.4322
0.90	5.1892	6.1842	6.9768	7.6614	8.2799	8.8629	9.4052	9.9148	10.3860	10.8323	12.8165	14.5538	16.1013	17.5104
0.95	5.8644	7.1863	8.2430	9.1850	10.0285	10.7794	11.4768	12.1313	12.7501	13.3489	16.0119	18.2850	20.2894	22.1026

q = 0.010 ν = 8

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.7455	0.0028	-0.5236	-0.9320	-1.2728	-1.5703	-1.8364	-2.0786	-2.3018	-2.5095	-3.3755	-4.0782	-4.6836	-5.2196
0.10	1.1658	0.5571	0.1134	-0.2417	-0.5404	-0.7968	-1.0227	-1.2276	-1.4160	-1.5908	-2.3254	-2.9091	-3.4036	-3.8415
0.15	1.4724	0.9486	0.5762	0.2692	0.0107	-0.2175	-0.4206	-0.6043	-0.7702	-0.9224	-1.5593	-2.0665	-2.4964	-2.8695
0.20	1.7295	1.2795	0.9601	0.7041	0.4813	0.2843	0.1086	-0.0519	-0.2001	-0.3374	-0.9026	-1.3437	-1.7159	-2.0415
0.25	1.9469	1.5811	1.3069	1.0873	0.9019	0.7388	0.5908	0.4525	0.3255	0.2077	-0.2883	-0.6813	-1.0033	-1.2842
0.30	2.1463	1.8595	1.6410	1.4582	1.3016	1.1639	1.0421	0.9307	0.8278	0.7320	0.3174	-0.0150	-0.2980	-0.5438
0.35	2.3359	2.1188	1.9563	1.8221	1.7030	1.5943	1.4961	1.4059	1.3225	1.2446	0.9203	0.6600	0.4319	0.2319
0.40	2.5202	2.3735	2.2628	2.1706	2.0903	2.0185	1.9530	1.8926	1.8363	1.7835	1.5463	1.3521	1.1861	1.0428
0.45	2.7015	2.6265	2.5694	2.5216	2.4797	2.4420	2.4075	2.3755	2.3455	2.3173	2.1950	2.0933	2.0049	1.9258
0.50	2.8852	2.8852	2.8852	2.8852	2.8852	2.8852	2.8852	2.8852	2.8852	2.8852	2.8852	2.8852	2.8852	2.8852
0.55	3.0728	3.1516	3.2125	3.2641	3.3098	3.3514	3.3897	3.4255	3.4592	3.4912	3.6326	3.7531	3.8604	3.9582
0.60	3.2657	3.4277	3.5538	3.6612	3.7567	3.8438	3.9244	3.9999	4.0713	4.1391	4.4406	4.6998	4.9315	5.1365
0.65	3.4702	3.7226	3.9203	4.0894	4.2404	4.3783	4.5064	4.6267	4.7405	4.8490	5.3173	5.7188	6.0787	6.4086
0.70	3.6899	4.0417	4.3186	4.5565	4.7694	4.9628	5.1376	5.3018	5.4572	5.6053	6.2657	6.8360	7.3475	7.8167
0.75	3.9319	4.3954	4.7620	5.0722	5.3449	5.5947	5.8270	6.0454	6.2523	6.4496	7.3296	8.0906	8.7932	9.4372
0.80	4.2072	4.8001	5.2580	5.6490	5.9994	6.3207	6.6197	6.9008	7.1672	7.4212	8.5692	9.5771	10.4653	11.2707
0.85	4.5378	5.2750	5.8498	6.3459	6.7909	7.1990	7.5787	7.9358	8.2806	8.6130	10.0897	11.3338	12.4458	13.4615
0.90	4.9602	5.8797	6.6113	7.2433	7.8102	8.3382	8.8367	9.3052	9.7489	10.1647	11.9976	13.5716	14.9751	16.2545
0.95	5.5843	6.8048	7.7773	8.6343	9.4104	10.1158	10.7598	11.3644	11.9363	12.4804	14.8939	16.9585	18.7938	20.4791

TABLE I (Continued)

q = 0.010 ν = 9

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.7329	-0.0003	-0.5199	-0.9330	-1.2808	-1.5850	-1.8560	-2.1020	-2.3289	-2.5404	-3.4309	-4.1518	-4.7728	-5.3241
0.10	1.1539	0.5461	0.1091	-0.2405	-0.5366	-0.7951	-1.0255	-1.2347	-1.4272	-1.6060	-2.3530	-2.9525	-3.4597	-3.9090
0.15	1.4497	0.9371	0.5646	0.2632	0.0075	-0.2167	-0.4176	-0.6001	-0.7679	-0.9233	-1.5738	-2.0897	-2.5270	-2.9119
0.20	1.6949	1.2643	0.9486	0.6914	0.4721	0.2781	0.1043	-0.0538	-0.1995	-0.3352	-0.9030	-1.3533	-1.7336	-2.0643
0.25	1.9076	1.5538	1.2908	1.0761	0.8902	0.7262	0.5788	0.4438	0.3187	0.2025	-0.2867	-0.6771	-1.0058	-1.2925
0.30	2.1038	1.8215	1.6110	1.4362	1.2856	1.1521	1.0308	0.9191	0.8157	0.7194	0.3108	-0.0177	-0.2962	-0.5399
0.35	2.2891	2.0767	1.9169	1.7846	1.6700	1.5664	1.4724	1.3860	1.3057	1.2306	0.9086	0.6469	0.4235	0.2264
0.40	2.4673	2.3256	2.2181	2.1276	2.0487	1.9780	1.9136	1.8541	1.7986	1.7465	1.5205	1.3342	1.1741	1.0315
0.45	2.6422	2.5699	2.5148	2.4687	2.4282	2.3918	2.3584	2.3275	2.2985	2.2712	2.1516	2.0517	1.9647	1.8868
0.50	2.8191	2.8191	2.8191	2.8191	2.8191	2.8191	2.8191	2.8191	2.8191	2.8191	2.8191	2.8191	2.8191	2.8191
0.55	2.9977	3.0726	3.1305	3.1795	3.2229	3.2623	3.2986	3.3325	3.3645	3.3948	3.5280	3.6412	3.7419	3.8336
0.60	3.1810	3.3346	3.4539	3.5549	3.6446	3.7263	3.8018	3.8726	3.9394	4.0029	4.2828	4.5221	4.7360	4.9311
0.65	3.3749	3.6126	3.7980	3.9564	4.0975	4.2252	4.3436	4.4546	4.5596	4.6595	5.1026	5.4832	5.8244	6.1372
0.70	3.5819	3.9117	4.1700	4.3899	4.5863	4.7659	4.9322	5.0878	5.2352	5.3757	6.0017	6.5424	7.0274	7.4723
0.75	3.8089	4.2411	4.5794	4.8701	5.1287	5.3656	5.5859	5.7929	5.9890	6.1760	7.0104	7.7314	8.3770	8.9680
0.80	4.0665	4.6145	5.0463	5.4171	5.7493	6.0539	6.3373	6.6038	6.8564	7.0973	8.1715	9.0965	9.9252	10.6853
0.85	4.3725	5.0625	5.6075	6.0778	6.4996	6.8866	7.2467	7.5853	7.9062	8.2116	9.5714	10.7448	11.7944	12.7533
0.90	4.7634	5.6358	6.3294	6.9286	7.4662	7.9592	8.4169	8.8469	9.2541	9.6420	11.3714	12.8571	14.1801	15.3866
0.95	5.3557	6.5128	7.4350	8.2312	8.9434	9.5959	10.2032	10.7737	11.3135	11.8271	14.1036	16.0504	17.7816	19.3565

q = 0.010 ν = 16

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.7100	-0.0002	-0.5209	-0.9452	-1.3093	-1.6313	-1.9218	-2.1873	-2.4334	-2.6635	-3.6412	-4.4384	-5.1226	-5.7315
0.10	1.1034	0.5317	0.1077	-0.2393	-0.5379	-0.8025	-1.0416	-1.2608	-1.4639	-1.6537	-2.4595	-3.1135	-3.6731	-4.1697
0.15	1.3753	0.9019	0.5496	0.2586	0.0075	-0.2155	-0.4173	-0.6026	-0.7744	-0.9351	-1.6194	-2.1740	-2.6490	-3.0688
0.20	1.5962	1.2058	0.9126	0.6706	0.4605	0.2731	0.1030	-0.0532	-0.1984	-0.3342	-0.9141	-1.3858	-1.7901	-2.1465
0.25	1.7875	1.4693	1.2302	1.0310	0.8580	0.7036	0.5632	0.4333	0.3125	0.1993	-0.2855	-0.6812	-1.0210	-1.3216
0.30	1.9613	1.7104	1.5207	1.3630	1.2254	1.1018	0.9891	0.8851	0.7882	0.6972	0.3048	-0.0175	-0.2951	-0.5412
0.35	2.1237	1.9375	1.7957	1.6772	1.5737	1.4807	1.3958	1.3173	1.2440	1.1747	0.8753	0.6283	0.4138	0.2226
0.40	2.2797	2.1557	2.0613	1.9822	1.9129	1.8502	1.7928	1.7396	1.6898	1.6429	1.4393	1.2701	1.1222	0.9898
0.45	2.4314	2.3689	2.3211	2.2809	2.2456	2.2137	2.1845	2.1574	2.1319	2.1079	2.0032	1.9155	1.8383	1.7689
0.50	2.5835	2.5835	2.5835	2.5835	2.5835	2.5835	2.5835	2.5835	2.5835	2.5835	2.5835	2.5835	2.5835	2.5835
0.55	2.7362	2.8000	2.8490	2.8905	2.9272	2.9604	2.9910	3.0195	3.0463	3.0718	3.1836	3.2785	3.3619	3.4376
0.60	2.8917	3.0212	3.1213	3.2062	3.2813	3.3490	3.4114	3.4698	3.5247	3.5769	3.8070	4.0033	4.1772	4.3357
0.65	3.0550	3.2545	3.4082	3.5387	3.6544	3.7597	3.8570	3.9480	4.0339	4.1152	4.4749	4.7830	5.0574	5.3081
0.70	3.2288	3.5020	3.7142	3.8950	4.0556	4.2015	4.3365	4.4630	4.5825	4.6961	5.1996	5.6318	6.0169	6.3683
0.75	3.4173	3.7727	4.0500	4.2860	4.4962	4.6880	4.8658	5.0321	5.1894	5.3391	6.0034	6.5727	7.0818	7.5472
0.80	3.6290	4.0786	4.4293	4.7296	4.9971	5.2414	5.4681	5.6808	5.8816	6.0721	6.9198	7.6484	8.3005	8.8964
0.85	3.8808	4.4424	4.8831	5.2605	5.5976	5.9054	6.1902	6.4575	6.7105	6.9515	8.0222	8.9430	9.7661	10.5185
0.90	4.1995	4.9059	5.4617	5.9387	6.3635	6.7523	7.1132	7.4519	7.7724	8.0777	9.4343	10.6000	11.6400	12.5896
0.95	4.6800	5.6082	6.3389	6.9668	7.5279	8.0415	8.5184	8.9657	9.3889	9.7918	11.5798	13.1126	14.4770	15.7200

TABLE I (Continued)

q = 0.010 ν = 36

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.6933	-0.0002	-0.5221	-0.9556	-1.3328	-1.6697	-1.9767	-2.2600	-2.5240	-2.7717	-3.8389	-4.7177	-5.4799	-6.1592
0.10	1.0695	0.5208	0.1065	-0.2384	-0.5394	-0.8090	-1.0550	-1.2822	-1.4941	-1.6933	-2.5521	-3.2601	-3.8740	-4.4209
0.15	1.3261	0.8777	0.5381	0.2548	0.0074	-0.2146	-0.4174	-0.6051	-0.7803	-0.9452	-1.6572	-2.2458	-2.7561	-3.2113
0.20	1.5329	1.1663	0.8879	0.6553	0.4518	0.2690	0.1019	-0.0529	-0.1975	-0.3336	-0.9236	-1.4125	-1.8370	-2.2164
0.25	1.7098	1.4143	1.1892	1.0008	0.8357	0.6872	0.5513	0.4254	0.3076	0.1966	-0.2847	-0.6851	-1.0337	-1.3456
0.30	1.8696	1.6386	1.4625	1.3145	1.1847	1.0679	0.9609	0.8616	0.7686	0.6810	0.3001	-0.0174	-0.2943	-0.5427
0.35	2.0184	1.8477	1.7174	1.6080	1.5118	1.4250	1.3454	1.2715	1.2022	1.1369	0.8522	0.6144	0.4064	0.2196
0.40	2.1604	2.0476	1.9614	1.8888	1.8251	1.7675	1.7147	1.6656	1.6196	1.5762	1.3862	1.2269	1.0872	0.9615
0.45	2.2977	2.2412	2.1979	2.1615	2.1294	2.1005	2.0739	2.0492	2.0259	2.0040	1.9081	1.8275	1.7566	1.6927
0.50	2.4344	2.4344	2.4344	2.4344	2.4344	2.4344	2.4344	2.4344	2.4344	2.4344	2.4344	2.4344	2.4344	2.4344
0.55	2.5714	2.6283	2.6721	2.7090	2.7415	2.7710	2.7981	2.8234	2.8472	2.8696	2.9683	3.0516	3.1252	3.1920
0.60	2.7101	2.8250	2.9134	2.9881	3.0541	3.1139	3.1689	3.2202	3.2685	3.3142	3.5150	3.6851	3.8357	3.9723
0.65	2.8548	3.0306	3.1661	3.2808	3.3821	3.4738	3.5584	3.6373	3.7116	3.7820	4.0920	4.3554	4.5884	4.8002
0.70	3.0080	3.2485	3.4342	3.5914	3.7304	3.8566	3.9731	4.0818	4.1842	4.2813	4.7086	5.0723	5.3954	5.6890
0.75	3.1740	3.4852	3.7256	3.9296	4.1102	4.2743	4.4257	4.5669	4.7001	4.8264	5.3842	5.8590	6.2813	6.6662
0.80	3.3599	3.7503	4.0529	4.3098	4.5373	4.7439	4.9349	5.1134	5.2818	5.4417	6.1470	6.7494	7.2849	7.7732
0.85	3.5790	4.0641	4.4405	4.7601	5.0437	5.3019	5.5404	5.7632	5.9735	6.1733	7.0566	7.8114	8.4833	9.0957
0.90	3.8549	4.4598	4.9296	5.3298	5.6850	6.0082	6.3073	6.5874	6.8515	7.1022	8.2129	9.1620	10.0079	10.7782
0.95	4.2675	5.0525	5.6645	6.1860	6.6502	7.0725	7.4636	7.8299	8.1759	8.5041	9.9591	11.2024	12.3100	13.3189

q = 0.010 ν = 144

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.6840	-0.0005	-0.5231	-0.9620	-1.3473	-1.6945	-2.0129	-2.3087	-2.5858	-2.8474	-3.9874	-4.9413	-5.7761	-6.5269
0.10	1.0502	0.5149	0.1057	-0.2383	-0.5405	-0.8130	-1.0632	-1.2955	-1.5132	-1.7188	-2.6154	-3.3666	-4.0252	-4.6179
0.15	1.2980	0.8639	0.5318	0.2526	0.0071	-0.2144	-0.4177	-0.6067	-0.7839	-0.9513	-1.6815	-2.2938	-2.8308	-3.3145
0.20	1.4966	1.1438	0.8739	0.6467	0.4470	0.2667	0.1011	-0.0529	-0.1973	-0.3337	-0.9294	-1.4292	-1.8678	-2.2630
0.25	1.6660	1.3829	1.1660	0.9835	0.8230	0.6780	0.5448	0.4210	0.3048	0.1950	-0.2846	-0.6876	-1.0415	-1.3604
0.30	1.8184	1.5979	1.4290	1.2868	1.1617	1.0486	0.9448	0.8482	0.7576	0.6719	0.2974	-0.0175	-0.2942	-0.5439
0.35	1.9598	1.7976	1.6733	1.5686	1.4764	1.3931	1.3166	1.2454	1.1786	1.1154	0.8391	0.6067	0.4023	0.2178
0.40	2.0942	1.9875	1.9057	1.8367	1.7760	1.7211	1.6707	1.6238	1.5797	1.5381	1.3558	1.2024	1.0674	0.9454
0.45	2.2237	2.1705	2.1296	2.0952	2.0649	2.0375	2.0124	1.9889	1.9669	1.9461	1.8550	1.7783	1.7107	1.6496
0.50	2.3523	2.3523	2.3523	2.3523	2.3523	2.3523	2.3523	2.3523	2.3523	2.3523	2.3523	2.3523	2.3523	2.3523
0.55	2.4810	2.5343	2.5753	2.6098	2.6403	2.6678	2.6931	2.7167	2.7388	2.7598	2.8515	2.9289	2.9972	3.0589
0.60	2.6109	2.7181	2.8005	2.8700	2.9312	2.9866	3.0376	3.0851	3.1297	3.1718	3.3568	3.5129	3.6506	3.7752
0.65	2.7460	2.9094	3.0350	3.1410	3.2344	3.3189	3.3967	3.4691	3.5372	3.6016	3.8840	4.1226	4.3330	4.5234
0.70	2.8884	3.1112	3.2824	3.4269	3.5544	3.6697	3.7759	3.8747	3.9677	4.0556	4.4412	4.7671	5.0548	5.3155
0.75	3.0424	3.3293	3.5499	3.7362	3.9006	4.0494	4.1862	4.3137	4.4335	4.5469	5.0448	5.4660	5.8382	6.1755
0.80	3.2139	3.5726	3.8484	4.0815	4.2869	4.4729	4.6442	4.8038	4.9539	5.0960	5.7201	6.2484	6.7155	7.1391
0.85	3.4156	3.8587	4.1996	4.4874	4.7415	4.9717	5.1837	5.3813	5.5672	5.7432	6.5168	7.1722	7.7520	8.2779
0.90	3.6681	4.2170	4.6394	4.9965	5.3120	5.5977	5.8611	6.1066	6.3377	6.5565	7.5189	8.3347	9.0567	9.7121
0.95	4.0432	4.7494	5.2938	5.7544	6.1615	6.5306	6.8708	7.1882	7.4869	7.7700	9.0151	10.0719	11.0078	11.8577

TABLE I (Continued)

q = 0.025 ν = 4

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.3492	-0.3733	-0.8541	-1.2250	-1.5346	-1.8011	-2.0386	-2.2549	-2.4547	-2.6410	-3.4289	-4.0688	-4.6234	-5.1195
0.10	0.7912	0.1577	-0.2690	-0.5982	-0.8692	-1.1016	-1.3076	-1.4937	-1.6635	-1.8194	-2.4759	-3.0055	-3.4544	-3.8528
0.15	1.1162	0.5595	0.1765	-0.1205	-0.3659	-0.5761	-0.7612	-0.9265	-1.0772	-1.2162	-1.7913	-2.2441	-2.6293	-2.9692
0.20	1.3953	0.9119	0.5715	0.3061	0.0837	-0.1062	-0.2736	-0.4238	-0.5602	-0.6855	-1.1982	-1.5989	-1.9309	-2.2218
0.25	1.6438	1.2339	0.9406	0.7065	0.5106	0.3422	0.1909	0.0558	-0.0667	-0.1791	-0.6409	-0.9957	-1.2899	-1.5450
0.30	1.8786	1.5422	1.2996	1.1011	0.9350	0.7892	0.6582	0.5407	0.4339	0.3351	-0.0745	-0.3898	-0.6496	-0.8722
0.35	2.1042	1.8463	1.6548	1.4991	1.3667	1.2484	1.1417	1.0452	0.9569	0.8754	0.5298	0.2604	0.0357	-0.1561
0.40	2.3275	2.1496	2.0166	1.9072	1.8130	1.7280	1.6509	1.5805	1.5154	1.4548	1.1960	0.9882	0.8134	0.6590
0.45	2.5482	2.4564	2.3870	2.3292	2.2787	2.2327	2.1907	2.1520	2.1158	2.0819	1.9360	1.8165	1.7119	1.6191
0.50	2.7754	2.7754	2.7754	2.7754	2.7754	2.7754	2.7754	2.7754	2.7754	2.7754	2.7754	2.7754	2.7754	2.7754
0.55	3.0069	3.1049	3.1810	3.2457	3.3031	3.3556	3.4042	3.4496	3.4925	3.5333	3.7139	3.8686	4.0054	4.1301
0.60	3.2476	3.4524	3.6131	3.7506	3.8732	3.9842	4.0869	4.1833	4.2744	4.3612	4.7500	5.0875	5.3901	5.6676
0.65	3.5065	3.8294	4.0817	4.2976	4.4908	4.6691	4.8357	4.9922	5.1405	5.2819	5.9126	6.4600	6.9549	7.4087
0.70	3.7874	4.2367	4.5916	4.9009	5.1782	5.4325	5.6691	5.8916	6.1024	6.3041	7.2121	7.9962	8.6987	9.3444
0.75	4.0965	4.6913	5.1685	5.5805	5.9501	6.2895	6.6088	6.9091	7.1937	7.4650	8.6742	9.7218	10.6619	11.5209
0.80	4.4483	5.2181	5.8322	6.3641	6.8459	7.2878	7.6989	8.0853	8.4512	8.7998	10.3625	11.7073	12.9106	14.0080
0.85	4.8765	5.8553	6.6402	7.3224	7.9342	8.4948	9.0159	9.5091	9.9763	10.4211	12.3976	14.0938	15.6027	16.9759
0.90	5.4290	6.6812	7.6874	8.5557	9.3354	10.0534	10.7200	11.3451	11.9365	12.5001	14.9953	17.1245	19.0084	20.7194
0.95	6.2752	7.9533	9.2900	10.4495	11.4852	12.4332	13.3123	14.1354	14.9121	15.6495	18.8998	21.6588	24.1024	26.3194

q = 0.025 ν = 5

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.3412	-0.3719	-0.8606	-1.2428	-1.5623	-1.8413	-2.0899	-2.3152	-2.5234	-2.7176	-3.5449	-4.2151	-4.7946	-5.3128
0.10	0.7654	0.1569	-0.2674	-0.5993	-0.8761	-1.1159	-1.3278	-1.5200	-1.6966	-1.8606	-2.5454	-3.0978	-3.5720	-3.9894
0.15	1.0723	0.5456	0.1751	-0.1192	-0.3645	-0.5769	-0.7652	-0.9353	-1.0909	-1.2337	-1.8310	-2.3039	-2.7053	-3.0599
0.20	1.3299	0.8789	0.5572	0.2999	0.0840	-0.1048	-0.2720	-0.4228	-0.5607	-0.6881	-1.2151	-1.6288	-1.9779	-2.2807
0.25	1.5602	1.1814	0.9063	0.6862	0.4983	0.3345	0.1890	0.0564	-0.0654	-0.1778	-0.6426	-1.0067	-1.3096	-1.5730
0.30	1.7719	1.4664	1.2412	1.0581	0.9009	0.7635	0.6407	0.5274	0.4236	0.3277	-0.0732	-0.3885	-0.6515	-0.8792
0.35	1.9749	1.7426	1.5703	1.4264	1.3032	1.1946	1.0963	1.0054	0.9218	0.8442	0.5169	0.2560	0.0365	-0.1547
0.40	2.1733	2.0155	1.8963	1.7978	1.7124	1.6365	1.5667	1.5018	1.4416	1.3853	1.1467	0.9514	0.7860	0.6414
0.45	2.3689	2.2878	2.2262	2.1748	2.1298	2.0895	2.0523	2.0176	1.9853	1.9549	1.8237	1.7156	1.6221	1.5374
0.50	2.5691	2.5691	2.5691	2.5691	2.5691	2.5691	2.5691	2.5691	2.5691	2.5691	2.5691	2.5691	2.5691	2.5691
0.55	2.7726	2.8585	2.9251	2.9817	3.0319	3.0775	3.1194	3.1585	3.1953	3.2303	3.3851	3.5176	3.6357	3.7432
0.60	2.9834	3.1609	3.2987	3.4165	3.5215	3.6173	3.7060	3.7890	3.8675	3.9422	4.2747	4.5614	4.8185	5.0543
0.65	3.2073	3.4840	3.7015	3.8875	4.0538	4.2060	4.3475	4.4805	4.6065	4.7266	5.2626	5.7261	6.1423	6.5242
0.70	3.4480	3.8350	4.1401	4.4029	4.6385	4.8545	5.0556	5.2447	5.4240	5.5950	6.3588	7.0191	7.6137	8.1590
0.75	3.7143	4.2249	4.6302	4.9803	5.2945	5.5828	5.8513	6.1038	6.3433	6.5716	7.5929	8.4762	9.2672	9.9909
0.80	4.0172	4.6724	5.1943	5.6455	6.0506	6.4225	6.7686	7.0942	7.4040	7.6993	9.0152	10.1480	11.1599	12.0838
0.85	4.3821	5.2139	5.8776	6.4516	6.9669	7.4410	7.8825	8.2974	8.6903	9.0645	10.7283	12.1561	13.4282	14.5874
0.90	4.8515	5.9121	6.7590	7.4925	8.1515	8.7552	9.3161	9.8427	10.3407	10.8145	12.9159	14.7130	16.3101	17.7626
0.95	5.5708	6.9829	8.1133	9.0884	9.9608	10.7583	11.4980	12.1912	12.8458	13.4678	16.2179	18.5605	20.6371	22.5221

TABLE I (Continued)

q = 0.025 ν = 6

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.3365	-0.3711	-0.8643	-1.2547	-1.5831	-1.8703	-2.1279	-2.3627	-2.5773	-2.7777	-3.6320	-4.3297	-4.9295	-5.4657
0.10	0.7493	0.1542	-0.2663	-0.6000	-0.8801	-1.1243	-1.3423	-1.5396	-1.7212	-1.8901	-2.6000	-3.1701	-3.6600	-4.0952
0.15	1.0446	0.5365	0.1722	-0.1181	-0.3636	-0.5773	-0.7676	-0.9401	-1.0985	-1.2455	-1.8597	-2.3510	-2.7650	-3.1310
0.20	1.2893	0.8595	0.5477	0.2958	0.0826	-0.1038	-0.2708	-0.4222	-0.5611	-0.6896	-1.2263	-1.6515	-2.0111	-2.3269
0.25	1.5059	1.1479	0.8859	0.6722	0.4908	0.3299	0.1861	0.0555	-0.0646	-0.1766	-0.6437	-1.0128	-1.3236	-1.5941
0.30	1.7052	1.4183	1.2049	1.0311	0.8807	0.7475	0.6279	0.5189	0.4177	0.3232	-0.0723	-0.3877	-0.6527	-0.8832
0.35	1.8958	1.6776	1.5153	1.3806	1.2640	1.1605	1.0671	0.9815	0.9009	0.8258	0.5088	0.2524	0.0360	-0.1536
0.40	2.0801	1.9338	1.8222	1.7295	1.6491	1.5773	1.5120	1.4517	1.3949	1.3418	1.1148	0.9295	0.7694	0.6286
0.45	2.2613	2.1863	2.1292	2.0815	2.0397	2.0022	1.9678	1.9358	1.9056	1.8771	1.7540	1.6521	1.5637	1.4849
0.50	2.4458	2.4458	2.4458	2.4458	2.4458	2.4458	2.4458	2.4458	2.4458	2.4458	2.4458	2.4458	2.4458	2.4458
0.55	2.6321	2.7106	2.7713	2.8228	2.8684	2.9099	2.9482	2.9840	3.0176	3.0495	3.1905	3.3109	3.4183	3.5162
0.60	2.8243	2.9861	3.1118	3.2191	3.3145	3.4016	3.4823	3.5578	3.6284	3.6957	3.9944	4.2516	4.4819	4.6931
0.65	3.0285	3.2805	3.4782	3.6464	3.7959	3.9327	4.0597	4.1790	4.2919	4.3996	4.8796	5.2928	5.6635	6.0038
0.70	3.2478	3.5992	3.8735	4.1094	4.3206	4.5142	4.6942	4.8636	5.0236	5.1760	5.8564	6.4447	6.9726	7.4568
0.75	3.4899	3.9496	4.3132	4.6268	4.9081	5.1651	5.4043	5.6293	5.8426	6.0460	6.9541	7.7385	8.4413	9.0848
0.80	3.7631	4.3510	4.8184	5.2209	5.5819	5.9131	6.2215	6.5116	6.7865	7.0486	8.2174	9.2245	10.1251	10.9479
0.85	4.0908	4.8359	5.4277	5.9391	6.3982	6.8193	7.2112	7.5797	7.9287	8.2612	9.7409	11.0124	12.1487	13.1849
0.90	4.5115	5.4585	6.2129	6.8651	7.4501	7.9863	8.4848	8.9530	9.3959	9.8176	11.6910	13.2971	14.7258	16.0258
0.95	5.1543	6.4125	7.4162	8.2824	9.0580	9.7676	10.4261	11.0437	11.6284	12.1841	14.6433	16.7397	18.5978	20.2853

q = 0.025 ν = 7

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.3344	-0.3700	-0.8681	-1.2645	-1.6001	-1.8929	-2.1561	-2.3966	-2.6189	-2.8250	-3.7023	-4.4205	-5.0394	-5.5908
0.10	0.7389	0.1538	-0.2649	-0.6005	-0.8841	-1.1318	-1.3536	-1.5556	-1.7409	-1.9132	-2.6425	-3.2279	-3.7310	-4.1782
0.15	1.0249	0.5301	0.1717	-0.1172	-0.3626	-0.5777	-0.7702	-0.9449	-1.1057	-1.2550	-1.8821	-2.3845	-2.8120	-3.1878
0.20	1.2617	0.8464	0.5411	0.2942	0.0826	-0.1029	-0.2695	-0.4214	-0.5613	-0.6912	-1.2355	-1.6698	-2.0367	-2.3597
0.25	1.4695	1.1251	0.8720	0.6634	0.4851	0.3279	0.1854	0.0556	-0.0640	-0.1753	-0.6447	-1.0186	-1.3346	-1.6113
0.30	1.6610	1.3856	1.1803	1.0118	0.8670	0.7371	0.6200	0.5128	0.4138	0.3213	-0.0717	-0.3868	-0.6538	-0.8873
0.35	1.8433	1.6345	1.4785	1.3496	1.2373	1.1373	1.0467	0.9635	0.8864	0.8136	0.5028	0.2512	0.0362	-0.1524
0.40	2.0183	1.8791	1.7729	1.6843	1.6072	1.5382	1.4753	1.4173	1.3633	1.3122	1.0930	0.9138	0.7585	0.6207
0.45	2.1903	2.1192	2.0650	2.0196	1.9799	1.9441	1.9114	1.8810	1.8526	1.8254	1.7078	1.6101	1.5251	1.4492
0.50	2.3648	2.3648	2.3648	2.3648	2.3648	2.3648	2.3648	2.3648	2.3648	2.3648	2.3648	2.3648	2.3648	2.3648
0.55	2.5407	2.6145	2.6717	2.7201	2.7630	2.8019	2.8378	2.8714	2.9030	2.9327	3.0638	3.1757	3.2752	3.3659
0.60	2.7215	2.8735	2.9907	3.0904	3.1790	3.2597	3.3345	3.4045	3.4707	3.5334	3.8116	4.0508	4.2638	4.4584
0.65	2.9132	3.1474	3.3307	3.4875	3.6268	3.7542	3.8724	3.9833	4.0883	4.1879	4.6301	5.0117	5.3540	5.6681
0.70	3.1170	3.4433	3.6991	3.9186	4.1150	4.2935	4.4595	4.6154	4.7631	4.9039	5.5320	6.0728	6.5567	7.0007
0.75	3.3415	3.7700	4.1081	4.3974	4.6563	4.8938	5.1147	5.3224	5.5193	5.7071	6.5398	7.2597	7.9059	8.4977
0.80	3.5962	4.1431	4.5738	4.9454	5.2787	5.5844	5.8683	6.1342	6.3861	6.6264	7.7000	8.6263	9.4550	10.2125
0.85	3.9013	4.5899	5.1363	5.6084	6.0302	6.4162	6.7754	7.1137	7.4345	7.7402	9.1014	10.2718	11.3164	12.2694
0.90	4.2911	5.1647	5.8604	6.4581	6.9946	7.4875	7.9459	8.3765	8.7840	9.1720	10.8956	12.3727	13.6876	14.8848
0.95	4.8839	6.0433	6.9634	7.7598	8.4731	9.1259	9.7321	10.3006	10.8379	11.3489	13.6116	15.5430	17.2563	18.8125

TABLE I (Continued)

q = 0.025 ν = 8

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.3315	-0.3696	-0.8702	-1.2715	-1.6124	-1.9115	-2.1794	-2.4245	-2.6513	-2.8632	-3.7606	-4.4948	-5.1301	-5.6946
0.10	0.7311	0.1531	-0.2646	-0.6008	-0.8864	-1.1371	-1.3619	-1.5671	-1.7564	-1.9321	-2.6753	-3.2757	-3.7899	-4.2471
0.15	1.0114	0.5252	0.1709	-0.1170	-0.3621	-0.5778	-0.7714	-0.9479	-1.1106	-1.2619	-1.9005	-2.4122	-2.8498	-3.2346
0.20	1.2434	0.8360	0.5361	0.2920	0.0823	-0.1028	-0.2692	-0.4211	-0.5613	-0.6919	-1.2422	-1.6837	-2.0578	-2.3869
0.25	1.4447	1.1097	0.8610	0.6568	0.4807	0.3251	0.1846	0.0554	-0.0640	-0.1751	-0.6452	-1.0225	-1.3426	-1.6239
0.30	1.6297	1.3634	1.1637	0.9986	0.8561	0.7293	0.6139	0.5081	0.4101	0.3186	-0.0716	-0.3864	-0.6543	-0.8896
0.35	1.8049	1.6042	1.4534	1.3285	1.2195	1.1217	1.0329	0.9511	0.8752	0.8040	0.4983	0.2497	0.0360	-0.1523
0.40	1.9735	1.8394	1.7375	1.6522	1.5778	1.5111	1.4503	1.3941	1.3417	1.2925	1.0783	0.9022	0.7503	0.6146
0.45	2.1390	2.0706	2.0185	1.9748	1.9365	1.9020	1.8705	1.8412	1.8137	1.7879	1.6747	1.5806	1.4985	1.4250
0.50	2.3063	2.3063	2.3063	2.3063	2.3063	2.3063	2.3063	2.3063	2.3063	2.3063	2.3063	2.3063	2.3063	2.3063
0.55	2.4747	2.5454	2.5999	2.6461	2.6870	2.7241	2.7583	2.7903	2.8204	2.8490	2.9735	3.0795	3.1737	3.2595
0.60	2.6475	2.7923	2.9041	2.9987	3.0827	3.1591	3.2298	3.2960	3.3585	3.4179	3.6813	3.9072	4.1092	4.2929
0.65	2.8302	3.0527	3.2262	3.3744	3.5064	3.6269	3.7387	3.8435	3.9427	4.0371	4.4548	4.8144	5.1359	5.4305
0.70	3.0239	3.3326	3.5748	3.7824	3.9678	4.1372	4.2938	4.4409	4.5801	4.7128	5.3029	5.8117	6.2657	6.6819
0.75	3.2364	3.6418	3.9614	4.2352	4.4795	4.7033	4.9112	5.1062	5.2910	5.4671	6.2498	6.9241	7.5300	8.0850
0.80	3.4774	3.9945	4.4017	4.7520	5.0652	5.3521	5.6190	5.8693	6.1057	6.3310	7.3369	8.2055	8.9817	9.6909
0.85	3.7660	4.4169	4.9316	5.3746	5.7717	6.1339	6.4708	6.7875	7.0881	7.3747	8.6508	9.7465	10.7248	11.6179
0.90	4.1349	4.9582	5.6115	6.1732	6.6761	7.1377	7.5675	7.9713	8.3534	8.7168	10.3306	11.7146	12.9473	14.0703
0.95	4.6940	5.7841	6.6470	7.3930	8.0619	8.6737	9.2411	9.7734	10.2767	10.7553	12.8761	14.6878	16.2967	17.7580

q = 0.025 ν = 9

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.3293	-0.3697	-0.8719	-1.2767	-1.6216	-1.9255	-2.1982	-2.4472	-2.6778	-2.8934	-3.8098	-4.5585	-5.2065	-5.7843
0.10	0.7247	0.1522	-0.2645	-0.6013	-0.8882	-1.1408	-1.3681	-1.5757	-1.7676	-1.9466	-2.7023	-3.3155	-3.8397	-4.3060
0.15	1.0007	0.5216	0.1699	-0.1169	-0.3622	-0.5783	-0.7726	-0.9501	-1.1141	-1.2670	-1.9143	-2.4348	-2.8798	-3.2737
0.20	1.2279	0.8280	0.5324	0.2900	0.0819	-0.1027	-0.2691	-0.4213	-0.5618	-0.6928	-1.2470	-1.6938	-2.0748	-2.4090
0.25	1.4248	1.0972	0.8527	0.6518	0.4775	0.3230	0.1834	0.0551	-0.0639	-0.1750	-0.6459	-1.0252	-1.3485	-1.6333
0.30	1.6055	1.3452	1.1502	0.9881	0.8478	0.7230	0.6094	0.5046	0.4074	0.3165	-0.0715	-0.3865	-0.6551	-0.8914
0.35	1.7757	1.5806	1.4333	1.3111	1.2047	1.1089	1.0218	0.9414	0.8666	0.7965	0.4949	0.2479	0.0358	-0.1521
0.40	1.9392	1.8092	1.7105	1.6274	1.5548	1.4897	1.4303	1.3753	1.3240	1.2758	1.0664	0.8932	0.7436	0.6101
0.45	2.0992	2.0331	1.9827	1.9405	1.9034	1.8700	1.8393	1.8109	1.7843	1.7592	1.6494	1.5575	1.4774	1.4056
0.50	2.2605	2.2605	2.2605	2.2605	2.2605	2.2605	2.2605	2.2605	2.2605	2.2605	2.2605	2.2605	2.2605	2.2605
0.55	2.4236	2.4918	2.5445	2.5892	2.6286	2.6644	2.6974	2.7283	2.7573	2.7848	2.9054	3.0078	3.0987	3.1815
0.60	2.5905	2.7302	2.8384	2.9298	3.0109	3.0846	3.1528	3.2167	3.2769	3.3341	3.5867	3.8026	3.9954	4.1710
0.65	2.7667	2.9819	3.1493	3.2922	3.4194	3.5347	3.6416	3.7417	3.8364	3.9265	4.3249	4.6665	4.9723	5.2525
0.70	2.9542	3.2520	3.4848	3.6833	3.8604	4.0223	4.1719	4.3117	4.4440	4.5700	5.1311	5.6151	6.0482	6.4448
0.75	3.1592	3.5490	3.8542	4.1162	4.3484	4.5610	4.7585	4.9440	5.1198	5.2872	6.0331	6.6755	7.2505	7.7765
0.80	3.3914	3.8859	4.2744	4.6071	4.9050	5.1778	5.4316	5.6701	5.8957	6.1105	7.0674	7.8908	8.6274	9.3000
0.85	3.6677	4.2889	4.7778	5.1992	5.5768	5.9226	6.2437	6.5454	6.8313	7.1032	8.3132	9.3526	10.2807	11.1280
0.90	4.0201	4.8032	5.4245	5.9601	6.4393	6.8785	7.2860	7.6687	8.0310	8.3759	9.9067	11.2199	12.3900	13.4563
0.95	4.5521	5.5886	6.4115	7.1206	7.7546	8.3350	8.8734	9.3782	9.8555	10.3096	12.3224	14.0428	15.5715	16.9615

TABLE I (Continued)

q = 0.025 ν = 16

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.3231	-0.3681	-0.8793	-1.2988	-1.6602	-1.9813	-2.2722	-2.5392	-2.7874	-3.0200	-4.0147	-4.8314	-5.5370	-6.1669
0.10	0.7033	0.1501	-0.2626	-0.6024	-0.8961	-1.1574	-1.3942	-1.6119	-1.8142	-2.0036	-2.8138	-3.4766	-4.0473	-4.5558
0.15	0.9649	0.5088	0.1674	-0.1155	-0.3606	-0.5790	-0.7773	-0.9599	-1.1296	-1.2887	-1.9694	-2.5258	-3.0053	-3.4311
0.20	1.1770	0.8018	0.5192	0.2848	0.0810	-0.1013	-0.2672	-0.4200	-0.5622	-0.6956	-1.2679	-1.7363	-2.1401	-2.4982
0.25	1.3601	1.0552	0.8253	0.6334	0.4664	0.3169	0.1807	0.0546	-0.0629	-0.1732	-0.6477	-1.0375	-1.3738	-1.6724
0.30	1.5262	1.2863	1.1046	0.9531	0.8207	0.7016	0.5930	0.4925	0.3988	0.3106	-0.0704	-0.3850	-0.6571	-0.8994
0.35	1.6816	1.5035	1.3680	1.2546	1.1554	1.0662	0.9847	0.9091	0.8385	0.7719	0.4831	0.2438	0.0356	-0.1505
0.40	1.8305	1.7122	1.6220	1.5463	1.4799	1.4200	1.3652	1.3143	1.2666	1.2217	1.0264	0.8637	0.7213	0.5937
0.45	1.9751	1.9155	1.8700	1.8317	1.7980	1.7676	1.7397	1.7138	1.6895	1.6665	1.5663	1.4824	1.4087	1.3423
0.50	2.1198	2.1198	2.1198	2.1198	2.1198	2.1198	2.1198	2.1198	2.1198	2.1198	2.1198	2.1198	2.1198	2.1198
0.55	2.2649	2.3254	2.3719	2.4113	2.4460	2.4774	2.5064	2.5334	2.5588	2.5829	2.6887	2.7783	2.8573	2.9291
0.60	2.4124	2.5351	2.6298	2.7100	2.7809	2.8451	2.9043	2.9595	3.0115	3.0609	3.2784	3.4636	3.6276	3.7769
0.65	2.5671	2.7557	2.9012	3.0247	3.1342	3.2337	3.3256	3.4115	3.4925	3.5692	3.9079	4.1974	4.4545	4.6890
0.70	2.7313	2.9900	3.1907	3.3615	3.5130	3.6505	3.7776	3.8967	4.0090	4.1158	4.5876	4.9913	5.3513	5.6801
0.75	2.9098	3.2460	3.5077	3.7301	3.9279	4.1081	4.2751	4.4309	4.5781	4.7180	5.3388	5.8710	6.3461	6.7793
0.80	3.1102	3.5346	3.8650	4.1472	4.3981	4.6266	4.8385	5.0371	5.2247	5.4030	6.1951	6.8734	7.4794	8.0326
0.85	3.3480	3.8773	4.2913	4.6445	4.9595	5.2470	5.5135	5.7634	5.9998	6.2247	7.2209	8.0759	8.8391	9.5356
0.90	3.6486	4.3127	4.8326	5.2782	5.6755	6.0388	6.3753	6.6906	6.9888	7.2725	8.5315	9.6110	10.5718	11.4483
0.95	4.1006	4.9693	5.6526	6.2390	6.7613	7.2388	7.6818	8.0969	8.4894	8.8628	10.5163	11.9308	13.1895	14.3355

q = 0.025 ν = 36

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.3190	-0.3671	-0.8849	-1.3160	-1.6918	-2.0283	-2.3353	-2.6191	-2.8840	-3.1328	-4.2079	-5.0977	-5.8717	-6.5637
0.10	0.6895	0.1487	-0.2613	-0.6033	-0.9020	-1.1701	-1.4149	-1.6414	-1.8529	-2.0518	-2.9122	-3.6242	-4.2433	-4.7969
0.15	0.9418	0.5007	0.1657	-0.1144	-0.3595	-0.5796	-0.7809	-0.9673	-1.1415	-1.3056	-2.0158	-2.6049	-3.1170	-3.5750
0.20	1.1449	0.7847	0.5107	0.2814	0.0804	-0.1004	-0.2659	-0.4192	-0.5627	-0.6978	-1.2841	-1.7714	-2.1956	-2.5754
0.25	1.3185	1.0285	0.8073	0.6219	0.4593	0.3129	0.1788	0.0543	-0.0622	-0.1720	-0.6493	-1.0468	-1.3937	-1.7047
0.30	1.4751	1.2487	1.0758	0.9304	0.8029	0.6879	0.5826	0.4848	0.3932	0.3067	-0.0697	-0.3840	-0.6588	-0.9054
0.35	1.6208	1.4537	1.3259	1.2186	1.1243	1.0390	0.9608	0.8882	0.8201	0.7558	0.4756	0.2411	0.0355	-0.1493
0.40	1.7597	1.6494	1.5650	1.4939	1.4314	1.3751	1.3233	1.2752	1.2300	1.1874	1.0009	0.8443	0.7070	0.5832
0.45	1.8939	1.8387	1.7964	1.7608	1.7294	1.7011	1.6751	1.6509	1.6282	1.6067	1.5128	1.4338	1.3643	1.3017
0.50	2.0275	2.0275	2.0275	2.0275	2.0275	2.0275	2.0275	2.0275	2.0275	2.0275	2.0275	2.0275	2.0275	2.0275
0.55	2.1616	2.2173	2.2601	2.2962	2.3280	2.3568	2.3833	2.4080	2.4312	2.4532	2.5495	2.6309	2.7027	2.7678
0.60	2.2972	2.4095	2.4959	2.5689	2.6333	2.6916	2.7453	2.7954	2.8425	2.8870	3.0827	3.2483	3.3948	3.5277
0.65	2.4387	2.6104	2.7426	2.8544	2.9532	3.0426	3.1250	3.2018	3.2741	3.3426	3.6440	3.8997	4.1260	4.3316
0.70	2.5883	2.8230	3.0040	3.1571	3.2924	3.4152	3.5284	3.6340	3.7335	3.8278	4.2428	4.5955	4.9085	5.1923
0.75	2.7503	3.0536	3.2877	3.4861	3.6617	3.8211	3.9681	4.1052	4.2344	4.3570	4.8975	5.3565	5.7640	6.1350
0.80	2.9315	3.3118	3.6059	3.8555	4.0764	4.2770	4.4623	4.6353	4.7985	4.9533	5.6344	6.2154	6.7319	7.2025
0.85	3.1450	3.6169	3.9824	4.2927	4.5678	4.8179	5.0487	5.2640	5.4670	5.6598	6.5118	7.2392	7.8861	8.4754
0.90	3.4135	4.0012	4.4571	4.8449	5.1884	5.5005	5.7891	6.0591	6.3138	6.5558	7.6258	8.5392	9.3519	10.0907
0.95	3.8144	4.5763	5.1686	5.6721	6.1196	6.5271	6.9042	7.2571	7.5901	7.9063	9.3051	10.4972	11.5575	12.5231

TABLE I (Continued)

q = 0.025 $\nu = 144$

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.3154	-0.3670	-0.8885	-1.3267	-1.7118	-2.0590	-2.3776	-2.6736	-2.9510	-3.2130	-4.3548	-5.3113	-6.1496	-6.9042
0.10	0.6803	0.1469	-0.2611	-0.6042	-0.9058	-1.1780	-1.4278	-1.6600	-1.8777	-2.0833	-2.9807	-3.7330	-4.3927	-4.9869
0.15	0.9271	0.4947	0.1638	-0.1146	-0.3594	-0.5804	-0.7833	-0.9720	-1.1489	-1.3161	-2.0460	-2.6587	-3.1964	-3.6808
0.20	1.1248	0.7736	0.5046	0.2783	0.0792	-0.1006	-0.2657	-0.4193	-0.5633	-0.6995	-1.2942	-1.7937	-2.2324	-2.6279
0.25	1.2935	1.0116	0.7957	0.6139	0.4539	0.3095	0.1767	0.0533	-0.0625	-0.1720	-0.6505	-1.0527	-1.4061	-1.7249
0.30	1.4452	1.2258	1.0576	0.9160	0.7914	0.6788	0.5753	0.4791	0.3887	0.3034	-0.0700	-0.3840	-0.6601	-0.9093
0.35	1.5859	1.4245	1.3007	1.1965	1.1047	1.0218	0.9456	0.8747	0.8082	0.7453	0.4700	0.2384	0.0346	-0.1494
0.40	1.7196	1.6134	1.5320	1.4634	1.4030	1.3484	1.2982	1.2515	1.2076	1.1662	0.9847	0.8319	0.6974	0.5759
0.45	1.8484	1.7954	1.7548	1.7206	1.6905	1.6632	1.6382	1.6149	1.5930	1.5723	1.4816	1.4052	1.3380	1.2772
0.50	1.9763	1.9763	1.9763	1.9763	1.9763	1.9763	1.9763	1.9763	1.9763	1.9763	1.9763	1.9763	1.9763	1.9763
0.55	2.1043	2.1573	2.1980	2.2324	2.2626	2.2900	2.3151	2.3386	2.3606	2.3814	2.4726	2.5495	2.6173	2.6787
0.60	2.2334	2.3400	2.4219	2.4909	2.5518	2.6069	2.6575	2.7046	2.7489	2.7909	2.9745	3.1296	3.2663	3.3900
0.65	2.3677	2.5301	2.6549	2.7602	2.8530	2.9369	3.0142	3.0861	3.1537	3.2176	3.4980	3.7348	3.9436	4.1327
0.70	2.5093	2.7306	2.9007	3.0442	3.1708	3.2853	3.3907	3.4888	3.5811	3.6684	4.0511	4.3744	4.6599	4.9184
0.75	2.6622	2.9473	3.1664	3.3513	3.5145	3.6621	3.7980	3.9245	4.0434	4.1560	4.6499	5.0676	5.4364	5.7706
0.80	2.8327	3.1888	3.4627	3.6940	3.8979	4.0825	4.2525	4.4108	4.5597	4.7006	5.3193	5.8428	6.3053	6.7245
0.85	3.0330	3.4729	3.8112	4.0969	4.3491	4.5774	4.7876	4.9836	5.1678	5.3423	6.1085	6.7572	7.3308	7.8507
0.90	3.2837	3.8286	4.2478	4.6020	4.9148	5.1981	5.4591	5.7024	5.9312	6.1479	7.1002	7.9068	8.6202	9.2673
0.95	3.6560	4.3569	4.8968	5.3534	5.7567	6.1222	6.4590	6.7731	7.0686	7.3486	8.5791	9.6224	10.5455	11.3834

q = 0.050 $\nu = 4$

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	0.0001	-0.6776	-1.1456	-1.5157	-1.8287	-2.1017	-2.3469	-2.5714	-2.7794	-2.9741	-3.8053	-4.4873	-5.0804	-5.6119
0.10	0.4016	-0.1761	-0.5780	-0.8949	-1.1606	-1.3919	-1.5988	-1.7872	-1.9603	-2.1206	-2.8015	-3.3564	-3.8325	-4.2568
0.15	0.6927	0.1928	-0.1587	-0.4371	-0.6706	-0.8734	-1.0540	-1.2174	-1.3675	-1.5069	-2.0916	-2.5601	-2.9618	-3.3183
0.20	0.9392	0.5098	0.2038	-0.0393	-0.2451	-0.4235	-0.5824	-0.7262	-0.8580	-0.9800	-1.4887	-1.8941	-2.2356	-2.5369
0.25	1.1572	0.7969	0.5357	0.3257	0.1483	-0.0062	-0.1454	-0.2712	-0.3862	-0.4927	-0.9364	-1.2863	-1.5810	-1.8393
0.30	1.3612	1.0683	0.8548	0.6792	0.5306	0.3999	0.2822	0.1757	0.0780	-0.0127	-0.3936	-0.6935	-0.9449	-1.1635
0.35	1.5559	1.3332	1.1668	1.0305	0.9140	0.8097	0.7153	0.6294	0.5503	0.4769	0.1658	-0.0813	-0.2899	-0.4708
0.40	1.7479	1.5950	1.4804	1.3860	1.3043	1.2305	1.1634	1.1018	1.0448	0.9916	0.7634	0.5784	0.4214	0.2830
0.45	1.9378	1.8589	1.7992	1.7494	1.7058	1.6664	1.6304	1.5970	1.5660	1.5367	1.4109	1.3073	1.2165	1.1356
0.50	2.1330	2.1330	2.1330	2.1330	2.1330	2.1330	2.1330	2.1330	2.1330	2.1330	2.1330	2.1330	2.1330	2.1330
0.55	2.3302	2.4137	2.4785	2.5335	2.5824	2.6265	2.6673	2.7054	2.7413	2.7755	2.9269	3.0566	3.1725	3.2785
0.60	2.5352	2.7077	2.8424	2.9576	3.0605	3.1545	3.2418	3.3237	3.4013	3.4752	3.8031	4.0848	4.3381	4.5716
0.65	2.7530	3.0237	3.2373	3.4211	3.5857	3.7356	3.8745	4.0052	4.1291	4.2473	4.7780	5.2367	5.6476	6.0248
0.70	2.9885	3.3691	3.6709	3.9290	4.1606	4.3737	4.5728	4.7602	4.9380	5.1074	5.8614	6.5154	7.1027	7.6406
0.75	3.2499	3.7541	4.1525	4.4983	4.8095	5.0952	5.3603	5.6097	5.8461	6.0717	7.0821	7.9534	8.7325	9.4445
0.80	3.5494	4.1940	4.7102	5.1572	5.5572	5.9243	6.2671	6.5899	6.8957	7.1872	8.4844	9.5990	10.5930	11.4992
0.85	3.9086	4.7297	5.3863	5.9531	6.4637	6.9322	7.3679	7.7771	8.1643	8.5329	10.1692	11.5701	12.8160	13.9497
0.90	4.3708	5.4204	6.2575	6.9831	7.6332	8.2282	8.7807	9.2988	9.7884	10.2539	12.3145	14.0724	15.6321	17.0488
0.95	5.0834	6.4796	7.5955	8.5565	9.4149	10.1986	10.9247	11.6044	12.2458	12.8547	15.5421	17.8265	19.8487	21.6827

TABLE I (Continued)

$q = 0.050$ $\nu = 5$

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	-0.0005	-0.6787	-1.1560	-1.5365	-1.8588	-2.1426	-2.3975	-2.6303	-2.8460	-3.0480	-3.9134	-4.6225	-5.2382	-5.7899
0.10	0.3932	-0.1740	-0.5779	-0.8995	-1.1714	-1.4094	-1.6220	-1.8160	-1.9951	-2.1622	-2.8690	-3.4447	-3.9418	-4.3831
0.15	0.6746	0.1899	-0.1568	-0.4357	-0.6715	-0.8777	-1.0621	-1.2299	-1.3844	-1.5275	-2.1321	-2.6186	-3.0352	-3.4051
0.20	0.9081	0.4978	0.2006	-0.0392	-0.2427	-0.4220	-0.5823	-0.7279	-0.8619	-0.9864	-1.5088	-1.9263	-2.2822	-2.5945
0.25	1.1147	0.7737	0.5229	0.3199	0.1460	-0.0068	-0.1436	-0.2689	-0.3845	-0.4918	-0.9419	-1.3007	-1.6037	-1.8697
0.30	1.3048	1.0308	0.8279	0.6616	0.5180	0.3915	0.2777	0.1730	0.0766	-0.0131	-0.3918	-0.6948	-0.9506	-1.1744
0.35	1.4863	1.2786	1.1238	0.9950	0.8840	0.7857	0.6963	0.6136	0.5371	0.4658	0.1633	-0.0804	-0.2877	-0.4697
0.40	1.6630	1.5224	1.4161	1.3280	1.2514	1.1831	1.1206	1.0625	1.0086	0.9580	0.7422	0.5643	0.4122	0.2784
0.45	1.8370	1.7649	1.7102	1.6644	1.6243	1.5883	1.5551	1.5243	1.4956	1.4685	1.3513	1.2543	1.1702	1.0944
0.50	2.0144	2.0144	2.0144	2.0144	2.0144	2.0144	2.0144	2.0144	2.0144	2.0144	2.0144	2.0144	2.0144	2.0144
0.55	2.1938	2.2694	2.3280	2.3777	2.4217	2.4618	2.4985	2.5327	2.5649	2.5955	2.7307	2.8463	2.9494	3.0432
0.60	2.3792	2.5348	2.6553	2.7581	2.8498	2.9334	3.0107	3.0831	3.1516	3.2167	3.5066	3.7555	3.9785	4.1831
0.65	2.5753	2.8170	3.0068	3.1690	3.3140	3.4466	3.5698	3.6852	3.7945	3.8988	4.3630	4.7635	5.1224	5.4510
0.70	2.7857	3.1232	3.3892	3.6179	3.8223	4.0098	4.1842	4.3476	4.5025	4.6502	5.3087	5.8769	6.3882	6.8570
0.75	3.0179	3.4631	3.8152	4.1189	4.3905	4.6396	4.8716	5.0893	5.2953	5.4918	6.3703	7.1297	7.8079	8.4279
0.80	3.2821	3.8517	4.3040	4.6938	5.0435	5.3635	5.6614	5.9415	6.2079	6.4618	7.5921	8.5624	9.4288	10.2212
0.85	3.5999	4.3209	4.8944	5.3886	5.8320	6.2397	6.6193	6.9760	7.3136	7.6343	9.0594	10.2832	11.3736	12.3667
0.90	4.0072	4.9242	5.6530	6.2840	6.8506	7.3692	7.8499	8.3009	8.7274	9.1331	10.9346	12.4742	13.8416	15.0834
0.95	4.6293	5.8458	6.8177	7.6548	8.4021	9.0850	9.7188	10.3132	10.8744	11.4075	13.7627	15.7652	17.5390	19.1486

$q = 0.050$ $\nu = 6$

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	-0.0011	-0.6787	-1.1618	-1.5506	-1.8811	-2.1723	-2.4350	-2.6756	-2.8976	-3.1054	-3.9962	-4.7287	-5.3625	-5.9304
0.10	0.3873	-0.1743	-0.5775	-0.9014	-1.1774	-1.4202	-1.6384	-1.8372	-2.0210	-2.1925	-2.9212	-3.5137	-4.0254	-4.4817
0.15	0.6619	0.1876	-0.1571	-0.4351	-0.6715	-0.8793	-1.0662	-1.2369	-1.3945	-1.5413	-2.1615	-2.6636	-3.0922	-3.4729
0.20	0.8886	0.4900	0.1981	-0.0396	-0.2427	-0.4214	-0.5819	-0.7283	-0.8634	-0.9893	-1.5221	-1.9503	-2.3157	-2.6388
0.25	1.0878	0.7579	0.5146	0.3152	0.1445	-0.0073	-0.1439	-0.2687	-0.3839	-0.4912	-0.9442	-1.3091	-1.6196	-1.8923
0.30	1.2709	1.0073	0.8107	0.6494	0.5097	0.3857	0.2736	0.1710	0.0756	-0.0136	-0.3912	-0.6949	-0.9530	-1.1805
0.35	1.4450	1.2456	1.0965	0.9727	0.8652	0.7696	0.6828	0.6032	0.5285	0.4587	0.1615	-0.0808	-0.2874	-0.4691
0.40	1.6123	1.4796	1.3778	1.2932	1.2194	1.1535	1.0934	1.0379	0.9858	0.9369	0.7272	0.5550	0.4060	0.2743
0.45	1.7760	1.7083	1.6567	1.6136	1.5757	1.5417	1.5105	1.4814	1.4539	1.4280	1.3155	1.2222	1.1410	1.0684
0.50	1.9420	1.9420	1.9420	1.9420	1.9420	1.9420	1.9420	1.9420	1.9420	1.9420	1.9420	1.9420	1.9420	1.9420
0.55	2.1105	2.1813	2.2360	2.2824	2.3235	2.3608	2.3952	2.4273	2.4574	2.4860	2.6121	2.7198	2.8155	2.9029
0.60	2.2837	2.4293	2.5418	2.6377	2.7230	2.8007	2.8727	2.9400	3.0033	3.0634	3.3306	3.5599	3.7640	3.9509
0.65	2.4672	2.6925	2.8690	3.0194	3.1532	3.2754	3.3889	3.4955	3.5956	3.6910	4.1159	4.4814	4.8092	5.1098
0.70	2.6633	2.9771	3.2225	3.4334	3.6211	3.7925	3.9519	4.1018	4.2433	4.3781	4.9796	5.4991	5.9649	6.3906
0.75	2.8794	3.2906	3.6145	3.8923	4.1411	4.3685	4.5800	4.7789	4.9674	5.1471	5.9486	6.6383	7.2558	7.8208
0.80	3.1238	3.6480	4.0618	4.4179	4.7370	5.0297	5.3020	5.5581	5.8008	6.0317	7.0591	7.9435	8.7336	9.4552
0.85	3.4167	4.0773	4.6007	5.0526	5.4580	5.8297	6.1747	6.4987	6.8054	7.0975	8.3966	9.5116	10.5053	11.4108
0.90	3.7902	4.6279	5.2944	5.8701	6.3847	6.8560	7.2940	7.7051	8.0939	8.4639	10.1051	11.5089	12.7566	13.8916
0.95	4.3590	5.4706	6.3549	7.1162	7.7973	8.4200	8.9977	9.5390	10.0503	10.5362	12.6846	14.5146	16.1358	17.6076

TABLE I (Continued)

q = 0.050 ν = 7

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	-0.0004	-0.6788	-1.1680	-1.5621	-1.8989	-2.1955	-2.4634	-2.7091	-2.9371	-3.1499	-4.0616	-4.8122	-5.4624	-6.0437
0.10	0.3846	-0.1728	-0.5765	-0.9043	-1.1838	-1.4297	-1.6513	-1.8540	-2.0413	-2.2161	-2.9614	-3.5677	-4.0915	-4.5586
0.15	0.6542	0.1865	-0.1556	-0.4334	-0.6715	-0.8819	-1.0712	-1.2440	-1.4037	-1.5527	-2.1845	-2.6968	-3.1365	-3.5260
0.20	0.8759	0.4861	0.1970	-0.0387	-0.2410	-0.4196	-0.5809	-0.7289	-0.8657	-0.9934	-1.5332	-1.9693	-2.3417	-2.6714
0.25	1.0691	0.7482	0.5102	0.3132	0.1436	-0.0066	-0.1425	-0.2669	-0.3821	-0.4896	-0.9477	-1.3171	-1.6322	-1.9102
0.30	1.2462	0.9912	0.7998	0.6420	0.5055	0.3830	0.2719	0.1700	0.0755	-0.0129	-0.3895	-0.6951	-0.9566	-1.1869
0.35	1.4139	1.2218	1.0775	0.9578	0.8531	0.7597	0.6747	0.5965	0.5238	0.4552	0.1605	-0.0796	-0.2856	-0.4674
0.40	1.5756	1.4470	1.3493	1.2677	1.1966	1.1328	1.0745	1.0207	0.9705	0.9230	0.7182	0.5497	0.4032	0.2726
0.45	1.7341	1.6686	1.6187	1.5768	1.5401	1.5071	1.4768	1.4488	1.4225	1.3976	1.2893	1.1992	1.1207	1.0503
0.50	1.8942	1.8942	1.8942	1.8942	1.8942	1.8942	1.8942	1.8942	1.8942	1.8942	1.8942	1.8942	1.8942	1.8942
0.55	2.0553	2.1229	2.1751	2.2192	2.2583	2.2938	2.3266	2.3571	2.3859	2.4132	2.5332	2.6355	2.7264	2.8092
0.60	2.2206	2.3590	2.4663	2.5575	2.6386	2.7123	2.7806	2.8445	2.9048	2.9615	3.2125	3.4278	3.6197	3.7952
0.65	2.3952	2.6097	2.7771	2.9200	3.0459	3.1607	3.2672	3.3671	3.4615	3.5512	3.9498	4.2931	4.6008	4.8827
0.70	2.5819	2.8798	3.1110	3.3088	3.4855	3.6465	3.7961	3.9365	4.0695	4.1961	4.7606	5.2460	5.6799	6.0776
0.75	2.7870	3.1749	3.4793	3.7401	3.9734	4.1871	4.3857	4.5724	4.7492	4.9177	5.6648	6.3088	6.8855	7.4132
0.80	3.0182	3.5108	3.8991	4.2335	4.5331	4.8076	5.0624	5.3010	5.5270	5.7424	6.7018	7.5278	8.2659	8.9402
0.85	3.2932	3.9136	4.4052	4.8291	5.2077	5.5540	5.8760	6.1785	6.4649	6.7377	7.9511	8.9930	9.9215	10.7671
0.90	3.6443	4.4307	5.0553	5.5916	6.0721	6.5122	6.9212	7.3051	7.6683	8.0140	9.5478	10.8587	12.0246	13.0856
0.95	4.1782	5.2195	6.0442	6.7552	7.3912	7.9729	8.5127	9.0186	9.4966	9.9503	11.9573	13.6686	15.1866	16.5652

q = 0.050 ν = 8

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	-0.0001	-0.6796	-1.1712	-1.5704	-1.9124	-2.2147	-2.4871	-2.7372	-2.9694	-3.1869	-4.1164	-4.8816	-5.5461	-6.1391
0.10	0.3817	-0.1718	-0.5770	-0.9058	-1.1872	-1.4362	-1.6609	-1.8669	-2.0577	-2.2356	-2.9941	-3.6131	-4.1469	-4.6231
0.15	0.6481	0.1856	-0.1546	-0.4333	-0.6723	-0.8833	-1.0737	-1.2481	-1.4098	-1.5608	-2.2035	-2.7247	-3.1731	-3.5706
0.20	0.8670	0.4816	0.1960	-0.0381	-0.2402	-0.4195	-0.5815	-0.7298	-0.8671	-0.9954	-1.5411	-1.9843	-2.3634	-2.6988
0.25	1.0564	0.7410	0.5054	0.3111	0.1431	-0.0062	-0.1415	-0.2662	-0.3818	-0.4898	-0.9494	-1.3221	-1.6416	-1.9240
0.30	1.2297	0.9800	0.7920	0.6359	0.5007	0.3801	0.2703	0.1693	0.0754	-0.0124	-0.3892	-0.6959	-0.9584	-1.1904
0.35	1.3932	1.2059	1.0646	0.9472	0.8445	0.7523	0.6684	0.5909	0.5188	0.4512	0.1598	-0.0788	-0.2849	-0.4676
0.40	1.5505	1.4254	1.3304	1.2508	1.1812	1.1187	1.0617	1.0089	0.9596	0.9133	0.7113	0.5445	0.4000	0.2709
0.45	1.7043	1.6408	1.5923	1.5517	1.5160	1.4839	1.4544	1.4271	1.4015	1.3774	1.2718	1.1838	1.1069	1.0380
0.50	1.8592	1.8592	1.8592	1.8592	1.8592	1.8592	1.8592	1.8592	1.8592	1.8592	1.8592	1.8592	1.8592	1.8592
0.55	2.0153	2.0806	2.1311	2.1737	2.2115	2.2457	2.2773	2.3067	2.3345	2.3608	2.4764	2.5747	2.6620	2.7415
0.60	2.1750	2.3086	2.4120	2.4998	2.5777	2.6485	2.7140	2.7753	2.8331	2.8881	3.1288	3.3347	3.5184	3.6863
0.65	2.3435	2.5499	2.7107	2.8478	2.9692	3.0792	3.1811	3.2766	3.3669	3.4528	3.8342	4.1623	4.4537	4.7202
0.70	2.5232	2.8092	3.0316	3.2210	3.3898	3.5440	3.6872	3.8215	3.9486	4.0696	4.6048	5.0650	5.4772	5.8547
0.75	2.7201	3.0928	3.3839	3.6336	3.8568	4.0609	4.2502	4.4268	4.5940	4.7533	5.4628	6.0742	6.6214	7.1222
0.80	2.9427	3.4141	3.7857	4.1053	4.3896	4.6493	4.8906	5.1173	5.3320	5.5365	6.4471	7.2309	7.9306	8.5694
0.85	3.2060	3.7996	4.2686	4.6696	5.0287	5.3576	5.6633	5.9504	6.2223	6.4812	7.6324	8.6194	9.4998	10.3027
0.90	3.5419	4.2928	4.8839	5.3933	5.8495	6.2672	6.6553	7.0196	7.3643	7.6919	9.1451	10.3897	11.4970	12.5051
0.95	4.0524	5.0399	5.8230	6.4977	7.1013	7.6531	8.1643	8.6437	9.0966	9.5272	11.4331	13.0591	14.5021	15.8111

TABLE I (Continued)

q = 0.050 ν = 9

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	-0.0004	-0.6796	-1.1750	-1.5781	-1.9237	-2.2299	-2.5065	-2.7601	-2.9957	-3.2165	-4.1620	-4.9400	-5.6156	-6.2198
0.10	0.3793	-0.1716	-0.5767	-0.9072	-1.1911	-1.4425	-1.6695	-1.8776	-2.0707	-2.2512	-3.0208	-3.6504	-4.1930	-4.6772
0.15	0.6431	0.1846	-0.1545	-0.4327	-0.6722	-0.8845	-1.0765	-1.2526	-1.4159	-1.5684	-2.2186	-2.7474	-3.2025	-3.6072
0.20	0.8591	0.4782	0.1949	-0.0383	-0.2397	-0.4189	-0.5811	-0.7301	-0.8682	-0.9975	-1.5485	-1.9964	-2.3810	-2.7211
0.25	1.0459	0.7349	0.5018	0.3094	0.1422	-0.0065	-0.1414	-0.2656	-0.3812	-0.4893	-0.9512	-1.3273	-1.6499	-1.9354
0.30	1.2167	0.9705	0.7852	0.6311	0.4972	0.3777	0.2688	0.1683	0.0749	-0.0127	-0.3886	-0.6960	-0.9603	-1.1943
0.35	1.3773	1.1933	1.0540	0.9381	0.8370	0.7461	0.6631	0.5866	0.5152	0.4481	0.1589	-0.0790	-0.2844	-0.4670
0.40	1.5315	1.4089	1.3157	1.2374	1.1689	1.1074	1.0511	0.9991	0.9504	0.9046	0.7056	0.5406	0.3974	0.2695
0.45	1.6819	1.6198	1.5724	1.5326	1.4977	1.4662	1.4373	1.4105	1.3854	1.3617	1.2582	1.1715	1.0957	1.0277
0.50	1.8332	1.8332	1.8332	1.8332	1.8332	1.8332	1.8332	1.8332	1.8332	1.8332	1.8332	1.8332	1.8332	1.8332
0.55	1.9855	2.0491	2.0982	2.1398	2.1765	2.2098	2.2405	2.2691	2.2961	2.3216	2.4337	2.5288	2.6131	2.6898
0.60	2.1410	2.2709	2.3714	2.4563	2.5316	2.6000	2.6633	2.7224	2.7782	2.8311	3.0645	3.2638	3.4414	3.6033
0.65	2.3048	2.5048	2.6600	2.7923	2.9099	3.0165	3.1152	3.2076	3.2949	3.3779	3.7456	4.0608	4.3422	4.5989
0.70	2.4790	2.7551	2.9705	3.1538	3.3171	3.4661	3.6042	3.7334	3.8556	3.9718	4.4877	4.9307	5.3266	5.6887
0.75	2.6692	3.0297	3.3114	3.5526	3.7673	3.9635	4.1457	4.3163	4.4773	4.6307	5.3127	5.8993	6.4235	6.9027
0.80	2.8841	3.3405	3.6989	4.0061	4.2805	4.5305	4.7628	4.9809	5.1872	5.3835	6.2567	7.0068	7.6771	8.2888
0.85	3.1393	3.7123	4.1635	4.5501	4.8957	5.2118	5.5051	5.7806	6.0413	6.2893	7.3912	8.3367	9.1799	9.9491
0.90	3.4641	4.1869	4.7563	5.2460	5.6837	6.0844	6.4559	6.8046	7.1344	7.4483	8.8402	10.0324	11.0932	12.0592
0.95	3.9553	4.9064	5.6584	6.3052	6.8828	7.4110	7.9009	8.3599	8.7937	9.2062	11.0319	12.5902	13.9735	15.2305

q = 0.050 ν = 16

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	-0.0002	-0.6803	-1.1870	-1.6050	-1.9668	-2.2894	-2.5824	-2.8523	-3.1035	-3.3394	-4.3536	-5.1908	-5.9173	-6.5675
0.10	0.3719	-0.1700	-0.5762	-0.9121	-1.2037	-1.4639	-1.7004	-1.9184	-2.1214	-2.3118	-3.1303	-3.8039	-4.3870	-4.9078
0.15	0.6272	0.1818	-0.1530	-0.4311	-0.6729	-0.8890	-1.0857	-1.2672	-1.4362	-1.5949	-2.2774	-2.8387	-3.3245	-3.7576
0.20	0.8339	0.4681	0.1920	-0.0377	-0.2379	-0.4172	-0.5807	-0.7316	-0.8723	-1.0046	-1.5742	-2.0432	-2.4492	-2.8107
0.25	1.0118	0.7153	0.4910	0.3037	0.1403	-0.0063	-0.1400	-0.2638	-0.3794	-0.4881	-0.9571	-1.3444	-1.6800	-1.9791
0.30	1.1729	0.9401	0.7634	0.6157	0.4865	0.3704	0.2642	0.1659	0.0740	-0.0124	-0.3868	-0.6970	-0.9664	-1.2070
0.35	1.3232	1.1509	1.0194	0.9093	0.8128	0.7260	0.6465	0.5729	0.5039	0.4389	0.1567	-0.0780	-0.2825	-0.4656
0.40	1.4671	1.3528	1.2655	1.1923	1.1280	1.0699	1.0167	0.9673	0.9210	0.8773	0.6872	0.5285	0.3896	0.2648
0.45	1.6065	1.5491	1.5051	1.4682	1.4356	1.4063	1.3794	1.3543	1.3308	1.3086	1.2117	1.1304	1.0589	0.9945
0.50	1.7458	1.7458	1.7458	1.7458	1.7458	1.7458	1.7458	1.7458	1.7458	1.7458	1.7458	1.7458	1.7458	1.7458
0.55	1.8858	1.9440	1.9888	2.0267	2.0601	2.0903	2.1182	2.1442	2.1686	2.1917	2.2933	2.3793	2.4551	2.5239
0.60	2.0278	2.1458	2.2367	2.3137	2.3818	2.4434	2.5001	2.5531	2.6029	2.6502	2.8584	3.0354	3.1920	3.3345
0.65	2.1765	2.3576	2.4972	2.6156	2.7204	2.8156	2.9035	2.9856	3.0630	3.1362	3.4594	3.7352	3.9800	4.2029
0.70	2.3342	2.5823	2.7745	2.9378	3.0825	3.2138	3.3352	3.4487	3.5558	3.6575	4.1065	4.4901	4.8315	5.1428
0.75	2.5054	2.8274	3.0775	3.2899	3.4785	3.6502	3.8091	3.9574	4.0975	4.2305	4.8196	5.3234	5.7725	6.1818
0.80	2.6974	3.1032	3.4185	3.6874	3.9262	4.1436	4.3450	4.5335	4.7115	4.8805	5.6298	6.2707	6.8423	7.3635
0.85	2.9249	3.4303	3.8246	4.1606	4.4599	4.7326	4.9851	5.2217	5.4452	5.6577	6.5985	7.4043	8.1224	8.7772
0.90	3.2120	3.8449	4.3393	4.7622	5.1385	5.4821	5.8002	6.0981	6.3795	6.6472	7.8331	8.8481	9.7512	10.5741
0.95	3.6431	4.4692	5.1167	5.6713	6.1648	6.6155	7.0331	7.4241	7.7935	8.1447	9.6990	11.0268	12.2064	13.2793

TABLE I (Continued)

q = 0.050 ν = 36

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	-0.0001	-0.6809	-1.1963	-1.6264	-2.0021	-2.3391	-2.6471	-2.9321	-3.1984	-3.4491	-4.5347	-5.4362	-6.2221	-6.9263
0.10	0.3669	-0.1690	-0.5758	-0.9159	-1.2134	-1.4807	-1.7252	-1.9517	-2.1633	-2.3627	-3.2268	-3.9448	-4.5705	-5.1311
0.15	0.6164	0.1799	-0.1520	-0.4300	-0.6734	-0.8923	-1.0927	-1.2785	-1.4522	-1.6160	-2.3266	-2.9178	-3.4332	-3.8951
0.20	0.8170	0.4610	0.1899	-0.0374	-0.2366	-0.4160	-0.5804	-0.7328	-0.8754	-1.0100	-1.5945	-2.0817	-2.5068	-2.8882
0.25	0.9886	0.7021	0.4834	0.2999	0.1389	-0.0061	-0.1391	-0.2625	-0.3781	-0.4872	-0.9616	-1.3578	-1.7041	-2.0149
0.30	1.1433	0.9196	0.7487	0.6051	0.4790	0.3653	0.2610	0.1642	0.0734	-0.0122	-0.3855	-0.6978	-0.9711	-1.2168
0.35	1.2872	1.1222	0.9960	0.8899	0.7966	0.7124	0.6351	0.5633	0.4960	0.4324	0.1551	-0.0773	-0.2812	-0.4646
0.40	1.4243	1.3154	1.2321	1.1619	1.1002	1.0445	0.9934	0.9458	0.9012	0.8590	0.6748	0.5200	0.3841	0.2617
0.45	1.5566	1.5022	1.4605	1.4253	1.3944	1.3665	1.3408	1.3169	1.2945	1.2733	1.1806	1.1026	1.0339	0.9720
0.50	1.6883	1.6883	1.6883	1.6883	1.6883	1.6883	1.6883	1.6883	1.6883	1.6883	1.6883	1.6883	1.6883	1.6883
0.55	1.8202	1.8750	1.9170	1.9525	1.9838	2.0121	2.0382	2.0625	2.0853	2.1068	2.2015	2.2814	2.3519	2.4158
0.60	1.9536	2.0640	2.1488	2.2205	2.2838	2.3410	2.3937	2.4428	2.4890	2.5327	2.7246	2.8869	3.0303	3.1604
0.65	2.0927	2.2612	2.3910	2.5007	2.5976	2.6853	2.7660	2.8413	2.9122	2.9792	3.2742	3.5242	3.7452	3.9459
0.70	2.2396	2.4699	2.6474	2.7975	2.9301	3.0503	3.1611	3.2644	3.3617	3.4539	3.8592	4.2033	4.5082	4.7851
0.75	2.3986	2.6961	2.9255	3.1197	3.2915	3.4473	3.5909	3.7249	3.8511	3.9707	4.4976	4.9452	5.3424	5.7036
0.80	2.5764	2.9490	3.2369	3.4810	3.6968	3.8926	4.0733	4.2421	4.4011	4.5518	5.2162	5.7817	6.2834	6.7399
0.85	2.7857	3.2476	3.6050	3.9079	4.1762	4.4199	4.6449	4.8550	5.0530	5.2409	6.0697	6.7756	7.4025	7.9731
0.90	3.0486	3.6233	4.0683	4.4463	4.7813	5.0856	5.3669	5.6298	5.8774	6.1124	7.1502	8.0348	8.8209	9.5357
0.95	3.4408	4.1845	4.7619	5.2529	5.6886	6.0845	6.4506	6.7929	7.1157	7.4220	8.7755	9.9287	10.9531	11.8845

q = 0.050 ν = 144

u \ b	1	2	3	4	5	6	7	8	9	10	15	20	25	30
0.05	-0.0001	-0.6814	-1.2023	-1.6403	-2.0253	-2.3727	-2.6915	-2.9878	-3.2656	-3.5280	-4.6729	-5.6329	-6.4750	-7.2336
0.10	0.3639	-0.1683	-0.5757	-0.9183	-1.2197	-1.4917	-1.7414	-1.9736	-2.1913	-2.3970	-3.2953	-4.0492	-4.7109	-5.3073
0.15	0.6100	0.1788	-0.1514	-0.4293	-0.6738	-0.8945	-1.0973	-1.2858	-1.4626	-1.6297	-2.3597	-2.9728	-3.5114	-3.9968
0.20	0.8071	0.4569	0.1887	-0.0372	-0.2359	-0.4154	-0.5802	-0.7336	-0.8775	-1.0134	-1.6078	-2.1073	-2.5462	-2.9420
0.25	0.9753	0.6943	0.4790	0.2977	0.1381	-0.0061	-0.1385	-0.2617	-0.3774	-0.4867	-0.9646	-1.3664	-1.7197	-2.0385
0.30	1.1264	0.9077	0.7401	0.5989	0.4746	0.3624	0.2592	0.1632	0.0730	-0.0122	-0.3848	-0.6983	-0.9741	-1.2231
0.35	1.2666	1.1058	0.9825	0.8786	0.7871	0.7045	0.6285	0.5578	0.4914	0.4287	0.1541	-0.0770	-0.2804	-0.4640
0.40	1.3999	1.2941	1.2129	1.1446	1.0844	1.0300	0.9799	0.9334	0.8897	0.8483	0.6674	0.5151	0.3810	0.2598
0.45	1.5282	1.4754	1.4350	1.4009	1.3708	1.3437	1.3187	1.2955	1.2737	1.2531	1.1627	1.0866	1.0196	0.9590
0.50	1.6555	1.6555	1.6555	1.6555	1.6555	1.6555	1.6555	1.6555	1.6555	1.6555	1.6555	1.6555	1.6555	1.6555
0.55	1.7830	1.8358	1.8763	1.9105	1.9407	1.9679	1.9929	2.0163	2.0382	2.0589	2.1497	2.2262	2.2937	2.3548
0.60	1.9115	2.0177	2.0992	2.1679	2.2285	2.2833	2.3337	2.3806	2.4247	2.4664	2.6492	2.8034	2.9393	3.0624
0.65	2.0453	2.2070	2.3311	2.4359	2.5282	2.6117	2.6886	2.7601	2.8273	2.8909	3.1698	3.4052	3.6128	3.8007
0.70	2.1862	2.4065	2.5757	2.7185	2.8443	2.9582	3.0630	3.1606	3.2523	3.3391	3.7196	4.0410	4.3245	4.5813
0.75	2.3384	2.6220	2.8400	3.0239	3.1861	3.3329	3.4680	3.5937	3.7120	3.8239	4.3147	4.7295	5.0957	5.4273
0.80	2.5080	2.8623	3.1347	3.3646	3.5674	3.7509	3.9198	4.0771	4.2250	4.3651	4.9795	5.4989	5.9577	6.3734
0.85	2.7073	3.1447	3.4811	3.7652	4.0158	4.2426	4.4515	4.6461	4.8290	5.0022	5.7625	6.4058	6.9741	7.4892
0.90	2.9567	3.4984	3.9151	4.2671	4.5778	4.8591	5.1182	5.3596	5.5866	5.8016	6.7457	7.5448	8.2513	8.8918
0.95	3.3268	4.0235	4.5599	5.0133	5.4136	5.7761	6.1101	6.4215	6.7144	6.9917	8.2105	9.2431	10.1563	10.9846

APPENDIX B

CONJUGATE PRIOR DISTRIBUTIONS

We briefly discussed the class of conjugate distributions in the introduction to this thesis. In this part of the appendix, we give explicit definitions for certain families of distributions. We include only those families which we have used in treating the problems of this paper. A more exhaustive listing of these distributions is given by Raiffa and Schlaiffer [7] .

Normal Distribution

The density function for the normal distribution is defined by

$$f(x|a, b) = \frac{1}{\sqrt{2\pi}\, b} e^{-\frac{(x-a)^2}{2b^2}} .$$

It is related to the standard normal distribution by the equation

$$F_N(x|a, b) = F_z\left(\frac{x-a}{b}\right) .$$

The mean and variance for this distribution are given by

$$E(x) = a,$$

$$V(x) = b^2.$$

This family of distributions is the conjugate family for the mean in a normal sampling distribution if the variance is known.

Beta Distribution

A random variable X is said to have a beta distribution with specifications (a, b) if it has the density function

$$f(x|a, b) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1} \qquad a, b > 0, \ 0 \leq x \leq 1$$

where

$$B(a, b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} .$$

The mean and variance of this distribution are given by

$$E(x) = \frac{a}{a+b}$$

$$V(x) = \frac{ab}{(a+b)^2 \ (a+b+1)} .$$

The distribution function has been tabulated by K. Pearson, Tables of the Incomplete Beta Function, Biometrika, London, 1934.

This distribution is the conjugate prior for p in the binomial distribution where n is known. Indeed, this is the conjugate prior for any sampling distribution $f(y|\theta)$ for which a kernal can be represented as

$$k(y|\theta) \propto \theta^r \ (1-\theta)^t \quad r, t > 0, \ 0 < \theta < 1.$$

The Gamma-1 Distribution

A random variable X is said to have a gamma-1 distribution if its density function is defined by

$$f(x|a, b) = \frac{b^a \ x^{a-1} \ e^{-bx}}{\Gamma(a)} \qquad a, b > 0, \ x \geq 0$$

where $\Gamma(a)$ is the complete gamma function

$$\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} dt.$$

The mean and variance for this distribution are

$$E(x) = \frac{a}{b} \text{ and } V(x) = \frac{a}{b^2} .$$

The distribution function for a gamma-1 variate is related to the chi-square distribution function by the equation

$$F_{\gamma 1}(x|a, b) = F_{\chi^2}(2bx|2a).$$

The family of gamma-1 distributions is the conjugate family of distributions for the parameters of the Poisson and the negative exponential distributions.

The Inverted-Gamma-2 Distribution

The density function for the inverted-gamma-2 distribution is defined by

$$f(x|a, b) = \frac{2\left(\frac{ab}{2x^2}\right)^{\frac{a+1}{2}} e^{-\frac{ab}{2x^2}}}{\Gamma\left(\frac{a}{2}\right)\left(\frac{ab}{2}\right)^{1/2}} \qquad a, b > 0, \ x > 0.$$

The distribution function for the inverted-gamma-2 distribution is related to the chi-square distribution function by the equation

$$F_{i\gamma 2}(x|a, b) = F_{\chi^2}\left(\frac{ab}{x^2} \mid a\right) .$$

The mean and variance for this distribution are given by

$$E(x) = \left(\frac{ab}{2}\right)^{1/2} \frac{\Gamma \frac{a-1}{2}}{\Gamma \frac{a}{2}} \qquad a > 1$$

$$V(x) = \frac{ab}{a-2} - E(x)^2 \qquad a > 2.$$

The mode is at

$$x_{mod} = \left[\frac{ab}{a+1}\right]^{1/2}.$$

This distribution is the conjugate prior distribution for the standard deviation of a normal distribution when the mean is known.

The Normal-Inverted-Gamma-2 Distribution

The density function for the normal-inverted-gamma-2 distribution is a bivariate density defined by

$$f(x, y \mid a, b, r, t) = \left[\frac{b}{2\pi y^2}\right]^{1/2} e^{-\frac{b(x-a)^2}{2y^2}} \frac{\left(\frac{rt}{2y^2}\right)^{\frac{r+1}{2}} e^{-\frac{rt}{2y^2}}}{\Gamma\left(\frac{r}{2}\right)\left(\frac{rt}{2}\right)^{1/2}}$$

$$r, t, b > 0, \quad -\infty < x < \infty, \quad y \geq 0.$$

The marginal distribution of x is related to Student's t- distribution. That is, if we take

$$w = \left(\frac{b}{t}\right)^{1/2} (x-a),$$

then w has a Student's t- distribution with r degrees of freedom. The marginal distribution of y is the inverted gamma-2 distribution with specifications (r, t). The conditional distribution of x for given y is the normal distribution with mean a and variance $\frac{y^2}{b}$. The conditional distribution of y for given x is an inverted-gamma-2 distribution with specifications

$$(r+1,\ t+\frac{b(x-a)^2}{r}).$$

The family of normal-inverted-gamma-2 distributions is the conjugate family of distributions for the mean and variance of the normal distribution when neither is known.

VITA

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Thesis: THE DISTRIBUTION OF CERTAIN FUNCTIONS OF PARAMETERS: PRIOR AND POSTERIOR

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