ON THE THEOREMS OF DE RHAM

By

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PREFACE

The author is deeply grateful for the patient and valuable direction given by Professor R. B. Deal, Jr. in the formulation and preparation of this thesis. Additional appreciation is also expressed for the many considerations, time and constructive comments given by Professors Milton Berg, W. Marsden and V. Troxel.

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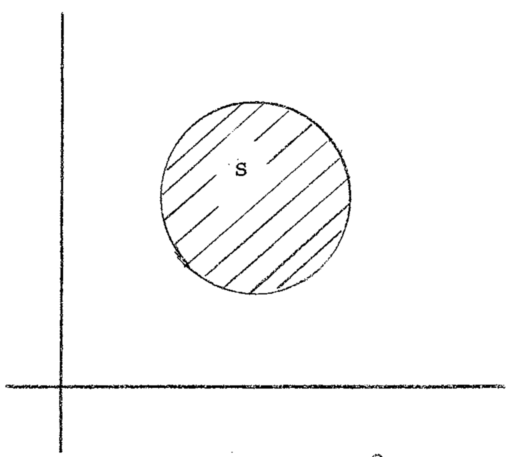
CHAPTER I

INTRODUCTION

In beginning college mathematics one is asked to solve many types of problems and one type is the following. Is there a closed differential 2-form α in Euclidean two spaces, E^2, such that

$$\int_s \alpha = c$$

where s is a surface and c is any constant? Clearly, any 2-form is closed in E^2. Then $\alpha = \frac{c}{A}\, dx\, dy$, A is the area of s, satisfied the α in the problem.



Suppose α is a 1-form on E^2 with $d\,\alpha = 0$. Then is α exact, that is, is there a β such that $\alpha = d\,\beta$? The answer to this problem is a result called Poincare's Lemma: Let α be a differential form defined in an open ball of radius r, B_r^n, contained in E^n. Let one assume further that $dw = 0$. Then, there exists a function f, defined in B_r^n such that $df = \alpha$. A proof of this is given in Chapter IV.

The problems we have looked at have been answered in Euclidean space and in the case of a space that is deformable to a point. Let one now consider a space that is not deformable to a point. Let the manifold M consist of E^3 with the origin removed,

$$M = E^3 - \{0\}.$$

Suppose W is a one-form on M such that $dW = 0$. Then is W exact? In this case one cannot use Poincare Lemma as clearly the manifold cannot be shrunk to a point. Nonetheless, $W = df$, where

$$f(x) = \int_{(1,0,0)}^{x} W.$$

The above integral is taken along any path C that avoids 0. To see this is independent of the path one uses Stokes Theorem, which is stated and proved in Chapter III. If C' is any other path, avoiding 0, in M from (1,0,0) to x, then the chain C - C' is the boundary of a piece of surface Σ in M and

$$\int_C W - \int_{C'} W = \int_{C-C'} W = \int_{\partial\Sigma} W = \int_{\Sigma} dW = 0$$

since $dW = 0$.

But now one asks the global question: Is there always a λ on all of M such that $dW = \lambda$? The answer in general is no.

The thing that the mathematician would now bring to mind would be when and under what conditions will the answer be yes. A mathematician, G. de Rham, answered these questions [6]. This paper develops the background necessary to prove the existence theorems of de Rham and gives examples of these theorems.

The idea of the Proof is due to A. Weil [10]. The method is the theory of Sheaves due to Leroy. The cohomology developed is a generalization of the classical Čech definition of cohomology.

CHAPTER II

HISTORY AND DEFINITIONS

The main purpose of this chapter is to establish the mathematical language and definitions necessary throughout the rest of the paper. Another purpose is to give some history and background on the development of geometry in general and the existence theorems of de Rham in particular.

Until the end of the eighteenth century, Euclidean geometry stood forth as the most solidly established body of truths known to man and as the necessary and indubitable geometry of space. Immanuel Kant affirmed that the laws of Euclidean geometry were necessary, he maintained that the space of Euclid is a fundamental intuition.

However, geometry underwent a profound revolution in the nineteenth century. The creation of non-euclidean geometry in the early part of the nineteenth century cast doubt on the Euclidean character of physical space and showed the mind is not restricted to think in Euclidean terms. Projective geometry was built up to a full-fledged independent subject. It was shown that the Euclidean geometry and several basic non-Euclidean geometries, namely, the hyperbolic goemetry of Gauss, Bolyai, and Lobatchevsky, the spherical or double elliptic geometry of Riemann, and the elliptic of Felix Klein, can be derived as a special case of projective geometry; however, some formulations of projective geometry exclude spherical geometry.

In 1897 Bertran Russell wrote an essay on the foundations of geometry. Since Russell lived in the shadow of Authur Cayley and Klein, we can understand why Russell believed projective geometry was all of geometry. Just as Russell's forerunners may have committed error because they did not know projective geometry, Russell could not have known twentiety-century developments. One of these is a new branch of geometry, topology, which generalizes on projective geometry as projective geometry in turn generalizes on Euclidean and the basic non-Euclidean.

Since the introduction of differentials by Newton and Leibniz in the Seventeenth century, there has been a large amount of written literature on differential geometry. In 1847 H. Grassman derived an algebra A for analyzing subspaces of vector spaces. The covariant tensor fields form a submolule of A (if non-restrictive), which inherits a multiplication from A, the exterior multiplication. Through E. Cartan the Grassman algebra has become an indispensable tool for dealing with submanifolds.

Much modern differential goemetry to a large degree has become differential topology, and the methods employed are a far cry from the tensor analysis of the differential geometry of the 1930's. This development, however, has not been abrupt as might be imagined. It has its roots in the movement toward differential geometry in the large to which mathematicians such as Hoff and Rinow, Cohn-Vossen, de Rham, Hodge, and Myers gave importance. The objectives of their work were to derive relationships between the topology of a manifold and its local differential goemetry. Other sources of inspiration were Cartan and M. Morse and his calculus of variations. One of the major new ideas was that of fibre bundle, which gave a global structure to a differentiable manifold more general than that included in older theories. Methods and results of differential geometry were applied with outstanding success to the theories of complex manifolds and algebraic varieties, and these in turn, have stimulated differential geometry. The discovery by Milnor of invariants of the differential structure of a manifold that are not topological invariants establish differential topology as a discipline of major importance.

G. de Rham has done much work in modern differential goemetry, we are able to determine the cohomology of a manifold by use of his theorems. De Rham's theorems state there are precisely two cohomology theories. Moreover, if our differentiable manifold is compact there is only one.

Now let us look at some definitions and properties necessary for the rest of the paper. An n-dimensional manifold which is a space not necessarily Euclidean space nor is it a domain in an Euclidean space, but which, from the viewpoint of a short-sighted observer living in the space, looks just like a domain of Euclidean space. As an example, consider the two spheres, S^2. This cannot be considered a part of the Euclidean plane E^2. However, to an observer on S^2, he can describe his immediate vicinity by coordinates and so he fails to distinguish between this and a small domain on E^2.

An n-dimensional manifold consists of a space M together with a collection of local coordinate neighborhoods U_1, U_2, . . . such that each point of M lies in at least one of these U. On each U is given a coordinate system,

$$x^1, \ldots, x^n$$

so that the values of the coordinates

$$(x^1(P), \ . \ . \ . \ , \ x^n(P)),$$

where P ranges over U, make up an open domain in Euclidean n-space E^n. Suppose that U with coordinate system

$$x^1, \ . \ . \ . \ , \ x^n$$

and V with coordinate system

$$y^1, \ . \ . \ . \ , \ y^n$$

overlap. We may express the V coordinates $y^1, \ . \ . \ . \ , \ y^n$ of a point P in terms of the U coordinates $x^1, \ . \ . \ . \ , \ x^n$ of this point

$$y^i = y^i(x^1, \ . \ . \ . \ , \ x^n), \ i = 1, \ . \ . \ . \ , \ n.$$

As a part of the definition, we assume that the functions are differentiable as often as we please.

A manifold M together with an equivalence class of differentiable structures on M is called a *differentiable manifold*.

An associate algebra $\bigwedge(V)$ over R containing the vector space V over R is called a *Grassman*, or *exterior*, algebra if

(i) $\bigwedge(V)$ contains the unit element 1 of R,

(ii) $\bigwedge(V)$ is generated by 1 and the elements of V,

(iii) If $x \in V$, $x \bigwedge x = 0$.

(iv) The dimension of $\bigwedge(V)$ is ^{n}.

All the elements, $e_{i_r} \in V$,

$$e_{i_1} \bigwedge . \ . \ . \bigwedge e_{i_p}, \qquad i_1 < . \ . \ . < i_p$$

for a fixed p span a linear subspace of $\bigwedge(V)$, which we denote by $\bigwedge^p(V)$ and whose elements are called *p-vectors*.

Let V^* be the dual space of V and consider the algebra $\bigwedge(V^*)$ over R. The linear space $\bigwedge^p(V^*)$ is called the *space of exterior p-forms* over V; its elements are called <u>p-forms</u>. We note at this point that $\bigwedge^0(V^*) = R$ and $\bigwedge^1(V^*) = V^*$ and so on.

Let M be a differentiable manifold of dimension n. Associated with each point $P \in M$, there exists the tangent space T_P and its dual $T_P{}^*$ of covariant vectors at a point P. Let U be a coordinate neighborhood containing P with local coordinates $u^1, \ldots, u^n$ and natural dual $du^1, \ldots, du^n$ for the space $T_p{}^*$. An element $\alpha(P) \in \bigwedge^p(T_P{}^*)$ has the following representation in U:

$$\alpha(P) = a_{(i_1 \ldots i_p)} du^{i_1}(P) \wedge \ldots \wedge du^{i_p}(P),$$

where the $a_{(i_1 \ldots i_p)}$ are of class r.

Then α is said to be a <u>differential form of degree P and class r</u> or simply a p-form.

Again, let M be a differentiable manifold of class $k \geq 2$. Then, there exists a map

$$d: \quad \bigwedge(T^*) \longrightarrow \bigwedge(T^*)$$

sending exterior forms of class r into exterior forms of class r+1 with the properties

(i) For p = 0 df is a covector

(ii) d is a linear map such that $d(\bigwedge^p(T^*)) \subset (\bigwedge^{p+1}(T^*))$

(iii) For $\alpha \in \bigwedge^p(T^*)$, $\beta \in \bigwedge^q(T^*)$, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta,$$

(iv) $d(df) = 0$.

It can be shown that d is unique as an operator, also

$$d\alpha = \frac{\partial \alpha}{\partial x^i} dx^i$$

The elements of $\bigwedge_c^p (T^*)$, the kernel of d, are called closed p-forms and the images $\bigwedge_c^p(T^*) = \bigwedge_d^p(T^*)$ of $\bigwedge^{p-1}(T^*)$ under d are called the exact p-forms. By considering the quotient space of the closed forms of degree p by the subspace of exact p-forms will be called the p-dimensional cohomology group of M obtained using differential forms and denoted by:

$$D^p(M) = \frac{\bigwedge_c^p (T^*)}{\bigwedge_e^p (T^*)} \quad .$$

A differentiable manifold M of dimension n is said to be orientiable if there exists over M a continuous differential form of degree n that is nowhere zero.

The carrier, carr α, of a differential form α is the closure of V written, $\ddot{V}$, where V is the set of points where α is not equal zero.

Now one will look at two examples of exterior products. First, consider the linear space based on the differentials dx, dy, . . . and, as is customary, omit the exterior multiplication sign between dx's, that is $dx \wedge dy$ is denoted by dx dy.

Let

$$\alpha = Adx + Bdy + Cdz$$

and

$$\beta = E\ dx + F\ dy + G\ dz$$

then

$$\alpha \wedge \beta = (BG - CF)\ dy\ dz + (CE - AG)\ dz\ dx + (AF - BF)\ dx\ dy,$$

illustrating the vector - or cross-product of two ordinary vectors.

Next, let us consider α as above and

$$\eta = P\ dy\ dz + Q\ dz\ dx + R\ dx\ dy$$

then

$$\alpha \wedge \eta = (AP + BQ + CR)\, dx\, dy\, dz,$$

illustrating the dot - or inner-product. Mostow, Sampson, and Meyer in their book have a complete development of exterior algebra and exterior products.

By a <u>locally finite open covering</u> $U = \{U_i\}$ of a manifold M we shall mean, for each $P \epsilon M$, P is contained in only a finite number of the U_i.

A covering $V = \{V_i\}$ of M is called a <u>refinement</u> of U if there is a map

$$\phi : V \longrightarrow U$$

defined by associating with each $V_i \epsilon V$ a set $U_i \epsilon U$ such that $V_i \quad U_i$. A refinement V of U is called a <u>strong refinement</u> if each $\bar{V}_i$ is compact and contained in some U_i. In this case we write $V \ll U$.

A covering $U = \{U_i\}$ of M is said to be <u>simple</u> if (a) it is strong locally finite and (b) every non-empty intersection $U_0 \cap \ldots \cap U_p$ of open sets of the covering is homeomorphic with a star shaped region in an n-dimensional affine space with a distinguished point. The n-dimensional affine space with a distinguished point is denoted by R^n.

Now, we shall discuss some of the properties that we shall need from singular homology theory. By a <u>P-simplex</u> $[\phi : S^p]$, p = 0, 1, 2, . . . on a differential manifold M is understood an Euclidean p-simplex S^p together with a differentiable map ϕ of S^p into M. Consider the ordered sequence of points $(P_0, P_1, \ldots, P_p)$, linearly independent, in $\{(x^1, x^2, \ldots, x^n, \ldots) \mid x_i \neq 0$ for only a finite number of $i\}$ denoted by $\Delta(P_0, \ldots, P_p)$ the convex hull containing them. By a <u>singular p-simplex</u> on M we mean a map ϕ of class 1 of

$\Delta(P_0, \ldots, P_p)$ into M. A singular p-chain is a map of the set of all singular p-simplexes into R usually written as a formal sum $\Sigma\, g_i t_i^p$, g_i an integer, with singular simplexes t_i^p indexed in some fixed manner.

The support of t^p is the set of points $\phi(\Delta(P_0, \ldots, P_p))$. A chain is called locally finite if each compact set meets only a finite number of supports with $g_i \neq 0$. We consider only locally finite chains. A chain is finite if there are only a finite number of non-vanishing g_i. The support of a chain $\Sigma\, g_i\, t_i^p$ is the union of the support of all the t_i^p where $g_i \neq 0$.

Assume we have the definition of the operator ∂, cycles, and boundaries from chapter three. Let S_p denote the vector space of all finite p-chains, S_p^c the subspace of p-cycles and S_p^b the space of boundaries of finite (p+1)-chains. The quotient

$$\frac{S_p^c}{S_p^b} = S\, H_p$$

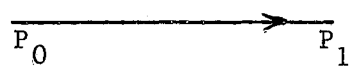
is called the p^{th} singular homology space or group of M.

In many parts of this paper we will use the circumflex to indicate omission, that is $(x^1, x^2, \ldots, \widehat{x^i}, \ldots, x^n)$ means $(x^1, x^2, \ldots, x^{i-1}, x^{i+1}, \ldots x^n)$.

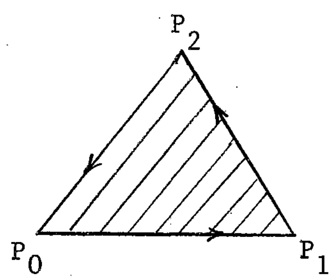
CHAPTER III

INTEGRATION AND STOKE'S THEOREM

We shall develop a method of integration by use of building blocks called Euclidian simplices of various dimensions; we shall omit the repetition of the adjective Euclidean in this part and it will be understood that everything takes place in Euclidean space. A 0-simplex is a single point denoted by (P_0). A 1-simplex is a directed line segment on a straight line. It is completely determined by its ordered pair of vertices (P_0, P_1).



A 3-simplex is a closed triangle with vertices taken in some definite order. It is completely determined by its ordered triple of vertices in the proper* order, (P_0, P_1, P_2).



Similarly, one has a 3-simplex based on an ordered quadruple (P_0, P_1, P_2, P_3) of four points, no three collinear. Geometrically it represents a tetrahedran and its interior.

Finally, an n-simplex is the closed conves hull $(P_0, \ldots, P_n)$

of independent points taken in a definite order. We mean by independent points that the n vectors $(P_1 - P_0), (P_2 - P_0), \ldots, (P_n - P_0)$ are linearly independent. The geometrical set so spanned consists of all points

$$P = t_0P_0 + t_1P_1 + \ldots + t_nP_n, \; t_1 \geq 0, \sum_{i-0}^{n} t_i = 1,$$

We might say all possible centroids of systems of non-negative masses $t_0, \ldots, t_n$ located at $P_0, \ldots, P_n$ respectively.

The boundary of a simplex S is a formal sum of simplices of one lower dimension with integer coefficients:

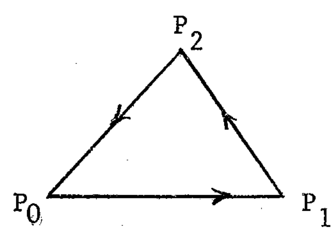
$$\partial(P_0, \ldots, P_n) = \partial S = \Sigma \, (-1)^i \, (P_0, \ldots, \hat{P_i}, \ldots, P_n)$$

Example 1: Consider 

then

$$\partial(P_0, P_1) = (-1)^0 \, (P_1) + (-1)^1 \, (P_0) = (P_0) - (P_1)$$

Example 2: Consider



then

$$\partial(P_0P_1P_2) = (-1)^0 \, (P_1P_2) + (-1)^1 \, (P_0P_2) + (-1)^2 \, (P_0,P_1)$$
$$= (P_0P_1) - (P_0P_2) + (P_1P_2),$$

where one thinks of each minus sign in ∂S as representing a reversal in the rotation sense.

An n-chain is a formal sum $C = \Sigma \, a^i S_i$ where the a^i are constants and S_i are n-simplices.

Since one would like ∂ to be a linear operation it will be defined by

$$\partial C = \Sigma\, a^i \partial S_i .$$

Looking at $\partial(\partial C) = \partial(\Sigma\, a^i \partial S_i) = \Sigma\, a^i\ \partial(\partial S_i)$, then looking at $\partial(\partial S_i) = \partial[\partial(P_0 \ldots P_n)] = \partial\Sigma(-1)^i\ (P_0, \ldots, \hat{P}_i, \ldots, P_n)$, which has $(P_0, \ldots, \hat{P}_i, \ldots, \hat{P}_j, \ldots, P_n)$ twice, once from $\partial(P_0, \ldots, \hat{P}_i, \ldots, P_n)$ and also from $\partial(P_0, \ldots, \hat{P}_j, \ldots P_n)$. In the first the sign is $(-1)^{i+j-1}$ and in the last $(-1)^{i+j}$; therefore, they differ in sign. From this one concludes $\partial(\partial S_i) = 0$, which implies $\partial(\partial C) = 0$. This gives one a basic result that the boundary of each chain itself has zero boundary.

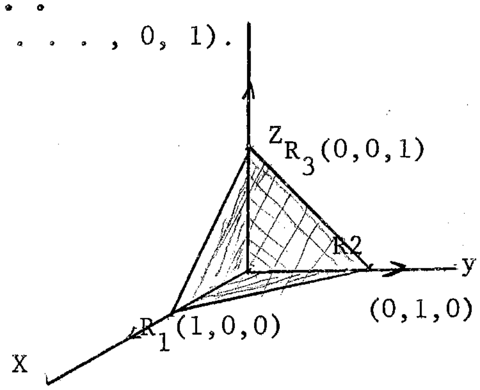
Given any two n-simplices $(P_0, \ldots, P_n)$, $(Q_0, \ldots, Q_n)$ there is a unique linear correspondence between them that preserve the ordering of the vertices. It is given by

$$\sum_{i=0}^{n} t_i P_i \qquad \sum_{i=0}^{n} t_i Q_i, \quad t_i \geq 0 \quad \sum_{i=0}^{n} t_i = 1.$$

The standard n-simples $\bar{S}^n$ is the simplex in E^n based on

$$R_0 = (0, 0, \ldots, 0)$$
$$R_1 = (1, 0, \ldots, 0)$$
$$R_2 = (0, 1, \ldots, 0)$$
$$\ldots$$
$$R_n = (0, \ldots, 0, 1).$$

Example: In E^3



Let ω be an n-form defined on an open set U of E^n and $\bar{S}^n \subset U$. We may write ω in the unique way

$$\omega = A(x^1 \ . \ . \ . \ x^n)\ dx^1 \wedge dx^2 \wedge \ . \ . \ . \wedge dx^n.$$

With the variables in their natural order now we may define

$$\int_{\bar{S}^n} \omega = \int_{\bar{S}^n} A(x^1 \ . \ . \ . \ x^n)\ dx^1 \wedge dx^2 \wedge \ . \ . \ . \wedge dx^n$$

where the right hand side is now standard ordinary n-fold integration.

Example: If ω = dy dx then

$$\int_{\bar{S}^2} dy \wedge dx = - \int_{\bar{S}^2} dx \wedge dy = - \int_0^1 \int_0^{1-y} dx \wedge dy$$

$$= - \int_0^1 (1-y)\ dy = 1\ [y - \frac{y^2}{2}]_0^1 = - \frac{1}{2}$$

Now consider an n-dimensional manifold M and shall define an n-simplex in M. Consider a smooth mapping

$$\phi : U \to M$$

where U is an n-dimensional neighborhood of $\bar{S}^n$ in Euclidean space.

Denote the preliminary simplex by

$$(S^n,\ U,\ \phi).$$

If one is given a second one,

$$(T^n, V, \psi)$$

it will be considered the same as the first provided

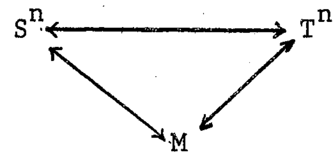
$$\phi\ (\Sigma\ t_i P_i) = \psi\ (\Sigma\ t_i Q_i),\ t_i \geq 0,\ \Sigma\ t_i = 1$$

where $S^n = (P_0, \ . \ . \ . \ , P_n)$, $T^n = (Q_0, \ . \ . \ . \ , Q_n)$.

We have set up the natural order preserving linear equivalence between S^n and T^n:

$$S^n \longleftrightarrow T^n$$

then ϕ (P) = ψ (Q) whenever P and Q are corresponding points.



The equivalents so generated will be called an n-simplex in M, denoted by σ^n.

The open neighborhoods U we have introduced merely serve to eliminate difficulties with differentiability on the boundary.

Let $\sigma^n = (S^n, U, \phi)$ and S^n have faces

$$t_o = (P_1, \ . \ . \ . \ . \ , P_n)$$

$$t_1 = (P_0, P_1, \ . \ . \ . \ . \ , P_n)$$

$$. \ . \ .$$

$$t_n = (P_0, P_1, \ . \ . \ . \ . \ , P_n)$$

and one has $\partial S^n = \Sigma \pm t_i$.

Now restrict ϕ to a neighborhood V_i of t_i such that V_i U and define faces of σ^n each represented by

$$T_i = (t_i, V_i, \phi)$$

and the corresponding boundary

$$\partial\sigma^n = \Sigma \pm T_i,$$

this is an (n - 1) - chain in M. By an n-chain C of M one means a formal sum

$$C = \sum_i a_i\sigma_i^n$$

with constant coefficients a_i and n-simplices σ_i^n.

One denotes by

$$C_n(M)$$

the set of all ordered singular differentiable n-chains on M. As before

$$\partial C = \Sigma\, a_i\, \partial\sigma_i^n ,$$

thus $\partial : C_n(M) \longrightarrow C_{n-1}(M)$, $n = 1, 2, \ldots$

Also as before in the Euclidean situation

$$\partial\,(\partial C) = 0.$$

Now consider a manifold of any dimension, a p-form ω on M and a p-chain C on M where

$$C = \Sigma\, a_i \sigma_i$$

where the a_i are constants and σ_i are P-simplices. As one would like $\int$ to be linear in both respects, one has

$$\int_C \omega = \Sigma\, a_i \int_{\sigma_i} \omega .$$

So we have now reduced the problem of defining

$$\int_{\sigma_i} \omega$$

One may represent σ_i in the form

$$(\bar{S}^P, U, \phi).$$

Now one defines

$$\int_{\sigma_i} \omega = \int_{\bar{S}^P} \phi^* \omega$$

where

$$\phi^* : F^P(M) \longrightarrow F^P(E^n).$$

This is ordinary integration as discussed above.

This is still not a satisfactory integration theory of differential forms over a differentiable manifold. What must be done is the piecing together of the local theory and making it global. To do this the following theorem of J. Dieudonne is of great importance.

Theorem 3.1: To a locally finite open covering $\{U_i\}$ of a differentiable manifold of class $k \geq 1$ there is associated a set of functions $\{g_i\}$ with the properties,

(i) Each g_j is of class k and satisfies the inequality

$0 \leq g_j \leq 1$

everywhere. Moreover, its carrier is compact and is contained in one of the open sets U_i.

(ii) $\sum_j g_j = 1$

(iii) Every point of M has a neighborhood met only by a finite number of carriers of g_j.

The g_j are said to form a partition of unity subordinated to $\{U_i\}$, that is, a partition of the function 1 into non-negative functions with small carriers. Property (iii) states that the partition of unity is locally finite, that is, each point $P \in M$ has a neighborhood met only by a finite number of carriers of g_j.

To show that the locally finite open covering $U = \{U_i\}$ of a differentiable manifold M there is associated a partition of unity. Consider:

(a) M is normal, that is, to every pair of disjoint closed sets, there exist disjoint open sets containing them

(b) Since M is normal, there exist locally finite open coverings $V = \{V_i\}$, $W^o = \{W_i^o\}$, $W = \{W_i\}$, and $W' = \{W_i'\}$ such that

$\bar{W}_i' \subset W_i \subset \bar{W}_i \subset W_i^o \subset \bar{W}_i^o \subset V_i \subset \bar{V}_i \subset U_i$

for each i.

Let one assume, with no loss of generality, that each U_i is contained in a coordinate neighborhood and has a compact closure.

In constructing a partition of unity we employ a smoothing function in E^n, that is, a function $g_\epsilon \geq 0$ of class k corresponding to an arbitrary $\epsilon > 0$ such that:

(i) Carr $(g_\epsilon) \subset \{r \leq \epsilon\}$ where r denotes the distance from the origin;

(ii) $g_\epsilon > 0$ for $r < \epsilon$;

(iii) $\int_{E^n} g_\epsilon\ (\mu^1, \mu^2, \ldots, \mu^n)\ d\mu^1 d\mu^2 \ldots d\mu^n = 1.$

For each U_i, let f_i be the continuous function,

$$f_i(P) = \begin{cases} 1, & P \epsilon W_i' \\ 0, & P \epsilon \text{ the complement of } W_i \end{cases}$$

$$0 \leq f_i(P) \leq 1 \qquad P \epsilon W_i - \bar{W}_i' \ .$$

Let $\mu = (\mu^1, \mu^2, \ldots, \mu^n)$ be a local coordinate system in U_i and define "distance" between points of U_i to be the ordinary Euclidean distance between the corresponding points of B_i where B_i is the ball in E^n homeomorphic with U_i. Let ϵ_i be chosen so small that a sphere of radius ϵ_i with center P is contained in U_i for all $P \epsilon V_i$ and does not meet W_i for $P \epsilon V_i - \bar{W}_i^o$.

Consider the functions

$$h_i(P) = h_i(\mu) = \int f_i(u)\ g_{\epsilon i}(\mu - v)\ dv,\ P \epsilon V_i,$$

since g_{ϵ_i} is of class k so is h_i of class k. Since $f_i(v)\ g_{\epsilon_i}(\mu - v) \geq 0$ for every $P \epsilon V_i$, this implies $h_i \geq 0$ also since for $P \epsilon W_i'$ $f_i(P) = 1$ and $g_{\epsilon_i} > 0$ for $r < \epsilon_i$ then $h_i(P) > 0$ where $P \epsilon W_i'$. If $P \epsilon V_i - \bar{W}_i^o$, then either $f_i(P) = 0$ if $V \epsilon V_i - \bar{W}_i^o$ or if $P \epsilon \bar{W}_i^o$ then $g_{\epsilon_i} = 0$ by choice of ϵ_i. In either case $h_i = 0$. If one defines h_i to be 0 in the complement of V_i, it is a function of class k on M.

In the above one has shown the following,

(i) h_i if of class k

(ii) $h_i \geq 0$, $h_i(P) > 0$, $P \epsilon W_i'$, $h_i(P) = 0$, $P \epsilon V_i - \bar{W}_i^o$

(iii) $W_i \subset \text{Carr}\,(h_i) \subset \bar{W}_i^o \subset U_i$.

Now define for each $P \epsilon M$ $h(P) = \sum_i h_i(P)$. One may do this since U is a locally finite covering. Since each h_i is of class k, so is the sum or h(P) is of class k. Also since W' is a covering of M, some $h_i(P) > 0$; therefore, $h(P) > 0$.

One may conclude that the function

$$g_i(P) = \frac{h_i\,(P)}{h(P)}$$

forms a partition of unity subordinated to the covering U, that is, a partition of the function 1 into non-negative functions with small carriers.

If M is an oriented manifold of dimension n, then there exists a unique functional which associates to a continuous differential form α of degree n with compact carrier a real number denoted by

$$\int_M \alpha$$

and called the integral of α over M with the following properties:

(i) $\int_M (\alpha + \beta) = \int_M \alpha + \int_M \beta$

(ii) If the carrier of α is contained in a coordinate neighborhood U with local coordinates $\mu^1, \mu^2, \ldots, \mu^n$ such that $d\mu^1 \wedge d\mu^2 \wedge \ldots \wedge d\mu^n > 0$ in U and $\alpha = a(\mu^1, \ldots, \mu^n)$ $d\mu' \wedge \ldots \wedge d\mu^n$, then $\int_M \alpha = \int_U a(\mu^1, \mu^2, \ldots, \mu^n)$ $d\mu^1 \wedge d\mu^2 \wedge \ldots \wedge d\mu^n$ where the n-fold integral on the right is the standard integration developed above.

In order to define the integral of an n-form α with compact carrier S, consider a locally finite open covering $\{U_i\}$. Since every point $P \epsilon S$ has a neighborhood met by only a finite number of carriers of the g_j, these neighborhoods for all $P \epsilon S$ form a covering of S. There exists a finite sub-covering, which tells one there is at most a finite number of non-zero g_j. As the Carr g_j is contained in a coordinate neighborhood, then $\int g_j \alpha$ is defined. Now

$$\int_M \alpha = \sum_j \int g_j \alpha.$$

The integral of α over M so defined is independent of the choice of neighborhood containing the Carr (g_j) as well as the covering $\{U_i\}$. Also, it is unique, covergent, and satisfies the properties (i) and (ii) above.

A domain D with <u>regular boundary</u> is a point set of M whose points may be classified as either interior or boundary points.

Now let D be a compact domain with regular boundary and let h be a real-valued function on M with the property that $h(P) = 1$ if $P \epsilon D$ and zero otherwise. Now define the integral of a $(p - 1)$ - form α over D,

$$\int_D \alpha = \int_M h\alpha.$$

Theorem 2.2: (Stokes' Theorem) Let ω be a p-form on a manifold M and D a $(P + 1)$ domain. Then

$$\int_{\partial D} \omega = \int_D d\omega.$$

Select a countable open covering of M by coordinate neighborhoods $\{U_i\}$ in such a way that either U_i does not meet ∂ D or U_i is a coordinate neighborhood of a boundary point P such that $U_i \cap D$ consists of those points $Q \epsilon U_i$ satisfying $\mu^n(Q) \geq \mu^n(P)$. Let $\{g_i\}$ be a partition of unity subordinated to this covering. Since D and its boundary are both compact, each meets only a finite number of carriers of g_j.

Therefore,

$$\int_{\partial D} \omega = \sum_j \int_{\partial D} g_j \ \omega$$

and

$$\int_D d\omega = \sum_j \int_D d(g_j \ \omega).$$

Since both sums are finite, one needs only establish that

$$\int_{\partial D} g_i \ \omega = \int_D d \ g_j \ \omega$$

for each i, which reduces to a p-form on a (p + 1)-chain c. Thus, one must show

$$\int_{\partial c} \omega = \int_c d\omega \ .$$

Now since c is the sum of (p + 1)-simplices with constant coefficients, it suffices to prove

$$\int_{\partial \sigma} \omega = \int_\sigma d\omega$$

where σ is a (p + 1)-simplex. According to a representation

$$(\bar{S}^{p+1}, U, \phi)$$

of σ one has from the definition

$$\int_\sigma d\omega = \int_{\bar{S}^{P+1}} \phi^* \ d\omega = \int_{\bar{S}^{P+1}} d(\phi^* \ \omega).$$

This reduces the problem to a Euclidean one. Let N be a p-form on a neighborhood U of $\bar{S}^{P+1}$ in E^{P+1}. To prove

$$\int_{\partial \bar{S}^{P+1}} N = \int_{\bar{S}^{P+1}} dN$$

consider $N = \sum A_i \ (x^1, x^2, \ldots, x^{P+1}) \ dx^1, \ldots, \widehat{dx^i}, \ldots,$ dx^{P+1}, therefore, one needs the formula for the case of N a monomial only. Since we may permute coordinates provided one is careful about signs, it suffices to take the case

$$N = A\, dx^1 \ldots dx^P.$$

Then

$$dN = (-1)^P \frac{\partial A}{\partial x^{P+1}}\, dx^1 \ldots dx^{P+1}.$$

Thus

$$\int_{\bar{S}^{P+1}} dN = (-1)^P \int_{\bar{S}^{P+1}} \frac{\partial A}{\partial x^{P+1}}\, dx^1 \ldots dx^{P+1}$$

$$= (-1)^P \int_{\{x^j \geq 0 \ \sum_o^p x^i \leq 1\}} dx^1 \ldots dx^P$$

$$\int_o^{1 - \sum_o^P X^i} \frac{\partial A}{\partial x^{P+1}}\, dx^{P+1}$$

$$= (-1)^P \int_{\{x^j \geq 0 \ \sum_o^p x^i \leq 1\}} [A(x^1, x^2, \ldots x^P,$$

$$1 - \sum_{i=o}^{p} x^i) - A(x^1, x^2, \ldots x^P, 0)]\, dx^1 \ldots dx^P.$$

Now investigate $\partial \bar{S}^{P+1}$,

$$\bar{S}^{P+1} = (R_0\ R_1 \ldots R_{P+1})$$

$$R_0 = (0, \ldots, 0)$$

$$R_1 = (1, 0, \ldots, 0)$$

$$R_2 = (0, 1, 0, \ldots, 0)$$

$$\ldots$$

$$R_{P+1} = (0, \ldots, 0, 1).$$

All R_i are points in E^{p+1}.

Therefore,

$$\partial\, \bar{S}^{P+1} = (R_1 \ . \ . \ . \ R_{P+1}) + (-1)^{P+1} (R_0 \ . \ . \ . \ R_p) + \text{other}$$

faces where N = 0 on the other faces since one of the $x^1 \ . \ . \ . \ x^P$ is constant there. Thus

$$\int_{\partial \bar{S}^{p+1}} N = \int_{(R_1 \ . \ . \ . \ R_{P+1})} N + (-1)^{p+1} \int_{(R_0 \ . \ . \ . \ R_P)} N .$$

The face $(R_0 \ . \ . \ . \ R_p)$ is the standard $\bar{S}^P$, on it $x^{P+1} = 0$ and so

$$(-1)^{p+1} \int_{(R_0 \ . \ . \ . \ R_p)} N = (-1)^{p+1} \int_{\bar{S}^P} A(x^1, x^2, \ . \ . \ . \ , x^P, 0)$$

$$dx^1 \ . \ . \ . \ dx^P,$$

which is the second term in the expansion for $\int dN$ above. The first term is obtained by projecting downward in the x^{P+1} direction

$$\int_{(R_1 \ . \ . \ . \ R_{P+1})} N = \int_{(R_1 \ . \ . \ . \ R_P R_0)} A(x^1, \ . \ . \ . \ , x^P, 1 - \Sigma x_i)$$

$$dx^1 \ . \ . \ . \ dx^P$$

$$= (-1)^P \int_{(R_0, R_1, \ . \ . \ . \ , R_P)} A(x^1, \ . \ . \ . \ , x^P, 1 - \sum_1^P x_i)$$

$$dx^1, \ . \ . \ . \ dx^P$$

$$= (-1)^P \int_{\bar{S}^P} A(x^1, \ . \ . \ . \ x^P, 1 - \sum_1^P x_i)$$

$$dx^1 \ . \ . \ . \ dx^P,$$

which is the first term in the expression for $\int dN$ above. Therefore, Stokes theorem is proven.

CHAPTER IV

COHOMOLOGY AND HOMOLOGY

The intuitive ideas of homology and cohomology are simple and straightforward. The idea is to study the nature of a manifold by defining chains of cells of different dimensions with coefficients in some group, ring, or field, a boundary operator or coboundary operator, and an algebraic structure on the collection that will yield certain invariants of the process that will then have geometric significance.

If C_n is the collection of chains of dimension n, or C^n the cochains, then these form an abelian group, module, or vector space depending on the coefficients; ∂_n is a homomorphism, or linear function, called a boundary operator, on C_n to C_{n-1}, and δ_n is a homomorphism, or linear function, called a coboundary operator, on C^n to C^{n+1}. Those chains B_n in C_n that are images of ∂_{n+1} are called bounding cycles, and those chains Z_n in C_n for which ∂_n gives the zero chain in C_{n-1} (ker ∂_n) are called cycles. Similarly for the coboundary cycles B^n are defined by $B^n = \delta_n C^{n-1}$ and the cocycles by $Z^n = \ker \delta_n$. In both cases it is always required that $\partial_n\partial_{n+1} = 0$ and $\delta_n\delta_{n-1} = 0$ so $B_n \subset Z_n$ and $B^n \subset Z^n$. Some of the invariants are the measure of how much cycles or cocycles of each dimension fail to be the bounding cycles or cobounding cocycles as determined by the factor groups, $H_n = Z_n/B_n$, called the homology groups, or $H^n = Z^n/B^n$, called the cohomology groups.

The boundary and coboundary operators are related in a manner similar to that of a linear operator and its adjoint. If γ^n is a cochain in C^n, that is, a homomorphism on C_n to the group of coefficients, σ_n in C_n and the notation $< \sigma_n, \gamma^n >$ is used for the value of γ^n at σ_n, similar to that for a vector in the dual space and its value at a vector in the space, then

$$< \partial_{n+1} \sigma_{n+1}, \gamma^n > = < \sigma_{n+1}, \delta_n \gamma^n >.$$

Notice the similarity of this and Stoke's theorem $\int_{\partial\sigma} \omega = \int_{\sigma} d\omega$ when written in the form $\int_{\sigma} \omega = < \sigma, \omega >$. If $\partial = \partial_n$, $\sigma = \sigma_{n+1}$, $\omega = \gamma^n$, and $d = \delta_n$, then Stoke's theorem could be written

$$< \partial\sigma, \omega > = < \sigma, d\omega > \text{ or } < \partial_{n+1} \sigma_{n+1}, \gamma^n > = < \sigma_{n+1}, \delta_n \gamma^n >.$$

This leads to the anticipation of an isomorphism between the geometric cohomology and that of exterior differential forms.

In the next two chapters we will develop a proof of the existence Theorem of de Rham. The idea of the proof is due to A. Weil. The method is the theory of sheaves due to Leroy.

The cohomology being developed is a straightforward generalization of the classical Čech definition of cohomology. One will use the idea of cohomology with 'coefficients' in a sheaf Γ, which is a generalization of Steenrods cohomology with 'local coefficients'.

Let $U = \{U_i\}$ be any countable open covering of a differentiable manifold M and consider chains and forms defined only in $U_i \cap U_j$.

The nerve of U, denoted by N(U) is the simplicial complex whose vertices are the elements of U and where any finite number of vertices $U_{i_0}, U_{i_1}, \ldots, U_{i_p}$ span a simplex of N(U) if and only if $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p} \neq \emptyset$. By a p-simplex $\sigma = \Delta(i_0, \ldots, i_p)$ one means an ordered finite set $(i_0, \ldots, i_p)$ of indices such that

$U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p} \neq \emptyset$. If $U_{i_0}, \ldots, U_{i_p}$ are the vertices of a p-simplex σ, then $\cap \sigma = U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p}$. For any open sets U and V, $V \subset U$, let ρ_{UV} denote the restriction map on differential forms,

$$\rho_{UV} : \Lambda^q(U) \longrightarrow \Lambda^q(V) \qquad q = 0, 1, 2, \ldots, n$$

defined by

$$\rho_{UV}(\alpha) = \alpha | V, \qquad \alpha \in \Lambda^q(U).$$

If U, V, and W are open sets such that $W \subset V \subset U$, then $\rho_{UW} = \rho_{VW}\rho_{UV}$.

A p-cochain of N(U) is a function f that assigns to each p-simplex σ an element of an abelian group or vector space $\Gamma(\cap \sigma)$. In the sequel $\Gamma(U)$ will be one of the following:

(i) R: the real numbers

(ii) $\Lambda^q = \Lambda^q(U)$: the space of q-forms over U

(iii) $\Lambda^q_c = \Lambda^q_c(U)$: the space of closed q-forms over U.

This generalizes the usual definition. This gives (a) for every open set U there is a vector space $\Gamma(U)$ and (b) if V U, then

$$\rho_{UV} : \Gamma(U) \longrightarrow \Gamma(V).$$

The value $f(i_0 \ldots i_p) \equiv f(\Delta(i_0, \ldots, i_p))$ of a p-cochain is an element of $\Gamma(U_{i_0} \cap \ldots \cap U_{i_p})$.

Much as was done in the Euclidian simplex, if $\sigma = \Delta(i_0, \ldots, i_p)$, let the faces of σ be the simplices $\sigma^j = \Delta(i_0 \ldots \hat{i}_j \ldots i_p)$, $j = 0, 1, \ldots, p$. Then, $\cap \sigma^j \subset \cap \sigma$ and there is a homomorphism

$$\rho_{\sigma^j \sigma} : \Gamma(\ \sigma^j) \longrightarrow \Gamma(\ \sigma)$$

defined by the restriction map above, that is $p_{\sigma^j\sigma}\, f(\sigma^j) = f(\sigma^j)/\cap \sigma$ is an element of the vector space $\Gamma(\cap \sigma)$.

If f and g are p-cochains of N(U) with values in the same abelian group $\Gamma(\cap \sigma)$, then cochains f + g and a . f, $a \in R$ are defined by

$$(f + g)\, \sigma = f(\sigma) + g(\sigma)$$

$$(a \cdot f)\ \sigma = a(f(\sigma))$$

for each simplex $\sigma \in N(U)$. Thus, the p-cochains form a vector space over the reals, will be denoted by

$$C^{P}\ (N(U),\ \Gamma).$$

The coboundary operator δ assigns a cochain δf to each p-cochain; f is defined by

$$(\delta f)(\sigma) = \sum_{j=0}^{p} (-1)^{j}\ \rho_{\sigma^{j}\sigma}\ f(\sigma^{i}),\ \sigma = \Delta(i_{o},\ .\ .\ .\ ,\ i_{p}).$$

Thus,

$$\delta\ :\ C^{P}\ (N(U),\ \Gamma) \longrightarrow C^{P+1}\ (N(U),\ \Gamma).$$

To show $\delta(\delta f) = 0$ one needs only remember that $\rho_{\sigma j^{i}\ \sigma^{j}}\ \rho_{\sigma^{j}\ \sigma} =$ $\rho_{\sigma j^{i}\ \sigma}$ and in a manner similar to that for boundary of Euclidian simplex one gets zero. In the usual way define p-dimensional cohomology group $H^{P}(N(U),\ \Gamma)$ as the quotient of $Z^{P}(N(U),\ \Gamma)$, the set of p-cochains whose coboundarys vanish, by $B^{P}(N(U),\ \Gamma)$, the set of p-cochains that are the coboundary of (p + 1)-cochains:

$$H^{P}\ (N(U),\ \Gamma) = \frac{Z^{P}\ (N(U),\ \Gamma)}{B^{P}\ (N(U),\ \Gamma)}\ .$$

In particular if M is connected

$$H^{P}\ (N(U),\ \Gamma) = \Gamma(M).$$

For, a 0-cochain f assigns to each $U_{i} \in U$ an element $\alpha_{U_{i}}$ of $\Gamma(U_{i})$. The condition $\delta f = 0$ requires that if $f(U_{j}) = \alpha_{U_{j}} \in \Gamma\ (U_{j})$, $U_{j} \in U$ and $U_{i} \cap U_{j} \neq \phi$ then

$$\rho_{U_{j}\ U_{i} \cap U_{j}}\ \alpha_{U_{i}} = \rho_{U_{i}\ U_{i} \cap U_{j}}\ \alpha_{U_{i}}\ .$$

Conversely, for any globally defined $\alpha \in {}^{q}(M)$, a 0-cochain satisfying $\delta f = 0$ is given by defining $f(U_{i}) = \rho_{MU_{i}}\alpha$, $U_{i} \in U$ and $f(\sigma) = 0$ for all other $\sigma \in N(U)$. That the map $\Gamma(M) \quad H^0(N(U), \Gamma)$ is a monomorphism follows from above.

A 1-cochain is defined by $f(U_i, U_j) = \alpha_{U_iU_j} \in \Gamma(U_i \cap U_j)$.
It is a cocycle if $\rho_{U_I \cap U_J, U_i \cap U_j \cap U_k} \alpha_{U_iU_j} - \rho_{U_k \cap U_j, U_i \cap U_j \cap U_k} \alpha_{U_kU_j}$

$$+ \rho_{U_k \cap U_i, U_i \cap U_j \cap U_k} \alpha_{U_kU_i} = 0$$

$$\alpha_{U_iU_j}, \alpha_{U_kU_j}, \text{ and } \alpha_{U_kU_i} \in \Gamma(U_i \cap U_j \cap U_k).$$

If $U_i = U_j = U_k$, we conclude that $\alpha_{U_iU_j} = \alpha_{U_jU_i}$; the cocycle $\alpha_{U_iU_j}$ is a co-boundary if it can be expressed as $\alpha_{U_j} - \alpha_{U_i}$.

In this part, we shall write $(\delta f)(\sigma) = \Sigma\, (-1)^j\, f(\sigma^j)$ for simplicity.

A covering $V = \{V_i\}$ of M is called a refinement of U if there is a map

$$\phi : V \longrightarrow U$$

defined by associating with each $V_i \in V$ a set $U_i \in U$ such that $V_i \subset U_i$.
If $\sigma = (V_0, \ldots, V_p) \in N(V)$, let $\phi\sigma = (\phi V_0, \ldots, \phi V_p)$. Then $\bigcap \phi\, \sigma^j \supset \sigma \neq \phi$ and $\phi\sigma$ is an element of N(U). Hence, there is a map

$$\phi : N(V) \longrightarrow N(U).$$

This map in turn induces a map $\tilde{\phi}$ sending each cochain $f \in C^P(N(U), \Gamma)$ to a cochain $\tilde{\phi} f \in C^P (N(U), \Gamma)$ where for each $\sigma \in N(V)$

$$\tilde{\phi} f(\sigma) = \rho_{\phi\sigma, \sigma} f(\phi\sigma)..$$

The map ϕ is not unique. However all such ϕ induce the same homorphisms

$$\phi^* : H^P(N(U), \Gamma) \longrightarrow H^P(N(V), \Gamma).$$

Moreover, if $W = \{W_i\}$ is a refinement of V, the combined homorphism

$$H^P(N(U), \Gamma) \quad H^P(N(V), \Gamma) \quad H^P(N(W), \Gamma)$$

is the same as the direct $H^P(N(U), \Gamma)$ $H^P(N(W), \Gamma)$.

One will use the notation ϕ_{UV} for ϕ^*.

The set of all converings of M is partically ordered by inclusion where V is contained in U if and only if V is a refinement of U. If V is a refinement of U, one writes $V < U$. If $W < V < U$, it can be shown

$$\phi_{UW} = \phi_{VW}\phi_{UV}$$

The direct limits

$$H^P(M, \Gamma) = \lim_{U} H^P(N(U), \Gamma)$$

of the groups $H^P(N(U),\Gamma)$ $P = 0, 1, \ldots$ are defined by the following: Two elements $h_i \in H^P(N(U_i), \Gamma)$, $i = 1, 2$ are said to be equivalent if there exists an element $h_3 \in H^P(N(U_3), \Gamma)$ with $U_3 < U_i$, $i = 1, 2$ such that $h_3 = \phi_{U_iU_3} h_i$, $i = 1, 2$; the direct limit is the set of these equivalence classes.

Now one develops a theory dual to the above. As before with every open set $U_i \in U$ one associates a vector space which is again denoted by $\Gamma(U_i)$. If $U_j \subset U_i$, then $\rho_{U_iU_j} : \Gamma(U_j) \longrightarrow \Gamma(U_i)$.

By a p-chain is meant a formal sum

$$g = \sum_{(i)} g(i_0, \ldots, i_p) \Delta (i_0, \ldots, i_p)$$

$$g(i_0, \ldots, i_p) \in \Gamma (U_{i_0} \cap \ldots \cap U_{i_p})$$

where $\Delta(i_0, \ldots, i_p)$ is a p-simplex on N*U) and (i) implies summation on $(i_0, \ldots, i_p)$. The coefficients of a p-chain lie in $\Gamma(U_{i_0} \cap \ldots \cap U_{i_p})$. In the applications Γ will be either

(i) R : the real numbers

(ii) $S_q(U)$: the space of finite singular chains with support in U or

(iii) $S_q^c(U)$: the subspace of finite singular cocycles.

A boundary operation ∂ mapping p-chains into (p-1)-chains is defined on p-simplices as follows:

$$\partial[\Delta(i_0, \ldots, i_p)] = \sum_{k=0}^{p} (-1)^k \quad \Delta(i_0, \ldots, i_k, \ldots, i_p);$$

and on p-chains by linear extension,

$$\partial g = \sum_{(i)} g(i_0, \ldots, i_p) \, \partial[\Delta(i_0, \ldots, i_p)]$$

where $g(i_0, \ldots, i_p)$ for the corresponding images ρ . . $g(i_0, \ldots, i_p)$. Denoting the coefficients of $\partial g(j_0, \ldots, j_{p-1})$ one gets $\partial g(j_0, \ldots, j_{p-1}) = \sum_{k=0}^{p} \sum_{i} (-1)^k \, g(j_0, \ldots, j_{k-1}, j_i, j_k, \ldots, j_{p+1})$ where i runs over all indices for which the corresponding intersection is not empty. In order for the sum to be finite one assumes the covering U to be locally finite.

Once again it can be shown $\partial\partial g = 0$. Then define the p-dimensional homology group $H_p(N(U), \Gamma)$ as the quotient of $Z_q(N(U), \Gamma)$, the set of q-chains whose boundary vanish, by $B_q(N(U), \Gamma)$, the set of q-chains that are the boundaries of (q+1)-chains,

$$H_q(N(U), \Gamma) = \frac{Z_q(N(U), \Gamma)}{B_q(N(U), \Gamma)} \quad .$$

Let $V = \{V_i\}$ be a refinement of U. Then as for the cohomology there is a map $\phi : V \longrightarrow U$ defined by associating with each $V_i \in V$ a set $U_i \in U$ such that $V_i \subset U_i$. To the p-chain g on V one may assign a chain $\tilde{\phi}$ g on U as follows:

$$\tilde{\phi} : \sum_{(i)} g(i_0, \ldots, i_p) \qquad \sum_{(i)} g(i_0, \ldots, i_p)$$

$$\Delta(\phi(i_0), \ldots, \phi(i_p)) = \phi(V_r).$$

Cycles are mapped into cycles and boundaries into boundaries. Hence γ induces a homomorphism

$$\phi_* : H_p(N(V), \Gamma) \quad H_p(N(U), \Gamma)$$

As before this homomorphism does not depend on ϕ but rather on V and U and so one denotes ϕ_* by ϕ_{UV}. Also, if $W < V < U$, $\phi_{WU} = \phi_{VU}\phi_{WV}$.

The inverse limits

$$H_p(M, \Gamma) = \lim_{U} H_p(N(U), \Gamma)$$

of the groups $H_p(N(U),\Gamma)$ $p = 0, 1, \ldots$ are defined as follows: Two elements $h_i \in H_p(N(U_i), \Gamma)$ $i = 1, 2$ are equivalent if there exists an element $h_3 \in H_p(N(U_3), \Gamma)$ with $U_3 < U_i$, $i = 1, 2$ such that $h_3 = \phi_{U_3U_i}h_i$, $i = 1, 2$; the inverse limit is the set of these equivalence classes. With the obvious definitions of addition and multiplication by a scalar $H_p(M, \Gamma)$ is a vector space.

A refinement V of U is called a strong refinement if each $\bar{V}_i$ is compact and contained in some U_j. One writes $V \ll U$ for V is a strong refinement of U and for the pair V_i and U_j we write $V_i \Subset U_j$.

Theorem 4.1: For a compact differentiable manifold M,

$$H^P(M, \Lambda^q) = \{0\}$$

for all $p > 0$, and $q = 0, 1, \ldots$ (we are not implying this is true for all Γ).

Let V be a locally finite strong refinement of the open covering U of M and $\{e_j\}$ a partition of unity subordinated to V. For an element $f \in C^P(N(V), \Lambda^q)$ let $f_j = e_jf$. Then $\delta f_j = \delta(e_jf) = e_j \delta f = (\delta f)_j$ and so if f is a cocycle so is f_j.

Let f be a p-cocycle, $p > 0$ by definition, $f = \Sigma f_j$ is a locally finite sum, now prove that each cocycle f_j is a coboundary, that is,

$f_j = \delta g_j$, where $g_j(V_0, \ldots, V_{p-1}) = 0$ if $V_0 \cap \ldots \cap V_{p-1}$ does not intersect V_j. This being the case $g = \Sigma g_j$ is well defined and $f = \Sigma f_j = \Sigma \delta g_j = \delta g$.

Consider a fixed j and put $g_j(V_0, \ldots, V_{p-1}) = f_j(V_j, V_0, \ldots, V_{p-1})$ if $V_j \cap V_0 \cap \ldots \cap V_{p-1} \neq \phi$ and $g_j = 0$ otherwise. In the first case $(\delta(g_j))(V_0, \ldots, V_p) = \Sigma (-1)^i f_j(V_j, V_0, \ldots, \widehat{V_i}, \ldots, V_p)$. Since f_j is a cocycle,

$$0 = (\delta f_j)(V_j, V_0, \ldots, V_p) = f_j(V_0, \ldots, V_p)$$

$$- \Sigma (-1)^i f_j(V_j, V_0, \ldots, \widehat{V_i}, \ldots, V_p);$$

hence, $f_j = \delta g_i$.

In the second case $V_j \cap V_0 \cap \ldots \cap V_p \neq \phi, f_j(V_0, \ldots, V_p) = 0$, but δg_i also vanishes for in $\delta g_j(V_0, \ldots, V_p) = \Sigma (-1)^j g_j(V_0, \ldots, \widehat{V_i}, \ldots, V_p)$ each term on the right is either zero, by definition of g_j, or else it is the restriction of $f_j(V_j, V_0, \ldots, \widehat{V_j}, \ldots, V_p)$ to the set $V_0 \cap \ldots \cap V_p$. Since e_j vanishes outside V_j, so must f_j; thus, the value is again zero.

One concludes that $f_j = \delta g_j$ in all cases; therefore, the proof is complete.

Theorem 4.2: For a differentiable manifold M

$$H^P(M, S_q) = \{0\}$$

for all $p > 0$, and $q = 0, 1, 2, \ldots$, moreover, in order that a 0-chain be a boundary it is necessary and sufficient that the sum of its coefficients be zero.

Consider all singular q-simplices. Divide these simplices into classes so that all those simplices in the j^{th} class are contained in U_j. For each cycle g construct a singular chain g_j by deleting those singular simplices not in the j^{th} class. One knows g_j is a cycle since $\partial(g_j) = (\partial g)_j$. Since $g = \Sigma\, g_j$ it suffices to show that g_j is a boundary. For simplicity take $j = 0$. Define a $(p + 1)$-chain h as follows,

$$h(i_0, \ldots, i_{P+1}) = \begin{cases} g_0(i_1, \ldots, i_{P+1}) \text{ if } i_0 = 0 \\ 0 \text{ if } i_0 \neq 0 \end{cases}$$

or

$$h = \sum_{(i)} g_i(i_1, \ldots, i_{P+1})\, \Delta(0, i_1, \ldots, i_{P+1}).$$

Now since

$$\partial h = \sum_{(i)} g_0(i_1, \ldots, i_{P+1})\, \Delta(i_1 \ldots i_{P+1}) - \sum_{(i)} \sum_{k=1}^{P+1} (-1)^k g_0(i_1, \ldots, i_{P+1})\, \Delta(0, i_0, \ldots, \hat{i}_k, \ldots, i_{P+1})$$

and

$$0 = \partial\, g_0 = \sum_{(i)} \sum_{k=0}^{P+1} (-1)^k g_0(i_1, \ldots, i_{P+1})\, \Delta(i_1, \ldots, i_k, \ldots, i_{P+1})$$

where

$$g_0 = \sum_{(i)} g_0(i_1, \ldots, i_{P+1})\, \Delta(i_1, \ldots, i_{P+1}),$$

$$g_0 = \partial\, h,$$

for by comparing the expression ∂g_0 with the last sum in ∂h they are the same; therefore, one concludes g_j is a boundary, so g is a boundary.

For $p = 0$

$$h = \Sigma\, g_0(i)\, \Delta\, (0, i)$$

and thus

$$\partial h = \Sigma\, g_0(i)\, \Delta\, (i) - \Sigma\, g_0\, (i)\, \Delta\, (0)$$

where $g_0 = \Sigma\, g_0(i)\, \Delta\, (i)$;

therefore, it is necessary that $\Sigma\, g_0(i)$ vanish in order for g_0 to be a boundary. On the other hand suppose $\Sigma\, g(i)$ vanishes, then since $\Sigma\, g(i) = \Sigma\Sigma\, g_j(i)$, then $\Sigma\, g_0(i) = 0$ by the choice of the g_j. Therefore, a 0-chain is a boundary, if and only if, the sum of its coefficients is zero.

Although an exact form is closed, the converse is not true. The following theorem is a partial converse called Poincare Lemma.

Theorem 4.3: On a star shaped region Δ in R^n every closed p-form, $p > 0$, is exact.

First define a homotopy operator

$$I : \bigwedge^p(\Delta) \longrightarrow \bigwedge^{p-1}(\Delta),\ p > 0$$

with the property that $dI\alpha + Id\alpha = \alpha$ for every p-form α defined in a neighborhood of Δ. Hence, if α is closed in Δ, then $Id_\alpha = 0$ and $\alpha = dI\alpha = d\beta$, where $\beta = I\alpha$.

Let $u^1, u^2, \ldots, u^r$ be a coordinate system at the origin. Let tu denote the vector $(tu^1, \ldots, tu^n)$, $0 \le t \le 1$. Then for $\alpha = a_{(i_1 \ldots i_p)}(u^1, u^2, \ldots, u^n)\, du^1 \ldots du^{i_p}$, put

$$I\alpha = \sum_{k=1}^{p} (-1)^{k-1} \int_0^1 t^{p-1} a_{(i_1 \ldots i_p)}(tu)\, dt \cdot u^{i_k} du^{i_1} \wedge \ldots \wedge \widehat{du^{i_k}} \wedge \ldots \wedge du^{i_p}.$$

Thus,

$$dI\alpha = P \int_0^1 t^{p-1} a_{(i_1 \ldots i_p)}(tu)dt \cdot du^{i_1} \wedge \ldots \wedge du^{i_p} + \sum_{k=1}^{p} \sum_{j=1}^{n} (-1)^{k-1} \int_0^1 t^p \frac{\partial a_{(i_1 \ldots i_p)}}{\partial u^j} \cdot t(u)_{dt} \cdot u^{i_k} du^j \wedge du^{i_1} \wedge \ldots \wedge \widehat{du^{i_k}} \wedge \ldots \wedge du^{i_p}.$$

Now looking at

$$Id\alpha = \sum_{j=1}^{n} \int_0^1 t^p \frac{\partial a_{(i_1 \ldots i_p)}}{\partial u^j} (tu)dt \cdot u^j du^{i_1} \wedge \ldots \wedge du^{i_p}$$

$$- \sum_{j=1}^{n} \sum_{k=1}^{p} (-1)^{k-1} \int_0^1 t^p \frac{\partial a_{(i_1 \ldots i_p)}}{\partial u^j}$$

$$(tu)dt \cdot u^{i_k} du^j \wedge du^{i_1} \wedge \ldots \wedge \widehat{du^{i_k}} \wedge \ldots \wedge du^{i_p}.$$

Thus by adding

$$dI\alpha + Id\alpha = [p\int_0^1 t^{p-1} a_{(i_1 \ldots i_p)}(tu)$$

$$+ \sum_{j=1}^{n} \int_0^1 t^p \frac{\partial a_{(i_1 \ldots i_p)}}{\partial u^j} (tu)dt]$$

$$\cdot du^{i_1} \wedge du^{i_2} \wedge \ldots \wedge du^{i_p}$$

$$= \int_0^1 \frac{\partial}{\partial t} [t^p a_{(i_1 \ldots i_p)}(tu)]dt \cdot du^{i_1} \ldots du^{i_p}$$

$$= a_{(i_1 \ldots i_p)}(u) \cdot du^{i_1} \wedge du^{i_2} \wedge \ldots \wedge du^{i_p}$$

$$= \alpha$$

In analogy with Poincare Lemma on states the following theorem without proof.

Theorem 4.4: The singular homology groups $H_p(\Delta)$, $p > 0$ of a star shaped region in R^n are trivial.

Let $f \in C^p(N(U), \wedge^q)$ and $g \in C_p(N(U), S_q)$ and define the inner product

$$(f(i_0, i_1, \ldots, i_p), g(i_0, i_1, \ldots, i_p))$$
$$= \int_{g(i_0, \ldots, i_p)} f(i_0, \ldots, i_p)$$

and $(f, g) = \sum_{(i)} (f(i_0, \ldots, i_p), g(i_0, \ldots, i_p))$.

Either f or g is assumed to be finite, in this case, the sum is finite. The elements $f \in C^p(N(U), \Lambda^q)$ and $g \in C_p(N(U), S_q)$ are said to be of type (p, q).

Theorem 4.5: For elements $f \in C^p(N(U), \Lambda^q)$ and $g \in C_{p+1}(N(U), S_q)$

$$(\delta f, g) = (f, \partial g).$$

Since inner product is linear in each variable, we may assume that

$g = g(0, \ldots, p+1)\,\Delta(0, 1, \ldots, p+1)$. Then, $(\delta f, g) =$

$$\sum_i \pm (-1)^i \int_{g(0, \ldots, p+1)} f(0, \ldots, \hat{i}, \ldots, p+1) \quad = (f, \partial g)$$

since $(\partial g)(0, \ldots, \hat{i}, \ldots, p+1) = (-1)^i\, g(0, \ldots, p+1)$.

Denote by d the operator on the cochain groups $C^p(N(U), \Lambda^q)$ defined by:

$$d : C^p(N(U), \Lambda^q) \longrightarrow C^p(N(U), \quad ^{q+1})$$

where to an $f \in C^p(N(U), \Lambda^q)$ one associates the element df whose values are obtained by applying the differential operator d to the forms $f(i_0, \ldots, i_p) \in \Lambda^q(U_{i_0} \cap \ldots \cap U_{i_p})$. It can be shown that $dd = 0$.

An operator

$$D : C_p(N(U), S_q) \longrightarrow C_p(N(U), S_{q-1})$$

is defined as follows: D is the operator replacing each element $g \in C_p(N(U), S_q)$ by its boundary. Clearly $DD = 0$.

Theorem 4.6: For elements $f \in C^p(N(U), \Lambda^q)$ and $g \in C_p(N(U), S_q)$, $(f, Dg) = (dg, g)$.

Theorem 4.7: $\delta d = d\delta$ and $\partial D = D\partial$.

CHAPTER V

De RHAM THEOREM

A covering U of M is said to be simple if, (a) it is strong locally finite and (b) every non-empty intersection $U_0 \cap U_1 \cap \ldots \cap U_p$ of open sets of the covering is homeomorphic with a star shaped region in R^n. It can be shown that such a covering exists.

Let $f_0 \in Z^0 (N(U), \bigwedge_c^q)$, $q_0 \in C_0 (N(U), S_q)$ and consider the system of equations

$$
\begin{aligned}
f_0 &= df_1 & Dg_0 &= \delta g_1 \\
\delta f_1 &= df_2 & Dg_1 &= \delta g_2 \\
\delta f_2 &= df_3 & Dg_2 &= \delta g_3 \\
&\vdots & &\vdots \\
\delta f_{q-1} &= df_q & Dg_{q-1} &= \delta g_q .
\end{aligned}
$$

f_i, i = 1, 2, . . . is of type (i - 1, q - i) and q_i, i = 1, 2, . . . is of type (i, q-i). In the event there exist cochains f_i and chains g_i satisfying these relations it follows from theorem 4.6 that

$$
\begin{aligned}
(f_0, g_0) &= (df_1, g_0) = (f_1, Dg_0) = (f_1, \delta g_1) \\
&= (\delta f_1, g_1) = (df_2, g_1) = (f_2, Dg_2) \\
&= (f_2, \delta g_2) = \ldots = (\delta f_{q-1}, g_{q-1}) \\
&= (df_q, g_{q-1}) = (f_q, Dg_{q-1}) = (f_q, \delta g_q) \\
&= (\delta f_q, g_q)
\end{aligned}
$$

Since $d\delta f_q = \delta df_q = \delta\delta f_{q-1} = 0$, the coefficients of δf_q are constants. It follows that δf_q may be identified with a cocycle z^q with constant coefficients.

For a chain of type (p, 0) let D_o be the operator denoting addition of the coefficients in each singular 0-chain. Evidently, $\partial D_o = D_o\partial$ and $D_oD = 0$. Thus, since $\partial g_q = Dg_{q-1}$, $\partial D_o g_q$ vanishes, that is $D_o g_q$ is a cycle Z_q, one concludes that

$$(\delta f_q, g_q) = (Z^q, 1_q)$$

or

$$\int_\Gamma \alpha = (Z^q, 1_q).$$

The problem of computing the period of a closed q-form over a q-cycle has been reduced to that of integrating a closed 0-form over a 0-chain.

If f_0 is closed, then there exists a (q-1)-form f_1 such that $f_0 = df_1$, and since δf_1 is closed, there exist f_2 such that $\delta f_1 = df_2$, etc. The dual argument shows that $g_i \epsilon C_i\ (N(U), S_{q-i})$ exists.

Suppose that a cocycle Z^q, of type (q, 0) with constant coefficients, and a cycle L_q, of type (q, 0) are given. Since U is strongly finite, $H^P(N(U), \Lambda^q) = \{0\}$. Therefore, there exists f_q such that $Z^q = \delta f_q$; now since Z^q has constant coefficients, $d\delta f_q$ must vanish. Since $H^q(N(U), \Lambda^q) = \{0\}$ and $\delta d = d\delta$, then there exists f_{q-1} such that $df_q = \delta f_{q-1}$. By a continuation of this process one gets $f_{q-2}, \ldots,$ f_1. From above $d\delta f_1 = \delta df_1 = 0$ implying that if one lets $f_0 = df_1$ it is a cocycle. Now

$$df_0 = \alpha df_1 = 0,$$

implying f_0 is a closed q-form. In a similar manner $g_0 \epsilon C_p(N(U), S_q^c)$ can be constructed from 1_q.

We have shown that cochains f_i of type (i - 1, q = i) exist satisfying the system of equations above. Let

$$A_i = \{f_i \quad d\delta f_i = 0\}$$

$$X_i = \{f_i \quad df_i = 0\}$$

$$Y_i = \{f_i \quad \delta f_i = 0\}$$

The values of f_i on N(U) are (q-i)-forms. d is a map that maps homomorphically onto

$$d : A_i \longrightarrow Z^{i-1} (N(U), \textstyle\bigwedge_c^{q-i+1}), \quad 2 \leq i \leq q$$

$$d : X_i \longrightarrow \{0\} \qquad 2 \leq i \leq q$$

$$d : Y_i \longrightarrow B^{i-1} (N(U), \textstyle\bigwedge_c^{q-i+1}) \quad 2 \leq i \leq q$$

Since $q - i + 1 > 0$, we can apply Poincare Lemma to d operating on A_i and get it to be a homomorphism onto. For $f_i \epsilon Y_i$, $\delta f_i = 0$. Since the cohomology is trivial for $i > 0$, there exists f' such that $f_i = \delta f'$ from which $df_i = d\delta f' = \delta df' = 0$, which implies $df_i \epsilon B^{i-1}(N(U), \bigwedge_c^{q-i+1})$. To show d is onto, let f' be an element of $B^{i-1}(N(U), \bigwedge_c^{q-i+1})$. Since $f' = \delta f_i$ for some $f_i \epsilon C^{i-1}(N(U), \bigwedge_c^{q-i})$ and since $q - i + 1 > 0$ we can again apply Poincare Lemma, $f_i = df''$. Now $f' = \delta df'' = d\delta f''$ and since $\delta(\delta f') = 0$, $\delta f' \epsilon Y_i$, d is onto. That d is a homomorphism from X_i onto $\{0\}$ is clear. We note X_i is the kernel of the homomorphism.

The following isomorphisms are a consequence of the above:

$$\frac{A_i}{X_i} \cong Z^{i-1} (N(U), \textstyle\bigwedge_c^{q-i+1})$$

$$\frac{X_i + Y_i}{X_i} \cong \frac{Y_i}{X_i \cap Y_i} \cong B^{i-1} (N(U), \textstyle\bigwedge_c^{q-i+1})$$

thus we know

$$\frac{A_i/X_i}{(X_i + Y_i)/X_i} \cong \frac{A_i}{X_i + Y_i} \cong \frac{Z^{i-1}\ (N(U), \Lambda_c^{q-i+1})}{B^{i-1}\ (N(U), \Lambda_c^{q-i+1})}$$

$$\cong H^{i-1}\ (N(U), \Lambda_c^{q-i+1})$$

Consider

$$\delta : A_i \longrightarrow Z^i\ (N(U), \Lambda_c^{q-i})$$

$$\delta : X_i \longrightarrow B^i\ (N(U), \Lambda_c^{q-i})$$

$$\delta : Y_i \longrightarrow \{0\}.$$

Thus by using an argument similar to the above one for d we can get δ to be a homomorphism onto for $1 \le i \le q - i$ and we may conclude that

$$\frac{A_i}{Y_i} \cong Z^i\ (N(U), \Lambda_c^{q-i+1})$$

$$\frac{X_i + Y_i}{Y_i} \cong \frac{X_i}{X_i \cap Y_i} \cong B^i\ (N(U), \Lambda_c^{q-i})$$

and thus

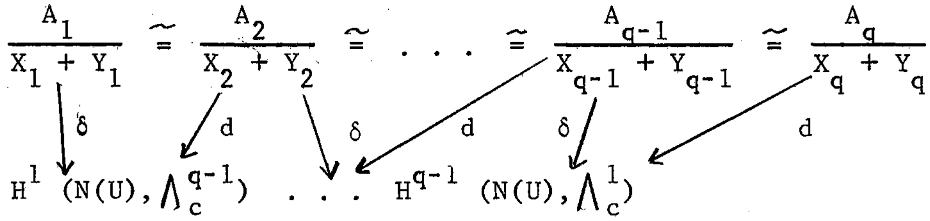
$$\frac{A_i/Y_i}{(X_i + Y_i)/Y_i} \cong \frac{A_i}{X_i + Y_i} \cong \frac{Z^i\ (N(U), \Lambda_c^{q-i})}{B^i\ (N(U), \Lambda_c^{q-i})}$$

$$\cong H^i\ (N(U), \Lambda_c^{q-i}).$$

We have shown the following

$$\frac{A_i}{X_i + Y_i} \cong H^i\ (N(U), \Lambda_c^{q-i}) \cong \frac{A_{i-1}}{X_{i-1} + Y_{i-1}} \quad .$$

We represent this by the following diagram



We must show the following two things:

$$\frac{A_1}{X_1 + Y_1} \cong D^q \cong \frac{\Lambda_c^q}{\Lambda_e^q} \text{ and } \frac{A_q}{X_q + Y_q} \cong H^q\ (N(U),\ R),$$

which will give us

$$D^q \cong H^q\ (N(U),\ R).$$

Let $f \epsilon A_1$ then $df_1 \epsilon Z^0\ (N(U), \Lambda_c^q)$ and therefore may be identified with a closed q-form. As before one needs only consider d operating on Y_1. If $f \epsilon Y$, then dy represents an exact form. On the other hand by Poincare's Lemma a closed q-form may be represented df and an exact form as df with $\delta f = 0$, thus

$$\frac{A_1}{X_1 + Y_1} \cong D^q.$$

Let $f_q \epsilon A_q$ then $d\delta f_q = 0$, which implies δf_q has constant coefficient and thus an element of Z^q (N(U), R). Now as before one needs only consider δ on X_q. But an element $x \epsilon X_q$ has constant coefficients which implies $\delta x \epsilon B^q$ (N(U), R). Now

$$\frac{A_q}{X_q + Y_q} \cong H^q\ (N(U),\ R)$$

which gives

$$D^q \cong H^q\ (N(U),\ R).$$

By a dual argument it can be shown that the singular homology is dual to the groups H_q (N(U), R).

Let g_i be of type (i, q-i) exist satisfying the system of equations (2.1.1). Now let

$$A_i^1 = \{g_i \mid D\partial g_i = 0\}$$

$$X_i^1 = \{g_i \mid Dg_i = 0\}$$

$$Y_i^1 = \{g_i \mid \partial g_i = 0\}$$

$$D : A_i^1 \longrightarrow Z_i\ (N(U),\ S^c_{q-i-1})$$

$$D : X_i^1 \longrightarrow \{0\}$$

$$D : Y_i^1 \longrightarrow B_i\ (N(U),\ S^c_{q-i-1})$$

and we note D is a mapping homomorphically onto

$$\partial : A_i^1 \longrightarrow Z_{i-1}\ (N(U),\ S^c_{q-i})$$

$$\partial : X_i^1 \longrightarrow B_{i-1}\ (N(U),\ S^c_{q-i})$$

$$\partial : Y_i^1 \longrightarrow \{0\}$$

and note ∂ is a mapping homomorphically onto.

This gives

$$\begin{array}{ccccccccc}
\dfrac{A_1'}{X_1' + Y_1'} & \cong & \dfrac{A_2'}{X_2' + Y_2'} & \cong & \cdots & \cong & \dfrac{A_{q-2}'}{X_{q-2}' + Y_{q-2}'} & \cong & \dfrac{A_{q-1}'}{X_{q-1}' + Y_{q-1}'} \\
\searrow D & \swarrow \partial & & \searrow D & & \partial \swarrow & \downarrow D & & \\
& H_1\ (N(U),\ S^c_{q-2}) & & & \cdots & & H_{q-2}\ (N(U),\ S^c_{q-1}) & &
\end{array}$$

Let ∂_o be the operator denoting addition of the coefficients of each chain in $C_0\ (N(U),\ S_q)$. Let Z_{00} be the space annihilated by ∂_0 and put $H_{00} = Z_{00}/B_0$.

Then,

$$\begin{array}{ccc}
\dfrac{A_0'}{X_0' + Y_0'} & \cong & \dfrac{A_1'}{X_1' + Y_1'} \\
\partial_0 \downarrow & & \searrow \partial_0 \\
S_q^C \,/\, S_q^b & & H_{00}\ (N(U),\ S^c_{q-1})
\end{array}$$

and on the other end

$$A'_{q-1} \cong \frac{A'_q}{X'_q + Y'_q}$$

$$D: A'_{q-1} \longrightarrow H_{q-1}(N(U), S^c_0), \quad \partial: \frac{A'_q}{X'_q + Y'_q} \longrightarrow H_{q-1}(N(U), S^c_0), \quad D_0: \frac{A'_q}{X'_q + Y'_q} \longrightarrow H_q(N(U), R).$$

Then from the complete sequence we get

$$\frac{S^c_q}{S^b_Q} \cong H_q\ (N(U),\ R).$$

Now we will extend De Rham's isomorphism theorem from a simple covering to that of any covering.

Theorem 5.1: For any covering $U = \{U_i\}$ of a differentiable manifold M there exists a covering $W = \{W_i\}$ by means of coordinate neighborhoods with the properties (a) $W < U$ and (b) there exists a map $\phi : W_i \longrightarrow U_i$ such that $W_{i_0} \cap \ldots \cap W_{i_p} \neq \phi$ implies $W_{i_o} \cup \ldots \cup W_{i_p} \subset U_{i_0} \cap \ldots \cap U_{i_p}$.

First there exists locally finite coverings V and U' such that $V \ll U' < U$. For any point $P \epsilon M$, there exists a ball W(P) around P such that

(i) $P \epsilon U'_i$ implies $W(P)\quad U'_i$,

(ii) $P \epsilon V_i$ implies $W(P)\quad V_i$

(iii) $P \epsilon \overline{V_i}$ implies $W(P)\quad \overline{V_i} \neq \phi$.

For since P is an element of only a finite number of U'_l and V'_j, (i) and (ii) are satisfied. If $P \epsilon V_0 \epsilon V$, then either $\overline{V_i} \cap V_0 = \phi$ or $\overline{V} \cap V_0 \neq \phi$. In the first case, (iii) is fulfilled. As in the second case, since V is locally finite there is only a finite number of V_i such that $V_i \cap \overline{V} \neq \phi$ and by choosing W(P) small (iii) may be satisfied.

Let $W_i = W(P)$ be a covering of M by coordinate neighborhoods. Then there exists open sets V_i with $P_i \epsilon V_i$ and by (ii) above $W_i \subset V_i \Subset U_i' \cap U_i$, which implies part (a) of the theorem. Suppose that $W_i \cap W_j = \phi$; then $W_i \cap \overline{V_j} \neq \phi$. By (iii) $P_j \epsilon \overline{V_j} \subset U_j'$ and so by (i) $W_j \subset \overline{V_j} \subset U_j$, thus by symmetry $W_i \cup W_j \subset U_i \cap U_j$ and by an inductive process we get part (b) of the theorem.

Let $\overline{A_i}$ be the direct limit of the $A_i = A_i(U)$ and $\overline{X_i}$, $\overline{Y_i}$ the corresponding direct limits.

Theroem 5.2: The maps d and δ induce homomorphisms

$$\bar{d} : \overline{A_i} \longrightarrow H^{i-1} (N(U), \Lambda_c^{q-i+1})$$

$$\bar{\delta} : \overline{A_i} \longrightarrow H^{i} (N(U), \Lambda_c^{q-i})$$

Moreover these maps are homomorphisms onto.

Let $f_i \epsilon A_i(U)$, $\bar{d}f_i$ and $\bar{\delta}f_i$ are defined as the cohomology classes containing df_i and δf_i respectively. They are well defined from the notion of direct limits. To show $\bar{\delta}$ and $\bar{d}$ are onto, let $Z \epsilon Z^{i-1} (N(U), \Lambda_c^{q-i+1})$ and W be a refinement of U as in the above theorem:

$$\phi : W_j \longrightarrow U_j$$

then the values of $\phi^* Z$ are defined on $W_0 \cap \ldots \cap W_{i-1} \subset U_0$ and may be extended to W_0. By Poincare's Lemma there exists $y \epsilon C^{i-1} (N(W), \Lambda^{q-i})$ for which $\phi^* Z = dy$ on W_0 and consequently on $W_0 \cap \ldots \cap W_{i-1}$, thus $\bar{d}$ is onto. Since the cohomology is trivial, any $z \epsilon Z^i (N(U), \Lambda_c^{q-i})$ is of the form δy, $y \epsilon C^{i-1} (N(U), \Lambda^{q-i})$, the element y represents any element $\overline{A_i}$; thus, $\bar{\delta}$ is onto.

Theorem 5.3:

$$\text{Kernel } \bar{d} = \text{Kernel } \bar{\delta} = \overline{X_i} + \overline{Y_i}$$

Let us first consider the images of $x_i(U_j) + y_i(U_j)$ under d and δ

$$d[x_i(U_j) + y_i(U_j)] = d[x_i(U_j)] + d[y_i(U_j)]$$
$$= 0 + d[y_i(U_j)]$$
$$= d[y_i(U_j)]$$
$$\delta[x_i(U_j) + y_i(U_j)] = \delta[x_i(U_j)] + \delta[y_i(U_j)]$$
$$= \delta[x_i(U_j)] + 0$$
$$= \delta[x_i(U_j)].$$

Since we are working in the space of closed forms, the lemma is true for $\bar{d}$. Now considering $\delta[x_i(U_j)]$ there exists a refinement W of U, as in the above proof, such that $\phi x_i(U) = dz(W)$. Thus $\delta x_i(U_j)$ will be in the same equivalence class as $\delta dz(W) = d\delta Z(W) = 0$; therefore, $x_i(U_j) + y_i(U_j) \in$ kernel $\bar{\delta}$.

On the other hand let dZ(U) represent $\{0\}$. Then for a suitable refinement ψ, $\psi dz = \delta U$ where $dU = 0$, since U is closed. Now by Poincare's Lemma, for further refinement of ϕ, $\phi U = dV$. From the above we get

$$d(\phi\psi Z - \delta V) = \phi\psi Z - \delta dV$$
$$= \phi\delta U - \phi\delta U$$
$$= 0.$$

And now by considering $\phi\psi Z = (\phi\psi Z - \delta V) + \delta V$, Z is an element of $\overline{X_i} + \overline{Y_i}$ since clearly $\phi\psi Z - \delta V \epsilon X_i$ and $\delta V \epsilon \overline{X_i}$. Analogous reasoning applies to $\bar{\delta}$.

Theorem 5.4: (de Rham's isomorphism theorem) Let M be a compact differential manifold then

$$\frac{\overline{A_i}}{\overline{X_i} + \overline{Y_i}} \cong H^{i-1}(N(U), \Lambda_c^{q-i+1}).$$

This is a direct result of the previous theorems and elementary theorem from algebra about isomorphisms.

The rank of $H_p(M, R)$ as a vector space over R is called the p^{th} betti number $b_p(M)$ of the differentiable manifold M. Thus $b_p(M)$ or just b_p is the dimension of the vector space H_p that is, the maximum number of p-cycles over R, linearly independent of the bounding p-cycles.

Since the p^{th} betti number b_p of M is the dimension of the group $H^p(M)$, it follows that $b_p(M)$ is equal to the number of linearly independent closed differential forms of degree p module the exact forms of degree p.

Let W be a closed p-form. To each p-cycle Z on M corresponds a period of W

$$\int_Z W = (W, Z).$$

If Z happens to be a boundary, $Z = b = \partial C$ the period vanishes, since by Stokes theorem

$$\int_Z W = \int_b W = \int_{\partial C} W = \int_C dW = \int_C 0 = 0.$$

Because of this there is a relation between periods.

Lemma 5.1: Whenever cycles $Z_1, \ldots$ are related by:

$$\Sigma\, a_i Z_i = \text{boundary},$$

then

$$\Sigma\, a_i \int_{Z_i} W = 0.$$

Let us stop at this point and consider an example in the 1-dimensional case. The existence theorems of de Rham's are concerned with the periods of a closed differential form over the singular cycles of a compact differentiable form over the singular cycles of a compact differentiable manifold. Let α be a 1-form and Γ a singular 1-cycle. We shall show how the period

$$\int_{\Gamma} \alpha$$

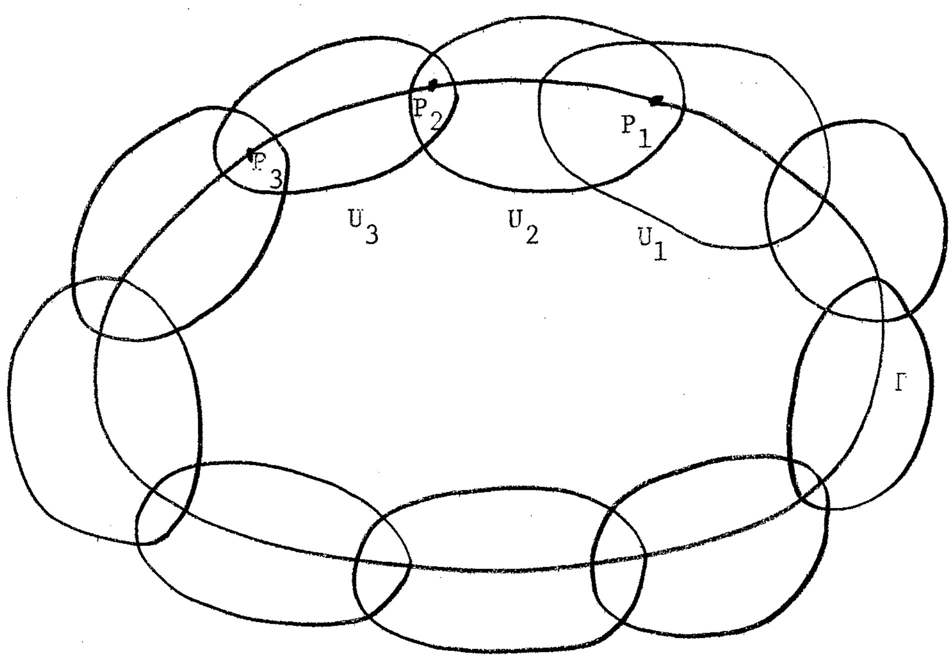
is related to an indefinite integral.

Let $U = \{U_i\}$ be a countable open covering of M by coordinate neighborhoods such that U_i correspond to an open ball in R^n. Subdivide Γ until each 1-simplex is contained in some U_i. Then

$$\Gamma = \Sigma\, \Gamma_i$$

where each Γ_i is a chain in some U_i. Each $\partial\, \Gamma_i$ is a 0-chain which may also be subdivided into parts each of which belong to a U_i.

Example:



Let Γ be the closed curve. Then α has an integral in each U_i. By Poincare's Lemma $\alpha = df_i$ in each U_i for some function f_i depending on α and U_i, thus

$$\int_{\Gamma}\alpha = \Sigma\,[f_i(P_{i+1}) - f_i(P_i)] = \Sigma\,(f_{i-1} - f_i)\,(P_i)$$

by regrouping. So in this way we are able to reduce the integration to the trivial case of integrating closed 0-forms over 0-chains.

Similarly the problem of computing the period of a closed q-form α, with compact carrier, over a singular q-cycle Γ is now considered.

One may again write $\Gamma = \Sigma\,\Gamma_i$ with Γ_i contained in U_i. If α_i denotes the restriction of α to U_i and f_0 the 0-cochain whose values are α_i, that is, $f_0(U_i) = \alpha_i$, then denote by g_0 the chsin whose coefficients are Γ_i,

$$\int_{\Gamma}\alpha = (f_0,\ g_0).$$

One notes at this point the independence of the subdivision.

Let Γ and Γ' be q-cycles such that $\partial\,\Gamma = \partial\,\Gamma'$. Now choose a common finite open covering, $U = \{U_i\}$, and let α be a closed q-form, then in every U_i $\alpha = dy_i$ by Poincare's Lemma.

Therefore

$$(\alpha,\ \Gamma) = \Sigma(dy_i,\ \Gamma_i) = \Sigma(y_i,\ \partial\,\Gamma_i)$$

and

$$(\alpha,\ \Gamma') = \Sigma(dy_i,\ \Gamma_i') = \Sigma(y_i,\ \partial\,\Gamma_i)$$

by Stokes theorem. If one considers

$$\Sigma\ (y_i,\ \partial\,\Gamma_i) = \Sigma\int_{\partial\,\Gamma_i} y_i = \int_{\partial\,\Gamma}\beta$$

since the inner product is linear. In the same manner

$$\Sigma\ (y_i,\ \partial\,\Gamma_i) = \int_{\partial\,\Gamma}\beta$$

therefore

$$(\alpha,\ \Gamma) = (\alpha,\ \Gamma').$$

Theorem 5.5: (De Rham's first theorem) Let $\{\Gamma_q^i\}$ (i = 1, 2, . . . , $b_q(M)$) be a basis of the singular q-cycles modulo the singular boundaries of a compact manifold M and W_q^i (i = 1, . . . , $b_q(M)$) be bq arbitrary real constants. Then, there exists a regular, closed q-form α on M having the W_q^i as periods, that is

$$\int_{\tau_q^i} \alpha = W_q^i, \quad i = 1, \ldots, b_q.$$

$$\tau_q^i \in \Gamma_q^i$$

Due to the isomorphism theorem, one needs only establish this for cycles and cocycles, with real coefficients, on the nerve of a given covering U.

Let L be a linear functional $Z_q(N(U), R)$ that vanishes on $B_q(N(U), R)$. Now extend L to $C_q(N(U), R)$ in the following way: Let $\{\xi_i\}$ be a basis of $C^q(N(U), R)/Z_q(N(U), R)$ let $\xi_i' \epsilon \xi_i$. Then every $\xi \epsilon\ C_q(N(U), R)$ has a unique representation in the form

$$\xi = \sum r_i \xi_i + \Gamma', \ \Gamma \epsilon\ Z_q(N(U), R), \ r_i \ \epsilon\ R.$$

Now the extension of L to $C_q(N(U), R)$ is complete by putting $L(\xi) = L(\Gamma')$.

There exists a unique cochain x ϵ $C_q(N(U), R)$ such that $(x, \xi) = L(\xi)$, x would be the cochain whose values are $L(\Delta(i_0, \ldots, i_q))$. Now

$$(\delta\ x, \ \xi) = (x, \ \partial\xi) = L(\partial\xi) = 0$$

by theorem [4.5] and L vanishes on $B_q(N(U), R)$. ξ is an arbitrary chain, therefore δx vanishes. Now $(x, \delta\xi) = (dx, \xi) = (\delta x, \xi) = 0$ therefore $dx = 0$.

Theorem 5.6: (De Rham's second Theorem) A closed form is exact if and only if all of its periods vanish.

Let us suppose that $(x, \partial\xi) = 0$ for all $\xi \epsilon C_{p+1}(N(U), R)$. We now consider the cochain x and its properties. Let L be a linear functional on B_{q-1} (N(U), R) defined by

$$L(\partial N) = (x, N), \quad N \epsilon C_p(N(U), R).$$

Since from above $\partial N = \partial N'$ implies $(x, N) = (x, N')$, L is well defined. Extend L to all the (q-1)-chains. We may find a y such that

$$(y, \beta) = L(\beta), \ \beta \epsilon C_{q-1}(N(U), R).$$

Therefore since $y \epsilon C^{q-1}(N(U), R)$, we may consider

$$\begin{aligned}(x - \delta y, N) &= (x, N) - (\delta y, N)\\ &= (x, N) - (y, \partial N)\\ &= L(\partial N) - L(\partial N)\\ &= 0\end{aligned}$$

since L is linear and by theorem [4.5]. Since this holds for all $N \epsilon C_q(N(U), R)$, then δx vanishes; hence, x is a coboundary. If all of the periods of x vanish, then

$$(x, N) = 0$$

and

$$\begin{aligned}(x, N) &= (\delta y, N)\\ &= (y, \partial N)\\ &= (dy, N).\end{aligned}$$

Since this is true for all N, then $x = dy$. On the other hand if $x = dy$, then clearly for all q-cycles N

$$(x, N) = (dy, N) = 0 \ .$$

We note at this time that our work has been with respect to an orientable manifold M. Although we did not state de Rham's Existence Theorems for orientable manifolds, they are valid only on orientable manifolds [6].

It is now time to explore some examples of De Rham's theorems. Let us take M the unit circle, S^1, in E^2. We may take the central angle θ (mod 2π) as parameter. A 1-form

$$\omega = f(\theta)\, d\theta, \text{ where } f(\theta+2n\pi) = f(\theta),$$

is exact if there is a periodic function g such that $f(\theta) = \frac{dg}{d\theta}$.

Now

$$\int_{S^1} \omega = \int_0^{2\pi} \frac{dg}{d\theta}\, d\theta = g(2\pi) - g(0) = 0.$$

The above shows that the condition is necessary. If the integral vanishes, then we may set

$$g(\theta) = \int_0^{\theta} f(t)\, dt,$$

and this relation is well defined θ mod 2π. Then, the condition is sufficient.

Any 1-form on M is closed. Let $\omega = kd\theta$ where k is a constant. Then we may ask can we find k such that for any real number a

$$\int_{S^1} \omega = \int_{S^1} kd\theta = a?$$

Clearly

$$\int_{S^1} d\theta = \frac{a}{k}$$

would be the same as

$$\int_{S^1} kd\theta = a$$

$$\int_{S^1} d\theta = \int_0^{2\pi} d\theta = 2\pi = \frac{a}{k}$$

if and only if $k = \frac{a}{2\pi}$. Therefore, we have De Rham's second theorem on S^1.

As another example take the cylinder $(-1, 1) \times S^1 = \{(t, \theta) \mid -1 < t < 1, \theta \bmod 2\pi\}$. Let c' denote the unit circle in the xy-plane.

Theorem: Let ω be a closed 1-form on the cylinder. Then ω is exact if and only if

$$\int_{c'} \omega = 0.$$

If f is a 0-form on the cylinder and $\omega = df$, then

$$\int_{c'} \omega = \int_{c'} df = \int_{\partial c'} f = 0;$$

therefore, the condition is necessary.

Now consider the mapping

$$\phi: (-1,1) \times E^1 \longrightarrow (-1,1) \times S^1$$

$$\phi(t, \theta) = (t, \cos\theta, \sin\theta),$$

which gives a covering of the cylinder by the infinite strip. Then let ω be a closed 1-form on the cylinder such that

$$\int_{c'} \omega = 0.$$

If $0 < t < 1$, then the 2-chain

$$c^2 = [0, t] \times S^1$$

has boundary

$$\partial c^2 = \{t\} \times S^1 - c' ;$$

hence

$$\int_{\{t\}\times S^1} \omega = \int_{\{t\}\times S^1} \omega - \int_{c'} \omega = \int_{\partial c^2} \omega = \int_{c^2} d\omega = 0$$

which implies that the integral of ω taken over any circle parallel to c' vanishes. With this, consider the form $\phi^*\omega$, a 1-form on the infinite strip, which from a differentiable structure alone, is indistinguishable from E^2. We know $d(\phi^*\omega) = \phi^*(d\omega) = \phi^*(0) = 0$, and hence $\phi^*\omega$ is a closed 1-form on the strip. By the converse of the Poincare Lemma, there exists a function g on the strip such that $\phi^*\omega = dg$.

Is there a function f on the cylinder satisfying $\phi^*f = g$? Clearly, for this condition it is necessary and sufficient that g be periodic of period 2π in θ. But

$$g(t, \theta+2\pi) - g(t, \theta) = \int_{\theta}^{\theta+2\pi} \frac{dg(t, s)}{ds}\, ds$$

$$= \int_{[t] X [\theta, \theta+2\pi]} dg = \int_{[t] X [\theta, \theta+2\pi]} \phi^*\omega$$

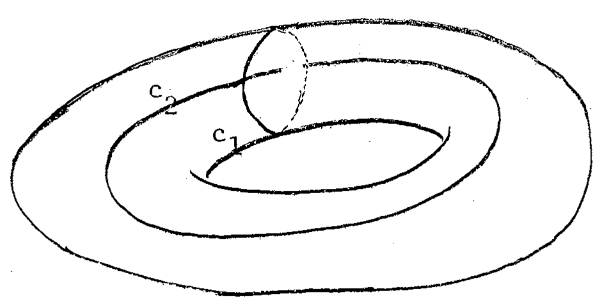
$$= \int_{[t] X [\theta, \theta + 2\pi]} \omega = 0.$$

Thus, g has the required periodicity, so there is a function f on the cylinder satisfying $\phi^*f = g$. Hence, $dg = d\phi^*f = \phi^*(df)$, and $\phi^*\omega = \phi\ df$. Since ϕ^* is locally 1-1 with a smooth inverse; hence ϕ^* is 1-1 and

$\omega = df$.

Let us now turn to some examples without any proof but just an application of De Rham's theorems.

Let us consider a torus, Σ, in E^3. The only significant 2-cycle is Σ itself. By De Rham's first theorem, a 2-form α on Σ is exact if and only if $\int_{\Sigma} \alpha = 0$.



There are two significant one-cycles c_1 and c_2. Here c_1 and c_2 cross once. De Rham's first theorem asserts that if ω is a closed 1-form on Σ, then ω is an exact differential if and only if

$$\int_{c_1} \omega = \int_{c_2} \omega = 0.$$

De Rham's second theorem asserts that if real numbers a_1, a_2 are given there exists a closed 1-form α such that

$$\int_{c_1} \alpha = a_1 \quad , \qquad \int_{c_2} \alpha = a_2 .$$

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