A COMPARISON OF THE EFFECTIVENESS OF AN ABSTRACT AND A CONCRETE APPROACH IN TEACHING OF INTEGERS TO SIXTH GRADE STUDENTS

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PREFACE

The purpose of this research was to compare differences in achievement between equivalent groups of sixth grade students. One group was taught manipulation of signed numbers from a concrete, number line approach, while the other group received instruction in an abstract, algebraic approach. The conception of this research grew out of elementary teacher training classes in which an algebraic approach was used to establish the structure of our number system, and is a direct result of some thoughts and contentions of the writer that appeared in a recent nationally known journal.

Several acknowledgments are in order. First, I want to acknowledge the marvelous cooperation and constant encouragement I received from Dr. W. Ware Marsden, chairman of my committee, and Dr. Vernon Troxel, who, as thesis adviser, worked closely with me in the reporting of this research. The cooperation of Mr. Cecil Smith, Elementary Curriculum Coordinator of the Pittsburg, Kansas School System, made it possible to conduct the research in the public school classroom and to obtain extensive test results on the subjects involved. The experimental teachers, Mrs. Virginia Romondo and Mr. Harold Fisher, possessed the enthusiasm and interest necessary to the success of such an investigation.

I also wish to acknowledge the love and understanding of my wife, Barbara, who throughout the ordeal of course work, research, writing, and minimal salaries, maintained constant encouragement. And to my boys -- Doug, Kent, and Glenn -- I promise to make up for lost time from playing catch and going places together.

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INTRODUCTION

During the last fifteen years the mathematics curriculum has undergone radical revisions at every level. Courses formerly reserved for graduate study are finding their way into the undergraduate curriculum. Many topics, traditionally thought of as college level, are now being introduced at the secondary level. Algebraic and geometric concepts are being introduced very early, generally in the junior high school. Now, however, even more emphasis is being placed on generalizations and intuitive understanding of algebraic and geometric concepts in the upper elementary grades. The movement of more and more material, typically thought to be complicated and abstract, downward into the lower grades has caused many educators to wonder where and when the trend will stop.

There exists very little chance that the curriculum in mathematics will become the static body of material that it was before the revision began. Even as this period of change is leveling out and teachers are becoming more adequately prepared and, therefore, more confident of their performance in the classroom, the need for further revision is being indicated by such publications as the Goals for School Mathematics: A Report of the Cambridge Conference on School Mathematics. In the foreword of this Report, Francis Keppel, United

States Commissioner of Education, makes the following comments;

> Most of the new curricula reforms have . . . tended to create such new courses as existing teachers, after enjoying the benefits of brief retraining, can competently handle. They have done so fully aware that they are thus setting an upper limit, and an upper limit that is uncomfortably close.
>
> If the matter were to end there, the result might well be disastrous. New curricula would be frozen into the educational system that would come to possess, in time, all the deficiencies of curricula that are now being swept away. And in all likelihood, the present enthusiasm for curriculum reform will have long since been spent; the "new" curricula might remain in the system until, like the old, they become not only inadequate but in fact intolerable. Given the relative conservatism of the educational system, and the tendency of the scholar to retreat to his own direct concerns, the lag may well be at least as long as it has been during the first half of this century.
>
> The present report is a bold step toward meeting this problem. It is characterized by a complete impatience with the present capacities of the educational system. It is not only that most teachers will be completely incapable of teaching much of the mathematics set forth in the curricula proposed here; most teachers would be hard put to comprehend it. No brief period of retraining will suffice. Even the first grade curriculum embodies notions with which the average teacher is totally unfamiliar.
>
> None the less, these are the curricula toward which the schools should be aiming. If teachers cannot achieve them today, they must set their courses so that they may begin to achieve them in ten years, or twenty years, or thirty. If this is what the teacher of the future must know, the schools of education of the present must begin at once to think how to prepare those teachers. There must still be short-term curriculum reforms; they must look upon themselves as constituting a stage toward the larger goals; and they must at all costs be consistent with those larger goals. (1, p. viii)

In the process of accomplishing the goals suggested by Keppel the educator will encounter many questions to be answered and problems to be solved. The Cambridge Conference intended merely to establish the goals; the Report was purely a discussion document and not a prescription. The attainment of these goals will involve facing the problems of teacher education, and the "many fine points of pedagogic techniques" (1, p.3) that were admittedly ignored by the Conference. Problems of pedagogy as well as psychological problems are only a few of the areas of concern. Some problems have always existed, while others have been brought to light with the new developments.

A pedagogical question of long standing is: Does a child comprehend material better when it is presented through a concrete approach rather than through an abstract presentation? Most educators agree that material presented through a concrete experience is accepted and understood more readily than when presented through an abstract approach (2), (3), (4). It has been stated by Piaget (5), and implied by others, that there are certain ideas and degrees of abstraction that cannot be mastered until a child reaches a certain level of maturity (6), (7). However, in the more modern programs algebraic, geometric, and other abstract concepts are being introduced at an early level in elementary curricula. How soon, then, can concepts presented abstractly be comprehended as adequately as those presented concretely? Somewhere along the path of education in mathematics the student must progress from the concrete to the abstract.

> Cooperation of mathematicians, psychologists, and classroom teachers must smooth the learner's path from intuitive, informal, exploratory, discovery procedures to symbolic, formal, deductive procedures. (8, p. 99)

The report of the Cambridge Conference on School Mathematics, which took place in June, 1963, sets the stage for future development in mathematics. The goals set forth for mathematics education during the next fifteen to twenty years seem impossible until one stops to consider what has been accomplished during the past fifteen years. Much of the material suggested for the elementary level is highly sophisticated and involves abstract concepts.

It is not always possible to correlate mathematics with everyday experiences. However, the National Council of Teachers of Mathematics' 24th Yearbook, The Growth of Mathematical Ideas, concludes: ". . . students seldom demand that learning be useful if it is successful." (9, p. 416) It goes on to add:

> This is not to suggest that learning that has no practical value in the everyday world is less valuable than that which does not have such ready use. It is only to say that practicality is not, for all students, an essential criterion of effective learning, that is, learning which produces motivating satisfactions in the learner. (9, p. 416)

Students do and can study mathematics simply for the pure pleasure it provides them.

Statement of the Problem

The purpose of this research was to compare the acquired abilities of two groups of children introduced to the same topic through different approaches, one abstract, the other concrete. By controlling, insofar as was possible, the different variables involved, an attempt was made to arrive at a decision based upon objective data that might shed light on the thought processes of students at the intermediate grade levels. The question considered in this investigation was: Do sixth grade students understand the addition and subtraction of integers better when the material is presented through a concrete, or visual procedure or when the material is presented through abstract, or algebraic, procedures?

"In experimental studies the condition that is varied," (10, p. 139) or "the variable manipulated by the experimenter," (11, p. 39) is referred to as the independent variable. For this study, the independent variable was the approach used to introduce and perform operations with integers.

The dependent variable is referred to as " . . . the variable that is being predicted (10, p. 136) and is " . . . a presumed result of variation in the independent variable," (11, p. 39). The test scores on the quizzes used to measure the effects of the two approaches are the dependent variable.

Constructs or intervening variables " . . . are terms invented to account for internal and directly unobservable psychological processes that in turn account for behavior," (11, p. 44). The constructs of mathematical abilities, motivation, and heredity, to name only a few, are not always observable and are difficult to measure. For this study these constructs were considered to have negligible effect due to the random process of assigning the subjects to the different approaches.

Definition of Terms

The terms "abstract" and "concrete" are ambiguous terms that mean different things to different people. Mathematics, as a subject area, is probably one of the most abstract in nature. Such statements as " . . . learning is a process in which the concrete and the abstract interact," (4, p. 35) " . . . sound learning proceeds from concrete experience to abstraction, (6, p. 273) and " . . . concrete experience will help to make meaningful to pupils the highly abstract subject matter of arithmetic" (4, p. 38) point out the correlation and interrelation of the two concepts. In this study the two terms are used to indicate different approaches used in the presentation of a body of common material.

The consideration of Dale's "Cone of Experience" (4, p. 43) should help to visualize the distinction between the two approaches. Each division of the Cone " . . . represents a stage between the two extremes - between direct experience and pure abstraction. The cone device, then, is a visual metaphor of learning experiences - arranged in the order of increasing abstractness as one proceeds from direct experience." (4, p. 42) A demonstration is placed on the fourth band from the bottom,

> . . . because it is frequently a process of observing, thus differing from the activities on the first three bands, which essentially involve <u>doing</u>. Nevertheless, a demonstration often can be and is followed by doing on the part of the observer. (4, p. 138)

"A demonstration is a visualized explanation of an important fact or idea or process." (4, p. 48) In this experiment the number line was used to explain visually the operations of addition and subtraction in the approach using <u>concrete</u> methods.

At the apex of the cone are found the verbal and visual symbols. "As we enter (this) stage on the cone, we no longer have realistic reproduction of the thing itself but an abstract representation." (4, p. 52) "We have abstracted everything from the original except the meaning of terms, and on this meaning we have reached more or less common agreement." (4, p. 53) The algebraic procedure referred to as the ordered pair approach was the <u>abstract</u> procedure used in this experiment.

In defining the terms "abstract" and "concrete," the following phrases used by Good (12) in his <u>Dictionary of Education</u> to define abstract and concrete concepts, abstract and concrete problems, and the concept of reasoning, can serve as a point of reference for most educators. With reference to concrete, the phrases dealing with " . . . concepts within the experience of the individual" (12, p. 252), and " . . . an idea or image . . . that can be perceived by the senses" (12, p. 90) leads to the following functional definition.

<u>Concrete Approach</u>. The instructional approach that used visual procedures (the number line) with which the subjects of this experiment had had experience.

The following functional definition for abstract was formulated by considering the phrases " . . . an idea or aggregation of ideas that has been acquired as a symbol" (12, p. 90), " . . . not associated with any particular object or concrete experience, involving adequate responses " . . . dealing with content that is in no way related to the previous life experiences of the individual." (12, p. 309)

<u>Abstract Approach</u>. The instructional approach that used algebraic procedures (ordered pairs) which were completely unfamiliar to the subjects of this experiment

Other terms that need to be defined for this study are as follows:

<u>Ordered Pairs</u>. An algebraic representation of the Integers, such that (a,b) is associated with the result of a - b, and in which the order is important (i.e. $(a,b) \neq (b,a)$).

Integers. The positive and negative integers and zero (i. e. 0, ±1, ±2, ±3, ±4, . . .).

Whole Numbers. The positive integers and zero (i. e. 0, 1, 2, 3, 4, . . .).

Standard Numerals. The integers written in more standard or acceptable form indicating their actual meaning with a sign (i. e. -2, +4, 0 etc.).

Significant Difference or Statistically Significant. This means that a certain two quantities that are being compared differ by more than can reasonably be attributed to chance variation.

Achievement. The post-test scores on the tests entitled "Suggested Quiz" prepared by the Greater Cleveland Mathematics Program and used to measure the abilities of both groups to manipulate integers.

Previous and Related Research

The many new elementary mathematics programs present topics and terms that traditionally have been reserved for more advanced students. It is only natural for concerned educators to question the ability of young pupils to understand these concepts and terms, and the research has reflected this concern. Studies seem to indicate that many young children can learn more mathematics than has generally been expected of them (13, #19), (13, #105), (13, #114), but because of differing abilities (13, #106) and differing backgrounds (13, #28), (13, #128), not all of them can learn the same things at the same age. Therefore, the crucial question seems to be, ". . . what mathematics should what children learn at what age? And on this question, very little research is available." (13, p. 1)

In the process of reviewing the literature and research in the area of mathematics education, a research design suggested by Robert M. Gagne (14, p. 123-125) in 1961 was found by the writer. The approach suggested by Gagne appeared feasible for use in this study. Professor Gagne, then at Princeton University and presently with the University of California at Berkeley, responded to an inquiry by the writer regarding the suggested study by stating: "I certainly know of no one who is specifically undertaking a study like that described . . . " * He went on to indicate that if he were to consider such a study today, he would make some changes in the proposed method.

> In particular, I would want to think again about what kind of 'aptitude' test should be used. It seems to me quite possible that commercially available tests might not get at the ability involved very well, and that one might need to develop a test as part of the research.

This point had also occurred to the investigator for this study. The necessity to instigate the research at an earlier level was deemed essential since some recognized authorities are introducing integers to children much earlier than has been customary.

The rationale in working with signed numbers is quite varied as indicated by many recent publications and available elementary mathemathics series.

Cohen (15), a former mathematics specialist on the Madison Project at Webster College, proposed the "Postman Stories" model as devised by

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*Quotation from a letter dated October 26, 1966, from Dr. Robert M. Gagne, School of Education, University of California, Berkeley.

the Madison Project as a remarkably clear and understandable way to interpret both addition and subtraction. In this approach verbal stories are translated into numerical manipulations of integers. Coon (16) used the number line not only to add and subtract but also to introduce multiplication as the process of successive additions. Overholt (17) illustrated the possibility of using the number line, number pairs, and positions in two-dimensional space to convey the operations with integers. A study in which ordered pairs were used to add rational numbers (18) is the only source, other than a descriptive article (19) by the writer of this dissertation, that suggested the ordered pair approach might be feasible with the elementary age child. This approach has been used at higher levels to stress the structural or deductive nature of our number system.

Pupil understanding of arithmetical concepts and procedures has been facilitated in recent years by varied concrete representations and visual teaching devices. The number line has been used for a long time to teach negative numbers in algebra. It has recently become rather well known in arithmetic in such areas as:

> . . . counting, in determining the sum or difference of two amounts, in reading numerals, in giving a notion of the nature of the number series, and in giving background experience for later work in addition and subtraction of numbers written with numerals. (20, p. 21)

The number line has certainly proved to be a valuable teaching aid for the graphical representation of relationships between numbers of all kinds and for showing that number series extend indefinitely in either direction (positive or negative) from a given point.

In the U.S. Department of Health, Education, and Welfare pamphlet on General Facilities and Equipment, (21) a survey of equipment, materials and teaching aids deemed of most value to classroom teachers in secondary mathematics, the following question was posed:

> Is there a tendency toward a greater use of visual aids, instruments, gadgets, models, and the like in order to motivate and encourage pupils to discover relationships for themselves and seek the general principles? Or is there a tendency to use models and other visual aids in order to make teaching more meaningful to many pupils who may or may not be able to work with symbolic abstractions alone? (21, p. 35-36)

Games, pictures, counting devices, and other supplementary materials have shown an increase in use during the past twenty years, particularly in the early grades (13, #154). However, expensive, commercially produced, number aids were found (13, #66) to be no more effective than inexpensive materials selected by the teacher.

Brownell (22) presented a thorough discussion of three theories of the teaching of arithmetic that considered both content and the methodology of teaching. He referred to the three theories as the drill theory, the incidental learning theory, and the meaning theory. Although both the drill theory and incidental learning theory have been rejected in idea, if not in practice, there is general agreement that arithmetic should be taught meaningfully. In 1922, Thorndike (23) presented a rational discussion stressing the importance of teaching for meaning and understanding. Studies by Howard (24), Betz (25), Canton (26), and Weaver (27) have shown that meaningful teaching results in learning that is superior to that achieved under other kinds of teaching. However, the real question left to be researched is: What do we mean when we say something is taught meaningfully?

Glennon contended that: "Meanings or understandings are not 'rules' to be studied as such by children, but may be formulated by them as an outgrowth of meaningful learning." (28, p. 22). Although full agreement on essential meanings is not to be found, the most definitive source is found in Buckingham (29).

Emphasis on drill as a learning process has undergone considerable modification in recent years. Even the word itself is being replaced with the more descriptive word "practice." Practice, according to Burton (30), has two essential phases:

a) Integrative phase - The phase in which perception of the meaning is developed.

b) Repetitive phase - The phase in which precision is developed.

Brownell and Hendrickson have the following comment about repetitive practice:

> The fundamental method of teaching some factual material, most symbols, and arbitrary associations in general, remains, as always, the administration of repetitive practice. (31, p. 101)

Practice, or drill, still plays an important part in the field of teaching. There is an important distinction between practice and a program of reteaching. The purpose of practice is to fix skills and abilities already learned, while the purpose of reteaching is to provide the pupil the opportunity to relearn through systematic, condensed instruction.

Although it is quite difficult to separate objectives of mathematical learning into neat, discrete categories, Glennon (28) has indicated three major areas:

a) Cognitive - This category includes subject matter understandings, concepts, and facts.

b) Affective - This category includes attitudes, appreciations, and other emotional factors in the learning process.

c) Psychomotor - This category includes skills and abilities which require both mind and muscle.

The discussion thus far has dealt primarily with the cognitive and psychomotor categories of learning. However, the affective category should also be considered. Many studies in recent years have found this category to have a major effect upon learning. Attitudes toward mathematics and motivation within the classroom were of particular concern in this study. As Travers has commented:

> . . . achievement is also a function of motivation and that the presence or absence of numerous external conditions affect learning; for example, the way in which the teacher interacts with the pupil. (10, p. 40)

It was impossible in this study to do much in the way of controlling or measuring these variables. However, consideration of the following related studies indicate some reasons for assumptions made later.

Poffenberger and Norton (32) determined that positive or negative attitudes of college freshmen toward mathematics are attributed more to attitudes expressed by parents than to the influence of previous teachers.

Chase (33) found that when reading was disliked by students it was because it was too easy, while if arithmetic was disliked, the reason given was it is too hard. Although girls normally dislike arithmetic more than boys do (33), Mosher (34) determined that this sex difference in subject preference did not occur until the secondary school level, eliminating the consideration of this problem from the realm of this study.

Theoretical Implications of Research

In the undertaking of any research, theory is the beginning point from which hypotheses are generated. Theory as developed by a scientist is a set of generalizations believed to have some value in predicting important events. Hypotheses derived from formulated theory are simply statements of some consequences that can be expected if the theory is true. Theories used in educational research involve generalizations about some aspect of education. They are generalizations that are generally ". . . based on information and are often substantiated by research, but they do not yet have the certainty, usefulness, or status of laws." (10, p. 25)

The 'new' mathematics movement, as a whole, has completely ignored the available evidence from psychological research and the best procedures known to psychologists. The reform movements are generally supported by practical educators

> . . . who are eager to meet the new challenges facing the schools and cannot wait until psychologists have completed their research and have come to agreement about the proper theoretical basis for classroom learning. (35, p. 290)

Guba and Getzels (36) believe that a theoretical framework sharpens research objectives by suggesting that variables should be eliminated, simplifies the complex task of interpreting meaningful and, even more, nonsignificant results, and makes research cumulative from one study to the next.

There exists no comprehensive theory of learning or teaching universally accepted by all educators, least of all, mathematics educators. Since this situation exists, there is accepted by most, micro-theories considered pertinent to this study will be derived and examined.

Paradigms are models, patterns, or schemata, but not theories. However, they are ways of thinking and patterns for research that, when carried out, could lead to the development of theory. This study used the conceptual framework developed by Gage (37) to guide the group of authors for the Handbook of Research on Teaching. He identified three major classes of variables and subcategories within each variable. They are:

a) Central Variables - This refers to behavior or characteristics of teachers. This class includes:
   1) teaching methods,
   2) instruments and media of teaching, and the
   3) teacher's personality and characteristics
b) Relevant Variables - This refers to the antecendents, consequents, or concurrents of the central variables. This includes the categories of:
   1) social interaction within the classroom, and the
   2) social background of teaching.
c) Site Variables - This refers to the site or situations in which other variables are studied. The site variables include:
   1) grade level, and
   2) subject matter.

It is within this conceptual framework that this study was conducted. The site variables were controlled. The central variables were explained with respect to teaching methods, discussed with reference to media of teaching, and the teacher's personality and characteristics were controlled to minimize their effects of this experiment. The relevant variables

" . . . are neither necessary nor sufficient to characterize a piece of research as research on teaching." (37, p. vii) The relevant variables, although noted and discussed, were generally disregarded as having an insignificant effect on this experiment.

The following list of statements has been gleaned from the relevant research and literature as being necessary for effective learning in mathematics, and were considered as postulates for this study.

1. Motivation of the student is necessary to the learning situation.
2. The learner should be involved in discovering ideas in mathematics as an active participant rather than as a passive listener.
3. Mathematics should be presented in a meaningful manner moving from the crete to the point where the learner makes generalizations.
4. Emphasis should be placed on meaning rather than mere manipulation; practice, however, should be provided to develop precision and fix skills already learned.

From these postulates the micro-theory for this study has been formulated. The central point of contention, as seen by the investigator, is the question of meaningful teaching and what is abstract as compared to concrete. Bernstein comments, "No one quarrels with the idea of proceeding from the concrete to the abstract." (6, p. 273) However, he goes on to add, "The issue is whether a starting point which is concrete to the teacher is concrete to the learner." (6, p. 273) Fehr states that:

> Physical objects and special structural systems of sensory aids have been commercialized and advertised but they must be critically evaluated before use, and in general they are not necessary for learning mathematics. (38, p. 87)

Also, Travers commented that " . . . although there is little evidence to justify their use," (10, p. 144) visual and auditory aids are being extensively used.

> Even elementary-school children, when they have been properly prepared, can manage a great deal of conceptualized experience through the abstract language of visual symbols. (4, p. 53)

Dale adds that symbols are not " . . . dry and lifeless abstractions. They are as rich and vibrant as the meanings they stand for." (4, p. 309)

Therefore, the theorem proposed by this study is:

> If a topic is presented in a meaningful manner, although using abstract, algebraic procedures and the learner is provided practice to develop skills, then the learner will be able to generalize as well as if the topic had been presented by concrete, visual procedures.

Hypothesis to be Tested

The fact that integers are being introduced concretely at the intermediate grade level in many modern elementary series (39), (40), (41), (42) seems to indicate that a fifth or sixth grade student can cope with mathematical ideas concerning the integers when presented in this manner. Whether these ideas can be developed as effectively through an abstract approach has not been determined. Therefore, the null hypothesis for this experiment was as follows: No significant difference exists between the tested achievement of two groups of sixth graders who were introduced to the study of integers through two different approaches.

Plan of the Study

The sixth grade level was determined to be the logical level at which to conduct this research. Although at the present time the integers are being introduced intuitively at earlier grade levels in many modern elementary programs, the sixth grade was determined to be the level at which the subjects of this research would first be introduced to integers. Since the integers as a number system had never been introduced in the educational background of these subjects, their prior understanding was judged to be nil. Hence, it was unnecessary to give a pretest, and all subjects were judged to have started the experiment with equal understanding of integers.

The two approaches generally differed widely, although at times they used common techniques of presentation. Both approaches were directed toward the same outcome - The ability to add and subtract signed numbers as standard numerals.

One group, which will be referred to as group C (concrete), used the Greater Cleveland Mathematics Program, Intermediate Series, Book 7, 1964-65 edition. In this program the integers are introduced as the difference of two whole numbers, and the number line is heavily relied upon to promote the understanding of standard numerals used to represent these differences. The number line is then used to develop understanding of the operations of addition and subtraction. The concepts of closure, commutativity, associativity, addition of zero (not specified as the additive identity), and opposites, which are referred to as additive inverses, are developed to varying degrees of understanding.

Operations illustrated on the number line are then discussed, and generalizations are determined regarding the manipulations of signed numbers.

The other group, which will be called group A (abstract), were introduced to materials that involve abstract algebraic rules. This method of presentation, commonly referred to as the ordered pair approach, involved the student in algebraic manipulations with ordered pairs.* The operations of addition and subtraction are defined by algebraic rules involving ordered pairs where the possible outcomes are discussed and generalizations determined with regard to manipulations of signed numbers.

Group A used material developed by the investigator based upon reference material (43), (44), (45) generally considered more advanced in the normal sequence of mathematical development. The integers were introduced as ordered pairs of whole numbers, which are associated with the result of their differences. Hence, the two approaches started with approximately the same content, but approach A then developed the operations of addition and subtraction through abstract algebraic methods without relying upon concrete interpretation. The concepts of closure, commutativity, associativity, additive identity, and additive inverse were also developed through abstract interpretation rather than through the use of concrete procedures.

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*A discussion of this approach may be found in an article entitled "Introducing the Integers as Ordered Pairs," written by the author of this dissertation, and published in School Science and Mathematics, Official Journal of the Central Association of Science and Mathematics Teachers, Inc., March, 1966, p. 277-282.

The experimental materials prepared by the investigator closely paralleled in format the control group's material. When group C had a page of drill work, group A also had a page; but while group C was using the number line to establish concepts, group A was using the abstract procedures discussed above.

Scope of the Study

The population from which the sample was selected consisted of all sixth grade students in the public schools of Pittsburg, Kansas. It was necessary to work with groups already established as a result of administrative scheduling.

Certain factors such as the availability of two classes in the same building, capabilities of the teachers, and willingness of the teachers to experiment had underlying effects upon the final selection. Comparisons of the groups to verify their equivalence was made in terms of these factors: I.Q. scores from the Otis Quick-Scoring Mental Ability Test and arithmetic computation scores from the Metropolitan Achievement Tests, (Intermediate Battery).

The data for this study were collected during the second semester of the school year 1965-66. Four classes of sixth grade students, two classes from the Westside Elementary School and two from the Lincoln Elementary School in Pittsburg, Kansas, were involved in the experiment with these materials. A single teacher in each school presented both approaches, the experimental group material to one class and the control group material to the other class. The teachers, the one a man and the other a woman, had received training to help them present the materials for both approaches. Since one of the two classes in each school was the instructor's homeroom class, one instructor presented the control group material to his homeroom class; the other instructor presented the experimental material to his homeroom class. Which instructor presented which material to his homeroom class was determined by the flip of a coin. Since both approaches involved new and different materials from those used in the classrooms, the so-called Hawthorne effect should have been of no advantage to either group in this experiment.

Pittsburg, with nineteen to twenty thousand people, is a typical midwestern, college town. Emphasis is placed on education and the teachers as a group are outstanding. A laboratory school is maintained by Kansas State College on its campus. However, this experiment was conducted in the Pittsburg Public Schools to obtain a more typical classroom situation with typical students.

CHAPTER II

PERSONNEL AND EXPERIMENTAL PROCEDURES

Information concerning the population from which the sample for this study was drawn, the test instruments used, and the statistical treatment and methods employed to test the hypothesis of this study will be given in this chapter.

Personnel for the Study

In the previous chapter in the section on Scope of the Study, we indicated the process of selecting the sample for this study. It was necessary for the investigator to work with existing classes, during the 1965-66 school year, as a result of administrative scheduling. Since the experimental treatment involved one teacher teaching two classes, each by means of a different approach, the selection of the sample narrowed to the school, teacher, and number of pupils as indicated in Table I. Other factors that entered into the consideration and final selection of the sample were the availability of two classes in the same school and the willingness of administration and teachers to be involved.

One man and one woman were selected to teach two classes, one by each approach. Mrs. Virginia Romondo, sixth grade teacher at Lincoln School in Pittsburg, Kansas received both her bachelor's and master's degrees at Kansas State College of Pittsburg and has taught for seven

TABLE I

SCHOOL, EXPERIMENTAL TEACHER, TEACHER, AND NUMBER OF PUPILS INVOLVED IN THE STUDY

Class I.D.	School	Experimental Teacher	Teacher	No. of Pupils
1	Westside	Fisher	Elsten-Fisher	23
2	Lincoln	Romondo	Goble-McCallum	18
3	Westside	Fisher	Porter	24
4	Lincoln	Romondo	Romondo	22

years. Mr. Harold Fisher is building principal and sixth grade teacher at Westside. He teaches a half day and attends to administrative responsibilities the other half. Mr. Fisher also received his bachelor's and master's degrees at Kansas State College and has taught for eighteen years. Both teachers are recognized as outstanding teachers in the Pittsburg School System. Mr. Fisher and Mrs. Romondo were students in an on-campus class the semester prior to the experiment. The class entitled "Modern Mathematics for Elementary Teachers" was taught by the investigator for this study. Both approaches for introducing the integers as used in this study were presented to the class, thereby giving the two experimental teachers a working knowledge of both approaches.

Tests Used in the Study

The chief purpose of testing is to permit judgments to be made concerning the individuals being tested. If those judgments are to have any real merit, they must be based on dependable scores--which, in turn, must be earned on dependable tests. If the measuring instrument is unreliable, any judgments based on it are necessarily of doubtful worth.

> It is a statistical and logical fact that no test can be valid unless it is reliable; knowing the reliability of a test in a particular situation, we know the limits beyond which validity in the situation cannot rise. (46, p. 2)

In reporting the following tests used in this study, special attention will be given to the reliability of each.

Three tests were used to provide data needed for this study. They are the Metropolitan Achievement Tests (Intermediate Battery), the Otis Quick-Scoring Mental Ability Tests (Beta Form), and the "Suggested Quizzes" at the end of each section of material covered in the Greater Cleveland Mathematics Program. From the first two sources of data, information was available on each student's I.Q. and achievement level early in his sixth grade year. The quizzes were used to measure the students' achievement for both approaches.

The Metropolitan Achievement Tests and the Otis Quick-Scoring Mental Ability Test were given as group tests and administered to the subjects of this study when they were in the fifth grade. The quizzes, used to measure student achievement in the particular topic under consideration, were given at the end of the experiment, in late March of 1966. The following is a description of each test used:

1. Metropolitan Achievement Tests

The Metropolitan Achievement Tests (MAT) are a coordinated series of measures of achievement in the important skill and content areas of the elementary and junior high school curriculum. The main purposes of the test as envisioned by its authors were to contribute to teacher understanding and analysis of pupils' achievement and to provide dependable data for evaluation of pupil growth.

The Tests are organized into five levels; the Intermediate Battery, for use in grades 5 and 6, was used for this study. At each level MAT includes tests covering the most important knowledge or skill areas in the grades for which that level is intended. There are four regularly published forms for the Intermediate Battery. All the forms are comparable as to difficulty and content, are equally good measures of the respective subjects, and yield comparable results. The form A was used for all subjects for this particular study.

The publishers of MAT have attempted to insure the validity of their tests by basing them on thoroughgoing analyses of the textbooks, courses of study, expert formulations of the goals of instruction in the various elementary branches and by subjecting the content to rigorous experimental tryout prior to publication. Data on the reliability of the different tests of the Intermediate Battery are furnished, with two tests being of particular concern to this study. The data in Table II consist of split-half coefficients computed separately for pupils in each of several schools and the standard errors of measurement in raw score terms as furnished by the publishers.

The Arithmetic Computation subtest was used in this study in an attempt to establish the equivalence of the two groups under consideration. Arithmetic Computation is a 48-item test which covers fundamental operations with whole numbers, decimals and fractions, through fractional parts of numbers, reading of graphs, and addition and subtraction of denominate numbers.

TABLE II

RELIABILITY COEFFICIENTS AND STANDARD ERRORS OF

MEASUREMENT FOR SUBTESTS*

	Test	r_{11}** Range	Mdn.	*** Range	Mdn.
1.	Word Knowledge	.88-.95	.94	3.0-3.4	3.1
2.	Reading	.89-.92	.90	2.5-2.8	2.6
3.	Spelling	.91-.96	.92	2.6-3.5	3.0
4.	Language:				
	Part A Usage	.78-.84	.81	1.9-2.5	2.2
	Part B Parts of Speech	.64-.77	.72	1.3-1.3	1.3
	Part C Punctuation and Capitalization	.80-.88	.83	2.1-2.4	2.2
	Total (Parts A-C)	.87-.91	.89	3.3-3.5	3.3
5.	Language Study Skills	.76-.85	.79	2.0-2.4	2.2
6.	Arithmetic Computation	.82-.94	.88	2.1-2.7	2.4
7.	Arithmetic Problem Solving and Concepts	.90-.95	.92	2.2-2.5	2.4
8.	Social Studies Information	.86-.87	.87	3.3-3.5	3.4
9.	Social Studies Study Skills	.64-.77	.73	2.2-2.5	2.2
10.	Science	.87-.90	.89	2.8-3.3	3.0

* Taken from Directions for Administering, Metropolitan Achievement Tests, Intermediate Battery. Walter N. Duast, General Editor. Harcourt Brace and World, Inc., New York (1959), p. 20.

** Values reported are ranges and medians of four independent estimates of corrected split-half coefficients. Each estimate is based on a random sample (N = 100) of grade 6.1 pupils from a single school system, the four systems being chosen to typify high, low, and average performance on the test.

*** Standard error of measurement in terms of raw score.

The I.Q. scores obtained from the Otis Quick-Scoring Mental Ability Test were also used for the purpose of verifying the equivalence of the two groups to be studied.

2. Otis Quick-Scoring Mental Ability Tests

The Otis Quick-Scoring Mental Ability Tests comprise three tests: Alpha, Beta, and Gamma. The Beta test, designed for grades 4 through 9, was used to measure the mental ability of the subjects for this study. The purpose of each of these tests is to measure mental ability, the thinking power or the degree of maturity of the mind.

Although it is impossible to measure mental ability directly, it is possible to measure the effects mental ability has had in enabling the pupil to acquire certain knowledge and mental skill. The ability to answer some types of questions depends less upon schooling and more upon mental ability. In constructing the test, the aim was to choose that kind of question which depends as little as possible on schooling and as much as possible on thinking.

There are six forms of the Beta test. Form A, which was used to test the subjects of this study, is self-administering and hand scored.

The reliability and validity of the Beta test have been established in several different ways. The average of the coefficients of correlation between two forms of the test is .79, while the average of the coefficients of correlation between odd and even items of a single test corrected by the Spearman-Brown Formula is .86.

Another measure of reliability, which is entirely independent of the degree of heterogeneity of the group, is the standard error of measurement; for 465 pupils in grade 4 to 9 this was determined to be 4.0 points. This means, a pupil's score will be in error not more than 4.0 points in 66 2/3 per cent of the cases.

The degree to which a test measures the ability it is designed to measure is referred to as the validity of the test. The purpose of the Beta test is to obtain some measure (such as the I.Q.) that indicates the probable rate of progress the pupil will make in school, thus indicating the appropriate criterion for validity. The determination of the validity of each item consisted of comparing the number of passes of that item by a group of pupils who were making rapid progress through school with the number of passes of the item by a group of pupils who were making slow progress through school. Only those items were used by the authors of the Beta test which showed a distinct gain in number of passes of the rapid-progress pupils over the number of passes of the slow-progress pupils. Each item justified its inclusion, therefore, because it definitely contributed to the capacity of the test to measure brightness as reflected in rate of progress through school.

3. "Suggested Quiz" from the Greater Cleveland Mathematics Program

The Greater Cleveland Mathematics Program (GCMP) is an on-going experimental program being conducted by the Educational Research Council of Greater Cleveland. GCMP has written and revised its materials and tests throughout several years of experimentation. The programs in the grades one through three have achieved final acceptance by the Educational Research Council and research is available on the established tests that accompany the materials. A report on research includes test reliability, item difficulty, and the relationship between various sub-test scores, the total scores, and the Lorge-Thorndike I.Q. scores. (47) This research indicates the extensive testing program carried on by the GCMP and should indicate some validity and reliability to quizzes used from later programs.

However, results are not available on tests for the intermediate grade level materials for the Greater Cleveland Mathematics Program as yet. Reliability of these quizzes were measured, using the data of this study, by the Kuder-Richardson Formula #20. This method, often employed to determine test reliability, or interitem consistency, is based upon the consistency of the subjects' responses to all items in the test. This technique employs an examination of performance on each item found from a single administration of a single test. The reliability coefficient (r_{11}) of the whole test is determined by the formula:

$$r_{11} = \left(\frac{n}{n-1}\right) \frac{\sigma_1^2 - \Sigma pq}{\sigma_t^2}$$

where n is the number of test items in the test, σ_t the standard deviation of total scores on the test, and Σpq is the sum of the products of the proportion of persons who pass (p) and the proportion who do not pass (q) for all items. Such reliability coefficient provides a measure of both equivalence and homogeneity. Whereas the split-half coefficient is based upon a planned split designed to yield equivalent sets of items, the Kuder-Richardson reliability coefficient is actually the mean of all split-half coefficients, resulting from differing splittings of a test. The reliability coefficient for this test was determined to be 0.88.

Abstract Approach Materials

The writing of the materials to be used by the experimental group using abstract concepts was accomplished in the following manner.

It was necessary to decide approximately how much material should be developed to constitute a legitimate experiment. A survey of several existing modern elementary school mathematics programs indicated that a logical development first introduced the "new" numbers, compared them, and then proceeded to add, subtract, multiply and divide them. (39) (40) (41) (42) There appeared to be natural breaks between comparison of the newly introduced numbers and the operations with numbers, and again between the operations of addition and subtraction and of multiplication and division. Division correlates easily with multiplication; subtraction goes hand-in-hand with addition. Several programs terminated the study of integers after the introduction of addition and subtraction. (39) (40) (41) Others continued with the other operations after an intervening lesson or two on other topics. (42) Addition and subtraction are inversely related and are logically developed together. Hence, a development through the operations of addition and subtraction was determined to be the information to be introduced and to be of sufficient depth to constitute a legitimate experiment.

The following model (Figure 1) was constructed and used by the investigator as a guide in writing the experimental materials. The foremat for the abstract was similar to that of the concrete materials. Instructions were worded and arranged similarly, and comparable amounts of practice exercises were provided.

The investigator then used the abstract materials with a group of in-service elementary teachers enrolled in an extension class in "Modern Mathematics for Elementary Teachers" in Baxter Springs, Kansas. The primary purposes for using the material were to proof read for errors, to determine if logical development progressed to desired conclusions, and to have in-service teachers help adjust the vocabulary used in the directions to the sixth grade level of comprehension.

Several of the teachers in this extension class used the material in their classrooms. Following are their comments and the inferences drawn by the investigator:

> The problems which are worked were worked by a "top" fourth grade student who saw my paper on my desk and began asking questions. He worked on his own <u>entirely</u> except for answering the one question as to what "<" and ">" meant."

The investigator had apparently not made it clear that the materials were not programmed materials, but needed teacher explanations and directed practice periods. This same teacher commented, "I feel the better fourth grade pupil could handle this."[*]

Another added:

> In my opinion this would be somewhat difficult for the average sixth grade student without some oral explanation from the teacher.[*]

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[*]Statements by members of the Baxter Springs extension class pertaining to the abstract materials written by the investigator to introduce integers.

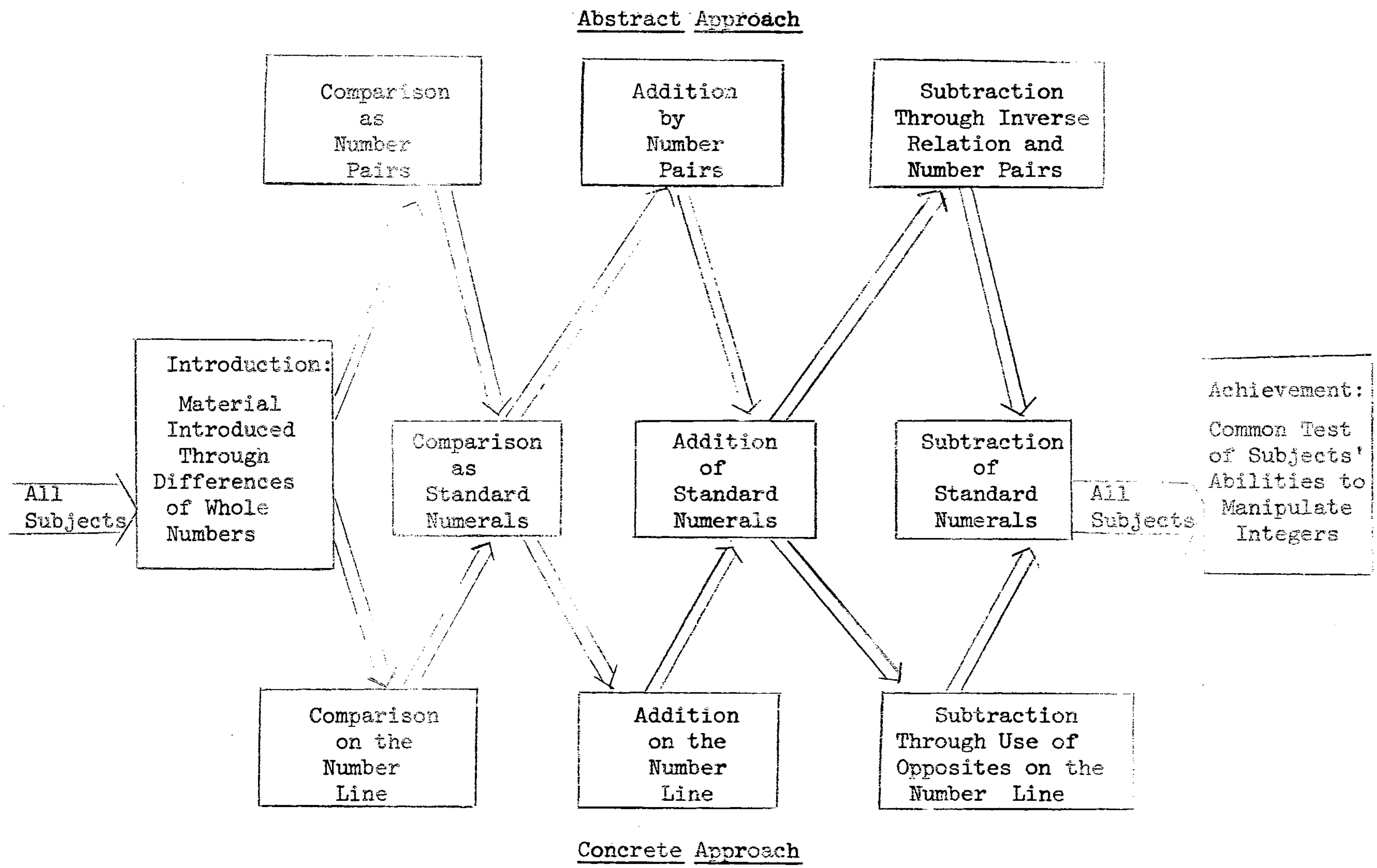

Figure 1. Model of Approaches Used in This Experiment

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The investigator realized this and agreed that not only oral explanations but many examples were necessary, as was true with the concrete materials also.

One teacher commented on the instructions:

> The instructions are well written but the average child would not be able to understand what he is to do without a period of drill and explanation from the teacher.*

The difficulties in understanding the explanations were overcome during this experimental period and a revision of the materials was made before the actual experiment was conducted. (See Appendix B for the Abstract materials).

Basis for Equating Groups

The basis for equating the experimental groups for this investigation was the equivalent groups methods.

> When the purpose of an experiment is to determine the amount of change due directly to an [experimental factor], the equivalent-groups method is valid.
>
> (1) Where the total net change in the trait or traits in question produced by irrelevant factors is negligible, or where the amount of such change is measured and discounted by the use of a control experimental factor.
>
> (2) Where it is readily possible to equate groups. (48, p. 29)

The distinctive features of this method are that there is more than one group, normally as many as there are experimental factors and that all groups are equivalent. The equivalent-groups method is superior to the one-group method in that the carry-over from one experimental

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* Statements by members of the Baxter Springs extension class pertaining to the abstract materials written by the investigator to introduce integers.

factor to another is avoided by applying each treatment to a different group rather than following one experimental factor with another on the same group.

Equivalence of groups does not indicate that all subjects participating in the experiment be equivalent, but that the groups should be equivalent. To be equivalent the various groups must have like means and like variability, but it is not required that there be an equal number of subjects in each group.

Since it was necessary for the investigator of this study to work with classes already established as a result of administrative scheduling consideration was given to the mean scores of the established classes to determine the best experimental grouping. Table III indicates the different classes and their mean scores from available tests that seem pertinent to this study.

TABLE III

MEAN SCORES ON PERTINENT AVAILABLE TESTS FOR THE CLASSES INVOLVED IN THE EXPERIMENT

Class	No. Students	Arith Comp.**	I.Q.***
1	23*	49.04	98.86
2	18	51.55	106.17
3	24	47.21	104.25
4	22	53.18	106.32

* No. Metropolitan Achievement Test Scores on two pupils.

** Arithmetic Computation Raw Scores from _Metropolitan Achievement Tests_.

*** I.Q. Score determined from the _Otis Quick-Scoring Mental Ability Test_.

The experimental design of this study involved the experimental teacher at each school teaching both approaches, one to each class. Therefore, the experimental groups had to be formed from associating class 1 with class 2 (See Table I) for one approach, and class 3 and class 4 to comprise the other experimental group; or class 1 with 4, and 2 with 3. However, since one of the two classes for each experimental teacher was his own homeroom, it was determined advisable to use one approach for one homeroom and the other approach for the other homeroom. Therefore, it was necessary to have classes 1 and 2 form one experimental group, and classes 3 and 4 the other experimental group. Table IV indicates the mean scores for the two experimental groups on the Arithmetic Computation Test from the _Metropolitan Achievement Tests_, and the I. Q. scores obtained from the _Otis Quick-Scoring Mental Ability Test_.

TABLE IV

MEAN SCORES ON ARITHMETIC COMPUTATION AND I. Q.

FOR THE EXPERIMENTAL GROUPS

Groups	Arithmetic Computation	I. Q.
1-2	50.13	102.15
3-4	50.07	105.24

The Arithmetic Computation score was considered the more important of the two arithmetic scores available from the MAT as an indication of desired ability to be equated since this study deals primarily with the achievement of the ability to manipulate signed numbers rather than concepts. The I. Q. score is an over-all indication of the subjects' abilities and is always considered a likely tool to compare the equivalence of two groups.

An analysis of the mean scores and the variances on the Arithmetic Computation and the I. Q. scores for the two experimental groups was made after the normally expected mortality occurred during the experimental testing period and is reported later in this study.

Experimental Treatment

The two schools determined as the site of this experiment had two sixth grade classes that were to be involved in this study. The experimental teacher for each school was to teach both approaches, each to a separate class, to minimize the effect of the teacher in favor of either approach. Since one class in each school was the experimental teacher's homeroom class, one used the concrete approach in his homeroom while the other used the abstract approach. Through this procedure it was believed possible to minimize the effects of previously established rapport in favor of either approach.

Once the experimental design had established the experimental grouping of the classes, the only question that remained was which group would use which approach. This was determined by a flip of the coin between the two experimental teachers. At the beginning of the experiment it was determined that Mr. Fisher at Westside would use the concrete approach with his homeroom and Mrs. Romondo would use the abstract approach with her homeroom. Table V indicates the experimental teacher, the school, and the homeroom designation for both approaches. The identification A_3 indicates the abstract approach (A) was used in class number 3 and HR is used to indicate the experimental teacher's homeroom.

TABLE V

EXPERIMENTAL CLASSES ASSIGNED TO DIFFERENT APPROACHES AND HOMEROOM DESIGNATION

Concrete			Abstract		
C_1	Fisher (HR)	Westside	A_3	Romondo (HR)	Lincoln
C_2	Romondo	Lincoln	A_4	Fisher	Westside

The experiment was conducted in the classroom from March 7 to March 25, 1966, a total of three weeks or 15 school days. Class time allotted to mathematics in each of the classes varied from 45 to 55 minutes per day, but each experimental teacher attempted to allow both approaches equal time each day. The classes C_1 and C_2 used the concrete or visual approach, which makes use of the number line. The <u>Greater Cleveland Mathematics Program</u>, Intermediate Series, Book 7, 1964-65 edition, pages 27-70 with selected pages omitted was used with this experimental group. The classes A_3 and A_4 used the abstract or algebraic approach material written by the investigator. (See Appendix B) Testing time was included in the stated 15 days and consisted of two quizzes, one near the middle (Thursday, March 17) and one at the conclusion. (See Appendix C) Testing sessions were conducted by the investigator under circumstances as similar as possible.

Statistical Methods

The hypothesis of this study was tested for statistical significance using the t-test with equated groups. The formula used was:

$$t = \frac{\overline{X}_C - \overline{X}_A - \mu_{\overline{X}_C - \overline{X}_A}}{\sigma_{\overline{X}_C - \overline{X}_A}} \quad ,$$

where $\sigma_{\overline{X}_C - \overline{X}_A} \sqrt{\dfrac{\sigma\, x_C^2}{n_C} + \dfrac{\sigma\, x_A^2}{n_A}}$,

and $\mu_{\overline{X}_C - \overline{X}_A} = \mu_{X_C} - \mu_{X_A} = 0$. (49, p. 147)

The subscripts C and A indicate that the statistical tool was applied to either the concrete (C) or abstract (A) experimental group's achievement scores. A two-tailed test of significance was made with the rejection level fixed in advance at the .05 level.

Assumptions

The following assumptions were made in the process of conducting this study:

1. That the sample of students included in the study was a representative sample of sixth grade students in Pittsburg, Kansas.

2. That the distribution of the scholastic abilities of the students included in the sample did not deviate seriously from a normal distribution.

3. That the Arithmetic Computation Scores on the *Metropolitan Achievement Tests* and the I. Q. scores from the *Otis Quick-Scoring Mental Ability Test* could be used to measure the scholastic ability of the students involved in the study, and could be used as a control variable in the experimental design.

4. That the two approaches would be relatively successful in teaching signed numbers.

5. That the materials for the different approaches were used in the classes to which they were assigned.

6. That the underlying philosophy for each approach was accepted by the experimental teachers and that the approach procedures were used only with the indicated approach.

7. That the "suggested quizzes" from the Greater Cleveland Mathematics Program could measure achievement in manipulation of signed numbers for both approaches.

Limitations of the Study

There were several limiting factors apparent in this study, and, as a result, certain restrictions must be placed on the findings and conclusions of the study.

The total number of students involved was limited to only 87. The data for eight students were lost to the study as a result of absences and lack of test data. When one considers the large number of sixth grade students in any one year, 79 seems to be a very small sample. The power of a test is defined as the probability of rejecting the null hypothesis when it is in fact false, and the power of a statistical test increases with an increase in the size of the sample.

The differences in the ability or influence of the individual teachers were eliminated or minimized in the experimental design. However, the small number of teachers and classes might be considered as a limitation of this study.

It was assumed that variations existed in the students from class to class within and between schools. The only comparisons made were in achievement in arithmetic computation and I. Q. scores, which consider few other characteristics. Little attempt was made to ascertain motivation of individual students, educational philosophy of individual teachers or schools, or the influence of geographical location within the city where the experiment was conducted. The author assumed that the effects of these factors were minimized by random assignment or by statistical treatment of the collected data.

The variable size of the experimental groups might be considered by some as a limitation of the study. However, Siegel comments:

> . . . two samples may be obtained by either of two methods: (a) they may each be drawn at random from two populations, or (b) they may arise from the assignment at random of two treatments to the members of some sample whose origins are arbitrary. In either case it is not necessary that the two samples be the same size. (50, p. 95)

Snedecor also agrees, "There is no necessity that the two groups be of the same size." (51, p. 80)

Summary

The personnel for the study were those students already formed into classes by administrative scheduling in the public schools in Pittsburg, Kansas. Selection of classes to form experimental groups depended upon several factors, with final criterion being the equivalence of the two groups. The statistical design involved two experimental teachers, each teaching two classes and both approaches, with one teaching the concrete approach to his homeroom class while the other taught the abstract approach to his homeroom.

A general discussion of the process used by the investigator to write the abstract materials included a comparison with the concrete material published by the GCMP.

Tests used in the study were as follows: 1) The Arithmetic Computation scores of the Metropolitan Achievement Tests, and 2) The I. Q. scores obtained from the Otis Quick-Scoring Mental Ability Tests were used as the basis for equating the experimental groups. 3) The "Suggested Quizzes" that follow the related materials of the Greater Cleveland Mathematics Program were used to measure the achievement of the subjects with reference to manipulating signed numbers.

The t-test for equivalent groups was used to determine the significance of the difference between the mean scores of the two groups on the dependent variable.

CHAPTER III

ANALYSIS OF THE DATA AND RESULTS OF THE STUDY

In this chapter an analysis of the data collected in the study will be presented, and the results of the test of the hypothesis will be stated.

The statistical data will be summarized in tabular form, and a short explanation will be given for each table. The raw data collected for each student included in the study appear in Appendices D and E.

Equivalent Groups

The two groups were equated on the basis of arithmetic computation scores on the Metropolitan Achievement Test and I. Q. scores on the Otis Quick-Scoring Mental Ability Test. Tables VI and VII present summaries of the data used in establishing the equivalence of the two groups used in this experiment. A comparison with Table III in Chapter II will indicate a difference of eight in the total number of subjects in the group using the concrete approach. Standardized test data were incomplete for two students in class C_1. Five students in class C_1 did not complete the experimental treatment as a result of illness; one student in class C_2 moved before the experiment was complete. Therefore, the experimental classes using the concrete approach numbered 16 and 17 for a total of 33 subjects. The classes using the abstract materials remained intact throughout the experiment.

TABLE VI

SUMMARY OF DATA FOR EQUIVALENT GROUPS

Items in Analysis	Concrete Group	Abstract Group
Number of Subjects	33	46
Mean, Arithmetic Computation	51.30	50.07
SD, Arithmetic Computation	9.44	7.36
Mean, I.Q. Scores	105.12	105.24
SD, I.Q. Scores	9.89	9.94
t-test Ratio, Arithmetic Computation	0.624	
t-test Ratio, I.Q. Scores	0.053	

The means and standard deviations of the two groups differed slightly on both measurements. This difference in the mean scores between the two groups was statistically analyzed by the t-test. The t-test ratio of 0.624 for arithmetic computation and 0.053 for I.Q. scores substantiated the assumption that no significant difference existed between the means of the two groups.

To establish the equivalence of experimental groups, it is also necessary to consider the variances of the two groups. McCall states: "To be equivalent the various groups must have like <u>means</u> and like <u>variability</u> among the subjects constituting each group." (48, p. 40) He goes on to comment ". . . it is not absolutely required that there be an equal number of subjects in each group. The essential is that the groups be equivalent as to means and variability." (48, p. 41) The assumption that no significant difference existed between the mean scores of the two groups was substantiated; the significance of the difference between the variances of the two groups was also determined.

The formula for testing for significance of the difference between the variances employs the variance ratio

$$F = \frac{s_g^2}{s_1^2}, \quad (52, p. 134)$$

where s_g^2 is the variance in the sample group with the greater variance and s_1^2 is the variance in the sample group with the lesser variance. Table VI includes the variances of the two groups and calculated F ratios.

TABLE VII

VARIANCES OF CONCRETE AND ABSTRACT GROUPS ON
ARITHMETIC COMPUTATION AND IQ SCORES
AND CALCULATED F RATIOS

Measurements	Concrete Group	Abstract Group
Variances, Arithmetic Computation	89.25	54.19
Variances, I.Q. Scores	97.94	98.83
F ratio, Arithmetic Computation	$F_{32,45} = 1.646$	
F ratio, I.Q. Scores	$F_{45,32} = 1.009$	

Since the choice of the numerator in this formula for F is determined by the greater numerical magnitude and not logical consideration, this " . . . arbitrary placing of the larger variance in the numerator doubles the probability of obtaining deviations about the mean." (53, p. 193) Therefore, the 5 per cent and 1 per cent levels indicated in the F-table must be interpreted at the 10 per cent and 2 per cent levels.

The subscripts of F indicate the degrees of freedom, the first being the number of degrees of freedom associated with the numerator, and the second is the number of degrees associated with the denominator. The table of F with 32 (n_1-1) degrees of freedom across and 45 (n_2-1) degrees of freedom down revealed, by interpolation, F = 1.71 at the 5 per cent and 2.12 at the 1 per cent level of significance. For the purpose of evaluating the variance ratio, then F - 1.71 is at the 10 per cent level and F = 2.12 is at the 2 per cent level. The obtained F-value of 1.646 for the variances on the arithmetic computation scores between sample groups, though approaching the arbitrarily established significance level, is not significant since it does not exceed either value found from the table. In a similar computation, the obtained value of $F_{45,32}$= 1.009 for the variances on I.Q. scores between sample groups is not significant since it does not exceed the values of F = 1.75 at the 10 per cent level or F - 2.23 at the 2 per cent level.

The assumption that no significant difference existed between the two groups on both the arithmetic computation scores and the I.Q. scores has been substantiated by the statistical tools, the t-test and the F ratio. Therefore, the two experimental groups were considered to be equivalent at the beginning of the experiment since no significant difference existed between either their mean scores or their variances.

Tests of Stated Hypothesis

The hypothesis tested in this study was concerned with the achievement of equivalent groups in manipulations and operations with signed numbers at the sixth grade level.

This hypothesis in null form is as follows: No significant difference exists between the tested achievement of two groups of sixth graders who were introduced to the study of integers through two different approaches. To test this hypothesis a statistical comparison of the achievement of the two equivalent groups of students was made.

All members of one group of 33 students studied signed numbers from GCMP materials using the concrete, or visual, approach while the other group, 46 in number, received their training in signed numbers using abstract, or algebraic, materials prepared by the writer.

Two quizzes, one near the middle and one at the conclusion of the experiment, established the achievement score for the subjects in both groups. The achievement mean and the standard deviation of each group on scores obtained from the quizzes and the t-test ratio comparing the achievement of the two groups are given in Table VIII.

The t-test ratio for the achievement scores made by the two groups was 0.57. This low t-test ratio indicates that for these two equivalent groups of sixth grade students the difference of 0.76 between the achievement means was not statistically significant.

Therefore, the null hypothesis is not rejected and any group differences found in student achievement is accepted as the result of chance.

The conclusion was drawn from these data that sixth grade students taught manipulation of signed numbers from an abstract, algebraic procedure achieve equally as well as students taught manipulation of signed numbers from a concrete, number line approach when equivalent groups are compared.

TABLE VIII

SUMMARY OF ACHIEVEMENT SCORES AND T-TEST RATIO

Items in Analysis	Concrete Group	Abstract Group
Number of Subjects	33	46
Mean, Achievement Scores	28.30	27.54
SD, Achievement Scores	5.65	6.19
Difference Between Means	0.76	
t-test Ratio	0.57	

Summary

The results of the statistical analysis of the data collected for the study have been presented in Chapter III.

The t-test and F ratio were used to determine if significant differences existed between the means and variances of the two groups on computation scores from the Metropolitan Achievement Test and I.Q. scores from the Otis Quick-Scoring Mental Ability Test at the beginning of the experiment. The low t-test ratios and F ratios substantiated the assumption that no significant differences existed between the two groups.

The t-test was used to compare differences in achievement between equivalent groups of sixth grade students. One group was taught manipulation of signed numbers from a concrete, number line approach, while the other group received instruction in an abstract, algebraic approach.

No statistically significant difference was found in the achievement of the two groups.

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

The major objective of this study was to compare achievement in manipulation of signed numbers when different approaches were used. The study was limited to a small population from one city. Generalizations are limited to the population that was samples.

Summary and Conclusions of the Study

This study was designed to present experimental evidence secured at the sixth grade level that might help to 1) determine if student achievement in manipulation of signed numbers is equivalent when taught by means of a concrete, visual approach and by means of an abstract, algebraic approach, and 2) provide information that could be used by educators and interested citizens to evaluate present and future trends in mathematics education.

The null hypothesis was stated that no significant difference in student achievement in manipulation of signed numbers would be found at the .05 level of confidence between equivalent groups using the abstract and concrete approach.

The equivalent groups design was used, and groups established by administrative scheduling were equated " . . . as much alike as possible." (54, p. 232) After the experiment was conducted and normal attrition occurred, an analysis of existing groups was conducted to determine significance of differences. The t-test ratio of 0.624 for arithmetic computation scores and 0.053 for I. Q. scores substantiated the assumption that no significant difference existed between the means of the two groups. The F ratio was used to test for significance of differences between variances. An F ratio of 1.646 on arithmetic computation and 1.009 on I. Q. scores, with the appropriate degrees of freedom, was far short of significance at the 2 per cent level of confidence. This analysis indicated that the two groups did not differ significantly.

A comparison was made of the achievement scores between the two equivalent groups. The calculated t-test value for these data was 0.57, far under the .05 level of confidence specified for significance in the design of this study. Therefore, the null hypothesis was not rejected.

On the basis of this research and subject to the specified limitations, the following conclusions were made by the writer:

1. Sixth grade students taught manipulation of signed numbers from an abstract, algebraic approach achieved as well as those taught by means of a concrete, visual approach when students with equated abilities were compared.

2. This result should alleviate some of the concern of educators over the introduction of algebraic procedures at earlier levels in the educational process.

3. It can also be interpreted as some indication of the attainability of the very ambitious recommendations for curriculum reform as presented in the Cambridge Report, <u>Goals for School Mathematics</u>.

Recommendations for Further Study

There are a number of problems that have been suggested by this study that are of importance to mathematics education. The writer hopes that this study will provide some background for other studies in the same area of interest. The writer therefore makes the following recommendations as the result of this study:

1. Research that covers a longer time interval and other populations should be conducted.

2. More research is needed to determine at what grade levels abstract or concrete materials can be used to the best advantage for students and teachers.

3. Research should be conducted to determine the type of student that can profit most from the abstract or the concrete approach.

4. Research into the varying degrees of abstract and concrete materials that can be used cooperatively in accomplishing desired goals should be conducted.

5. Research should be conducted on levels of understanding and concept formation with regard to concrete and abstract materials.

6. Research should be conducted concerning the previous experiences of the teacher and the effects of teacher attitude upon the successful usage of a particular approach.

7. Research should be conducted to determine significant factors that influence students' attitudes toward different teaching approaches.

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APPENDIX A

SCHOOLS AND TEACHERS PARTICIPATING IN THE STUDY

Letter Identification	Name of School	Location	Teacher
C_1	Westside Elementary	Pittsburg, Kansas	Fisher (HR)*
C_2	Lincoln Elementary	Pittsburg, Kansas	Romondo
A_3	Westside Elementary	Pittsburg, Kansas	Fisher
A_4	Lincoln Elementary	Pittsburg, Kansas	Romondo (HR)*

*(HR) - Homeroom

APPENDIX B

MATERIALS WRITTEN BY THE INVESTIGATOR TO BE USED BY THE EXPERIMENTAL GROUP USING THE ABSTRACT APPROACH

INTRODUCTION TO THE SET OF INTEGERS

The differences of the following whole numbers are all whole numbers. For example, 5 - 1 is 4; 4 - 0 is 4; 16 - 7 is 9; and many others.

Determine the following differences of two whole numbers.

1) 6 - 3 is ____________

2) 2 - 0 is ____________

3) 9 - 4 is ____________

4) 12 - 10 is ____________

5) 15 - 4 is ____________

6) 13 - 13 is ____________

7) 8 - 1 is ____________

8) 22 - 6 is ____________

9) 31 - 16 is ____________

10) 134 - 15 is ____________

11) 213 - 152 is ____________

12) 1009 - 246 is ____________

13) 2432 - 1312 is ____________

14) 5000 - 436 is ____________

Every whole number can be written as the difference of two whole numbers. For example, 4 is 5 - 1; 17 is 23 - 6; 9 is 9 - 0; and others.

Write each whole number in the form of a difference of two whole numbers.

1) 2 is ________ - ________

2) 13 is ________ - ________

3) 7 is ________ - ________

4) 16 is ________ - ________

5) 21 is ________ - ________

6) 15 is ________ - ________

7) 27 is ________ - ________

8) 31 is ________ - ________

9) 29 is ________ - ________

10) 54 is ________ - ________

11) 128 is ________ - ________

12) 78 is ________ - ________

13) 101 is ________ - ________

14) 92 is ________ - ________

Each whole number can be written as the difference of two whole numbers in many ways. For example, 4 is 5 - 1, 12 0 8, 54 - 50, 4 - 0, and many others. Write the following whole numbers as several differences.

1) 6 is ________, and ________, and ________, and ________.

2) 12 is ________, and ________, and ________, and ________.

3) 1 is ________, and ________, and ________, and ________.

4) 19 is ________, and ________, and ________, and ________.

5) 0 is ________, and ________, and ________, and ________.

6) 23 is ________, and ________, and ________, and ________.

7) 47 is ________, and ________, and ________, and ________.

For each whole number n, the difference n - 0 is the <u>simplest form</u>. For example, 4 is 4 - 0, 13 is 13 - 0, 2 is 2 - 0, and others. Write the following whole numbers as difference in the <u>simplest form</u>.

1) 9 is ________________________.

2) 5 is ________________________.

3) 15 is ________________________.

4) 0 is ________________________.

5) 18 is ________________________.

6) 24 is ________________________.

7) 32 is ________________________.

8) 48 is ________________________.

9) 22 is ________________________.

10) 74 is ________________________.

11) 129 is ________________________.

12) 231 is ________________________.

13) 1001 is ________________________.

14) 2131 is ________________________.

Let us agree that (a,b) means the same as a - b, where a and b stand for any whole number. For example: (4,2) means (- 2.

Other examples:

a) (6,1) means <u>6 - 1.</u>

b) (9,0) means <u>9 - 0.</u>

Express the following as the difference of the indicated whole numbers.

Exercises:

1) (8,4) means ______ - ______

2) (2,1) means ______ - ______

3) (0,0) means ______ - ______

4) (17,9) means ______ - ______

5) (9,2) means ______ - ______

6) (6,6) means ______ - ______

7) (23,5) means ______ - ______

8) (31,12) means ______ - ______

9) (48,6) means ______ - ______

10) (21,20) means ______ - ______

11) (41,11) means ______ - ______

12) (72,41) means ______ - ______

13) (14,9) means ______ - ______

14) (52,14) means ______ - ______

15) (29,9) means ______ - ______

Every whole number can be written as the difference of two whole numbers or as the pair of whole numbers separated by a comma, called a <u>number pair</u>. For example, 6 is 7 - 1 or (7,1); 17 is 23 - 6 or (23,6); and others.

Express the following whole numbers as the difference of two whole numbers and as a number pair.

1) 3 is ___, or (__,__)

2) 4 is ___, or (__,__)

3) 14 is ___, or (__,__)

4) 17 is ___, or (__,__)

5) 25 is ___, or (__,__)

6) 29 is ___, or (__,__)

7) 33 is ___, or (__,__)

8) 39 is ___, or (__,__)

9) 43 is ___, or (__,__)

10) 51 is ___, or (__,__)

11) 62 is ___, or (__,__)

12) 79 is ___, or (__,__)

13) 108 is ___, or (__,__)

14) 142 is ___, or (__,__)

The number pair (a,b) can be associated with the whole number which results from a - b. For example: (4,1) ⟶ 3, since 4 - 1 is 3.

Associate the following number pairs with the whole number which results.

1) (2,0) ⟶ ________, since 2 - 0 is ________.

2) (6,2) ⟶ ________, since 6 - 2 is ________.

3) (7,4) ⟶ ________, since ___-___ is ________.

4) (12,5) ⟶ ________, since ___-___ is ________.

5) (13,8) ⟶ ________

6) (8,0) ⟶ ________

7) (15,6) ⟶ ________

For every whole number n, the number pair (n,0) is in the simplest form and is associated with n which results from n - 0. For example: (5,0) ⟶ 5, since 5 - 0 is 5.

Write the following number pairs in simplest form and the resulting whole number. For example: (4,2) ⟶ (2,0) ⟶ 2, since 2 - 0 is 2.

1) (9,8) ⟶ (1,0) ⟶ ___, since ___-___ is ___.

2) (11,4) ⟶ (__,0) ⟶ ___, since ___-___ is ___.

3) (7,1) ⟶ (6,0) ⟶ ___.

4) (8,2) ⟶ (__,0) ⟶ ___.

5) (12,12) ⟶ (__,0) ⟶ ___.

6) (17,8) ⟶ (__,0) ⟶ ___.

Although every whole number can be written as the difference of two whole numbers, the difference of two whole numbers is not always a whole number.

For Example: (2,4) or 2 - 4
(0,5) or 0 - 5 .

We have discussed the fact before and have stated that the set of whole numbers is closed for addition, since the sum of any two whole numbers is always a whole number.

For Example: 3 + 5 = 8
17 + 23= 40 .

What about the set of whole numbers for multiplication? Does the product of the following examples result in a whole number?

Examples: 2 x 4 = ___
6 x 9 = ___
12 x 7 = ___

Can you think of any two whole numbers whose product is not a whole number? (Yes or No) If yes, what is your example? Discuss it with your teacher. What about 0 x 8, does the product of zero times another whole number give you a whole number? (Yes or No) If the product of two whole numbers always results in a whole number, then the set of whole numbers is said to be (closed, not closed) for multiplication.

The set of whole numbers is not closed for subtraction as it is indicated in the example differences above. 2 - 4 or (2,4) does not have an answer in the set of whole numbers. It is this type of problem for which we wish to establish an answer.

We agreed earlier that the number pair (n,0) is the simplest form of the different number pairs which result in n. Similarly, the number pair (0,n) is the simplest form of the different number pairs with the same result of 0 - n, whatever that is.

Let us agree to name the difference 0 - 1 as "$^{-}1$" (read: "negative one"). Let us agree to name the difference 0 - 2 as "$^{-}2$" (read: "negative two"). In general, 0 - n is ^{-}n. Abbreviations such as $^{-}1$, $^{-}2$, and ^{-}n are examples of the <u>standard</u> <u>numerals</u> for the differences we have been unable to determine until now. These "new numbers," the result of differences such as 0 - n, are called <u>negative</u> <u>integers</u>.

Write standard numerals for each of these negative integers.

1. (0,1) ⟶ (0 - 1) is ________.
2. (0,5) ⟶ (0 - 5) is ________.
3. (0,3) ⟶ (0 - 3) is ________.
4. (0,7) ⟶ ________.
5. (0,14) ⟶ ________.
6. (0,99) ⟶ ________.

Let us agree to name the difference 1 - 0 as "$^{+}1$" (read: "positive one"). Let us agree to name the difference 2 - 0 as "$^{+}2$" (read: "positive two"). In general, n - 0 is ^{+}n. The numeral $^{+}2$ corresponds to the whole number 2 as the result of 2 - 0 and either may be used for the standard numeral for the differences such as n - 0. These differences are called <u>positive</u> <u>integers</u>.

Write standard numerals for each of the differences or number pairs.

1. 4 - 1 = ________.
2. 5 - 3 = ________.
3. 4 - 0 = ________.
4. (8,4) ⟶ ________.
5. (3,2) ⟶ ________.
6. (11,6) ⟶ ________.
7. (6,6) ⟶ ________.

The positive integers, the negative integers, and the number 0 are all called <u>integers</u>.

Determine the simplest form of the following number pairs and the standard numeral, positive integer, negative integer or zero.

1. (3,1) ⟶ (2,0) ⟶ $^{+}2$
2. (6,2) ⟶ (__,0) ⟶ 4
3. (7,7) ⟶ (0,0) ⟶ _____
4. (11,6) ⟶ (__,0) ⟶ _____
5. (2,6) ⟶ (0,__) ⟶ $^{-}4$
6. (7,10) ⟶ (__,__) ⟶ _____
7. (3,7) ⟶ (__,__) ⟶ _____
8. (5,6) ⟶ (__,___) ⟶ _____

In each row, circle the differences or number pairs that are equal to the integer in the box at the beginning of the row.

$\boxed{4}$ (0 - 4), (8,4), (3,7), (2 - 5), (8 - 3), (6,2), (7 - 3)

$\boxed{^{-}1}$ (0,1), (1 - 0), (1 - 2), (3,1), (3 - 2), (8,7), (5,4)

$\boxed{^{-}3}$ (3 - 0), (5,2), (0,3), (3 - 6), (5,3), (4 - 7), (2,5)

$\boxed{3}$ (5,4), (3 - 6), (7,4), (3 - 0), (6 - 3), (7,5), (8 - 5)

$\boxed{6}$ (9 - 6), (3 - 9), (0,6), (12 - 6), (10,16), (1,7), (8,2)

$\boxed{^{-}5}$ (1,6), (5 - 0), (7 - 2), (0 - 5), (4,6), (10 - 15), (2,7)

$\boxed{0}$ (5,7), (6 - 6), (3,3), (4 - 3), (0 - 1), (1,1), (11 - 10)

Write the standard numeral for each of the following differences or number pairs.

1. (1 - 2) = ________
2. (7,10) ⟶ ________
3. (5,2) ⟶ ________
4. (8 - 11) = ________
5. (3,7) ⟶ ________
6. (6 - 8) = ________
7. 5 - 5 = ________
8. 6 - 5 = ________
9. (5,6) ⟶ ________
10. (3 - 4) ⟶ ________
11. (9,3) ⟶ ________
12. (5,7) ⟶ ________

ORDER IN THE SET OF INTEGERS

Since we do not as yet know how to add or subtract with integers, we will investigate these operations by using number pairs. Let us first discuss this approach in considering the equality or order of two integers through number pairs.

Questions for consideration:

a) Are the number pairs (3,1) and (9,7) equal?

b) Is the result of the number pair (4,7) greater than (7,4)?

c) Is (1,6) less than (1,5)?

Consider the two number pairs (3,1) and (9,7) from Question a) above.

In simplest form (3,1) ——— (2,0)
and (9,7) ——— (2,0)

Therefore, (3,1) and (9,7) are obviously equal since the whole number resulting from 2 - 0 is the same.

Therefore, we can find the simplest form of each number pair and take their difference to determine when two number pairs are equal.

For example: Is (7,4) = (3,0)?
Is (7 - 4) = (3 - 0)?
Yes, 3 = 3

Determine whether the following two given number irs are equal or unequal by taking their difference. For Example:

(6,1)	(8,3)
6 - 1	8 - 3
5 = or ≠	5

1. (2,1) ____ - ____ = or ≠ ____ - ____ (9,8)

2. (4,3) ____ - ____ = or ≠ ____ - ____ (11,9)

3. (14,6) ____ - ____ = or ≠ ____ - ____ (8,1)

4. (17,5) ____ - ____ = or ≠ ____ - ____ (13,1)

5. (13,3) ______ = or ≠ ______ (10,0)

6. (15,7) ______ = or ≠ ______ (10,3)

Another way to determine if two number pairs are equal is to consider the sum of their individual numbers in a certain manner. For example:

a) $(n,0)$ is equal to $(n,0)$ since,
$n + 0$ equals $0 + n$.

b) $(3,1)$ is equal to $(9,7)$ since
$3 + 7$ equals $1 + 9$.

or, in general: $(a,b) = (c,d)$ if $a + d = b + c$.

Determine whether the given number pairs are equal or unequal by using the formula we developed: $(a,b) = (c,d)$ if $a + d = b + c$.

1. (6,1), (8,3): 6 + 3 = or ≠ 1 + 8, therefore (6,1) = or ≠ (8,3).
2. (4,2), (8,6): 4 + 6 ______ 2 + 8, therefore (4,2) ______ (8,6).
3. (7,4), (3,0): 7 + 0 ______ 4 + 3, therefore (7,4) ______ (3,0).
4. (5,2), (3,1): 5 + 1 ______ 2 + 3, therefore (5,2) ______ (3,1).
5. (9,1),(13,5): 9 + 5 ______ 1 + 13,therefore (9,1) ______ (13,5).
6. (14,7),(9,2):14 + 2 ______ 7 + 9, therefore (14,7)______ (9,2).
7. (6,3), (5,3): 6 + 3 ______ 3 + 5, therefore (6,3) ______ (5,3).

Notice: The formula $(a,b) = (c,d)$ if $a + d = b + c$, allows us to determine when two negative integers represented as number pairs are also equal.

8. (1,4), (4,7): 1 + 7 ______ 4 + 4, therefore (1,4) ______ (4,7).
9. (9,13),(1,5): 9 + 5 ______13 + 1, therefore (9,13)______ (1,5).
10. (2,3), (4,6): 2 + 6 ______ 3 + 4, therefore (2,3) ______ (4,6).

The formula (a,b) _____ (c,d) if a + d _____ b + c allows us to compare two number pairs to determine which is the larger. In problem number 4 above, since $5 + 1 \neq 2 + 3$, therefore $(5,2) \neq (3,1)$, but obviously $5 + 1 > 2 + 3$, so therefore, $(5,2) > (3,1)$. Hence we can conclude that not only:

$$(a,b) = (c,d) \text{ if } a + d = b + c,$$

but also

$$(a,b) > (c,d) \text{ if } a + d > b + c.$$

We can also determine in a similar manner that:

$$(a,b) < (c,d) \text{ if } a + d < b + c.$$

Determine whether the following two number pairs are equal, or if not equal, which is the larger by the above formulas. Circle the correct sign. For example:

(3,1), (8,6): 3+6 = or $<$ or $>$ 1+8, therefore (3,1) = or $<$ or $>$ (8,6).
1) (5,2), (8,2): 5+2 = or $<$ or $>$ 2+8, therefore (5,2) = or $<$ or $>$ (8,2).
2) (1,0), (6,6): 1+6 = or $<$ or $>$ 0+6, therefore (1,0) = or $<$ or $>$ (6,6).
3) (6,1), (8,6): 6+6 = or $<$ or $>$ 1+8, therefore (6,1) = or $<$ or $>$ (8,6).
4) (14,10), (6,4): 14+4 = or $<$ or $>$ 10+6, therefore (14,10) = or $<$ or $>$ (6,4).
5) (11,4), (8,1): 11+1 = or $<$ or $>$ 4+8, therefore (11,4) = or $<$ or $>$ (8,1).
6) (8,2), (4,0): ____ = or $<$ or $>$ ____, therefore (8,2) = or $<$ or $>$ (4,0).

<u>Notice</u>: We are comparing two number pairs by comparing the sums of their parts which are whole numbers. We can compare two whole numbers to determine equality or which is larger, therefore allowing us to compare the two number pairs.

7) (7,1) ____ (5,2) since 7 + 2 ____ 1 + 5.
8) (4,1) ____ (5,1). Do the sums mentally compare.)
9) (6,5) ____ (8,6).
10) (10,4) ____ (7,3).

The formulas: $(a,b) = (c,d)$ if $a + d = b + c$

$(a,b) > (c,d)$ if $a + d > b + c$

and $(a,b) < (c,d)$ if $a + d < b + c$

can also be used to compare two negative integers as number pairs.

Determind which of the follosing number pairs are larger by the formulas above.

1. (1,4), (3,5): 1+5 = or < or > 4+3, therefore (1,4) = or < or > (3,5).
2. (2,4), (5,5): ___+___ __________ ___+___, therefore (2,4) __________ (5,5).
3. (3,7), (2,6): ___+___ __________ ___+___, therefore (3,7) __________ (2,6).
4. (1,3), (6,7): ___+___ __________ ___+___, therefore (1,3) __________ (6,7).
5. (5,4), (4,5): ___+___ __________ ___+___, therefore (5,4) __________ (4,5).
6. (0,4), (1,3): ___+___ __________ ___+___, therefore (0,4) __________ (1,3).
7. (2,6), (3,5): ___+___ __________ ___+___, therefore (2,6) __________ (3,5).

Notice: Although as yet we do not know how the standard numerals for negative integers compare, we can compare the number pairs by comparing the sums of their parts as whole numbers.

8. (1,4) ____ (2,3), since 1 + 3 _____ 4 + 2.
9. (0,1) ____ (2,2). (Do the sums mentally and compare)
10. (2,5) ____ (4,6).

Let us attempt to compare the integers as standard numerals by considering the number pair first and then writing the standard numeral which represents the number pair.

Examples: a) (2,1) < (4,0) since 2 + 0 < 1 + 4;

therefore 1 < 4. {Accepted from previous understanding of whole numbers.}

b) (6,1) > (4,4) since 6 + 4 ≥ 1 + 4;

therefore 5 > 0. [Accepted from previous understanding of whole numbers.]

c) (1,2) < (3,3) since 1 + 3 < 2 + 3;

therefore $^{-}1 < 0$. [$^{-}1$ and all ^{-}n can be determined as less than 0.]

d) (1,4) < (2,3) since 1 + 3 < 4 + 2;

therefore $^{-}3 < {}^{-}1$. [May seem to be unusual results, but see if comparison holds on following exercises.]

Determine the order of the following number pairs and compare the resulting integers in standard form. Use the symbols =, < , or > .

1) (5,1) ___ (4,3), since 5 + 3 ___ 1 + 4; therefore 4 ___ 1.
2) (6,5) ___ (3,3), since 6 + 3 ___ 5 + 3; therefore 1 ___ 0.
3) (1,3) ___ (2,2), since 1 + 2 ___ 3 + 2; therefore $^{-}2$ ___ 0.
4) (2,4) ___ (0,1), since 2 + 1 ___ 4 + 0; therefore $^{-}2$ ___ $^{-}1$.
5) (2,5) ___ (1,4), since 2 + 4 ___ 5 + 1; therefore $^{-}3$ ___ $^{-}3$.
6) (1,5) ___ (0,6), since 1 + 6 ___ 5 + 0; therefore $^{-}4$ ___ $^{-}6$.
7) (0,7) ___ (2,8), since ___+___ ___ ___+___; therefore $^{-}7$ ___ $^{-}6$.
8) (1,9) ___ (7,8), since ___+___ ___ ___+___; therefore $^{-}8$ ___ $^{-}1$.
9) (6,5) ___ (5,6), since ___+___ ___ ___+___; therefore 1 ___ $^{-}1$.

<u>Notice</u>: The negative number with the larger value disregarding the negative sign ($^{-}$) is the smaller of the two negative numbers.

10) (1,4) ___ (2,3), since 1 + 3 ___ 4 + 2; therefore $^{-}3$ ___ $^{-}1$.

11) (0,5) ___ (0,3); therefore $^{-}5$ ___ $^{-}3$.

12) (2,6) ___ (5,6); therefore $^{-}4$ ___ $^{-}1$.

Use your developed abilities through number pairs to determine the correct sign (=, <, or >) between the following standard numeral for integers.

1. 4 ___ 5.
2. 1 ___ 0.
3. $^{-}1$ ___ 0.
4. $^{-}1$ ___ $^{-}2$.
5. $^{-}3$ ___ $^{-}2$.
6. $^{-}4$ ___ 4.
7. 1 ___ $^{-}1$.
8. $^{-}7$ ___ $^{-}8$.
9. 4 ___ 0.
10. $^{-}3$ ___ $^{-}5$.
11. 5 ___ $^{-}5$.
12. $^{-}10$ ___ $^{-}11$.
13. 0 ___ $^{-}2$.
14. 14 ___ $^{-}1$.

ADDITION OF INTEGERS

These "new numbers, the set of integers, contain the familiar whole numbers as a subset. We can add and multiply with whole numbers, and if we choose our problems carefully, we can subtract and divide. The question arises as to whether we can do any of these familiar operations with all of the integers.

Problem: To find a procedure for computing the sum of two integers.

We will again attempt to determine a way to add integers by first adding <u>number pairs</u> with familiar results of whole numbers. This will give us a clue as to how to add number pairs resulting in negative integers. Eventually, we will be able to add integers in standard form.

Let us investigate the sum of $2 + 3$. 2 can be represented as a number pair in simplest form as $(2,0)$ and 3 as $(3,0)$. Since we know that $2 + 3 = 5$, we would like for $(2,0) + (3,0)$ to equal $(5,0)$. It appears that if we add the first number of each number pair, we get the first number of our answer, which is in the form of a number pair.

Combine the following number pairs in simplest form and then check by considering the standard form of the numerals.

For example:

$(4,0) + (1,0) = (4 + 1,0) = (5,0).$

Check: ${}^{+}4 + {}^{+}1 = {}^{+}5$ or simply $4 + 1 = 5$.

1) $(2,0) + (5,0) = (\underline{\quad + \quad},0) = (\underline{\quad},0).$

Check: ${}^{+}2 + {}^{+}5 = \underline{\qquad}$ or simply $2 + 5 = \underline{\quad}$.

2) $(7,0) + (13,0) = (\underline{\quad + \quad},0) = (\underline{\quad},0).$

Check: ${}^{+}7 + {}^{+}13 = \underline{\qquad}$ or simply $7 + 13 = \underline{\quad}$.

3) $(21,0) + (14,0) = (\underline{\quad + \quad},0) = (\underline{\quad},0).$

Check: ${}^{+}21 + {}^{+}14 = \underline{\qquad}$ or simply $21 + 14 = \underline{\quad}$.

4) $(3,0) + (24,0) = (\underline{\quad + \quad},0) = (\underline{\quad},0).$

Check: ${}^{+}3 + {}^{+}24 = \underline{\qquad}$ or simply $3 + 24 = \underline{\quad}$.

5) $(17,0) + (42,0) = (\underline{\quad + \quad},0) = (\underline{\quad},0).$

Check: ${}^{+}17 + {}^{+}42 = \underline{\qquad}$ or simply $17 + 42 = \underline{\quad}$.

6) $(51,0) + (37,0) = (\underline{\quad + \quad},0) = (\underline{\quad},0).$

Check: ${}^{+}51 + {}^{+}37 = \underline{\qquad}$ or simply $51 + 37 = \underline{\quad}$.

If number pairs are not in simplest form, perhaps we still add the first elements of each number pair to arrive at the first number of the answer. In a similar manner, perhaps we add the second elements of each number pair to arrive at the second number of the answer.

For example: $(5,1) + (3,2) = (5 = 3, 1 + 2) = (8,3).$
$(5,1)$ in simplest form is $(4,0)$
$(3,2)$ in simplest form is $(1,0)$, and
$(8,3)$ in simplest form is $(5,0)$.

Therefore: $(5,1) + (3,2) = (5 + 3, 1 + 2)$ or $(8,3)$ is true
since $(4,0) + (1,0) = (5,0)$.

Add the following number pairs in the manner indicated above and check by changing the number pair to simplest form and to the standard form of the numeral.

1) (4,2) + (5,1) = (4 + 5, _____) = (9, __)

Check: (4,2) in simplest form is (__,0)

(5,1) in simplest form is (__,0) and

(9, __) in simplest form is (__,0).

Therefore: (4,2) + (5,1) = (4 + 5, _____) = (9, __) is true

since (2,0) +(__,0) = (__,0) or simply

$^{+}2 + {}^{+}4 = {}^{+}6$.

2) (6,1) + (3,2) = (__+__, __+__) = (__,__)

since (5,0) + (__,0) = (__,0) or simply

5 + $^{+}$__ = $^{+}$___.

3) (9,2) + (6,3) = (__+__, __+__) = (__,__)

since (7,0) + (3,0) = (__,0) or simply

$^{+}$___ + $^{+}$___ = $^{+}$___.

4) (13,1) + (10,4) = (__+__, __+__) = (__,__)

since (12,0) + (6,0) = (__,0) or simply

___ + ___ = ___.

5) (13,4) + (28,14) = (__+__, __+__) = (__,__)

since (9,0) + (14,0) = (__,0) or simply

___ + ___ = ___.

6) (14,3) + (21,2) = (__+__, __+__) = (__,__)

(__,0) + (__,0) = (__,0)

$^{+}11 + {}^{+}19$ = ___.

7) (52,2) + (12,4) = (_____, _____) = (__,__)

(__,0) + (__,0) = (__,0)

___ + ___ = ___.

If we wish to add any two number pairs the rule to follow apparently is:

$$(a,b) + (c,d) = (a + c,\ b + d)$$

where (a,b) and (c,d) are number pairs not necessarily in simplest form.

Find the sum of the following number pairs by using the rule $(a,b) + (c,d) = (a + c,\ b + d)$ and check the related integers in standard form.

1) (6,3) + (8,2) = (___ + ___, ___ + ___) = (___,___),
   $^{+}3$ + $^{+}6$ = $^{+}9$

2) (5,5) + (6,1) = (___ + ___, ___ + ___) = (___,___),
   0 + $^{+}5$ = $^{+}$___

3) (14,6) + (12,5) = (___ + ___, ___ + ___) = (___,___),
   $^{+}8$ + $^{+}7$ = $^{+}$___

4) (5,1) + (14,4) = (___ + ___, ___ + ___) = (___,___),
   $^{+}4$ + $^{+}10$ = ___

5) (15,3) + (10,10) = (___ + ___, ___ + ___) = (___,___),
   $^{+}12$ + 0 = ___.

6) (21,0) + (16,3) = (___ + ___, ___ + ___) = (___,___),
   $^{+}$___ + $^{+}13$ = $^{+}$___.

7) (37,4) + (16,2) = (______,______), (Perform sums in your head.
   $^{+}$___ + $^{+}$___ = $^{+}$___.

8) (18,0) + (42,17) = (______,______)
   18 + $^{+}$___ = $^{+}$___.

9) (1,1) + (19,2) = (______,______)
   ___ + ___ = ___.

10) (25,1) + (21,5) = (______,______)
   ___ + ___ = ___.

11) (30,3) + (35,2) = (______,______)
   ___ + ___ = ___.

12) (20,10) + (51,9) = (______,______)
   ___ + ___ = ___.

CLOSURE PROPERTY

This new form of addition defined by (a,b) + (c,d) = (a + c, b + d) adds two number pairs and the result is a number pair. When we add two whole numbers and the sum is a whole number, we have the Closure Property, or simply say the set of whole numbers is closed for addition. Apparently the set of integers is also closed for addition because when we add two integers together as number pairs, we get a number pair which results in an integer.

Use the rule for addition to find the sums of the following integers as ordered pairs and then check the results of the sum as two positive integers. For example:

(9,4) + (3,0) = (9+3, 4+0) = (12,4), therefore $^+5 + ^+3 = ^+8$.

1) (12,8) + (5,1) = (_+_,_+_) = (_,_), therefore $^+$_ + $^+$_ = $^+$_.

2) (8,6) + (9,3) = (_+_,_+_) = (_,_), therefore $^+$_ + $^+$_ = $^+$_.

3) (9,2) + (4,4) = (_+_,_+_) = (_,_), therefore __ + __ = __.

4) (13,1) + (14,6) = (_+_,_+_) = (_,_), therefore __ + __ = __.

5) (14,5) + (24,1) = (_+_,_+_) = (_,_), therefore __ + __ = __.

6) (18,4) + (16,0) = (_+_,_+_) = (_,_), therefore __ + __ = __.

7) (10,2) + (10,5) = (___,___), therefore, ___ + ___ = ___.

8) (34,0) + (2,2) = (___,___), therefore, ___ + ___ = ___.

9) (25,10) + (20,10) = (___,___), therefore, ___ + ___ = ___.

10) (32,4) + (8,1) = (___,___), therefore ___ + ___ = ___.

Notice: The sum of two number pairs which represents positive integer results in a number pair which represents a positive integer. In fact, you will notice that the sum of two positive integers results in positive integers, which compares favorably to the result of two similar whole numbers.

Determine the sums of the following positive integers by relating them to whole numbers.

11) $^{+}4 + {}^{+}5 = {}^{+}9$, since $4 + 5 = 9$.

12) $^{+}2 + {}^{+}11 =$ __, since $2 + 11 =$ __.

13) $^{+}14 + {}^{+}22 =$ __, since $14 + 22 =$ __.

14) $^{+}8 + {}^{+}17 =$ ___.

15) $^{+}6 + {}^{+}13 =$ ___.

16) $^{+}16 + {}^{+}24 =$ ___.

What is the sum of zero and any other whole number? For example:

$0 + 4 =$ ____. $6 + 0 =$ ____. $231 + 0 =$ ____.

What then is the sum of zero and any positive integer?

Consider:

$(8,4) + (3,3)$ where $(3,3)$ results in zero

$(8,4) + (3,3) = (8 + 3, 4 + 3) = (11,7)$, or

$^{+}4 + 0 = {}^{+}4$

Find the sums of the following positive integers and zero.

17) $^{+}11 + 0 =$ ____.

18) $0 + {}^{+}32 =$ ____.

19) $0 + {}^{+}24 =$ ____.

20) $^{+}6 + 0 =$ ____.

ADDITIVE IDENTITY PROPERTY

Zero, when added to any other whole number, results in that same whole number. This is referred to as the Additive Identity Property and zero is the Identity Element for addition. If zero is still to be the identity element for addition in the set of integers, then zero added to any other integer should result in that integer.

Add the following integers as ordered pairs by means of the definition $(a,b) + (c,d) = (a + c, b + d)$. Compare the results in standard numeral form.

1) $(3,3) + (2,4) = (3+2, 3+4) = (5,7)$; therefore $0 + {}^{-}2 = {}^{-}2$.
2) $(1,5) + (2,2) = (1+2,$ ___+___$) = (3,$ ___$)$; therefore ${}^{-}4 + 0 =$ ___.
3) $(3,6) + (4,4) =$ (___+___, ___+___) = (___, ___); therefore ${}^{-}3 + 0 =$ ___.
4) $(4,5) + (9,9) =$ (___+___, ___+___) = (___, ___); therefore ${}^{-}1 + 0 =$ ___.
5) $(6,6) + (0,9) =$ (___+___, ___+___) = (___, ___); therefore $0 + {}^{-}9 =$ ___.
6) $(5,5) + (6,8) =$ (___, ___); therefore $0 +$ ___ = ___.
7) $(2,8) + (9,9) =$ (___, ___); therefore ___ $+ 0 =$ ___.
8) $(1,14) + (10,10) =$ (___, ___); therefore ___ $+ 0 =$ ___.

<u>Notice</u>: The sum of zero and any negative integers results in that same negative integer.

9) $0 + {}^{-}2 =$ <u>2</u>. (Consider results in Exercise 1 above)

10) ${}^{-}1 + 0 =$ ___.

11) ${}^{-}12 + 0 =$ ___.

12) $0 + {}^{-}15 =$ ___.

13) ${}^{-}9 + 0 =$ ___.

14) $0 + {}^{-}25 =$ ___.

15) ${}^{-}31 + 0 =$ ___.

16) ${}^{-}6 +$ ___ $= {}^{-}6$

17) ___ $+ {}^{-}4 = {}^{-}4$.

Add the following number pairs by the rule $(a,b) + (c,d) = (a + c, b + d)$. Consider the standard numeral form to determine the result of the sum of two negative integers. For example:

$(2,4) + (0,1) = (2+0, 4+1) = (2,5)$; therefore ${}^{-}2 + {}^{-}1 = {}^{-}3$.

1) $(1,5) + (3,6) = (1+3,$ ___+___$) = (4,$ ___$)$; therefore ${}^{-}$___ + ${}^{-}$___ = ___.
2) $(2,11) + (6,8) =$ (___+___, ___+___) = (___, ___); therefore ${}^{-}$___ + ${}^{-}$___ = ___.
3) $(4,9) + (2,9) =$ (___+___, ___+___) = (___, ___); therefore ___ + ___ = ___.
4) $(3,14) + (3,9) =$ (___+___, ___+___) = (___, ___); therefore ___ + ___ = ___.
5) $(4,7) + (4,12) =$ (___+___, ___+___) = (___, ___); therefore ___ + ___ = ___.

<u>Notice</u>: When adding two negative integers as standard numerals we seem to obtain a negative integer for our answer.

6) (1,15) + (5,9) = (_+_,_+_) = (_,_); therefore __ + __ = ___.

7) (5,14) + (12,24) = (_+_,_+_) = (_,_); therefore __ + __ = ___.

8) (2,9) + (3,19) = (_+_,_+_) = (_,_); therefore __ + __ = ___.

Notice anything else about the sum of two negative integers besides the result being negative? Did you observe that the addition of two negative integers was similar to adding two positive integers only since these were both negative apparently the answer is also negative. Use this ability to combine the following negative integers in standard numeral form.

9) $^-2 + {}^-1 =$ ____. (Refer to the example above.)

10) $^-5 + {}^-7 =$ ____. (Refer to Problem 3 preceding)

11) $^-3 + {}^-4 =$ ____.

12) $^-6 + {}^-11 =$ ____.

13) $^-14 + {}^-4 =$ ____.

14) $^-4 + {}^-3 =$ ____.

15) $^-11 + {}^-6 =$ ____.

16) $^-4 + {}^-14 =$ ____.

COMMUTATIVE PROPERTY

We have previously studied the Commutative Property with whole numbers which indicated that two whole numbers could be added in either order; or a + b = b + a, where a and b are any two whole numbers. Consider the previous pairs of problems 11) and 14), 12) and 15), and 13) and 16). Apparently the addition of two negative integers is also Commutative.

Add the following number pairs by the rule (a,b) + (c,d) = (a + c, b + d) and compare the result of the sum of a positive and a negative integer.

1) (4,2) + (1,5) = (4 + 1, 2 + 5) = (5,7); therefore
2 + $^{-}4$ = $^{-}2$.

2) (7,3) + (2,3) = (__+__,__+__) = (__,__); therefore
___ + ___ = ____.

3) (5,10) + (6,3) = (__+__,__+__) = (__,__); therefore
___ + ___ = ____.

4) (4,7) + (8,2) = (__+__,__+__) = (__,__); therefore
___ + ___ = ____.

5) (12,4) + (6,8) = (__+__,__+__) = (__,__); therefore
___ + ___ = ____.

6) (13,3) + (1,6) = (__,__); therefore ___ + ___ = ____.

7) (2,16) + (12,3) = (__,__); therefore ___ + ___ = ____.

Notice: Look at the exercises above and see if the sum of a positive and negative integer is always positive or always negative. The answer can apparently be either positive or negative. See if you can tell how to determine which sign it will be.

8) (11,0) + (1,5) = (__,__); therefore ___ + ___ = ____.

9) (2,20) + (9,1) = (__,__); therefore ___ + ___ = ____.

10) (4,25) + (10,4) = (__,__); therefore ___ + ___ = ____.

The sign of the answer is apparently always the same as the sign of the larger of the two numbers to be added disregarding the signs. Check this out in the following exercises.

11) (5,2) + (6,20) = (__,__); therefore ___ + ___ = ____.

12) (3,5) + (15,2) = (__,__); therefore ___ + ___ = ____.

13) (6,21) + (11,6) = (__,__); therefore ___ + ___ = ____.

The answer is not the sum of the two numbers with the sign of the larger, but seems to be the difference of the two numbers disregarding the sign. For example:

$$^{+}2 + {}^{-}4 = {}^{-}(4 - 2) \text{ or } 2^{*}$$

*Answer is negative since 4 > 2 and 4 is negative.

*Answer is difference of 2 and 4. (Compare answer with Exercise 1 above.)

14) $^{+}4 + {}^{-}1 = {}^{+}(4 - 1) = {}^{+}$____. (Consider Exercise 2 above.)

15) $^{-}5 + {}^{+}3 = {}^{-}($___ - ___$) = {}^{-}$____.

16) $^{+}8 + {}^{-}2 = {}^{+}($___ - ___$) = {}^{+}$____.

17) $^{+}11 + {}^{-}4 = {}^{+}($___ - ___$) = {}^{+}$____.

18) $^{+}9 + {}^{-}14 = ($___ - ___$) =$ ____.

19) $^{-}3 + {}^{+}6 = ($___ - ___$) =$ ____.

20) $^{+}7 + {}^{-}12 = ($___ - ___$) =$ ____.

Use this new ability to add the following integers.

21) $^{+}15 + {}^{-}10 =$ ____.

22) $^{-}22 + {}^{+}9 =$ ____.

23) $^{-}36 + {}^{+}14 =$ ____.

24) $^{-}17 + {}^{+}24 =$ ____.

25) $^{+}13 + {}^{-}13 =$ ____.

26) $^{-}5 + {}^{-}30 =$ ____.

27) $^{+}18 + {}^{-}8 =$ ____.

28) $^{-}45 + {}^{+}23 =$ ____.

29) $^{-}16 + {}^{+}15 =$ ____.

30) $^{+}41 + {}^{-}21 =$ ____.

To state what we have developed in generalized terms about adding integers in standard numeral form, we devise the following statements.

a) Sum of two positive integers:

$$^{+}m + {}^{+}n = {}^{+}(m + n)$$

b) Sum of two negative integers:

$$^{-}m + {}^{-}n = {}^{-}(m + n)$$

c) Sum of a positive and negative integer:

$$^{-}m + {}^{+}n = {}^{-}(m - n), \text{ where } m > n.$$

$$^{-}m + {}^{+}n = {}^{+}(n - m), \text{ where } n > m.$$

For example:
a) $^{+}4 + {}^{+}3 = {}^{+}(4 + 3) = {}^{+}7$
b) $^{-}2 + {}^{-}6 = {}^{-}(2 + 6) = {}^{-}8$
c) $^{-}8 + {}^{+}5 = {}^{-}(8 - 5) = {}^{-}3$, since $8 > 5$.
$^{-}7 + {}^{+}9 = {}^{+}(9 - 7) = {}^{+}2$, since $9 > 7$.

1) $^{+}2 + {}^{+}7 = {}^{+}(\underline{\ \ +\ \ }) = {}^{+}\underline{\qquad}$.	11) $^{+}7 + {}^{+}2 = \underline{\qquad}$.
2) $^{-}3 + {}^{-}8 = {}^{-}(\underline{\ \ +\ \ }) = {}^{-}\underline{\qquad}$.	12) $^{-}8 + {}^{-}3 = \underline{\qquad}$.
3) $^{+}9 + {}^{+}5 = \underline{{}^{+}\qquad}$.	13) $^{+}5 + {}^{+}9 = \underline{\qquad}$.
4) $^{-}12 + {}^{-}6 = \underline{{}^{-}\qquad}$.	14) $^{-}6 + {}^{-}12 = \underline{\qquad}$.
5) $^{-}11 + {}^{+}9 = \underline{{}^{-}(11-9)} = \underline{{}^{-}\qquad}$.	15) $^{+}9 + {}^{-}11 = \underline{\qquad}$.
6) $^{+}13 + {}^{-}6 = \underline{{}^{+}(13-6)} = \underline{{}^{+}\qquad}$.	16) $^{-}6 + {}^{+}13 = \underline{\qquad}$.
7) $^{-}5 + {}^{+}14 = \underline{{}^{+}(\ \ -\ \ )} = \underline{{}^{+}\qquad}$.	17) $^{+}14 + {}^{-}5 = \underline{\qquad}$.
8) $^{+}9 + {}^{-}15 = \underline{{}^{-}(\ \ -\ \ )} = \underline{{}^{-}\qquad}$.	18) $^{-}15 + {}^{+}9 = \underline{\qquad}$.
9) $^{+}8 + {}^{-}8 = \underline{\qquad}$.	19) $^{-}8 + {}^{+}8 = \underline{\qquad}$.
10) $^{-}16 + {}^{+}8 = \underline{\qquad}$.	20) $^{+}8 + {}^{-}16 = \underline{\qquad}$.

COMMUTATIVE PROPERTY

Observe by comparing the results of Exercises 1) - 11), 2) - 12), 3) - 13), etc. above that addition of integers is apparently commutative. That is: $a + b = b + a$, where a and b are integers.

ADDITIVE INVERSE PROPERTY

Consider the following sums of particular pairs of positive and negative integers, first as number pairs combined by the rule $(a,b) + (c,d) = (a + c, b + d)$ and then as standard numerals.

1) $(3,1) + (5,7) = (\_\_\_,\_\_\_)$; therefore $^{+}2 + {}^{-}2 = \_\_\_\_$.

2) $(2,9) + (10,3) = (\_\_\_,\_\_\_)$; therefore $\_\_ + \_\_ = \_\_\_\_$.

3) $(10,5) + (3,8) = (\_\_\_,\_\_\_)$; therefore $\_\_ + \_\_ = \_\_\_\_$.

4) $(12,3) + (4,13) = (\_\_\_,\_\_\_)$; therefore $\_\_ + \_\_ = \_\_\_\_$.

<u>Notice</u>: Observe that the sum of a positive and negative integer does not always fall in one of the two categories we indicated previously.

$$^{-}m + {}^{+}n = {}^{+}(n - m) \text{ if } n > m, \text{ or}$$

$$^{-}m + {}^{+}n = {}^{-}(m - n) \text{ if } m > n.$$

In the exercises above $m = n$; then apparently $^{-}m + {}^{+}n = 0$ if $m = n$.

5) $(8,4) + (6,10) = (\_\_\_,\_\_\_)$; therefore $\_\_ + \_\_ = \_\_\_\_$.

6) $(2,12) + (14,4) = (\_\_\_,\_\_\_)$; therefore $\_\_ + \_\_ = \_\_\_\_$.

7) $(28,14) + (0,14) = (\_\_\_,\_\_\_)$; therefore $\_\_ + \_\_ = \_\_\_\_$.

The negative integer ^{-}m is said to be the <u>opposite</u> of the positive integer ^{+}m, and the positive integer ^{+}n is said to be the <u>opposite</u> of the negative integer ^{-}n. The <u>additive inverse property</u> states that an integer and its opposite added together give zero. $^{-}m + {}^{+}m = 0$ or $^{+}n + {}^{-}n = 0$.

What is the opposite of each integer named below:

1) 3,___	6) $^{-}123$,___
2) $^{-}1$,___	7) 57,___
3) 4,___	8) $^{-}2$,___
4) 5,___	9) $^{-}15$,___
5) 0,___	10) $^{-}1965$,___

Write the standard numeral for each sum.

11) $(^{+}2) + (^{-}2$_ = ______.

12) $(^{-}3) + (^{+}3) =$ ______.

13) $4 + (^{-}4) =$ ______.

14) $(^{-}5) + 5 =$ ______.

15) $(^{-}8) + (^{+}8) =$ ______.

16) $(^{-}7) + (^{+}7) =$ ______.

17) $15 + (^{-}15) =$ ______.

18) $(^{+}32) + (^{-}32) =$ ______.

19) $(^{-}17) + (^{+}17) =$ ______.

20) $(^{-}23) + 23 =$ ______.

REVIEW

1) $^{+}2 + {^{+}5} =$ ____. $^{-}2 + {^{-}5} =$ ____. $^{+}2 + {^{-}5} =$ ____. $^{-}2 + {^{+}5} =$ ____.

2) $^{+}3 + {^{+}8} =$ ____. $^{-}3 + {^{-}8} =$ ____. $^{+}3 + {^{-}8} =$ ____. $^{-}3 + {^{+}8} =$ ____.

3) $^{+}6 + {^{+}9} =$ ____. $^{-}6 + {^{-}9} =$ ____. $^{+}6 + {^{-}9} =$ ____. $^{-}6 + {^{+}9} =$ ____.

4) $^{+}8 + {^{+}7} =$ ____. $^{-}8 + {^{-}7} =$ ____. $^{+}8 + {^{-}7} =$ ____. $^{-}8 + {^{+}7} =$ ____.

5) $^{+}7 + {^{+}3} =$ ____. $^{-}7 + {^{-}3} =$ ____. $^{+}7 + {^{-}3} =$ ____. $^{-}7 + {^{+}3} =$ ____.

6) $^{+}4 + {^{+}5} =$ ____. $^{-}4 + {^{-}5} =$ ____, $^{+}4 + {^{-}5} =$ ____. $^{-}4 + {^{+}5} =$ ____.

7) $9 + 5 =$ ____. $^{-}9 + {^{-}5} =$ ____. $9 + {^{-}5} =$ ____. $^{-}9 + 5 =$ ____.

8) $3 + 9 =$ ____. $^{-}3 + {^{-}9} =$ ____. $3 + {^{-}9} =$ ____. $^{-}3 + 9 =$ ____.

Compute the following sums of three addends. (Add the integers in the parentheses first.)

9) $(^{+}3 + {^{+}4}) + {^{+}5} =$ __?__
$(^{+}7) + {^{+}5} =$ ______

10) $(^{+}3 + {^{-}4}) + {^{+}5} =$ __?__
(___)+ $^{+}5 =$ ______

11) $(^{-}3 + {^{+}4}) + {^{+}5} =$ __?__
(___)+ $^{+}5 =$ ______

12) $(^{-}3 + {^{-}4}) + {^{+}5} =$ __?__
(___)+ $^{-}5 =$ ______

13) $(6 + {^{-}7}) + 3 =$ __?__
(___)+ $3 =$ ______

14) $^{+}3 + (^{+}4 + {^{+}5}) =$ __?__
$^{+}3 + (^{+}9) =$ ______

15) $^{+}3 + (^{-}4 + {^{+}5}) =$ __?__
$^{+}3 +$ (___) = ______

16) $^{-}3 + (^{+}4 + {^{+}5}) =$ __?__
$^{-}3 +$ (___) = ______

17) $^{-}3 + (^{-}4 + {^{-}5}) =$ __?__
$^{-}3 +$ (___) = ______

18) $6 + (^{-}7 + 3) =$ __?__
$6 +$ (___) = ______

ASSOCIATIVE PROPERTY

Compare the sums of the three addends preceding and notice that apparently the way the addends are associated does not affect the sum. Therefore, the integers apparently obey the Associate Property as did the whole number.

Complete each of the following equations:

1) $4 + 0 = \_\_\_\_$.

2) $^{-}6 + 0 = \_\_\_\_$.

3) $0 + {}^{-}10 = \_\_\_\_$.

4) $^{+}11 + 0 = \_\_\_\_$.

5) $0 + 50 = \_\_\_\_$.

6) $14 + \_\_\_\_ = 14$.

7) $^{-}7 + \_\_\_\_ = {}^{-}7$.

8) $\_\_\_\_ + {}^{-}13 = {}^{-}13$.

9) $^{-}83 + \_\_\_\_ = {}^{-}83$.

10) $35 + \_\_\_\_ = 35$.

11) If one addend is ____, the sum is the same as the other addend.

Compute each of the following sums.

12) $^{-}6 + {}^{+}6 = \_\_\_\_$.

13) $^{+}4 + {}^{-}4 = \_\_\_\_$.

14) $9 + {}^{-}9 = \_\_\_\_$.

15) $13 + {}^{-}13 = \_\_\_\_$.

16) $^{+}20 + {}^{-}20 = \_\_\_\_$.

17) $3 + {}^{-}3 = \_\_\_\_$.

18) $^{-}16 + {}^{+}16 = \_\_\_\_$.

19) $65 + {}^{-}65 = \_\_\_\_$.

20) $^{-}62 + \_\_\_\_ = 0$.

21) $^{+}3 + \_\_\_\_ = 0$.

22) $\_\_\_\_ + {}^{-}9 = 0$.

23) $^{+}30 + \_\_\_\_ = 0$.

24) $100 + \_\_\_\_ = 0$.

25) $^{-}15 + \_\_\_\_ = 0$.

26) $^{-}18 + \_\_\_\_ = 0$.

27) $^{+}11 + \_\_\_\_ = 0$.

28) What do you notice about the sum? ______________________________

29) Each addend is the __________________________ of the other.

Write the opposite for each of the following integers.

30) 6, ____ 33) $^{-}16$, ____

31) $^{-}13$, ____ 34) 0, ____

32) $^{-}9$, ____ 35) $^{+}100$, ____

SUBTRACTION OF INTEGERS

In the last section we determined how to add integers in standard form by considering the results of the sums of number pairs which represented these integers. We developed a rule to add number pairs by considering them first in simplest form and determining familiar results for positive integers. Let us use the definition for addition:

$$(a,b) + (c,d) = (a + c,\ b + d)$$

and the definition of order or equality established earlier:

$$(a,b) = (c,d) \text{ if } a + d = b + c$$

to develop a rule for subtracting number pairs. Again through the use of number pairs, we will establish a way to subtract integers in standard numeral form.

Problem: To find a procedure for computing the difference of two integers.

We would like a rule similar to the one for addition. It would subtract number pairs such as $(a,b) - (c,d)$ and get an answer comparable to the answer for the sum which is in terms of a, b, c, and d. Let us say: $(a,b)-(c,d)=(m,n)$ and then determine values for m and n.

We can write this subtraction problem in terms of a related addition problem:

$$(a,b) - (c,d) = (m,n) \text{ if } (a,b) = (m,n) + (c,d).$$

This compares to related number facts with whole numbers:

$$x - y = z \text{ if } x = z + y.$$

We know how to add integers as number pairs, so if we add the right side of:

$$(a,b) = (m,n) + (c,d)$$

we get: $$(a,b) = (m + c,\ n + d).$$

Using the rule for equality of two integers established earlier, we know that:

$$(a,b) = (m + c,\ n + d) \text{ if } a + n + d = b + m + c.$$

The question to be answered now is when is:

$$a + n + d = b + m + c?$$

If we were to let $n = b + c$, and $m = a + d$, then

$$a + n + d = b + m + c.$$

That is, when we made these substitutions for m and n:

$a + (b + c) + d = b + (a + d) + c$, and use the commutative property for the whole numbers on the right, obviously:

$$a + b + c + d = a + b + c + d.$$

Since this is true when $m = a + d$, and $n + b + c$, then:

$(a,b) - (c,d) = (m,n)$ can now be written as $(a,b) - (c,d) = (a + d,\ b + c)$

We have now developed a rule for subtracting integers as ordered pairs using previously accepted facts:

$$(a,b) - (c,d) = (a + d,\ b + c)$$

Use the rule we developed to determine the differences of some familiar integers as number pairs. Also consider the results as integers in standard numeral form. For example:

$(6,3) - (4,2) = (6 + 2,\ 3 + 4) = (8,7)$, therefore ${}^{+}3 - {}^{+}2 = {}^{+}1$.

This compares favorably to the whole number results for 3 - 2. Consider the following:

1) $(6,1) - (5,4) = (\underline{\quad + \quad}, \underline{\quad + \quad}) = (\underline{\quad}, \underline{\quad})$; therefore
$^+5 - {}^+1 = \underline{\qquad}$.

2) $(9,2) - (10,5) = (\underline{\quad + \quad}, \underline{\quad + \quad}) = (\underline{\quad}, \underline{\quad})$; therefore
$\underline{\qquad} - \underline{\qquad} = \underline{\qquad}$.

3) $(7,3) - (4,2) = (\underline{\quad}, \underline{\quad})$; therefore
$\underline{\qquad} - \underline{\qquad} = \underline{\qquad}$.

4) $(10,2) - (6,3) = (\underline{\quad}, \underline{\quad})$; therefore
$\underline{\qquad} - \underline{\qquad} = \underline{\qquad}$.

5) $(7,1) - (6,5) = (\underline{\quad}, \underline{\quad})$; therefore
$\underline{\qquad} - \underline{\qquad} = \underline{\qquad}$.

<u>Notice</u>: The results of the exercises above are familiar results that can be compared to subtracting a whole number from a larger whole number. Watch the following exercises for unusual results.

6) $(4,2) - (6,3) = (\underline{\quad + \quad}, \underline{\quad + \quad}) = (\underline{\quad}, \underline{\quad})$; therefore
$^+2 - {}^+3 = \underline{\qquad}$.

7) $(5,4) - (8,1) = (\underline{\quad + \quad}, \underline{\quad + \quad}) = (\underline{\quad}, \underline{\quad})$; therefore
$\underline{\qquad} - \underline{\qquad} = \underline{\qquad}$.

8) $(6,3) - (5,0) = (\underline{\quad}, \underline{\quad})$; therefore $\underline{\qquad} - \underline{\qquad} = \underline{\qquad}$.

9) $(8,4) - (9,1) = (\underline{\quad}, \underline{\quad})$; therefore $\underline{\qquad} - \underline{\qquad} = \underline{\qquad}$.

10) $(8,6) - (12,3) = (\underline{\quad}, \underline{\quad})$; therefore $\underline{\qquad} - \underline{\qquad} = \underline{\qquad}$.

We cannot subtract a large <u>whole</u> <u>number</u> from a smaller <u>whole</u> <u>number</u> and obtain a whole number. But according to Exercises 6) through 10) above, a large positive integer from a smaller positive integer results in a negative integer. It appears that if $m > n$ then $^+n - {}^+m = {}^-(m - n)$. Check this out in the following exercises.

For example:

$(7,4) - (8,2) = (9,12)$; therefore

$^{+}3 - {}^{+}6 = {}^{-}(6 - 3)$ or $^{-}3$.

11) $(9,2) - (18,10) =$ (___,___); therefore

____ - ____ = $^{-}$(___ - ___) or ____.

12) $(11,10) - (11,1) =$ (___,___); therefore

____ - ____ = $^{-}$(___ - ___) or ____.

13) $(6,2) - (12,3) =$ (___,___); therefore

____ - ____ = $^{-}$(___ - ___) or ____.

14) $(9,6) - (16,4) =$ (___,___); therefore

____ - ____ = $^{-}$(___ - ___) or ____.

Compute the following subtraction problems using the apparent fact that if $m \quad n$ then $^{+}n - {}^{+}m = {}^{-}(m - n)$, and the following addition problems using methods established earlier.

1) $^{+}5 - {}^{+}2 =$ ____	2) $^{+}5 + {}^{-}2 =$ ____
3) $^{+}2 - {}^{+}3 =$ ____	4) $^{+}2 + {}^{-}3 =$ ____
5) $^{+}1 - {}^{+}6 =$ ____	6) $^{+}1 + {}^{-}6 =$ ____
7) $^{+}17 - {}^{+}6 =$ ____	8) $^{+}17 + {}^{-}6 =$ ____
9) $^{+}4 - {}^{+}7 =$ ____	10) $^{+}4 + {}^{-}7 =$ ____

Notice anything about the equations $^{+}5 - {}^{+}2 = {}^{+}3$ and $^{+}5 + {}^{-}2 = {}^{+}3$. Observe that the integer $^{-}2$ is the ________________ of the integer $^{+}2$. Consider the similar facts in relations equations 3) and 4), 5) and 6), etc. In each exercise the operation was changed from ____________ to ________________. It is apparent from these exercises that to subtract an integer, we can add its ____________________.

$^{+}a - {}^{+}b = {}^{+}a +$ ____.

CLOSURE PROPERTY

The set of whole numbers is <u>not closed</u> for the operation subtraction. You recall this means we cannot always subtract any two whole numbers and obtain a whole number answer. The set of integers, however, is <u>closed</u> for subtraction. The difference of any two integers always results in an integer. This is apparent from our rule (a,b) - (c,d) = (a + d,b + c) which subtracts any two integers (in the form of a number pair) and results in an integer (also in the form of a number pair).

Use the rule (a,b) - (c,d) = (a + d,b + c) to determine the difference of the following number pairs which result in both positive and negative integers.

1) (1,6) - (4,2) = (__+__,__+__) = (__,__); therefore ____ - ____ = ____.

2) (1,0) - (2,5) = (__+__,__+__) = (__,__); therefore ____ - ____ = ____.

3) (2,8) - (1,5) = (__+__,__+__) = (__,__); therefore ____ - ____ = ____.

4) (7,3) - (3,9) = (__,__); therefore ___ - ___ = ____.

5) (10,2) - (3,4) = (__,__); therefore ___ - ___ = ____.

6) (3,10) - (2,7) = (__,__); therefore ___ - ___ = ____.

7) (1,5) - (2,11) = (__,__); therefore ___ - ___ = ____.

8) (4,7) - (7,1) = (__,__); therefore ___ - ___ = ____.

9) (5,14) - (11,15)= (__,__); therefore ___ - ___ = ____.

10) (7,10) - (12,11)= (__,__); therefore ___ - ___ = ____.

Consider the results of the previous exercises to work the following subtraction problems and compare their results with the addition problems on the right.

1) $^{-}5 - {}^{+}2 =$ ____.	$^{-}5 + {}^{-}2 =$ ____.
2) $^{+}1 - {}^{-}3 =$ ____.	$^{+}1 + {}^{+}3 =$ ____.
3) $^{-}6 - {}^{-}4 =$ ____.	$^{-}6 + {}^{+}4 =$ ____.
4) $^{-}4 - {}^{-}6 =$ ____.	$^{-}4 + {}^{+}6 =$ ____.
5) $^{+}8 - {}^{-}1 =$ ____.	$^{+}8 + {}^{+}1 =$ ____.
6) $^{+}2 - {}^{-}5 =$ ____.	$^{+}2 + {}^{+}5 =$ ____.
7) $^{+}4 - {}^{-}9 =$ ____.	$^{+}4 + {}^{+}9 =$ ____.
8) $^{-}3 - {}^{+}6 =$ ____	$^{-}3 + {}^{-}6 =$ ____.
9) $^{-}9 - {}^{-}4 =$ ____.	$^{-}9 + {}^{+}4 =$ ____.
10) $^{-}3 - {}^{+}1 =$ ____.	$^{-}3 + {}^{-}1 =$ ____.

It should be apparent from the exercises above that if a and b are either positive or negative integers, in general:

$$a - b = a + {}^{-}b.$$

??? COMMUTATIVE PROPERTY FOR SUBTRACTION ???

The question arises, since the set of integers is closed for subtraction, perhaps subtraction of integers is Commutative. However, consideration of the results of the subtraction problems 1) - 6), 2) - 10), and 3) - 4), obviously subtraction of integers is <u>not</u> commutative.

REVIEW

Complete each of the following equations:

1) $^{-}5 - {}^{=}2 = {}^{-}5 +$ ___ = ___. 2) $^{-}8 - {}^{-}2 = {}^{-}8 +$ ___ = ___.

3) $3 - {}^{-}1 = 3 +$ ___ = ___. 4) $4 - {}^{-}4 = 4 +$ ___ = ___.

5) $3 - 7 = 3 +$ ___ = ___.

Compute each sum and difference

6) $^{-}2 + 3 =$ ___. $^{-}2 - {}^{+}3 =$ ___. $2 - {}^{-}3 =$ ___.

$^{-}2 + 3 =$ ___. $^{-}2 - {}^{-}3 =$ ___ $2 + 3 =$ ___.

7) $^{-}8 + 5 =$ ___. $8 + {}^{-}5 =$ ___ $^{-}8 + {}^{-}5 =$ ___.

$5 + {}^{-}8 =$ ___. $^{-}5 + 8 =$ ___. $^{-}5 + {}^{-}8 =$ ___.

8) $6 - {}^{+}8 =$ ___. $^{-}6 - {}^{+}8 =$ ___. $^{+}6 + {}^{-}8 =$ ___.

$6 - 8 =$ ___. $^{-}6 - 8 =$ ___. $6 + {}^{-}8 =$ ___.

Recall that for every subtraction equation, there is an inversely related addition equation. Solve the following equations using this fact.

For example:

a) $^{+}6 + n = {}^{+}3$ if $n = {}^{+}3 - {}^{+}6$, or $n = {}^{-}3$.
b) $x + {}^{-}3 = {}^{+}9$ if $x = {}^{+}9 - {}^{-}3$, or $x = {}^{+}12$.
c) $w - {}^{+}6 = {}^{-}4$ if $w = {}^{-}4 + {}^{+}6$, or $w = {}^{+}2$.

1) $n - {}^{+}3 = {}^{-}5$ if $n =$ ___ + ___, or $n =$ ___.

2) $^{+}4 + x = {}^{-}6$ if $x =$ ___ - ___, or $x =$ ___.

3) $9 + \square = {}^{+}2$ if $\square =$ ___ - ___, or $\square =$ ___.

4) $\triangle - {}^{-}6 = {}^{-}3$ if $\triangle =$ ___ + ___, or $\triangle =$ ___.

5) $n + {}^{-}2 = {}^{+}5$ if $n =$ ___ - ___, or $n =$ ___.

6) $w - {}^{+}4 = {}^{+}9$ if $w =$ ___ + ___, or $w =$ ___.

7) $x + {}^{-}3 = {}^{+}5$ if $x =$ ___ - ___, or $x =$ ___.

8) $n - {}^{-}2 = {}^{-}1$ if $n =$ ___ + ___, or $n =$ ___.

9) $^{-}6 + n = {}^{+}4$ if $n =$ ___ - ___, or $n =$ ___.

10) $n - {}^{+}3 = 0$ if $n =$ ___ + ___, or $n =$ ___.

APPENDIX C

TEST ITEMS USED IN TESTING COMPUTATIONAL SKILLS WITH SIGNED NUMBERS OF THE SUBJECTS IN BOTH GROUPS

UNIT TEST

Write the standard numeral for the opposite of each integer.

1) $^-8$ ___	2) $^-2$ ___	3) 3 ___
4) -3 ___	5) 7 ___	6) 19 ___
7) 0 ___	8) $^-19$ ___	9) $^-10$ ___

Compute the sums.

10) $^+2 + {}^+3 =$ ___	11) $^+2 + {}^-3 =$ ___
12) $^-2 + {}^+3 =$ ___	13) $^-2 + {}^-3 =$ ___

14) The sum of two positive integers is always ____________________.

15) The sum of two negative integers is always ____________________.

16) - 17) The sum of a positive integer and a negative integer will be ____________________ if the positive integer is further from zero and will be ____________________ if the negative integer is further from zero.

Compute

18) $^-5 + 8 + {}^-13 + {}^-8 =$ _______

19) $^-23 + 17 + 18 + {}^-35 =$ _______

UNIT TEST II

Compute the sums.

1) $-2 + 8 =$ ___	2) $^-12 + {}^-8 =$ ___	3) $^-13 + 5 =$ ___
4) $17 + {}^-8 =$ ___	5) $23 + {}^-16 =$ ___	

Compute the differences.

6) $^-2 - 3 =$ ___	7) $2 - {}^-3 =$ ___	8) $5 - 17 =$ ___
9) $^-5 - {}^-17 =$ ___	10) $^-13 - 18 =$ ___	

Solve the equations.

11) $^{-}2 + x = 3$

$x =$ ___

12) $^{-}8 + w = {}^{-}9$

$w =$ ___

13) $n - {}^{-}5 = 4$

$n =$ ___

14) $^{+}2 + x = {}^{+}5$ $x +$ _____

15) $^{+}2 + x = {}^{-}1$ $x =$ _____

16) $^{-}2 + x = {}^{+}1$ $x =$ _____

17) $^{-}2 + x = {}^{-}5$ $x =$ _____

APPENDIX D

STANDARD SCORES ON THE ARITHMETIC COMPUTATION SECTION OF THE METROPOLITAN ACHIEVEMENT TESTS (INTERMEDIATE BATTERY) AND THE I.Q. SCORES FROM THE OTIS QUICK-SCORING MENTAL ABILITY TEST FOR THE 79 SUBJECTS IN BOTH GROUPS

Group Number	Subject Number	Arithmetic Computation	I. Q.
C_1	1	58	101
	2	51	102
	3	32	87
	4	52	105
	7*	63	107
	8	57	113
	9	50	104
	10	57	94
	11	44	98
	12	57	108
	15*	68	117
	16	58	108
	17	31	84
	19*	48	95
	20	50	103
	21	40	127
C_2	1	51	98
	3	60	119
	4	37	108
	5	33	95
	6	54	101
	7	54	105
	8	41	97
	9	62	114
	10	65	131
	11	46	100
	12	57	106
	13	42	98
	14	55	107
	15	52	103
	16	63	116
	17	55	106
	18	50	112
A_3	1	42	111
	2	55	111
	3	39	91
	4	58	106
	5	52	112
	6	44	98
	7	55	106
	8	38	100
	9	57	106
	10	68	116

*Subject number omitted indicate subject lacking test data.

Group Number	Subject Number	Arithmetic Computation	I. Q.
	11	40	99
	12	50	97
	13	36	118
	14	50	104
	15	41	96
	16	33	75
	17	56	119
	18	48	105
	19	48	102
	20	44	105
	21	45	92
	22	46	113
	23	48	111
	24	40	109
A_4	1	57	109
	2	56	95
	3	41	113
	4	56	107
	5	47	91
	6	57	105
	7	40	88
	8	44	101
	9	54	91
	10	47	102
	11	44	100
	12	50	110
	13	67	116
	14	47	91
	15	60	115
	16	63	125
	17	68	114
	18	52	109
	19	46	112
	20	60	120
	21	56	107
	22	58	118

APPENDIX E

ACHIEVEMENT SCORES ON TEST OF COMPUTATIONAL SKILLS FOR 79 SUBJECTS INCLUDED IN BOTH GROUPS

Group Number	Subject Number	Raw Score
C_1	1	29
	2	23
	3	13
	4	2;
	7	35
	8	32
	9	26
	10	28
	11	27
	12	29
	15	33
	16	31
	17	9
	19	24
	20	21
	21	30
C_2	1	29
	3	33
	4	27
	5	26
	6	32
	7	26
	8	30
	9	33
	10	32
	11	32
	12	31
	13	31
	14	32
	15	34
	16	32
	17	32
	18	31
A_3	1	28
	2	31
	3	19
	4	14
	5	33
	6	33
	7	28
	8	19
	9	25
	10	35
	11	23
	12	16
	13	24
	14	20
	15	15
	16	15
	17	28
	18	23
	19	20
	20	21
	21	24
	22	23
	23	26
	24	23
A_4	1	34
	2	29
	3	33
	4	32
	5	29
	6	33
	7	27
	8	30
	9	32
	10	34
	11	26
	12	31
	13	36
	14	29
	15	36
	16	33
	17	36
	18	30
	19	32
	20	35
	21	30
	22	34

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