A NEW AXIOMATIC APPROACH FOR THE STEENROD SQUARING OPERATIONS

By

GEORGE MALCOLM BUTLER

Bachelor of Science
Oklahoma State University
Stillwater, Oklahoma
1963

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1964

Submitted to the faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
DOCTOR OF PHILOSOPHY
July, 1968

A NEW AXIOMATIC APPROACH FOR THE STEENROD

SQUARING OPERATIONS

Thesis Approved:

Hiroshi Uehara
Thesis Adviser

E. K. McLachlan

William J. Tiederman, Jr.

John E. Hoffman

N N Durham
Dean of the Graduate College

ACKNOWLEDGEMENTS

I am indebted to many people for their guidance and encouragement during my graduate studies. I am especially obligated to my advisor, Professor Hiroshi Uehara, and wish to express my gratitude for his interest, encouragement, and guidance during my graduate career. I am also indebted to Professor Nobuo Shimada for his guidance in my research.

I acknowledge my obligation to Professors McLachlan, Hoffman, and Weibelt, as my teachers and as members of my advisory committee, for their time and effort on my behalf. I am grateful to Professor L. Wayne Johnson for his encouragement throughout my graduate career and for financial support in the form of a Graduate Teaching Assistantship.

I am grateful to my wife, Lucia, who has understood and been a constant help.

INTRODUCTION

In 1947 Steenrod (9) introduced the Steenrod squaring operations as a sequence of homomorphisms

$$sq^{i} : H^{n}(K,L;Z_{2}) \rightarrow H^{n+i}(K,L;Z_{2})$$

defined for finite simplicial pairs (K,L) and all $n \geq 0$. He used it for the homotopy classification of continuous maps. In 1953 Serre (5) gave an axiomatization of these squaring operations valid for all pairs (X,A) by using spectral sequences. In Cohomology Operations by N.E. Steenrod (8) there appears a proof also of the existence and uniqueness of the squaring operations. This proof is long involved and difficult for a newcomer to the field to follow. In this paper we will present a proof of the existence and uniqueness of the Steenrod squaring operations that is direct, short and much different.

The main part of the paper is Chapter III where the cohomology suspension is shown to be an isomorphism in small dimensions. The tool used to do this is Brown's (1) generalization of the Eilenberg-Zilber theorem for fiber spaces in terms of the twisted tensor product.

In Chapter I we state the axioms for the squaring operations and consider some necessary preliminaries. In Chapter II we show the existence of the squaring operations and prove the uniqueness theorem. In both Chapters I and II we anticipate the results of Chapter III and assume the cohomology suspension to be an isomorphism in small dimensions. In Chapter IV a summary of the paper is given and a problem for further research is suggested. Numbers appearing in parentheses, ( ), refer to an entry in the bibliography.

TABLE OF CONTENTS

CHAPTER I

PRELIMINARIES

Axioms for the Steenrod Squaring Operations

The Steenrod squaring operations are a sequence of homomorphisms $sq^0, sq^1, \dots, sq^i, \dots$

$$sq^i : H^n(X,Y;Z_2) \to H^{n+i}(X,Y;Z_2)$$

defined for all pairs (X,Y) of topological spaces and integers n, $i \geqq 0$. The sq^i satisfy the following axioms:

1. (Naturality) If $f : (X,Y) \to (A,B)$ is a continuous mapping then $f^* sq^i = sq^i f^*$

2. (Dimension) sq^0 = identity map, $sq^i(x) = 0$ if $\deg(x) < i$ and $sq^i(x) = x^2$ if $\deg(x) = i$. Here $\deg(x) = i$ means that x is in $H^i(X,Y;Z_2)$.

3. (Cartan Formula) For x in $H^p(X,Y,Z_2)$ and y in $H^q(A,B;Z_2)$

$$sq^i(x \times y) = \sum_{k=0}^{i} sq^k(x) \times sq^{i-k}(y).$$

This is the axiomatization as given by Steenrod (8).

Complexes, Homology, and Cohomology

Let R be a commutative ring with unity, Z the additive group of integers, and Z_m the cyclic group of integers modulo m.

<u>Definition 1.1</u>: A chain complex K of R-modules is a family $\{K_n, \partial_n\}$ of R-modules K_n and R-homomorphisms $\partial_n : K_n \to K_{n-1}$, defined for all in integers, such that $\partial_n \partial_{n+1} = 0$ for each n.

Definition 1.2: The homology $H(K)$ is the family of R-modules $H_n(K) = \text{Ker}\partial_n/\text{Im}\partial_{n+1}$. The members of $\text{Ker}\partial_n$ are called n-cycles and the members of $\text{Im}\partial_{n+1}$ are called bounding cycles. If x is an n-cycle then cls(x) is the member of $H_n(K)$ which contains x.

Definition 1.3: The chain complex $K \otimes_R L$ is the family

$$\{\sum_{i=0}^{n} K_i \otimes_R L_{n-i}, \sum_{i=0}^{n} \partial_i \otimes 1 + (-1)^i\, 1 \otimes \partial_{n-i}\}.$$

In this definition we agree $\partial_i(x) = 0$ if x is in K_j or L_j and $i \neq j$.

Remark 1.1: If x is a n-cycle K and y is a m-cycle of L then $x \otimes y$ is a $(m + n)$-cycle of $K \otimes_R L$. Also the tensor product of a n-cycle and bounding cycle is a bounding cycle and the tensor product of two bounding cycles is a bounding cycle. Hence for a n-cycle x and a m-cycle y

$$p(\text{cls}(x) \otimes \text{cls}(y)) = \text{cls}(x \otimes y)$$

is a well determined homology class in $K \otimes_R L$. So p defines a homomorphism

$$p\colon H_m(K) \otimes_R H_n(L) \to H_{m+n}(K \otimes_R L).$$

Theorem 1.1: (The Kunneth Tensor Formula) If K and L are chain complexes of R-modules satisfying $\text{Ker}\partial_n : K_n \to K_{n-1}$ and if $H_n(K)$ are projective R-modules for all n, then for each n

$$p\colon \sum_{i=0}^{n} H_i(K) \otimes_R H_{n-i}(L) \to H_n(K \otimes_R L)$$

is an isomorphism of R-modules.

A proof appears in (3). Note if R is a field then the hypothesis of this theorem is satisfied.

Let K be a chain complex of R-modules and G be an R-module. For each n $\text{Hom}(K_n,G)$ is a R-module. We have the following sequence of R-modules and homomorphisms:

$$\text{Hom}(K,G): \ldots \to \text{Hom}(K_{n-1},G) \xrightarrow{\delta_{n-1}} \text{Hom}(K_n,G) \xrightarrow{\delta_n} \text{Hom}(K_{n+1},G) \to \ldots$$

where $\delta_n(f) = f\partial_{n+1}$. We see that $\delta_{n+1}\,\delta_n = 0$ for all n.

Definition 1.4: The n^{th} cohomology of K with coefficients in G, $H^n(K,G)$, is the R-module

$$\text{Ker}\ \delta^n/\text{Im}\delta^{n-1}.$$

Theorem 1.2: Let K be a complex of free abelian groups and A be any abelian group. Then for each n the abelian groups $H^n(K,A)$ and $\text{Hom}(H_n(K),A) \oplus \text{Ext}(H_{n-1}(K),A)$ are isomorphic.

This is a consequence of the Universal Coefficient Theorem (3).

Let (X,Y) be a pair of topological spaces with Y a subspace of X. Let S(X) denote the singular chain complex of X. The chain complex of the pair (X,Y) is defined to be S(X)/S(Y). It is well known that S(X) and S(X)/S(Y) are free abelian groups. If Y = $\emptyset$ then we define S(Y) = 0.

Definition 1.5: Let K be a chain complex of R-modules and G be an R-module. The homology H(K,G) of K with coefficients in G is $H(K \otimes_R G)$. G is regarded as the trivial chain complex. Namely G = $\{G_n, \partial_n\}$ where $G_0 = G$, $G_i = 0$ if $i \neq 0$, and $\partial_n = 0$ for all n. For an abelian group A the homology of the pair (X,Y) with coefficients in A is $H(S(X)/S(Y) \otimes_Z A)$.

Let G be a R-module. We can also regard G as a Z-module. Hence H(X,Y;G) is defined. The R-module structure of G can be used to define an R-module structure for each member of the family H(X,Y:G). First for each n $(S(X)/S(Y))_n \otimes_Z G$ can be made an R-module by defining $r(x \otimes g) = x \otimes rg$ where r is in R, x is in $(S(X)/S(Y))_n$ and g is in G. With this definition of scalar multiplication the boundary operators $\{\partial_n \otimes 1\}$ of the chain complex of Z-modules, $(S(X)/S(Y) \otimes_Z G$ are each R-linear homomorphisms. Thus $(S(X)/S(Y) \otimes_Z G)$ is a chain complex of R-modules. Hence $H(X,Y;G)$ is a family of R-modules.

Definition 1.6: The nth cohomology $H^n(X,Y;A)$ of the pair (X,Y) with coefficients in a R-module A is $H^n(S(X)/S(Y),A)$.

Eilenberg - MacLane Spaces

Definition 1.7: Let G be an abelian group and n a natural number. An Eilenberg-Maclane space of type (G,n) is a topological space whose nth homotopy group is isomorphic to G and whose other homotopy groups are trivial.

Theorem 1.3: For each pair (G,n) an Eilenberg-Maclane space of type (G,n) exists which is a C.W. complex. Furthermore any such two Eilenberg-MacLane spaces of type (G,n) have the same homotopy type.

This is Corollary 2.10.2 of (10).

We will be concerned with the case $G = Z_2$. For each $n \geqq 1$ let X_n denote one Eilenberg-MacLane complex of type (Z_2,n). The proof of Theorem 1.3 demonstrates that X_n is connected.

Proposition 1.1: $H_0(X_n) \cong Z$, $H_n(X_n) \cong Z_2$, and $H_i(X_n) \cong 0$ if $0 < i < n$.

If $n > 1$ this follows from the Hurewitz theorem. The case $n = 1$ follows from Remark 2.10.9 of (10). Since X_n has one path component it follows that $H_0(X_n) \cong Z$ for all n.

Proposition 1.2: $H^i(X_n,Z_2) \cong Z_2$ if $i = 0$, n and $H^i(X_n,Z_2) \cong 0$ if $0 < i < n$.

From Theorem 1.2 we have

$H^i(X_n,Z_2) \cong \mathrm{Hom}(H_i(X_n), Z_2) \oplus \mathrm{Ext}(H_{i-1}(X_n),Z_2)$.

From Proposition 1.1 we see that $H_{i-1}(X_n)$ is either trivial or free on one generator. Thus the right hand summand is trivial when $i \leqq n$. The conclusion follows now from Proposition 1.1.

Theorem 1.4: Let (X,Y) be a pair of C.W. complexes, $n \geqq 1$, $*$ be in X_n, and x be in $H^n(X,Y;Z_2)$. Let ι' be the generator of $H^n(X_n, * \cdot Z_2) \cong Z_2$. There is a mapping $f(X,Y) \to (X_n,*)$ such that $f^*(\iota_n') = x$ and furthermore any other mapping with this property is homotopic to f. Similarly if ι_n is the generator of $H^n(X_n,Z_2)$, X is a C.W. complex, and x is a member of $H^n(X,Z_2)$, then there is a map $f\colon X \to X_n$ such that $f^*(\iota_n) = x$.

This is Corollary 2.8.10 of (10).

Definition of the Cohomology Suspension

The cohomology suspension is a homomorphism

$$\sigma^*\colon H^i(X_n,*;Z_2) \to H^{i-1}(X_{n-1},Z_2)$$

defined for $i, n \geqq 2$. Let EX_n be the space of paths in X_n based at $*$, i.e. $EX_n = \{\alpha : I_r \to X_n : \alpha(r) = *\}$. Here r is any non-negative real number and I_r is the closed interval from 0 to r. In Chapter III a suitable topology will be defined for EX_n. Let $p\colon EX_n \to X_n$ be defined by $p(\alpha) = \alpha(0)$. We see that $p^{-1}(*) = \Omega X_n$ is the space of loops based at $*$. It is well known that ΩX_n has the same homotopy type as X_{n-1}. Hence it is an Eilenberg-MacLane space of type $(Z_2,n-1)$. Let $h\colon X_{n-1} \to \Omega X_n$ be a homotopy equivalence. It is also known that EX_n is acyclic. Consider the following diagram where the middle row is the long exact cohomology sequence of the pair $(EX_n,\Omega X_n)$.

$$\begin{array}{ccccccccc}
 & & & H^{i-1}(X_{n-1},Z_2) & & & & & \\
 & & & \uparrow h^* & \delta^* & & & & \\
\cdots \to & H^{i-1}(EX_n,Z_2) & \to & H^{i-1}(\Omega X_n,Z_2) & \to & H^i(EX_n,\Omega X_n,Z_2) & \to & H^i(EX_n,Z_2) & \to \cdots \\
 & & & & & \uparrow p^* & & & \\
 & & & & & H^i(X_n, *;Z_2) & & &
\end{array}$$

Now $H^k(EX_n Z_2) \cong 0$ when $k \geqq 1$, thus the coboundary δ^* is an isomorphism for $i \geqq 2$. Define $\sigma^* = h^* (\delta^*)^{-1} p^*$. Note that h^* is an isomorphism since h is a homotopy equivalence.

The Geometric Realization of a Semi-Simplicial Complex

Let (X,Y) be a pair of topological spaces. Using Milner's construction in (4) a pair (X^*,Y^*) of C. W. complexes is constructed. The pair (X^*, Y^*) is the geometric realization of the pair (S(X),S(Y)) of semi-simplicial complexes. If X is itself a C. W. complex then we take $X^* = X$.

<u>Theorem 1.5</u>: There is a mapping $j:(X^*,Y^*) \to (X,Y)$ such that for all $n \geqq 0$ $j^* : H^n(X,Y;Z_2) \to H^n(X^*,Y^*;Z_2)$ is an isomorphism. Furthermore each map f: $(X,Y) \to (A,B)$ induces a mapping $\overline{f} : (X^*,Y^*) \to (A^*,B^*)$ such that $j\overline{f} = fj$.

These facts are shown in (4).

CHAPTER II

UNIQUENESS AND EXISTENCE THEOREMS

The Uniqueness of the Steenrod Squares

In this section we will assume that the Steenrod squaring operations exist and satisfy the axioms. It is shown in Chapter III that the cohomology suspension

$$\sigma^*: H^i(X_n,*;Z_2) \to H^{i-1}(X_{n-1};Z_2)$$

for each $n > 1$ and $1 < i < 2n$ is an isomorphism. Henceforth all cohomology groups will have coefficients in Z_2.

Lemma 2.1: If $\delta^*: H^n(A) \to H^{n+1}(X,A)$ is the coboundary map for the pair (X,A) then for each i $sq^i\delta^* = \delta^* sq^i$.

This is Lemma 1.2 of (7).

Lemma 2.2: Let $n > 1$. Consider the mapping $(\sigma^*)^{-1}: H^{n-1}(X_{n-1}) \to H^n(X_n,*)$. If $0 < i < n$ then $sq^i(\sigma^*)^{-1}(\iota_{n-1}) = (\sigma^*)^{-1}sq^i(\iota_{n-1})$.

Recall that $\sigma^* = h^*(\delta^*)^{-1}p^*$. Therefore $(\sigma^*)^{-1} = (p^*)^{-1}(\delta^*)(h^*)^{-1}$. The lemma follows now by application of Lemma 2.1 and the naturality axiom.

Theorem 2.1: The Steenrod squaring operations are uniquely determined for all pairs (X,Y) of topological spaces.

The proof is by mathematical induction. Let P be the set of natural numbers for which n in P implies sq^i: $H^j(X,Y) \to H^{i+j}(X,Y)$ is uniquely determined for all $i,j \leqq n$ and all pairs of topological spaces. From the dimension axiom we see that 0 and 1 are in P. Suppose that $n > 1$ and $n-1$ is in P. We need to show that n is in P. From the dimension axiom and the induction hypothesis we see that sq^i: $H^j(X,A) \to H^{j+i}(X,A)$ is uniquely determined when $j < n$, $i \leqq n$ and when $j = n$, $i = 0,n$. Therefore suppose $j = n$, $0 < i < n$. Let x be in $H^n(X,Y)$. We construct the following diagram

$$(X,Y) \xleftarrow{j} (X^*,Y^*) \xrightarrow{f} (X_n,*)$$

where j is the mapping described in Theorem 1.5 and f is a mapping such that $f^*(\iota'_n) = j^*(x)$. Thus $x = (j^*)^{-1} f^*(\iota'_n)$. Hence $sq^i(x) = sq^i(j^*)^{-1} f^*(\iota'_n)$. Therefore it is necessary that $sq^i(x) = (j^*)^{-1} f^* sq^i(\iota_{n-1})$ because sq^i satisfies the naturality axiom. Now $\iota' = (\sigma^*)^{-1}(\iota_{n-1})$. Thus $sq^i(\iota'_n) = sq^i(\sigma^*)^{-1}(\iota_{n-1}) = (\sigma^*)^{-1}sq^i(\iota_{n-1})$ by Lemma 2.2. But $sq^i(\iota_{n-1})$ is uniquely determined by the induction hypothesis hence so is $sq^i(\iota'_n)$ and consequently so is $sq^i(x)$. Therefore n is in P.

The Existence of the Steenrod Squaring Operations

In this section we show the existence of the Steenrod squaring operations. Again we will use the result in Chapter III that for $n > 1$ and $1 < i < 2n$, σ^*: $H^i(X_n,*) \to H^{i-1}(X_{n-1})$ is an isomorphism.

First the Steenrod squaring operations will be defined as a set theoretic function and then will be shown to satisfy the axioms. Let (X,Y) be a pair of topological spaces and let x be in $H^n(X,Y)$. Define

$sq^0(x) = x$, $sq^n(x) = x^2$, and $sq^i(x) = 0$ if $i > n$. In these instances it is clear the squaring operations satisfy the axioms. It remains to define $sq^i(x)$ for $0 < i < n$ and verify that the axioms are satisfied. Mathematical induction will be used to do this. Let P be the set of natural numbers for which n in P implies that $sq^i(x)$ is defined for $0 < i < n$ and satisfies the axioms. We see that 0 and 1 are members of P. Suppose now that $n > 1$, $n-1$ is a member of P and $0 < i < n$. Construct the following diagram:

$$(X,Y) \xleftarrow{j} (X^*,Y^*) \xrightarrow{f} (X_n,*)$$

where j is the mapping of Theorem 1.5 and f is selected so that $f^*(\iota_n') = j^*(x)$. Thus $x = j^{*-1}f^*(\iota_n')$.

$$\text{Define } sq^i(x) = (j^*)^{-1}f^*(\sigma^*)^{-1}sq^i(\iota_{n-1}).$$

We see that $sq^i(x)$ is well defined because $(\sigma^*)^{-1} : H^{i-1}(X_{n-1}) \to H^i(X_n,*)$ is an isomorphism for $i < 2n$ and $sq^i(\iota_{n-1})$ is defined by the induction hypothesis. We see immediately that the dimension axiom is satisfied. The remainder of this chapter will be a verification that $sq^i(x)$ satisfies the remainder of the axioms.

<u>Theorem 2.2</u>: If $n \geqq 2$ and $0 < i < n$ then $sq^i(x)$ satisfies the naturality axiom.

Let $g : (S,T) \to (X,Y)$ be a mapping of pairs. Let $(S,T) \xleftarrow{j_1} (S^*,T^*)$ $(X,Y) \xleftarrow{j_2} (X^*,Y^*)$ be the maps given by Theorem 1.5 and $\bar{g} : (S^*,T^*) \to (X^*,Y^*)$ be the map induced by g. We need to show that $sq^i\, g^*(x) = g^*\, sq^i(x)$, where x is in $H^n(X,Y)$, $n \geqq 2$. By definition

$$sq^i\, g^*(x) = (j_1^*)^{-1} k^* (\sigma^*)^{-1} sq^i(\iota_{n-1}) \tag{A}$$

$$\text{where } k^*(\iota_n') = j_1^*\, g^*(x). \tag{B}$$

Also by definition

$$g^* sq^i(x) = g^*(j_2^*)^{-1} f^*(\sigma^*)^{-1} sq^i(\iota_{n-1}) \tag{C}$$

$$\text{where } f^*(\iota_n') = j_2^*(x). \tag{D}$$

From (D) above we see $\bar{g}^* f^*(\iota_n') = \bar{g}^* j_2^*(x)$ but $\bar{g}^* j_2^*(x) = j_1^* g^*(x)$ by Theorem 1.5. Comparing this with (B) we concluded that k and $f\bar{g}$ are homotopic by Theorem 1.4. Consequently substituting $\bar{g}^* f^*$ for k^* in (A) we have

$$\begin{aligned} sq^i g^*(x) &= (j_1^*)^{-1}\, \bar{g}^* f^*(\sigma^*)^{-1} sq^i(\iota_{n-1}) \\ &= g^*(j_2^*)^{-1} f^*(\sigma^*)^{-1} sq^i(\iota_{n-1}) \text{ by Theorem 1.5} \\ &= g^* sq^i(x) \text{ from (B).} \end{aligned}$$

We will next verify that sq^i is a homomorphism. To do this we need to know the n^{th} Z_2 - cohomology group of a space in terms of its n^{th} Z_2-homology group.

<u>Proposition 2.1</u>: Let X be a topological space. For each n $H^n(X,Z_2) \cong \mathrm{Hom}_{Z_2}(H_n(X,Z_2),Z_2)$.

We will define a mapping φ and show that it is an isomorphism. Let $f\colon S(X)_n \to Z_2$ be a cocycle and cls(f) be the member of $H^n(X,Z_2)$ which contains f. By definition $H_n(X,Z_2) = \mathrm{Ker}(\partial n \otimes 1)/\mathrm{Im}(\partial_{n+1} \otimes 1)$. We see that

$$\mathrm{Ker}(\partial_n \otimes 1) = \{x \otimes 1 : \partial_n(x) \text{ is in } 2(S(X)_{n-1})\}.$$

Define

$$\varphi \ \mathrm{cls}(f) \ (\mathrm{cls}(x \otimes 1)) = f(x).$$

We see that

$$\varphi \ \mathrm{cls}(f) \ (\mathrm{cls}(x \otimes 1) + \mathrm{cls}(y \otimes 1)) = \varphi \ \mathrm{cls}(f) \ \mathrm{cls}((x + y) \otimes 1)$$

$$= f(x + y) = f(x) + f(y) = \varphi \ \mathrm{cls}(f) \ (\mathrm{cls}(x \otimes 1) + \mathrm{cls}(f)(\mathrm{cls}(y \otimes 1)).$$

Similarly

$$\varphi \ \mathrm{cls}(f+g)(\mathrm{cls}(x \otimes 1) = \varphi \ \mathrm{cls}(f)(\mathrm{cls}(x \otimes 1)) + \varphi \ \ \mathrm{cls}(g)(\mathrm{cls}(x \otimes 1).$$

We need to show this definition is independent of the choice of representatives for cls(f) and $\mathrm{cls}(x \otimes 1)$. Therefore suppose $\mathrm{cls}(x \otimes 1) = \mathrm{cls}(y \otimes 1)$. Then

$$\mathrm{cls}(x \otimes 1) - \mathrm{cls}(y \otimes 1) = \mathrm{cls}((x\text{-}y) \otimes 1) =$$

$$\mathrm{cls}(\partial_{n+1} \otimes 1)(z \otimes 1) = \mathrm{cls}(\partial_{n+1} \otimes 1) \text{ for some } z \text{ in } S(X)_{n+1}.$$

Therefore

$$\varphi \ \mathrm{cls}(f)(\mathrm{cls}(x \otimes 1)) - \varphi \ \mathrm{cls}(f)(\mathrm{cls}(y \otimes 1)) =$$

$$\varphi \ \mathrm{cls}(f)(\mathrm{cls}(x \otimes 1) - \mathrm{cls}(y \otimes 1)) = \varphi \ \mathrm{cls}(f)(\partial_{n+1} z \otimes 1)$$

$$= f\partial_{n+1} z = 0 \text{ because } f \text{ is a cocycle.}$$

Therefore it follows that φ cls(f) is a well defined member of

$\mathrm{Hom}_{Z_2}(H_n(X,Z_2),Z_2)$. Now suppose $\mathrm{cls}(f) = \mathrm{cls}(g)$. We have that $\mathrm{cls}(f-g) = \mathrm{cls}(\delta h) = \mathrm{cls}(h\partial_n)$ for some cochain h. Therefore

$$\varphi\ \mathrm{cls}(f)\ \mathrm{cls}(x \otimes 1) - \varphi\ \mathrm{cls}(g)\ \mathrm{cls}(x \otimes 1) =$$

$$\varphi\ (\mathrm{cls}(f) - \mathrm{cls}(g))\ \mathrm{cls}(x \otimes 1) = \varphi\ \mathrm{cls}(f-g)\ \mathrm{cls}\ (x \otimes 1)$$

$$= \varphi\ \mathrm{cls}(h\partial_n)\ \mathrm{cls}(x \otimes 1) = h\partial_n(x) = h(2y) = 0.$$

Therefore φ is a Z_2-linear map of $H^n(X,Z_2)$ and $\mathrm{Hom}_Z(H_n(X,Z_2),Z_2)$.

We show next that φ is a surjection. Consider an element f in the range of φ. Let G be the subgroup of $S(X)_n$ whose members are x such that $\partial_n(x)$ is in $2(S(X)_{n-1})$. A homomorphism f' is defined on G by $f'(x) = f(\mathrm{cls}(x \otimes 1))$. Since $S(X)_n$ is a free abelian group f' can be extended to a map $\overline{f}: S(X)_n \to Z_2$. We see that $\overline{f}$ is a cocycle because

$$\overline{f}(\partial z) = f'(\partial z) = f\ \mathrm{cls}(\partial z \otimes 1) = 0.$$

Also

$$\varphi\ \mathrm{cls}(\overline{f})\ \mathrm{cls}(x \otimes 1) - \overline{f}(x) = f'(x) = f(\mathrm{cls}(x \otimes 1) \text{ so } \varphi\ (\overline{f}) = f.$$

We need to show next that φ is an injection. Suppose $\varphi \mathrm{cls}(f)\ \mathrm{cls}(x \otimes 1) = f(x) = 0$ for all x such that $\partial_n(x)$ is in $2(S(X)_{n-1})$. In particular $f(x) = 0$ for all x in Ker ∂_n. We have the following diagram:

$$\begin{array}{ccc} S(X)n & \xrightarrow{f} & Z_2 \\ \downarrow \partial_n & \nearrow h & \\ S(X)_{n-1} & & \end{array} \qquad \text{where Ker } \partial_n \subset \text{Ker } f.$$

Therefore there is a homomorphism which makes the diagram commute. Hence $f = h\partial_n$ and thus f is a coboundary, cls(f) = 0, and φ is an injection.

<u>Proposition 2.2:</u> $H_n(X \times Y, Z_2) \cong \sum_{i=0}^{n} H_i(X,Z_2) \otimes_{Z_2} H_{n-i}(Y,Z_2)$.

Let f : $S(X \times Y) \to S(X) \otimes S(Y)$ be a chain equivalence. It follows then that

$$f \otimes 1 \; : \; S(X \times Y) \otimes Z_2 \to S(X) \otimes S(Y) \otimes Z_2$$

is a chain equivalence ($\otimes$ means $\otimes_Z$).

Consider the mapping

$$g \; : \; S(X) \otimes S(Y) \otimes Z_2 \to (S(X) \otimes Z_2) \otimes_{Z_2} (S(Y) \otimes Z_2)$$

given by $g(x \otimes y \otimes 1) = (x \otimes 1) \otimes (y \otimes 1)$.

Clearly g is an isomorphism of Z_2 - modules.

It follows too that g is a chain mapping because

$$\partial g(x \otimes y \otimes 1) = \partial((x \otimes 1) \otimes (y \otimes 1))$$

$$= (\partial x \otimes 1) \otimes (y \otimes 1) + (x \otimes 1) \otimes (\partial y \otimes 1)$$

and

$$g\partial(x \otimes y \otimes 1) = g((\partial x \otimes y \otimes 1) + (x \otimes \partial y \otimes 1))$$

$$= (\partial x \otimes 1) \otimes (y \otimes 1) + (x \otimes 1) \otimes (\partial y \otimes 1).$$

Hence g is a chain equivalence. The conclusion follows now by application of Theorem 1.1 to the complex which is the range of g.

In the following let Hom mean Hom_{Z_2} and $\otimes$ mean $\otimes_{Z_2}$.

Proposition 2.3: Let V and W be Z_2-modules, then $\mathrm{Hom}(V \otimes W, Z_2) \cong \mathrm{Hom}(V,Z_2) \otimes \mathrm{Hom}(W,Z_2)$.

By Theorem 3.1 of (3) we have that

$$\mathrm{Hom}(V \otimes W,Z_2) \cong \mathrm{Hom}(V, \mathrm{Hom}(W,Z_2)).$$

Now $\mathrm{Hom}(W,Z_2) \cong \sum_{\alpha} Z_2(\alpha)$ for some set of indexs α because $\mathrm{Hom}(W,Z_2)$ is a vector space over Z_2.

Hence it follows

$$\mathrm{Hom}(V \otimes W,Z_2) \cong \mathrm{Hom}(V,\sum_{\alpha} Z_2)$$

$$\cong \sum_{\alpha} \mathrm{Hom}(V,Z_2) \cong \sum_{\alpha} (\mathrm{Hom}(V,Z_2) \otimes Z_2))$$

$$\cong \mathrm{Hom}(V,Z_2) \otimes \sum_{\alpha} Z_2 \cong \mathrm{Hom}(V,Z_2) \otimes \mathrm{Hom}(W,Z_2).$$

Lemma 2.3: $H^n(X \times Y, Z_2) \cong \sum_{i=0}^{n} H^i(X,Z_2) \otimes H^{n-i}(Y,Z_2)$.

By Proposition 2.1 we have that

$$H^n(X \times Y, Z_2) \cong \mathrm{Hom}\,(H_n(X \times Y, Z_2), Z_2).$$

By Proposition 2.2

$$H_n(X \times Y, Z_2) \cong \sum_{i=0}^{n} H_i(X,Z_2) \otimes H_{n-i}\,(Y,Z_2).$$

Substituting this in the expression above and applying Proposition 2.3 we have

$$H^n(X \times Y, Z_2) \cong \sum_{i=0}^{n} \mathrm{Hom}(H_i(X,Z_2),Z_2) \otimes \mathrm{Hom}(H_{n-i}(Y,Z_2),Z_2).$$

$$\cong \sum_{i=0}^{n} H^i(X,Z_2) \otimes H^{n-i}(Y,Z_2) \text{ by Proposition 2.1.}$$

Lemma 2.3 is also true in the relative case. If by the product of pairs (X,A), (Y,B) we mean

$$(X,A) \times (Y,B) = (X \times Y, X \times B \cup A \times Y)$$

then the Eilenberg-Zilber theorem holds for these pairs whenever $\{X \times B, A \times Y\}$ is an excisive couple in $X \times Y$. Namely the chain complexes $S((X,A) \times (Y,B))$ and $S(X,A) \otimes S(Y,B)$ are chain equivalent, (cf (6), p.234). We have therefore

Lemma 2.4: $H^n((X,A) \times (Y,B), Z_2) \cong$

$$\sum_{i=0}^{n} H^i(X,A,Z_2) \otimes H^{n-i}(Y,B;Z_2) \text{ whenever}$$

$\{X \times B, Y \times A\}$ is an excisive couple in $X \times Y$.

Lemma 2.5: Let $\mu: X_n \times X_n \to X_n$ be such that $\mu^*(\iota_n) = 1 \times \iota_n + \iota_n \times 1$, then if z is in $H^i(X_n)$, $n < i < 2n$, $\mu^*(z) = 1 \times z + z \times 1$.

By Lemma 2.3 and Proposition 1.2 we have that

$$H^i(X_n \times X_n) \cong H^0(X_n) \otimes H^i(X_n) \oplus H^i(X_n) \otimes H^0(X_n).$$

Therefore $\mu^*(z) = x \times 1 + 1 \times y$ for some x and y in $H^i(X_n)$. Consider the following diagram

$$X_n \xrightarrow{k_1, k_2} X_n \times X_n \xrightarrow{\mu} X_n$$

where k_1 and k_2 are the injections into the first and second coordinates respectively. We have

$$(\mu\ k_1)^*(z_n) = k_1^*(1 \times z_n + z_n \times 1) = 0 + z_n.$$

Hence $\mu\ k_1$ is homotopic to the identity mapping by Theorem 1.4. Similarly

$$(\mu k_2)^*(z_n) = k_2^*\ (1 \times z_n + z_n \times 1) = z_n + 0.$$

Hence μk_2 is homotopic to the identity mapping.

Therefore

$$z = (id)^*(z) = k_1^*\ \mu^*(z) = k_1^*(1 \times x + y \times 1) = x$$

and similarly

$$z = (id)^*(z) = k_2^*\ \mu^*(z) = k_2^*(1 \times x + y \times 1) = y.$$

Hence $\overset{*}{\mu}(z) = 1 \times z + z \times 1.$

<u>Corollary</u>: Let $\bar{p}_1, \bar{p}_2 : (X_n \times X_n, *\times *) \to (X_n, *)$ be the projections into the first and second coordinates respectively and $\bar{\mu} : (X_n, \times X_n, *\times *) \to (X_n, *)$ be such that $\bar{\mu}^*(z_n') = \bar{p}_1^*(z_n') + \bar{p}_2^*(z_n')$. Then if $n < i < 2n$ and x is in $H^i(X_n, *)$, $\bar{\mu}^*(x) = \bar{p}_1^*(x) + \bar{p}_2^*(x)$.

Consider the following diagram

$$\begin{array}{ccccc}
X_n & \xleftarrow{p_1 p_2} & (X_n \times X_n) & \xrightarrow{\mu} & X_n \\
\downarrow j & & \downarrow i & & \downarrow j \\
(X_n, *) & \xleftarrow{\bar{p}_1 \bar{p}_2} & (X_n \times X_n, *\times *) & \xrightarrow{\bar{\mu}} & (X_n, *)
\end{array}$$

where p_1, p_2 are the projections into the first and second coordinates respectively, i, j are the injections and $\mu^*(z_n) = z_n \times 1 + 1 \times z_n = p_1(z_n) + p_2(z_n)$.

We see that the left-hand square of the diagram commutes. Considering now the right hand portion we have

$$i^* \bar{\mu}^*(z_n') = i^*(\bar{p}_1^*(z_n') + \bar{p}_2^*(z_n'))$$

$$= (\bar{p}_1 i)^*(z_n') + (p_1^* z)(z_n')$$

$$= (jp_1)^*(z_n') + (jp_2)^*(z_n')$$

$$= p_1^*(z_n) + p_2^*(z_n)$$

$$= \mu^* j^*(z_n').$$

Therefore we conclude that $\bar{\mu} i$ and $j\mu$ are homotopic by Theorem (1.4). Now we know that the map i^* is an isomorphism in dimensions larger than zero. Let $x \epsilon H^i(X_n,*)$, $0 < i < 2n$. We have

$$i^*(\bar{p}_1^*(x) + \bar{p}_2^*(x)) = p_1^* j^*(x) + p_2^* j^*(x)$$

by the commutivity of the left-hand portion of the diagram. Hence

$$i^* \bar{\mu}^*(x) = \mu^* j^*(x) \text{ from above}$$

$$= p_1^* j^*(x) + p_2^* j^*(x) \text{ by Lemma 2.5.}$$

Therefore we conclude that

$$\bar{\mu}(x) = \bar{p}_1^*(x) + \bar{p}_2^*(x).$$

<u>Theorem 2.3</u>: Let $n > 1$ and $0 < i < n$ then $sq^i: H^n(X,Y) \to H^{n+i}(X,Y)$ is a homomorphism.

Let x and y be in $H^n(X,Y)$, then by definition

$$sq^i(x + Y) = (j^*)^{-1} k^*(\sigma^*)^{-1} sq^i(z_{n-1}) \tag{A}$$

where $j^*(X^*,Y^*) \to (X,Y)$ is the map of Theorem 1.5 and $k^*(z_n') = (j^*)^{-1}$ $(x + y)$. Consider the following diagram

$$(X^*,A^*) \xrightarrow{d} (X^* \times X^*, A^* \times A^*) \xrightarrow{f \times g} (X_n \times X_n, * \times *) \xrightarrow{\bar{\mu}} (X_n,*)$$

where $f^*(\iota_n') = (j^*)^{-1}(x)$, $g^*(\iota_n') = (j^*)^{-1}(y)$, $\bar{\mu}^*(\iota_n') = \bar{p}_1^*(\iota_n') + \bar{p}_2^*(\iota_n')$, and $d(p) = (p,p)$. We note that $\bar{p}_1(f \times g)d = f$ and $\bar{p}_2(f \times g)d = g$.

We have now that

$$\begin{aligned} d^*(f \times g)^* \bar{\mu}^*(\iota_n') &= d^*(f \times g)^*(\bar{p}_1^*(\iota_n') + \bar{p}_2^*(\iota_n')) \\ &= f^*(\iota_n') + g^*(\iota_n') \\ &= (j^*)^{-1}(x + y). \end{aligned}$$

Therefore in (A) above we can take $k = \bar{\mu}(f \times g)d$.

Hence we have

$$\begin{aligned} sq^i(x+y) &= (j^*)^{-1}d^*(f \times g)^* \bar{\mu}^*(\sigma^*)^{-1}sq^i(\iota_{n-1}) \\ &= (j^*)^{-1}d^*(f \times g)^*(\bar{p}_1^*(\sigma^*)^{-1}sq^i(\iota_{n-1}) + \bar{p}_2^*(\sigma^*)^{-1}sq^i(\iota_{n-1})) \\ &= (j^*)^{-1}f^*(\sigma^*)^{-1}sq^i(\iota_{n-1}) + (j^*)^{-1}g^*(\sigma^*)^{-1}sq^i(\iota_{n-1}) \\ &= sq^i(x) + sq^i(y). \end{aligned}$$

We will show next that sq^i, $0 < i < n$, satisfies the Cartan formula. The proof of this will be preceded by several lemmas.

<u>Lemma 2.6</u>: Let I be the unit interval and $\bar{I} = \{o,1\}$ its boundary. Let I be the generator of $H^1(I, \bar{I}) \cong Z_2$ and 1 be the element of $H^0(\bar{I})$ corresponding to the point 1 of $\bar{I}$. If $\delta: H^i(\bar{I} \times A) \to H^{i+1}((I,\bar{I}) \times A)$ is the coboundary map for the pair $(I,\bar{I}) \times A$ then $\delta(1 \times y) = I \times y$ for each y in $H^i(A)$.

The proof of this appears in Lemma 1.2 of (7).

<u>Lemma 2.7</u>: Let $n \geqq 2$ and $0 < i < n$, then the following diagram commutes

$$\begin{array}{ccc} H^{n-1}(\Omega X_n) & \xrightarrow{\delta^*} & H^n(EX_n,\Omega X_n) \\ \downarrow sq^i & & \downarrow sq^i \\ H^{n+i-1}(\Omega X_n) & \xrightarrow{\delta^*} & H^{n+i}(EX_n,\Omega X_n). \end{array}$$

By definition we have that for $n \geqq 2$, $0 < i < n$

$$sq^i(\iota_n') = (\sigma^*)^{-1} sq^i(\iota_{n-1}). \quad \text{Thus}$$

$$sq^i \sigma^{*-1}(\iota_{n-1}) = (\sigma^*)^{-1} sq^i(\iota_{n-1}). \quad \text{So}$$

$$sq^i(p^*)^{-1}\delta^*(h^*)^{-1}(\iota_{n-1}) = (p^*)^{-1}\delta^* h^{*-1} sq^i(\iota_{n-1}).$$

By using naturality we have

$$(p^*)^{-1} sq^i\, \delta^*(h^*)^{-1}(\iota_{n-1}) = (p^*)^{-1}\, \delta^* sq^i(h^*)^{-1}(\iota_{n-1}).$$

Now $(p^*)^{-1}$ is an isomorphism so we can cancel it yielding

$$sq^i\, \delta^*(h^*)^{-1}(\iota_{n-1}) = \delta^*\, sq^i(h^*)^{-1}(\iota_{n-1}).$$

$H^{n-1}(\Omega X_n) \cong Z_2$ and h^* is an isomorphism so $(h^*)^{-1}(\iota_{n-1})$ generates $H^{n-1}(\Omega X_n)$, thus the lemma follows.

<u>Definition 2.1</u>: Suppose A_1, A_2 are subsets of a space X. $\{A_1,A_2\}$ is an excisive couple in X if the inclusion chain map of $S(A_1) + S(A_2)$ and $S(A_1 \cup A_2)$ induces an isomorphism of homology. Here + means group sum.

Let (X,A) and (Y,B) be pairs of topological spaces. Suppose x is in $H^n(X,A)$ and y is in $H^m(Y,B)$. Their cross product, $x \times y$, is defined and is a member of $H^{n+m}((X,A) \times (Y,B))$ provided $\{A \times Y, X \times B\}$ is an excisive couple in $X \times Y$(cf. (6), pp. 249-255). We see from Definition 2.1 that if $A = \emptyset$ or $B = \emptyset$ then $\{A \times Y, X \times B\}$ is an excisive pair in $X \times Y$.

The following two lemmas are standard; their proofs appear in (6), page 189.

Lemma 2.8: Let A_1, A_2 be subsets of a space X. $\{A_1, A_2\}$ is an excisive couple in X if the excision map $(A_1, A_1 \cap A_2) \subset (A_1 \cup A_2, A_2)$ induces an isomorphism of singular homology.

Lemma 2.9: Let $U \subset A \subset X$ be such that $cl(U) \subset$ interior A. Here cl(U) means the closure of U. Then the excision map $(X - U, A - U) \subset (X, A)$ induces an isomorphism of singular homology.

The following lemma will be useful in the proof of the Cartan formula.

Lemma 2.10: Let (X,A), (Y,B) be pairs of spaces. Suppose A is finite and X is locally contractable. Suppose too that if x is a point of X and U is a contractable neighborhood of x then the homotopy can be chosen to leave x fixed. Suppose further that X is Hausdorff and normal and B is closed, then $\{X \times B, A \times Y\}$ is an excisive couple in $X \times Y$.

Since A is finite and X is Hausdorff there is a finite collection $O_1, \ldots, O_k$ of open sets of X which are pairwise disjoint, which each contain exactly one point of A, and such that each point of A is in one of them. Further if $O = O_1 \cup \ldots \cup O_k$ then there is a homotopy f_t: $O \to O$ such that f_o = identity, $f_1(O) = A$ and f_t restricted to A is the identity for each t in I. Consider the following diagram

$$(A \times Y, A \times B) \underset{k}{\overset{i}{\rightleftarrows}} (A \times Y \cup O \times B, A \times B \cup O \times B) \overset{i'}{\rightarrow}$$

$$(A \times Y \cup X \times B, X \times B)$$

where i and i′ are the inclusion maps. k is defined as follows. Note that $A \times y \;\; O \times B = A \times B$. Define $k_1 : A \times Y \to A \times Y$ to be the identity map. Define $k_2 : O \times B \to A \times Y$ by $k_2 = f_1 \times 1$. k_1 and k_2 coincide on $A \times Y \cup O \times B$ and $A \times Y$ and $O \times B$ are each closed subsets of $A \times Y$ $O \times B$, consequently k_1, k_2 defines a map $k : A \times Y \;\; O \times B \to A \times y$. We see that ki = identity. We will show that ik is homotopic to the identity mapping. Define $g_t : A \times Y \cup O \times B \to A \times Y \cup O \times B$ by g_t restricted to $A \times Y$ to be the identity map and g_t restricted to $O \times B$ to be $f_t \times 1$. These maps agree on $A \times Y \cap O \times B = A \times B$ for each t in I. Therefore these two maps define a map $g_t : A \times Y \cup O \times B \to A \times Y \cup O \times B$ with $g_1 = ik$ and g_0 = identity. Therefore i is a homotopy equivalence.

Consider now the mapping i'. We will use Lemma 2.9 to show that it induces an isomorphism of singular homology. Let $X' = A \times Y \cup X \times B$, $A' = X \times B$, and $U = (X - O) \times B$. We see that $U \subset A' \subset X'$. Now $cl(U) = U$ because U itself is closed being the product of closed sets. Also it is true that

$$(X - O) \times B \subset \text{interior } X \times B$$

because by the normality of X there is an open set O' containing $X - O$ containing no points of A. Consequently $O' \times B = (A \times Y \cup X \times B) \cap (O' \times Y)$ is an open set of $A \times Y \;\; X \times B$. Therefore $U = cl(U) \subset O' \times B \subset$ interior $X \times B$. Thus we conclude by Lemma 2.9 that i' induces isomorphism of singular homology because $X' - U = A \times Y \cup O \times B$ and $A' - U = A \times B \cup O \times B$. Consequently $i'i$ induces isomorphism of homology and therefore it follows by Lemma 2.8 that $\{A \times Y, X \times B\}$ is an excisive couple in $X \times Y$.

<u>Lemma 2.11</u>: Let $h : X_{n-1} \to \Omega X_n$ be a homotopy equivalence and $g : \Omega X_n \to X_{n-1}$ be a homotopy inverse for h. The maps $(1 \times g)^* : H^i((I, \bar{I}) \times X_{n-1}) \to$

$H^i((I,\overline{I}) \times \Omega X_n)$ and $(1 \times h)^* : H^i((I,\overline{I}) \times \Omega X_n) \to H^i((I,\overline{I}) \times X_{n-1}$ are isomorphisms.

By Lemma 2.4

$H^i((I,\overline{I}) \times X_{n-1}) \cong H^1(I,\overline{I}) \otimes H^{i-1}(X_{n-1})$ and $H^i((I,\overline{I}) \times \Omega X_n) \cong H^1(I,\overline{I}) \otimes H^{i-1}(\Omega X_{n-1})$.

Therefore each element of $H^i((I,\overline{I}) \times X_{n-1})$ has the form $I \times y$ for some y in $H^{i-1}(X_{n-1})$ and $(1 \times h)^*(1 \times g)^*(I \times y) = I \times (gh)^*(y) = I \times y$. Also each element of $H^i((I,\overline{I}) \times \Omega X_n)$ has the form $I \times z$ for some z in $H^{i-1}(\Omega X_n)$ and $(1 \times g)^*(1 \times h)^*(I \times z) = I \times (hg)^* z = I \times z$. Therefore $(1 \times g)^*$ and $(1 \times h)^*$ are inverses of each other, and as a result each is an isomorphism.

<u>Lemma 2.12</u>: Let $n \geqq 2$ and $0 < i < n$. If I is the generator of $H^1(I,\overline{I})$, then $sq^i(I \times \iota_{n-1}) = I \times sq^i(\iota_{n-1})$.

First we note that $I \times \iota_{n-1}$ is defined. Define a map φ: $(I \times \Omega X_n, \overline{I} \times X_n) \to (EX_n, \Omega X_n)$ by $\varphi(t,\alpha) = \alpha$ restricted to $[0,tr]$ where α is a map $\alpha:[0,r] \to X_n$ with $\alpha(0) = \alpha(r) = *$. Associated with this map is the following commutative diagram

$$\begin{array}{ccc} H^j(\overline{I} \times \Omega X_n) & \xrightarrow{\delta^*_1} & H^{j+1}((I,\overline{I}) \times \Omega X_n \\ \uparrow P_1^* & & \uparrow \varphi^* \\ H^j(\Omega X_n) & \xrightarrow{\delta^*} & H^{j+1}(EX_n,\Omega X_n) \end{array}$$

where $P_1 = \varphi$ restricted to $\overline{I} \times \Omega X_n$. We see $P_1(\{0\} \times \Omega X_n) =$ constant and $P_1(\{1\} \times \Omega X_n)$ is the projection into the second coordinate and is a homeomorphism. Therefore for $j \geqq 1$ and y in $H^j(\Omega X_n)$ we have $P_1^*(y) = 1 \times y$ where

1 is the element of $H^0(\bar{I})$ corresponding to the point 1 in $\bar{I}$. Thus

$$\delta_1^* P_1^*(y) = \delta_1^*(1 \times y) = I \times y.$$

Now let $0 < i < n$ and take $x = g^*(\iota_{n-1})$ where g is a homotopy inverse for the homotopy equivalence $h: X_{n-1} \to \Omega X_n$.

We have

$$\begin{aligned} sq^i(I \times x) &= sq^i(\delta_1^* P_1^* x) \\ &= sq^i(\varphi^* \delta^* x) \\ &= \varphi^* \delta^* \, sq^i(x) \\ &= I \times sq^i(x). \end{aligned}$$

The lemma follows now because

$$sq^i(I \times x) = sq^i(I \times g^*(\iota_{n-1})) = sq^i(1 \times g)^*(I \times \iota_{n-1}) =$$

$$(1 \times g^*) \, sq^i(I \times \iota_{n-1})$$

and $I \times sq^i(x) = I \times sq^i g^*(\iota_{n-1}) = I \times g^* sq^i(\iota_{n-1}) =$

$$(1 \times g)^* I \times sq^i(\iota_{n-1}).$$

Therefore $I \times sq^i(\iota_{n-1}) = sq^I(I \times \iota_{n-1})$ because $(1 \times g)^*$ is an injection by Lemma 2.11.

Lemma 2.13: Let $n \geqq 2$, ι'_{n-1} be the generator of $H^{n-1}(X_{n-1},*)$. If $0 < i < n$, then $sq^i(I \times \iota'_{n-1}) = I \times sq^i(\iota'_{n-1})$.

We see by Lemma 2.10 that $I \times \iota'_{n-1}$ is defined. The identity map $j: X_{n-1} \to (X_{n-1},*)$ induces the map $j^*: H^i(X_{n-1},*) = H^i(X_{n-1})$ which is an isomorphism for $i > 0$. Consider the following diagram:

$$(I,\bar{I}) \times X_{n-1} \xrightarrow{1 \times j} (I,\bar{I}) \times (X_{n-1},*).$$

Now $H^k((I,\bar{I}) \times X_{n-1}) \cong H^1(I,\bar{I}) \otimes H^{k-1}(X_{n-1})$ and $H^k((I,\bar{I}) \times (X_{n-1},*) \cong H^1(I,\bar{I}) \otimes H^{k-1}(X_{n-1},*)$. The map $(1 \times j)^*$ corresponds to $1 \otimes j^*$ in this identification so $(1 \times j)^*$ is an isomorphism when $k > 1$. Therefore if $n \geqq 2$ and $0 < i < n$, we have

$$(1 \times j)^* sq^i(I \times \iota'_{n-1}) = sq^i(1 \times j)^*(I \times \iota'_{n-1}) = sq^i(I \times \iota_{n-1})$$

$$= I \times sq^i \iota_{n-1} \text{ by Lemma 2.12 and}$$

$$(1 \times j)^* I \times sq^i \iota'_{n-1} = I \times j^*(sq^i \iota'_{n-1}) = I \times sq^i \iota_{n-1}.$$

Hence the conclusion follows since $(1 \times j)^*$ is an injection.

Lemma 2.14: Let $p \geqq 1$. The map $g:(I,\bar{I}) \times X_{p-1} \to (X_p,*)$ determined by $g^*(\iota'_p) = I \times \iota_{p-1}$ has the property that $g^*: H^i(X_p,*) \to H^i((I,\bar{I}) \times X_{p-1})$ is an injection for $i < 2p$. In case $p = 1$ we take $X_0 = \{*\}$.

For any p we have that

$$H^0((I,\bar{I}) \times X_{p-1}) \cong H^0(I,\bar{I}) \otimes H^0(X_{p-1}) = 0 \text{ and}$$

$$H^0(X_p,*) = 0.$$

Thus g^* is always an isomorphism in dimension 0. For $p = 1$ we need only consider g^* in dimension 1. Consider therefore $g^*: H^1(X_1,*) \to H^1((I,\bar{I}) \times X_0)$. Now $H^1(X_1,*) \cong Z_2$ and is generated by ι'_1. Also $H^1((I,\bar{I}) \times X_0) \cong Z_2 \otimes Z_2$ and is generated by $I \times 1$. By definition $g^*(\iota'_1) = I \times 1$ so g^* is an injection in dimension 1 for $p = 1$. Suppose now that $p \geqq 2$. Consider first the case $i = 1$. We have $g^*: H^1(X_p,*) \to H^1((I,\bar{I}) \times X_{p-1})$. If $p \geqq 2$ then $H^1(X_p,*) \cong H_1(X_p) = 0$ by Proposition 1.2. Thus g^* is necessarily an injection in dimension 1. Now suppose

$p \geqq 2$ and $i \geqq 2$. Consider the following diagram

$$\begin{array}{ccccc}
H^{i-1}(\bar{I} \times \Omega X_p) & \xrightarrow{\delta_1^*} & H^i((I,\bar{I}) \times \Omega X_p) & \xrightarrow{(1 \times h)^*} & H^i((I,\bar{I}) \times X_{p-1}) \\
\uparrow p_1^* & & \uparrow \varphi^* & & \uparrow g^* \\
H^{i-1}(\Omega X_p) & \xrightarrow{\delta^*} & H^i(EX_p,\Omega X_p) & \xleftarrow{p^*} & H^i(X_p,*).
\end{array}$$

The left-hand portion of the diagram appears in Lemma 2.12 and is known to be commutative. For the right-hand portion we see that

$(1 \times h)^* \varphi^* p^*(\iota_n') = (1 \times h)^* \delta_1^* p_1^* (\delta^*)^{-1} p^*(\iota_p')$ because the diagram commutes

$= I \times h^*(\delta^*)^{-1} p^* \iota_p'$ by Lemma 2.12

$= I \times \sigma^*(\iota_p')$ by definition of the cohomology suspension

$= I \times \iota_{p-1}$ because the cohomology suspension is an isomorphism in dimensions less than $2p$. Also we have $g^*(\iota_p') = I \times \iota_{p-1}$, therefore by Theorem 1.4 the maps g and $p\varphi(1 \times h)$ are homotopic. Now $(1 \times h)^*$ is an isomorphism by lemma 2.11. Also p^* is an isomorphism when $2 \leqq i < 2p$ because the cohomology suspension is an isomorphism in these dimensions. Consider now the map $\varphi^* = \delta_1^* p_1^* \delta^{*-1}$ for dimensions $i \geqq 2$. It was shown in Lemma 2.8 that for each x in $H^i(\Omega X_p)$, $\delta_1^* p_1^*(x) = I \times x$. The range of $\delta_1^* p_1^*$ is $H^i((I,\bar{I}) \times \Omega X_p) \cong H^i(I,\bar{I}) \otimes H^{i-1}(\Omega X_p)$ so we conclude that $I \times x = 0$ only when $x = 0$. Therefore $\delta_1^* p_1^*$ is an injection when $i \geqq 2$ hence so is φ^*. We have shown that $g^* = (1 \times h)^* \varphi^* p^*$ therefore g^* is an injection when $p \geqq 2$ and $2 \leq i < 2p$.

<u>Therorem 2.4</u>: (The Cartan Formula) Let (X,A) and (Y,B) be pairs of topological spaces such that $\{A \times Y, X \times B\}$ is an excisive pair in $X \times Y$. Let x be in $H^p(X,A)$ and y be in $H^q(Y,B)$ with $p + q = n \geqq 2$. If $0 < i < n$ then $sq^i(x \times y) = \sum_{k=0}^{i} sq^k(x) \times sq^{i-k}(y)$.

Let $p \geqq 1$ and construct the following diagram:

$$(I,\bar{I}) \times X_{p-1} \times (V,C) \xrightarrow{g \times 1} (X_p,*) \times (V,C)$$

where $g^*(\iota_p') = I \times \iota_{p-1}$, $X_0 = \{*\}$ and (V,C) is a pair of C.W. complexes. We note that the cross product operation is defined for the cohomology of the pairs appearing in the diagram. This follows from Lemma 2.10 and the fact that every C.W. complex is locally contractable and the homotopy can be chosen to satisfy Lemma 2.10, (11, p. 230). Let v be in $H^q(V,C)$ and $p + q = n \geqq 2$. Consider now $sq^i(\iota_p' \times v)$, $0 < i < n$. It is a member of

$$H^{n+i}((X_p,*) \times (V,C)) \cong \sum_{k=0}^{n+i} H^k(X_p,*) \otimes H^{n+i-k}(V,C).$$

Therefore $$sq^i(\iota_p' \times v) = \sum_{k=-p}^{q+i} \lambda_{p+k,q+i-k}$$

where $\lambda_{p+k,q+i-k}$ is in $H^{p+k}(X_p,*) \otimes H^{q+i-k}(V,C)$.

We have now that

$$\begin{aligned}
(g \times 1)^* sq^i(\iota_p' \times v) &= sq^i(g \times 1)^*(\iota_p' \times v) \\
&= sq^i(g^*(\iota_p' \times v)) \\
&= sq^i(I \times \iota_{p-1} \times f) \\
&= I \times sq^i(\iota_{p-1} \times v) \text{ by Lemma 2.9} \\
&= I \times (\sum_{k=0}^{i} sq^k(\iota_{p-1}) \times sq^{i-k}(v)) \text{ by the induction hypothesis} \\
&= \sum_{k=0}^{i} (I \times sq^k(\iota_{p-1}) \times sq^{i-k}(v)).
\end{aligned}$$

Similarly we have that

$$(g \times 1)^*(\sum_{k=0}^{i} sq^k(\iota_p) \times sq^{i-k}(v)) = \sum_{k=0}^{i} g^* sq^k(\iota_p') \times sq^{i-k}(v)$$

$$= \sum_{k=0}^{i} I \times sq^k(\iota_{p-1}) \times sq^{i-k}(v).$$

Now from Lemma 2.10 we know that g^* is an injection in dimension less than 2p. It follows therefore $(g \times 1)^*$ is an injection whenever g^* is an injection because the cohomology groups are all vector spaces over Z_2 and $(g \times 1)^*$ corresponds to $g^* \otimes 1$. Therefore we see that

$$\lambda_{p+k,q+i-k} = 0 \text{ when } k < 0 \text{ and}$$

$$\lambda_{p+k,q+i-k} = sq^k(\iota_p') \times sq^{i-k}(v) \text{ when } 0 \leqq k < p.$$

Here we will agree that $sq^k = 0$ whenever k is a negative integer so that the above formulas always makes sense. Note that i may be less than k. Therefore we have that

$$sq^i(\iota_p' \times v) = \sum_{k=0}^{p-1} sq^k(\iota_p') \times sq^{i-k}(v) + \sum_{k=p}^{q+i} \lambda_{p+k,q+i-k}. \tag{A}$$

Note that if we take n = p then q = 0 and (A) becomes

$$sq^i(\iota_n' \times v) = sq^i(\iota_n') \times v, \ 0 < i < n. \tag{B}$$

Consider now the following diagram

$$(V,C) \times (I,\overline{I}) \times X_{q-1} \xrightarrow{1 \times g} (V,C) \times (X_q,*)$$

where $g^*(\iota_q') = I \times \iota_{q-1}$, (V,C) is again a pair of C.W. complexes, and $X_0 = \{*\}$. Let p + q = n and v be in $H^p(V,C)$. By the same procedure as above we argue that

$$(1 \times g)^* sq^i(v \times \iota_q) = (1 \times g)^* \sum_{k=0}^{i} sq^k(v) \times sq^{i-k}(\iota_q).$$

Suppose that $sq^i(v \times \iota_q') = \sum_{k=-p}^{q+i} \mu_{p+k,q+i-k}$

where $\mu_{p+k,q+i-k}$ is in $H^{p+k}(V,C) \otimes H^{q+i-k}(X_q,*)$.

Now g^* is an injection in dimensions less than 2q so

$\mu_{p+k,q+i-k} = sq^k(v) \times sq^{i-k}(\iota_q)$ if $-q + i < k \leqq i$

and $\mu_{p+k,q+i-k} = 0$ if $i < k \leqq q+i$. Therefore we have that

$$sq^i(v \times \iota_q') = \sum_{k=-p}^{-q+i} \mu_{p+k,q+i-k} + \sum_{k=-q+i+1}^{i} sq^k(v) \times sq^{i-k}(\iota_q'). \qquad (C)$$

Note in this case if we take q=n, then p=0 and (C) becomes

$$sq^i(v \times \iota_q') = v \times sq^i(\iota_q'),\ 0 < i < n. \qquad (D)$$

Now let $p + q = n$ with $p,q \geqq 1$. Consider $sq^i(\iota_p' \times \iota_q')$. From (A) we see that it is the sum of terms and that terms for the index k are specified for $-p \leqq k \leqq p-1$. From (C) we see that the last 2q-1 terms are specified, namely those terms for which $-q+i+1 \leqq k \leqq i+q$. But $-q + i + 1 = i + 1 + p - p - q < n + 1 - n + p = p + 1$. Therefore $-q + i + 1 \leqq p$. Thus the terms are specified for all values of the index k. Hence

$$sq^i(\iota_p' \times \iota_q') = \sum_{k=0}^{i} sq^k(\iota_p') \times sq^{i-k}(\iota_q'). \qquad (E)$$

The Cartan formula follows now from (B), (D), (E) and Theorem 1.5. Consider first the case $p,q \geqq 1$. Let (X,A), (Y,B) be pairs of spaces such that $\{A \times Y, X \times A\}$ is an excisive couple in $X \times Y$. Suppose x is in $H^p(X,A)$ and y is in $H^q(Y,B)$. Construct the following diagram

$$(X,A) \times (Y,B) \xleftarrow{j_1 \times j_2} (X^*,A^*) \times (Y^*,B^*) \xrightarrow{f \times g} (X_p,*) \times (X_q,*)$$

where the middle pairs are the geometric realizations of the first pairs and j_1, j_2 are the maps of Theorem 1.5. Also f and g are selected so that

$f^*(\iota_p') = j_1^*(x)$ and $g^*(\iota_q') = j_2^*(y)$. It follows from the fact that $\{A \times Y, X \times A\}$ is an excisive couple that $\{A^* \times Y^*, X^* \times A^*\}$ is an excisive couple in $X^* \times Y^\times$ and consequently that $(j_1 \times j_2)^*$ is an isomorphism, (cf.(6),p.493,497).

We have

$$(j_1 \times j_2)^* sq^i(x \times y) = sq^i(j_1 \times j_2)^*(x \times y) =$$

$$sq^i(j_1^*(x) \times j_2^*(y)) = sq^i(f^*(\iota_p') \times g^*(\iota_q')) =$$

$$sq^i(f \times g)^*(\iota_p' \times \iota_q') = (f \times g)^* sq^i(\iota_p' \times \iota_q') =$$

$$(f \times g)^* \sum_{k=o}^{i} sq^k(\iota_p') \times sq^{i-k}(\iota_q') \text{ by (E).}$$

Also

$$(j_1 \times j_2)^* \sum_{k=0}^{i} sq^k(x) \times sq^{i-k}(y) = \sum_{k=0}^{i} sq^k\, j_1^*(x) \times sq^{i-k} j_2^*(y)$$

$$= \sum_{k=0}^{i} sq^k\, f^*(\iota_p') \times sq^{i-k} g^*(\iota_q') = (f \times g)^* \sum_{k=0}^{i} sq^k(\iota_p') \times sq^{i-k}(\iota_q').$$

Therefore we conclude $sq^i(x \times y) = \sum_{k=0}^{i} sq^k(x) \times sq^{i-k}(y)$.

In the case $p = 0$, $q = n$ (or $p = n$, $q = 0$) a similar argument using (B) (or (D)) shows that

$$sq^i(x \times y) = x \times sq^i(y) \text{ if } p = 0,\ q = n \text{ and}$$

$$sq^i(x \times y) = sq^i(x) \times y \text{ if } p = n,\ q = 0.$$

CHAPTER III

THE COHOMOLOGY SUSPENSION.

The cohomology suspension is a map

$$\sigma^* : H^i(X_n,*) \to H^{i-1}(X_{n-1})$$

defined for each $n \geqq 2$ and $i \geqq 2$. Using Brown's generalization of the Eilenberg-Zilber Theorem in terms of the twisted tensor product (1) we will show that the cohomology suspension is an isomorphism for each $n \geqq 2$ and $2 \leqq i < 2n$. The cohomology suspension has been defined in Chapter I. We will still write $H^i(K)$ for $H^i(K,Z_2)$.

Path Spaces

Let R^+ denote the non-negative real numbers and I_r denote the closed interval from 0 to r, r being in R^+. The space of paths P(B) in a topological space B is defined by

$$P(B) = \{(\alpha,r) : \alpha : I_r \to B, r \text{ in } R^+\}.$$

Let $h : P(B) \to B^I \times R^+ (I = I_1)$ be given by $h(\alpha,r) = (\alpha',r)$ where $\alpha'(t) = \alpha(tr)$, t in I. B^I is given the compact open topology and $B^I \times R$ is given the product topology.

<u>Proposition 3.1</u>: h is an injection.

Let $(\alpha,r) \neq (\beta,s)$. Suppose $r \neq s$. Then $(\alpha',r) \neq (\beta',s)$ since r and s are different. Suppose next that $r = s$ and $\alpha \neq \beta$. Then $\alpha(t_0) \neq \beta(t_0)$ for some t_0 in I_r. There are two cases to consider, namely $r = 0$ and $r \neq 0$.

Case 1. $r = 0$. Then $\alpha(0) \neq \beta(0)$. In this case $\alpha'(t) = \alpha(0)$ for all t in I and $\beta'(t) = \beta(0)$ for all t in I hence $\alpha' \neq \beta'$.

Case 2. $r \neq 0$. Let $t = t_0/r$. Then $\alpha'(t) = \alpha((t_0/r)r) = \alpha(t_0)$ and $\beta'(t) = \beta((t_0/r)r) = \beta(t_0)$. Therefore $\alpha'(t) \neq \beta'(t)$ and it follows that $\alpha' \neq \beta'$. Hence we have shown that h is an injection.

P(B) is given a topology by requiring that h be homeomorphism of P(B) and its image.

It is possible to define a multiplication for certain pairs of paths in P(B). Paths (α,r) and (β,s) such that $\alpha(r) = \beta(0)$ are multiplied as follows:

$$(\alpha,r)(\beta,s) = (\gamma,r + s) \text{ where}$$

$$\gamma(t) = \alpha(t) \text{ if } 0 \leqq t \leqq r \text{ and}$$

$$\gamma(t) = \beta(t-r) \text{ if } r \leqq t \leqq r + s.$$

Usually we will surpress r and s and write $\alpha\beta$ for the multiplication of the paths (α,r) and (β,s).

Let $b \in B$. We will let e_b denote the pair $(e_b,0)$ where $e_b(0) = b$. Then $e_{\alpha(0)}$ and $e_{\alpha(r)}$ are respectively a left and right identity for (α,r) with respect to the multiplication defined above.

E(B) and $\Omega(B)$ will denote respectively the subspaces of P(B) consisting of all paths ending at b and the subspace of all paths beginning and ending at b. The multiplication in P(B) defines an associative multiplication with unit in $\Omega(B)$ and defines an action of $\Omega(B)$ on the right of E(B).

Fiber Spaces

Suppose $p : X \to B$ is continuous.

Let $U_p \subset P(B) \times X$ be defined by

$$U_p = \{(\alpha, x) \quad \alpha(r) = p(x)\}.$$

A lifting function for the map p is a map $\lambda : U_p \to X$ such that $p\,\lambda(\alpha, x) = \alpha(o)$. A lifting function λ is transitive if

i) $\lambda(e_b, x) = x$ for x in X, $b = p(x)$

ii) $\lambda(\alpha\beta, x) = \lambda(\alpha, \lambda(\beta, x))$ when $\alpha\beta$ is defined and (β, x) is in U_p.

<u>Definition 3.1</u>: A transitive fiber space is a quadruple (X, B, p, λ) where $p : X \to B$ and λ is a transitive lifting function for p.

Consider the quadruple (EX_n, X_n, p, λ) where $p : EX_n \to X_n$ is defined by $p(\alpha) = \alpha(o)$ and $\lambda : U_p \to EX_n$ is given by $\lambda(\alpha, \beta) = (\alpha\beta)$.

<u>Proposition 3.2</u>: (EX_n, X_n, p, λ) is a transitive fiber space.

By definition we have

$$U_p = \{(\alpha, \beta) : \alpha(r) = p(\beta) = \beta(0)\}$$

so the composition of paths $\alpha\beta$ is always defined and hence λ is well defined. In this case λ is also a transitive lifting function for p because

i) if α is in EX_n then $\lambda(e_{\alpha(0)}, \alpha) = e_{\alpha(0)}(\alpha) = \alpha$ and

ii) $\lambda(\alpha\beta,\gamma) = (\alpha\beta)\gamma = \alpha(\beta\gamma) = \lambda(\alpha,\lambda(\beta,\gamma))$

whenever $\alpha\beta$ is defined and (β,γ) is in U_p. Note that (β,γ) being in U_p is equivalent to $\beta\gamma$ being defined. Thus (EX_nX_n,p,λ) is a transitive fiber space.

Twisted Tensor Products

The reader is referred to Homology by S. MacLane (3) for the definitions of DGA module, DGA algebra, DGA module over a DGA algebra, and DGA coalgebra. DGA means differential graded augmented. In all cases we will assume the ground ring to be the field Z_2. Hom will mean Hom_{Z_2} and $\otimes$ will mean $\otimes_{Z_2}$.

Let K be a DGA coalgebra with $d : K \to K \otimes K$ as coproduct. Let G, N, and H be Z_2-modules and $u : G \otimes N \to H$ be a Z_2 -homomorphism. Let U be in Hom(K,G), V be in Hom(K,N), and c be in $K \otimes N$.

Definition 3.2: We define the cup product $U \smile V$ and the cap product $c \frown U$ as follows:

$$U \smile V = u(U \otimes V)\, d$$

$$c \frown U = (1 \otimes u)(1 \otimes U \otimes 1)(d \otimes 1)(c)$$

where 1 denotes the appropriate identity map. We see that $U \smile V$ is a member of Hom(K,H) and that $c \frown U$ is a member of $K \otimes H$.

Definition 3.3: Let K be a DGA coalgebra and A be a DGA algebra. A twisting cochain is a member φ of Hom(K,A) such that if $\varphi = \Sigma\varphi_q$ then

1) φ_q is in $\mathrm{Hom}(K_q,A)$, $\varphi_0 = 0$, $\varphi_q(K_q) \subset A_{q-1}$

2) $\epsilon\varphi_1 = 0$ and $\partial\varphi_q = \varphi_{q-1}\,\partial + \sum_{k=1}^{q-1} \varphi_k \smile \varphi_{q-k}$

where $a : A \to Z_2$ is the augmentation and the cup product is formed using the multiplication in A.

Definition 3.4: Let K be a DGA coalgebra, A an DGA algebra, and L an DGA A-module. Let $\varphi : K \to A$ be a twisting cochain. We define a Z_2 - homomorphism $\partial_\varphi : K \otimes L \to K \otimes L$ as follows:

$$\partial_\varphi(k \otimes h) = \partial k \otimes h + k \otimes \partial h + (k \otimes h) \frown \varphi \qquad \text{(B)}$$

where k is in K, h is in L, and the cap product is formed using the pairing $A \otimes L \to L$ defined by the A-module structure of L. We see from Definition 3.2 that ∂_φ can be written as

$$\partial_\varphi = \partial \otimes 1 + 1 \otimes \partial + (1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1)$$

where $d : K \to K \otimes K$ is the coproduct and $u : A \otimes L \to A$ is scalar multiplication. Since ∂ and φ each lower dimension by one and u, d preserve dimension we see that ∂_φ lowers dimension by one in $K \otimes L$.

Proposition 3.3: Let ∂_φ be as defined in definition 3.3 then $\partial_\varphi \partial_\varphi = 0$.

From (B)

$$\begin{aligned}
\partial_\varphi\partial_\varphi = {} & (\partial \otimes 1 + 1 \otimes \partial + (1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1))(\partial \otimes 1 + 1 \otimes \partial + \\
& (1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1)) \\
= {} & \partial\partial \otimes 1 + 1 \otimes \partial\partial + \partial \otimes \partial + \partial \otimes \partial + \\
& (\partial \otimes 1)(1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1) + && \text{(i)} \\
& (1 \otimes \partial)(1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1) + && \text{(ii)} \\
& (1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1)(\partial \otimes 1) + && \text{(iii)} \\
& (1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1)(1 \otimes \partial) + && \text{(iv)} \\
& (1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1)(1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1). && \text{(v)}
\end{aligned}$$

Because $\partial\partial = 0$ and Z_2 is the ground ring we see that the sum of the first four terms is 0. The remaining five terms have been assigned numbers so that we may easily identify them.

From (i) we have

$$(\partial \otimes 1)(1 \otimes u)(1 \otimes \varphi\ 1)(d \otimes 1) = (1 \otimes u)(\partial \otimes 1 \otimes 1)(1 \otimes \varphi \otimes 1)(d \otimes 1).$$

From (iii) we have

$$(1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1)(\partial \otimes 1) = (1 \otimes u)(1 \otimes \varphi \otimes 1)(\partial \otimes 1 \otimes 1 + 1 \otimes \partial \otimes 1)(d \otimes 1)$$

because the coproduct d is a chain mapping,

$$= (1 \otimes u)(1 \otimes \varphi \otimes 1)(\partial \otimes 1 \otimes 1)(d \otimes 1) + (1 \otimes u)(1 \otimes \varphi \otimes 1)(1 \otimes \partial \otimes 1)(d \otimes 1).$$

$$= (1 \otimes u)(\partial \otimes \varphi \otimes 1)(d \otimes 1) + (1 \otimes u)(1 \otimes \varphi\partial \otimes 1)(d \otimes 1)$$

$$= (1 \otimes u)(\partial \otimes 1 \otimes 1)(1 \otimes \varphi \otimes 1)(d \otimes 1) + (1 \otimes u)(1 \otimes \varphi\partial \otimes 1)(d \otimes 1).$$

Therefore we have (i) + (iii) $= (1 \otimes u)(1 \otimes \varphi\partial \otimes 1)(d \otimes 1).$

From (v)

$$(1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1)(1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1) =$$

$$(1 \otimes u)(1 \otimes \varphi \otimes 1)(1 \otimes 1 \otimes u)(d \otimes 1 \otimes 1)(1 \otimes \varphi \otimes 1)(d \otimes 1) =$$

$$(1 \otimes u)(1 \otimes 1 \otimes u)(1 \otimes \varphi \otimes 1 \otimes 1)(1 \otimes 1 \otimes \varphi \otimes 1)(d \otimes 1 \otimes 1)(d \otimes 1) =$$

$$(1 \otimes u)(1 \otimes u \otimes 1)(1 \otimes \varphi \otimes \varphi \otimes 1)(1 \otimes d \otimes 1)(d \otimes 1)$$

because u and d are associative.

$$= (1 \otimes u)(1 \otimes u(\varphi \otimes \varphi)d \otimes 1)(d \otimes 1) =$$

$(1 \otimes u)(1 \otimes (\varphi \smile \varphi) \otimes 1)(d \otimes 1)$ by definition 3.2

where we use the pairing $u : A \otimes L \to L$ to form the cup product

$= (1 \otimes u)(1 \otimes \partial\varphi + \varphi\partial \otimes 1)(d \otimes 1)$ from Definition 3.3

$= (1 \otimes u)(1 \otimes \varphi\partial \otimes 1)(d \otimes 1) + (1 \otimes u)(1 \otimes \partial\varphi \otimes 1)(d \otimes 1).$

Therefore we have

$$(i) + (iii) + (v) = (1 \otimes u)(1 \otimes \partial\varphi \otimes 1)(d \otimes 1).$$

Looking at this last term we have

$(1 \otimes u)(1 \otimes \partial\varphi \otimes 1)(d \otimes 1) = (1 \otimes u)(1 \otimes \partial \otimes 1)(1 \otimes \varphi \otimes 1)(d \otimes 1)$

$= ((1 \otimes u)(1 \otimes 1 \otimes \partial) + (1 \otimes \partial)(1 \otimes u))(1 \otimes \varphi \otimes 1)(d \otimes 1)$

because u is a chain mapping

$= (1 \otimes u)(1 \otimes 1 \otimes \partial)(1 \otimes \varphi \otimes 1)(d \otimes 1) + (1 \otimes \partial)(1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1)$

$= (1 \otimes u)(1 \otimes \varphi \otimes \partial)(d \otimes 1) + (ii) =$

$(1 \otimes u)(1 \otimes \varphi \otimes 1)(1 \otimes 1 \otimes \partial)(d \otimes 1) + (ii) =$

$(1 \otimes u)(1 \otimes \varphi \otimes 1)(d \otimes 1)(1 \otimes \partial) + (ii) =$

$(iv) + (ii).$

Therefore $(i) + (ii) + (iii) + (iv) + (v) = 0.$

Definition 3.5: Let K be a DGA coalgebra, A be a DGA algebra, and L be a DGA A-module. Let φ: $K \to A$ be a twisting cochain. The twisted tensor product of K and L with respect to the twisting cochain φ is the DGA Z_2-module $K_\varphi \otimes L$ defined as follows: with respect to grading and augmentation $K_\varphi \otimes L = K \otimes L$. The differentiation ∂_φ on $K_\varphi \otimes L$ is defined in definition 3.4. From Proposition 3.3 we see that ∂_φ is a differentiation.

The Topology of (EX_n, X_n, p) in Terms of the Twisted Tensor Product

Let $*$ be in X_n. Let $S(X_n)$ denote the chain complex with Z_2 coefficients generated by singular simplices taking the vertices of the standard simplex into $*$. Since X_n is arcwise connected it follows by a well known theorem that $S(X_n)$ is chain equivalent to the complex of chains with Z_2-coefficients generated by all singular simplicies whose image is in X_n.

$S(X_n)$ is a coalgebra if the coproduct is defined as follows. Let σ be a singular simplex of dimension q whose vertices are all mapped to $*$. Suppose $0 \leqq k \leqq q$. Let $\sigma(0,1,\ldots,k)$ be the singular k-simplex defined by

$$\sigma(0,1,\ldots,k)(t_0,\ldots,t_k) = \sigma(t_0,\ldots,t_k,0,\ldots,0).$$

Let $\sigma(k,\ldots,q)$ be the singular (q-k)-simplex defined by

$$\sigma(k,\ldots,q)(t_0,\ldots,t_{q-k}) = \sigma(0,\ldots,0,t_0,\ldots,t_{q-k}).$$

Here $(t_0,\ldots,t_p)$ is the usual representation for a member of the standard p-simplex. We see that $\sigma(0,\ldots,k)$ and $\sigma(k,\ldots,q)$ are members of $S(X_n)$. Define $d(\sigma)$ by

$$d(\sigma) = \sum_{k=0}^{q} \sigma(0,\ldots,k) \otimes \sigma(k,\ldots,q).$$

This defines d on the generators of $S(X_n)$ hence by extending linearly we have a Z_2-homomorphism $d\colon S(X_n) \to S(X_n) \otimes S(X_n)$. It is well known that $S(X_n)$ is an associative DGA coalgebra with d as the coproduct.

The transitive fiber space (EX_n, X_n, p, λ) has for its fiber $p^{-1}(*) = \Omega X_n$. Let the continuous map $m\colon \Omega X_n \times \Omega X_n \to \Omega X_n$ be defined by $m(\alpha,\beta) = \alpha\beta$. Let

$$g : S(\Omega X_n) \otimes S(\Omega X_n) \to S(\Omega X_n \times \Omega X_n)$$

be the Eilenberg-Zilber map. Here $S(\Omega X_n)$ and $S(\Omega X_n \times \Omega X_n)$ is the chain complex with Z_2-coefficients generated by all singular simplices.
We have the following diagram

$$S(\Omega X_n) \otimes S(\Omega X_n) \xrightarrow{g} S(\Omega X_n \times \Omega X_n) \xrightarrow{m_\#} S(\Omega X_n)$$

where $m_\#$ is the Z_2-homomorphism induced by m.
Let 1 denote the 0-simplex in $S(\Omega X_n)$ whose image is e_*. Then $S(\Omega X_n)$ is a DGA algebra under the multiplication $m_\# g$.

<u>Theorem 3.1</u>: There is a twisting cochain
$\Phi : S(X_n) \to S(\Omega X_n)$ which satisfies

i) If w is a constant simplex in $S(X_n)$, $\Phi(w) = 0$.

This is Theorem 4.1 of (1).

For the transitive fiber space (EX_n, X_n, p, λ) the lifting function $\lambda : U_p \to EX_n$ defines a map $\overline{\lambda} : \Omega X_n \times \Omega X_n \to \Omega X_n$ by taking $\overline{\lambda}$ to be the restriction of λ to $\Omega X_n \times \Omega X_n$. Notice that $\overline{\lambda} = m$ (defined above). Therefore we can use λ to define a DGA $S(\Omega X_n)$-module structure on the DGA Z_2-module $S(\Omega X_n)$. We see that this $S(\Omega X_n)$-module structure is just the structure obtained by regarding the DGA algebra $S(\Omega X_n)$ as a DGA $S(\Omega X_n)$-module.

Using the twisting cochain described in Theorem 3.1 we can form the twisted tensor product

$$S(X_n)_\Phi \otimes S(\Omega X_n).$$

The following theorem is the main theorem of Brown's (1) and gives the relation between $S(X_n)$ (the base) $S(\Omega X_n)$(the fiber), and $S(EX_n)$(the total space). Let $S(EX_n)$ denote the chains with Z_2-coefficients generated by those singular simplices whose vertices are mapped to ΩX_n. It is well known that $S(EX_n)$ is chain equivalent to the chain complex with Z_2-coefficients generated by all singular simplices whose image is in EX_n.

Theorem 3.2: Let Φ be the twisting cochain in Theorem 3.1. There is a chain equivalence

$$\psi : S(X_n)_\Phi \otimes S(\Omega X_n) \to S(EX_n).$$

This is Theorem (4.1) of (1). The definition of ψ is given in the proof of this theorem, but will be omitted because it is complicated and will not be needed.

Let $D \subset S(X_n)$ be the subcomplex consisting of all degenerate chains. See (3), p. 236 for the definition and properties of D. Let $i:\Omega X_n \to EX_n$ be the inclusion mapping and let

$h' : S(\Omega X_n) \to S(X_n)_\Phi \otimes S(\Omega X_n)$ and

$\pi' : S(X_n)_\Phi \otimes S(\Omega X_n) \to S(X_n)$ be defined as follows:

Let 1 be the zero simplex of $S(X_n)$ and let a be the augmentation of $S(\Omega X_n)$.

$$h'(S) = 1 \otimes X \qquad S \text{ in } S(\Omega X_n)$$

$$\pi'(T \otimes S) = a(S)T \qquad T \text{ in } S(X_n),\ S \text{ in } S(\Omega X_n)$$

Lemma 3.1: $\psi : S(X_n)_\Phi \otimes S(\Omega X_n)$ can be chosen so that $\psi h' = i_\#$ and $P_\# \psi = \pi'(\text{mod } D)$.

This is Lemma 7.4 of 1. We will assume hereafter that ψ is always chosen to satisfy this lemma.

Let $S_n(X_n)$ denote the chains generated by singular simplexes taking the n-1 skeleton of the standard simplex into * and let $j:S_n(X_n) \to$

$S(X_n)$ be the inclusion map. Let $\Phi' = \Phi j : S_n(X_n) \to S(\Omega X_n)$. Φ' is obviously a twisting cochain. Let 1 be the identity map on $S(\Omega X_n)$.

Lemma 3.2: $\psi(j \otimes 1) : S_n(X_n)_{\Phi'} \otimes S(\Omega X_n) \to S(EX_n)$ is a chain equivalence and $\Phi'_q = 0$ for $q < n$.

This is Corollary 4.3 of (1).

Let h' and π' be as in Lemma 3.1. Define $\pi = \pi'(j \otimes 1)$. We see that $\mathrm{Im}(h')$ is contained in $S_n(X_n) \otimes S(\Omega X_n)$ so take $h: S(\Omega X_n) \to S_n(X_n)_{\Phi'} \otimes S(\Omega X_n)$ to be h' with its range restricted.

Lemma 3.3: $\psi(j \otimes 1)h = i_{\#}$ and $p_{\#}\, \psi(j \otimes 1) = \pi \pmod D$.

This is an immediate consequence of Lemma 3.1.

$$p_*: H_i(EX_n, \Omega X_n; Z_2) \to H_i(X_n, * ; Z_2) \text{ is an Isomorphism}$$
$$\text{for } 2 \leqq i < 2n,\ n \geqq 2.$$

Henceforth we will assume that all homology groups have Z_2-coefficients. From Lemma 3.3 we have that $\mathrm{Im}(h) = 1 \otimes S(\Omega X_n)$ is a subcomplex of $S_n(X_n)_{\Phi'} \otimes S(\Omega X_n)$, that $\psi(j \otimes 1)(1 \otimes (S\Omega X_n))$ is contained in $S(\Omega X_n)$, and $\psi(j \otimes 1)$ restricted to $1 \otimes S(\Omega X_n)$ is a chain equivalence of $1 \otimes S(\Omega X_n)$ and $S(\Omega X_n)$. We have proved therefore

Lemma 3.4: The chain equivalence $\psi(j \otimes 1)$ induces a chain equivalence $\psi' : S_n(X_n)_{\Phi'} \otimes S(\Omega X_n)/1 \otimes S(\Omega X_n) \to (S(EX_n)/S(\Omega X_n))$. ψ' maps the equivalence class containing y to the equivalence class containing $\psi(j \otimes 1)(y)$.

Let W $S_n(X_n)$ be the subcomplex generated by the constant singular simplexes. Denote by w_q the constant simplex of dimension q.

Lemma 3.5: $W \otimes S(\Omega X_n)$ is a subcomplex of $S_n(X_n)_{\Phi'} \otimes S(\Omega X_n)$.

$W \otimes S(\Omega X_n)$ is generated by elements of the form $w_q \otimes T$, T in $S(\Omega X_n)$. We need to show that $\partial_{\Phi'}(w_q \otimes T)$ is a member of $W \otimes S(\Omega X_n)$. By Definition 3.4

$$\partial_{\Phi'}(w_q \otimes T) = \partial w_q \otimes T + w_q \otimes \partial T + w_q \otimes T \frown \Phi'$$

$$= \partial w_q \otimes T + w_q \otimes \partial T + (1 \otimes m_{\#}g)(1 \otimes \Phi' \otimes 1)(d \otimes 1)(w_q \otimes T).$$

By definition $d(w_q) = \sum_{i=0}^{q} w_i \otimes w_{q-i}$ therefore

$$\partial_{\Phi'}(w_q \otimes T) = \partial_q \otimes T + w_q \otimes \partial T + \sum_{i=0}^{q} w_i \otimes \Phi'(w_{q-i})(T)$$

(We write $(T)(T')$ for $m_{\#}g(T \otimes T')$). But by Theorem 3.1 $\Phi'(w) = 0$, hence $\partial_{\Phi'}(w_q \otimes T) = \partial w_q \otimes T + w_q \otimes \partial T$ which is in $W \otimes S(\Omega X_n)$.

We have $1 \otimes S(\Omega X_n) \subset W \otimes S(\Omega X_n)$, so the identity map induces a chain map

$$I : S_n(X_n)_{\Phi'} \otimes S(\Omega X_n)/1 \otimes S(\Omega X_n) \to S_n(X_n)_{\Phi'} \otimes S(\Omega X_n)/W \otimes S(\Omega X_n).$$

I is an onto mapping so the following is a short exact sequence of complexes:

$$0 \to \text{Ker } I \xrightarrow{i} \text{Domain } I \xrightarrow{I} \text{Range } I \to 0 \qquad \text{(A)}$$

where i is the inclusion map.

Lemma 3.6: $I_* : H(\text{Domain } I) \to H(\text{Range } I)$ is an isomorphism.

Associated with the short exact sequence (A) is the long exact sequence

$$\ldots \to H_n(\text{Ker} I) \xrightarrow{(i_n)_*} H_n(\text{Domain } I) \xrightarrow{(I_n)_*} H_n(\text{Range } I) \xrightarrow{\partial_*} H_{n-1}(\text{Ker } I)) \to \ldots$$

We see therefore that I_* will be an isomorphism if $H(\text{Ker } I) = 0$. From the definition of I

$$\text{Ker } I = W^{*} \otimes S(\Omega X_n)$$

where $(W^{*})_0 = 0$ and $(W^{*})_i = W_i$ if $i > 0$. The differential for $W^{*} \otimes S(\Omega X_n)$ is $\partial_{\Phi'}$, but in this case it is also the usual differential for the tensor product of two complexes. This is a consequence of the proof of Lemma 3.5 and the fact that $\partial(W_1) = 0$. Hence by Theorem 1.1 $H(W^{*} \otimes S(\Omega X_n)) \cong H(W^{*}) \otimes H(S(\Omega X_n))$. But $H(W^{*}) = 0$, hence $H(\text{Ker } I) = 0$.

The Z_2-module $S_n(X_n) \otimes S(\Omega X_n)/W \otimes S(\Omega X_n)$ is isomorphic to the Z_2-module $(S_n(X_n)/W) \otimes S(\Omega X_n)$ by the correspondence $\Lambda[S \otimes T] = [S] \otimes T$ where S is in $S_n(X_n)$, T is in $S(\Omega X_n)$, and $[\ \ ]$ is the appropriate equivalence class. Let $\bar{\partial}$ denote the differential for Domain (Λ). It is induced by $\partial_{\Phi'}$. We can use Λ to define a differentiation ∂' on Range (Λ) by

<u>Definition 3.6</u>:
$$\begin{aligned}\partial'([S] \otimes T) &= \Lambda\, \bar{\partial}\, \Lambda^{-1}([S] \otimes T) \\ &= \Lambda\, \bar{\partial}[S \otimes T] \\ &= \Lambda[\partial_{\Phi'}(S \otimes T)] \\ &= \Lambda[\partial S \otimes T + S\ \partial T + S \otimes T \frown \Phi'] \\ &= [\partial S] \otimes T + [S] \otimes \partial T + \Lambda[S \otimes T \frown \Phi'].\end{aligned}$$

where S is in $S_n(X_n)$ and T is in $S(\Omega X_n)$. It is clear that $\partial'\partial' = 0$ and $\Lambda\partial' = \partial'\Lambda$. We have proved therefore

<u>Lemma 3.7</u>: Λ is a chain equivalence of the complexes $S_n(X_n)_{\Phi'} \otimes S(\Omega X_n)/W \otimes S(\Omega X_n)$ and $(S_n(X_n)/W) \otimes S(\Omega X_n)$ with the differentiation of Definition 3.6.

Next we will compute the homology of the complex $(S_n(X_n)/W) \otimes S(\Omega X_n)$ in dimensions smaller than $2n$. The computation rests on the following lemmas.

<u>Lemma 3.8</u>: If $q < 2n$ then $\partial\Phi'_q = \Phi'_{q-1}\partial$.

This is a consequence of Lemma 3.2 because since Φ' is a twisting cochain we have

$$\partial\Phi' = \Phi'_{q-1}\,\partial + \sum_{i=1}^{q-1} \Phi'_i \smile \Phi'_{q-i}.$$

But $\Phi'_i = 0$ if $i < n$ by Lemma 3.2. The conclusion follows now because if $q < 2n$ then either $i < n$ or $q-i < n$.

From Proposition 1.1 we conclude that $H_n(S_n(X_n)) \cong Z_2 (n > 1)$. From Theorem 1.3 and Proposition 1.1 we conclude that $H_{n-1}(S(\Omega X_n)) \cong Z_2$. Denote by e_n a fixed fundamental n-cycle of $S_n(X_n)$, i.e. e_n is such that $\text{cls}(e_n) \neq 0$.

<u>Lemma 3.9</u>: Let $n \geqq 2$. $\Phi'_n(e_n)$ is a fundamental cycle of $S(\Omega X_n)$.

We see from Lemma 3.8 that Φ'_n is a chain mapping in dimension n. Hence it is sufficient to show that there exists one n-cycle in $S_n(X_n)$ which is mapped to a fundamental (n-1)-cycle in $S(\Omega X_n)$ by Φ'.

By Lemma 3.2: $S_n(X_n)_{\Phi'} \otimes S(\Omega X_n)$ is acyclic. Let x be a fundamental (n-1) - cycle of $S(\Omega X_n)$. Consider the chain $1 \otimes x$.

$\partial_{\Phi'}(1 \otimes x) = \partial 1 \otimes x + 1 \otimes \partial x + (1 \otimes x)\frown\Phi' = 0.$

Thus $1 \otimes x$ is an (n-1)-cycle. There exists an n-chain y of $S_n(X_n) \otimes S(\Omega X_n)$ such that $\partial_{\Phi'}(y) = x$. We can suppose

$$y = \sum_{i=0}^{n-1} w_i \otimes T_{n-i} + \sum_\alpha s_\alpha \otimes T_\alpha$$

where T_{n-i} is an (n-i)-chain of $S(\Omega X_n)$, each s_α is an n-simplex in $S_n(X_n)$, and each T_α is a zero simplex in $S(\Omega X_n)$. Therefore

$$\partial_{\Phi'}(y) = \sum_{i=0}^{n-1}(\partial w_i \otimes T_{n-i} + w_i \otimes \partial T_{n-i} + (w_i \otimes T_{n-i})\frown\Phi')$$

$$+ \sum_\alpha(\partial s_\alpha \otimes T_\alpha + s_\alpha \otimes \partial T_\alpha + s_\alpha \otimes T_\alpha\frown\Phi').$$

Now $(w_i \otimes T_{n-i}) \frown \Phi' = 0$, $0 \leqq i \leqq n-1$ and $\partial T_\alpha = 0$ for all α. Also for each n-simplex s_α we have

$$d(s_\alpha) = \sum_{i=0}^{n} s_\alpha(0,\ldots,i) \otimes s_\alpha(i,\ldots,n)$$

$$= 1 \otimes s_\alpha + s_\alpha \otimes 1 + \sum_{i=1}^{n-1} w_i \otimes w_{n-i}.$$

Therefore

$$\partial_{\Phi'}(y) = \sum_{i=0}^{n-1} \partial w_i \otimes T_{n-i} + w_i \otimes \partial T_{n-i}$$

$$+ \sum_\alpha(\partial s_\alpha \otimes T_\alpha + 1 \otimes \Phi'(s_\alpha)T_\alpha + s_\alpha \otimes \Phi'(1)\, T_\alpha + \sum_{i=1}^{n-1} w_i \otimes \Phi'(w_{n-i})\, T_\alpha)$$

$$= \sum_{i=0}^{n-1} \partial w_i \otimes T_{n-i} + w_i \otimes \partial T_{n-i} + \sum_\alpha \partial s_\alpha \otimes T_\alpha + 1 \otimes \Phi'(s_\alpha)\, T_\alpha.$$

But $\partial_{\Phi'}(y) = 1 \otimes x$ also. Therefore we conclude

$$1 \otimes x = 1 \otimes \partial T_n + \sum_\alpha 1 \otimes \Phi'(s_\alpha)T_\alpha \text{ and hence}$$

$$x = \partial T_n + \sum_\alpha \Phi'(s_\alpha)T_\alpha.$$ Now x is a fundamental cycle and ∂T_n is a bound so $\sum_\alpha \Phi'(s_\alpha)T_\alpha$ is a fundamental cycle. Let 1 be the unit in $S(\Omega X_n)$. Therefore the 0-simplex whose image is the path e_* is 1. Since ΩX_n is arcwise connected there is a one-simplex T'_α such that $\partial T'_\alpha = 1 + T_\alpha$ for each α. Now

$$\Phi'(s_\alpha)T_\alpha + \Phi'(s_\alpha) = \Phi'(s_\alpha)(T_\alpha + 1) =$$

$$\Phi'(s_\alpha)(\partial T'_\alpha) = 0 + \Phi'(s_\alpha)(\partial T')$$

$$= \Phi'(\partial s_\alpha) + \Phi'(s_\alpha)(\partial T'_\alpha) \text{ by Lemma 3.2}$$

$$= \partial\Phi'(s_\alpha) + \Phi'(s_\alpha)(\partial T'_\alpha) \text{ by Lemma 3.8}$$

$$= \partial(\Phi'(s_\alpha)(T'_\alpha)).$$ Therefore

$\Phi'(s_\alpha)T_\alpha$ and $\Phi'(s_\alpha)$ are members of the same homology class.

Thus $\sum_\alpha \Phi'(s_\alpha) = \Phi(\Sigma(s_\alpha))$ is a fundamental (n-1) cycle.

The proof of the lemma is complete if we can show Σs_α is an n-cycle in $S_n(X_n)$. Now $\partial(\Sigma s_\alpha) = 0$ or w_{n-1}. If $\partial(\Sigma s_\alpha) = w_{n-1}$ then n must be even. In that case $\Sigma s_\alpha + w_n$ is a cycle and $\Phi'(\Sigma s_\alpha + w_n) = \Phi'(\Sigma s_\alpha)$. Thus for each $n \geqq 2$ there is at least one n-cycle whose image under Φ' is a fundamental (n-1) cycle.

Let $\partial^{\otimes}$ denote the usual differentiation for the tensor product $(S_n(X_n)/W) \otimes S(\Omega X_n)$.

<u>Lemma 3.10</u>: $\partial_i' = \partial_i^{\otimes}$ for $i < 2n$.

$(S_n(X_n)/W) \otimes S(\Omega X_n)$ is generated by elements $\sigma \otimes \tau$ where σ is a p-simplex in $S_n(X_n)/W$ and τ is a q-simplex in $S(\Omega X_n)$. Suppose $p + q < 2n$. Then

$$d(\sigma) = \sum_{i=0}^{p} \sigma(0,\ldots,i) \otimes \sigma(i,\ldots,p) = \sum_{i=0}^{n-1} w_i \otimes \sigma(i,\ldots,p) + \sum_{i=n}^{p} \sigma(0,\ldots,i) \otimes w_i .$$

Therefore by definition

$$\begin{aligned} \partial'(\sigma \otimes \tau) &= [\partial\sigma] \otimes \tau + [\sigma] \otimes \partial\tau + \Lambda[\sigma \otimes \tau \frown \Phi'] \\ &= [\partial\sigma] \otimes \tau + [\sigma] \otimes \partial\tau + \\ &\Lambda[\sum_{i=0}^{n-1} w_i \otimes \Phi'\sigma(i,\ldots,p)\ \tau + \sum_{i=n}^{p} \sigma(0,\ldots,i) \otimes \Phi'(w_{p-i})\ \tau] \\ &= [\partial\sigma] \otimes \tau + [\sigma] \otimes \partial\tau + 0 + 0 = \partial^{\otimes}(\sigma \otimes \tau). \end{aligned}$$

Let e_n denote a fundamental cycle in $S_n(X_n)$ and $\bar{e}_n$ denote the corresponding fundamental cycle in $S_n(X_n)/W$. Let $e_{n-1} = \Phi'(e_n)$. We showed in Lemma 3.9 that e_{n-1} is a fundamental cycle in $S(\Omega X_n)$. Let $Z_2(\bar{e}_n \otimes e_{n-1})$ denote the subspace of $(S_n(X_n)/W) \otimes S(\Omega X_n)$ generated by $\bar{e}_n \otimes e_{n-1}$.

<u>Lemma 3.11</u>: $\operatorname{Im}\partial_{2n}' = \operatorname{Im}\partial_{2n}^{\otimes} \oplus Z_2(\bar{e}_n \otimes e_{n-1})$.

We will show first that the set on the left is a subset of the one on the right. Let σ be a p-simplex of $S_n(X_n)/W$ and τ be a q-simplex of $S(\Omega X_n)$. Suppose $p + q = 2n$. If $p < 2n$ we see from the proof of Lemma 3.10 that $\partial'_{2n}(\sigma \otimes \tau) = \partial^{\otimes}_{2n}(\sigma \otimes \tau)$. Therefore suppose $p = 2n$. We wish to compute $\partial'_{2n}(\sigma \otimes \tau)$. We have

$$d(\sigma) = \sum_{i=0}^{2n} \sigma(0,\ldots,i) \otimes \sigma(i,\ldots,2n) =$$

$$\sum_{i=0}^{n-1} w_i \otimes \sigma(i,\ldots,2n) + \sum_{i=n+1}^{2n} \sigma(0,\ldots,i) \otimes w_{2n-1} + \sigma(0,\ldots,n) \otimes \sigma(n,\ldots,2n).$$

By definition

$$\begin{aligned}\partial'(\sigma \otimes \tau) &= [\partial\sigma] \otimes \tau + [\sigma] \otimes \tau + \Lambda[\sigma \otimes \tau \frown \Phi'] \\ &= \partial^{\otimes}_{2n}(\sigma \otimes \tau) + \Lambda[\sigma(0,\ldots,n) \otimes \Phi'\sigma(n,\ldots,2n)\ \tau] \\ &= \partial^{\otimes}_{2n}(\sigma \otimes \tau) + [\sigma(0,\ldots,n)] \otimes \Phi'\ \sigma(n,\ldots,2n)\ \tau.\end{aligned}$$

Now $[\sigma(0,\ldots,n)]$ is a cycle in $S_n(X_n)/W$ and $\Phi\sigma(n,\ldots,2n)\ \tau$ is a cycle in $S(\Omega X_n)$. This follows because every n-chain in $S_n(X_n)/W$ is a cycle and because

$$\begin{aligned}\partial(\Phi'\sigma(n,\ldots,2n)\ \tau) &= (\partial\Phi'(\sigma n,\ldots,2n))\tau + \Phi'\sigma(n,\ldots,2n)\partial\tau \\ &= \Phi'(\partial\sigma(n,\ldots,2n))\tau + 0 = 0 = 0.\end{aligned}$$

Therefore we can write

$$[\sigma(0,\ldots,n)] = \epsilon\, \bar{e}_n + \partial v \text{ where } \epsilon = 0 \text{ or } 1$$

and v is some $(n + 1)$-chain. Also

$$\Phi'\sigma(n,\ldots,2n)\tau = \epsilon'\ e_{n-1} + \partial v' \text{ where } \epsilon' = 0 \text{ or } 1 \text{ and}$$

v' is some n chain. Therefore

$$\partial'(\sigma \otimes \tau) = \partial^{\otimes}_{2n}(\sigma \otimes \tau) + \epsilon\epsilon'\ \bar{e}_n \otimes e_{n-1} + \epsilon\ \bar{e}_n \otimes \partial v' + \partial v \otimes \epsilon' \bar{e}_{n-1} + \partial v \otimes \partial v^1$$

$= \partial^{\otimes}_{2n}(\sigma \otimes \tau + \epsilon\, \bar{e}_n \otimes v' + \epsilon' v \otimes e_{n-1} + \partial v \otimes v') + \epsilon\epsilon'(\bar{e}_n \otimes e_{n-1})$.

Therefore $\partial'(\sigma \otimes \tau)$ is a member of $\mathrm{Im}\partial^{\otimes}_{2n} \oplus Z_2(\bar{e}_n \otimes e_{n-1})$.

We need now to show the set on the right is a subset of the one on the left. First we need to do a preliminary computation. Let ι_n be the generator of $H^n(X_n)$. It is well known that there is a space Y and a member y in $H^n(Y)$ such that $y^2 \neq 0$. Choose a map $f{:}Y \to X_n$ so that $f^*(\iota_n) = y$. Then $f^*(\iota_n^2) = f^*(\iota_n)f^*(\iota_n) = y^2 \neq 0$. We conclude therefore that $\iota_n^2 \neq 0$. In Proposition 2.1 we have shown that $H^i(X_n,Z_2)$ and $\mathrm{Hom}_{Z_2}(H_i(X_n,Z_2),Z_2)$ are isomorphic as vector spaces over Z_2. Therefore we will regard ι_n^2 as a member of $\mathrm{Hom}_{Z_2}(H_{2n}(X_n,Z_2),Z_2)$. Because $\iota_n^2 \neq 0$, there is a 2n-cycle, e_{2n}, of $S_n(X_n)$ for which $\iota_n^2(\mathrm{cls}(e_{2n})) = 1$. Also any $f \in \mathrm{Hom}_{Z_2}(H_i(X_n,Z_2),Z_2)$ determines a cocycle $\bar{f}$ of $\mathrm{Hom}_{Z_2}(S_n(X_n)_i,Z_2)$ as follows: If σ is a cycle in $S_n(X_n)_i$ define $\bar{f}(\sigma) = f(\mathrm{cls}(\sigma))$. The cycles are a subspace of $S_n(X_n)_i$ so $\bar{f}$ can be extended to a Z_2-linear map of $S_n(X_n)_i$. Thus we will write $\bar{\iota}_n$ or $(\overline{\iota_n^2})$ for this cocycle determined by ι_n or ι_n^2. Consider now the chain $e_{2n} \otimes 1$. Let u: $Z_2 \otimes Z_2 \to Z_2$ be the Z_2-homomorphism defined by $u(1 \otimes 1) = 1$. We have

$$d(e_{2n}) = \sum_{i=0}^{n-1} w_i \otimes s_i + \sum_{i=n+1}^{2n} s_i \otimes w_i + \sum_k \sigma_k \otimes \sigma'_k$$

for some i-chains, s_i of $S_n(X_n)$ and some n-simplexes σ_k and σ'_k of $S_n(X_n)$. We can assume too that σ_k and σ'_k are n-cycles for each k. This is because of the fact that if σ_k(or σ'_k) is not a cycle then $\sigma_k + w_n$ is. So

$\sigma_k = \epsilon_k e_n + \partial v'_k$ and $\sigma'_k = \epsilon'_k\, e_n + \partial v'_k$ where ϵ_k, $\epsilon_k{}' = 0$ or 1 and v_k, v'_k are n + 1 chains of $S_n(X_n)$. Thus

$$d(e_{2n}) = \sum_{i=0}^{n-1} w_i \otimes s_i + \sum_{i=n+1}^{2n} s_i \otimes w_i + \sum_k(\epsilon_k e_n + \partial v_k) \otimes (\epsilon'_k e_n + \partial v'_k)$$

$$= \sum_{i=0}^{n-1} w_i \otimes s_i + \sum_{i=n+1}^{2n} s_i \otimes w_i + \sum_k \epsilon_k \epsilon_k'(e_n \otimes e_n)$$

$$+ \sum_k(\epsilon_k e_n \otimes \partial v_k' + \partial v_k \otimes \epsilon_k' e_n + \partial v_k \otimes \partial v_k').$$

We note that $\bar{\iota}_n(e_n) = 1$ because e_n is a fundamental n-cycle and that $\bar{\iota}_n(\partial v_k) = \delta(\bar{\iota}_n) = 0$, $\bar{\iota}_n(\partial v_k') = \delta(\bar{\iota}_n) = 0$. We define $\bar{\iota}_n(\sigma) = 0$ if σ is a chain of dimension other than n. By the definition of the cup product operation we know that

$$\bar{\iota}^2(e_{2n}) = \mu(\bar{\iota}_n \otimes \bar{\iota}_n)\ d(e_{2n}).$$

In this case we have

$$\bar{\iota}_n^2(e_{2n}) = \mu(\bar{\iota}_n \otimes \bar{\iota}_n)(\sum_{i=0}^{n-1} w_i \otimes s_i + \sum_{i=n+1}^{2n} s_i \otimes w_i) +$$

$$\mu \sum_k \epsilon_k\ \epsilon_k'\ \bar{\iota}_n(e_n) \otimes \bar{\iota}_n(e_n) +$$

$$\mu(\sum_k \epsilon_k \bar{\iota}_n(e_n) \otimes \bar{\iota}_n(\partial v_k') + \bar{\iota}_n(\partial v_k) \otimes \epsilon_k'\ \bar{\iota}_n(e_n) + \bar{\iota}_n(\partial v_k) \otimes \bar{\iota}_n(\partial v_k'))$$

$$= 0 + \sum \epsilon_k\ \epsilon_k'(1)(1) + 0 = \sum_k \epsilon_k\ \epsilon_k' = 1.$$ Hence

$$d(e_{2n}) = \sum_{i=0}^{n-1} w_i \otimes s_i + \sum_{i=n+1}^{2n} s_i \otimes w_i + e_n \otimes e_n$$

$$+ \sum_k (\epsilon_k e_n \otimes \partial v_k' + \partial v_k \otimes \epsilon_k' e_n + \partial v_k \otimes \partial v_k').$$

We have therefore by definition

$$\partial_{2n}'(e_{2n} \otimes 1) = \partial e_{2n} \otimes 1 + e_{2n} \otimes \partial 1 + \wedge[e_{2n} \otimes 1 \frown \Phi']$$

$$= \wedge(\sum_{i=0}^{n-1} w_i \otimes \Phi'(s_i) + \sum_{i=n+1}^{2n} s_i \otimes \Phi'(w_i) + e_n \otimes \Phi' e_n +$$

$$\sum_k(\epsilon_k e_n \otimes \Phi' \partial v_k' + \partial v_k \otimes \Phi' \epsilon_k'\ e_n + \partial v_k \otimes \Phi' \partial v_k')) =$$

$$\bar{e}_n \otimes e_{n-1} + \sum_k \epsilon_k\ \bar{e}_n \otimes \partial\Phi' v_k' + [\partial v_k] \otimes \Phi' \epsilon_k' e_n$$

$$+[\partial v_k] \otimes \partial\Phi' v_k'$$ by Lemma 3.8

$$= \bar{e}_n \otimes e_{n-1} + \partial^{\otimes}_{2n} \sum_k (\epsilon_k \bar{e}_n \otimes \Phi v'_k + [v_k] \otimes \Phi' \; \epsilon'_k e_n$$

$$+ [\partial v_k] \otimes \Phi' v'_k).$$

$$= \bar{e}_n \otimes e_{n-1} + \partial^{\otimes}_{2n}(x) \text{ where } x \text{ is the argument of } \partial^{\otimes}_{2n}$$

in the line above. We also see $\partial'_{2n}(e_{2n} \otimes 1) = \bar{e}_n \otimes e_{n-1} + \partial'_{2n}(x)$ because the terms of x have homogeneous degrees (n,n), (n + 1, n-1) and (n,n) respectively.

We will show now that $\mathrm{Im}\partial^{\otimes}_{2n} \oplus Z_2(\bar{e}_n \otimes e_{n-1})$ is a subset of $\mathrm{Im}\partial'_{2n}$. Let $p + q = 2n$, σ be a p-simplex of $s_n(X_n)/W$ and τ be a q-simplex of $S(\Omega X_n)$. Consider an element of the form $\partial^{\otimes}_{2n}(\sigma \otimes \tau) + \epsilon(\bar{e}_n \otimes e_{n-1})$ where $\epsilon = 0$ or 1. To complete the proof it is sufficient to show that it is a member of $\mathrm{Im}\partial'_{2n}$. There are four possibilities to consider.

i) $p < 2n$, $\epsilon = 0$. In this case

$$\partial^{\otimes}_{2n}(\sigma \otimes \tau) = \partial'_{2n}(\sigma \otimes \tau).$$

ii) $p < 2n$, $\epsilon = 1$. We have

$$\begin{aligned}\partial^{\otimes}_{2n}(\sigma \otimes \tau) + \bar{e}_n \otimes e_{n-1} &= \partial'(\sigma \otimes \tau) + \bar{e}_n \otimes e_{n-1} \\ &\quad + \partial'_{2n}(x) + \partial'_{2n}(x) \\ &= \partial'_{2n}(\sigma \otimes \tau) + \partial'_{2n}(e_{2n} \otimes 1) + \partial'_{2n}(x). \\ &= \partial'_{2n}(\sigma \otimes \tau + e_{2n} \otimes 1 + x).\end{aligned}$$

iii) $p = 2n$, $\epsilon = 0$. We have already shown that

$$\partial'_{2n}(\sigma \otimes \tau) = \partial^{\otimes}_{2n}(\sigma \otimes \tau) + \partial'_{2n}(y) + \epsilon'(\bar{e}_n \otimes e_{n-1})$$

for some y and $\epsilon' = 0$ or 1. Therefore

$$\begin{aligned}\partial^{\otimes}_{2n}(\sigma \otimes \tau) &= \partial'_{2n}(\sigma \otimes \tau) + \partial'_{2n}(y) \text{ if } \epsilon' = 0 \\ &= \partial'_{2n}(\sigma \otimes \tau + y) \text{ and} \\ \partial^{\otimes}_{2n}(\sigma \otimes \tau) &= \partial'_{2n}(\sigma \otimes \tau) + \bar{e}_n \otimes e_{n-1} + \partial'_{2n}(y) \text{ if } \epsilon' = 1 \\ &= \partial'_{2n}(\sigma \otimes \tau + y) + \partial'_{2n}(e_{2n} \otimes 1 + x)\end{aligned}$$

$$= \partial'_{2n}(\sigma \otimes \tau + y + e_{2n} \otimes 1 + x).$$

iv) $p = 2n$, $\epsilon = 1$. From (iii) we have

$$\begin{aligned}\partial^{\otimes}_{2n}(\sigma \otimes \tau) + \bar{e}_n \otimes e_{n-1} &= \partial'_{2n}(\sigma \otimes \tau) + \partial'_{2n}(y) + \epsilon'(\bar{e}_n \otimes e_{n-1}) + \bar{e}_n \otimes e_{n-1} \\ &= \partial'_{2n}(\sigma \otimes \tau + y) \text{ if } \epsilon' = 1 \\ &= \partial'_{2n}(\sigma \otimes \tau + y + e_{2n} \otimes 1 + x) \text{ if } \epsilon' = 0\end{aligned}$$

Lemma 3.12: The map $r: H_i(S_n(X_n)/W) \otimes H_0(S(\Omega X_n) \to H_i((S_n(X_n)/W) \otimes S(\Omega X_n))$ defined by $r(\mathrm{cls}(x) \otimes \mathrm{cls}(y)) = \mathrm{cls}(x \otimes y)$ is an isomorphism for $i < 2n$.

Let $i < 2n-1$. Then by definition

$$H_i((S_n(X_n)/W) \otimes S(\Omega X_n)) = \mathrm{Ker}\ \partial'_i/\mathrm{Im}\partial'_{i+1} = \mathrm{Ker}\ \partial^{\otimes}_i/\mathrm{Im}\partial^{\otimes}_{i+1}$$

by Lemma 3.10. The Kunneth Tensor formula (Theorem 1.1) gives that for $i < 2n-1$

$$r : \sum_{k=0}^{i} H_k(S_n(X_n)/W) \otimes H_{i-k}(S(\Omega X_n)) \to H_i((S_n(X_n)/W) \otimes S(\Omega X_n))$$

is an isomorphism. In this case the domain of r is $H_i(S_n(X_n)/W) \otimes H_0\ S(\Omega X_n)$ because each of the terms in the direct sum is trivial except for the value of the index $k = i$.

Let $i = 2n-1$. By definition

$$\begin{aligned}H_{2n-1}((S_n(X_n)/W) \otimes S(\Omega X_n)) &= \mathrm{Ker}\ \partial'_{2n-1}/\mathrm{Im}\partial'_{2n} \\ &= \mathrm{Ker}\ \partial^{\otimes}_{2n-1}/\mathrm{Im}\partial^{\otimes}_{2n} \oplus Z_2(\bar{e}_n \otimes e_{n-1})\end{aligned}$$ by Lemma 3.10 and Lemma 3.11.

Consider the following sequence:

$$0 \to Z_2(\bar{e}_n \otimes e_{n-1}) \xrightarrow{i} \mathrm{Ker}\ \partial^{\otimes}_{2n-1}/\mathrm{Im}\partial^{\otimes}_{2n} \xrightarrow{j} \mathrm{Ker}\ \partial^{\otimes}_{2n-1}/\ \mathrm{Im}\partial'_{2n} \to 0$$

where i is the map induced by the inclusion map and j is the projection. The map j is a surjection. Now by the Kunneth Tensor formula (Theorem 1.) we have a map

$$r' : \sum_{k=0}^{2n-1} H_k(S_n(X_n)/W) \otimes H_{2n-1-k}(S(\Omega X_n)) \to$$

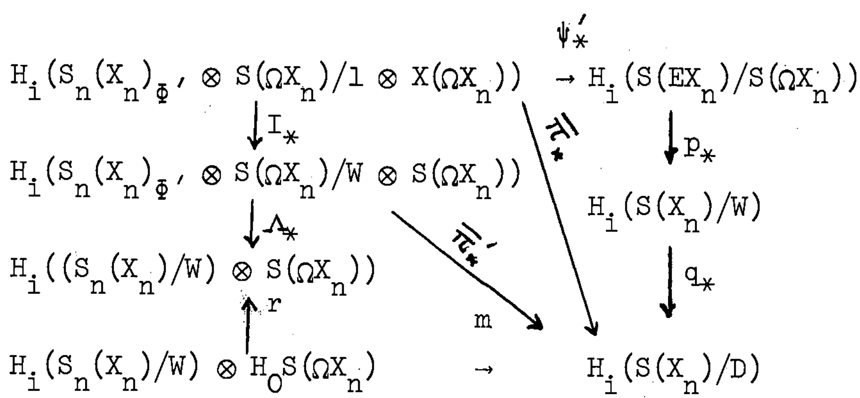
$$\operatorname{Ker} \partial^{\otimes}_{2n-1} / \operatorname{Im} \partial^{\otimes}_{2n} \text{ which is an isomorphism.}$$

In this case the domain of r' is

$$H_n(S_n(X)/W) \otimes H_{n-1}(S(\Omega X_n)) \oplus H_{2n-1}(S_n(X_n)/W) \otimes H_0(S(\Omega X_n)).$$

Therefore the above is a short exact sequence. Thus we conclude that the lemma is true for $i = 2n-1$.

Consider now the following diagram for $2 \leq i < 2n$:



where ψ' is the isomorphism induced by ψ'(Lemma 3.1), I_* is the isomorphism of Lemma 3.6, Λ_* is the isomorphism of Lemma 3.7, and r is the isomorphism of Lemma 3.12. Define $\bar{\pi}_*$ as follows: Let $\pi : S_n(X_n) \otimes S(\Omega X_n) \to S(X_n)$ be the map of Lemma 3.3. We see that $\pi(1 \otimes S(\Omega X_n)) = \{ 0,1 \}$. Therefore π induces a map

$$\bar{\pi} : S_n(X_n) \otimes S(\Omega X_n) / 1 \otimes S(\Omega X_n) \to S(X_n)/D$$

in dimensions larger than zero and is a chain map in dimensions larger than one. Define $\bar{\pi}_*$, to be the map induced by $\bar{\pi}$. The map q_* is the one induced by the identity map on $S(X_n)$. Each constant simplex is degenerate in the dimensions larger than zero, hence the identity on $S(X_n)$ induces a map $q\colon S(X_n)/W \to S(X_n)/S$ in dimensions larger than zero which is a chain mapping in dimensions larger than one. It is well known that q_* is an isomorphism in dimensions larger than one. To define the map m, we use that fact that the inclusion map $i\colon S_n(X_n) \to S(X_n)$ induces an isomorphism $i_*\colon H_i(S_n(X_n)/W \to H_i(S(X_n)/D)$ for $i \geqq 2$. We recall that $H_0(S(\Omega X_n)) \cong Z_2$. Let S be in $S_n(X_n)$ such that [S] is an i-cycle in $S_n(X_n)/W$. We define $m(\text{cls}[S] \otimes 1) = \text{cls}[S]$, where 1 is the nontrivial member of $H_0(S(\Omega X_n))$. Here [ ] again means the appropriate equivalence class. Since the map i_* defined above is an isomorphism it follows that m is an isomorphism.

From Lemma 3.3 we know that the right hand portion of this diagram commutes, i.e. $\bar{\pi}_* = q_* p_* \psi'_*$.

The map $\bar{\pi}'_*$ is defined as follows. By definition $\pi(W \otimes S(\Omega X_n))$ D in dimensions larger than zero because $\pi(W_q \otimes T) = a(T)W_q$ which is a degenerate chain if $q > 0$. Therefore π induces a chain mapping in dimensions larger than one. We have

$$\bar{\pi}'\ S_n(X_n)_\Phi \otimes S(\Omega X_n)/W \otimes S(\Omega X_n) \to S(X_n)/D.$$

Define $\bar{\pi}'_*$ to be the map induced $\bar{\pi}'$.

<u>Lemma 3.13</u>: The upper left hand portion of the diagram commutes, that is $\bar{\pi}'_*\ I_* = \bar{\pi}_*$.

Let [y] be an i-cycle in the domain of I with y in $S_n(X_n) \otimes S(\Omega X_n)$. We have

$$\bar{\pi}'_*\ I_*\ \text{cls}[y] = \bar{\pi}'_*\ \text{cls}(I[y]) = \bar{\pi}'_*\ \text{cls}[y]$$

$= \text{cls}(\bar{\pi}'\ [y]) = \text{cls}[\pi y]$. Also

$$\bar{\pi}_* \mathrm{cls}[y] = \mathrm{cls}(\bar{\pi}\,[y] = \mathrm{cls}[\pi y].$$

Lemma 3.14: The lower left hand portion of the diagram commutes, that is $\bar{\pi}'_* \Lambda_*^{-1} r = m$. Therefore $\bar{\pi}'_*$ is an isomorphism, $\bar{\pi}_*$ is an isomorphism, and consequently p_* is an isomorphism.

Let $\mathrm{cls}[S] \otimes \mathrm{cls}(T)$ be a generator of $H_i(S_n(X_n)/W \otimes H_0(S(\Omega X_n))$ where S is in $S_n(X_n)$, [S] is an i-cycle of $S_n(X_n)$, and T is a non-trivial zero cycle of $S(\Omega X_n)$. We have

$$\bar{\pi}'_* \Lambda_*^{-1} r(\mathrm{cls}[S] \otimes \mathrm{cls}(T)) = \bar{\pi}'_* \Lambda_*^{-1} \mathrm{cls}([S] \otimes T)$$
$$= \bar{\pi}'_* \mathrm{cls}([S \otimes T]) = \mathrm{cls}(\bar{\pi}'[S \otimes T]) = \mathrm{cls}\,[\pi(S \otimes T)]$$
$$= \mathrm{cls}[a(T)\,S] = \mathrm{cls}[S].$$ Also by definition

$$m(\mathrm{cls}[S] \otimes \mathrm{cls}(T)) = \mathrm{cls}[S].$$

Theorem 3.2: The cohomology suspension $\sigma^*: H^i(X_n,*) \to H^{i-1}(X_{n-1})$ is an isomorphism for $n \geqq 2$ and $2 \leqq i < 2n$.

From Chapter I we have by definition that $\sigma^* = h^*\,(\delta^*)^{-1} p^*$. We have already seen that h^* and $(\delta)^{-1}$ are isomorphisms in the correct dimensions. Now $p^* : H^i(X_n,*) \to H^i(EX_n, \Omega X_n)$ is the dual of the map $p_* : H_i(EX_n,\Omega X_n) \to H_i(X_n,*)$ because of Proposition 2.1. Therefore p^* is an isomorphism if p_* is, but p_* is an isomorphism when $2 \leqq i < 2n$, $n \geqq 2$ by Lemma 3.14.

CHAPTER IV

SUMMARY AND CONCLUSIONS

This paper is concerned with finding a new approach to the axiomatization for the Steenrod squaring operations. Using Brown's generalization of the Eilenberg-Zilber theorem for fiber spaces, a proof is given that the cohomology suspension

$$\sigma^*: H^i(X_n,*;Z_2) \to H^{i-1}(X_{n-1},Z_2)$$

is an isomorphism for $2 \leqq i < 2n$ if $n \geqq 2$. The Steenrod squaring operations can then be defined inductively and the axioms verified by classical methods.

For each prime $p > 2$ there is a sequence

$$P^i: H^n(X,A;Z_p) \to H^{n+2i(p-1)}(X,A;Z_p)$$

known as the pth reduced powers. An axiomatization similar to that of the squaring operations has been given in (8) for the pth reduced powers. The author believes that the method used here for the axiomatization of the Steenrod squares can be generalized to give that of the pth reduced powers.

BIBLIOGRAPHY

1. Brown, E. H. "Twisted Tensor Products, I." Ann. Math. Vol. 69 (1959), 223-246.

2. Hilton, P. J. and Wylie, S. Homology Theory. New York: Academic Press, 1960.

3. MacLane, Saunders. Homology. Berlin: Springer-Verlag, 1963.

4. Milnor, John W. "The Geometric Realization of a Semi-Simplicial Complex." Ann. Math. Vol. 65. (1957), 357-362.

5. Serre, J. P. "Cohomology Mod 2 of Eilenberg-MacLane Complexes." Comm. Math. Helv. Vol. 27. (1953), 198-232.

6. Shimada, N. Seminar Notes, Oklahoma State University, Fall 1966 and Spring 1967.

7. Spanier, E. H. Algebraic Topology. St. Louis: McGraw-Hill, 1966.

8. Steenrod, Norman E. Cohomology Operations. Princeton: Princeton University Press, 1962.

9. Steenrod, Norman E. "Products of Cocycles and Extensions of Mappings." Ann. Math. Vol. 48 (1947), 290-320.

10. Whitehead, George W. Homotopy Theory. Cambridge: The M. I. T. Press, 1966.

VITA

George M. Butler

Candidate for the Degree of

Doctor of Philosophy

Thesis: A NEW AXIOMATIC APPROACH FOR THE STEENROD SQUARING OPERATIONS

Major Field: Mathematics

Biographical:

Personal Data: Born in Stillwater, Oklahoma, May 29, 1940, the son of Jack Weldon and Martha Grant Butler.

Education: Graduated from Hillcrest High School, Dallas, Texas in 1958; received the Bachelor of Science degree from Oklahoma State University with major in Mathematics, June, 1963; received the Master of Science degree from Oklahoma State University with major in Mathematics, in June 1964; completed requirements for Doctor of Philosophy degree in July, 1968.

Professional Experience: Graduate Assistant in Department of Mathematics, Oklahoma State University, January, 1963-May, 1964, September, 1965, August, 1967; Graduate Assistant in Department of Mathematics, Louisiana State University, September,1964-August, 1965; Assistant Professor of Mathematics, Louisiana Polytechnic Institute, September, 1967.

Professional Organizations: Member of Pi Mu Epsilon, Mathematical Association of America, and American Mathematical Society.