DEAL CURVATURE OF HYPERSURFACES

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PREFACE

This author is deeply grateful for the patient and valuable direction given by Professor R. B. Deal, Jr. in the formulation and preparation of this thesis. Additional appreciation is also expressed for the many considerations, time, and constructive comments given by Professors John E. Hoffman, Vernon Troxel, and Kenneth E. Wiggins.

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CHAPTER I

INTRODUCTION

Differential geometry has a long history as a branch of mathematics but a large portion of the knowledge produced belongs to the realm of contemporary mathematics. Much of this new material is scattered throughout the research journals.

Mathematics, in general has been expanding in all areas at a fabulous rate during the past half century. At the same time, one of the most striking trends in contemporary mathematics is the constantly increasing interrelationship among its various branches. Thus, as a possible means to alleviate some of the resulting pedagogical problems, one needs to study some of the latest developments, reexamine the traditional areas of mathematics in light of these developments and clarify and condense the material presently required for undergraduates by pointing out the important ideas and techniques being used.

The various concepts and computational techniques that are currently in use in differential geometry and emphasized in this paper are:

1) exterior differential calculus of E. Cartan; and
2) covariant differentiation $\nabla_X Y$ for vector fields X and Y.

Purpose of Study

It is the purpose of this study to develop many of the basic concepts and techniques that are currently being used as research tools in modern differential geometry with the following objectives in mind:

1.) The material necessary for a thorough understanding of recent research papers in differential geometry by Shiing-Shen Chern and Richard Lashof on curvature of manifolds will be presented.

2.) The exterior algebra of forms will be used to derive a measure of curvature of a hypersurface called the Deal curvature.

Procedure

The basic definitions and theorems will be presented in a setting familiar to the advanced undergraduate student of mathematics. In fact, it will be shown that many of the basic ideas used are just generalizations of concepts presently being used in elementary calculus. An algebra, called the exterior algebra of forms, will be utilized in conjunction with the method of moving frames as developed by Cartan.

After a careful study of some recent research papers in differential geometry (in particular, see [4] and [5]) where many of the modern research tools are used, the author of this thesis developed the necessary background material for an understanding of the topics being presented in these papers. The notes on differential geometry by N. Hicks [10] influenced the material in Chapter II on differentiable manifolds, and the notation is that due to Barrett O'Neill [15]. The development of the exterior algebra of forms follows that presented by Mostow, Sampson, and Meyer [13].

These methods are then used to study a geometrical object, such as a surface, and a measure of curvature of a surface in E^3, introduced by R. B. Deal, Jr., (see [6]), is presented in this modern setting and then generalized to a hypersurface in E^n. Many other applications of exterior forms have been given by H. Flanders [8].

Brief History

The discipline was well launched after the formulation of the analytic geometry of R. Descartes (1596-1650) and the calculus by G. Leibniz (1646-1716) and I. Newton (1643-1727). Many of the isolated results on curves and surfaces were contributed by L. Euler (1707-1783). In France, G. Monge (1746-1818) founded an extensive school of geometry that influenced much of the development of differential geometry.

It was C. F. Gauss (1777-1855) who transformed the theory of surfaces into its modern systematic mold. He recognized the fundamental significance of intrinsic geometry. His main work in differential geometry is his treastise of 1827, *Disquisetiones generals circa superfices curvas*.

A development of intrinsic geometry independent of imbedding was given by B. Riemann (1826-1866) in 1854. Riemann dropped the restriction of two dimensions and laid the foundations for "Riemannian geometry" that has been extensively developed. These results were not published until 1868, after Riemann's death.

Felix Klein (1849-1926) and his "Erlangen program" had more influence, at first, than that of Riemann's work. Klein defined a geometry as being a theory of invariants of a group of transformations. For example, Euclidean geometry would be a theory of invariants of a group of rigid motions.

Around the turn of the 20th century G. Ricci and T. Levi-Civita developed the tensor calculus, which became a powerful tool of differential geometry. Einstein's theory of relativity created much acitvity in the further development of Riemannian geometry during this time.

During the early part of the 20th century, E. Cartan (1869-1951) utilized the earlier work of H. Grassman (1809-1877) in 1847 (on the algebra of subspaces of vector spaces) to systemitize the study of differentials. When the Frenet formulas were discovered (by F. Frenet in 1847, and independently by J. Serret in 1851), the theory of surfaces in E^3 was already a richly developed branch of geometry. The success of the Frenet approach to curves led G. Darboux (in 1887) to adope the "method of moving frames" to the theory of surfaces. Then, it was Cartan who brought the method to full generality. His essential idea was very simple: To each point of the object under study (a curve, a surface, Euclidean space itself, . . .) assign a frame; then using orthonormal expansion express the rate of change of the frame in terms of the frame itself. This, of course, is what the Frenet formulas do in the case of a curve. Cartan, also, introduced the notion of connections in fibre bundles. This notion has been given a modern formulation, first by E. Ehresmann [7], and has been utilized by S. S. Chern [4] and others.

CHAPTER II

BASIC DEFINITIONS AND THEOREMS

The goal of modern differential geometry is a study of differentiable manifolds using the tools of analysis. Thus, one wants to use calculus on a manifold and that calculus is the same as the one used on Euclidean space. For this reason we start with Euclidean space, mapping from E^n to E^m, tangent vectors, vector fields, and derivatives of these objects. Then a definition of a manifold is given and the previous definitions on E^n are extended to the manifold. Let R denote the real numbers.

<u>Definition</u> <u>2.1</u>. <u>Euclidean</u> <u>n-space</u> is the pair (S, d), where $S = \{(p_1, p_2, \ldots, p_n) \mid p_i \in R, i = 1, 2, \ldots, n\}$ and d is a mapping, $d : S \times S \longrightarrow R$, defined by $d(p, q) = [\sum_{i=1}^{n}(p_i - q_i)^2]^{\frac{1}{2}}$, where $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$ belong to S.
Thus $(S, d) = E^n$.

By the dot product of points $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ in E^n we mean the real number $p \cdot q = \sum_{i=1}^{n} p_i q_i$. The dot product is an inner product. That is, the dot product has the following properties:

1.) Bilinearity $(ap + bq) \cdot r = ap \cdot r + bq \cdot r$
$r \cdot (ap + bq) = ar \cdot p + br \cdot q$

2.) Symmetry $p \cdot q = q \cdot p$

3.) Positive definite $p \cdot p \geq 0$, $p \cdot p = 0$ if and only if $p = 0$.

If $p = (p_1, p_2, \ldots, p_n)$ then

$$\|p\| = (p_1^2 + p_2^2 + \ldots + p_n^2)^{\frac{1}{2}},$$

called the norm of p. Thus, the norm is a real valued function on E^n, and it has the following properties:

1.) $\|p + q\| \leq \|p\| + \|q\|$

2.) $\|ap\| = |a| \ \|p\|$ where $|a|$ is the absolute value of the real number a.

Thus, $d(p,q) = \|p - q\|$.

The mapping d gives some structure to the set S and E^n is a metric space with metric d. Therefore, E^n is a topological space with the usual topology. Now, add additional structure by making E^n into an R-module (vector space over the reals) with the following definitions of vector addition and scalar multiplication:

$$p + q = (p_1, p_2, \ldots, p_n) + (q_1, q_2, \ldots, q_n)$$

$$= (p_1+q_1, p_2+q_2, \ldots, p_n+q_n)$$

$$\alpha_p = \alpha(p_1, p_2, \ldots, p_n) = (\alpha p_1, \alpha p_2, \ldots, \alpha p_n)$$

for all p, q in E^n and all α in R.

Definition 2.2. Let $x_1, x_2, \ldots, x_n$ be the real valued functions on E^n such that for each $p=(p_1, p_2, \ldots, p_n)$ in E^n, $x_i(p) = p_i$ for $i = 1, 2, \ldots, n$.

The functions $x_1, x_2, \ldots, x_n$ are called the natural coordinate functions of E^n.

Definition 2.3. A real-valued function f on E^n is differentiable (or of class C^∞) provided all partial derivatives of f, of all orders, exist and are continuous. Notation: $f \in C^\infty(E^n, R)$.

Next, we define a tangent vector at a point in E^n. Then, we define a directional derivative (with respect to a tangent vector) of a real valued function which generalizes the usual directional derivative in elementary calculus. This generalization will allow us to define a tangent vector on a manifold as a linear mapping on real-valued functions.

Definition 2.4. A tangent vector v_p to E^n consists of two points of E^n: its vector part v and its point of application p.

We think of a tangent vector v_p as the arrow from p to p + r. Tangent vectors v_p and w_q are equal if and only if v = w and p = q. Tangent vectors with the same vector parts but different points of application are called parallel.

Definition 2.5. Let p be a point of E^n. The set $T_p(E^n) = \{v_p | v_p$ is a tangent vector to E^n at p$\}$ is called the tangent space of E^n at p.

We can make $T_p(E^n)$ into a vector space by defining $v_p + w_p$ to be $(v + w)_p$ and $c(v_p)$ to be $(cv)_p$. These operations on each tangent space make $T_p(E^n)$ a vector space isomorphic to E^n. We need only show that the mapping $v \rightarrow v_p$ is a linear isomorphism from E^n to $T_p(E^n)$.

Definition 2.6. A vector field V on E^n is a function that assigns to each point of p of E^n a tangent vector V(p) to E^n at p.

Let $\mathfrak{X} = \{V \mid V$ is a vector field on $E^n\}$ and $\mathfrak{F} = \{f \mid f \in C^\infty(E^n, R)\}$. Then we can make $\mathfrak{X}$ into an $\mathfrak{F}$-module (vector space over $\mathfrak{F}$) by the usual pointwise principle:

$$(V + W)(p) = V(p) + W(p)$$

and

$$(fV)(p) = f(p)V(p) \text{ for all } p.$$

Definition 2.6. Let $U_1, U_2, \ldots, U_n$ be the vector fields on E^n such that

$$U_1(p) = (1, 0, \ldots, 0)_p$$
$$U_2(p) = (0, 1, 0, \ldots, 0)_p$$
$$\vdots$$
$$U_n(p) = (0, 0, \ldots, 1)_p \text{ for each } p \text{ in } E^n.$$

We call $\{U_1, U_2, \ldots, U_n\}$ the <u>natural frame field</u> on E^n.

<u>Lemma 2.1</u>. If V is a vector field on E^n, there exist uniquely determined real-valued functions v_i, $i = 1, 2, \ldots, n$, on E^n such that

$$V = \sum_{i=1}^{n} v_i U_i \quad .$$

The functions v_i are called the <u>Euclidean coordinate functions</u> of V.

Proof: By definition of V, $V : E^n \longrightarrow T_p(E^n)$ so the vector part of $V(p)$ may be denoted as $(v_1(p), \ldots, v_n(p))$ where the v_i are real-valued functions on E^n (since they depend on the point p). Thus,

$$V(p) = (v_1(p), v_2(p), \ldots, v_n(p))_p$$
$$= v_1(p)(1,0,0, \ldots, 0)_p + \ldots + v_n(p)(0,0, \ldots, 1)_p$$

at each point p. But by definition of addition and scalar multiplication of vector fields, V and $\sum v_i U_i$ have the same tangent vector at each point p. Hence, $V = \sum v_i U_i$.

<u>Definition 2.7</u>. A vector field V on E^n is <u>differentiable</u> if and only if its Euclidean coordinate functions are differentiable (in the sense of Definition 2.3).

<u>Definition 2.8</u>. Let f be a differentiable real-valued function on E^n, and let v_p be a tangent vector at $p \in E^n$. Then

$$v_p [f] = \frac{d}{dt} (f(p + tv)) \Big|_{t=0}$$

is called the <u>derivative of f with respect to v_p</u>.

<u>Lemma 2.2</u>. If $v_p = (v_1, v_2, \ldots, v_n)_p \in T_p(E^n)$ then

$$v_p\,[f] = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(p).$$

Proof: Let $p = (p_1, p_2, \ldots, p_n)$. Then,

$$p + tV = (p_1 + tv_1, p_2 + tv_2, \ldots, p_n + tv_n)$$

and since $\dfrac{d\,(p_i + tv_i)}{dt} = v_i$, we have

$$\frac{d}{dt}\,(f(p + tv))\,\Big|_{t=0} = \Sigma\, v_i \frac{\partial f}{\partial x_i}(p).$$

The main properties of the directional derivative are given in the following theorem and the proof is a direct application of Lemma 2.2.

Theorem 1. If $f, g \in C^\infty(E^n, R)$, $v_p, w_p \in T_p(E^n)$ and $a, b \in R$, then

1.) $(av_p + bw_p)[f] = av_p[f] + bw_p[f]$

2.) $v_p[af + bg] = av_p[f] + bv_p[g]$

3.) $v_p[fg] = v_p[f] \cdot g(p) + f(p) \cdot v_p[g]$.

We again apply the pointwise principle and take directional derivatives of vector fields so we have the following:

Corollary 2.1. If $V, W \in \mathfrak{X}$ and $f, g, h \in C^\infty(E^n, R)$, $a, b \in R$, then

1.) $(fV + gW)[h] = fV[h] + gW[h]$

2.) $V[af + bg] = aV[f] + bV[g]$

3.) $V[fg] = V[f] \cdot g + f \cdot V[g]$.

<u>Definition 2.9</u>. A <u>curve in E^n</u> is a differentiable function α : $I \rightarrow E^n$ from an open interval I into E^n. Thus $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_i = x_i \circ \alpha$ are the Euclidean coordinate functions of α.

Definition 2.10. Let $\alpha : I \longrightarrow E^n$ be a curve in E^n with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. For each real number $t \in I$, the velocity vector of α at t is the tangent vector

$$\alpha'(t) = \left(\frac{d\alpha_1(t)}{dt}, \frac{d\alpha_2(t)}{dt}; \ldots, \frac{d\alpha_n(t)}{dt}\right)_{\alpha(t)}$$

at the point $\alpha(t)$ in E^n.

Lemma 2.3. Let $\alpha : I \longrightarrow E^n$ be a curve in E^n and $f \in C^\infty(E^n, R)$. Then

$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t)$$

Proof: Since $\alpha'(t) = (\frac{d\alpha 1}{dt}, \ldots, \frac{d\alpha n}{dt})$, we have by Lemma 2.2

$$\alpha'(t)[f] = \sum_{i=1}^{n} \alpha_i'(t) \frac{\partial f}{\partial x_i}(\alpha(t)).$$

Now $f(\alpha) = f(\alpha_1, \ldots, \alpha_n)$ and hence $\frac{df(\alpha)}{dt}(t) = \Sigma \frac{\partial f}{\partial x_i}(\alpha(t)) \cdot \frac{d\alpha(t)}{dt}$ by chain rule for composite functions. Thus $\alpha'(t)[f] = \frac{df(\alpha)}{dt}(t)$.

Definition 2.11. Given a function $F : E^n \longrightarrow E^m$ let $f_1, f_2, \ldots, f_m$ denote the real-valued functions on E^n such that

$$F(p) = (f_1(p), f_2(p), \ldots, f_m(p))$$

for all points p in E^n. The functions f_i are called Euclidean coordinate functions of F and we write $F=(f_1, f_2, \ldots, f_m)$.

The function F is differentiable, or $C^\infty(E^n, E^m)$, provided its coordinate functions are differentiable, or $C^\infty(E^n, R)$. A differentiable function $F : E^n \longrightarrow E^m$ is called a mapping from E^n to E^m. Notice that the f_i in $F = (f_1, f_2, \ldots, f_m)$ are the composition mappings: $f_i(p) = x_i(F(p))$.

Definition 2.12. If $\alpha : I \longrightarrow E^n$ is a curve in E^n and $F : E^n \longrightarrow E^m$ is a mapping (differentiable function), then the composite function

$\beta = F(\alpha) : I \quad E^m$ is a curve in E^m called the image of α under F.

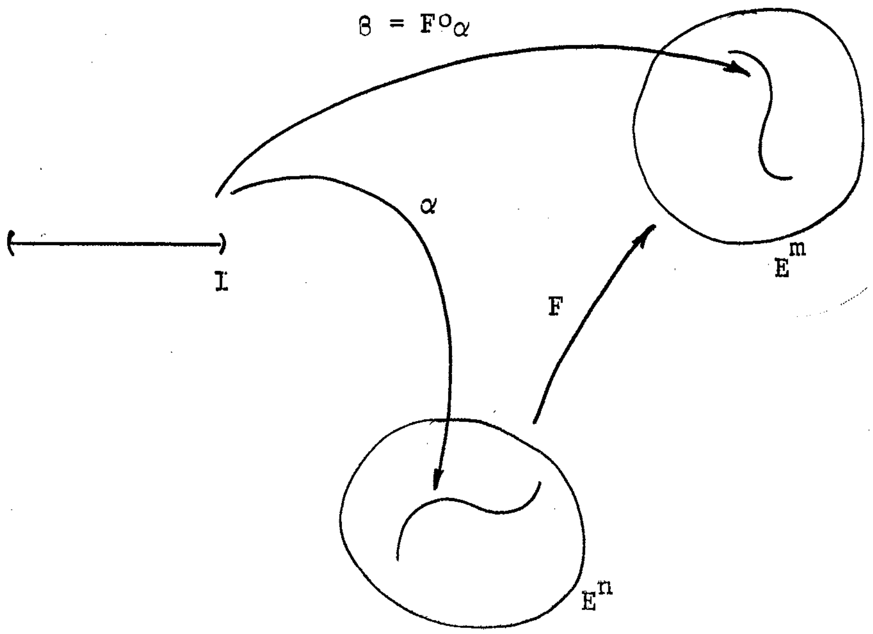

Figure 2.1

Definition 2.12. Let $F : E^n \rightarrow E^m$ be a mapping. If v is a tangent vector to E^n at p, let $F_*(v)$ be the initial velocity of the curve $\beta : t \rightarrow F(p + tv)$ in E^m. The resulting function $F_* : T_p(E^n) \longrightarrow T_{F(p)}(E^m)$ is called the derivative map F_* of F.

Thus F_* is the function that assigns to each tangent vector v in E^n at p a tangent vector $F_*(v)$ to E^m at F(p). Consider the tangent vector v as the initial velocity of the curve $\alpha : t \rightarrow p + tv$. Now the image of α under the mapping F is the curve β such that

$$\beta(t) = F(\alpha(t)) = F(p + tv).$$

And so from the definition above we have

$$F_*(v) = \beta'(0) = \left(\frac{d(F(p + tv))}{dt} \Big|_{t=0} \right)_{F(p)}.$$

The figure below describes the case where n = m = 3.

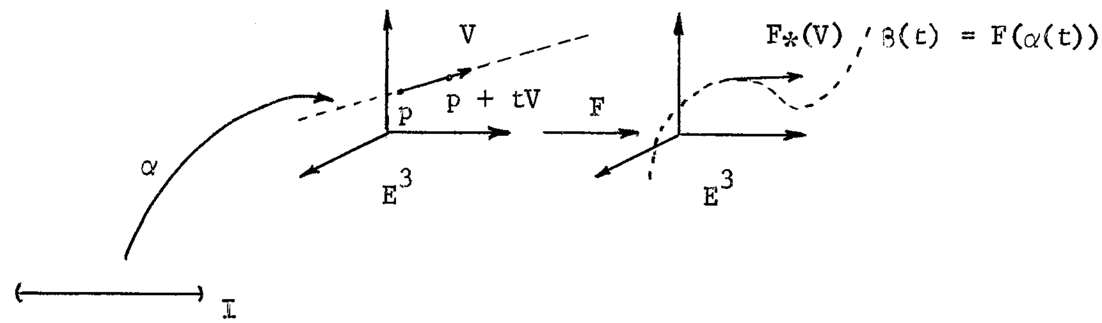

Figure 2.2

Theorem 2.2. Let $F : E^n \longrightarrow E^m$ be a mapping with $F = (f_1, f_2, \ldots, f_m)$. If $v \in T_p(E^n)$, then

$$F_*(v) = (v[f_1], v[f_2], \ldots, v[f_m])_{F(p)}.$$

Proof: Given $v \in T_p(E^n)$ we have from definition of F_* that $\beta(t) = F(p + tv) = (f_1(p + tv), f_2(p + tv), \ldots, f_m(p + tv))$ and $\beta'(0) = F_*(v)$. But by definition of velocity vector,

$$\beta'(0) = \frac{d}{dt}(F(p+tv))\Big|_{t=0} = \Big(\frac{d}{dt} f_1(p+tv)\Big|_{t=0},$$

$$\frac{d}{dt} f_2(p+tv)\Big|_{t=0}, \ldots, \frac{d}{dt} f_m(p+tv)\Big|_{t=0}\Big)$$

$$= (v[f_1], v[f_2], \ldots, v[f_m])_{F(p)}.$$

The following corollary shows the strong link between the calculus and linear algebra.

Corollary 2.2. Let $F = (f_1, f_2, \ldots, f_m)$ be a mapping from E^n to E^m. Then at each point p of E^n, the derivative map $F_{*p} : T_p(E^n) \longrightarrow T_{F(p)}(E^m)$ is a linear transformation. (The proof is immediate since $V_p[f_i]$ is linear)

Now since F_* is a linear transformation from $T_p(E^n)$ to $T_{F(p)}(E^m)$ we express F_* in the following matrix form:

$$F_* \cdot \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_i}{\partial x_1} & \frac{\partial f_i}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

where $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ is the transpose of $U_i(p) = (0, 0, \ldots, 1, \ldots, 0)_p$

and the 1 is in the i^{th} slot. Thus

$$F_* = \left(\frac{\partial f_j}{\partial x_i}\right) \qquad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m.$$

This last matrix is called the <u>Jacobian matrix</u> of F at p.

<u>Theorem 2.3</u>. Let $F : E^n \longrightarrow E^m$ be a mapping. If $\beta = F(\alpha)$ is the image in E^m of the curve α in E^n, then $\beta' = F_* (\alpha')$. (This says that F_* preserves velocities).

Proof: If $F = (f_1, \ldots, f_m)$ then

$$\beta = F(\alpha) = (f_1(\alpha), f_2(\alpha), \ldots, f_m(\alpha)) = (\beta_1, \ldots, \beta_m).$$

By Theorem 2.2

$$F_* (\alpha'(t)) = (\alpha'(t)[f_1], \alpha'(t)[f_2], \ldots, \alpha'(t)[f_m]).$$

By Lemma 2.3 we have $\alpha'(t)[f_i] = \frac{df_i(\alpha)}{dt}(t) = \frac{d\beta_i}{dt}(t)$, hence

$$F_* (\alpha'(t)) = (\frac{d\beta_1(t)}{dt}, \frac{d\beta_2(t)}{dt}, \ldots, \frac{d\beta_m(t)}{dt})\ \beta(t).$$

Therefore $\beta' = F_*(\alpha')$.

Definition 2.13. A mapping $F : E^n \longrightarrow E^m$ is regular if and only if for each point p of E^n the derivative map is one-to-one.

A mapping that has an inverse mapping is called diffeomorphism (remember that by a mapping we mean differentiable function). Thus a diffeomorphism is necessarily both one-to-one and onto, but a mapping which is one-to-one and onto need not be a diffeomorphism. (Consider the mapping $F : E^2 \longrightarrow E^2$ defined by $F = (u^3, v - u)$. Then $F^{-1} = (u^{1/3}, v + u^{1/3})$ is not differentiable at $u = 0$.

Theorem 2.4. Let $F : E^n \longrightarrow E^n$ be a mapping such that F_{*p} is one-to-one at some point p. Then there is an open set U containing p such that the restriction of F to U is a diffeomorphism $U \longrightarrow V$ onto an open set V.

Definition 2.14. A set $e_1, e_2, \ldots, e_n$ of n mutually orthogonal unit vectors ($e_i \cdot e_j = \delta_{ij}$) tangent to E^n at p is called a frame at the point p.

Theorem 2.5. Let $e_1, \ldots, e_n$ be a frame at the point p of E^n. If v is any tangent vector to E^n at p then

$$v = (v \cdot e_1)\, e_1 + (v \cdot e_2)\, e_2 + \ldots + (v \cdot e_n)\, e_n.$$

We call the above process (which works in any inner-product space) the orthonormal expansion of v in terms of the frame $e_1, e_2, \ldots, e_n$.

If we let $e_i = U_i$ then

$$V = (v^1, v^2, \ldots, v^n) = \sum_{i=1}^{n} v^i U_i$$

$$w = (w^1, w^2, \ldots, w^n) = \sum_{i=1}^{n} w^i U_i$$

and in terms of these Euclidean coordinates

$$V \cdot W = \sum v^i w^i .$$

Now if we use the frame $e_1, \ldots, e_n$, we have

$$V = \sum a_i e_i \quad (a_i = V \cdot e_i)$$

$$W = \sum b_i e_i \quad (b_i = W \cdot e_i) ,$$

but the dot product is given by the same simple formula $V \cdot W = \sum a_i b_i$, since $V \cdot W = (\sum a_i e_i) \cdot (\sum b_j e_j) - \sum (a_i b_j) \; e_i \cdot e_j = \sum a_i b_i \, \zeta_{ij}$

$$= \sum a_i b_i .$$

It is for this reason that we use frames and the advantage becomes enormous when applied to more complicated geometric situations.

Definition 2.14. Let $e_1, \ldots, e_n$ be a frame at a point p of E^n. The n x n matrix A whose rows are Euclidean coordinates of these n vectors is called the attitude matrix of the frame.

Thus, if $e_i = (a_{i1}, a_{i2}, \ldots, a_{in})_p$ then

$$A = (a_{ij}) \qquad 1 \leq i,j \leq n.$$

Notice that the rows of A are orthonormal since

$$\sum_k a_{ik} a_{jk} = e_i \cdot e_j = \delta_{ij} \text{ for } 1 \leq i,j \leq n.$$

By definition, this means that A is an orthogonal matrix. Hence $A^t A = I$, where I is the n x n identity matrix, and ${}^tA$ is the transpose of A. This means that ${}^tAA = I$ and so ${}^tA = A^{-1}$, the inverse of A.

Definition 2.15. A vector field on a curve $\alpha : I \longrightarrow E^n$ is a function that assigns to each number t in I a tangent vector Y(t) to E^n at the point $\alpha(t)$.

Thus for each $t \in I$, we can write

$$Y(t) = (y_1(t), \ldots, y_n(t))$$

$$= \sum y_i(t) \, U_i \, (\alpha(t)).$$

To differentiate a vector field on α we need only differentiate its

Euclidean coordinate functions, thus giving a new vector field on α. That is, $Y(t) = (y_1(t), y_2(t), \ldots, y_n(t)) = \Sigma\, y_i(t)\, U_i(\alpha(t))$

$$Y'(t) = \Sigma \frac{dy_i(t)}{dt} U_i(\alpha(t)).$$

It is easy to show that we have the following properties:

$$(aY + bZ)' = aY' + bZ' \ , \ a, b \in R, \text{ the reals}$$

and

$$(fY)' = \frac{df}{dt}Y + fY', \quad (Y \cdot Z)' = Y' \cdot Z + Y \cdot Z'.$$

We say that a vector field Y on a curve is <u>parallel</u> provided its Euclidean coordinate functions are constants (i.e., $Y(t) = (c_1, c_2, c_3) = \Sigma\, c_i U_i$ for all t.

<u>Lemma 2.4</u>.

(1) A curve α is constant $\Leftrightarrow \alpha' = 0$.

(2) A nonconstant curve α is a straight line $\Leftrightarrow \alpha'' = 0$.

(3) A vector field Y on a curve is parallel $\Leftrightarrow Y' = 0$.

<u>Definition 2.16</u>. Let W be a vector field on E^n and let v be a tangent vector to E^n at the point p. Then the <u>covariant derivative</u> of W with respect to v is the tangent vector

$$\nabla_v W = W(p + tv)'\ (0)$$

at the point p.

Thus, $\nabla_v W$ measures the initial rate of change of W(p) as p moves in the v direction (See fig. 2.3).

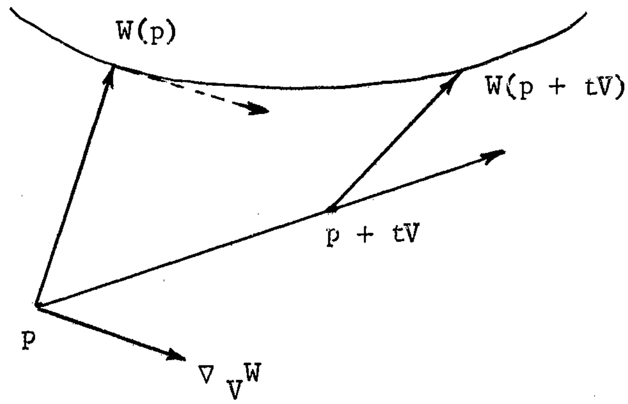

Figure 2.3

Lemma 2.5. If $W = \sum w_i U_i$ is a vector field on E^n and v is a tangent vector at p, then

$$\nabla_v W = \sum_{i=1}^{n} v\,[w_i]\,U_i(p).$$

Proof: Since $W(p + tv) = \sum w_i(p + tv)\,U_i(p + tv)$ for the restriction of W to the curve $t \longrightarrow p + tv$ and to differentiate all we have to do is differentiate the Euclidean coordinates. But the derivatives of $w_i(p + tv)$ at t=0 is $v\,[w_i]$. Hence,

$$\nabla_v W = \sum_{i=1}^{n} v\,[w_i]\,U_i(p).$$

Thus, to apply ∇_v to a vector field we apply v to its Euclidean coordinates. We use the linearity and Leibnizian properties of the directional derivative to derive the corresponding properties of the covariant derivative.

Theorem 2.6. Let v and w be tangent vectors to E^n at p, and a, $b \in R$, $f \in C^\infty(E^n, R)$, let Y and Z be vector fields on E^n. Then

1.) $\nabla_{av + bw} Y = a\nabla_v Y + b\nabla_w Y$ for all a, b, $\in R$.

2.) $\nabla_v(aY + bZ) = a\nabla_v Y + b\nabla_v Z$

3.) $\nabla_v(fY) = v[f]Y_{(p)} + f(p)\nabla_v Y$

4.) $v[Y \cdot Z] = \nabla_v Y \cdot Z(p) + Y(p) \cdot \nabla_v Z$

Proof:

1.) $\nabla_{av + bw} Y = \Sigma\ (av + bw)\ [y_i]\ U_i(p)$

$$= a\ \Sigma\ v[y_i]\ U_i(p) + b\ \Sigma\ w[y_i]U_i(p)$$

$$= a\nabla_v Y + b\nabla_w Y$$

2.) $\nabla_v(aY + bZ) = \Sigma\ v[aY_i + bZ_i]\ U_i(p)$

$$= a\ \Sigma\ v[Y_i]\ U_i(p) + b\ \Sigma\ v[Z_i]\ U_i(p)$$

$$= a\nabla_v Y + b\nabla_v Z$$

3.) $\nabla_v(fY) = \Sigma\, v[fy_i]\ U_i(p) = \Sigma\ v_p[f]y_i(p)U_i(p) + \Sigma\ f(p)v[y_i]U_i(p)$

$$= v_p[f]\Sigma\ y_i(p)U_i(p) + f(p)\Sigma\ v[y_i]U_i(p)$$

$$= v[f]Y(p) + f(p) \cdot \nabla_v Y.$$

4.) $\nabla_v(Y \cdot Z) = v[Y \cdot Z] = v[\Sigma\ y_i z_i]$

$$= \Sigma\ v[y_i] \cdot z_i(p) + \Sigma\ y_i(p)v[Z_i]$$

$$= \nabla_v Y \cdot Z(p) + Y(p) \cdot \nabla_v Z$$

Note: The properties above are also sufficient conditions for ∇_v. For suppose we are given a covariant differentiation satisfying the conditions 1 through 4 above. Then for the vector field W and tangent vector v at p we have:

$$\nabla_v W = \nabla_v(\Sigma\ w_i U_i(p))$$

$$= \Sigma\ \nabla_v(w_i U_i(p)) = \Sigma\ (v[w_i]U_i(p) + w_i(p)\nabla_v U_i(p))$$

$$= \Sigma\ \nabla_v[w_i]U_i(p) = W(p + tv)'\ \big|_{t=0}\ \text{ since}$$

$$\nabla_v U_i(p) = 0 \quad (\text{i.e., } v[c] = 0).$$

Using the pointwise principle, we can take the covariant derivative of a vector field W with respect to a vector field V. The properties of the preceding theorem take the following form:

Corollary 2.3. Let V, W, Y, and Z be vector fields on E^n. Then

(1.) $\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z$ for all reals a, b.

(2.) $\nabla_{fV + gW} Y = f\nabla_V Y + g\nabla_W Y$, for all functions f and g.

(3.) $\nabla_V(fY) = V[f]Y + f\nabla_V Y$, for all $f \in C^\infty (E^n, R)$

(4.) $V[Y \cdot Z] = \nabla_V Y \cdot Z + Y \cdot \nabla_V Z$.

Definition 2.17. If $W = \Sigma\, w_i U_i$ is a vector field on E^n, the covariant differential of W is defined to be $\nabla W = \Sigma\, dw_i U_i$. Thus $\Delta W : T_p(E^n) \quad T_p(E^n)$ such that $(\nabla W)_v = \Sigma\, dw_i(v)\, U_i(p) = \nabla_v W$. Notice that dw_i is a linear mapping on tangent vectors.

Definition 2.18. The bracket (or Lie product) of two vector fields V and W is the vector field $[V, W] = \nabla_V W - \nabla_W V$.

Theorem 2.7. If $f, g \in C^\infty (E^n, R)$ then

1.) $[V, W][f] = V[W[f]] - W[V[f]]$

2.) $[V, W] = - [W, V]$

3.) $[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0$

4.) $[fV, gW] = fV[g]W - gW[f]V + fg[V, W]$.

Differentiable Manifolds

Let M be a set of points. An m-coordinate pair on M is a pair (ϕ, M_1) consisting of a subset M_1 of M and a 1 to 1 map $\phi : M_1 \quad E^n$ such that $\phi(M_1)$ is open in E^m. One m-coordinate pair (ϕ, M_1) is C^∞ related to another m-coordinate pair (θ, M_2) if and only if the maps $\phi \circ \theta^{-1}$ and $\theta \circ \phi^{-1}$ are C^∞ maps wherever they are defined (i.e., domains of definition must be open).

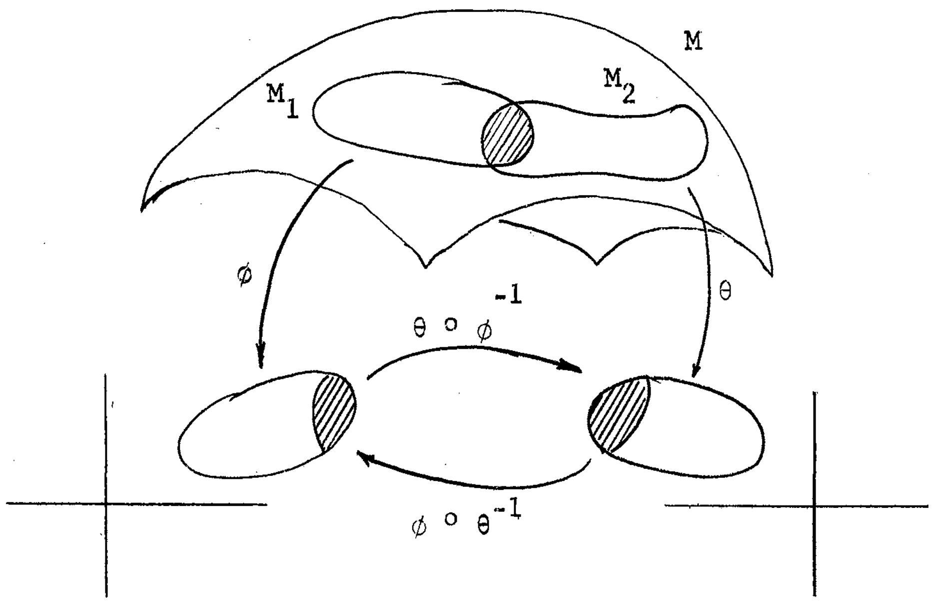

Figure 2-4

A C^{∞} m-subatlas on M is a collection of m-coordinate pairs (ϕ_h, M_h), each of which is C^{∞} related to every other member of the collection, and the union of the sets M_h is M. A maximal collection of C^{∞} related m-coordinate pairs is called a C^{∞} m-atlas. If a C^{∞} m-atlas contains a C^{∞} m-subatlas, we say the subatlas generates the atlas.

Definition 2.19. An m dimensional C^{∞} manifold (or a C^{∞} m-manifold) is a set M together with a C^{∞} m-atlas.

An atlas on a set M is called a differentiable structure on M. Each m-coordinate pair (ϕ, M_1) on a set M induces a set of m real valued functions on M_1 defined by $x_i = u_i \circ \phi$ for i = 1, 2, . . . , m where the u_i are the natural coordinate slot functions of E^m (i.e., $u_i : E^m \longrightarrow R$ by $u_i(p) = p_i$ where $p = (p_1, p_2, \ldots, p_i, \ldots, p_m)$ is in E^m). The functions $x_1, \ldots, x_m$ are called coordinate functions (or a coordinate system) and M_1 is called the domain of the coordinate system.

We list some examples:

Example 1. Let M be E^n with a C^r n-subatlas equal to the pair (ϕ, E^n) where ϕ is the identity map on E^n.

Example 2. Let M be any open set of E^n and let a C^r n-subatlas be the pair (ϕ, M) where ϕ is the identity map of E^n restricted to M.

Example 3. Let M_1 be the 1-dimensional C^1 manifold of example 1. That is, let $M_1 = R$ and ϕ the identity map. Let $M_2 = R$ and with the C^1 1-subatlas (x^3, R), where x is the identity mapping on R. Then $M_1 \neq M_2$ since $x^{1/3}$ is not C^1 at the origin (i.e., $x \circ (x^3)^{-1} = x \circ x^{1/3} = x^{1/3}$).

This example shows that the same set of points may have different differentiable structures.

Example 4. Let g be a C^∞ real valued function on E^{n+1}, with $n > 0$, and suppose $dg \neq 0$ on the set $N = \{p\epsilon E^{n+1} | \ g(p) = 0\}$. Then N is a C^∞ n-manifold when a C^∞ n-subatlas is chosen as follows: At each point $p \in M$, choose a partial derivative of g that doesn't vanish, say the i^{th} one, apply the implicit function theorem to obtain a neighborhood of p (relative topology on M) which projects in a 1-1 way into the $U_i = 0$ hyperplane of E^{n+1}.

Example 5. Let V ve a vector space over R with Let $\{e_1, \ldots, e_n\}$ be a basis of V. The group of all non-singular matrices (a_{ij});

$$ae_i = \sum_{i=1}^{n} a_{ij}e_j \quad , \quad i = 1, 2, \ldots, n$$

called the general linear group and denoted by GL(n,R).

Now map

$$GL(n, R) \longrightarrow E^{n^2}$$

defined by: (a_{ij}) $(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, a_{22}, \ldots, a_{nn})$. The image is open since it is the inverse image of an open set (using the determinant map : $E^{n^2} \longrightarrow R$ which is continuous).

<u>Definition 2.20</u>. Let M be a fixed C^∞ n-manifold. An <u>open set in M</u> is a subset A of M such that $\phi\,(A \cap U)$ is open in E^n for every n-coordinate pair (ϕ, U).

With the above defintion for open sets the manifold M becomes a topological space.

<u>Definition 2.21</u>. Let M be a C^∞ m-dimensional manifold and N an n-dimensional C^∞ manifold. If A M, A open, then F : $A \longrightarrow N$ is C^∞ (A, N) at $p \in A$ if and only if $g \circ F \circ \phi^{-1} \in C^\infty\,(\phi(A \cap M_1), E^n)$ at $\phi(p)$ for all coordinate pairs (ϕ, M_1) at $p \in M$ and all $g \in C^\infty\,(N_1, R)$, $F(p) \in N_1$. (See figure below).

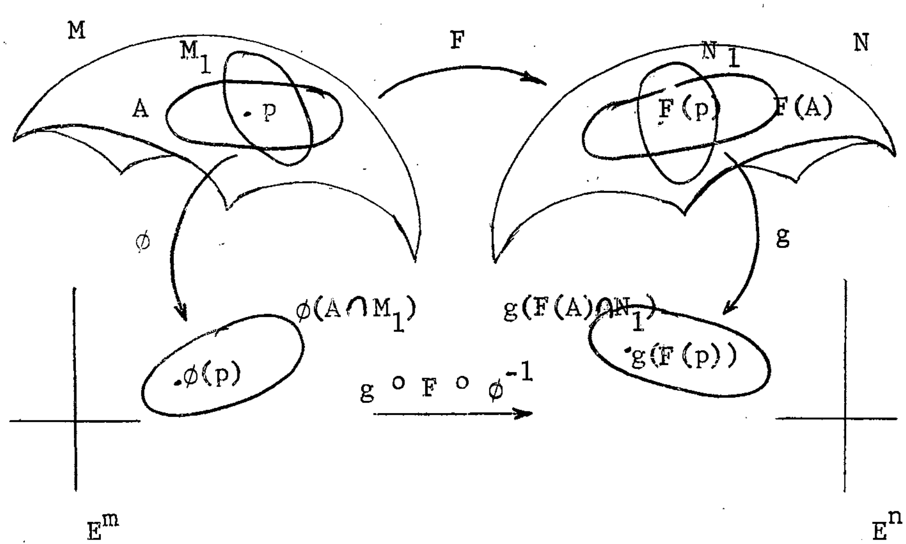

Figure 2.5

Note: The definition above includes the special cases where $M = E^n$ and ϕ is identify map (or where $N = E^n$, and g is identify map).

Definition 2.22. Let I be an open interval of the real line and M a C^∞ m-manifold. A differentiable mapping $\alpha : I \longrightarrow M$ is called a curve in M.

Lemma 2.6. If α is a curve, $\alpha : I \longrightarrow M$, whose image lies in M, where (ϕ, M_1) is an m-coordinate pair of the C^∞ m-dimensional manifold M, then there exist unique differentiable functiona $a_1, a_2, \ldots, a_m$ on I such that

$$\alpha(t) = \phi^{-1}(a_1(t), a_2(t), \ldots, a_m(t)) \text{ for all } t \in I.$$

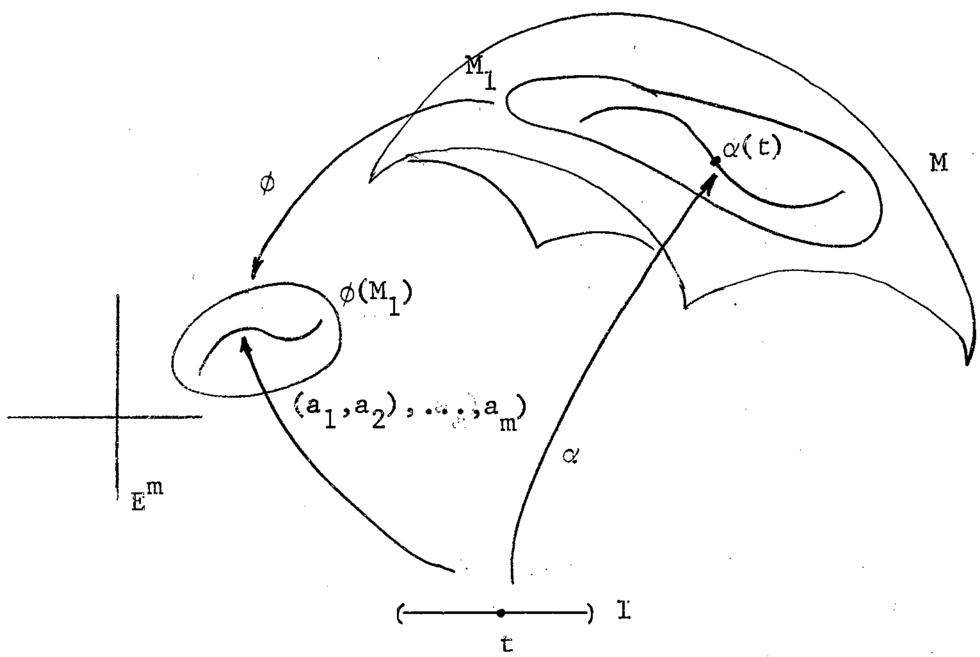

Figure 2.6

For the proof of the above Lemma we need only consider the Euclidean coordinate functions of the mapping $\phi \circ \alpha$ and the uniqueness comes from the following:

$$(a_1, a_2, \ldots, a_n) = \phi \circ \alpha = \phi \circ \phi^{-1} (b_1, b_2, \ldots, b_m)$$
$$= (b_1, b_2, \ldots, b_m).$$

Definition 2.23. Let M be a C^∞ m-manifold. Let $\mathfrak{F} = \{f \mid f \phi C^\infty (M, R)\}$. A tangent vector at $p \in M$ is a linear mapping $v : \mathfrak{F} \longrightarrow R$ such that $v[fg] = v[f] \cdot g(p) + f(p) \cdot v[g]$.

Definition 2.24. The tangent space, $T_p(M)$, to M at p is the set of all tangent vectors v at p.

The tangent space $T_p(M)$ is a vector space over R where $(v + w)[f] = v[f] + w[f]$ and $(av)[f] = a \cdot v[f]$ for all $v, w \in T_p(M)$, $f \in F$ and $a \in R$.

Let $x_1, \ldots, x_m$ be a coordinate system about $p \in M$. We define coordinate vectors $(\frac{\partial}{\partial x_i})_p$ by

$$\left(\frac{\partial}{\partial x_i}\right)_p [f] = \frac{\partial(f \circ \phi^{-1})}{\alpha U_i} (\phi(p))$$

where $x_i = U_i \circ \phi$, $i = 1, 2, \ldots, m$.

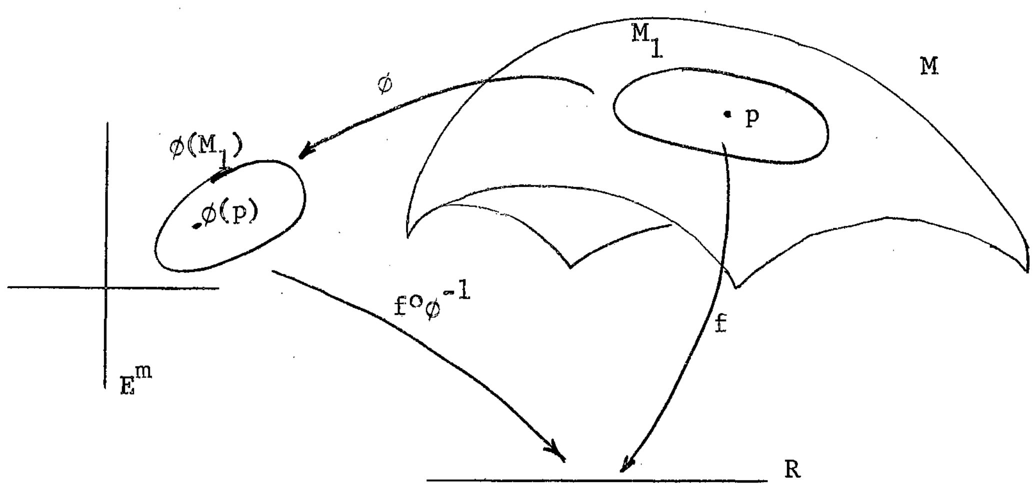

Figure 2.7

Notice that the $\left(\frac{\partial}{\partial x_i}\right)_p$ are tangent vectors since

1.) $$\left(\frac{\partial}{\partial x_i}\right)[af + bg] = \frac{\partial}{\partial u_i}((af + bg) \circ \phi^{-1})\ (\phi(p))$$

$$= \frac{\partial}{\partial u_i}(af \circ \phi^{-1} + bg \circ \phi^{-1})\ \phi(p)$$

$$= \frac{a\partial}{\partial u_i}(f \circ \phi^{-1})(\phi(p)) + \frac{b\partial}{\partial u_i}(g \circ \phi^{-1})(\phi(p))$$

$$= a\left(\frac{\partial}{\partial x_i}\right)_p [f] + b\left(\frac{\partial}{\partial x_i}\right)_p [g]$$

2.) $$\left(\frac{\partial}{\partial x_i}\right)_p [fg] = \frac{\partial}{\partial u_i}(fg \circ \phi^{-1})\ (\phi(p))$$

$$= \frac{\partial}{\partial u_i}((f \circ \phi^{-1})(g \circ \phi^{-1}))\ (\phi(p))$$

$$= \frac{\partial}{\partial u_i}(f \circ \phi^{-1})(\phi(p)) \cdot g(p) + f(p) \cdot \frac{\partial}{\partial u_i}(g \circ \phi^{-1})(\phi(p))$$

$$= \left(\frac{\partial}{\partial x_i}\right)_p [f] \cdot g(p) + f(p)\left(\frac{\partial}{\partial x_i}\right)_p [g] .$$

Lemma 2.7. Let $x_1, \ldots, x_m$ be a coordinate system about $p \in M$ with $x_i(p) = 0$ for all i. Then for every function of $f \in C^\infty(M_1, R)$, M, an open subset of M with $p \in M_1$, there exist m functions $f_1, \ldots, f_m$ in $C^\infty(M, R)$ with $f_i(p) = \left(\frac{\partial}{\partial x_i}\right)_p [f]$ and $f = f(p) + \sum_i x_i f_i$ in M_1.

In the above lemma we need only consider the map ϕ belonging to the x_i and let $F = f \circ \phi^{-1}$. F is defined over some ball $B = \{q \in E^m \mid d(0, q) < r\}$. Let $(a_1, \ldots, a_m) \in B$. Then let $F_i = \int_0^1 \frac{\partial F}{\partial u_i}(a_1, \ldots, a_{i-1}, ta_i, 0, \ldots, 0)\,dt$. Then set $f_i = F_i \circ \phi$.

Theorem 2.8. Let M be a C^∞ m-manifold and let $x_1, \ldots, x_m$ be a coordinate system about $p \in M$. Then if $v \in T_p(M)$, $v = \sum_i v\ [x]\left(\frac{\partial}{\partial x_i}\right)_p$

and the coordinate vectors $\left(\frac{\partial}{\partial x_i}\right)_p$ from the base for $T_p(M)$.

For Theorem 2.8. let $y_i = x_i - x_i(p)$ if $x_i(p) \neq 0$ for all i. Then for any $v \in T_p(M)$ and $f \in C^\infty(N_p, R)$ we use Lemma 2.7 with respect to the coordinate system $y_1, \ldots, y_m$ and note that $\left(\frac{\partial f}{\partial y_i}\right)_p = \left(\frac{\partial f}{\partial x_i}\right)_p$. Also, if c is a constnat map, $v[c] = 0$ and so $v[f] = v[c] + v[\sum_i y_i f_i]$ $= \Sigma [v[y_i] \cdot f_i(p) + y_i(p) \cdot v[f_i]] = \Sigma[v[x_i - x_i(p)] \cdot f_i(p) +$ $(x_i - x_i(p))(p) \cdot v[f_i]] = \Sigma\, v[x_i] \cdot \left(\frac{\partial f}{\partial x_i}\right)_p$. Thus we have the required representation. Now if $v = \Sigma\, a_i \frac{\partial}{\partial x_i} = 0$ then $0 = v[x_j] =$ $\Sigma\, a_i \frac{\partial x_j}{\partial x_i} = a_j$ so the coordinate vectors are independent and span $T_p(M)$ so this space has dimension m.

With the above definition of tangent vector and the representation theorem we have for all the corresponding definitions given earlier on E^n a counterpart defined on manifolds.

Definition 2.25. A vector field, V on a subset M_1 of a C^∞ m-manifold M, is a mapping that assigns to each $p \in A$ a tangent vector $v_p \in T_p(M)$.

A vector field V is C^∞ on M_1 if and only if M_1 is open and for all $f \in C^\infty(B, R)$, the function $V[f](p) = V_p[f]$ is C^∞ on M_1 B. If V and W are C^∞ vector fields on M_1 M, their bracket $[V, W]$ is a C^∞ vector field on M_1 defined by $[V, W]_p[f] = V_p[W[f]] - W_p[V[f]]$. Thus we have all the properties listed earlier in Theorem 2.7 and we note in particular that $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ for all i and j since cross partial derivatives of C^∞ functions are equal.

Let M and N be C^∞ manifolds of dimension m and n respectively. If F is a C^∞ mapping, $F : M \longrightarrow N$. Then we call the induced map $F_* : T_p(M) \longrightarrow T_{F(p)}(N)$ the Jacobian map of F and F_* is defined by

$$F_* V_{F(p)}[g] = V_{F(p)}[g \circ F], \text{ for } v \in T_{F(p)}(N)$$

and $g \in C^\infty(N_1, R)$ with $F(p) \in N_1$ (an open subset of N). Notice that $F_*(v)$ is a tangent vector at F(p) and that F_* is linear. We get a matrix representation of F_* by selecting coordinate systems $x_1, \ldots, x_m$ at p and $y_1, \ldots, y_n$ at F(p), computing F_* on the basis $V_i = \frac{\partial}{\partial x_i}$ at p. Thus if $w_j = \frac{\partial}{\partial y_j}$ is a base at F(p) we have

$$F_* V_i = \sum_j (F_* V_i)\, y_j w_j \quad \text{hence}$$

$$((F_* V_i) y_j) = \frac{\partial (y_j \circ f)}{\partial x_i} \quad \text{for } 1 \le i \le m,\ 1 \le j \le n.$$

The generalization of the definition of covariant differentiation or a connection on any C^∞ manifold is the existence of an operator ∇ which satisfies the four conditions below and assigns to C^∞ vector fields V and W a C^∞ field $\nabla_V W$:

1.) $\nabla_V (W + Z) = \nabla_V W + \nabla_V Z$

2.) $\nabla_{V+W}(Z) = \nabla_V(Z) + \nabla_W(Z)$

3.) $\nabla_{fV}(W) = f(p)\ \nabla_V w$

4.) $\nabla_V (fW) = V[f]\ W_p + f(p)\ \nabla_V (W)$.

CHAPTER III

TENSORS AND FORMS

This chapter presents the exterior algebra of forms and an operator on these forms called the exterior derivative. The exterior algebra is a subalgebra of a tensor algebra over a finite dimensional vector space U. (In later application U becomes the tangent space $T_p(M)$ to a manifold M at $p \in M$, U^* is the dual space, the space of forms on M.)

Tensor Products

<u>Definition 3.1</u>. Let $U_1, \ldots, U_r, W$ be vector spaces over field R. A mapping $f : U_1 \times U_2 \times \ldots \times U_r \longrightarrow W$ is called r-linear (or multilinear) if $f(x_1, \ldots, x_r)$ is linear in each of the r-entires, that is, if : $f(x_1, \ldots, x_i + x_i', \ldots, x_r) = f(x_1, \ldots, x_i, \ldots, x_r) + f(x_1, \ldots, x_i', \ldots, x_r)$ and $f(x_1, \ldots, cx_i, \ldots, x_r) = cf(x_1, \ldots, x_i, \ldots, x_r)$ for all $x_i, x_i' \in U_i$ and all $c \in R$.

<u>Example 3.1</u>. Let $U_1 = U_2 = R^n = \{(x_1, \ldots, x_n) \mid x_i \in R\}$. Let $f : U_1 \times U_2 \longrightarrow R$ defined by $f(x,y) = \Sigma x_i y_i$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. Then f is bilinear.

<u>Example 3.2</u>. Let U be a vector space over field R, dim U = m with base $\{U_1, \ldots, U_m\}$. Let V be a vector space over field R, dim V = n with base $\{V_1, \ldots, V_n\}$. Let P be a vector space over field R,

$\dim P = mn$ with base $\{P_{ij}\}$, $i = 1, \ldots, m$; $j = 1, \ldots, n$.

Define $f : U \times V \longrightarrow P$ by $f(x,y) = x^i y^j p_{ij}$ where $x = x^i U_i$, $y = y^i V_j$.

Now $f(x_1 + x_2, y) = (x_1^i + x_2^i)\, y^j\, p_{ij} = x_1^i y^j p_{ij} + x_2^i y^j p_{ij} = f(x_1, y)$

$$+ f(x_2, y)$$

$$f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$$

$$f(cx, y) = f(x, cy) = cf(x, y).$$

Note: $f(U_i, V_j) = p_{ij}$, hence $f(U \times V)$ spans P. Also, let $g : U \times V \longrightarrow L$, where g is bilinear, L is any other vector space. Define $g_1 : P \longrightarrow L$, g_1 linear $g_1(P_{ij}) = g(U_i, V_j)$. This defines g_1 uniquely on P by linearity. Now $g = g_1 \circ f$ since $g_1 \circ f(x, y) = g_1(f(x,y))$

$$= x^i y^j (p_{ij}) = x^i y^j g(U_i, V_j) = g(x^i U_i, y^j V_j)$$

$$= g(x, y).$$

__Definition 3.2__. By a __Tensor Product__ of two vector spaces U, V on R is meant a vector space P over R equipped with a fixed bilinear mapping $f : U \times V \longrightarrow P$ having the following properties:

(i) the image $f(U \times V)$ spans P.

(ii) if $g : U \times V \xrightarrow[\text{bilinear}]{} L$ then $\exists$ linear mapping

$$g_1 : P \longrightarrow L \ni g = g_1 \circ f$$

$$\begin{tikzcd} U \times V \arrow[r, "f"] \arrow[dr, "g"'] & P \arrow[d, "g_1"] \\ & L \end{tikzcd}$$

Figure 3.1

Properties:

1.) The linear mapping $g_1 : P \longrightarrow L$ in definition 2 is uniquely determined by the bilinear mapping g.

2.) Let $\{P, f\}$ and $\{P', f'\}$ be two tensor products of vector spaces U and V. Then there is one and only one linear mapping $h : P \quad P'$ such that $f' = h \circ f$ and h is an isomorphism.

$$\begin{tikzcd} UXV \arrow[r, "f"] \arrow[dr, "f'"'] & P \arrow[d, "h"] \\ & P' \end{tikzcd}$$

From 2.) We see that any two tensor products of U and V are canonically isomorphic. Thus by $U \otimes V$ we denote any one of the $\{P, f\}$ and for the given mapping $f : U \times V \longrightarrow U \otimes V$ we use the following: $f(x, y) = x \otimes y$. From the bilinearity of f we have

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$$
$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$$
$$(cx) \otimes y = x \otimes (cy) = c(x \otimes y)$$

3.) Every element $t \in U \otimes V$ can be expressed in at least one way as a sum $t = \sum_{i=1}^{s} x_i \otimes y_i$ where x_i in U, $y_i \in V$, $(i = 1, 2, \ldots, s)$.

4.) If U and V have dimension m and n respectively then $U \otimes V$ has dim $m \cdot n$, and if $\{U_1, \ldots, U_m\}$, $\{V_1, \ldots, V_n\}$ are bases for U and V, respectively, then the elements $U_i \otimes V_j$ in $U \otimes V$ form a base.

5.) $U \otimes V \cong V \otimes U$

Now given vector spaces U, V, W, over R, we can apply the tensor-product operation twice, for many, for example, $(U \otimes V) \otimes W$.

In a similar way we can form repreated tensor products with any number of factors. For our purposes we will be interested in repeated tensor products of a single vector space U.

6.) $(U \otimes V) \otimes W = U \otimes (V \otimes W)$

7.) Let $U_1, \ldots, U_r$ be vector spaces over R. The mapping $f : U_1 \times \ldots \times U_r \longrightarrow U_1 \otimes \ldots \otimes U_r$ defined by $f(x_1, \ldots, x_r) = x_1 \otimes \ldots \otimes x_r$, $x_i \in U_i$ is r-linear and its image spans $U_1 \otimes \ldots \otimes U_r$. If $g : U_1 \times \ldots \times U_r \longrightarrow L$ is any r-linear mapping into a vector space L then there is a unique linear mapping $g' : U_1 \otimes \ldots \otimes U_r \longrightarrow L$ such that

$$g(x_1, \ldots, x_r) = g'(x_1 \otimes \ldots \otimes x_r)$$

(proof is by induction on r)

The Tensor Algebra of a Vector Space

Let J denote an arbitrary set, and suppose that there is assigned to each element j in J a vector space U_j over R. Let S denote the set of all mappings f that assign to each j in J a vector f(j) in U_j in such a way that

(i) f(j) is the zero vector in U_j for all but a finite number of j in J.

We make S into a vector space over R as follows:

(ii) $(f + f')(j) = f(j) + f'(j)$

$(cf)(j) = c \cdot f(j)$

for any f, f' in S, any j in J, and any c in R. With these operations S is a vector space.

Definition 3.3. The space S is called the direct sum of the family $\{U_j\}$.

Let x_j be an element in U_j. Denote by x_j the element of S defined by

(iii) $$x_j'(i) = \begin{cases} x_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Now since

$$(x_j^1 + x_j^2)'\ (i) = \begin{cases} x_j' + x_j^2 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and

$$(cx_j')'\ (i) = \begin{cases} cx_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

the mapping $U_j \longrightarrow S$ defined by $x_j \longrightarrow x_j'$ is a linear mapping which maps U_j isomorphically onto a subspace U_j' of S. Now let f be any element of S and write $f(j) = x_j$, so that x_j is in U_j. By condition (i), all but a finite number of the x_j are zero. Let $x_{j_1}, \ldots, x_{j_r}$ be those which are not zero. From (ii) and (iii) it follows that

(iv) $f = x_{j_1}' + \ldots + x_{j_r}'$.

Conversely, given any elements $x_{j_1}, \ldots, x_{j_r}$ in $U_{j_1}, \ldots, U_{j_r}$ (iv) defines an element f in S.

Finally, we simplify our notation by writing x_j instead of the mapping x_j'. Thus we have the following:

Any element of the direct sum can be expressed as a finite sum

$$x_{j_1} + \ldots + x_{j_r}$$

with x_{j_1} in $U_{j_1}, \ldots, x_{j_r}$ in U_{j_r}. Furthermore the expression is unique, provided the x's are nonzero and provided the elements $j_1, \ldots, j_r$ of J are distinct.

Now let U be an n-dimensional vector space over the field R. Let

U^* be the dual vector space. We introduce the following notation:

(v) $$U^p_q = \underbrace{U \otimes \cdots \otimes U}_{p} \otimes \underbrace{U^* \otimes \cdots \otimes U^*}_{q}$$

$$= (\otimes^p U) \otimes (\otimes^q U^*)$$

In particular, $U^p_o = \otimes^p U$, $U^o_q = \otimes^q U^*$.

Thus $U^1_o = U$, $U^o_1 = U^*$, and we further define $U^o_o = R$.

From all these vector spaces we now build a giant vector space T(U), namely, their direct sum

(vi) $T(U)$ = direct sum of all U^p_q $(p, q = 0, 1, 2, \ldots)$

The elements of T(U) are called <u>tensors</u> on U. As we have just seen each U^p_q can be regarded as a subspace of T(U). The elements of U^p_o are called contravariant tensors of type (p, 0), elements of U^o_q are called covariant tensors of type (0, q); elements of U^p_q, $p, q > 0$ are called mixed tensors of type (p, q).

From (v) we have

$$U^p_q = U^p_o \otimes U^o_q \quad \text{provided } p > 0,\ q > 0.$$

(Even if p or q is zero, we can still regard the above formula as correct. For example, if $q = 0$ the right hand side is $U^p_o \otimes R = U^p_o$.) T(U) is a vector space over R, and now we show that it can be made into a ring.

We define a product $T(U) \times T(U) \longrightarrow T(U)$ by

$$(t, t') \longrightarrow t \otimes t' = \sum_{i,j} t_i \otimes t'_j$$

where $t = \sum_{i=1} t_i$, $t' = \sum_{j=1} t'_j$, $t, t' \in T(U)$.

Now, the system T(U), with the product given is an associative algebra. Every element of T(U) can be expressed as a finite sum of elements of the type

$$x_1 \otimes \cdots \otimes x_p \otimes y_1^* \otimes \cdots \otimes y_q^*$$

$$(x_i \in U^i{}_o,\ U_j \text{ in } U^o{}_j)$$

and the product in T(U) of two such elements is given by

$$(x_1 \otimes \cdots \otimes x_p \otimes y_1^* \otimes \cdots \otimes y_q^*) \otimes (z_1 \otimes \cdots \otimes z_r) \otimes (w_1^* \otimes \cdots \otimes w_s^*)$$

$$= x_1 \otimes \cdots \otimes x_p \otimes z_1 \otimes \cdots \otimes z_r \otimes y_1^* \otimes \cdots \otimes y_q^* \otimes w_1^* \otimes \cdots \otimes w_s^*.$$

Furthermore the contravariant tensors in T(U) form a subalgebra

$$T_o(U) = \text{direct sum of } U^p{}_o,\ p = 0, 1, 2, \ldots) \text{ of } T(U)$$

and the covariant tensors form a subalgebra

$$T^o(U) = \text{direct sum of } U^o{}_q\ (q = 0, 1, 2, \ldots) \text{ of } T(U).$$

Definition 3.4. $T_o(U)$ is called the contravariant tensor algebra over U, and $T^o(U)$ is called the covariant tensor algebra over U; T(U) is called the tensor algebra over U.

Exterior Algebra of U

Let U be a vector space over R, and $T_o(U)$ = direct sum $U^p{}_o$ p = 0, 1, 2, . . . $T_o(U)$ has the product operation $\otimes$ so let S denote the ideal of $T_o(U)$ generated by all elements of the type

$$x \otimes x, \quad x \in U.$$

By this we mean: S consists of all elements in $T_o(U)$ which can be obtained from the elements of the type $x \otimes x$ by a finite number of the three operations in $T_o(U)$. (addition, scalar multiplication, tensor product by arbitrary elements).

Now $t_1, t_2 \in S$ $\ t_1 + t_2$ and $t_1 - t_2$ $\ S$, hence S is a subgroup of $T_o(U)$, regarded as an abelian group, so we can form the quotient group $T_o(U)/S$, consisting of all the cosets of S. Every coset of S can be written (in many ways) as $t + S$, $t \in T_o(U)$ (for any t in the coset). We make $T_o(U)/S$ into a vector space by

$$(t_1 + S) + (t_2 + S) = (t_1 + t_2) + S,$$
$$c(t + S) = ct + S.$$

These operations are independent of the representatives chosen for suppose:

If $t_1 + S = t_1' + S$ and $t_2 + S = t_2' + S$, then

$$\begin{aligned}(t_1 + t_2) + S &= (t_1' + S_1 + t_2' + S_2) + S \\ &= (t_1' + t_2') + (S_1 + S_2) + S \\ &= (t_1' + t_2') + S\end{aligned}$$

and similarly

$$t_1 + S = t_1' + S \Rightarrow t_1 - t_1' \in S \text{ and so}$$
$$c(t' - t) \in S, \text{ thus } ct' + S = ct + S$$

showing that the operations defined above are independent of the t chosen. We now make $T_o(U)/S$ into an algebra by defining a product operation, called the <u>wedge product</u>, in $T_o(U)/S$ by the rule:

$$(t_1 + S) \wedge (t_2 + S) = (t_1 \otimes t_2) + S$$

The right hand side depends only on the cosets and not the representatives chosen. For if t_1' is in $t_1 + S$ and if t_2' is in $t_2 + S$, then both $t_1' - t_1$ and $t_2' - t_2$ are in S. (i.e., $t_1' + S_1 = t_1 + S_2$ and so $t_1' - t_2 = S_2 = S_1 \in S$). Thus, from the definition of S the products $(t_1' - t_1) \otimes t_2'$ and $t_1 \otimes (t_2' - t_2)$ must also be in S. Hence so is their sum $t_1' \otimes t_2' - t_1 \otimes t_2$, and therefore $t_1' \otimes t_2' + S = t_1 \otimes t_2 + S$.

<u>Definition 3.5</u>. The quotient algebra $T_o(U)/S$ is called the <u>exterior algebra</u> of U (or the Grassman algebra of U).

We shall denote it by $\wedge U$. We now examine its structure. Let

$P : T_o(U) \longrightarrow U$ be the canonical mapping

$$P : t \longrightarrow t + S.$$

P is linear with Ker P = S, since $P(at_1 + bt_2) + S = at_1 + S + at_2 + S$

$$= a(t_1 + S) + b(t_2 + S)$$
$$= aP(t_1) + bP(t_2)$$

$$P(t) = 0 \Rightarrow t + S = S \text{ or } t - 0 = t \in S.$$

Also:

$$P(t_1 \otimes t_2) = (t_1 \otimes t_2) + S = (t_1 + S) \wedge (t_2 + S)$$
$$= P(t_1) \wedge P(t_2).$$

Thus we have the result that P is a homomorphism of algebras.

$$P : U^p{}_o \longrightarrow \text{subspace of } \wedge U \text{ call it } \wedge^p U \text{ (i.e., } P(U^p{}_o) = \wedge^p U).$$

From the definition of S we know that S = ker P contains no elements of $R = U^o{}_o$ or $U^1{}_o = U$. Hence, P maps R isomorphically onto $\wedge^o U$ and P maps $U^1{}_o$ isomorphically onto $\wedge^1 U$. Thus we identify $\wedge^o U$ with R and $\wedge^1 U$ with U. From this we have

(vii) $P(c) = c$ for all $c \in R$,

$P(x) = x$ for all $x \in U$.

Now since U^p is spanned by elements of the type $x_1 \otimes \ldots \otimes x_p$ $(x_i \in U)$ we have

(viii) $P(x_1 \otimes \ldots \otimes x_p) = P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_p)$
$= x_1 \wedge x_2 \wedge \ldots \wedge x_p$

Hence, $\wedge^p U$ is spanned by elements of the type

(ix) $x_1 \wedge x_2 \wedge \ldots \wedge x_p$, with $x_i \in U$.

Elements of $\wedge^p U$ are said to have <u>degree p</u>.

Since $x \otimes x \in S$ for all x in U we have $P(x \otimes x) = 0$, or $P(X) \wedge P(X) = 0$. But by (vii) above we have

(x) $x \wedge x = 0$ for all $x \in U$.

The mapping $\underbrace{U \times U \times \ldots \times U}_{p} \rightarrow \bigwedge{}^p U$ defined by

$$\text{(xi)} \quad (x_1, x_2, \ldots, x_p) \rightarrow x_1 \wedge \ldots \wedge x_p$$

is p-linear. This follows from the following diagram:

$$\begin{tikzcd}
U \times U \times \ldots \times U \arrow[r, "f", "\text{p-linear}"'] \arrow[dr, "p \circ f"'] & U^p_o = \overbrace{U \otimes \ldots \otimes U}^{p} \arrow[d, "p \text{ (linear)}"] \\
& \bigwedge{}^p U
\end{tikzcd}$$

Figure 3.3

Then from (xi) we have that if x, y are in U, then

$$(x + y) \wedge (x + y) = x \wedge x + x \wedge y + y \wedge x + y \wedge y$$

But from (x) $x \wedge x = y \wedge y = (x + y) \wedge (x + y) = 0$ so we have $x \wedge y + y \wedge x = 0$ or

$$\text{(xii)} \quad x \wedge y = -y \wedge x.$$

From (xii) we have

$$\text{(xiii)} \quad x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_p} = \operatorname{sign} \begin{pmatrix} 1 & \ldots & p \\ i_1 & \ldots & i_p \end{pmatrix}$$

$$x_1 \wedge \ldots \wedge x_p \; (x_i \in U)$$

showing that the expression is skew-symmetric in its entries, we also have

$$\text{(xiv)} \quad x_1 \wedge \ldots \wedge x_p = 0 \text{ if any two } x_i \text{ are identical } (x_i \in U).$$

For suppose $x_j = x_k$ then by suitable permutation we can put x_j and x_k adjacent. But $x_j \wedge x_k = 0$ and since the permutation can at most change the sign we have the result.

Using (xii) repeatedly we have

(xv) $x_1 \wedge \ldots \wedge x_p \wedge y_1 \wedge \ldots \wedge y_q = (-1)^{pq}\, y_1 \wedge \ldots \wedge y_q \wedge x_1 \wedge \ldots \wedge x_p$

for any x_i, y_i in U. Since any $u \in \bigwedge^p U$ and $v \in \bigwedge^q U$ can be written as a linear combination of elements $x_1 \wedge \ldots \wedge x_p$ and $y_1 \wedge \ldots \wedge y_q$ we have

(xvi) $U \wedge V = (-1)^{pq}\, V \wedge U$ for $U \in \bigwedge^p U$, $V \in \bigwedge^q V$

Notice: $x \wedge x$ not necessarily zero if $x \in \bigwedge^2 U$. For example let $x = x_1 \wedge x_2 + x_3 \wedge x_4$ then

$$\begin{aligned} x \wedge x &= (x_1 \wedge x_2 + x_3 \wedge x_4) \wedge (x_1 \wedge x_2 + x_3 \wedge x_4) \\ &= (x_1 \wedge x_2) \wedge x_3 \wedge x_4 + (x_3 \wedge x_4) \wedge (x_1 \wedge x_2) \\ &= x_1 \wedge x_2 \wedge x_3 \wedge x_4 + x_1 \wedge x_2 \wedge x_3 \wedge x_4 \\ &= 2(x_1 \wedge x_2 \wedge x_3 \wedge x_4) \neq 0 \end{aligned}$$

for $x_1, x_2, x_3, x_4 \in U$, x_i are linearly independent.

Also we have $\bigwedge^p U = 0$ if $p > n = \dim U$ (Since $\bigwedge^p U$, $p > n$, is spanned by elements of the type $x_1 \wedge \ldots \wedge x_p$ ($x_i \in U$) and every such element is zero if $p > n$).

Hence, $\bigwedge U = \bigwedge^o U \oplus \bigwedge^1 U \oplus \ldots \oplus \bigwedge^n U$.

<u>Theorem 3.1</u>. $x_1 \wedge \ldots \wedge x_p = 0$ if and only if $x_1, x_2, \ldots, x_p$ are linearly dependent ($x_i \in U$).

We now compute the dimension of the $\bigwedge^p(U)$. Let $B = \{e_1, \ldots, e_n\}$ be a base for U. Thus $\dim U = n$. If $x_i = x^j{}_i\, e_i$, $i = 1, 2, \ldots, p$, then

$$\begin{aligned} x_1 \wedge \ldots \wedge x_p &= x_1^{j_1} e_{j_1} \wedge \ldots \wedge x_p^{j_p} e_{j_p} \\ &= x_1^{j_1} \ldots x_p^{j_p}\, e_{j_1} \wedge \ldots \wedge e_{j_p}. \end{aligned}$$

Thus $\bigwedge^p U$ is spanned by the elements of the form

$e_{j_1} \wedge \ldots \wedge e_{j_p}$.

But from (viii) and (ix) we see that $\wedge^p U$ is spanned by $e_{j_1} \wedge \ldots \wedge e_{j_p}$ with $1 \leq j_1 < j_2 < \ldots < j_p \leq n$. There are $\binom{n}{p}$ elements $e_{j_1} \wedge \ldots \wedge e_{j_p}$ such that $2 \leq j_1 < j_2 < \ldots < j_p \leq n$. These elements are linearly independent in $\wedge^p U$. For suppose

$$\text{(xvii)} \qquad \sum_{j_1 < \ldots < j_p} c^{j_1 \ldots j_p} \; e_{j_1} \wedge \ldots \wedge e_{j_p} = 0$$

for some scalars $c^{j_1 \ldots j_p}$. Let $K_{p+1}, \ldots, K_n$ be distinct integers from 1 to n and form the exterior product of the left member above (xvii) with the element $e_{K_{p+1}} \wedge \ldots \wedge e_{K_n}$. All terms $e_{j_1} \wedge \ldots \wedge e_{j_p} \wedge e_{K_{p+1}} \wedge \ldots \wedge e_{K_n}$ $(j_1 < \ldots < j_p)$ vanish except the ones for which $j_1, \ldots, j_p$ is the complementary set of indices $K_1, \ldots, K_p$ corresponding to $K_{p+1}, \ldots, K_n$, so that $K_1, \ldots, K_n$ is a permutation of $1, \ldots, n$. Thus the product of the left side of (xvii) and $e_{K+1} \wedge \ldots \wedge e_n$ reduces to the single term $e^{K_1 \ldots K_p} \, e_{K_1} \wedge \ldots \wedge e_{K_n}$ (no summation) and this must be zero by (xvii). But $e_{K_1} \wedge \ldots \wedge e_{K_n}$ is not zero, by Theorem 3.1, thus $e^{K_1 \ldots K_p} = 0$.

Hence, we have proved that

$$\text{(xviii)} \quad \dim \wedge^p U = \binom{n}{p} \quad p = 0, 1, 2, \ldots, n.$$

and that

the elements $e_{j_1} \wedge \ldots \wedge e_{j_p}$ with $i \leq j_1 < \ldots < j_p \leq n$ form a base in $\wedge^p(U)$ if $\{e_1, \ldots, e_n\}$ is a base in U $(p > 0)$.

Thus we have

$$\dim \bigwedge U = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = (1+1)^n = 2^n.$$

Differentiable Forms on a Manifold

We now let $U = T_p^*(M)$, the dual space to the tangent space $T_p(M)$ at a point p in a C^∞ m-manifold M. The 0-forms on M are the differentiable real-valued functions $f : M \longrightarrow R$.

<u>Definition 3.6</u>. A 1-form ϕ at $p \in M$ is an element in $T_p^*(M)$. Thus $\phi_p(av + bw) = a\phi_p(v) + b\phi_p(w)$ for all a, b $\in$ R, v, w $\in T_p(M)$. Notice that by definition $\phi_p(v)$ is a real number for all $v \in T_p(M)$.

<u>Definition 3.7</u>. If f is a differentiable real-valued function on some open set containing a point p of a C^∞ m-manifold M then the differential df of f is the 1-form such that

$$df(v)_p = v_p[f] \quad \text{for all } v \in T_p(M).$$

If $x_1, x_2, \ldots, x_m$ is a local coordinate system in a neighborhood of $p \in M$, then the differentials $(dx_1)_p, (dx_2)_p, \ldots, (dx_m)_p$ form a basis for $T_p^*(M)$. In fact, they form a dual basis of the basis

$$\left(\frac{\partial}{\partial x_1}\right)p, \left(\frac{\partial}{\partial x_2}\right)p, \ldots, \left(\frac{\partial}{\partial x_m}\right)p \quad \text{for } T_p(M).$$

Note: Let $M = E^1$ with coordinate system x. Then any tengent vector v at p is of the form $(x_2 - x_1)_p = (\Delta x)_p = \Delta x \cdot 1_p$ and so $dx(\Delta x_p) = (\Delta x)_p[x] = (\Delta x \cdot 1_p)[x] = \Delta x \cdot (1_p)[x] = \Delta x \cdot \frac{dx}{dx} = \Delta x$, since $1_p = \left(\frac{d}{dx}\right)_p$. See Figure 3.2 below.

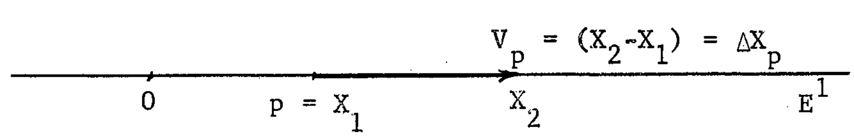

Figure 3.2

In a neighborhood of p, every 1-form ω can be uniquely written as

$$\omega = \sum_i f_i \, dx_i$$

where the $f_i(x_1, \ldots, x_m)$ are functions defined in the neighborhood of p and are called components of ω with respect to $(x_1, \ldots, x_m)$. The 1-form ω is called differentiable if the f_i are differentiable.

A 1-form can be defined as an $\mathfrak{F}(M)$-linear mapping of the $\mathfrak{F}(M)$-module $\mathfrak{X}(M)$ into $\mathfrak{F}(M)$. The two definitions are related by $(\omega(V))_p = \langle \omega_p, V_{(p)} \rangle$, $V \in \mathfrak{X}(M)$, $p \in M$, where $\langle , \rangle$ denotes the value of the first entry on the second entry as a linear functional on $\mathfrak{X}(M)$.

Let $\bigwedge T_p^*(M)$ be the exterior algebra over $T_p^*(M)$. An r-form ω is a mapping that assigns an element of $\bigwedge^r(T_p^*(M))$ to each point p of M. In terms of local coordinates $x_1, \ldots, x_m$ can be expressed uniquely as a sum

$$\omega = \sum_{i_1 < i_2 < \ldots < i_r} f_{i_1 \ldots i_r} \, dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_r}.$$

The r-form ω is called differentiable if the components $f_{i_1, \ldots i_r}$ are all differentiable. By an r-form we shall mean a differentiable r-form.

We denote by $D^r(M)$ the totality of differentiable r-forms on M for each $r = 0, 1, \ldots, m$. Thus $D^o(M) = \mathfrak{F}(M)$. Let $D(M) = \sum_{r=0}^{m} D^r(M)$, then with respect to the exterior product, D(M) forms an algebra over the field of real numbers.

Exterior Differentiation

The exterior derivative of a p-form on M is a mapping $d : D^r(M) \longrightarrow D^{r+1}(M)$ such that

1.) $d(w + n) = dw + dn$ for all $w \in D^n(M)$, $n \in D^s(M)$

2.) $d(\lambda \wedge \mu) = d\lambda \wedge \mu + (-1)^r \lambda \wedge d\mu$ where $\lambda \in D^r(M)$

3.) $d(dw) = 0$ for all $w \in D$.

4.) $df = \Sigma \frac{\partial f}{\partial x_i} dx_i$, for $f \in D^o(M)$.

The conditions above completely characterize d and in terms of local coordinates if

$$\omega = \sum_{i_1 < i_2 < \dots < i_r} f_{i_1 \dots i_r} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_r}$$

then

$$d\,\omega = \Sigma\, df_{i_1 \dots i_r}\, dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_r}.$$

Let $F : M \longrightarrow N$ be a differentiable mapping where M is a C^∞ m-manifold N is a C^∞ n-manifold. Since the Jacobian F_* maps tangent vectors on M into tangent vectors on N, it induces a map F^* of forms on N to forms on M. If g is a real-valued C^∞ function on N, then $F^*(g) = g^oF$ is a C^∞ real-valued function on M. Hence

$$F^* : D^o(N) \longrightarrow D^o(M)$$

Now if ϕ is a 1-form on N then define $F^*(\phi)(v) = \phi(F_*(v))$ for all $v \in T_p(M)$, and if η is a 2-form on N then define

$$F^*(\eta)(v,w) = \eta(F_* v, F_* w) \text{ for all pairs } (v,w) \in T_p(M) \times T_p(N).$$

In general, then we define $F^*(\mu) \in D^r(M)$ by

$$F^*(\mu)(v_1, v_2, \dots, v_r) = \mu(F_* v_1, F_* v_2, \dots, F_* v_r)$$

where $v_i \in T_p(M)$, $\mu \in D^r(N)$.

<u>Theorem 3.1</u>. Let $F : M \longrightarrow N$ be a differentiable mapping of a C^∞ m-manifold M into C^∞ n-manifold N and let ω and η be forms on N. Then

1.) $F^*(\omega + \eta) = F^*\omega + F^*\eta$

2.) $F^*(\omega \wedge \mu) = F^*\omega \wedge F^*\eta$

3.) $F^*(d\omega) = d(F^*\omega)$.

CHAPTER IV

ON THE DEAL CURVATURE OF HYPERSURFACES

This chapter utilizes the exterior algebra of forms and the moving frames as developed by E. Cartan to study hypersurfaces in Euclidean n-dimensional space. The Deal Curvature of a surface on E^3 is given and then extended to hypersurfaces in E^n.

The covariant differential, ∇W, of a vector field (see Definition 2.17) will be used in terms of the natural frame field U_1, U_2, U_3 to yield a vector field with 1-form coefficients. Then orthonormal expansion in terms of the moving frame e_1, e_2, e_3 is used to introduce the connection forms for the moving frame e_1, e_2, e_3.

Moving Frames in E^3

To each point p in E^3 we attach a right-handed orthonormal frame e_1, e_2, e_3 and suppose the vector fields e_i are differentiable. Let $X = (x^1, x^2, x^3) = x^1U_1 + x^2U_2 + x^3U_3$ denote the positioning vector of the point p. Now since

$$\begin{aligned} \nabla_V X &= \nabla_V(x^1U_1) + \nabla_V(x^2U_2) + \nabla_V x^3U_3 \\ &= v[x^1]U_1 + v[x^2]U_2 + v[x^3]U_3 \quad \text{(by Lemma 2.5)} \\ &= dx^1(v)U_1 + dx^2(v)U_2 + dx^3(v)U_3 \\ &\qquad \text{(by Definition 3.7)} \end{aligned}$$

where $v \in T_p(E^3)$, we have $\nabla X = (dx^1, dx^2, dx^3)$. Thus in the following sections we will use E. Cartan's notation and express the covariant differential by dX. Thus, $dX = (dx^1, dx^2, dx^3)$ and if we express dX in terms of the frame e_1, e_2, e_3 by expanding U_1, U_2, U_3 in terms of the e_i and then collecting terms we have:

$$dX = \sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3$$

where the σ_i are one-forms. We do the same for each e_i:

$$de_i = \omega_{i1} e_1 + \omega_{i2} e_2 + \omega_{i3} e_3 \qquad (i = 1, 2, 3)$$

where the ω_{ij} are one-forms. Since $e_i \cdot e_j = \delta_{ij}$, we have

$$de_i \cdot e_j + e_i \cdot de_j = 0,$$

and so

$$\omega_{ik} + \omega_{ki} = 0.$$

And, in particular, $\omega_{ii} = 0$.

We introduce the following matrix notation:

$$e = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad \sigma = (\sigma_1, \sigma_2, \sigma_3), \ \Omega = (\omega_{ij}).$$

Thus, we have $dX = \sigma e$, $de = \Omega e$, and $\Omega + {}^t\Omega = 0$, where ${}^t\Omega$ is the transpose of the matrix Ω.

Now, since $d(dX) = (d(dx^1), d(dx^2), d(x^3)) = 0$, we have

$$0 = d(dX) = d(\sigma e) = d\sigma \cdot e + \sigma \cdot de$$

$$= d\sigma \cdot e - \sigma\Omega e = (d\sigma - \sigma\Omega)e,$$

but the e_i are linearly independent, so

$$d\sigma = \sigma\Omega.$$

Also, $d(de) = 0$, and so

$$0 = d\Omega \cdot e - \Omega de = d\Omega e - \Omega^2 e,$$

therefore, $d\Omega = \Omega^2$.

In summary, then we have

Structure equations	Integrability conditions
$dX = \sigma e$	$d\sigma = \sigma\Omega$
$de = \Omega e$	$d\Omega = \Omega^2$
$\Omega + {}^t\Omega = 0$	

Notice that if we let

$$U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}$$

where $U_1(p) = (1, 0, 0)_p$, $U_2(p) = (0, 1, 0)_p$, $U_3 = (0, 0, 1)_p$ then

$$e_i = \Sigma\, b_{ij}U_j, \qquad e = BU$$

where $B = (b_{ij})$ is an orthogonal matrix:

$$I = e\,{}^te = BU\,{}^tU\,{}^tB = BI\,{}^tB = B\,{}^tB.$$

Thus,

$$dX = (dx^1, dx^2, dx^3)U = \sigma e = \sigma BU,$$

and so

$$(dx^1, dx^2, dx^3) = \sigma B,$$

hence

$$dx^1 \wedge dx^2 \wedge dx^3 = |B|\, \sigma_1 \wedge \sigma_2 \wedge \sigma_3.$$

But ${}^tBB = I$ so $|B|^2 = 1$, $|B| = \pm 1$. Since e is a right-handed system, $|B| = +1$ and so

$$dx \wedge dy \wedge dz = \sigma_1 \wedge \sigma_2 \wedge \sigma_3$$

and thus $\sigma_1 \wedge \sigma_2 \wedge \sigma_3$ is the volume element for E^3.

Surfaces in E^3

Let M be smooth surface in E^3 with X as positioning vector. We choose a moving orthonormal frame at each point p of M such that e_3 is normal to the surface. Then e_1 and e_2 span the tangent plane at each point p.

Since X is constrained to the surface M, dX must lie in the tangent plane and so σ_3 must be zero. Thus

$$dX = (dx^1, dx^2, dx^3) = \sigma_1 e_1 + \sigma_2 e_2.$$

and $\sigma_1 \wedge \sigma_2$ represents the element of area on M.

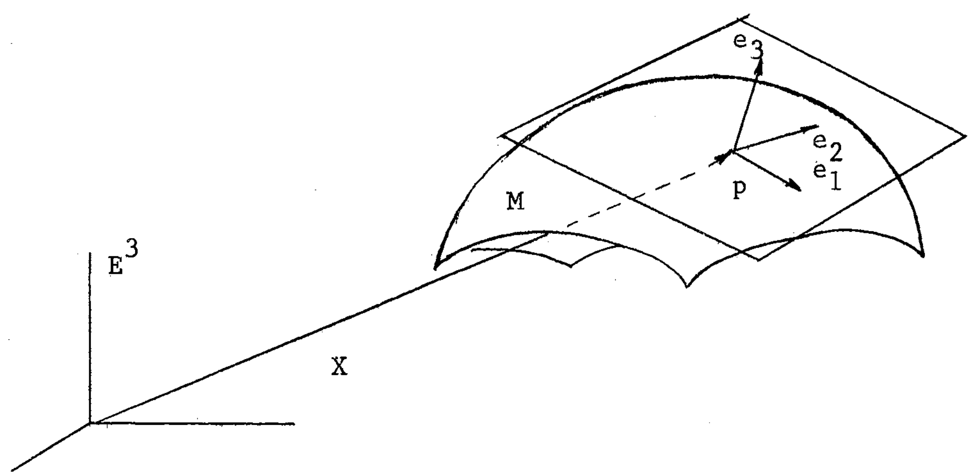

Figure 4.1

Since Ω is skew-symmetric we have

$$\Omega = \begin{pmatrix} 0 & \bar{\omega} & -\omega_1 \\ -\bar{\omega} & 0 & -\omega_2 \\ \omega_1 & \omega_2 & 0 \end{pmatrix}.$$

Therefore, the structure and integrability equations reduce to

$$dX = \sigma_1 e_1 + \sigma_2 e_2 \qquad d\sigma_1 = \bar{\omega} \wedge \sigma_2$$

$$de_1 = \bar{\omega} e_2 - \omega_1 e_3 \qquad d\sigma_2 = -\bar{\omega} \wedge \sigma_1$$

$$de_2 = -\bar{\omega}e_1 - \omega_2 e_3$$
$$de_3 = \omega_1 e_1 + \omega_2 e_2$$

$$\sigma_1 \wedge \omega_1 + \sigma_2 \wedge \omega_2 = 0$$
$$d\bar{\omega} + \omega_1 \wedge \omega_2 = 0$$
$$d\omega_1 = \bar{\omega} \wedge \omega_2 .$$

The elements of Ω are called the connection forms for M with respect to the frame e_1, e_2, e_3. The equation $d\bar{\omega} + \omega_1 \wedge \omega_2 = 0$ is called the Guass equation and

$$d\omega_1 + \omega_1 \wedge \omega_2 = 0$$
$$d\omega_2 = \bar{\omega} \wedge \omega_1$$

are called the Codazzi equations.

Also, since $dx \wedge dy = \sigma_1 \wedge \sigma_2$, the element of area on M is given by the form $\sigma_1 \wedge \sigma_2$. Now as X moves over the surface M, e_3 moves over the surface $S^2 = \{(x^1, x^2, x^3) | (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$. S^2 is called the spherical image of M and since e_1, e_2, are orthogonal to e_3, they lie in the tangent plane to S^2 and form a frame on S^2. Thus, since

$$de_3 = \omega_1 e_1 + \omega_2 e_2 ,$$

$\omega_1 \wedge \omega_2$ is the element of area on S^2 (i.e., de_3 plays same role to S^2 as dX does for the surface M). From Chapter III we know that there is only one linearly independent 2-form on M so

$$\omega_1 \wedge \omega_2 = K\sigma_1 \wedge \sigma_2$$

where K is a scalar called the <u>Gaussian</u> <u>curvature</u> of M at p.

Also, $\sigma_1 \wedge \omega_2 - \sigma_2 \wedge \omega_1$ is a 2-form on M and so

$$\sigma_1 \wedge \omega_2 - \sigma_2 \wedge \omega_1 = 2H\, \sigma_1 \wedge \sigma_2 .$$

We call H the <u>mean</u> <u>curvature</u> of M at p. The one-forms ω_1, ω_2 are linear combinations of σ_1 and σ_2 and since $\sigma_1 \wedge \omega_1 + \sigma_2 \wedge \omega_2 = 0$, we have a symmetry in the coefficients:

$$\omega_1 = p\sigma_1 + q\sigma_2$$
$$\omega_2 = q\sigma_1 + r\sigma_2 .$$

From this we have

$$\sigma_1 \wedge \omega_2 = r\sigma_1 \wedge \sigma_2$$
$$\sigma_2 \wedge \omega_1 = -p\sigma_1 \wedge \sigma_2$$

and so by adding the last two equations above we have

$$\sigma_1 \wedge \omega_2 - \sigma_2 \wedge \omega_1 = (p + r)\ \sigma_1 \wedge \sigma_2.$$

Therefore, $2H = p + r$ or $H = (p + r)/2$. Also,

$$\omega_1 \wedge \omega_2 = (p\sigma_1 + q\sigma_2) \wedge (q\sigma_1 + r\sigma_2)$$
$$= (pr - q^2)\ \sigma_1 \wedge \sigma_2$$

and hence, $K = pr - q^2$.

We call the matrix

$$S = \begin{pmatrix} p & q \\ q & r \end{pmatrix}$$

the shape operator of the surface M and the Gaussian curvature K and the mean curvature H of M are given by

$$K = \det S = pr - q^2$$
$$H = \frac{1}{2} \text{ trace } S = \frac{1}{2}\ (p + r).$$

The characteristic roots of the symmetric matrix S are called the principal curvatures k_1, k_2 of M. Thus

$$| S - \lambda I | = 0 \text{ implies}$$
$$(p - \lambda)(r - \lambda) - q^2 = 0 \text{ or } pr - q^2 - (p + r)\lambda + \lambda^2 = 0$$

and so $K = k_1 k_2$

$$2H = k_1 + k_2.$$

Notice that from $d\bar{\omega} + \omega_1 \wedge \omega_2 = 0$ we have

$$d\bar{\omega} + K\sigma_1 \wedge \sigma_2 = 0.$$

Thus we know the Gaussian curvature once we know $\bar{\omega}$, σ_1, σ_2. But from the relations

$$d\sigma_1 = \bar{\omega} \wedge \sigma_2$$
$$d\sigma_2 = -\bar{\omega} \wedge \sigma_1$$

we have $d\sigma_1 + d\sigma_2 = \bar{\omega} \quad (\sigma_2 - \sigma_1)$ which means that we know $\bar{\omega}$ once σ_1 and σ_2 are known. That is

$$d\sigma_1 = a\sigma_1 \wedge \sigma_2$$
$$d\sigma_2 = b\sigma_1 \wedge \sigma_2$$

are determined and so we have

$$d\sigma_1 + d\sigma_2 = (a\sigma_1 + b\sigma_2) \wedge (\sigma_2 - \sigma_1) = \bar{\omega} \wedge (\sigma_2 - \sigma_1).$$

Hence, $\bar{\omega} = a\sigma_1 + b\sigma_2$. Thus the Gaussian curvature is completely determined analytically by σ_1 and σ_2. This contains the theorem of Gauss that curvature is an intrinsic invariant of M.

On Deal Curvature of Surfaces

We consider the following two-form on M:

$$p\omega_1 \wedge \sigma_2 - r\omega_2 \wedge \sigma_1 + 2q\omega_2 \wedge \sigma_2.$$

Since the space of all two-forms on M is one dimensional we have

$$p\omega_1 \wedge \sigma_2 - r\omega_2 \wedge \sigma_1 + 2q\omega_2 \wedge \sigma_2 = K_D \sigma_1 \wedge \sigma_2$$

where K_D is a scalar called the <u>Deal Curvature</u> of M at p.

By direct computation

$$p\omega_1 \wedge \sigma_2 - r\omega_2 \wedge \sigma_1 + 2q\omega_2 \wedge \sigma_2 = (p^2 + r^2 + 2q^2)\sigma_1 \wedge \sigma_2.$$

Also,

$$[(2H)^2 - 2K]\sigma_1 \wedge \sigma_2 = [(p + r)^2 - 2(pr - q^2)]\sigma_1 \wedge \sigma_2$$
$$= [p^2 + r^2 + 2q^2]\sigma_1 \wedge \sigma_2$$

so,

$$K_D = (2H)^2 - 2K.$$

Thus, in terms of the principal curvatures k_1 and k_2 we have

$$K_D = (k_1 + k_2)^2 - 2(k_1 k_2)$$
$$= k_1^2 + k_2^2.$$

Example: Let M be a surface of a sphere with radius a. Then

$$X = (a \sin \phi \cos \theta, a \cos \phi \sin \theta, a \cos \phi)$$

and so

$$dX = (a \cos \phi \cos \theta, a \cos \phi \sin \theta, - a \sin \phi)\, d\phi$$
$$+ (-a \sin \phi \sin \theta, a \sin \phi \cos \theta, 0)\, d\theta$$
$$= (a\, d\phi)\, e_1 + (a \sin \phi\, d\theta)\, e_2,$$

where

$$e = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \cos \phi \sin \theta & \cos \phi \cos \theta & -\sin \phi \\ -\sin \theta & \cos \theta & 0 \\ \sin \phi \cos \theta & \sin \phi \sin \theta & \cos \phi \end{pmatrix}$$

Therefore

$$\sigma_1 = a d\phi , \quad \sigma_2 = a \sin \phi\, d\theta , \quad \sigma_3 = 0.$$

Thus

$$de_3 = (\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi)\, d\phi$$
$$+ (-\sin \phi \sin \theta, \sin \phi \cos \theta, 0)\, d\theta$$
$$= d\phi e_1 + \sin \phi\, d\theta e_2$$

and so $\omega_1 = d\phi$, $\omega_2 = \sin \phi\, d\theta$.

Therefore we have

$$\omega_1 \wedge \omega_2 = d\phi \quad \sin \phi\, d\theta = \frac{1}{a} \quad \sigma_1 \wedge \frac{1}{a} \quad \sigma_2 = \frac{1}{a^2} \quad \sigma_1 \wedge \sigma_2.$$

This gives $\frac{1}{a^2}$ as the Gaussian curvature of a sphere of radius a. Now since

$$d\phi = p\, a\, d\phi + q\, a \sin \phi\, d\theta$$

$$\sin\phi\, d\theta = q\, a\, d\phi + r\, a \sin\phi\, d\theta$$

we have $p = r = \frac{1}{a}$, $q = 0$, and hence

$$K_D = p^2 + r^2 + 2q^2 = \frac{1}{a^2} + \frac{1}{a^2} = \frac{2}{a^2}$$

Deal Curvature of Hypersurfaces

A hypersurface is an n-dimensional manifold M embedded in E^{n+1}. Let X denote the moving point p on M, and let n be the unit normal at each point in M. Consider the mapping $X \to n$ on M into S^n. The tangent space $T_p(M)$ is an n-dimensional Euclidean space, so we pick an orthonormal basis $e_1, e_2, \ldots, e_n$. Thus, at X, the vectors $e_1, e_2, \ldots, e_n$, n form on orthonormal basis of E^{n+1}. Now since dX is in the tangent space we have

$$dX = \sigma_1 e_1 + \sigma_2 e_2 + \ldots + \sigma_n e_n, \ \sigma_i \text{ are one-forms on M.}$$

From $e_i \cdot e_k = \delta_{ik}$, $e_1 \cdot n = 0$, $n \cdot n = 1$

we have $de_i \cdot e_k + e_i \cdot de_k = 0$, $de_i \cdot n + e \cdot dn = 0$, $n \cdot dn = 0$, and

$$de_i = \Sigma\, \omega_{ij}\, e_j - \omega_i n$$

$$dn = \Sigma\, \omega_i e_i$$

where ω_{ij} and ω_i are one-forms on M and

$$\omega_{ij} + \omega_{ji} = 0.$$

Thus in matrix notation we have

$$e = \begin{pmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{pmatrix}, \ \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n), \ \Omega = (\omega_{ij}), \ \omega = (\omega_1, \omega_2, \ldots, \omega_n).$$

Therefore

$$dX = \sigma\, e$$

$$d\begin{pmatrix} e \\ n \end{pmatrix} = \begin{pmatrix} \Omega & -{}^t\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} e \\ n \end{pmatrix}$$

$$\Omega + {}^t\Omega = 0$$

And since

$$0 = d(dX) = (d\sigma)e - \sigma(de) = (d\sigma)e = \sigma(\Omega e - {}^t\omega n)$$
$$= (d\sigma - \sigma\Omega)e + (\sigma {}^t\omega)n,$$

we have $d\sigma = \sigma\Omega$ and $\sigma^t\omega = 0$.

Now,

$$0 = d[d\begin{pmatrix} e \\ n \end{pmatrix}] = \begin{pmatrix} d\Omega & -d^t\omega \\ d\omega & 0 \end{pmatrix}\begin{pmatrix} e \\ n \end{pmatrix} - \begin{pmatrix} \Omega & -{}^t\omega \\ \omega & 0 \end{pmatrix} d \begin{pmatrix} e \\ n \end{pmatrix}$$
$$= \begin{pmatrix} d\Omega & -d^t\omega \\ d\omega & 0 \end{pmatrix}\begin{pmatrix} e \\ n \end{pmatrix} - \begin{pmatrix} \Omega & -{}^t\omega \\ \omega & 0 \end{pmatrix}^2 \begin{pmatrix} e \\ n \end{pmatrix}$$
$$= \begin{pmatrix} d\Omega - \Omega^2 + {}^t\omega\omega & -{}^td\omega + \Omega^t\omega \\ d\omega - \omega\Omega & 0 \end{pmatrix} \begin{pmatrix} e \\ n \end{pmatrix}$$

and so, $d\Omega - \Omega^2 + {}^t\omega\omega = 0$, $d\omega = \omega\Omega$.

We define a skew-symmetric matrix of two forms:

$$\Theta = (\theta_{ij}) = d\Omega - \Omega^2$$

$$d\sigma = \sigma\Omega \quad , \quad d\omega = \omega\Omega$$

$$\Omega + {}^t\Omega = 0$$

$$\sigma^t\omega = 0$$

$$\Theta + {}^t\omega\omega = 0$$

or in terms of individual elements of the matrices we have

$$d\sigma_j = \Sigma\, \sigma_i \wedge \omega_{ij}$$

$$\omega_{ij} + \omega_{ji} = 0$$

$$\Sigma\, \sigma_i \wedge \omega_i = 0$$

$$d\omega_j = \Sigma\, \omega_i \wedge \omega_{ij}$$

$$e_{ij} + \omega_i \wedge \omega_j = 0.$$

The σ_i form a basis for one-forms on M, hence we have

$$\omega_i = \Sigma\, b_{ij}\, \sigma_j.$$

Because $\Sigma\, \sigma_i \wedge \omega_i = 0$, the b_{ij} must be symmetric,

$$b_{ij} = b_{ji}.$$

The mean curvature and Gaussian curvature are defined by

$$H = \frac{1}{n} \Sigma\, b_{ii}, \quad K = | b_{ij} | .$$

Since $\sigma_1 \wedge \ldots \wedge \sigma_n$ is the n-dimensional volume element on M and $\omega_1 \wedge \ldots \wedge \omega_n$ is the corresponding quantity for S^n, K represents the ratio of volumes, volume of spherical image over volume of M, due to

$$\omega_1 \wedge \ldots \wedge \omega_n = (\Sigma\, b_{1j}\, \sigma_j) \wedge \ldots \wedge (\Sigma\, b_{nj}\, \sigma_j)$$
$$= | b_{ij} |\, \sigma_1 \wedge \ldots \wedge \sigma_n = K \sigma_1 \wedge \ldots \wedge \sigma_n$$

Now consider the following n-form on M:

$$\sum_{i=1}^{n} (-1)^{n-1}\, b_{ii}\, \omega_i \wedge \sigma_1 \wedge \ldots \wedge \hat{\sigma}_i \wedge \ldots \wedge \sigma_n$$
$$-2 \sum_{i=1}^{n-1} [(\sum_{j=i+1}^{n} (-1)^{n-1}\, b_{ij}) \omega_i \wedge \sigma_1 \wedge \ldots \wedge \sigma_i \wedge \ldots \wedge \sigma_n]$$
$$= K_D\, \sigma_1 \wedge \ldots \wedge \sigma_n .$$

The scalar K_D is called the <u>Deal curvature</u> of M.

As noted above

$$\omega_1 = b_{11}\, \sigma_1 + b_{12}\, \sigma_2 + \ldots + b_{1n}\, \sigma_n$$
$$\omega_2 = b_{21}\, \sigma_1 + b_{22}\, \sigma_2 + \ldots + b_{2n}\, \sigma_n$$
$$\vdots$$
$$\omega_n = b_{n1}\, \sigma_1 + b_{22}\, \sigma_2 + \ldots + b_{nn}\, \sigma_m$$

so we consider the following:

$$b_{11} \omega_1 \wedge \sigma_2 \wedge \sigma_3 \wedge \ldots \wedge \sigma_n = b_{11}^2\, \sigma_1 \wedge \ldots \wedge \sigma_m$$
$$- b_{22} \omega_2 \wedge \sigma_1 \wedge \sigma_3 \wedge \ldots \wedge \sigma_n = b_{22}^2\, \sigma_1 \wedge \ldots \wedge \sigma_n$$
$$\vdots$$
$$(-1)^{i-1} b_{ii} \omega_i \wedge \sigma_1 \wedge \ldots \wedge \sigma_{i-1} \wedge \sigma_{i+1} \wedge \ldots \wedge \sigma_n = b_{ii}^2 \sigma_1 \wedge \ldots \wedge \sigma_n$$
$$\vdots$$

$$(-1)^{n-1} b_{nn}\omega_n \wedge \sigma_1 \wedge \ldots \wedge \sigma_{n-1} = b_{nn}^2 \sigma_1 \wedge \ldots \wedge \sigma_n$$

$$2(b_{22} + \ldots + b_{nn})\omega_1 \wedge \sigma_2 \wedge \ldots \wedge \sigma_n = 2b_{11}b_{22}\sigma_1 \wedge \ldots \wedge \sigma_n$$

$$-2(b_{33} + \ldots + b_{nn})\omega_2 \wedge \sigma_1 \wedge \sigma_3 \wedge \ldots \wedge \sigma_n = 2b_{22}b_{33}\sigma_1 \wedge \ldots \wedge \sigma_n$$

$$\vdots$$

$$(-1)^{i-1} 2(b_{i+1,i+1} + \ldots + b_{nn})\omega_i \wedge \sigma_1 \wedge \sigma_2 \wedge \ldots \wedge \sigma_i \wedge \ldots \wedge \sigma_n = 2b_{ii}b_{i+1,i+1}\, \sigma_1 \wedge \ldots \wedge \sigma_n$$

$$\vdots$$

$$(-1)^{n-1} 2b_{nn}\omega_{n-1} \wedge \sigma_1 \wedge \ldots \wedge \sigma_{n-2} \wedge \sigma_n = 2b_{n-1,n-1}b_{n,n}\, \sigma_1 \wedge \ldots \wedge \sigma_n.$$

By adding the above equations we have

$$\sum_{i=1}^{n} (-1)^{i-1} b_{ii}\omega_i \wedge \sigma_1 \wedge \ldots \wedge \hat{\sigma}_i \wedge \ldots \wedge \sigma_n + 2 \sum_{i=1}^{n-1} \Big(\sum_{j=i+1}^{n} (-1)^{i-1} b_{jj} \Big) \omega_i \wedge \sigma_1 \wedge \ldots \wedge \hat{\sigma}_i \wedge \ldots \wedge \sigma_n$$

$$= \Big(\sum_{i=1}^{n} b_{ii}^2 + 2\sum_{i=1}^{n-1} b_{ii}b_{i+1,i+1} \Big) \sigma_1 \wedge \ldots \wedge \sigma_n$$

$$= (nH)^2\, \sigma_1 \wedge \ldots \wedge \sigma_n.$$

Now we consider the det $b_{ij} = \begin{vmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \\ b_{n1} & \ldots & b_{nn} \end{vmatrix}$,

and the following expressions derived from $\omega_i = \Sigma\, b_{ij}\, \sigma_j$.

$$-2\tilde{K} = -2\Big\{ \sum_{i=1}^{n} \Big[\sum_{j=i+1}^{n} (-1)^{i-1} b_{jj} \Big] \omega_i \wedge \sigma_1 \wedge \ldots \wedge \hat{\sigma}_i \wedge \ldots \wedge \sigma_n$$

$$- \sum_{i=1}^{n-1} \Big(\sum_{j=i+1}^{n} (-1)^{i-1} b_{ij} \Big) \omega_i \wedge \sigma_1 \wedge \ldots \wedge \sigma_i \wedge \ldots \wedge \sigma_n \Big\}$$

$$= -2\{(\sum_{i=1}^{n} b_{ii}b_{i+1,i+1})\ \sigma_1\wedge\ .\ .\ .\wedge\sigma_n$$

$$-\sum_{i=1}^{n-1}(\sum_{j=i+1}^{n} b_{ij}^2)\ \sigma_1\ \ .\ .\ .\ \ \sigma_n\}.$$

Thus from the form giving Deal curvature we have

$$\sum_{i=1}^{n}(-1)^{i-1}\ b_{ii}\omega_i\wedge\sigma_1\wedge .\ .\ .\wedge\sigma_i\wedge .\ .\ .\wedge\sigma_n$$

$$-2\sum_{i=1}^{n-1}[(\sum_{j=i+1}^{n-1}(-1)^{i-1}\ b_{ij})\ \omega_i\wedge\sigma_1\wedge .\ .\ .\wedge\hat{\sigma}_i\wedge .\ .\ .\wedge\sigma_n]$$

$$= [(nH)^2 - 2\tilde{K}\ ]\ \sigma_1\wedge .\ .\ .\wedge\sigma_n = K_D\ \sigma_1\wedge .\ .\ .\wedge\sigma_n.$$

It can be shown that in terms of the principal curvatures $k_1, \ldots, k_n$ (characteristic roots of the symmetric matrix (b_{ij})) that $K_D = k_1^2 + k_2^2 + \ldots + k_n^2$.

Example: Consider the surface of revolution given by

$$X(u^1,u^2) = (u^2\cos u^1,\ u^2\sin u^1,\ h(u^2)).$$

Let the frame be given by

$$e_1 = (-\sin u^1,\ \cos u^1,\ 0)$$

$$e_2 = (\cos u^1,\ \sin u^1, h'(u^2))\ \frac{1}{(1+h'(u^2)^2)^{\frac{1}{2}}}$$

$$e_3 = (h'(u^2)\cos u^1,\ h'(u^2)\ \sin u^1,\ -1)(1/(1+h'(u^2)^2)^{\frac{1}{2}}).$$

Then

$$dX(u^1,\ u^2) = u^2du^1e_1 + (1+h'(u^2)^2)^{\frac{1}{2}}du^2e_2$$

$$= \sigma_1e_1 + \sigma_2e_2$$

and

$$de_3 = (h'(u^2)du^1/(1+h'(u^2)^2)^{\frac{1}{2}})e_1 + (h''(u^2)du^2/(1+h'(u^2)^2))e_2$$

$$= \omega_1e_1 + \omega_2e_2.$$

Therefore

$$\omega_1 \wedge \omega_2 = (h'(u^2)h''(u^2)/(1+h'(u^2)^2)^{3/2})du^1 \wedge du^2$$

$$= (h'(u^2)h''(u^2)/u^2(1+h'(u^2)^2)^2)\sigma_1 \wedge \sigma_2$$

$$= K\ \sigma_1 \wedge \sigma_2$$

where K is the Gaussian curvature of $X(u^1,u^2)$.

Also, since

$$\omega_1 = p\sigma_1 + q\sigma_2$$

$$\omega_2 = q\sigma_1 + r\sigma_2 \quad ,$$

we have

$$p = h'(u^2)/(u^2(1+h'(u^2)^2)^{\frac{1}{2}}$$

$$q = 0$$

$$r = h''(u^2)/(1+h'(u^2)^2)^{3/2},$$

and so

$$p\omega_1 \wedge \sigma_2 - r\omega_2 \wedge \sigma_1 = (p^2+r^2)\ \sigma_1 \wedge \sigma_2$$

$$= (h'(u^2)^2(1+h'(u^2)^2)^2$$

$$+ (u^2)^2 h(u^2)^2)/(u^2)^2(1+h'(u^2)^2)^3 \sigma_1 \wedge \sigma_2$$

$$= K_D\ \sigma_1 \wedge \sigma_2$$

where K_D is the Deal curvature of the surface of revolution given by $X(u^1,u^2)$. If we let

$$h(u^2) = u^2, \quad 0 < u^2 < 1,$$

we have the surface of a cone and the Gaussian curvature vanishes;

$$K = 0$$

while the Deal curvature is given by

$$K_D = 1/2(u^2)^2.$$

If we let

$$h(u^2) = (a^2-(u^2)^2)^{\frac{1}{2}}$$

then the surface of revolution given by

$$X(u^1,u^2) = (u^2\cos u^1,\ u^2\sin u^1,\ (a^2-(u^2)^2)^{\frac{1}{2}})$$

is a sphere with radius a. Now since

$$h'(u^2) = -(u^2)/(a^2-(u^2)^2)^{\frac{1}{2}}, \quad h''(u^2) = -a^2/(a^2-(u^2)^2)^{3/2}$$

we have the Gaussian curvature of a sphere of radius a given by

$$\begin{aligned} K &= h'(u^2)h''(u^2)/(u^2)(1+h'(u^2)^2)^2 \\ &= 1/a^2. \end{aligned}$$

And, for the Deal curvature, K_D, of the sphere we have:

$$K_D = 2/a^2.$$

A SELECTED BIBLIOGRAPHY

1. Auslander, L., and Mackenzie, R. E., Introduction to Differentiable Manifolds, McGraw-Hill Book Company, N. Y., (1963).

2. Bishop, R. L., and Crittenden, R. J., Geometry of Manifolds, Academic Press Inc., N. Y., (1964).

3. Cartan, E., "La Methode du Repere Mobile," Actualities Sci. et Ind., (1935).

4. Chern, S. S., "Differential Geometry of Fiber Bundles," Proc. Int. Congress, (1950).

5. Chern, S. S., and Lashof, R. K., "The Total Curvature of Immersed Manifolds," Amer. J. Math., vol. 79 (1957) pp. 308-318.

6. Deal, R. B., Jr., Unpublished Doctoral Dissertation, University of Oklahoma, (1953).

7. Ehresmann, C., "Les Connexions Infinitesimales dans un Espace Fibre Differentiable," Colloque de Topologie, Bruxelles, (1950).

8. Flanders, H., Differential Forms with Applications to the Physical Sciences, Academic Press Inc., N. Y., (1963).

9. Goldberg, S. I., Curvature and Homology, Academic Press Inc., N. Y., (1962).

10. Hicks, N. J., Notes on Differential Geometry, D. Van Nostrand, N. Y., (1965).

11. Kobayashi, S., and Nomizu, K., Foundations of Differential Geometry I, John Wiley & Sons, Inc., N. Y., (1963).

12. Kreyszig, E., Differential Geometry, Oxford University Press, London, (1959).

13. Mostow, G. D., and Sampson, J. H., and Meyer, J. P., Fundamental Structures of Algebra, McGraw-Hill Inc., N. Y., (1963).

14. Nomizu, K., Lie Groups and Differential Geometry, Math. Soc. Japan, (1956).

15. O'Neill, Barrett, Elementary Differential Geometry, Academic Press Inc., N. Y., (1966).

16. Spivak, M., Calculus on Manifolds, W. A. Benjamin, Inc., N. Y., (1965).

17. Sternberg, S., Lectures on Differential Geometry, Prentice-Hall Inc., Englewood Cliffs, N. J., (1964).

18. Willmore, T. J., Introduction to Differential Geometry, Oxford University Press, Oxford, England, (1959).

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