NETWORK SYNTHESIS USING NULLATORS

AND NORATORS

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CHAPTER I

INTRODUCTION

1.1 Statement of the Problem

The nullator and the norator have been of much interest in active-

network studies since they were proposed in 1961. Network analysis and

synthesis with such elements have been reported.

The nullator is a two-terminal element for which the voltage across

and the current through this element are always zero. The norator is a

two-terminal element for which the voltage across and the current

through this element are arbitrary and can independently take on any

values which are determined by external circuits. The two elements are

shown schematically in Figure 1.1.1.
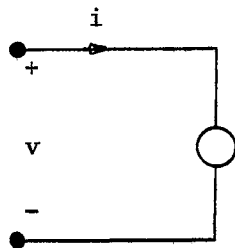
Nullator
$v = 0, \ i = 0$
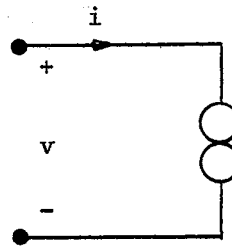
Norator
$v, \ i$ arbitrary

Figure 1.1.1. Definition of a Nullator and
a Norator

From the above definition, the behavior of the nullator and

norator seems paradoxical, and it appears unlikely that there exists any

meaningful physical realizations for them, when considered individually.

However, they are useful in active-network theory. For example, an

ideal transistor can be represented by a nullator-norator pair as shown

in Figure 1.1.2 (28). In addition, a negative impedance converter (NIC)

and an ideal gyrator can be realized by using nullator-norator pairs. A

two-port with a nullator as port 1 and a norator as port 2 is called a

nullor. The representation of an ideal transistor using a nullor also

has been reported (26).
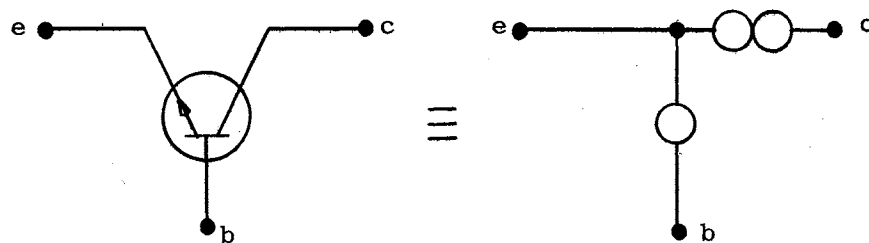
Figure 1.1.2.   Nullator-Norator Model of an
Ideal Transistor

Because of the extraordinary behavior exhibited by the nullator and

norator, the conventional node-mesh techniques of analysis of networks

containing such elements will fail. Thus, it requires certain modifica-

tions to conventional methods. Blackwell and Grigsby (5) have presented

a theorem discussing the topological configuration of a system contain-

ing components with both through and across variables specified. They

found that there exist two formulation trees, each consisting of proper components, if the system equations have a unique solution. This formulation can be applied in analyzing a network containing the nullator and norator.

The results presented in (5) do not cover the synthesis, but their formulation procedure gives an insight into the design of active networks containing nullators and norators. In this thesis, a systematic network synthesis procedure using resistors, capacitors and nullator-norator pairs is derived. The state model is used in the synthesis procedure.

1.2 Review of the Literature

Carlin and Youla (9) first proposed the nullator and norator in 1961 to demonstrate the use of negative resistors in network synthesis. Carlin (10) combined these two elements to form a two-port, the nullor, which is a unique active building block.

Since the nullator, norator and nullor were proposed, many papers have been written on these elements. Some of these papers are concerned with the analysis of networks containing such elements and others with the synthesis. In dealing with the synthesis, the task always is to obtain equivalent network using the nullator and norator for a particular physical device.

Davies (14) presented a topological method of analysis of networks containing the nullator and norator, whereas a simplified topological solution of such networks was presented by Brayshaw (7). Martinelli (26) presented some properties of the nullor, and described the possibility of synthesizing any rational immittance or scattering matrix by

means of only positive resistors, nullors, and lossless elements which can be inductors or capacitors. Davies (15) demonstrated the significance of nullators, norators and nullors in active-network theory, including a brief description of the synthesis of immittance matrices. Davies (16) also obtained nullator-norator models of controlled sources.

Antoniou (1) used the equivalence of an infinite-gain controlled source to a nullor to obtain new gyrator circuits. A procedure to generate all gyrators of a certain class using the nullator and norator was developed by Daniels (12), who also has presented a novel technique to derive the converter-inverter circuits using the nullator and norator (13). Bendik (4) used the nullator and norator as network elements to obtain gyrator circuits.

Martinelli (26) demonstrated the nullor model of an ideal transistor, whereas Myers (30) gave the nullator-norator T-model of an ideal transistor. Braun (6) obtained equivalent NIC networks using nullators and norators.

A synthesis procedure to realize non-reciprocal networks using unitors, which are essentially equivalent to three-terminal nullors, has been presented by Keen (23), who showed that classical passive network synthesis procedures can be unified and generalized by the introduction of unitors.

1.3 Technical Approach

The synthesis presented in this thesis is based upon a theorem which ensures the existence of a unique solution for networks containing nullators and norators. State-model approach is used to obtain the realization. The realization of the state model is achieved by first

considering the realization of the operator matrix which is normally

called the A-matrix. This obviously corresponds to source-free

networks.

The A-matrix considered is in the companion matrix form and its

realization is obtained by using the following steps.

(1) Start with an arbitrary network consisting of resistors,

capacitors, nullators and norators, without any sources.

(2) Write the cut-set, circuit, and component equations in

symbolic form.

(3) Derive the state model, thus obtaining the A-matrix in

symbolic form.

(4) Compare the given A-matrix in companion matrix form

with the A-matrix in symbolic form just derived to

obtain the topology and the element values of the

desired network.

(5) Simplify the realized network, if possible.

The above procedure results in a realization with a simple topology.

From this, a scheme to realize any entry of the A-matrix is derived.

This is used to obtain the realization of an arbitrary A-matirx. A

special case is the A-matrix in tridiagonal form, whose realization

resulted from using the above steps, possesses another simple topology.

Further, there exists another scheme to realize any entry of the A-

matrix in tridiagonal form.

The synthesis of transfer functions is considered next. Given a

transfer function, a corresponding state model is first derived. Now

that the A-matrix realization is accomplished, it remains to take into

account the input (source) and the output in realizing the state model.

By inserting the source into and obtaining the output from the realized

network for the A-matrix, the realization of the state model is achieved

(32). In some cases, controlled sources using the nullator-norator cir-

cuits may be required to do this.

Regarding the synthesis of immittance matrices, main efforts are

devoted to develop the synthesis procedure for short-circuit admittance

matrices. After some manipulations, a "new A-matrix" is obtained from

the state model corresponding to a short-circuit admittance matrix, and

this new A-matrix is readily realized by the technique developed

earlier. A simple modification in the realization of this new A-matrix

gives the realization of the short-circuit admittance matrix. For an

open-circuit impedance matrix, it is realized through the use of

gyrators connected with the realization of a short-circuit admittance

matrix which is taken to be the given open-circuit impedance matrix.

The results of the realizations of admittance and impedance

matrices are used in the synthesis of hybrid matrices. Realization of

voltage and current transfer-function matrices is similar to that of

scalar transfer functions.

The realization of the A-matrix in tridiagonal form is found useful

in the synthesis of gyrators using the state-model approach since the

short-circuit admittance matrix describing a gyrator can be regarded as

in tridiagonal form. It results in a three-transistor realization that

coincides with one obtained by using other methods (28).

Since it is well known that tridiagonal matrices are related to

ladder networks (25), the realization of the A-matrix in tridiagonal

form also finds application in realizing low-pass filters. By simply

inserting the source and identifying the output at the proper location

of the realized network for an A-matrix in tridiagonal form, its trans-
fer function can be determined to belong to that of a ladder network.
In this way, various kinds of transfer functions of low-pass filters can
be synthesized using networks containing the nullator and norator.

1.4 Organization of the Thesis

Chapter II presents the analysis of a class of networks containing
nullators and norators by means of a theorem. Based upon the conditions
described in this theorem, a state model is derived. Then, under some
topological constraints, this state model is simplifed such that it is
suitable for the use of synthesis.

Chapter III presents the synthesis of the A-matrix. This is impor-
tant in itself and is fundamental to the synthesis considered in the
next chapter.

Chapter IV presents the synthesis of transfer functions, immittance
matrices and gyrators.

Chapter V presents the synthesis of low-pass filters. A related
topic of RLC low-pass filters with prescribed terminal resistances is
also treated.

Chapter VI gives a summary and suggestions for further study.

CHAPTER II.

ANALYSIS -- THEOREM AND STATE MODELS

2.1   Introduction

In this chapter, analysis of a class of networks containing nulla-
tors and norators will be presented. The analysis is presented in
terms of a theorem which gives sufficient conditions for this class of
networks to have a unique solution of their network equations. This
theorem is derived from an analysis about systems containing through-
across type drivers by Blackwell and Grigsby (5). Corresponding to this
theorem, a state model for such a class of networks is derived. The
derivation of the state model is somewhat different from that used in
classical RLC networks because of the presence of nullators and norators;
the state model formulation requires two properly selected trees. Then,
the state model is to be simplified with respect to a subclass of net-
works, so that it can be used in developing the synthesis procedure in
later chapters.

2.2   Analysis of Networks Containing

Nullators and Norators

Davies (15) has given four methods to analyze networks containing
nullators and norators; namely, the nodal analysis method, the loop
analysis method, the practical method, and the topological method. He
has demonstrated that analysis of networks containing nullators and

norators is straightforward, and requires only slight modifications to
conventional methods. He has also given examples to show that signifi-
cant simplifications in active-network analysis may be achieved by
replacing conventional active elements by equivalent nullator-norator
models before commencing the analysis. Blackwell and Grigsby (5) pre-
sented a more generalized formulation for the analysis of double-driver
systems containing through-across drivers and no-specification compo-
nents, which determines the network topology when a unique solution of
the system equations exists. Since, for the nullator, both the across
and through variables are zero, and for the norator, the across and
through variables are arbitrary and independent, these two devices can
be identified as "through-across driver" and "no-specification
component", respectively. The results presented in (5) do not cover the
synthesis problem, but their formulation procedure gives an insight into
the design of active networks containing nullators and norators. The
following theorem is given in (5) and is presented here for ease of
reference. The elements used in this theorem are classified in Table
2.2.1, where the number of VC-elements is assumed to be equal to the num-
ber of N-elements, and the element variables are displayed in Table
2.2.2.

Theorem (Blackwell and Grigsby). Let a double-driver system, made
up of e elements and containing v vertices, have a one-part linear graph
G_b. If the system equations $A_a W_{vt} = 0$, $B_b M_{vt} = 0$, and $M_{vt} = Q_q W_{vt}$ have a
unique solution; then:

(1) there exists some tree T_1 of G_b which contains the
M-elements and the N-elements and for which the

TABLE 2.2.1

CLASSIFICATION OF ELEMENT TYPES

Classification	Description
VC-element	Both through and across variables specified
Q-element	Through and across variable related
M-element	The across variable specified (across driver)
W-element	The through variable specified (through driver)
N-element	No specifications

TABLE 2.2.2

DESCRIPTION OF ELEMENT TYPES

Matrix Variable	Description
M_m	The specified across variable associated with M-element
W_m	The through variable associated with M-element
M_w	The across variable associated with W-element
W_w	The specified through variable associated with W-element
M_n	The across variable associated with N-element
W_n	The through variable associated with N-element
M_{vc}	The specified across variable associated with VC-element
W_{vc}	The specified through variable associated with VC-element
M_q	The across variable associated with Q-element
W_q	The through variable associated with Q-element

W-elements and the VC-elements form a subgraph of

the cotree;

(2)   there exists some tree T_2 of G_b which contains the

M-elements and the VC-elements and for which the

W-elements and the N-elements form a subgraph of

the cotree,

where W_{vt} and M_{vt} denote vectors containing component matrix variables

in Table 2.2.2, respectively, and A_a, B_b, and Q_q denote associated

matrices.

Now, a theorem which covers the analysis and the synthesis of net-

works containing nullators and norators is presented. The element types

and symbols used in the theorem below and hereafter are shown in Table

2.2.3.

TABLE 2.2.3

ELEMENT TYPES AND SYMBOLS

Element Type	Symbol	No. of Elements	Voltage Variable Vector	Current Variable Vector
Nullator	VC	n_{vc}	V_{vc}	I_{vc}
Norator	N	n_n	V_n	I_n
Capacitor	C	n_c	V_c	I_c
Resistor	R	n_r	V_r	I_r
Inductor	L	n_ℓ	V_ℓ	I_ℓ
Voltage Source	V	n_v	V_v	I_v
Current Source	I	n_i	V_i	I_i

Theorem 2.2.1 Network Topology to Have a Unique Solution. Let a network have e elements and v vertices with the types of the elements as indicated in Table 2.2.3 and let the graph G_h corresponding to this network be connected. In this network, it is assumed that the number of the nullators, n_{vc}, is equal to the number of the norators, n_n. Let:

(i) there exists some tree T_1 of G_h which contains the voltage sources and the norators and for which the current sources and the nullators form a subgraph of the cotree;

(ii) there exists some tree T_2 of G_h which contains the voltage sources and the nullators and for which the current sources and the norators form a subgraph of the cotree.

Let the cut-set, circuit and component equations now be written as:

$$C_f I_{vt} = 0 \qquad (2.2.1)$$

$$B_f V_{vt} = 0 \qquad (2.2.2)$$

$$V_r - R_r I_r = 0 \qquad (2.2.3a)$$

$$V_c - Z_c I_c = 0 \qquad (2.2.3b)$$

$$V_\ell - Z_\ell I_\ell = 0 \qquad (2.2.3c)$$

where C_f and B_f represent, respectively, the cut-set matrix and the circuit matrix; I_{vt} and V_{vt} represent the current vector and the voltage vector containing component current variable vectors and voltage variable vectors as specified in Table 2.2.3, respectively; and R_r, Z_c, and Z_ℓ are diagonal matrices with the diagonal entries representing the

values of resistances, capacitances, and inductances, respectively.

Then, the set of Equations (2.2.1), (2.2.2), and (2.2.3) has a unique solution for some sets of element values of the resistors, capacitors, and inductors.

Proof. Equation (2.2.1), in general terms, can be written as

$$C_v I_v + C_n I_n + C_c I_c + C_r I_r + C_\ell I_\ell + C_{vc} I_{vc} + C_i I_i = 0 \qquad (2.2.4)$$

where C_v, C_n, ..., and C_i denote the submatrices of the cut-set matrix C_f. There are $v - 1$ equations in Equation (2.2.4). Since $I_{vc} = 0$ and I_i is specified, Equation (2.2.4) can be rewritten as

$$C_v I_v + C_n I_n + C_c I_c + C_r I_r + C_\ell I_\ell = -C_i I_i. \qquad (2.2.5)$$

Similarly, Equation (2.2.2) can be written as

$$B_c V_c + B_r V_r + B_\ell V_\ell + B_n V_n + B_i V_i = -B_v V_v \qquad (2.2.6)$$

where B_c, B_r, ..., and B_v denote the submatrices of the circuit matrix B_f. There are $e - v + 1$ equations in Equation (2.2.6), where V_v is specified.

Let the specified variables be expressed in the form

$$V_{vc} = 0 \qquad (2.2.7a)$$

$$I_{vc} = 0 \qquad (2.2.7b)$$

$$I_i = J_i \qquad (2.2.7c)$$

$$V_v = E_v \qquad (2.2.7d)$$

where J_i and E_v are known quantities.

Equations (2.2.5), (2.2.6), (2.2.7), and (2.2.3) can be written in matrix equation form:

$$
\begin{bmatrix}
C_v & C_n & C_c & C_r & C_\ell & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & B_i & B_n & B_r & B_c & B_\ell & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -R_r & \cdot & \cdot & \cdot & U & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -Z_c & \cdot & \cdot & \cdot & \cdot & \cdot & U & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -Z_\ell & \cdot & \cdot & \cdot & \cdot & U & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & U & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & U & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & U & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & U
\end{bmatrix}
\begin{bmatrix}
I_v \\ I_n \\ I_c \\ I_r \\ I_\ell \\ V_i \\ V_n \\ V_r \\ V_c \\ V_\ell \\ I_i \\ I_{vc} \\ V_v \\ V_{vc}
\end{bmatrix}
=
\begin{bmatrix}
-C_i J_i \\ -B_v E_v \\ 0 \\ 0 \\ 0 \\ J_i \\ 0 \\ E_v \\ 0
\end{bmatrix}
\tag{2.2.8}
$$

Consider the coefficient matrix in Equation (2.2.8). By hypothesis, some tree T_1 can be selected according to Condition (i). This is sufficient to ensure that the matrix $[C_v \ C_n]$ has maximum rank. Since $n_v + n_n \leq v - 1$, the rank of $[C_v \ C_n]$ is equal to $n_v + n_n$. Hence, the first $n_v + n_n$ columns of the coefficient matrix are linearly independent. Similarly, some tree T_2 can be selected according to Condition (ii), which implies that the matrix $[B_i \ B_n]$ has maximum rank equal to $n_i + n_n$. Note that $n_i + n_n \leq e - v + 1$. Thus, the $n_i + n_n$ columns corresponding to V_i and V_n are linearly independent. As a result, the only possible linear dependence that can occur is between the $n_c + n_r + n_\ell$

columns corresponding to I_c, I_r, and I_ℓ and the $n_r + n_c + n_\ell$ columns corresponding to V_r, V_c, and V_ℓ. In fact, this depends upon the entries in the submatrices R_r, Z_c, and Z_ℓ.

Consequently, with some sets of values of the resistors, capacitors, and inductors, all the columns of the coefficient matrix in Equation (2.2.8) are linearly independent and the coefficient matrix has maximum column rank. Since the coefficient matrix is a square matrix, it follows that it is nonsingular and the set of Equations (2.2.1), (2.2.2), and (2.2.3) has a unique solution. The theorem is now proved.

Theorem 2.2.1 provides a tool for derivation of the synthesis procedure using the state-model approach. The state model for a class of networks containing nullators and norators corresponding to Theorem 2.2.1 will be derived in the next section.

2.3 State Model of a Class of Networks

The state models of classical RLC networks have been extensively discussed (35). The fundamental cut-set matrix $[U \ C_{f1}]$ and the fundamental circuit matrix $[B_{f1} \ U]$ referring to a same tree of an RLC network are known to have the relationship as $C_{f1} = -B_{f1}^T$. In this section, the state model of a class of networks consisting of resistors, inductors, capacitors, nullators, norators, voltage sources, and current sources is derived. Although the inductors are not used in the synthesis procedures to be developed later, they are included in the state model for completeness.

The class of networks under consideration is assumed to satisfy Conditions (i) and (ii) of Theorem 2.2.1. Therefore, the first step in the analysis is to select two trees T_1 and T_2 stated in Theorem 2.2.1.

For reasons to be evident later, it is appropriate to write the fundamental cut-set matrix for tree T_1 and the fundamental circuit matrix for tree T_2. Some relationship does exist between the entries of these two matrices, but different from that mentioned in the preceding paragraph.

The general form of the state model of a network in symbolic form is

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + \Lambda\frac{d}{dt}u(t) \tag{2.3.1a}$$

$$y(t) = Cx(t) + Du(t) + \Gamma\frac{d}{dt}u(t) \tag{2.3.1b}$$

where x represents a state vector consisting of branch capacitor voltage variables and chord inductor current variables; y is the output vector which can be a current vector or a voltage vector or a combination of these two vectors; u is a vector consisting of specified drivers (excluding nullators); and the matrices A, B, Λ, C, D, and Γ are, in general, rectangular matrices, and some of them may be zero matrices in certain cases. The matrix A in Equation (2.3.1a) is of special interest and is called the operator matrix or the A-matrix of the network under consideration. Equation (2.3.1a) and Equation (2.3.1b) are referred, respectively, as the differential set and the algebraic set of the state model. But sometimes, for convenience, the former will be called the state equation, and the latter the output equation.

Before the derivation of the state model, it is to be noted that nothing has been said about topological locations of the resistors, inductors, and capacitors in the networks corresponding to Theorem 2.2.1. In general, the capacitors and the resistors are distributed

among the branches and the chords for both T_1 and T_2, and the distribution for T_1 may not be the same as that for T_2. However, the state model will be derived under the restriction that the distributions be the same for both T_1 and T_2.

The fundamental cut-set equations in symbolic form for T_1 are

$$
\begin{bmatrix}
U & & & & & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\
& U & & & & C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\
& & U & & & 0 & C_{32} & C_{33} & C_{34} & C_{35} \\
& & & U & & 0 & C_{42} & C_{43} & C_{44} & C_{45} \\
& & & & U & 0 & 0 & 0 & C_{54} & C_{55}
\end{bmatrix}
\begin{bmatrix}
I_v \\
I_{bc} \\
I_n \\
I_{br} \\
I_{b\ell} \\
\hline
I_{cc} \\
I_{cr} \\
I_{vc} \\
I_{c\ell} \\
I_i
\end{bmatrix}
= 0
\qquad (2.3.2)
$$

where I_{bc}, I_{br}, and $I_{b\ell}$ denote the current vectors of branch capacitors, branch resistors, and branch inductors, respectively; and I_{cc}, I_{cr}, and $I_{c\ell}$ denote the current vectors of chord capacitors, chord resistors, and chord inductors, respectively. Note that in Equation (2.3.2), $C_{51} = C_{52} = C_{53} = 0$, since each inductive branch defines a cut-set containing only current sources and inductors; and $C_{31} = C_{41} = C_{51} = 0$, since if the fundamental circuit equations are written for T_1, each capacitive chord defines a circuit containing only voltage sources and capacitors.

The fundamental circuit equations in symbolic form for T_2 are

$$
\begin{bmatrix}
B_{11} & B_{12} & 0 & 0 & 0 & \vdots & U & & & & \\
B_{21} & B_{22} & B_{23} & B_{24} & 0 & \vdots & & U & & & \\
B_{31} & B_{32} & B_{33} & B_{34} & 0 & \vdots & & & U & & \\
B_{41} & B_{42} & B_{43} & B_{44} & B_{45} & \vdots & & & & U & \\
B_{51} & B_{52} & B_{53} & B_{54} & B_{55} & \vdots & & & & & U
\end{bmatrix}
\begin{bmatrix}
V_v \\ V_{bc} \\ V_{vc} \\ V_{br} \\ V_{b\ell} \\ \text{---} \\ V_{cc} \\ V_{cr} \\ V_n \\ V_{c\ell} \\ V_i
\end{bmatrix} = 0
\qquad (2.3.3)
$$

where V_{bc}, V_{br}, and $V_{b\ell}$ denote the voltage vectors of branch capacitors, branch resistors, and branch inductors, respectively; and V_{cc}, V_{cr}, and $V_{c\ell}$ denote the voltage vectors of chord capacitors, chord resistors, and chord inductors, respectively. In Equation (2.3.3), $B_{13} = B_{14} = B_{15} = B_{25} = B_{35} = 0$, for the same reasons as above.

The component equations in symbolic form are

$$
\begin{bmatrix}
C_b & \\
& L_c
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix}
V_{bc} \\
I_{c\ell}
\end{bmatrix}
=
\begin{bmatrix}
I_{bc} \\
V_{c\ell}
\end{bmatrix}
\qquad (2.3.4a)
$$

$$
\begin{bmatrix}
L_b & \\
& C_c
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix}
I_{b\ell} \\
V_{cc}
\end{bmatrix}
=
\begin{bmatrix}
V_{b\ell} \\
I_{cc}
\end{bmatrix}
\qquad (2.3.4b)
$$

and

$$
\begin{bmatrix}
V_{br} \\
I_{cr}
\end{bmatrix}
=
\begin{bmatrix}
R_b & \\
& G_c
\end{bmatrix}
\begin{bmatrix}
I_{br} \\
V_{cr}
\end{bmatrix}
\qquad (2.3.4c)
$$

where C_b, L_c, L_b, C_c, R_b, and G_c are diagonal matrices with positive entries.

The state model can be derived by using Equations (2.3.2), (2.3.3), and (2.3.4). Eliminating the branch current variables I_{bc} and $I_{b\ell}$ and the chord voltage variables V_{cc} and $V_{c\ell}$ in Equations (2.3.4a) and (2.3.4b), and combining the resulting equations gives:

$$
\begin{bmatrix} C_b - C_{21} C_c B_{12} & 0 \\ 0 & L_c - B_{45} L_b C_{54} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} 0 & -C_{24} \\ -B_{42} & 0 \end{bmatrix} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} +
$$

$$
\begin{bmatrix} 0 & -C_{22} \\ -B_{44} & 0 \end{bmatrix} \begin{bmatrix} V_{br} \\ I_{cr} \end{bmatrix} + \begin{bmatrix} 0 & -C_{25} \\ -B_{41} & 0 \end{bmatrix} \begin{bmatrix} V_v \\ I_i \end{bmatrix} + \begin{bmatrix} C_{21} C_c B_{11} & 0 \\ 0 & B_{45} L_b C_{55} \end{bmatrix}
$$

$$
\cdot \frac{d}{dt} \begin{bmatrix} V_v \\ I_i \end{bmatrix} . \tag{2.3.5}
$$

Note that in obtaining Equation (2.3.5), the terms associated with V_{vc} and I_{vc} are dropped out because $V_{vc} = 0$ and $I_{vc} = 0$. Substituting I_{br} from Equation (2.3.2) and V_{cr} from Equation (2.3.3) in Equation (2.3.4c) results in

$$
\begin{bmatrix} V_{br} \\ I_{cr} \end{bmatrix} = \begin{bmatrix} U & R_b C_{42} \\ G_c B_{24} & U \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 0 & -R_b C_{44} \\ -G_c B_{22} & 0 \end{bmatrix} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} + \begin{bmatrix} 0 & -R_b C_{45} \\ -G_c B_{21} & 0 \end{bmatrix} \begin{bmatrix} V_v \\ I_i \end{bmatrix} \right\}
$$

$$
\tag{2.3.6}
$$

where the inverse can be found by using the generalized algorithm of Gauss (21) and is

$$
\begin{bmatrix} U & R_b C_{42} \\ G_c B_{24} & U \end{bmatrix}^{-1} = \begin{bmatrix} U + R_b C_{42}(U - G_c B_{24} R_b C_{42})^{-1} G_c B_{24} & -R_b C_{42}(U - G_c B_{24} R_b C_{42})^{-1} \\ -(U - G_c B_{24} R_b C_{42})^{-1} G_c B_{24} & (U - G_c B_{24} R_b C_{42})^{-1} \end{bmatrix}
$$

$$
\tag{2.3.7}
$$

Combining Equations (2.3.5), (2.3.6), and (2.3.7) yields

$$
\begin{bmatrix}
C_b - C_{21}C_cB_{12} & 0 \\
0 & L_c - B_{45}L_bC_{54}
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix}
V_{bc} \\
I_{c\ell}
\end{bmatrix}
=
$$

$$
\begin{bmatrix}
C_{22}\lambda G_cB_{22} & -C_{22}\lambda G_cB_{24}R_bC_{44} - C_{24} \\
-B_{44}R_bC_{42}\lambda G_cB_{22} - B_{42} & B_{44}[U + R_bC_{42}\lambda G_cB_{24}]R_bC_{44}
\end{bmatrix}
\begin{bmatrix}
V_{bc} \\
I_{c\ell}
\end{bmatrix}
+
$$

$$
\begin{bmatrix}
C_{22}\lambda G_cB_{21} & -C_{22}\lambda G_cB_{24}R_bC_{45} - C_{25} \\
-B_{44}R_bC_{42}\lambda G_cB_{21} - B_{41} & B_{44}[U + R_bC_{42}\lambda G_cB_{24}]R_bC_{45}
\end{bmatrix}
\begin{bmatrix}
V_v \\
I_i
\end{bmatrix}
+
$$

$$
\begin{bmatrix}
C_{21}C_cB_{11} & 0 \\
0 & B_{45}L_bC_{55}
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix}
V_v \\
I_i
\end{bmatrix}
\tag{2.3.8}
$$

where $\lambda = (U - G_cB_{24}R_bC_{42})^{-1}$.

The differential set of the state model can be obtained after taking the inverse of the coefficient matrix in the left-hand side of Equation (2.3.8). The existence of the inverse of this coefficient matrix can be proved by showing that the diagonal submatrices of this matrix, $C_b - C_{21}C_cB_{12}$ and $L_c - B_{45}L_bC_{54}$, are positive definite matrices. To do this, first note that B_{ij}'s in Equation (2.3.3) and C_{ij}'s in Equation (2.3.2) are related. These relationships are derived as follows.

The fundamental circuit equations in symbolic form for T_1 are:

$$
\begin{bmatrix}
-c_{11}^T & -c_{21}^T & 0 & 0 & 0 & U & & & & \\
-c_{12}^T & -c_{22}^T & -c_{32}^T & -c_{42}^T & 0 & & U & & & \\
-c_{13}^T & -c_{23}^T & -c_{33}^T & -c_{43}^T & 0 & & & U & & \\
-c_{14}^T & -c_{24}^T & -c_{34}^T & -c_{44}^T & -c_{54}^T & & & & U & \\
-c_{15}^T & -c_{25}^T & -c_{35}^T & -c_{45}^T & -c_{55}^T & & & & & U
\end{bmatrix}
\begin{bmatrix}
V_v \\ V_{bc} \\ V_n \\ V_{br} \\ V_{b\ell} \\ \hline V_{cc} \\ V_{cr} \\ V_{vc} \\ V_{c\ell} \\ V_i
\end{bmatrix} = 0 \qquad (2.3.9)
$$

where T denotes the transpose. Solving for V_{vc} from Equation (2.3.9) gives

$$
V_{vc} = c_{13}^T V_v + c_{23}^T V_{bc} + c_{33}^T V_n + c_{43}^T V_{br} \qquad (2.3.10)
$$

and it follows that

$$
V_n = (c_{33}^T)^{-1} V_{vc} - (c_{33}^T)^{-1} c_{13}^T V_v - (c_{33}^T)^{-1} c_{23}^T V_{bc} - (c_{33}^T)^{-1} c_{43}^T V_{br}.
$$

$$(2.3.11)$$

In Equation (2.3.11), it is assumed that c_{33}^{-1} exists. This is a necessary topological constraint. Solving for V_{cc}, V_{cr}, $V_{c\ell}$, and V_i from Equation (2.3.9) and substituting in V_n from Equation (2.3.11) yields

$$
V_{cc} = c_{11}^T V_v + c_{21}^T V_{bc}
$$

$$
V_{cr} = (c_{12} - c_{13} c_{33}^{-1} c_{32})^T V_v + (c_{22} - c_{23} c_{33}^{-1} c_{32})^T V_{bc} + (c_{33}^{-1} c_{32})^T V_{vc} + (c_{42} - c_{43} c_{33}^{-1} c_{32})^T V_{br}
$$

$$
V_{c\ell} = (c_{14} - c_{13} c_{33}^{-1} c_{34})^T V_v + (c_{24} - c_{23} c_{33}^{-1} c_{34})^T V_{bc} + (c_{33}^{-1} c_{34})^T V_{vc} + (c_{44} - c_{43} c_{33}^{-1} c_{34})^T V_{br} + c_{54}^T V_{b\ell}
$$

$$
V_i = (c_{15} - c_{13} c_{33}^{-1} c_{35})^T V_v + (c_{25} - c_{23} c_{33}^{-1} c_{35})^T V_{bc} + (c_{33}^{-1} c_{35})^T V_{vc} + (c_{45} - c_{43} c_{33}^{-1} c_{35})^T V_{br} + c_{55}^T V_{b\ell}
$$

Thus, Equation (2.3.9) can be rewritten as

$$
\begin{bmatrix}
-c_{11}^T & -c_{21}^T & 0 & 0 & 0 & & U & & & \\
(c_{13}c_{33}^{-1}c_{32}-c_{12})^T & (c_{23}c_{33}^{-1}c_{32}-c_{22})^T & -(c_{33}^{-1}c_{32})^T & (c_{43}c_{33}^{-1}c_{32}-c_{42})^T & 0 & & & U & & \\
(c_{13}c_{33}^{-1})^T & (c_{23}c_{33}^{-1})^T & -(c_{33}^{-1})^T & (c_{43}c_{33}^{-1})^T & 0 & & & & U & \\
(c_{13}c_{33}^{-1}c_{34}-c_{14})^T & (c_{23}c_{33}^{-1}c_{34}-c_{24})^T & -(c_{33}^{-1}c_{34})^T & (c_{43}c_{33}^{-1}c_{34}-c_{44})^T & -c_{54}^T & & & & & U \\
(c_{13}c_{33}^{-1}c_{35}-c_{15})^T & (c_{23}c_{33}^{-1}c_{35}-c_{25})^T & -(c_{33}^{-1}c_{35})^T & (c_{43}c_{33}^{-1}c_{35}-c_{45})^T & -c_{55}^T & U & & & &
\end{bmatrix}
\begin{bmatrix}
V_v \\ V_{bc} \\ V_{vc} \\ V_{br} \\ V_{b\ell} \\ V_{cc} \\ V_{cr} \\ V_n \\ V_{c\ell} \\ V_i
\end{bmatrix} = 0
$$

$$(2.3.12)$$

Now, the arrangement of the voltage variables in Equation (2.3.12) corresponds to tree T_2, and comparing this equation with Equation (2.3.3) results in:

$$B_{11} = -c_{11}^T; \quad B_{12} = -c_{21}^T; \quad B_{21} = (c_{13}c_{33}^{-1}c_{32} - c_{12})^T; \quad B_{22} = (c_{23}c_{33}^{-1}c_{32} - c_{22})^T;$$

$$B_{23} = -(c_{33}^{-1}c_{32})^T; \quad B_{24} = (c_{43}c_{33}^{-1}c_{32} - c_{42})^T; \quad B_{31} = (c_{13}c_{33}^{-1})^T; \quad B_{32} = (c_{23}c_{33}^{-1})^T;$$

$$B_{33} = -(c_{33}^{-1})^T; \quad B_{34} = (c_{43}c_{33}^{-1})^T; \quad B_{41} = (c_{13}c_{33}^{-1}c_{34} - c_{14})^T; \quad B_{42} = (c_{23}c_{33}^{-1}c_{34} - c_{24})^T;$$

$$B_{43} = -(c_{33}^{-1}c_{34})^T; \quad B_{44} = (c_{43}c_{33}^{-1}c_{34} - c_{44})^T; \quad B_{45} = -c_{54}^T; \quad B_{51} = (c_{13}c_{33}^{-1}c_{35} - c_{15})^T;$$

$$B_{52} = (c_{23}c_{33}^{-1}c_{35} - c_{25})^T; \quad B_{53} = -(c_{33}^{-1}c_{35})^T; \quad B_{54} = (c_{43}c_{33}^{-1}c_{35} - c_{45})^T; \quad B_{55} = -c_{55}^T$$

$$(2.3.13)$$

Thus, by expressing B_{45} and B_{12} in terms of C_{ij}, the diagonal sub-matrices become

$$C_b - C_{21} C_c B_{12} = C_b + C_{21} C_c C_{21}^T \qquad (2.3.14a)$$

$$L_c - B_{45} L_b C_{54} = L_c + C_{54}^T L_b C_{54} \qquad (2.3.14b)$$

which are obviously positive definite matrices and the inverse of the coefficient matrix in the left-hand side of Equation (2.3.8) exists.

Hence, the differential set of the state model is

$$\frac{d}{dt}\begin{bmatrix} V_{bc} \\ \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} (C_b - C_{21}C_cB_{12})^{-1}C_{22}(U - G_cB_{24}R_bC_{42})^{-1}G_cB_{22} \\ \\ -(L_c - B_{45}L_bC_{54})^{-1}[B_{44}R_bC_{42}(U - G_cB_{24}R_bC_{42})^{-1}G_cB_{22} + B_{42}] \end{bmatrix}$$

$$\begin{bmatrix} -(C_b - C_{21}C_cB_{12})^{-1}[C_{22}(U - G_cB_{24}R_bC_{42})^{-1}G_cB_{24}R_bC_{44} + C_{24}] \\ \\ (L_c - B_{45}L_bC_{54})^{-1}B_{44}[U + R_bC_{42}(U - G_cB_{24}R_bC_{42})^{-1}G_cB_{24}]R_bC_{44} \end{bmatrix}\begin{bmatrix} V_{bc} \\ \\ I_{c\ell} \end{bmatrix} +$$

$$\begin{bmatrix} (C_b - C_{21}C_cB_{12})^{-1}C_{22}(U - G_cB_{24}R_bC_{42})^{-1}G_cB_{21} \\ \\ -(L_c - B_{45}L_bC_{54})^{-1}[B_{44}R_bC_{42}(U - G_cB_{24}R_bC_{42})^{-1}G_cB_{21} + B_{41}] \end{bmatrix}$$

$$\begin{bmatrix} -(C_b - C_{21}C_cB_{12})^{-1}[C_{22}(U - G_cB_{24}R_bC_{42})^{-1}G_cB_{24}R_bC_{45} + C_{25}] \\ \\ (L_c - B_{45}L_bC_{54})^{-1}B_{44}[U + R_bC_{42}(U - G_cB_{24}R_bC_{42})^{-1}G_cB_{24}]R_bC_{45} \end{bmatrix}\begin{bmatrix} V_v^* \\ \\ I_i^* \end{bmatrix} +$$

$$\begin{bmatrix} (C_b - C_{21}C_cB_{12})^{-1}C_{21}C_cB_{11} & 0 \\ \\ 0 & (L_c - B_{45}L_bC_{54})^{-1}B_{45}L_bC_{55} \end{bmatrix}\frac{d}{dt}\begin{bmatrix} V_v^* \\ \\ I_i^* \end{bmatrix}$$

$$(2.3.15)$$

where V_v^* and I_i^* are terminal variables and are related to the driver variables V_v and I_i by $V_v^* = V_v$ and $I_i^* = I_i$.

The algebraic set of the state model also can be derived by using Equations (2.3.2), (2.3.3), and (2.3.4). From Equations (2.3.2) and (2.3.3),

$$\begin{bmatrix} I_v \\ V_i \end{bmatrix} = \begin{bmatrix} 0 & -C_{14} \\ -B_{52} & 0 \end{bmatrix}\begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} + \begin{bmatrix} 0 & -C_{11} \\ -B_{55} & 0 \end{bmatrix}\begin{bmatrix} V_{b\ell} \\ I_{cc} \end{bmatrix} + \begin{bmatrix} 0 & -C_{12} \\ -B_{54} & 0 \end{bmatrix}\begin{bmatrix} V_{br} \\ I_{cr} \end{bmatrix} + \begin{bmatrix} 0 & -C_{15} \\ -B_{51} & 0 \end{bmatrix}\begin{bmatrix} V_v \\ I_i \end{bmatrix}$$

$$(2.3.16)$$

Note again that $V_{vc} = 0$ and $I_{vc} = 0$. After substituting $V_{b\ell}$ and I_{cc} from Equation (2.3.4b), and V_{br} and I_{cr} from Equation (2.3.6) in Equation (2.3.16) and collecting terms, it results in

$$\begin{bmatrix} I_v \\ V_i \end{bmatrix} = \begin{bmatrix} C_{12}(U - G_c B_{24} R_b C_{42})^{-1} G_c B_{22} & -C_{12}(U - G_c B_{24} R_b C_{42})^{-1} G_c B_{24} R_b C_{44} - C_{14} \\ -B_{54} R_b C_{42}(U - G_c B_{24} R_b C_{42})^{-1} G_c B_{22} - B_{52} & B_{54}[U + R_b C_{42}(U - G_c B_{24} R_b C_{42})^{-1} G_c B_{24}] R_b C_{44} \end{bmatrix}\begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} +$$

$$\begin{bmatrix} C_{12}(U - G_c B_{24} R_b C_{42})^{-1} G_c B_{21} & -C_{12}(U - G_c B_{24} R_b C_{42})^{-1} G_c B_{24} R_b C_{45} - C_{15} \\ -B_{54} R_b C_{42}(U - G_c B_{24} R_b C_{42})^{-1} G_c B_{21} - B_{51} & B_{54}[U + R_b C_{42}(U - G_c B_{24} R_b C_{42})^{-1} G_c B_{24}] R_b C_{45} \end{bmatrix}\begin{bmatrix} V_v \\ I_i \end{bmatrix} +$$

$$\begin{bmatrix} C_{11} C_c B_{12} & 0 \\ 0 & B_{55} L_b C_{54} \end{bmatrix} \frac{d}{dt}\begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} + \begin{bmatrix} C_{11} C_c B_{11} & 0 \\ 0 & B_{55} L_b C_{55} \end{bmatrix} \frac{d}{dt}\begin{bmatrix} V_v \\ I_i \end{bmatrix}$$

$$(2.3.17)$$

where the derivative of the state variables, $\dfrac{d}{dt}\begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix}$, can be eliminated by using Equation (2.3.15).

Now, it follows that the algebraic set is given by

$$
\begin{bmatrix} I_v^* \\ V_i^* \end{bmatrix} = \begin{bmatrix} -C_{12}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{22} - C_{11}C_cB_{12}(C_b-C_{21}C_cB_{12})^{-1}C_{22}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{22} \\ B_{54}R_bC_{42}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{22} + B_{52} + B_{55}L_bC_{54}(L_c-B_{45}L_bC_{54})^{-1}B_{44}R_bC_{42}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{22} + B_{42}] \end{bmatrix}
$$

$$
\begin{bmatrix} C_{12}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{24}R_bC_{44} + C_{14} + C_{11}C_cB_{12}(C_b-C_{21}C_cB_{12})^{-1}[C_{22}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{24}R_bC_{44} + C_{24}] \\ -B_{54}[U+C_{42}R_b(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{24}]R_bC_{44} - B_{55}L_bC_{54}(L_c-B_{45}L_bC_{54})^{-1}B_{44}[U+R_cC_{42}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{24}]R_bC_{44} \end{bmatrix} \begin{bmatrix} V_{bc} \\ I_{\ell} \end{bmatrix}
$$

$$
\begin{bmatrix} -C_{12}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{21} - C_{11}C_cB_{12}(C_b-C_{21}C_cB_{12})^{-1}C_{22}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{21} \\ B_{54}R_bC_{42}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{21} + B_{51} + B_{55}L_bC_{54}(L_c-B_{45}L_bC_{54})^{-1}B_{44}R_bC_{42}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{21} + B_{41}] \end{bmatrix}
$$

$$
\begin{bmatrix} C_{12}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{24}R_bC_{45} + C_{15} + C_{11}C_cB_{12}(C_b-C_{21}C_cB_{12})^{-1}C_{22}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{24}R_bC_{45} + C_{25} \\ -B_{54}[U+R_bC_{42}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{24}]R_bC_{45} - B_{55}L_bC_{54}(L_c-B_{45}L_bC_{54})^{-1}B_{44}[U+R_bC_{42}(U-G_cB_{24}R_bC_{42})^{-1}G_cB_{24}]R_bC_{45} \end{bmatrix} \begin{bmatrix} V_v^* \\ I_i^* \end{bmatrix}
$$

$$
\begin{bmatrix} -C_{11}C_cB_{11} - C_{11}C_cB_{12}(C_b-C_{21}C_cB_{12})^{-1}C_{21}C_cB_{11} & 0 \\ 0 & -B_{55}L_bC_{55} - B_{55}L_bC_{54}(L_c-B_{45}L_bC_{54})^{-1}B_{45}L_bC_{55} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_v^* \\ I_i^* \end{bmatrix}
$$

$$(2.3.18)$$

where I_v^* and V_i^* are terminal variables and are related to the driver variables I_v and V_i by $I_v^* = -I_v$ and $V_i^* = -V_i$.

2.4 Simplified State Model

The state model derived in the preceding section, which includes the differential set in Equation (2.3.15) and the algebraic set in Equation (2.3.18), is for a class of networks containing resistors, inductors, capacitors, nullators, norators, and sources with the topology as stated in Theorem 2.2.1 and with the constraint that the RLC elements are distributed in both T_1 and T_2 in the same manner. This state model will be simplified by using additional restrictions on the topology. These restrictions are discussed below.

Earlier, it was mentioned that inductors will not be used in the synthesis. Furthermore, for simplicity, all the capacitors will be assumed to be contained in the branches and all the resistors will be assumed to be contained in the chords of both T_1 and T_2. Corresponding to this topology, the state model derived above can be simplified by substituting $I_{c\ell} = 0$, $C_c = 0$, and $R_b = 0$. Now, the state model for the above topology without any sources can be obtained by equating $V_v = 0$ and $I_i = 0$. It follows that the differential set of the state model for the above simplified portless network is given by

$$\frac{d}{dt} V_{bc} = C_b^{-1} C_{22} G_c B_{22} V_{bc} \qquad (2.4.1)$$

where B_{22} can be expressed in terms of C_{ij} using the relationships in Equation (2.3.13). Thus, Equation (2.4.1) can be written as

$$\frac{d}{dt} V_{bc} = C_b^{-1} C_{22} G_c (C_{23} C_{33}^{-1} C_{32} - C_{22})^T V_{bc}$$

$$= AV_{bc}. \qquad (2.4.2)$$

The synthesis of an A-matrix is considered in the next chapter, where the A-matrix in symbolic form in Equation (2.4.2) will be used in developing the synthesis procedure.

CHAPTER III

SYNTHESIS OF THE A-MATRIX

3.1   Introduction

Since a state model provides sufficient information for the characterization of the behavior of a given network at any instant of time, the state-model theory has received a rather thorough mathematical development and has practical applications.   In this chapter, network synthesis using state-model approach will be presented.

Given a scalar transfer function, $T(s)$ in s-domain with no poles at infinity, the corresponding general form of the state model in Equation (2.3.1) reduces to

$$\frac{d}{dt} x(t) = Ax(t) + Bu(t) \qquad (3.1.1a)$$

$$y(t) = Cx(t) + Du(t) \qquad (3.1.1b)$$

such that

$$T(s) = D + C(sU_n - A)^{-1}B \qquad (3.1.2)$$

where U_n represents an $n \times n$ identity matrix with n the degree of $T(s)$ (40); x denotes the state vector consisting of n state variables x_1, x_2, ..., and x_n; u and y, the input and output vectors, respectively; and A, B, C, D, the constant coefficient matrices of appropriate dimensions for linear, time-invariant networks.   Note that the state model

in Equation (3.1.1) is not unique. The derivation of a set of the

matrices {A, B, C, D} in Equation (3.1.1) can be readily obtained (32).

To realize T(s), one needs to realize the set of the matrices

{A, B, C, D} in Equation (3.1.1).

In Chapter II, the state model corresponding to a class of networks

with the network topology specified in Conditions (i) and (ii) of

Theorem 2.2.1 was derived. The simplified state model in Equation

(2.4.2) with respect to a subclass of networks defined previously will

be used in this chapter to obtain the synthesis procedure in realizing

networks containing nullators and norators. Synthesis of the A-matrix

is considered first and the results will be used later in the synthesis

of transfer functions and immittance matrices.

3.2 Preliminary Considerations

The A-matrix plays an important role in the state model of a net-

work. It was introduced by Bashkow (3) in 1957 as a new network

description. The use of the A-matrix in describing a network is conven-

ient because it relates the network in a natural way to a canonical

mathematical form. For example, the characteristic polynomial of a net-

work is given by $|sU - A|$, and its zeros give the natural frequencies of

this network. Because of its importance, a lot of work has been done on

the A-matrx. Bryant (8) obtained an explicit form of the A-matrix for

RLC networks. Dervisoglu (17) discussed the A-matrix for a restricted

class of active RLC networks.

The realization of an A-matrix by passive networks has been exten-

sively discussed. Rauch (34) has solved the problem for a special class

of RL and RC networks. Dervisoglu (18) has obtained the realization of

the A-matrix using a class of half-degenerate RLC networks. Silverman (36) presented the realizability of an A-matrix as a passive reciprocal network.

The synthesis of the A-matrix is now considered using a class of active RC networks containing nullators and norators, where the nullator and the norator appear as a pair with a common vertex. Since a pair of nullator and norator with a common vertex can be replaced by an ideal transistor, the realization is equivalent to using resistors, capacitors, and ideal transistors.

To realize an A-matrix, an arbitrary network consisting of resistors, capacitors, nullators, and norators, but no sources, is assumed. The arbitrary network is such that Conditions (i) and (ii) of Theorem 2.2.1 are satisfied. Furthermore, as assumed earlier, all capacitors are contained in the branches and all the resistors are contained in the chords for both T_1 and T_2.

The simplified state model in Equation (2.4.2) is in accordance with the topology as assumed above. For ready reference, Equation (2.4.2) is repeated below:

$$\frac{d}{dt} V_{bc} = C_b^{-1} C_{22} G_c (C_{23} C_{33}^{-1} C_{32} - C_{22})^T V_{bc} \qquad (3.2.1)$$

where the symbols were defined in Chapter II. Hence, the A-matrix in symbolic form is

$$A = C_b^{-1} C_{22} G_c (C_{23} C_{33}^{-1} C_{32} - C_{22})^T. \qquad (3.2.2)$$

The synthesis procedure is first considered when A in Equation (3.2.2) is in companion matrix form. Then, it will be extended to an arbitrary A-matrix. The special case of the A-matrix in tridiagonal

form deserves another section to discuss its realization.

3.3 Synthesis of an A-Matrix in

Companion Matrix Form

This is considered first because when the specification is in terms of a characteristic polynomial, such as the denominator of a transfer function, the corresponding A-matrix in companion matrix form can be obtained by inspection. Furthermore, the realization of an A-matrix in companion matrix form gives insight into the realization of an arbitrary A-matrix.

Consider the A-matrix in companion matrix form:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ -a_1 & -a_2 & -a_3 & \cdot & \cdot & \cdot & -a_{n-1} & -a_n \end{bmatrix}$$

(3.3.1a)

such that

$$\frac{d}{dt} x = A_c x$$

(3.3.1b)

wher a_1, a_2, ..., a_n are assumed to be positive constants and x consists of n state variables x_1, x_2, ..., and x_n. Its realization can be extended to the case that some of the a_i's in A_c are negative.

In order to make all the entries in the nth row of A_c positive, Equation (3.3.1b) is augmented by introducing a new state variable x_{n+1} such that

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & a_n & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \\ x_{n+1} \end{bmatrix}$$

$$= A_2 \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \\ x_{n+1} \end{bmatrix} \qquad\qquad (3.3.2)$$

where $\frac{d}{dt} x_{n+1} \equiv 0$. The reason for the above augmentation will be explained later. Note that Equation (3.3.2) is equivalent to Equation (3.3.1b).

The synthesis procedure is obtained by first equating the A-matrix in symbolic form in Equation (3.2.2) to the matrix A_2 in Equation (3.3.2). For obtaining the specific terms in Equation (3.2.2), A_2 is decomposed into a matrix triple product:

$$A_2 = \begin{bmatrix} U_n & 0 \\ 0 & J_{n+1} \end{bmatrix} \begin{bmatrix} U_n & 0 \\ 0 & D_1 \end{bmatrix} \begin{bmatrix} 0 & K \\ U_n & H \end{bmatrix}^T \qquad (3.3.3)$$

where U_n denotes a unit matrix of order n; J_{n+1}, an $1 \times (n+1)$ matrix $[1 \quad 1 \quad 1 \quad \ldots \quad 1]$; K, an $1 \times (n+1)$ matrix $[1 \quad 0 \quad 0 \quad \ldots \quad 0]$; $H = [0_1 \quad U_n]$ with 0_1 representing a zero column matrix; and

$D_1 = \text{diag}(a_1, a_2, \ldots, a_n, 1)$. The decomposition of A_2 into a triple product is simple and follows the algorithm below:

(1) Form the augmented $(2n + 2) \times (2n + 2)$ matrix A_3 as

$$A_3 = \left[\begin{array}{c|c} D_2 & -A_2 \\ \hline -A_2^T & D_3 \end{array} \right] \qquad (3.3.4)$$

where each of the matrices D_2 and D_3 represents a diagonal $(n + 1) \times (n + 1)$ submatrix whose diagonal entries are the sum of absolute values of other entries in the respective rows of A_3.

(2) Now, A_3 can be decomposed by using the Cederbaum's algorithm (11) in the form

$$A_3 = A_b \, D_4 \, A_b^T \qquad (3.3.5)$$

where

$$A_b = \left[\begin{array}{c} -A_{b1} \\ \hline A_{b2} \end{array} \right], \qquad D_4 = \left[\begin{array}{cc} U_n & 0 \\ 0 & D_1 \end{array} \right]$$

with

$$A_{b1} = \left[\begin{array}{cc} U_n & 0 \\ 0 & J_{n+1} \end{array} \right] \qquad (3.3.6)$$

and

$$A_{b2} = \left[\begin{array}{cc} 0 & K \\ U_n & H \end{array} \right]. \qquad (3.3.7)$$

Thus, the decomposition of A_2 in Equation (3.3.3) can be obtained from Equation (3.3.5).

Note that the decomposition in Equation (3.3.5) is essentially unique (11). However, the decomposition of A_2 into $A_2 = A_{b1} D_4 A_{b2}^T$ is not unique as the selection of D_2 and D_3 in A_3 in Equation (3.3.4) is not unique.

Now, comparing A_2 in Equation (3.3.3) with the matrix A in Equation (3.2.2) with the choice of C_b as a unit matrix for simplicity results in

$$C_{22} = \begin{bmatrix} U_n & O \\ O & J_{n+1} \end{bmatrix} = A_{b1} \qquad (3.3.8)$$

$$G_c = \begin{bmatrix} U_n & O \\ O & D_1 \end{bmatrix} = D_4 \qquad (3.3.9)$$

and

$$C_{23} C_{33}^{-1} C_{32} - C_{22} = \begin{bmatrix} O & K \\ U_n & H \end{bmatrix} = A_{b2} \qquad (3.3.10)$$

The choice of C_b as a unit matrix implies that each capacitor in the realization is of unit element value. Later, another solution will be described with each capacitor having element value other than unity.

The sum of the matrices A_{b1} and A_{b2} given in Equations (3.3.8) and (3.3.10), respectively, yields

$$C_{23} C_{33}^{-1} C_{32} = A_{b1} + A_{b2}. \qquad (3.3.11)$$

At this point, it should be noted from the fundamental cut-set equations in Equation (2.3.2) that C_{33} relates the locations of the nullators to the locations of the norators. Since each nullator and norator must exist as a pair, it is logical to set C_{33} to be a unit matrix, which

implies that every cut-set of a norator involves a nullator. This choice make the solution simple. Furthermore, this ensures that in the realization each nullator and norator form a pair so that it can be replaced by an ideal transistor.

Upon decomposition, Equation (3.3.11) can be written in the form, with C_{33} set to be a unit matrix,

$$C_{23}C_{32} = \begin{bmatrix} A_{b1} & U \end{bmatrix} \begin{bmatrix} U \\ \\ A_{b2} \end{bmatrix} \qquad (3.3.12)$$

where U is a unit matrix of appropriate order. A possible choice for C_{23} and C_{32} is

$$C_{23} = \begin{bmatrix} A_{b1} & U \end{bmatrix} \qquad (3.3.13)$$

$$C_{32} = \begin{bmatrix} U \\ \\ A_{b2} \end{bmatrix}. \qquad (3.3.14)$$

Obviously, the decomposition in Equation (3.3.12) is not unique and, therefore, there are many solutions. The above solution gives one particular realization.

Now, form the fundamental cut-set matrix:

$$\begin{bmatrix} U & 0 & C_{22} & C_{23} \\ \hline 0 & U & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} U & 0 & 0 & A_{b1} & A_{b1} & U \\ 0 & U & 0 & U & U & 0 \\ 0 & 0 & U & A_{b2} & 0 & U \end{bmatrix} \quad (3.3.15)$$

It can be shown that the above fundamental cut-set matrix is always realizable by simply constructing the graph (27) which is shown in

Figure 3.3.1. The resulting synthesized network is shown in Figure
3.3.2, where the element values of the conductors in mhos are obtained
from the matrix G_c in Equation (3.3.9) and each capacitor has unit ele-
ment value because the matrix C_b has been assumed to be a unit matrix.
Note that unless stated otherwise, for the element values in the figures
from here on, the conductors in mhos and capacitors in farads will be
used. Note that in Figure 3.3.1, the arrowhead on a line indicates the
direction of the current of the element associated with that line. This
will hold true for the network graph hereafter.
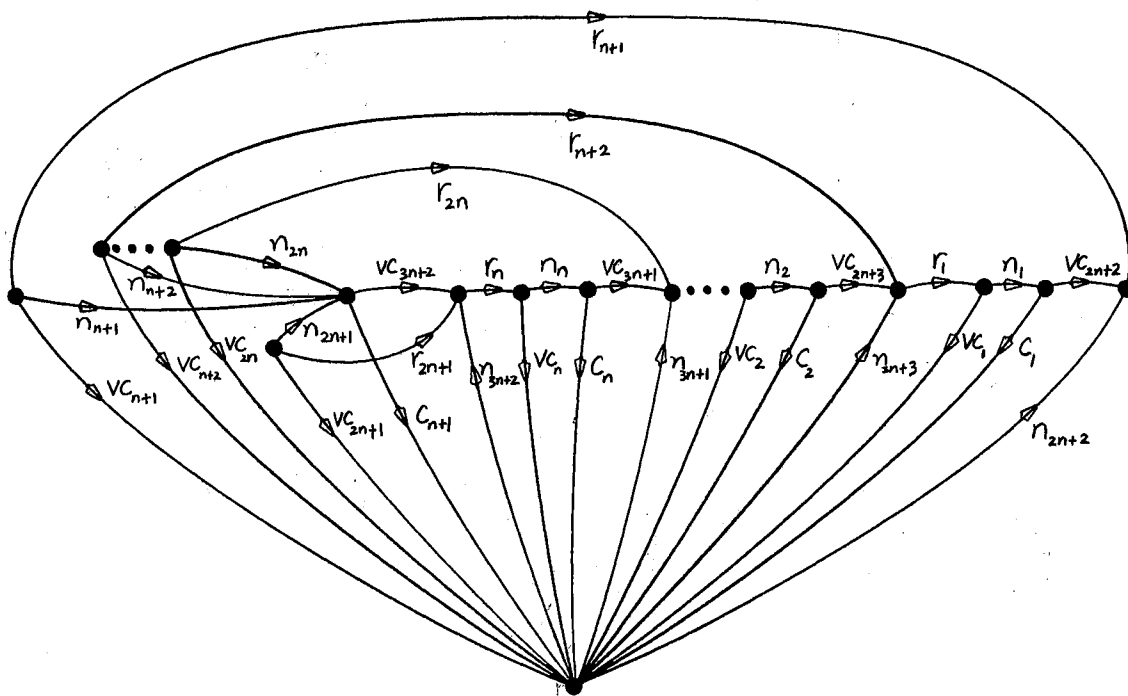
Figure 3.3.1. Network Graph
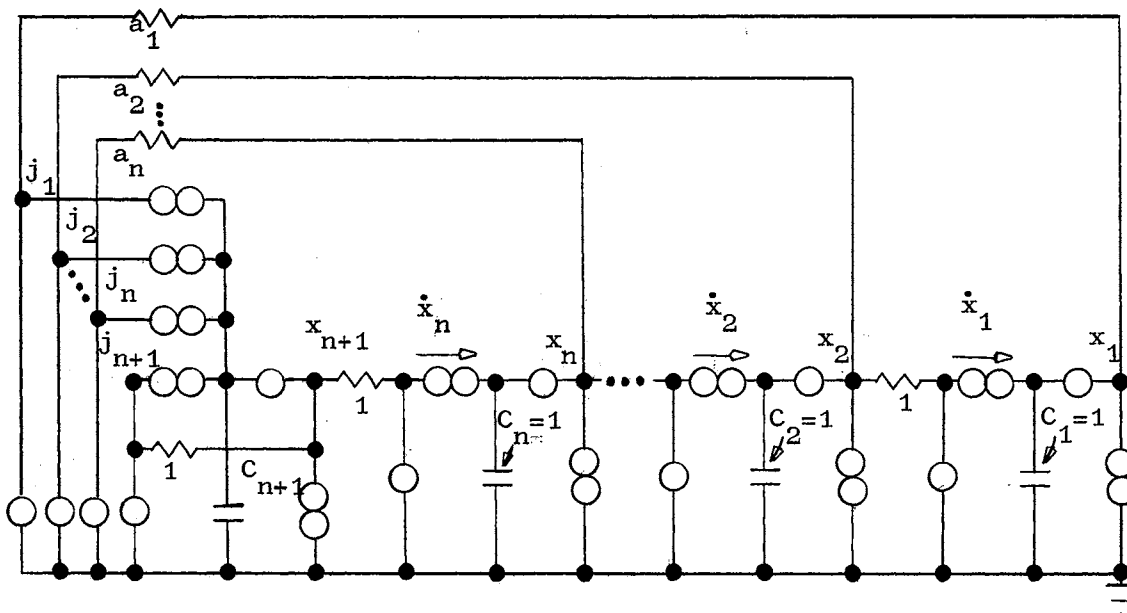
Figure 3.3.2. Synthesized Network

As shown in Figure 3.3.2, the state variable x_i and the derivative of the state variable x_i, $\dot{x}_i$ (dot denotes derivative with respect to time), can be considered to be the voltage across and the current through the capacitor C_i, $i = 1, 2, \ldots, n$. In addition, the introduced state variable x_{n+1} can be considered to be the voltage across the capacitor C_{n+1}. However, since $\frac{d}{dt} x_{n+1} \equiv 0$ as assumed previously in Equation (3.3.2), the capacitor C_{n+1} should be open-circuited.

After the capacitor C_{n+1} is removed, the network in Figure 3.3.2 can be simplified. It is observed that the nodes j_1, j_2, $\ldots$, j_n, and j_{n+1} can be combined to form only one node as far as the current-voltage operation at each of these nodes is concerned, since the adjoining nullators act as short circuits and open circuits. Further, the use of the equivalence shown in Figure 3.3.3, and the fact that a nullator in series with a norator is equivalent to an open circuit (30)

simplifies the network in Figure 3.3.2 to that shown in Figure 3.3.4.
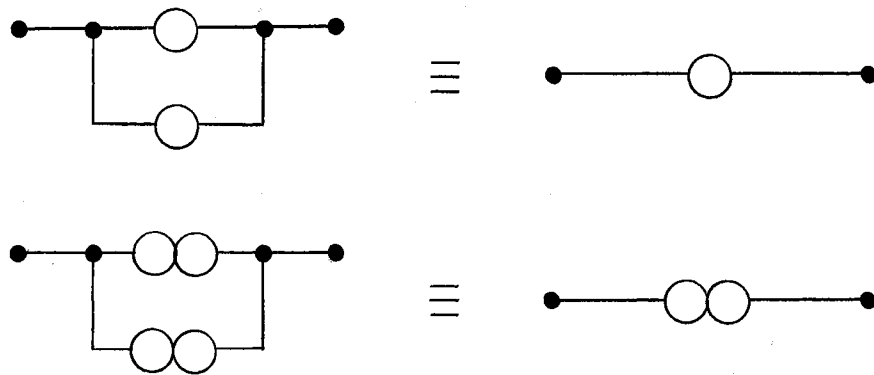
Figure 3.3.3.   Two Equivalences
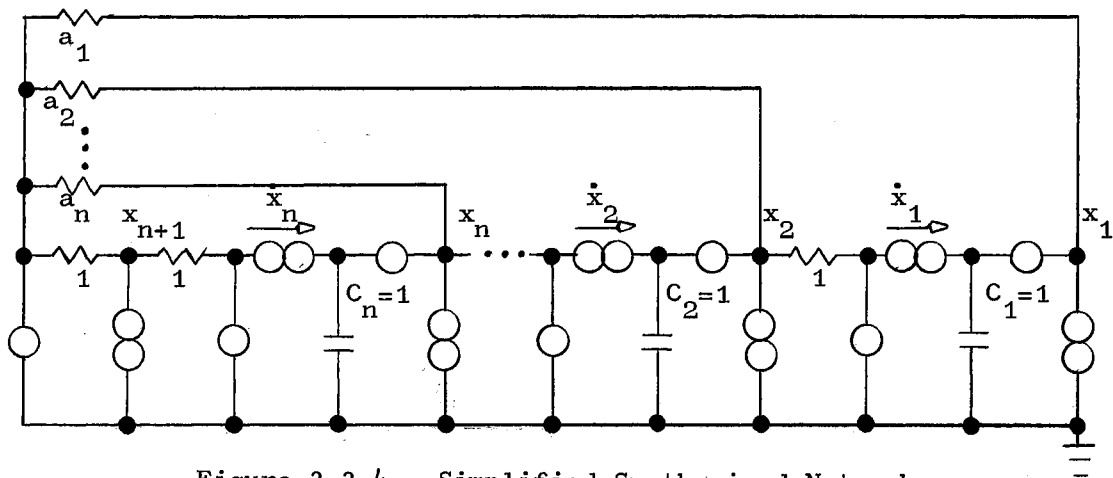
Figure 3.3.4.   Simplified Synthesized Network

The following example illustrates the synthesis procedure.

Example 3.3.1   Consider the third-order companion matrix

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \qquad (3.3.16a)$$

such that

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (3.3.16b)$$

where a_1, a_2, and a_3 are positive constants. The first step is to augment Equation (3.3.16b) such that all its entries in the coefficient matrix are positive. This gives

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad (3.3.17)$$

where x_4 is the introduced state variable. The equivalence of Equations (3.3.17) and (3.3.16b) is obvious. Decompose the coefficient matrix in Equation (3.3.17) into a matrix triple product corresponding to Equation (3.3.3):

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & a_1 & & & \\ & & & & a_2 & & \\ & & & & & a_3 & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

$$(3.3.18)$$

Now the matrices C_{22}, G_c, and A_{b2} can be identified from Equation (3.3.18) as:

$$C_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} = A_{b1} \qquad (3.3.19)$$

$$G_c = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & a_1 & & & \\ & & & & a_2 & & \\ & & & & & a_3 & \\ & & & & & & 1 \end{bmatrix} \qquad (3.3.20)$$

$$C_{23}C_{33}^{-1}C_{32} - C_{22} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} = A_{b2}. \qquad (3.3.21)$$

The submatrices C_{23} and C_{32} can be constructed from Equations (3.3.13), (3.3.14), (3.3.19), and (3.3.21):

$$C_{23} = \left[\begin{array}{ccccccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \qquad (3.3.22)$$

$$C_{32} = \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]. \qquad (3.3.23)$$

Note that C_{33} is assumed to be a unit matrix. Hence,

$$
\begin{bmatrix} U & O & | & C_{22} & | & C_{23} \\ & & |\overline{} & |\overline{} \\ O & U & | & C_{32} & | & C_{33} \end{bmatrix} =
\begin{bmatrix}
1 & & & & & & & & & & & | & 1 & & & & | & 1 & & 1 \\
& 1 & & & & & & & & & & | & & 1 & & & | & 1 & & & 1 \\
& & 1 & & & & & & & & & | & & & 1 & & | & & 1 & & 1 \\
& & & 1 & & & & & & & & | & & & & 1111 & | & & 1111 & & 1 \\
& & & & 1 & & & & & & & \overline{}| & \overline{1} & & & & \overline{}| & \overline{1} \\
& & & & & 1 & & & & & & | & & 1 & & & | & & 1 \\
& & & & & & 1 & & & & & | & & & 1 & & | & & & 1 \\
& & & & & & & 1 & & & & | & & & & 1 & | & & & & 1 \\
& & & & & & & & 1 & & & | & & & & & 1 | & & & & 1 \\
& & & & & & & & & 1 & & | & & & 1 & & | & & & & & 1 \\
& & & & & & & & & & 1 & | 1 & & 1 & & | & & & & & 1 \\
& & & & & & & & & & & 1 | & 1 & & 1 & & | & & & & & 1 \\
& & & & & & & & & & & 1 | & & 1 & & 1 | & & & & & 1
\end{bmatrix}
$$

$$(3.3.24)$$

Corresponding to the above fundamental cut-set matrix, the network graph is constructed in Figure 3.3.5.

Figure 3.3.5. Example Network Graph

After the simplifications described earlier in this section are
made, the network corresponding to the graph in Figure 3.3.5 is shown in
Figure 3.3.6.
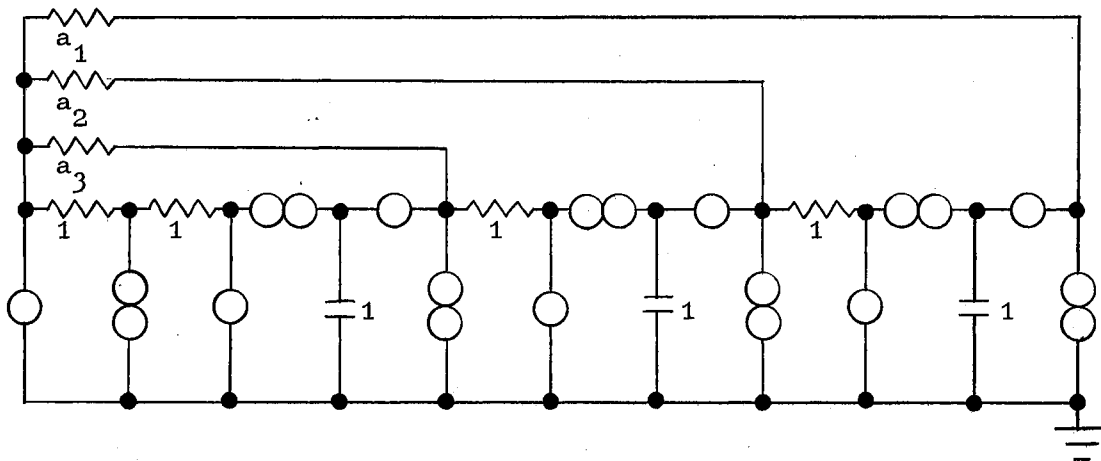
Figure 3.3.6. Example Synthesized Network

The reason for the augmentation in Equation (3.3.2) can now be
explained. If the decomposition procedure discussed earlier is applied
directly to A_c in Equation (3.3.1a), then the matrices A_{b1} and A_{b2} may
not be submatrices of a cut-set matrix (11). Therefore, the cut-set
matrix in Equation (3.3.15) may not be realizable. The augmentation
in Equation (3.3.2) allows for the direct realization of Equation
(3.3.1b).

It should be observed that for a given A-matrix, there are several
realizations corresponding to different sets of the matrices C_{22}, C_{23},
C_{32}, and C_{33}, and thus there is a choice in networks.

3.4 Building Blocks

Examination of the realization in Figure 3.3.4 reveals that there is a basic topology. This basic topology will be discussed in terms of building blocks. There are two such building blocks: one is for the operation of integration, which will be identified as an "integrator", and the other is for the operation of summation (or inversion), which will be identified as a "summer" (or an inverter), as shown in (a) and (b) of Figure 3.4.1, respectively. As far as their operations are concerned, these are similar to the integrator and the summer usually used in an analog-computer simulation, where an operational amplifier in conjunction with resistors and capacitors is implemented to achieve the operations of integration and summation.

$$(a) \quad \frac{d}{dt} x_o = a_0 x_1 \qquad\qquad (b) \quad x_o = -\frac{1}{a} \sum_{i=1}^{n} a_i x_i$$
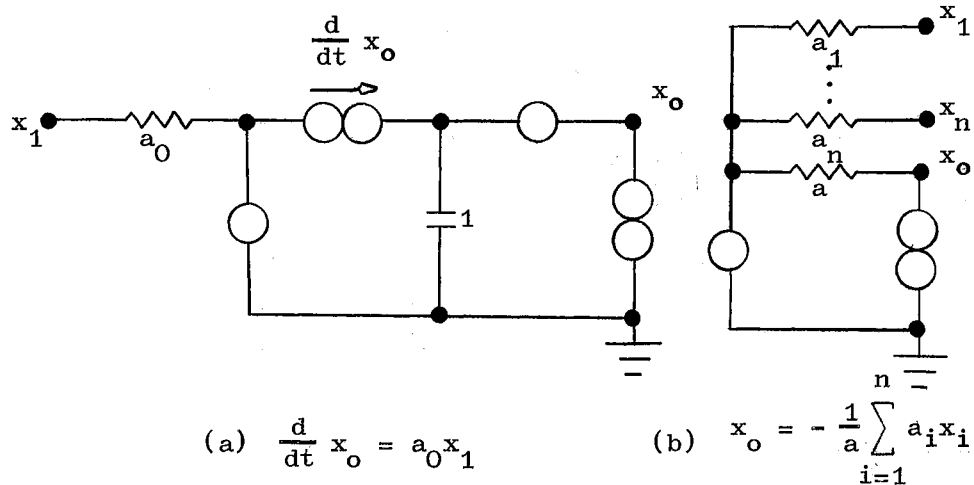
Figure 3.4.1. (a) Circuit for Integration; (b) Circuit for Summation (or Inversion)

In Figure 3.4.1, x_i represents an input or an output, and is a state variable corresponding to some capacitor voltage in the realization, and $\frac{d}{dt} x_o$ is the current through the capacitor in the integrator.

Note that for the operation of integration (Figure 3.4.1(a)), the output is of the same polarity as the input; whereas, for the operation of summation (or inversion) (Figure 3.4.1(b)), the output is out of phase with the input. Note further that the nullator and the norator appear as a pair in both the integrator and the summer, and, thus, allowing for the replacement of the nullator-norator pair by an ideal transistor.

The building block in Figure 3.4.1(a) may be extended to obtain a summing integrator as shown in Figure 3.4.2.

$$\frac{d}{dt} x_o = \sum_{i=1}^{n} a_i x_i$$
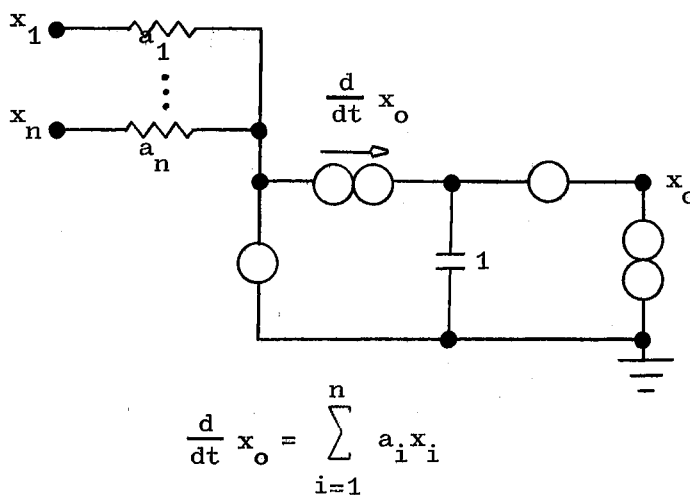
Figure 3.4.2. Summing Integrator

3.5 Use of the Building Blocks

When some of the a_i's in the matrix A_c of Equation (3.3.1) are negative, the first step of the synthesis procedure is again to augment the matrix A_c such that all its entries are positive. Note that if all the entries of A_c in Equation (3.3.1) are positive, there is no such need to augment it. For both of these two cases, a synthesis procedure similar to that discussed in Section 3.3 can be carried out to obtain the realization.

However, in view of the two building blocks shown in Figure 3.4.1 and the summing integrator in Figure 3.4.2, any A-matrix in companion matrix form can be synthesized immediately, without going through the detailed synthesis procedure. This is illustrated in the following example.

<u>Example 3.5.1</u> Consider the A-matrix in the following equation:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ b_1 & b_2 & -b_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (3.5.1)$$

where b_1, b_2, and b_3 are all positive constants.

Writing three state equations from Equation (3.5.1) gives:

$$\frac{d}{dt}\, x_1 = x_2 \qquad (3.5.2a)$$

$$\frac{d}{dt}\, x_2 = x_3 \qquad (3.5.2b)$$

$$\frac{d}{dt}\, x_3 = b_1 x_1 + b_2 x_2 - b_3 x_3. \qquad (3.5.2c)$$

With the aid of the two building blocks in Figure 3.4.1 and the summing

integrator in Figure 3.4.2, the realization is obtained readily by

using Equation (3.5.2), as shown in Figure 3.5.1.
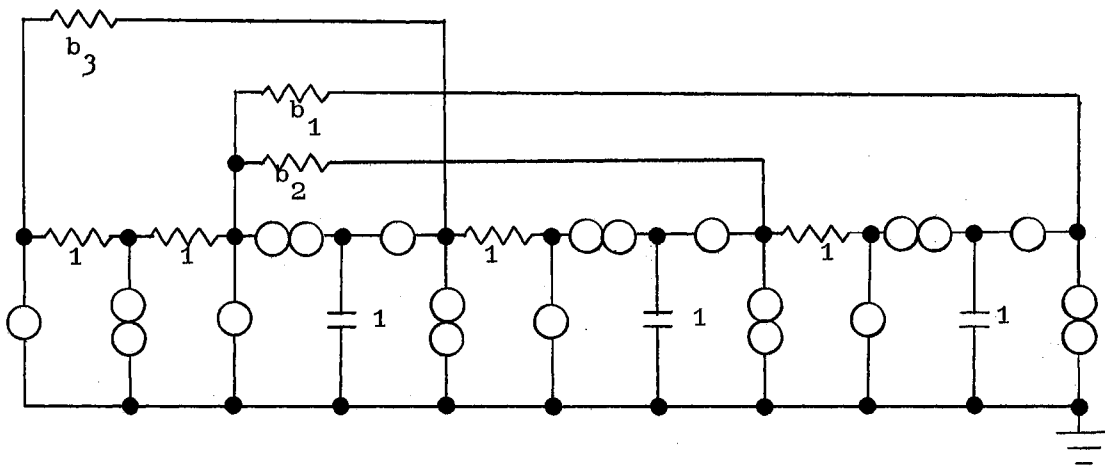
Figure 3.5.1. Example Synthesized Network

3.6 Change of Capacitor Element Values

In the realizations of Figure 3.3.4 and thereafter, each capacitor

has element value of unit magnitude. In this section, a method will be

described such that each capacitor can independently take on element

value different from unity if desired.

Consider the general network in Figure 3.3.4 for illustration.

The results will hold true for other networks discussed so far. The

method is to multiply both sides of Equation (3.3.1b) by a diagonal

matrix C_b = diag $(C_1, C_2, ..., C_n)$ consisting of capacitance values to

be determined.

Thus,

$$C_b \frac{d}{dt} x = C_b A_c x \qquad (3.6.1a)$$

or

$$
\begin{bmatrix}
C_1 & & & & \\
& C_2 & & & \\
& & \cdot & & \\
& & & \cdot & \\
& & & C_{n-1} & \\
& & & & C_n
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix}
x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
0 & C_1 & 0 & \cdots & 0 & 0 \\
0 & 0 & C_2 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & \cdots & 0 & C_{n-1} \\
-C_n a_1 & -C_n a_2 & -C_n a_3 & \cdots & -C_n a_{n-1} & -C_n a_n
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n
\end{bmatrix}
$$

$$(3.6.1b)$$

The coefficient matrix on the right-hand side of Equation (3.6.1) is now

taken to be the new A-matrix. Realizing this new A-matrix using the

technique developed in the preceding section yields the network shown in

Figure 3.6.1. The A-matrix of this network, of course, is the same as

that of the network in Figure 3.3.4, i.e., A_c. Each of the capacitances

C_1, C_2, ..., and C_n can take independently on any suitable value, and

some of the conductance values are accordingly changed. Note that if a

summer (or an inverter) is used, each of the two corresponding conduc-

tance values still has unit magnitude as shown in Figure 3.6.1. This

can be seen if the detailed synthesis procedure described in Section 3.3

is worked out to obtain the realization.
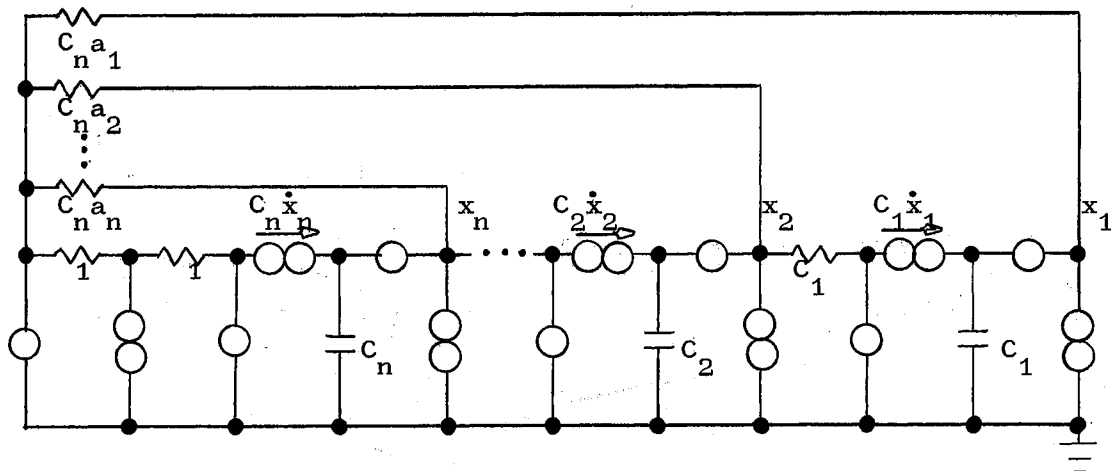
Figure 3.6.1. Synthesized Network Corresponding to
Figure 3.3.4

The above method can be considered to be the "current-scaling method" because it is the change of the current through each capacitor that makes every capacitor element value different from unity.

3.7 Synthesis of an Arbitrary A-Matrix

The above results can be extended to synthesize an arbitrary A-matrix. The two building blocks in Figure 3.4.1 can be combined to obtain a scheme for the realization of an arbitrary A-matrix. Figure 3.7.1 shows this scheme for drawing the circuit for a general entry a_{ij} in an A-matrix. The entry a_{ij} relates a state variable x_j to the derivative of other state variable x_i, $\dot{x}_i$. The pattern is repeated for each entry of the A-matrix.
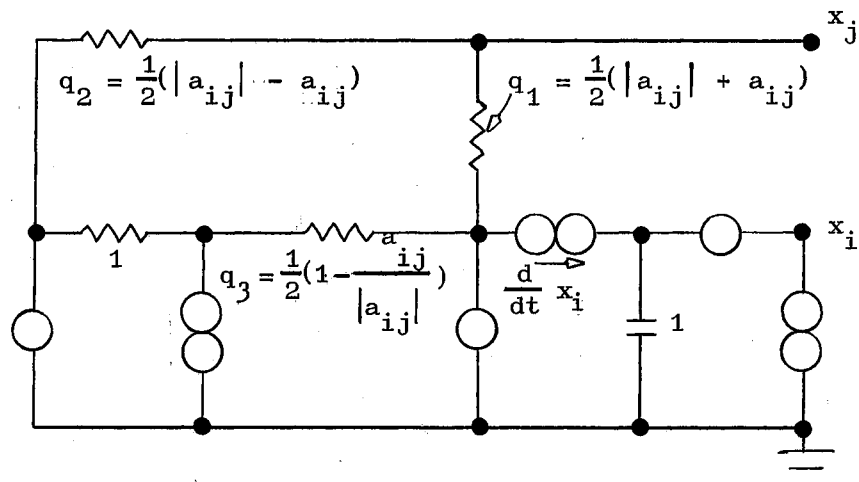
Figure 3.7.1. Scheme for Synthesizing Network

It can be seen that the scheme in Figure 3.7.1 consists of the two building blocks in Figure 3.4.1. However, since q_1, q_2, and q_3 represent conductances, when a_{ij} is negative, the branch of q_1 is open-circuited; whereas, when a_{ij} is positive, the branches of both q_2 and q_3 are open-circuited, thus causing the summer (or inverter) to be disconnected; and when a_{ij} is zero, the branches of q_1, q_2, and q_3 are all open-circuited.

The following example illustrates the synthesis of an arbitrary A-matrix using the scheme in Figure 3.7.1.

Example 3.7.1 Consider the source-free state equations:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 5 & 0 & 2 \\ -2 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (3.7.1)$$

where the A-matrix is as identified. It is desired to realize this

A-matrix.

By using the scheme in Figure 3.7.1 for every entry in this A-matrix, the realized network is shown in Figure 3.7.2.
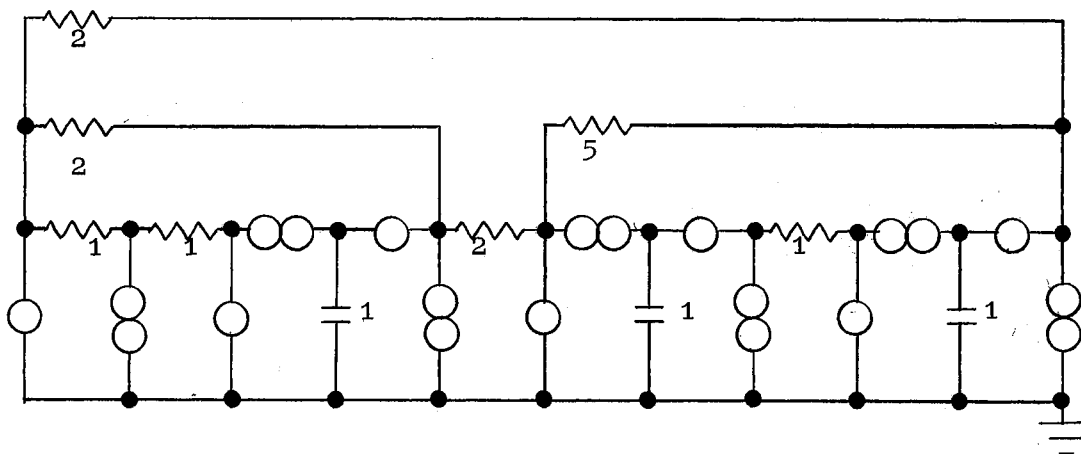
Figure 3.7.2.   Example Synthesized Network

In the scheme for synthesizing an arbitrary A-matrix in Figure 3.7.1, the element value of the capacitor is unity, and, therefore, in the realization, each capacitor has unit magnitude. If in the realization of an A-matrix, each of the capacitors with element value different from unity is desired, the following scheme in Figure 3.7.3 can be used to achieve this, where x_i and x_j have the same meaning as before. For example, the A-matrix in Example 3.7.1 can be synthesized by using this scheme. This is shown in Figure 3.7.4. The capacitances C_1, C_2, and C_3 can assume any suitable values.
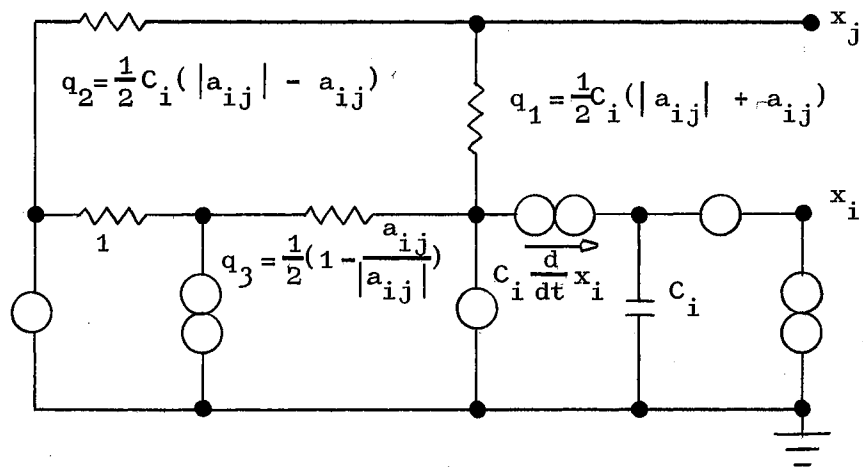
Figure 3.7.3. Scheme for Synthesizing Network
Corresponding to Figure 3.7.1
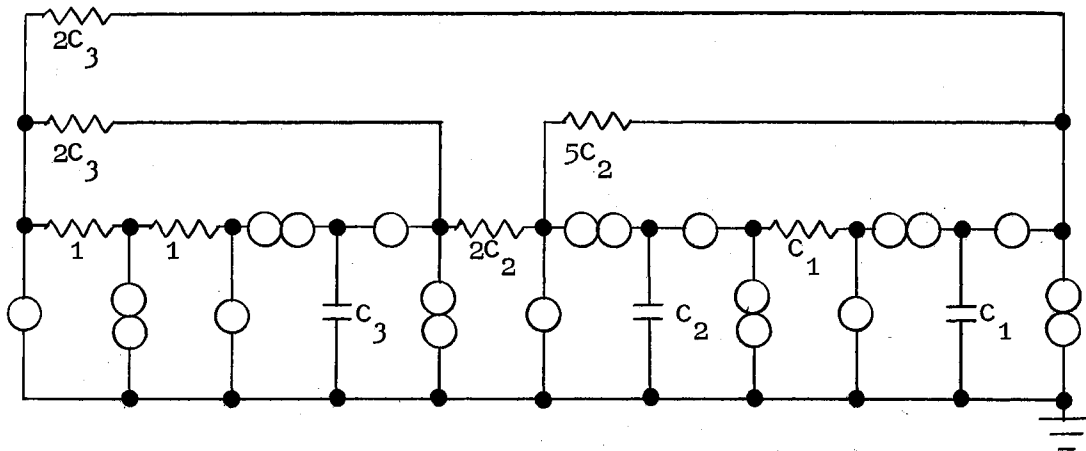
Figure 3.7.4. Example Synthesized Network
Corresponding to Figure 3.7.2

Perhaps it is worthwhile to make a comment about classical A-matrix

realizations using operational amplifiers. The realization in terms of

operational amplifiers first, and then replacing the operational

amplifiers by nullator-norator equivalents (28) results in a network composed of resistors, capacitors, and nullatator-norator pairs. In some cases, the operational-amplifier approach may require fewer nullator-norator pairs. For example, when a_{ij} is negative, a single pair is sufficient to realize it. However, from the operational amplifier equivalent circuit using a nullator-norator pair, it is known that the common vertex of the pair always corresponds to a ground potential. The nullator-norator synthesis used here does not have this restriction in general.

3.8 Synthesis of a Tridiagonal A-Matrix

Tridiagonal matrices are related to ladder networks (25) (39). A special case of interest is the tridiagonal A-matrix

$$A_T = \begin{bmatrix} -a_0 & a_1 & & & & & \\ -a_1 & 0 & a_2 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -a_{n-2} & 0 & a_{n-1} \\ & & & & & -a_{n-1} & -a_n \end{bmatrix} \qquad (3.8.1)$$

where all the a_i's are positive constants. It has been found that such a matrix is useful in the synthesis of a transfer function of a ladder network (20). This matrix can be derived from a Hurwitz polynomial by using the Navot's method (31) and some transformation (19).

The tridiagonal A-matrix A_T in Equation (3.8.1) can be realized by

using the method discussed in the preceding section. However, since the tridiagonal A-matrix has a special form, a simpler realization can be achieved with a network topology different from that previously obtained. This realization uses fewer number of elements compared to that by the method for an arbitrary A-matrix.

The procedure is similar to that developed in Section 3.3. However, it is not necessary to augment the tridiagonal A-matrix such that all its entries are positive because of its special form. The synthesis is obtained by first decomposing the tridiagonal A-matrix into a matrix triple product; then comparing with the A-matrix in symbolic form derived in Equation (3.2.2) to obtain the network topology and its element values.

The synthesis procedure is described below. It will be discussed, respectively, when the order n of A_T is even and odd.

(I) n even

The first step is to decompose A_T into a matrix triple product. One possible method of decomposition is the same as that given in Section 3.3. The decomposition is given by:

$$A_T = [E_{11} \ E_{33} \ E_{55} \ \cdots \ E_{(n-3)(n-3)} \ E_{(n-1)(n-1)} \mid E_{22} \ E_{44} \ E_{66} \ \cdots$$

$$E_{(n-2)(n-2)} \ E_{nn}] \cdot \text{diag} \ (a_0, \ a_1, \ a_2, \ \ldots, \ a_{n-1}, \ a_1, \ a_2, \ \ldots, \ a_n)$$

$$\cdot [F_{12} \ F_{24} \ F_{46} \ \cdots \ F_{(n-4)(n-2)} \ F_{(n-2)n} \mid F_{13} \ F_{35} \ F_{57} \cdots$$

$$F_{(n-3)(n-1)} \ F_{(n-1)n}]^T, \ n \neq 2 \qquad\qquad (3.8.2)$$

where E_{ii} represents an n X 2 submatrix with both of the two entries in the ith row as 1 and the remaining entries as zero; F_{ij}, $i \neq n-1$,

$j \neq n$, represents an $n \times 2$ submatrix with the entry in the ith row of the first column as -1 and the entry in the jth row of the second column as 1 and the remaining entries as zero, and $F_{(n-1)n}$ represents an $n \times 2$ submatrix with both nonzero entries in the $(n-1)^{th}$ row of the first column and in the nth row of the second column as -1. Note that F_{12} always exists. Note also that Equation (3.8.2) cannot be applied when $n = 2$ since for this case in the decomposition, there will be an F_{02} which is not defined. When $n = 2$, the decomposition is

$$A_T = [E_{11} | E_{22}] \cdot \text{diag} \ (a_0, \ a_1, \ a_1, \ a_2) \cdot [F_{12} | F_{(n-1)n}]^T. \quad (3.8.3)$$

The above decomposition in Equation (3.8.2) or (3.8.3) can be verified by multiplying out the three matrices in the right-hand side.

To illustrate the decomposition, consider a fourth-order A-matrix. It follows from Equation (3.8.2) that $A_T = [E_{11} \ E_{33} | E_{22} \ E_{44}] \cdot \text{diag} \ (a_0,$ $a_1, \ a_2, \ a_3, \ a_1, \ a_2, \ a_3, \ a_4) \cdot [F_{12} \ F_{24} | F_{13} \ F_{34}]^T$, or equivalently

$$\begin{bmatrix} -a_0 & a_1 & 0 & 0 \\ -a_1 & 0 & a_2 & 0 \\ 0 & -a_2 & 0 & a_3 \\ 0 & 0 & -a_3 & -a_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ & a_1 \\ & & a_2 \\ & & & a_3 \\ & & & & a_1 \\ & & & & & a_2 \\ & & & & & & a_3 \\ & & & & & & & a_4 \end{bmatrix}$$

$$\cdot \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}^T \cdot \quad (3.8.4)$$

Now, comparing the matrix A_T in Equation (3.8.2) with the A-matrix in symbolic form in Equation (3.2.2) with C_b again chosen to be a unit matrix yields

$$C_{22} = [E_{11} \ E_{33} \ E_{55} \ \cdots \ E_{(n-3)(n-3)} \ E_{(n-1)(n-1)} \mid E_{22} \ E_{44} \ E_{66} \ \cdots$$

$$E_{(n-2)(n-2)} \ E_{nn}] \tag{3.8.5}$$

$$G_c = \text{diag} \ (a_0, \ a_1, \ a_2, \ \ldots, \ a_{n-1}, \ a_1, \ a_2, \ \ldots, \ a_n) \tag{3.8.6}$$

and

$$C_{23} \ C_{33}^{-1} \ C_{32} - C_{22} = [F_{12} \ F_{24} \ F_{46} \ \cdots \ F_{(n-4)(n-2)} \ F_{(n-2)n}$$

$$\mid F_{13} \ F_{35} \ F_{57} \ \cdots \ F_{(n-3)(n-1)} \ F_{(n-1)n}].$$

$$\tag{3.8.7}$$

From Equations (3.8.5) and (3.8.7) with C_{33} set to be a unit matrix for the same reason as before, the following equation is obtained:

$$C_{23} \ C_{32} = [(E_{11} + F_{12}) \ (E_{33} + F_{24}) \ (E_{55} + F_{46}) \ \cdots \ (E_{(n-3)(n-3)}$$

$$+ F_{(n-4)(n-2)}) \ (E_{(n-1)(n-1)} + F_{(n-2)n})$$

$$\mid (E_{22} + F_{13}) \ (E_{44} + F_{35}) \ (E_{66} + F_{57}) \ \cdots$$

$$(E_{(n-2)(n-2)} + F_{(n-3)(n-1)}) \ (E_{nn} + F_{(n-1)n})]$$

$$= [H_1 \ H_2 \ H_3 \ \cdots \ H_{(n-2)/2} \ H_{n/2} \mid H_{(n+2)/2} \ H_{(n+4)/2} \ H_{(n+6)/2}$$

$$\cdots \ H_{n-1} \ H_n] \tag{3.8.8}$$

where $H_1 = E_{11} + F_{12}$, $H_2 = E_{33} + F_{24}$, $\ldots$, and $H_n = E_{nn} + F_{(n-1)n}$.

Equation (3.8.8) is then decomposed as

$$C_{23}C_{32} = [(H_1H_2 \cdots H_{n/2}) \quad (H_{(n+2)/2}H_{(n+4)/2} \cdots H_n)]$$

$$= [(H_1H_2 \cdots H_{n/2}) \quad U] \begin{bmatrix} U & O \\ O & (H_{(n+2)/2}H_{(n+4)/2} \cdots H_n) \end{bmatrix}.$$

$$(3.8.9)$$

Therefore, a choice for C_{23} and C_{32} is

$$C_{23} = [(H_1 \ H_2 \cdots H_{n/2}) \quad U] \tag{3.8.10}$$

$$C_{32} = \begin{bmatrix} U & O \\ O & (H_{(n+2)/2}H_{(n+4)/2} \cdots H_n) \end{bmatrix} \tag{3.8.11}$$

It should be pointed out that the first column of H_1, or of C_{23} in Equation (3.8.10) is a zero column and will be deleted. After deleting this column, the matrix H_1 will be denoted by H_1', and C_{23} by C_{23}'. The corresponding first row of C_{32} in Equation (3.8.11) also will be deleted. After deleting this row, the matrix C_{32} will be denoted by C_{32}' and is

$$C_{32}' = \begin{bmatrix} U' & O' \\ O & (H_{(n+2)/2} \ H_{(n+4)/2} \cdots H_n) \end{bmatrix}$$

where U' and O' are the corresponding unit matrix and zero matrix, U and O, in Equation (3.8.11) with the first row deleted.

The fundamental cut-set matrix can now be obtained as

$$C_T = \begin{bmatrix} U & 0 & C_{22} & C'_{23} \\ 0 & U & C'_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} U & 0 & 0 & E_{11}E_{33}\cdots E_{(n-1)(n-1)}E_{22}E_{44}\cdots E_{nn} & H'_1H_2\cdots H_{n/2} & U \\ 0 & U & 0 & U' & O' & U & O \\ 0 & 0 & U & O & H_{(n+2)/2}H_{(n+4)/2}\cdots H_n & O & U \end{bmatrix}$$

$$(3.8.12)$$

Again, it can be shown that the above fundamental cut-set matrix is always realizable by simply constructing the graph. The resulting graph and the synthesized network are shown in Figures 3.8.1 and 3.8.2, respectively, where the element values of the conductors are obtained from G_c in Equation (3.8.6) and all the capacitances are of unit magnitude because the matrix C_b is assumed to be a unit matrix. Dashed lines are indicated in Figure 3.8.2 to enclose the fundamental blocks L_1, L_2, ..., L_{n-1}. Among these blocks, there are only two basic types. For example, L_1 and L_2 form these two basic types. The block L_i is of the same form as L_1 if i is odd and of the same form as L_2 if i is even.
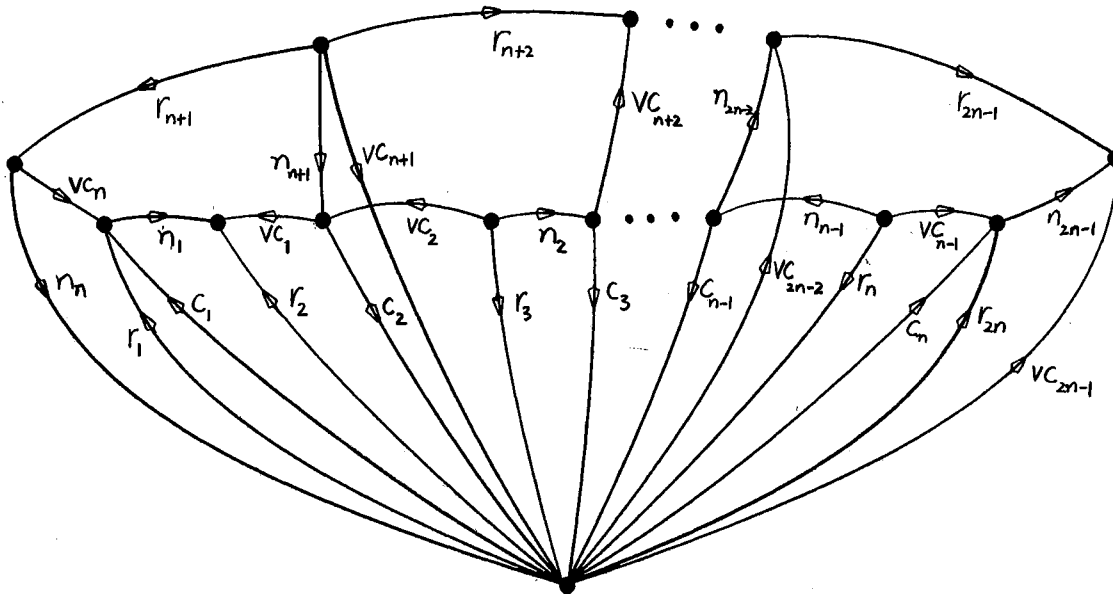
Figure 3.8.1. Network Graph With n Even

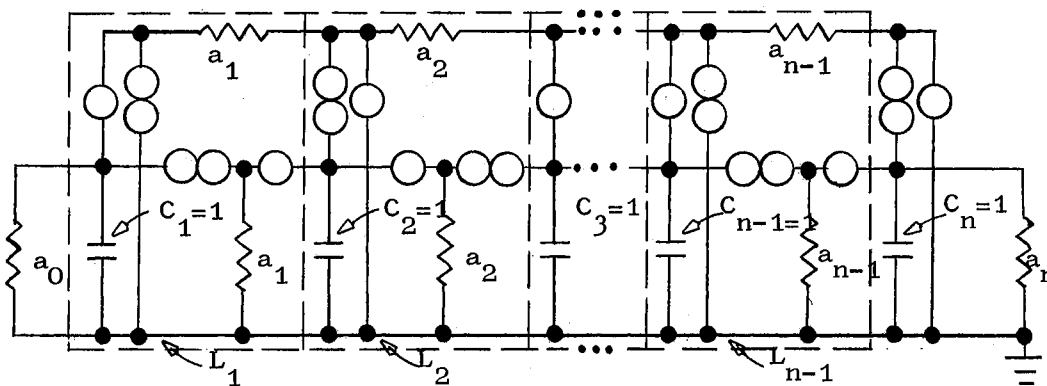
Figure 3.8.2. Synthesized Network

When $n = 2$, it proceeds in the same way as above, and the realization will be as shown in Figure 3.8.2 with $n = 2$.

(II) n odd

A decomposition of A_T with n odd in Equation (3.8.1) into a matrix

triple product is

$$A_T = [E_{11} \ E_{33} \ E_{55} \ \cdots \ E_{(n-2)(n-2)} \ E_{nn}$$

$$E_{22} \ E_{44} \ E_{66} \ \cdots \ E_{(n-3)(n-3)} \ E_{(n-1)(n-1)}]$$

$$\cdot \ \text{diag} \ (a_0, \ a_1, \ a_2, \ \ldots, \ a_n, \ a_1, \ a_2, \ \ldots, a_{n-1})$$

$$\cdot \ [F_{12} \ F_{24} \ F_{46} \ \cdots \ F_{(n-3)(n-1)} \ F_{(n-1)n}$$

$$F_{13} \ F_{35} \ F_{57} \ \cdots \ F_{(n-4)(n-2)} \ F_{(n-2)n}]^T, \ n \neq 1 \qquad (3.8.13)$$

where E_{ii} and F_{ij} are as defined in Case (I) above. The case for $n = 1$

is trivial, and is not considered here. Note that F_{12} always exists.

For illustration, consider the case of the decomposition of A_T in

Equation (3.8.13) when $n = 5$. It follows that $A_T = [E_{11} \ E_{33} \ E_{55} | E_{22} \ E_{44}]$

$\cdot \ \text{diag} \ (a_0, \ a_1, \ a_2, \ a_3, \ a_4, \ a_5, \ a_1, \ a_2, \ a_3, \ a_4)[F_{12} \ F_{24} \ F_{45} | F_{13} \ F_{35}]^T$,

or equivalently

$$
\begin{bmatrix}
-a_0 & a_1 & & & \\
-a_1 & 0 & a_2 & & \\
 & -a_2 & 0 & a_3 & \\
 & & -a_3 & 0 & a_4 \\
 & & & -a_4 & -a_5
\end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\cdot \
\begin{bmatrix}
a_0 & & & & & & & & & \\
 & a_1 & & & & & & & & \\
 & & a_2 & & & & & & & \\
 & & & a_3 & & & & & & \\
 & & & & a_4 & & & & & \\
 & & & & & a_5 & & & & \\
 & & & & & & a_1 & & & \\
 & & & & & & & a_2 & & \\
 & & & & & & & & a_3 & \\
 & & & & & & & & & a_4
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1
\end{bmatrix}^T
$$

$$(3.8.14)$$

Now, comparing Equation (3.8.13) with Equation (3.2.2), setting C_b to be a unit matrix, results in

$$C_{22} = [E_{11}\, E_{33}\, E_{55} \cdots E_{(n-2)(n-2)}\, E_{nn} \mid E_{22}\, E_{44}\, E_{66} \cdots E_{(n-3)(n-3)}$$

$$E_{(n-1)(n-1)}] \tag{3.8.15}$$

$$G_c = \text{diag}\,(a_0,\, a_1,\, a_2,\, \ldots,\, a_n,\, a_1,\, a_2,\, \ldots,\, a_{n-1}) \tag{3.8.16}$$

and

$$C_{23}\,C_{33}^{-1}\,C_{32} - C_{22} = [F_{12}\, F_{24}\, F_{46} \cdots F_{(n-3)(n-1)}\, F_{(n-1)n}$$

$$\mid F_{13}\, F_{35}\, F_{57} \cdots F_{(n-4)(n-2)}\, F_{(n-2)n}] \tag{3.8.17}$$

Substituting C_{22} from Equation (3.8.15) into Equation (3.8.17) with C_{33} set as a unit matrix gives

$$C_{23}\,C_{32} = [(E_{11} + F_{12})\, (E_{33} + F_{24})\, (E_{55} + F_{46}) \cdots (E_{(n-2)(n-2)}$$

$$+ F_{(n-3)(n-1)})\, (E_{nn} + F_{(n-1)n}) \mid (E_{22} + F_{13})$$

$$(E_{44} + F_{35})\, (E_{66} + F_{57}) \cdots (E_{(n-3)(n-3)}$$

$$+ F_{(n-4)(n-2)})\, (E_{(n-1)(n-1)} + F_{(n-2)n})]$$

$$= [K_1\, K_2\, K_3 \cdots K_{(n-1)/2}\, K_{(n+1)/2} \mid K_{(n+3)/2}\, K_{(n+5)/2}$$

$$K_{(n+7)/2} \cdots K_{n-1}\, K_n] \tag{3.8.18}$$

where $K_1 = E_{11} + F_{12}$, $K_2 = E_{33} + F_{24}$, $\ldots$, and $K_n = E_{(n-1)(n-1)}$

$$+ F_{(n-2)n}.$$

Equation (3.8.18) is then decomposed as

$$C_{23} \, C_{32} = [(K_1 \; K_2 \; \cdots \; K_{(n+1)/2}) \; (K_{(n+3)/2} \; \cdots \; K_{n-1} K_n)]$$

$$= [(K_1 \; K_2 \; \cdots \; K_{(n+1)/2}) \quad U] \begin{bmatrix} U & 0 \\ 0 & (K_{(n+3)/2} \; \cdots \; K_{n-1} \; K_n) \end{bmatrix}.$$

$$(3.8.19)$$

Therefore, choose C_{23} and C_{32} as

$$C_{23} = [K_1 \; K_2 \; \cdots \; K_{(n+1)/2}) \quad U] \qquad (3.8.20)$$

$$C_{32} = \begin{bmatrix} U & 0 \\ 0 & (K_{(n+3)/2} \; \cdots \; K_{n-1} \; K_n) \end{bmatrix}. \qquad (3.8.21)$$

Note that in Equation (3.8.20) the first column of K_1, or of C_{23}, and the second column of $K_{(n+1)/2}$ or the $(n+1)^{th}$ column of C_{23} are both zero columns and, therefore, these two columns can be deleted without losing any information. The resulting K_1 and $K_{(n+1)/2}$ will be identified by K_1' and $K_{(n+1)/2}'$, respectively, and the resulting C_{23} will be identified by C_{23}'. Furthermore, the corresponding first and $(n+1)^{th}$ rows of C_{32} in Equation (3.8.21) will also be deleted. After these two rows are deleted, C_{32} will be identified by C_{32}' and is

$$C_{32}' = \begin{bmatrix} U' & 0' \\ 0 & K_{(n+3)/2} \; K_{(n+5)/2} \; \cdots \; K_n \end{bmatrix}.$$

Now, combine C_{22}, C_{23}', C_{32}' obtained above and C_{33} which is assumed to be a unit matrix to form the fundamental cut-set matrix

$$
C_T = \begin{bmatrix} U & 0 & C_{22} & C'_{23} \\ \hline 0 & U & C'_{32} & C_{33} \end{bmatrix}
$$

$$
= \begin{bmatrix} U & 0 & 0 & E_{11}\,E_{33}\cdots E_{nn} & E_{22}\,E_{44}\cdots E_{(n-1)(n-1)} & K'_1\,K'_2\cdots K'_{(n+1)/2} & U \\ \hline 0 & U & 0 & U' & 0' & U & 0 \\ 0 & 0 & U & 0 & K_{(n+3)/2}\,K_{(n+5)/2}\cdots K_n & 0 & U \end{bmatrix}
$$

$$(3.8.22)$$

The above fundamental cut-set matrix can be shown to be realizable by constructing the graph. The resulting graph and the synthesized network are shown in Figures 3.8.3 and 3.8.4, respectively. Again, dashed lines are used to enclose the fundamental blocks L_1, L_2, ..., L_{n-1}. The statements about the fundamental blocks in Case (I) hold true for Figure 3.8.4. That is, there are two basic types of the fundamental blocks, such as L_1 and L_2; the block L_i is of the same form as L_1 if i is odd and of the same form as L_2 if i is even. The difference between the two synthesized networks of Figures 3.8.4 and 3.8.2 lies in the right-most pair of nullator-norator across the capacitor C_n, where one is switched around to get the other.

It may be observed that since the realization in Figure 3.8.2 or 3.8.4 has fundamental building blocks, it can be obtained just by inspection of the given tridiagonal A-matrix. Nevertheless, for illustration of the synthesis procedure, an example is given below.

Example 3.8.1 Consider the tridiagonal A-matrix

$$
A_P = \begin{bmatrix} -0.08 & 0.709 & 0 \\ -0.709 & 0 & 1.53 \\ 0 & -1.53 & -1.92 \end{bmatrix}
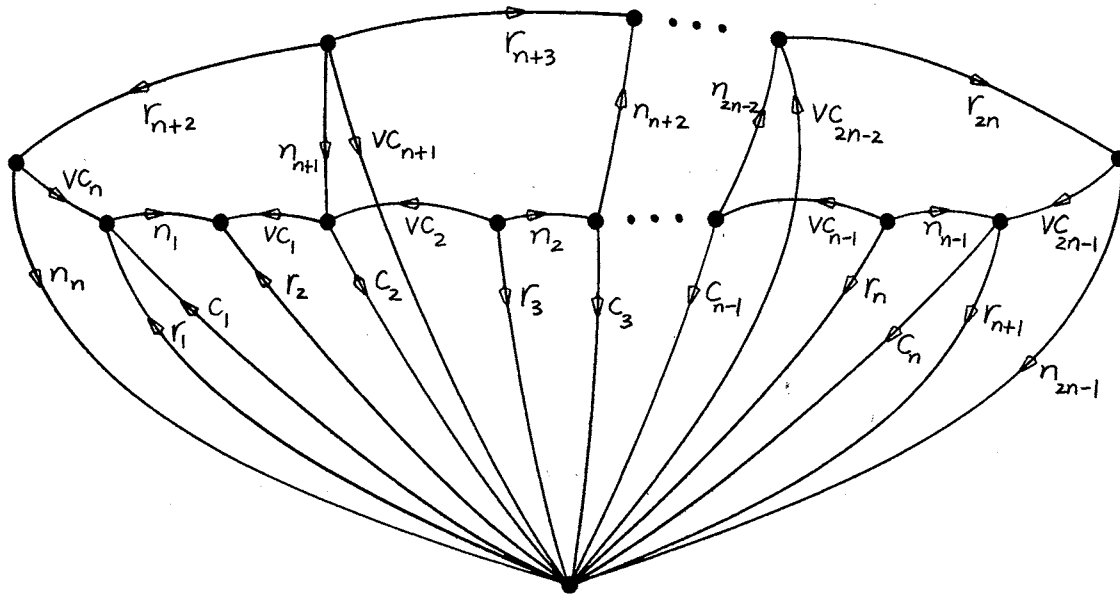$$

$$(3.8.23)$$

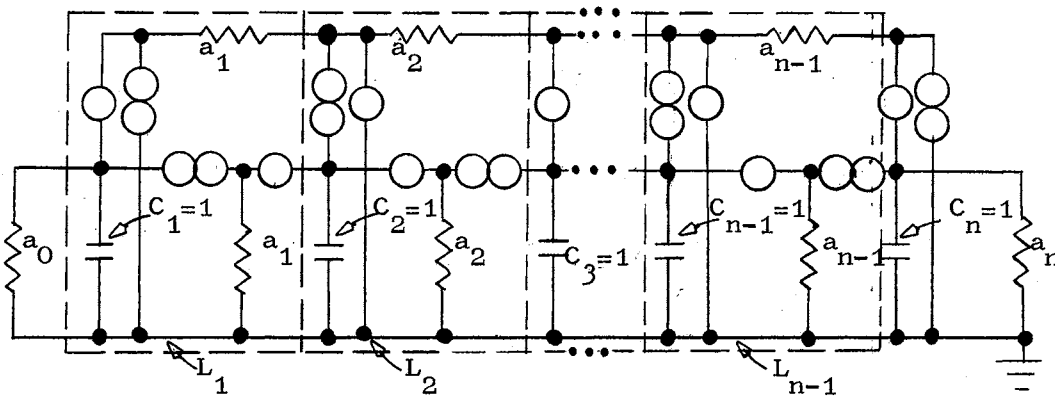Figure 3.8.3. Network Graph With n Odd

Figure 3.8.4. Synthesized Network

where the corresponding characteristic polynomial is $P(s) = |sU - A_p|$

$= s^3 + 2s^2 + 3s + 1$, which is strictly Hurwitz. First, decompose A_p into a matrix triple product, using Equation (3.8.13) and is

$$A_p = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.08 & & & & & \\ & 0.709 & & & & \\ & & 1.53 & & & \\ & & & 1.92 & & \\ & & & & 0.709 & \\ & & & & & 1.53 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}^T$$

$$(3.8.24)$$

By comparing Equation (3.8.24) with Equation (3.2.2), setting C_b as a unit matrix, the following equations are obtained:

$$C_{22} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad (3.8.25)$$

$$G_c = \begin{bmatrix} 0.08 & & & & & \\ & 0.79 & & & & \\ & & 1.53 & & & \\ & & & 1.92 & & \\ & & & & 0.709 & \\ & & & & & 1.53 \end{bmatrix} \qquad (3.8.26)$$

$$C_{23}C_{33}^{-1}C_{32} - C_{22} = \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}. \qquad (3.8.27)$$

Combining Equations (3.8.25) and (3.8.27) with C_{33} chosen to be a unit matrix results in

$$C_{23}C_{32} = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

deleted (column, rows)

$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3.8.28)$$

Hence, choose C_{23} and C_{32} as

$$C_{23} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad (3.8.29)$$

$$C_{32} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3.8.30)$$

Now, form the fundamental cut-set matrix:

$$
\begin{bmatrix}
\bar{U} & | & 0 & | & C_{22} & | & C_{23} \\
\hline
0 & | & U & | & C_{32} & | & C_{33}
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & & & & & & & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & & \bigcirc & & & & & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & & & & & & & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
& & & 1 & 0 & 0 & 0 & 0 & & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
& & & 0 & 1 & 0 & 0 & 0 & & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
& \bigcirc & & 0 & 0 & 1 & 0 & 0 & & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
& & & 0 & 0 & 0 & 1 & 0 & & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
& & & 0 & 0 & 0 & 0 & 1 & & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$(3.8.31)$$

Figures 3.8.5 and 3.8.6, respectively, show the graph and the synthesized network corresponding to this fundamental cut-set matrix.

Figure 3.8.5. Example Network Graph
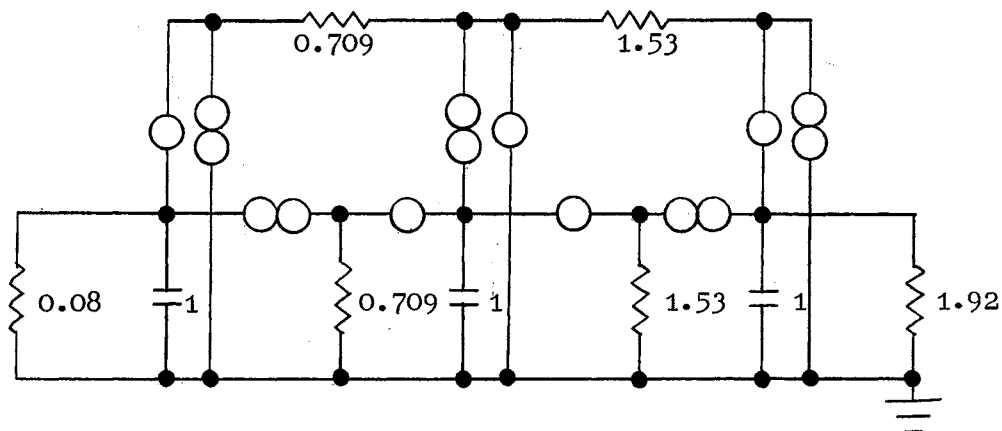
Figure 3.8.6.   Example Synthesized Network

It is noted that in the tridiagonal A-matrix A_T in Equation

(3.8.1), n even or odd, the entries a_{ij} and a_{ji}, $j = i + 1$, $i = 1$, 2, ...,

or (n-1), have the same value, but a_{ij} is positive and a_{ji} is negative.

As a matter of fact, for these two entries, a_{ij} and a_{ji}, as long as one

is positive and the other is negative, the resulting tridiagonal

A-matrix will possess the same characteristic polynomial and will have

the same realization as the tridiagonal A-matrix in Equation (3.8.1).

For example, each of the following two matrics has the same characteris-

tic polynomial as the matrix A_p in Equation (3.8.23), and their realiza-

tions, respectively, are the same as that of A_p, which is shown in

Figure 3.8.6.   This can be proved by carrying out the detailed synthesis

procedure.

$$A_{p_1} = \begin{bmatrix} -0.08 & -0.709 & 0 \\ 0.709 & 0 & 1.53 \\ 0 & -1.53 & -1.92 \end{bmatrix}$$

$$A_{p_2} = \begin{bmatrix} -0.08 & -0.709 & 0 \\ 0.709 & 0 & -1.53 \\ 0 & 1.53 & -1.92 \end{bmatrix}$$

Note further that corresponding to the two entries a_{ij} and a_{ji} mentioned in the preceding paragraph, there are two conductances a_i which are not distinguished in the fundamental block L_i of the general synthesized network shown in Figure 3.8.2 or 3.8.4. In fact, the a_{ij}'s and the a_{ji}'s may not be equal, but if a tridiagonal matrix is derived from a characteristic polynomial which is strictly Hurwitz, the relationship $a_{ij}a_{ji} = f_i$, $j = i + 1$, $i = 1$, 2, ..., n-1, should be held. Note that f_i is obtained from the Navot's method (31) of deriving a tridiagonal matrix. This will be discussed later in Subsection 5.4.1. From the network graph shown in Figure 3.8.1 or 3.8.3, it can be seen that the fundamental block L_i has the configuration shown in Figure 3.8.7, where a_{ij} may or may not be equal to a_{ji}.

i - odd i - even
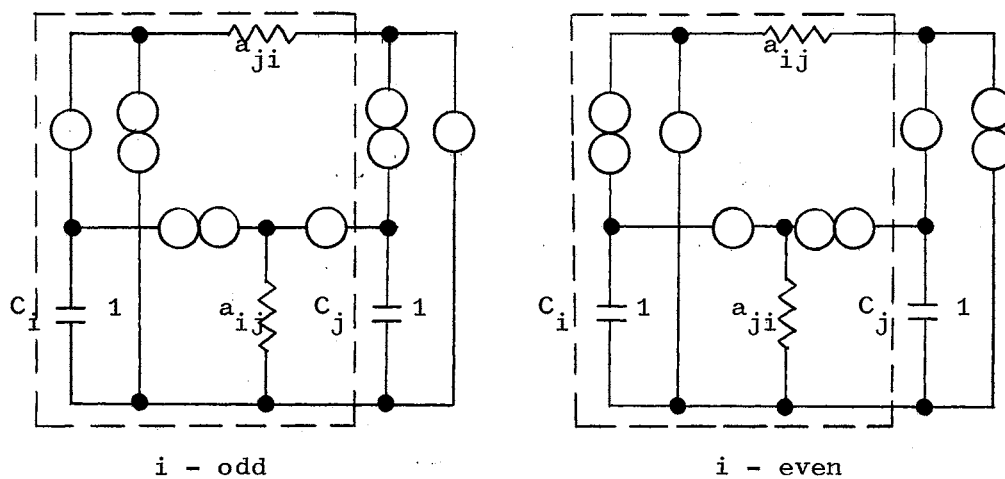
Figure 3.8.7. Fundamental Blocks

3.9 Comparison of the Synthesis Procedures

If the tridiagonal A-matrix A_T of Equation (3.8.1) is synthesized

using the procedure developed in Section 3.7, the number of nullator-

norator pairs used is greater than that using the procedure in Section

3.8. In general, for a tridiagonal A-matrix of order n as in Equation

(3.8.1), n even or odd, the former procedure needs 3n pairs of nullator-

norator; whereas, the latter procedure needs (2n-1) pairs of nullator-

norator; that is, the former procedure needs n+1 more pairs of

nullator-norator than the latter procedure. In view of the equivalent

ideal transistor realization of a nullator-norator pair, the use of the

procedure in Section 3.8 to synthesize a given tridiagonal A-matrix is

desirable.

For the purpose of comparison, the realization of A_p in Equation

(3.8.23) by the procedure of Section 3.7 is shown in Figure 3.9.1 where

9 pairs of nullator-norator are used. However, in the network of

Figure 3.8.6 obtained by using the procedure of Section 3.8 only 5 pairs

of nullator-norator are used.
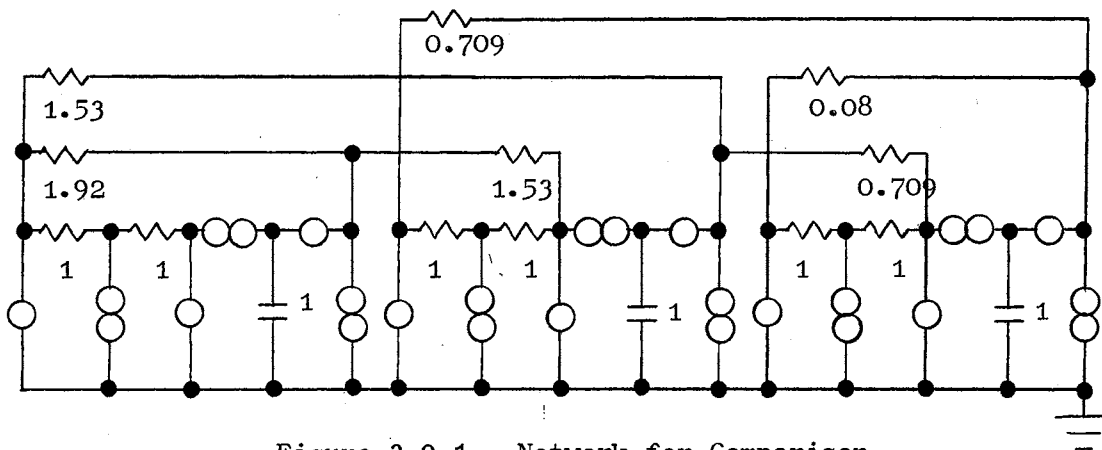
Figure 3.9.1. Network for Comparison

It is noted that the specification may be given in terms of a characteristic polynomial instead of an A-matrix. From this characteristic polynomial, the A-matrix in companion matrix form can be obtained easily by inspection; whereas, the derivation of the tridiagonal A-matrix is more involved. However, the realization of the latter uses fewer number of nullator-norator pairs than that of the former. Generally, for a characteristic polynomial of degree n, it needs 2n+1 pairs of nullator-norator to realize the corresponding A-matrix in companion matrix form, and 2n-1 pairs to realize the corresponding tridiagonal A-matrix. Furthermore, the tridiagonal A-matrix is considered to be the best as far as the sensitivity of the elements in the realized network is concerned (38).

The procedures discussed in the preceding sections are for source-free networks. They will be extended to include sources. These are the cases of realization of transfer functions, immittance matrices, and gyrators and are considered in the next chapter.

CHAPTER IV

SYNTHESIS OF TRANSFER FUNCTIONS, IMMITTANCE

MATRICES, AND GYRATORS

4.1 Introduction

The realization of an A-matrix has been considered in the preceding

chapter. It will be used in the synthesis of transfer functions,

immittance matrices, and gyrators by means of state models in this

chapter. The transfer functions to be synthesized are voltage-ratio

transfer functions, current-ratio transfer functions, transfer admit-

tance functions, and transfer impedance functions. The immittance

matrices to be synthesized are short-circuit admittance matrices, open-

circuit impedance matrices, hybrid matrices, voltage transfer-function

matrices and current transfer-function matrices. The synthesis of

transfer functions is considered first and is given in the next section.

4.2 Synthesis of Transfer Functions

Efforts are directed to synthesize voltage transfer functions. The

results will be extended to other types of transfer functions.

Consider the voltage-ratio transfer function

$$T(s) = \frac{V_o(s)}{V_{in}(s)} = d + \frac{b_n s^{n-1} + b_{n-1} s^{n-2} + \cdots + b_2 s + b_1}{s^n + a_n s^{n-1} + \cdots + a_2 s + a_1} \qquad (4.2.1)$$

where $V_o(s)$ and $V_{in}(s)$ are the Laplace-transformed load and source

voltage, respectively; a_1, a_2, ..., b_1, b_2, ..., b_n, and d are real

constants. Corresponding to Equation (4.2.1), many forms of the state

model exist. Perhaps the simplest one is the following (22) with the

A-matrix in compansion matrix form:

$$
\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & 0 & 1 \\
-a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}
+
\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u
\qquad (4.2.2a)
$$

$$
y = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}
+ du
\qquad (4.2.2b)
$$

where $u = v_{in}$ and $y = v_o$, the input function and the output function in

the time domain.

As mentioned previously, to realize the transfer function in

Equation (4.2.1), one needs to realize the corresponding state model

such as in Equation (4.2.2); that is, to realize the set of matrices

{A,B,C,D} in Equation (4.2.2). The realization of an A-matrix has been achieved and the realization of a state model can be obtained with the use of the two building blocks shown in Figure 3.4.1 and the summing integrator in Figure 3.4.2, in addition to the A-matrix realization.

The synthesis procedure is illustrated in the following example.

Example 4.2.1 Consider the voltage-ratio transfer function

$$\frac{V_2(s)}{V_1(s)} = \frac{s^2 + 2}{s^2 + 0.1s + 1.2} \tag{4.2.3a}$$

which can be rewritten as

$$\frac{V_2(s)}{V_1(s)} = 1 + \frac{-0.1s + 0.8}{s^2 + 0.1s + 1.2} \tag{4.2.3b}$$

The state model corresponding to Equation (4.2.3) is, from Equation (4.2.2),

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1.2 & -0.1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{4.2.4a}$$

$$y = \begin{bmatrix} 0.8 & -0.1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u \tag{4.2.4b}$$

where $u = v_1$ and $y = v_2$.

The realization of the state model in Equation (4.2.4) can be obtained by first realizing the A-matrix. This can be done by inspection from Figure 3.3.4 and the realization is shown in Figure 4.2.1.
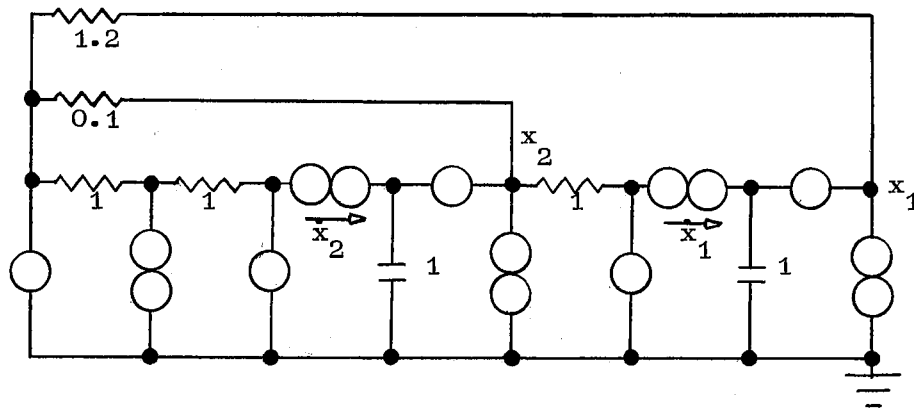
Figure 4.2.1. Example A-Matrix Realization

Now, the source u must be incorporated into the realization of the A-matrix in Figure 4.2.1, and the algebraic set of the state model, i.e., the output equation, in Equation (4.2.4b) must be realized.

It is seen from Equation (4.2.4a) that the source u is related to $\frac{d}{dt} x_2$ which represents the current through the capacitor C_2. This will be implemented by using a summing integrator shown in Figure 3.4.2 in the same way as a voltage source is applied to the operational amplifier of an integrator in an analog-computer simulation diagram. To realize the output equation in Equation (4.2.4b), two summers shown in Figure 3.4.1(b) will be used, the first being served as an inverter. The realization of the transfer function in Equation (4.2.3) is shown in Figure 4.2.2.

Earlier it was pointed out that there exist several realizations for a given A-matrix and, therefore, it follows that there are several realizations corresponding to a given transfer function.

If the transfer functions to be synthesized are of types other than the voltage-ratio transfer functions, then various kinds of controlled

sources (transducers) may be implemented at the source end and/or the

output end when incorporating the source to and obtaining the output

from the realization of the A-matrix. The representation of various

controlled sources using nullator-norator circuits can be found in a

book written by Mitra (28).
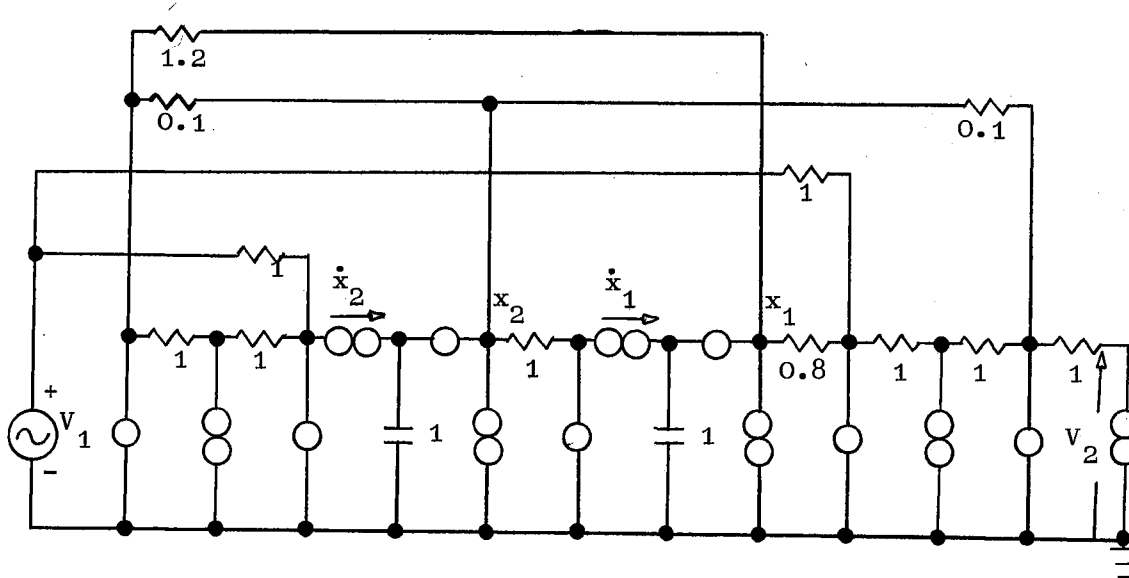
Figure 4.2.2. Example Synthesized Network

In the case of a transfer admittance function, a current-to-current

transducer (CCT) which is an ideal current-controlled current source is

needed at the output end. For a transfer impedance function, a current-

to-voltage transducer (CVT) which is an ideal current-controlled voltage

source is required at the source end. For a current-ratio transfer

function, the CVT and CCT are needed at the source end and the output

end, respectively.

The reasons for the possible use of these controlled sources are given as follows. At the source end, a summing integrator shown in Figure 3.4.2 will be used, and the inputs to this summing integrator always are voltages. Thus, if the source is a voltage source, no controlled sources are required; but if the source is a current source, a CVT is needed at the source end. At the output end, a summer shown in Figure 3.4.1(b) will be used, and the output of this summer is a voltage. Hence, if the output is a voltage, no controlled sources are required, but if the output is a current, a CCT is needed at the output end (another summer served as an inverter will be used, if necessary).

The following example illustrates the synthesis of a current-ratio transfer function.

Example 4.2.2   Consider the current-ratio transfer function

$$\frac{I_2(s)}{I_1(s)} = \frac{4(s^2 + 2)}{s^2 + s + 1} \qquad (4.2.5a)$$

which can be rewritten as

$$\frac{I_2(s)}{I_1(s)} = 4 + \frac{-4s + 4}{s^2 + s + 1} . \qquad (4.2.5b)$$

A state model corresponding to Equation (4.2.5) can be derived as

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad (4.2.6a)$$

$$y = \begin{bmatrix} 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 4u \qquad (4.2.6b)$$

where $u = i_1$ and $y = i_2$. The realizations of the A-matrix and of the state model in Equation (4.2.6) are shown in Figures 4.2.3 and 4.2.4, respectively.
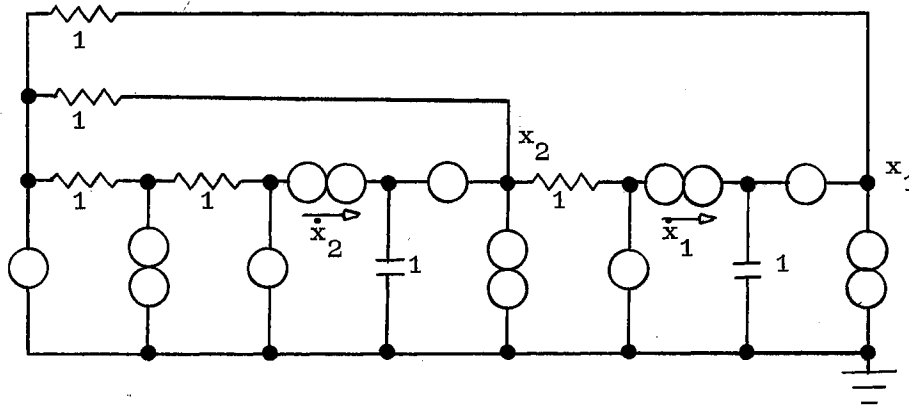
Figure 4.2.3. Example A-Matrix Realization
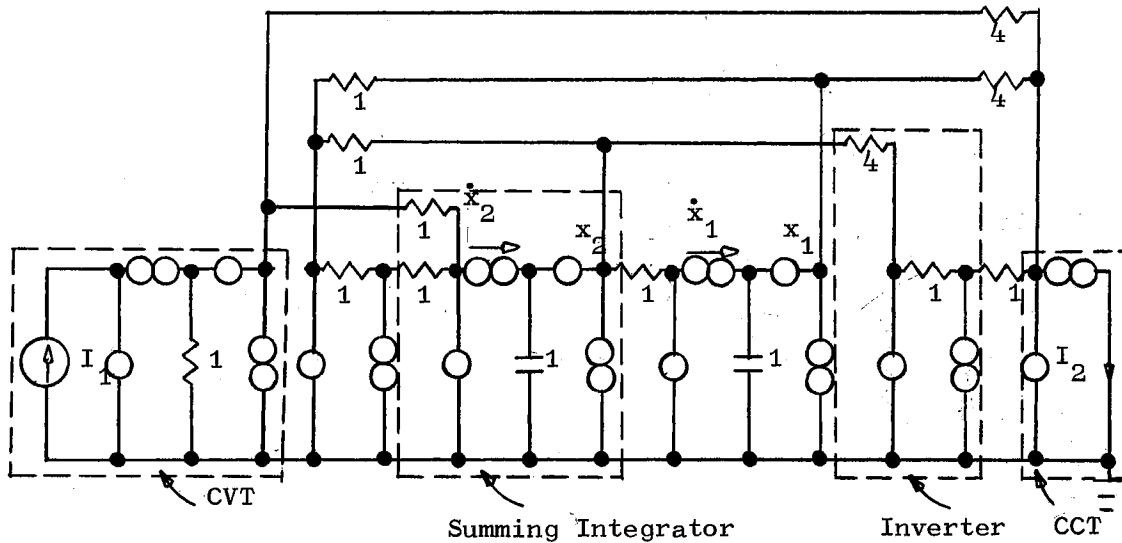
Figure 4.2.4. Example Synthesized Network

Note that in Figure 4.2.4 a CVT is used on the source end and a CCT is used on the output end. The CVT is applied to the summing integrator through a conductor of unit element value and the CCT just acts as a current-collecting device.

The synthesis of short-circuit admittance matrices using resistors, capacitors, and nullator-norator pairs will be considered in the next section.

4.3 Synthesis of Short-Circuit
Admittance Matrices

A short-circuit admittance matrix $Y(s) = [y_{ij}(s)]$ is defined by

$$
\begin{bmatrix} I_1(s) \\ I_2(s) \\ \cdot \\ \cdot \\ \cdot \\ I_n(s) \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ \cdot \\ \cdot \\ \cdot \\ V_n(s) \end{bmatrix} \qquad (4.3.1a)
$$

or more compactly by

$$
I(s) = Y(s) V(s) \qquad (4.3.1b)
$$

where y_{ij} in general are functions of s, the Laplace-transformed variable.

The first step of the synthesis procedure is to find a state model corresponding to Equation (4.3.1). For the time being, it is assumed that the short-circuit admittance matrix $Y(s)$ in Equation (4.3.1) has no

poles at infinity and, thus, the state model has the general form as in

Equation (3.1.1) which is repeated below for ready reference:

$$\frac{d}{dt} x(t) = Ax(t) + Bu(t) \qquad (4.3.2a)$$

$$y(t) = Cx(t) + Du(t) \qquad (4.3.2b)$$

where x is the state vector corresponding to some capacitor voltages, u

is the input voltage-source vector, and y is the output source-current

vector. As mentioned in Section 3.1, the derivation of a state model

from a short-circuit admittance matrix is somewhat more involved and can

be found in (32).

The state model in Equation (4.3.2) is then modified such that its

synthesis is reduced to the synthesis of a new A-matrix, to be called a

"modified A-matrix". The technique developed previously can be applied

to realize this modified A-matrix.

To achieve this, u(t) in Equation (4.3.2) is taken to be some other

capacitor voltage vector u'(t), and y(t) is taken to be the correspond-

ing capacitor currents $\frac{d}{dt}$ u'(t), assuming capacitors of unit element

value. Therefore, Equation (4.3.2) is written as

$$\frac{d}{dt} x(t) = Ax(t) + B u'(t) \qquad (4.3.3a)$$

$$\frac{d}{dt} u'(t) = -Cx(t) - D u'(t) . \qquad (4.3.3b)$$

The minus signs before the matrices C and D in Equation (4.3.3b) are

introduced because of the fact that the orientations of the source

currents y(t) in Equation (4.3.2b) are opposite to those of the capaci-

tor currents $\frac{d}{dt}$ u'(t) in Equation (4.3.3b) if the polarities of the

voltage sources u(t) and of the capacitor voltages u'(t) are kept the same.

It follows from Equation (4.3.3) that

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ u'(t) \end{bmatrix} = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \begin{bmatrix} x(t) \\ u'(t) \end{bmatrix} \qquad (4.3.4)$$

where u'(t) is now part of the new state vector and the coefficient matrix

$$A' = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \qquad (4.3.5)$$

is to be considered as the "modified A-matrix". This modified A-matrix can be synthesized using the technique developed in Section 3.7. The realization of the given short-circuit admittance matrix is obtained by simply replacing the capacitors identified with u'(t) in the realization of this modified A-matrix by voltage sources.

The following example illustrates the synthesis of a short-circuit admittance matrix.

Example 4.3.1 Consider the short-circuit admittance matrix

$$Y(s) = \frac{1}{s^2 + s + 1} \begin{bmatrix} 3s^2 + s + 2 & -(s + 1) \\ -2(s - 1) & 3s^2 + s + 3 \end{bmatrix}. \qquad (4.3.6)$$

A state model corresponding to this admittance matrix can be derived as

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (4.3.7a)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (4.3.7b)$$

where u_1, u_2 are voltage sources and y_1, y_2 are output currents.

The modified A-matrix in Equation (4.3.5) can now be formed from Equation (4.3.7):

$$A' = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 1 \\ \hline 1 & 1 & -3 & 0 \\ 0 & 2 & 0 & -3 \end{array}\right] . \qquad (4.3.8)$$

This modified A-matrix can be synthesized using the scheme in Figure 3.7.1. In its realization, replacing the capacitors corresponding to u_1' and u_2' by voltage source V_1 and V_2, respectively, yields the synthesized network of the given short-circuit admittance matrix of Equation (4.3.6) as shown in Figure 4.3.1.

If the matrix $Y(s)$ to be realized is a constant matrix, which means that in the realization there are no energy-storage elements, such as capacitors, then in the state model corresponding to this $Y(s)$, there will be no state vector, and only the algebraic set such as Equation (4.3.2b) exists. In other words, only the matrix D of the set of matrices $\{A, B, C, D\}$ needs to be considered. Then, in this case, the modified A-matrix defined in Equation (4.3.5) is

$$A' = -D . \qquad (4.3.9)$$
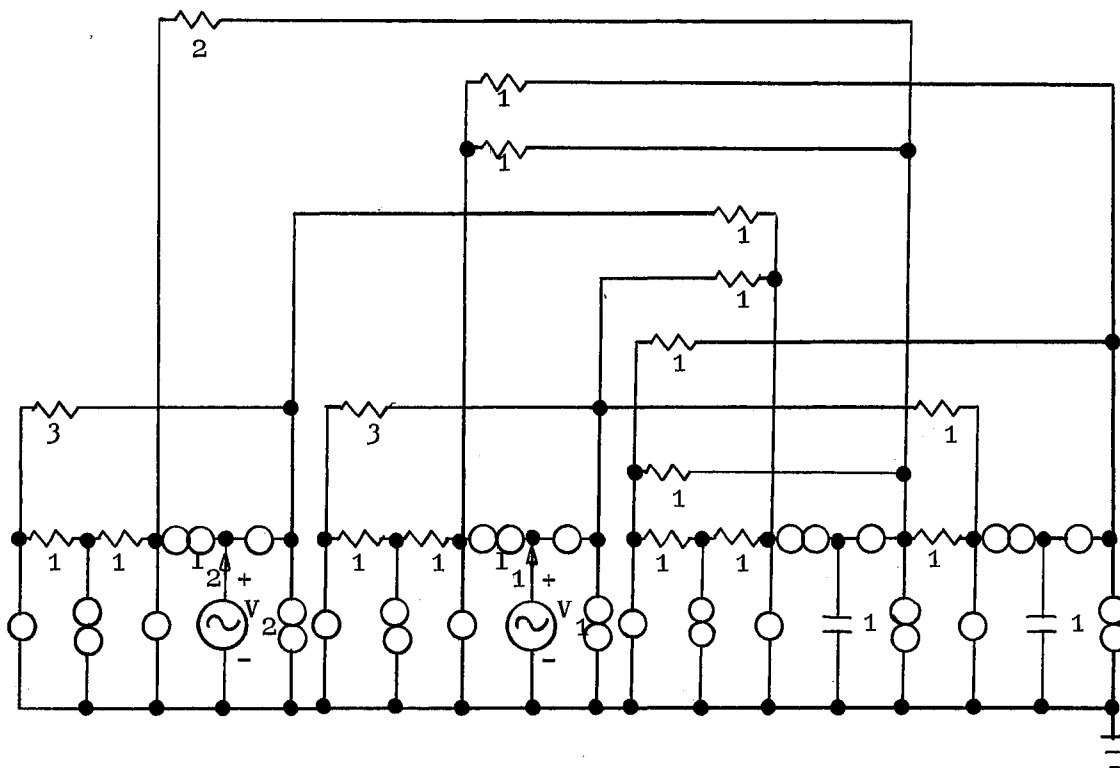
Figure 4.3.1. Example Synthesized Network

The realization of the constant matrix Y(s) is obtained when every capacitor in the realization of the modified A-matrix in Equation (4.3.9) is replaced by a voltage source.

The synthesis of driving-point admittance functions is of considerable interest (37). The above method can be used to obtain the realization. This is illustrated in the following example.

Example 4.3.2 Consider the realization of

$$Y_1(s) = -\frac{1}{s}.$$ (4.3.10)

The corresponding state model can be derived as

$$\frac{d}{dt} x = u \tag{4.3.11a}$$

$$y = -x . \tag{4.3.11b}$$

Hence, the modified A-matrix is formed from Equation (4.3.11):

$$A' = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{4.3.12}$$

In the realization of this modified A-matrix, which can be obtained by using the scheme in Figure 3.7.1, replacing the capacitor corresponding to u' by a voltage source results in the realization of the given admittance function as shown in Figure 4.3.2.
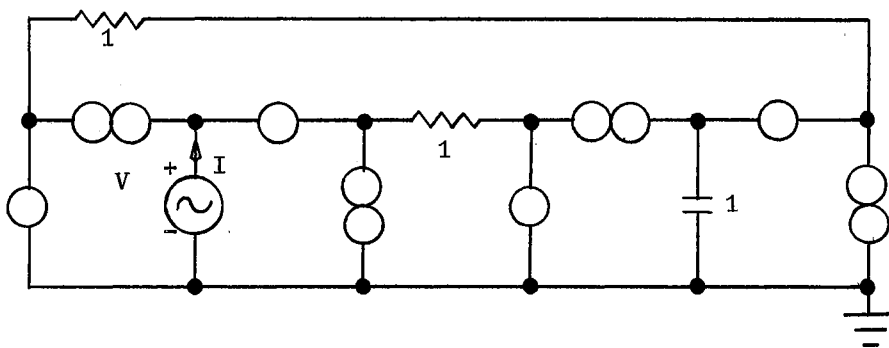
Figure 4.3.2. Example Synthesized Network

In the above discussion $Y(s)$ is assumed to be finite when s goes to infinity. When $Y(s)$ has a pole at infinity, then it can be written as $Y(s) = Y'(s) + sH_r$, where $Y'(s)$ has no pole at infinity and H_r is a real matrix. Note that there can be only simple poles at infinity for physically realizable networks. The realization of $Y(s)$ can be obtained by realizing $Y'(s)$ and sH_r separately and combining the two realizations with transformer coupling. The realization of transformers can be obtained in terms of two cascaded gyrators (32) and the realization of gyrators is discussed in the next section.

4.4 Synthesis of Gyrators Using
State-Model Approach

The ideal gyrator is a lossless, nonreciprocal multiport device which is described by the short-circuit admittance matrix

$$Y(s) = \begin{bmatrix} 0 & G \\ -G^T & 0 \end{bmatrix} \tag{4.4.1a}$$

such that

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & G \\ -G^T & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{4.4.1b}$$

where G is called the gyration conductance matrix.

Ideal gyrators can be used as impedance inverters; when terminated by an inductance (capacitance), a two-port gyrator can simulate a capacitance (inductance) because of its impedance inversion property. They can also be used as isolators and circulators under proper terminations and when embedded in an RC network, can give rise to complex

natural frequencies. In addition, two cascaded multiport gyrators can simulate a multiport transformer. Practically, ideal gyrators are used in microwave systems as well as at low frequencies when their transistor realizations make them significant for integrated circuit synthesis. It is known that the networks employing gyrators can be made less sensitive with respect to parameter variation compared to other active devices (28).

As a consequence, the realization of gyrators is important. Synthesis of the two-port gyrators has been extensively discussed (28) (32).

A good summary on realization of two-port gyrators is presented in (28). Some of the methods use controlled source models and then use the nullator-norator equivalent circuits for controlled sources. Here, a systematic synthesis procedure using resistors and nullator-norator pairs will be presented.

The matrix Y in Equation (4.4.1a) is a constant matrix. Therefore, as described in Section 4.3, the modified A-matrix defined in Equation (4.3.5) is

$$A' = -D$$

$$= \begin{bmatrix} 0 & -G \\ G^T & 0 \end{bmatrix}. \qquad (4.4.2)$$

The realization of the above modified A-matrix is illustrated in the following example.

Example 4.4.1 Consider the realization of the two-port gyrator matrix

$$Y(s) = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} \qquad (4.4.3)$$

where g is the gyration conductance. In this case, the A' in Equation (4.4.2) is

$$A' = \begin{bmatrix} 0 & -g \\ g & 0 \end{bmatrix}. \qquad (4.4.4)$$

First, this A-matrix is considered to be an arbitrary A-matrix, and the technique developed in Section 3.7 can be applied to synthesize it. Using the scheme in Figure 3.7.1 to obtain the realization of this A-matrix, then replacing the two capacitors by voltage sources V_1 and V_2 gives the realization for the two-port gyrator as shown in Figure 4.4.1.
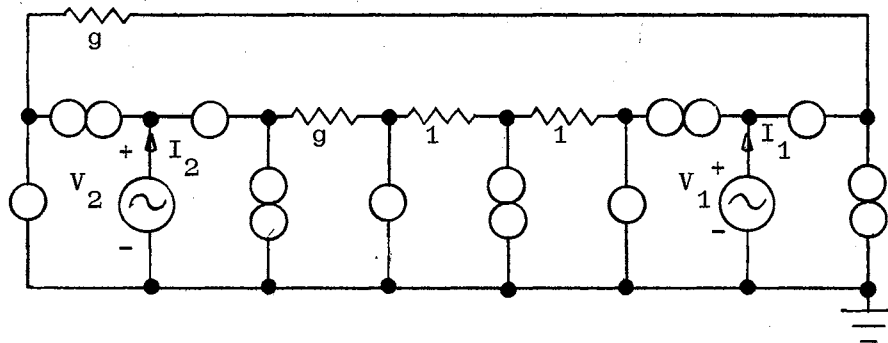
Figure 4.4.1. Synthesized Network for a Gyrator

Next, note that the matrix Y in Equation (4.4.3) is a skew-symmetric matrix, and can be considered as a tridiagonal matrix of the form shown in Equation (3.8.1). Hence, the technique developed in Section 3.8 can be applied to synthesize it. From Figure 3.8.2, with $n = 2$, replacing the capacitors by voltage sources as needed, the realization is shown in Figure 4.4.2, where the two conductors at both ends

as denoted by a_o and a_n in the general synthesized network of Figure

3.8.2 disappear because $a_o = 0$ and $a_n = 0$ from Equation (4.4.4). This

coincides with one of the classical realizations (28).
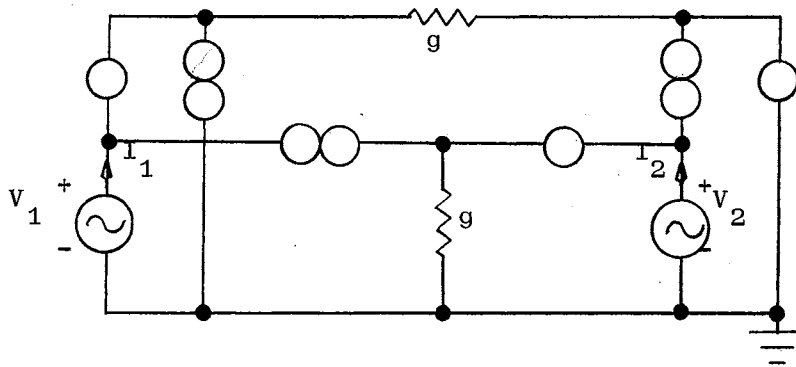
Figure 4.4.2.   Another Synthesized Network
for a Gyrator

It is observed that the realization in Figure 4.4.1 uses five

nullator-norator pairs; whereas, the realization in Figure 4.4.2 uses

only three nullator-norator pairs. Since a nullator-norator pair can

be replaced by an ideal transistor, as far as the number of nullator-

norator pairs used is concerned, the realization in Figure 4.4.2 is

more desirable than that in Figure 4.4.1.

To illustrate the use of the equivalent nullator-norator model of

an ideal transistor, two possible transistorized conversions of the

realization in Figure 4.4.2 are shown in (a) and (b) of Figure 4.4.3.
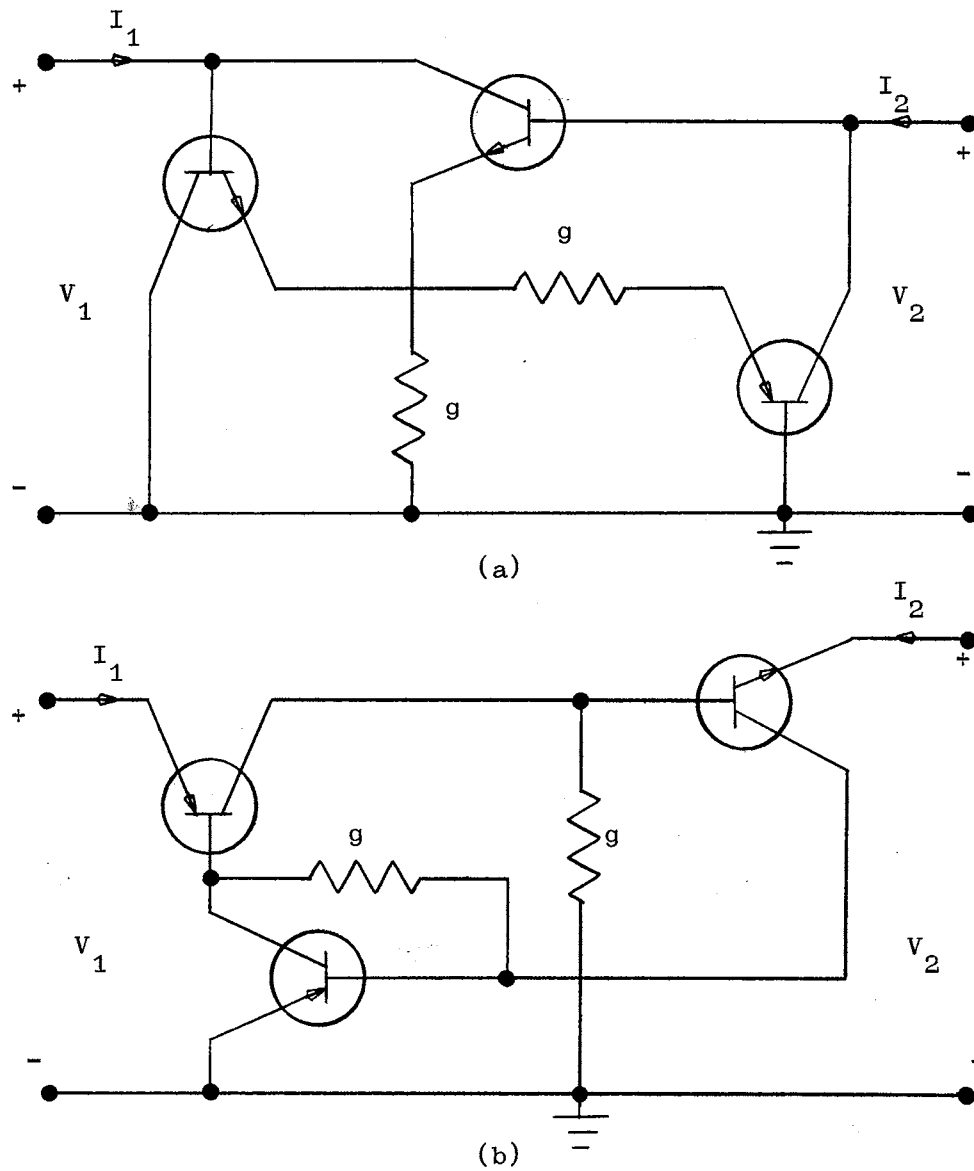
Figure 4.4.3. Equivalent Three-Transistor Realizations of Figure 4.4.2

4.5   Synthesis of Open-Circuit

Impedance Matrices

The results obtained in Sections 4.3 and 4.4 can be used to realize

an open-circuit impedance matrix $Z(s) = [z_{ij}(s)]$ defined by

$$
\begin{bmatrix}
V_1(s) \\
V_2(s) \\
\cdot \\
\cdot \\
\cdot \\
V_n(s)
\end{bmatrix}
=
\begin{bmatrix}
z_{11} & z_{12} & \cdots & z_{1n} \\
z_{21} & z_{22} & \cdots & z_{2n} \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
z_{n1} & z_{n2} & \cdots & z_{nn}
\end{bmatrix}
\begin{bmatrix}
I_1(s) \\
I_2(s) \\
\cdot \\
\cdot \\
\cdot \\
I_n(s)
\end{bmatrix}
\tag{4.5.1a}
$$

or more compactly by

$$
V(s) = Z(s)\, I(s) \; . \tag{4.5.1b}
$$

The synthesis of $Z(s)$ in Equation (4.5.1) will be achieved by first

realizing this $Z(s)$ as a short-circuit admittance matrix, and then

connecting the realized n-port network by n two-port gyrators (32).

In Figure 4.5.1, a multiport gyrator is shown to connect to a net-

work N_t which is described by a short-circuit parameter equation

$$
I''(s) = Y(s)\, V''(s) \; . \tag{4.5.2}
$$
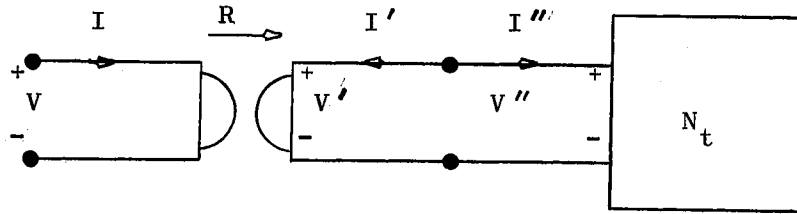
Figure 4.5.1.   Block Diagram for Realizing an
                 Open-Circuit Impedance Matrix

The multiport gyrator is described by the set of equations:

$$\begin{bmatrix} V(s) \\ V'(s) \end{bmatrix} = \begin{bmatrix} 0 & -R \\ R^T & 0 \end{bmatrix} \begin{bmatrix} I(s) \\ I'(s) \end{bmatrix}. \qquad (4.5.3)$$

where R is the gyration resistance matrix and will be taken as a unit

matrix of order n for simplicity. By using the relationships $V' = V''$

and $I' = -I''$, it is easy to derive from Equations (4.5.2) and (4.5.3)

that

$$V(s) = RY(s)R^T I(s) = Y(s)I(s) \qquad (4.5.4)$$

Now, comparing Equation (4.5.1b) with the above equation gives

$$Y(s) = Z(s) . \qquad (4.5.5)$$

The synthesis procedure developed in Section 4.3 can be applied to

realize this admittance matrix.

As to the realization of the multiport gyrator corresponding to

Equation (4.5.3), it is noted that Equation (4.5.3) can be decomposed

into n equations, each of which describes a two-port gyrator with the

gyration resistance equal to unity. Therefore, the realization of this

multiport gyrator is reduced to the realization of n two-port gyrators. The result in Example 4.4.1 of the preceding section can be used for each of these n two-port gyrators although the specification is in terms of open-circuit parameter equation here.

After connecting the n two-port gyrators to the realized network for the admittance matrix in Equation (4.5.5), the realization of the given $Z(s)$ is obtained. Note that $Z(s)$ may have a pole at infinity. The above procedure can still be applied, since the realization of an admittance matrix with a pole at infinity has been discussed.

The case of constant impedance matrix proceeds in a similar way.

The following example illustrates the synthesis of an impedance matrix.

<u>Example 4.5.1</u> Consider the open-circuit impedance matrix

$$Z(s) = \begin{bmatrix} 1 & 0 \\ \dfrac{2}{s^2 + s + 1} & 1 \end{bmatrix}. \tag{4.5.6}$$

From Equation (4.5.5),

$$Y(s) = Z(s) = \begin{bmatrix} 1 & 0 \\ \dfrac{2}{s^2 + s + 1} & 1 \end{bmatrix}. \tag{4.5.7}$$

To realize this $Y(s)$, the state model is first derived:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.5.8a}$$

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (4.5.8b)
$$

where u_1 and u_2 are voltage sources, and y_1 and y_2 are output currents. Synthesizing this state model with the technique of Section 4.3, and then connecting two two-port gyrators in the way shown in Figure 4.5.1, the realization can be obtained. This is shown in Figure 4.5.2.
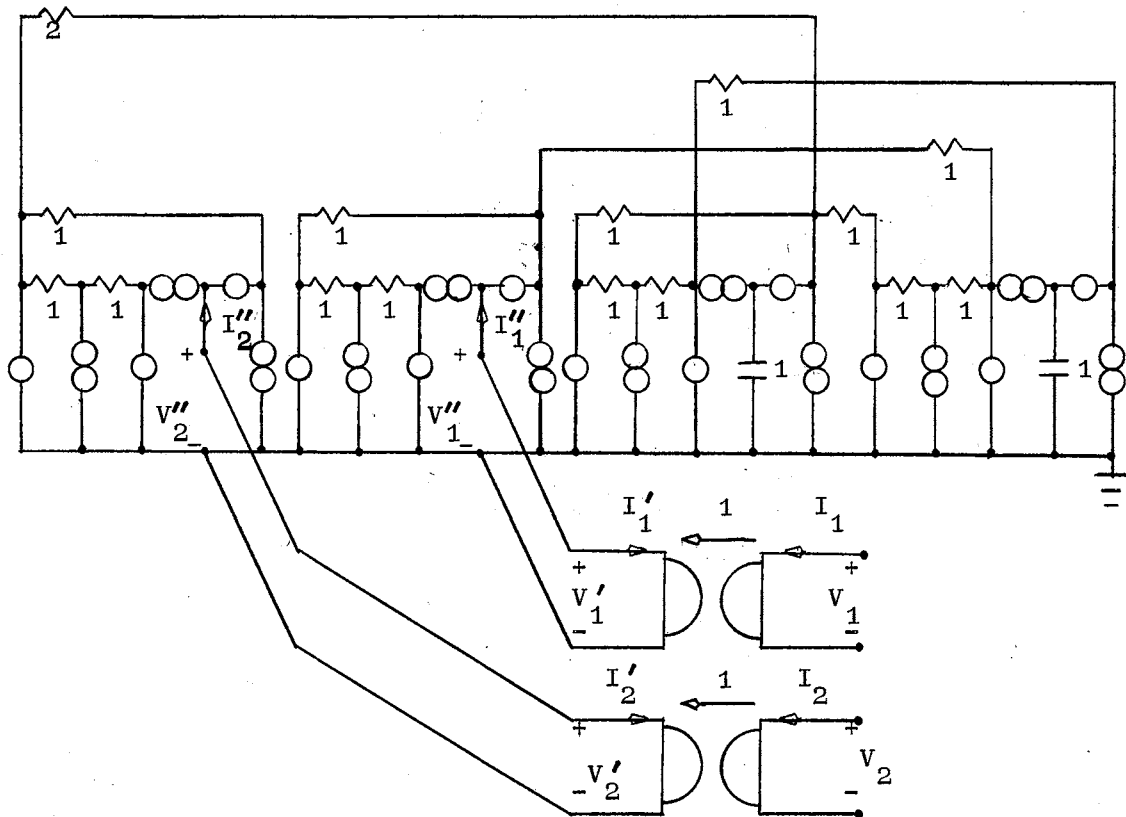
Figure 4.5.2. Example Synthesized Network

Note that if a given open-circuit impedance matrix is nonsingular, it may also be realized by first taking its inverse to obtain the short-circuit admittance matrix, then realizing the resulting admittance matrix by the technique developed earlier.

4.6 Synthesis of Hybrid Matrices

Synthesis of short-circuit admittance matrices and synthesis of open-circuit impedance matrices have been discussed earlier. They will be applied here to realize a hybrid matrix $H(s)$ defined by

$$\begin{bmatrix} V_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} \begin{bmatrix} I_1(s) \\ V_2(s) \end{bmatrix} = H(s) \begin{bmatrix} I_1(s) \\ V_2(s) \end{bmatrix} \quad (4.6.1)$$

where $H(s)$ may have, in general, a pole at infinity. The realization of the $H(s)$ is considered by using two cascaded networks: one is described by an open-circuit impedance parameter equation and the other by a short-circuit admittance parameter equation, both of which are derived from Equation (4.6.1). Derivation of an impedance matrix and an admittance matrix from the hybrid matrix $H(s)$ is not unique; there are many solutions. One is given by

$$\begin{bmatrix} V_1(s) \\ V_3(s) \end{bmatrix} = \begin{bmatrix} Z_{11}(s) & Z_{13}(s) \\ Z_{31}(s) & Z_{33}(s) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_3(s) \end{bmatrix} = Z(s) \begin{bmatrix} I_1(s) \\ I_3(s) \end{bmatrix} \quad (4.6.2a)$$

and

$$\begin{bmatrix} I_4(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} Y_{44}(s) & Y_{42}(s) \\ Y_{24}(s) & Y_{22}(s) \end{bmatrix} \begin{bmatrix} V_4(s) \\ V_2(s) \end{bmatrix} = Y(s) \begin{bmatrix} V_4(s) \\ V_2(s) \end{bmatrix} \quad (4.6.2b)$$

with

$$V_3(s) = V_4(s)$$

$$I_3(s) = -I_4(s)$$

$$Y_{44}(s) = 0$$

$$Z_{33}(s) = 0$$

$$Z_{11}(s) = H_{11}(s)$$

$$-Z_{13}(s)Y_{42}(s) = H_{12}(s)$$

$$Y_{24}(s)Z_{31}(s) = H_{21}(s)$$

$$Y_{22}(s) = H_{22}(s) .$$

(4.6.3)

From Equation (4.6.3), the Z_{ij}'s and Y_{ij}'s in Equation (4.6.2) can be found. Of course, they are not unique.

Realizing, respectively, the open-circuit impedance matrix $Z(s)$ and the short-circuit admittance matrix $Y(s)$ in Equation (4.6.2) by applying the techniques developed previously, and then cascading the two realized networks with $V_3(s) = V_4(s)$ and $I_3(s) = -I_4(s)$ yields the realization of the hybrid matrix $H(s)$.

The cases for hybrid matrices in g-parameters follows in a similar way.

The following example illustrates the synthesis procedure.

Example 4.6.1 Consider the constant hybrid matrix H in the equation

$$
\begin{bmatrix} V_1 \\ \\ I_2 \end{bmatrix}
=
\begin{bmatrix} v_{11} \\ v_{12} \\ \hline i_{21} \\ i_{22} \end{bmatrix}
=
\left[\begin{array}{cc|cc} 0 & -1 & 2 & -1 \\ 1 & -2 & 4 & -2 \\ \hline 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right]
\begin{bmatrix} i_{11} \\ i_{12} \\ \hline v_{21} \\ v_{22} \end{bmatrix}
=
\left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array} \right]
\begin{bmatrix} I_1 \\ \\ V_2 \end{bmatrix} .
$$

(4.6.4)

From this equation, a pair of z-parameters equation and y-parameter

equation corresponding to Equations (4.6.2a) and (4.6.2b), respectively,

can be determined by the use of Equation (4.6.3). There are many such

pairs. One is given by

$$
\begin{bmatrix} v_{11} \\ v_{12} \\ \hline v_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -2 & -2 \\ \hline -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{11} \\ i_{12} \\ \hline i_3 \end{bmatrix}
\qquad (4.6.5a)
$$

and

$$
\begin{bmatrix} i_4 \\ \hline i_{21} \\ i_{22} \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 \\ \hline -3 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_4 \\ v_{21} \\ v_{22} \end{bmatrix}.
\qquad (4.6.5b)
$$

The constant impedance matrix and constant admittance matrix in

Equation (4.6.5) are then realized by the techniques of Sections 4.5

and 4.3, respectively. Cascading these two realized networks with

$v_3 = v_4$ and $i_3 = -i_4$ gives the realization of the hybrid matrix H in

Equation (4.6.4) as shown in Figure 4.6.1.

4.7 Synthesis of Voltage and Current
Transfer-Function Matrices

State-model synthesis discussed in the previous sections can be

applied to synthesize a voltage or current transfer-function matrix.

To realize such a matrix, a state model is first derived. Then the

realization of the given voltage or current transfer-function matrix is

achieved by realizing the set of matrices $\{A, B, C, D\}$ in the state

model according to the discussion in Section 4.2.
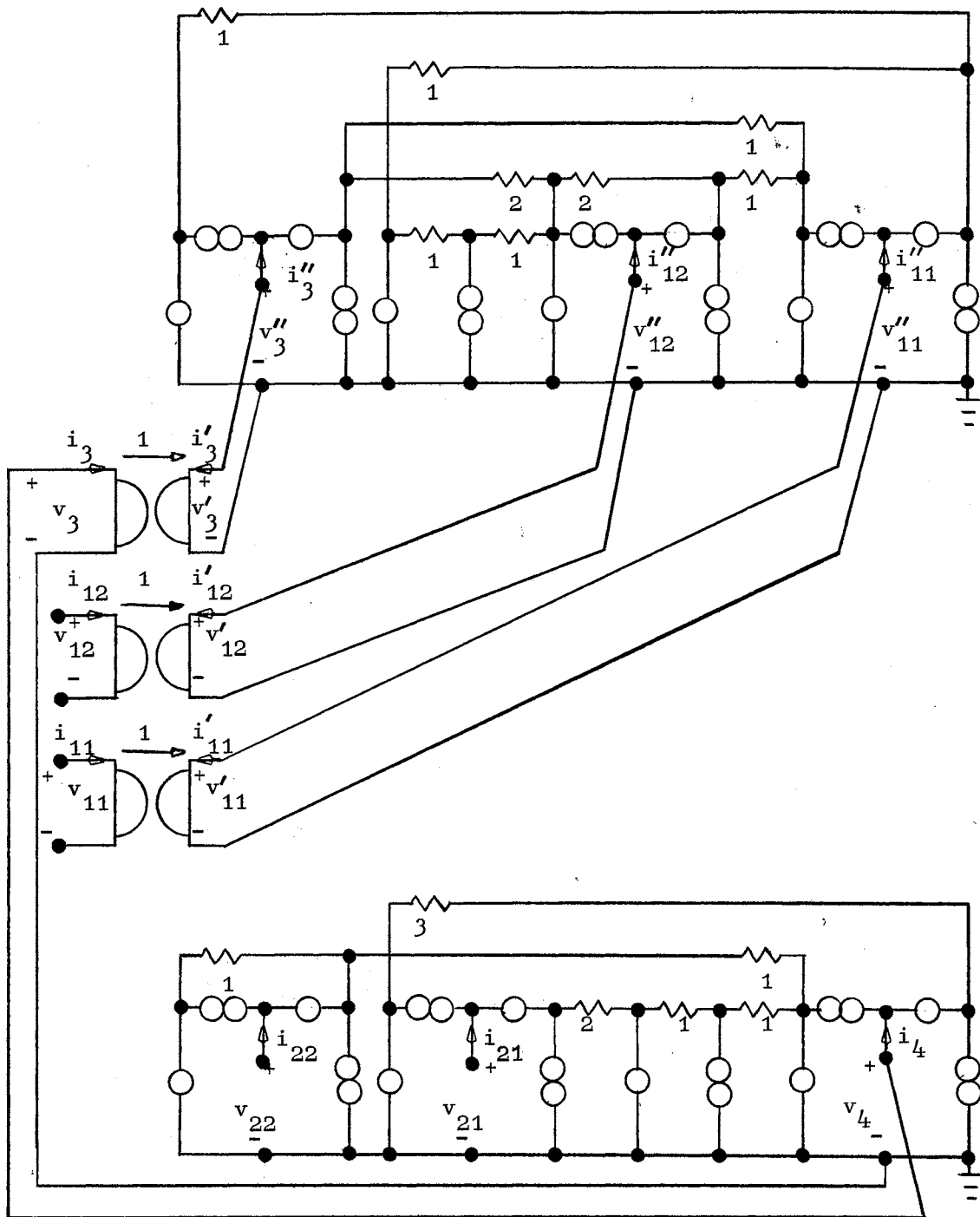
Figure 4.6.1.   Example Synthesized Network

It is illustrated in the following example.

Example 4.7.1 Consider the voltage transfer-function matrix

$$T_v(s) = \begin{bmatrix} \dfrac{1}{s + 1} \\[2ex] \dfrac{1}{(s + 1)(s + 2)} \end{bmatrix}. \qquad (4.7.1)$$

The state model corresponding to this matrix can be found as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad (4.7.2a)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (4.7.2b)$$

where u is the input voltage source, and y_1 and y_2 are the output voltages.

Realizing the A-matrix in Equation (4.7.2a), then incorporating the source and obtaining the output by using Equation (4.7.2b), yields the realization of the state model in Equation (4.7.2) as shown in Figure 4.7.1, which is equivalent to the realization of the voltage transfer-function matrix in Equation (4.7.1).

For the case of current transfer-function matrices, various kinds of controlled sources using nullator-norator circuits may be required in the realization as discussed in Section 4.2.
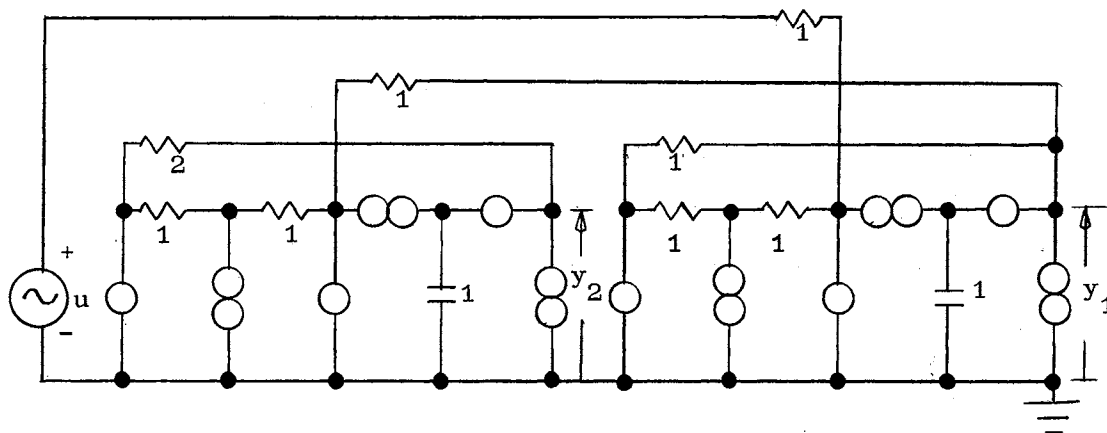
Figure 4.7.1. Example Synthesized Network

CHAPTER V

SYNTHESIS OF LOW-PASS FILTERS

5.1 Introduction

In Chapter III, the synthesis of a tridiagonal A-matrix was dis-
cussed. It has been recognized that tridiagonal matrices are related to
ladder networks (25) such as low-pass filters. Indeed, if a transfer
function corresponding to the realized network for a tridiagonal A-
matrix is derived, it will fall into the class of transfer functions of
ladder networks. This fact will be used in the synthesis of low-pass
filters with networks containing nullators and norators in the present
chapter. The related topic of prescribed source and load resistance
in doubly terminated low-pass filters will also be discussed.

5.2 Transfer Functions of Low-Pass Filters

The class of low-pass filter transfer functions with all the zeros
lying at infinity is of much interest. An example of such functions is
given by

$$Y_{21}(s) = \frac{I_2(s)}{V_1(s)} = \frac{p_o}{D_n(s)}, \quad D_n(s) = s^n + d_{n-1}s^{n-1} + \dots + d_o, \quad p_o > 0$$

$$(5.2.1)$$

where the input variable and output variable are defined in Figure 5.2.1
with R_1 and R_2 representing resistances in ohms. These functions are

important because they do correspond to the low-pass filters such as
Butterworth, Chebyshev, etc. The synthesis of such functions by doubly
terminated ladder networks has been extensively discussed (37).
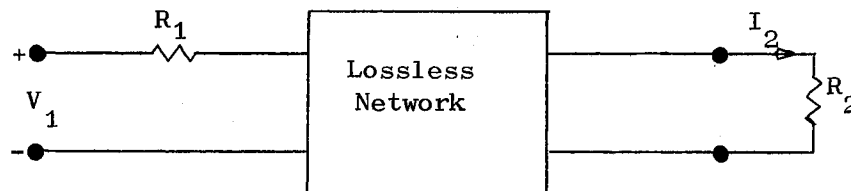
Figure 5.2.1. Network Configuration
for $Y_{21}(s)$

Synthesis of transfer functions has been considered in Section
4.2, where the realization of the characteristic polynomial of a trans-
fer function is achieved by synthesizing the corresponding A-matrix in
companion matrix form. The transfer function in Equation (5.2.1) can
be synthesized employing the techniques discussed in that section.
However, as pointed out in Section 3.9, for the same characteristic
polynomial, the realization of the corresponding tridiagonal A-matrix
uses fewer number of nullator-norator pairs than that of the correspond-
ing A-matrix in companion matrix form. Hence, a synthesis procedure
which makes use of the realization of a tridiagonal A-matrix is
presented below.

First, a brief review of the realization of a tridiagonal A-matrix
in Chapter III and its relation to classical low-pass ladder networks is
given. The synthesis of a tridiagonal A-matrix of the form shown in

Equation (3.8.1) was discussed in Section 3.8. The realized networks were shown in Figure 3.8.2 for n even and Figure 3.8.4 for n odd, where n represents the order of the tridiagonal A-matrix. It will be demonstrated qualitatively that the network shown in Figure 3.8.2 or 3.8.4 is equivalent to a classical low-pass ladder network.

The general form of an RLC low-pass ladder network is a lossless network terminated in resistance at both ends such as the one shown in Figure 5.2.1. The lossless network has shunt capacitance branches and series inductance branches. If every inductor of this lossless network is replaced by the grounded-gyrator equivalence (32) which, in turn, is replaced by the equivalent nullator-norator model obtained in Figure 4.4.2, the resulting network will have the same configuration as Figure 3.8.2 or 3.8.4, depending on n being even or odd, respectively.

When introducing a source, voltage source or current source, to the network realizing a tridiagonal A-matrix and specifying the output, it can be shown that the transfer function does correspond to a low-pass filter transfer function. This will be described in the following.

5.2.1 Case I

For this case, the source is a current source and the output is a voltage as shown in Figure 5.2.2, where the realization of the tridiagonal A-matrix in Figure 3.8.2 with n even is used. The current source is incorporated across the conductor a_o, and the output is taken from the conductor a_n.
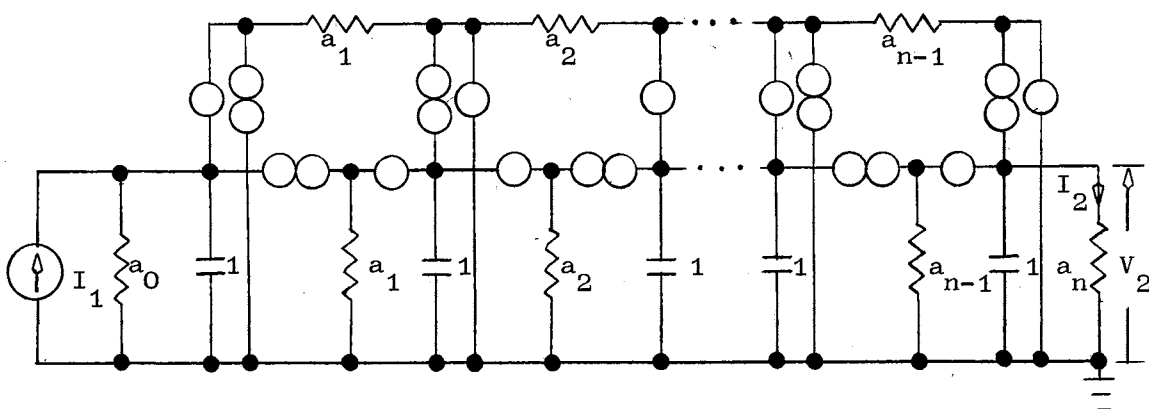
Figure 5.2.2. Network for Determining the Transfer Function

The transfer function is to be derived by using the state-model approach. Since the above network satisfies Conditions (i) and (ii) of Theorem 2.2.1, and furthermore all the capacitors can be selected in the branches and all the conductors (resistors) can be selected in the chords for both T_1 and T_2, the state equation can be obtained directly by using the general state equation derived in Equation (2.3.15) with the simplication that $V_v = 0$, $C_c = 0$ and $R_b = 0$, and is

$$\frac{d}{dt} V_{bc} = (C_b^{-1} C_{22} G_c B_{22}) V_{bc} + (-C_b^{-1} C_{25}) I_1 \qquad (5.2.2)$$

where the symbols are as defined in Chapter II. Note that in Equation (5.2.2), C_b has been assumed to be a unit matrix for simplicity in Chapter III and therefore Equation (5.2.2) reduces to

$$\frac{d}{dt} V_{bc} = (C_{22} G_c B_{22}) V_{bc} + (-C_{25}) I_1. \qquad (5.2.3)$$

By inspection of Figure 5.2.2, the output equation is

$$V_2 = P_1 V_{bc} \qquad (5.2.4)$$

where P_1 is an $1 \times n$ matrix $[0 \ 0 \ . \ . \ . \ 0 \ 1]$. From Equations (5.2.4) and

(5.2.3) the transfer function can be derived as

$$\frac{V_2}{I_1} = -P_1(sU - C_{22}G_cB_{22})^{-1}C_{25} \qquad (5.2.5)$$

where s is the Laplace-transformed variable and U represents an identity

matrix.

If the submatrices of the cut-set matrix, C_{22} and C_{25}, and the

submatrix of the circuit matrix, B_{22}, are evaluated from the graph of

Figure 5.2.2, it will be found that the transfer function is given by

$$\frac{V_2}{I_1} = \frac{(-1)^{(\frac{n}{2} + 1)} \prod\limits_{i=1}^{n-1} a_i}{D_n(s)} , \quad D_n(s) = |sU - A_1|, \ n - \text{even} \qquad (5.2.6)$$

where A_1 is the A-matrix in Equation (5.2.3) with a general form as A_T

in Equation (3.8.1), and is repeated below for convenience:

$$A_1 = \begin{bmatrix} -a_o & a_1 & & & & & \\ -a_1 & 0 & a_2 & & & & \\ & -a_2 & 0 & a_3 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -a_{n-2} & 0 & a_{n-1} \\ & & & & & -a_{n-1} & -a_n \end{bmatrix} . \qquad (5.2.7)$$

It is important to note that the a_i's in the numerator of Equation

(5.2.6) are from the lower off-diagonal entries in Equation (5.2.7). It

should be pointed out that because of the arbitrarily assumed orienta-

tion of each element in the graph of the network of Figure 5.2.2, which

is constructed to calculate C_{22}, C_{25}, and B_{22}, the matrix A_1 may be of a form with different sign distribution from that in Equation (5.2.7), but having the same characteristic polynomial as $D_n(s)$.

In Equation (5.2.6) when V_2/I_1 is negative, it means that the orientation of the output V_2 is opposite to that as assumed in Figure 5.2.2.

Now, by introducing a current source in the network of Figure 3.8.4, if the same procedure as above is followed, the transfer function can be derived as

$$\frac{V_2}{I_1} = \frac{(-1)^{(n-1)/2} \prod_{i=1}^{n-1} a_i}{D_n(s)}, \quad D_n(s) = |sU - A_1|, \quad n - \text{odd} \qquad (5.2.8)$$

where A_1 is defined in Equation (5.2.7) with n odd.

5.2.2 Case II

In this case, the network has a current source and a current output, and the network configuration shown in Figure 5.2.2 can also be used for n even. By making use of the result in Equation (5.2.6), the transfer function for this case is derived as

$$\frac{I_2}{I_1} = \frac{a_n V_2}{I_1}$$

$$= \frac{(-1)^{(n/2+1)} \prod_{i=1}^{n} a_i}{D_n(s)}, \quad D_n(s) = |sU - A_1|, \quad n - \text{even}. \qquad (5.2.9)$$

Similarly, using Equation (5.2.8) for n odd, it results in

$$\frac{I_2}{I_1} = \frac{a_n V_2}{I_1}$$

$$= \frac{(-1)^{(n-1)/2} \prod_{i=1}^{n} a_i}{D_n(s)}, \quad D_n(s) = \left| sU - A_1 \right|, \quad n - odd. \quad (5.2.10)$$

5.2.3 Case III

For this case, the source is a voltage source and the output is a voltage. In order to use the results derived in Case I, the network configuration for this case will be obtained by applying the Thevenin's theorem to Figure 5.2.2. It is shown in Figure 5.2.3 for n even.
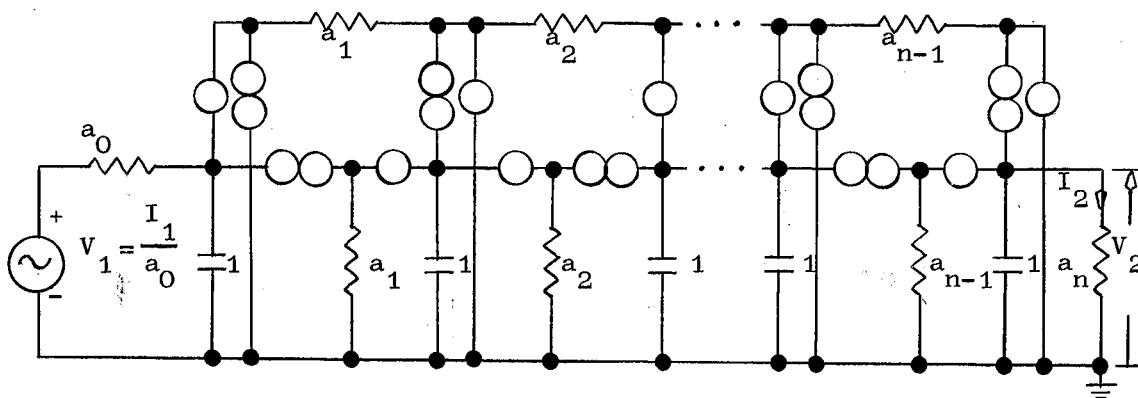
Figure 5.2.3. Network for Determining the Transfer Function

Then, the transfer function for n even is

$$\frac{V_2}{V_1} = \frac{V_2}{I_{1/a_o}}$$

$$= \frac{a_o V_2}{I_1}$$

$$= \frac{(-1)^{(n/2+1)} \prod\limits_{i=0}^{n-1} a_i}{D_n(s)}, \quad D_n(s) = \left| sU - A_1 \right|, \quad n - \text{even}. \quad (5.2.11)$$

In a similar way, the transfer function for n odd is

$$\frac{V_2}{V_1} = \frac{a_o V_2}{I_1}$$

$$= \frac{(-1)^{(n-1)/2} \prod\limits_{i=0}^{n-1} a_i}{D_n(s)}, \quad D_n(s) = \left| sU - A_1 \right|, \quad n - \text{odd}. \quad (5.2.12)$$

5.2.4   Case IV

For this case the source is a voltage source and the output is a current with the same network configuration as shown in Figure 5.2.3 for n even. From this network, the transfer function is

$$\frac{I_2}{V_1} = \frac{a_n V_2}{I_{1/a_o}}$$

$$= \frac{a_o a_n V_2}{I_1}$$

$$= \frac{(-1)^{n/2+1} \prod\limits_{i=0}^{n} a_i}{D_n(s)}, \quad D_n(s) = \left| sU - A_1 \right|, \quad n - \text{even.} \quad (5.2.13)$$

Similarly, when n is odd the transfer function is

$$\frac{I_2}{V_1} = \frac{a_o a_n V_2}{I_1}$$

$$= \frac{(-1)^{(n-1)/2} \prod\limits_{i=0}^{n} a_i}{D_n(s)}, \quad D_n(s) = \left| sU - A_1 \right|, \quad n - \text{odd.} \quad (5.2.14)$$

The above transfer functions will be used in the synthesis of low-pass filters in the next section.

5.3 Synthesis of Low-Pass Filters

Topology of the low-pass filters has already been derived. What remains to do is to calculate its element values for a given transfer function. As a matter of fact only conductor element values need to be determined.

The transfer functions derived in Equations (5.2.6), (5.2.8), (5.2.9), (5.2.10), (5.2.11), (5.2.12), (5.2.13), and (5.2.14) will be used to find the element value of every conductor in the low-pass filter. To achieve this, the steps below will be taken, and the details will be discussed following these steps.

(1) Draw the appropriate network configuration discussed in the preceding section, corresponding to the type of the given transfer function.

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(2) Find a tridiagonal A-matrix by Navot's method (31) from the denominator $D_n(s)$, i.e., characteristic polynomial, of the given transfer function, two of whose entries are made to contain an undetermined factor.

(3) Calculate this undetermined factor such that the transfer function has the proper numerator constant, thus obtaining all the entries of the tridiagonal A-matrix, from which the element values of the conductors are obtained.

Now to describe these steps suppose a transfer admittance function such as Equation (5.2.1) with n even is given. This is a Case IV transfer function and the network configuration in Figure 5.2.3 can be used. In order to adjust for the numerator constant of the given transfer function, a tridiagonal A-matrix in the following form

$$A_s = \begin{bmatrix} -a_o & f_1 & & & & \\ -1 & 0 & f_2 & & & \\ & -1 & 0 & f_3 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & -1 & 0 & f_{n-1}/k \\ & & & & & -k & -a_n \end{bmatrix} \quad (5.3.1)$$

is derived such that

$$\left| sU - A_s \right| = D_n(s) \quad (5.3.2)$$

where k is a positive constant to be determined and $a_o = f_o$, $a_n = f_n$ and f_i, $i = 0, 1, 2, \ldots, n$, are obtained from Navot's method which will be

given in Subsection 5.4.1. For each constant c, $0 < c \leq \alpha$, $\alpha =$ min $\{|D_n(j\omega)|^2,\ 0 \leq \omega < \infty\}$, there is a tridiagonal A-matrix such as A_s in Equation (5.3.1). Choose any one. The network at this stage is shown in Figure 5.3.1 which uses the fundamental blocks of Figure 3.8.7, since for this case the two entries a_{ij} and a_{ji}, $j = i + 1$, $i = 1, 2, \ldots$, or (n-1), are not equal.

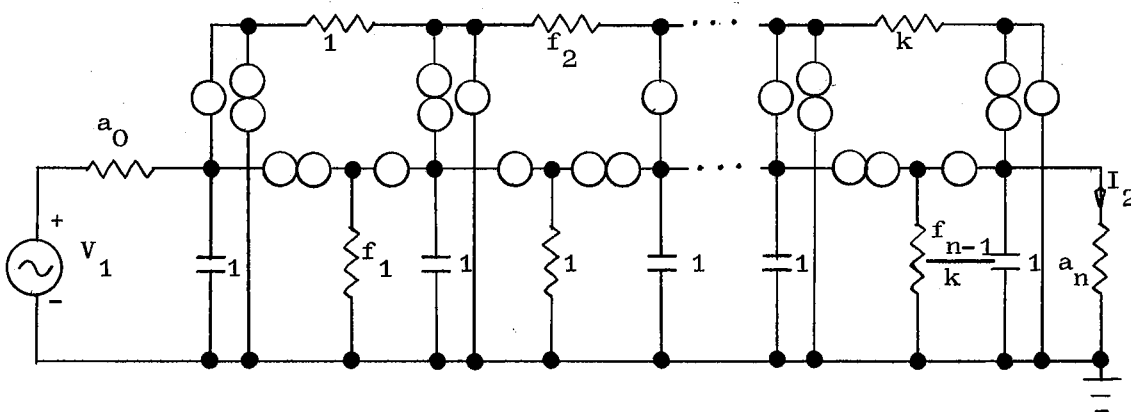Figure 5.3.1. Network for Synthesizing a Transfer Function

Comparing the numerator constant of the given transfer function in Equation (5.2.1) with that in Equation (5.2.13), disregarding the sign, yields

$$p_o = \prod_{i=0}^{n} a_i \qquad (5.3.3)$$

where as pointed out earlier, the a_i, $i = 1, 2, \ldots$, (n-2), (n-1) are from the lower off-diagonal entries in Equation (5.2.7). Therefore,

from Equation (5.3.1), $a_i = 1$, $i = 1, 2, \ldots, (n-2)$, and $a_{n-1} = k$. It follows from Equation (5.3.3) that

$$p_o = a_o k a_n \qquad (5.3.4)$$

and thus

$$k = \frac{p_o}{a_o a_n} .$$

By inserting the value of k above into Figure 5.3.1, the synthesized network will be obtained.

A numerical example is given below to illustrate the procedure.

Example 5.3.1 Suppose that it is desired to realize the transfer function

$$Z_{21}(s) = \frac{V_2}{I_1} = \frac{6}{s^4 + 7s^3 + 39s^2 + 115s + 154} . \qquad (5.3.5)$$

This is a Case I transfer function, and the network in Figure 5.2.2 will be used. By Navot's method, a tridiagonal A-matrix can be derived as

$$A_e = \begin{bmatrix} -2 & 4 & 0 & 0 \\ -1 & 0 & 9 & 0 \\ 0 & -1 & 0 & \frac{16}{k} \\ 0 & 0 & -k & -5 \end{bmatrix} . \qquad (5.3.6)$$

From Equation (5.2.6) with $n = 4$, the numerator constant is $(-a_1 a_2 a_3)$ where the negative sign means that the output voltage has a polarity opposite to that assumed. Equating the absolute value of this to the numerator constant of the given transfer function in Equation (5.3.5) gives

$$a_1a_2a_3 = 6$$

where, from Equation (5.3.6), $a_1 = 1$, and $a_2 = 1$, and $a_3 = k$. It follows that

$$k = 6.$$

By using the general network in Figure 5.2.2 with n = 4, and using the fundamental blocks of Figure 3.8.7 for determining the conductance element values, the realization is obtained as shown in Figure 5.3.2.
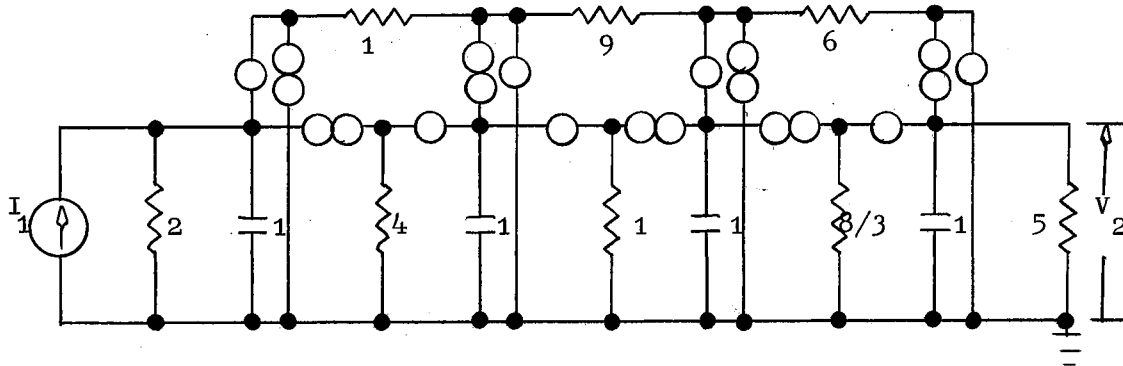
Figure 5.3.2. Example Synthesized Network

It should be noted that in classical RLC low-pass filter synthesis, there is a constraint on the numerator constant of the transfer function (20); whereas, in the present case, there is no such constraint because active elements are used in the realization.

In the design of classical RLC doubly terminated low-pass filters, it is not uncommon that in addition to the transfer function, both the

source resistance and load resistance are prescribed. This will be
considered in the next section.

5.4 Synthesis of RLC Low-Pass Filters With
Prescribed Source and Load Resistance

In this section, a constant, c, which ensures the realized RLC
doubly terminated low-pass ladder network has the prescribed source and
load resistance is given. This constant arises from Navot's method of
deriving a tridiagonal matrix from a strictly Hurwitz polynomial and is
expressed in terms of terminating resistances. Thus, the synthesis of
RLC low-pass ladder network with prescribed terminating resistances,
which uses a tridiagonal matrix, can be worked out in a straightforward
manner.

In a recent paper, Fowler and Yarlagadda (20) presented a transfer
function synthesis procedure using state-space approach. Given a trans-
fer function as in Equation (5.2.1), their synthesis procedure yields
several sets of element values for the realized ladder network. Each
set of the element values corresponds to a parameter c, where

$$0 < c \leq \alpha, \quad \alpha = \min \{|D_n(j\omega)|^2, \ 0 \leq \omega < \infty\}. \tag{5.4.1}$$

Note that the $D_n(s)$ is assumed to be a strictly Hurwitz polynomial. It
is the purpose of the following discussions to derive a value of c such
that the realized network has the prescribed terminating resistances.
It is shown that this c is related to the transmission coefficient t(s)
which is used in the classical network synthesis.

5.4.1 Necessary Background

In (20), the synthesis procedure uses a tridiagonal matrix derived by Navot's method (31). The method is described in the following. In deriving this matrix, an auxiliary polynomial

$$G_n(s) = s^n + g_{n-1}s^{n-1} + \ldots + g_o \qquad (5.4.2)$$

is used such that

$$G_n(s)G_n(-s) = D_n(s)D_n(-s) - c \qquad (5.4.3)$$

where c is a constant bounded by the constraint in Equation (5.4.1). Forming the continued fraction expansion gives

$$\frac{D_n(s) - G_n(s)}{(d_{n-1} - g_{n-1})D_n(s)} = \cfrac{1}{s + f_o + \cfrac{f_1}{s + \cfrac{f_2}{s + \cdot \\ \quad \cdot \\ \quad \cdot \\ + \cfrac{f_{n-1}}{s + f_n}}}} \qquad (5.4.4)$$

A tridiagonal matrix

$$K_t = \begin{bmatrix} -f_o & -f_1 & & & & \\ 1 & 0 & -f_2 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1 & 0 & -f_{n-1} \\ & & & & 1 & -f_n \end{bmatrix} \qquad (5.4.5)$$

is obtained such that

$$\left| sU - K_t \right| = D_n(s)$$

$$= s^n + (f_o + f_n)s^{n-1} + (\sum_{j=1}^{n-1} f_j + f_o f_n)s^{n-2} + \dots + (\prod_{k=1}^{n/2} f_{2k-1} + \prod_{\ell=0}^{n/2} f_{2\ell}).$$

$$(5.4.6)$$

By a simple approach, a ladder network shown in Figure 5.4.1, with $m + r = n$, $m = r = n/2$, is derived when n is even where R_1 and R_2 are resistances in ohms.

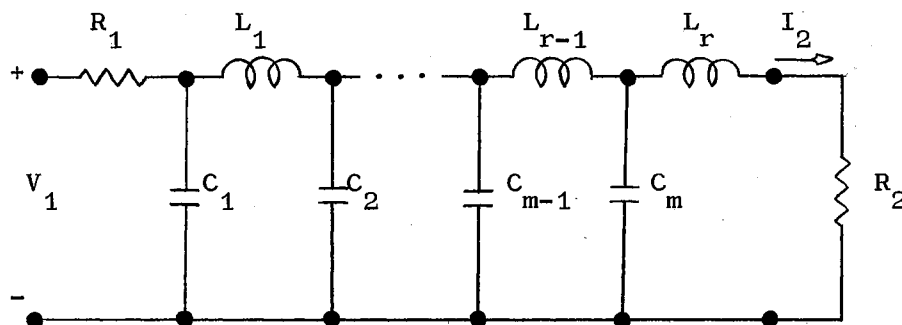Figure 5.4.1. Ladder Network With n Even

The element values of this ladder network are related to f_i by the following equations (20):

$$f_o = (R_1 C_1)^{-1}, \quad f_n = R_2 L_r^{-1}, \quad f_1 = (C_1 L_1)^{-1}, \quad f_2 = (C_2 L_1)^{-1}$$

$$(5.4.7)$$

$$f_3 = (C_2 L_2)^{-1}, \quad \dots, \quad f_{n-2} = (C_m L_{r-1})^{-1}, \quad f_{n-1} = (C_m L_r)^{-1}.$$

When n is odd, the network in Figure 5.4.1 can be used with $L_r = 0$, $m + r - 1 = n$, $m = r = (n+1)/2$. The element values are related to f_i by

$$f_o = (R_1 C_1)^{-1}, \quad f_n = (R_2 C_m)^{-1}, \quad f_1 = (C_1 L_1)^{-1}, \quad f_2 = (C_2 L_1)^{1}$$

(5.4.8)

$$f_3 = (C_2 L_2)^{-1}, \quad \ldots \quad , f_{n-2} = (C_{m-1} L_{r-1})^{-1}, \quad f_{n-1} = (C_m L_{r-1})^{-1}.$$

For both cases, n even and odd, there are $(n+1)$ equations with $(n+2)$ unknowns. In addition to these equations, there is another constraint, that is

$$\frac{1}{R_1 + R_2} = \frac{p_o}{d_o}$$

(5.4.9)

which is obtained by considering $s = 0$ in the network shown in Figure 5.4.1 and in Equation (5.2.1). Hence, the element values can be solved from Equations (5.4.7) and 5.4.9) or from Equations (5.4.8) and (5.4.9) in a simple manner (20).

Note that the element values are not unique. For each c satisfying Equation (5.4.1), there exists a particular set of source and load resistance.

5.4.2 Derivation of c

A short-circuit transfer admittance function in Equation (5.2.1) is given with n even and it is desired to find a c satisfying Equation (5.4.1) such that the realized doubly terminated ladder network has

prescribed source and load resistance which satisfy Equation (5.4.9).
Further, it is assumed that neither of these two resistances is
identically zero.

Equations (5.4.2), (5.4.3), (5.4.4), and (5.4.6) are used to solve
for c. Substituting $D_n(s)$ from Equation (5.4.6) and $G_n(s)$ from Equation
(5.4.2) in Equation (5.4.4) and simplifying, the following equation
results.

$$s^{n-1} + \frac{\sum\limits_{j=1}^{n-1} f_j + f_0 f_n - g_{n-2}}{f_0 + f_n - g_{n-1}} s^{n-2} + \cdots + \frac{\prod\limits_{k=1}^{n/2} f_{2k-1} + \prod\limits_{\ell=0}^{n/2} f_{2\ell} g_0}{f_0 + f_n - g_{n-1}}$$

$$= s^{n-1} + f_n s^{n-2} + \cdots + \prod_{p=1}^{n/2} f_{2p} \ . \tag{5.4.10}$$

Equating the corresponding coefficients in Equation (5.4.10) results in
(n-1) equations. Two of these are

$$\frac{\sum\limits_{j=1}^{n-1} f_j + f_0 f_n - g_{n-2}}{f_0 + f_n - g_{n-1}} = f_n \tag{5.4.11}$$

and

$$\frac{\prod\limits_{k=1}^{n/2} f_{2k-1} + \prod\limits_{\ell=0}^{n/2} f_{2\ell} -g_0}{f_0 + f_n - g_{n-1}} = \prod_{p=1}^{n/2} f_{2p} \ . \tag{5.4.12}$$

In addition, since

$$D_n(s)D_n(-s) = s^{2n} + [2(\sum_{j=1}^{n-1} f_j + f_0 f_n) - (f_0 + f_n)^2] s^{2(n-1)} + \cdots$$

$$+ (\prod_{k=1}^{n/2} f_{2k-1} + \prod_{\ell=0}^{n/2} f_{2\ell})^2 \tag{5.4.13}$$

and

$$G_n(s)G_n(-s) = s^{2n} + (2g_{n-2} - g_{n-1}^2)s^{2(n-1)} + \cdots + g_0^2. \tag{5.4.14}$$

Equation (5.4.3) yields n equations by comparing the corresponding coefficients. Two of these are

$$2(\sum_{j=1}^{n-1} f_j + f_0 f_n) - (f_0 + f_n)^2 - 2g_{n-2} + g_{n-1}^2 = 0 \tag{5.4.15}$$

and

$$c = (\prod_{k=1}^{n/2} f_{2k-1} + \prod_{\ell=0}^{n/2} f_{2\ell})^2 - g_0^2. \tag{5.4.16}$$

Equations (5.4.11), (5.4.12), (5.4.15), and (5.4.16) are sufficient to solve for c. In the above discussion, only two equations are considered in each case when comparing the coefficients. If these equations are satisfied, then the remaining equations will be automatically satisfied. From Equations (5.4.11), (5.4.12), and (5.4.16), it follows that

$$g_{n-1} = f_n - f_0 \tag{5.4.17}$$

and

$$g_o = \prod_{k=1}^{n/2} f_{2k-1} - \prod_{\ell=0}^{n/2} f_{2\ell} \; . \qquad (5.4.18)$$

Substituting Equation (5.4.18) in Equation (5.4.16) results in

$$c = 4 \left(\prod_{k=1}^{n/2} f_{2k-1} \right) \left(\prod_{\ell=0}^{n/2} f_{2\ell} \right) = 4 \prod_{i=0}^{n} f_i \; . \qquad (5.4.19)$$

By using the expressions for f_i from Equation (5.4.7), Equation (5.4.19) becomes

$$c = \frac{4R_2}{R_1 \left(\prod\limits_{i=1}^{m} C_i \right)^2 \left(\prod\limits_{j=1}^{r} L_j \right)^2} \; . \qquad (5.4.20)$$

The numerator constant p_o in Equation (5.2.1) is related to the element values by

$$p_o = \frac{1}{R_1 \left(\prod\limits_{i=1}^{m} C_i \right) \left(\prod\limits_{j=1}^{r} L_j \right)} \; . \qquad (5.4.21)$$

Substituting Equation (5.4.21) in Equation (5.4.20) gives

$$c = 4 p_o^2 R_1 R_2 \; . \qquad (5.4.22)$$

When n is odd, the values for c and p_0 in terms of the element values are

$$c = \frac{4}{R_1 R_2 (\prod\limits_{i=1}^{m} C_i)^2 (\prod\limits_{j=1}^{r-1} L_j)^2} \quad, \qquad p_0 = \frac{1}{R_1 R_2 (\prod\limits_{i=1}^{m} C_i)(\prod\limits_{j=1}^{r-1} L_j)} \quad .$$

$$(5.4.23)$$

From these, c can be derived for n odd, in terms of p_0, R_1, and R_2 and the expression is the same as the one in Equation (5.4.22).

Note that the resistances R_1 and R_2 appear together as a sum in Equation (5.4.9) and as a product in Equation (5.4.22). Therefore, in some cases, the realized network may have source and load resistance interchanged. In these cases, it is a simple matter to obtain the correct source and load resistance by using the reciprocity theorem.

If $p_0/D_n(s)$ in Equation (5.2.1) is a transfer impedance function $Z_{21}(s)$, or a voltage-ratio transfer function $T_{21}(s)$, then by a procedure similar to above, c is given, respectively, by

$$c = \frac{4p_0^2}{R_1 R_2} \qquad\qquad (5.4.24)$$

or

$$c = \frac{4p_0^2 R_1}{R_2} \quad . \qquad\qquad (5.4.25)$$

Note that c in Equations (5.4.22), (5.4.24), and (5.4.25) must satisfy Equation (5.4.1). It can be shown that this is always true if $D_n(s)$ is a polynomial with α occurring at $\omega = 0$, such as the Butterworth, the Bessel polynomial, and the Chebyshev polynomial for n odd. Furthermore, this c is related to the transmission coefficient $t(s)$ which is

used to derive the synthesis procedure in the classical network
synthesis. The relation is

$$|t(j\omega)|^2 = \frac{c}{|D_n(j\omega)|^2} . \qquad (5.4.26)$$

Equation (5.4.26) can be derived by comparing the expression for c and
the expression for t(s) (37). This provides some link between the syn-
thesis procedure in (20) and classical techniques (37).

The above results in Equations (5.4.22), (5.4.24), and (5.4.25) are
derived by using an algebraic manipulation. Actually they can be de-
duced from physical considerations. For $s = j\omega$, Equation (5.4.3) is the
same as the Feldtkeller equation (33) which can be written in different
forms (33) (37). The Feldtkeller equation describes power relations in
a lossless two-port teminated by resistances. Rewriting Equation
(5.4.3) gives

$$\frac{G_n(s)G_n(-s)}{c} + 1 = \frac{D_n(s)D_n(-s)}{c} . \qquad (5.4.27)$$

In filter theory literature (33) the function $D_n(s)/\sqrt{c}$ is known
as the effective transmission factor and the function $G_n(s)/\sqrt{c}$ is known
as the characteristic function. The properties of these functions (33)
for general lossless two-port are well known. By definition (33)

$$\frac{D_n(s)}{\sqrt{c}} = \frac{V_1}{I_2\sqrt{4R_1 R_2}} = \frac{D_n(s)}{p_o\sqrt{4R_1 R_2}} . \qquad (5.4.28)$$

Thus, it follows that $c = 4p_o^2 R_1 R_2$. Equations (5.4.24) and (5.4.25) can
be derived in a similar manner.

For illustration, a numerical example is given below.

Example 5.4.1 Realize the transfer admittance function

$$Y_{21}(s) = \frac{5}{s^4 + 2s^3 + 10s^2 + 10s + 17} \tag{5.4.29}$$

with prescribed $R_1 = \frac{16}{5}$ ohms and $R_2 = \frac{1}{5}$ ohm. The prescribed R_1 and R_2 satisfy the constraint in Equation (5.4.9). Calculate the value of the constant c according to Equation (5.4.22), and obtain

$$c = 4p_o^2 R_1 R_2 = 4 \times 5^2 \times \frac{16}{5} \times \frac{1}{5} = 64. \tag{5.4.30}$$

From Equation (5.4.1)

$$\alpha = \min\{ |D_n(j\omega)|^2, 0 \le \omega < \infty \} = 17^2 = 289. \tag{5.4.31}$$

Therefore, the inequality $0 < c \le \alpha$ is satisfied.

By Navot's method with the constant c = 64, a tridiagonal matrix can be derived as

$$A_t = \begin{bmatrix} -1 & 2 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -2 & -1 \end{bmatrix}. \tag{5.4.32}$$

From a procedure presented in (20), the realized network is shown in Figure 5.4.2.
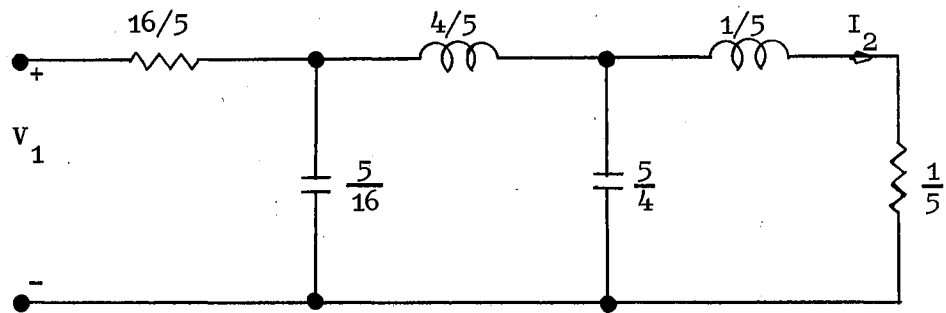
Figure 5.4.2.   Example Synthesized Network

CHAPTER VI

SUMMARY AND SUGGESTIONS FOR FURTHER STUDY

6.1 Summary

A new approach to network synthesis using resistors, capacitors, and nullator-norator pairs has been presented. The synthesis procedure is systematic and straightforward. A theorem is derived, which gives sufficient conditions for networks containing nullators and norators to have a unique solution. These sufficient conditions are given in terms of the topological locations of the nullators and the norators with respect to the other elements in the network as well as their element values. Then the synthesis procedure is developed based upon these conditions.

Synthesis of transfer functions, short-circuit admittance matrices, open-circuit impedance matrices, and hybrid matrices is considered. The synthesis procedure is different from other techniques. The synthesis is by means of state models. In each case, a state model is obtained from the complex frequency domain specification. The synthesis of the A-matrix, which corresponds to a source-free state model, is first considered. Then, the results are extended to the realization of a more general state model, thus obtaining the realization of transfer functions, immittance matrices, and hybrid matrices. Therefore, in the synthesis, the state model plays an important role.

Some of the important aspects of this synthesis are:

(1) The nullators and norators are used directly.

(2) A simple topology exists in the realization.

(3) Several realizations may exist in each case.

(4) The realization can be obtained almost by inspection,
without carrying out the detailed synetheis procedure,
because of the simple topology.

(5) The nullator and the norator appear as a pair with one
vertex in common in the realization, so that it can be
replaced by an ideal transistor.

(6) The realized network is, in general, not minimal in
the sense of the number of elements used.

One of the basic results obtained earlier is used in the low-pass
filter synthesis. Some additional results are derived in the calcula-
tion of a constant c. This constant arises from Navot's method of
deriving a tridiagonal matrix from a strictly Hurwitz polynomial, and
allows for the realization of a doubly terminated low-pass filter using
the state-space approach.

6.2 Suggestions for Further Study

Since the realized network is not minimal in general, further study
will be required to reduce the number of elements used, either resistors
or nullator-norator pairs, or both. This might be accomplished by
judicious selection of the free parameters.

Another area of further study is concerned with the realization of
open-circuit impedance matrices, hybrid matrices, and other types of
matrices without using gyrators.

The practical aspects of the active network synthesis, such as the problems of biasing, stability, sensitivity, etc., are not considered in this thesis. Furthermore, the realized networks have been assumed to use ideal transistors. The synthesis procedure developed needs to be modified to allow for the use of practical transistors. Further study is suggested on these topics. The work done by Murray-Lasso and others (24) (29) should be helpful in these investigations.

BIBLIOGRAPHY

(1) Antoniou, A. "New Gyrator Circuits Obtained by Using Nullors."
       Electronics Letters, Vol. 4, No. 5 (March, 1968), 87-88.

(2) Balabanian, N. Network Synthesis. Englewood Cliffs, N. J.:
       Prentice-Hall, 1958.

(3) Bashkow, T. R. "The A-Matrix, New Network Description." IRE
       Transactions on Circuit Theory, Vol. CT-4 (September, 1957),
       117-119.

(4) Bendik, J. "Equivalent Gyrator Networks With Nullators and
       Norators." IEEE Transactions on Circuit Theory, Vol. CT-14
       (March, 1967), 98.

(5) Blackwell, W. A., and L. L. Grigsby. "Systems of One-Port Linear
       Components Containing Through-Across Type Drivers." Journal
       of the Franklin Institute, Vol. 278, No. 4 (October, 1964),
       237-247.

(6) Braun, J. "Equivalent NIC Networks With Nullators and Norators."
       IEEE Transactions on Circuit Theory, Vol. CT-12 (September,
       1965), 441-442.

(7) Brayshaw, G. S. "Simplified Topological Solutions of Networks
       Containing Nullators and Norators." Electronics Letters,
       Vol. 4, No. 13 (June, 1968), 276.

(8) Bryant, P. R. "The Explicit Form of Bashkow's A-Matrix." IRE
       Transactions on Circuit Theory, Vol. CT-9 (September, 1962),
       303-306.

(9) Carlin, H. J., and D. C. Youla. "Network Synthesis With Negative
       Resistors." Proc. Inst. Radio Engrs., Vol. 49, No. 5 (May,
       1961), 907-920.

(10) Carlin, H. J. "Singular Network Elements." IEEE Transactions on
       Circuit Theory, Vol. CT-11 (March, 1964), 67-72.

(11) Cederbaum, I. "Applications of Matrix Algebra to Network Theory."
       IRE Transactions on Circuit Theory, Vol. CT-6 (May, 1959),
       127-137.

(12) Daniels, R. W. "The Synthesis of Fundamental Gyrators." IEEE
       Transactions on Circuit Theory, Vol. CT-16 (November, 1969),
       543-544.

(13) Daniels, R. W. "A Method for Generating Some Active Circuits." *IEEE Transactions on Circuit Theory*, Vol. CT-18 (May, 1971), 397-399.

(14) Davies, A. C. "Topological Solution of Networks Containing Nullators and Norators." *Electronics Letters*, Vol. 2, No. 3 (March, 1966), 90-92.

(15) _____. "The Significance of Nullators, Norators and Nullors in Active-Network Theory." *The Radio and Electronic Engineer*, Vol. 34 (November, 1967), 259-267.

(16) _____. "Nullator-Norator Equivalent Networks for Controlled Sources." *IEEE Proceedings*, Vol. 55, No. 5 (May, 1967), 722.

(17) Dervisoglu, A. "Bashkow's A-Matrix for Active RLC Networks." *IEEE Transactions on Circuit Theory*, Vol. CT-11 (September, 1964), 404-407.

(18) _____. "The Realization of the A-Matrix of a Certain Class of RLC Networks." *IEEE Transactions on Circuit Theory*, Vol. CT-13 (June, 1966), 164-170.

(19) Fowler, E. "A United State-Space Approach to RLCT Two-Port Transfer Function Synthesis." (Unpub. Ph. D. Thesis, Oklahoma State University, May, 1969.)

(20) Fowler, E., and R. Yarlagadda. "A State-Space Approach to RLCT Two-Port Transfer-Function Synthesis." *IEEE Transactions on Circuit Theory*, Vol. CT-19 (January, 1972), 15-20.

(21) Gantmacher, F. R. *The Theory of Matrices*. New York: Chelsea, 1959.

(22) Kalman, R. E. "Mathematical Description of Linear Dynamical Systems." *SIAM Journal Control*, Vol. 1 (1963), 152-192.

(23) Keen, A. W. "Immittance Synthesis in the Non-Reciprocal Domain." *IEEE Proceedings*, Vol. 110, No. 12 (December, 1963), 2118-2124.

(24) Kozemchak, E. B., and M. A. Murray-Lasso. "Computer-Aided Circuit Design by Singular Imbedding." *The Bell System Technical Journal*, Vol. 48, No. 1 (January, 1969), 275-315.

(25) Marshall, T. G. "Primitive Matrices for Doubly Terminated Ladder Networks." *Proceedings 4th Allerton Conference on Circuit and System Theory*, University of Illinois, Urbana (1966), 935-943.

(26) Martinelli, G. "On the Nullor." *IEEE Proceedings*, Vol. 53, No. 3 (March, 1965), 332.

(27) Mayeda, W. "Necessary and Sufficient Conditions for Realizability of Cut-Set Matrices." IRE Transactions on Circuit Theory, Vol. CT-7 (March, 1960), 79-81.

(28) Mitra, S. K. Analysis and Synthesis of Linear Active Networks. New York: John Wiley & Sons, 1969.

(29) Murray-Lasso, M. A., and W. D. Baker. "Computer Design of Multistage Transistor Bias Circuits." Proceedings of the Fifth Annual Allerton Conference on Circuit and System Theory. Monticello, Illinois (October, 1967).

(30) Myers, B. R. "Nullor Model of the Transistor." IEEE Proceedings, Vol. 53, No. 7 (July, 1965), 758.

(31) Navot, I. "The Synthesis of Certain Subclasses of Tridiagonal Matrices With Prescribed Eigenvalues." SIAM Journal of Applied Mathematics, Vol. 15, No. 2 (March, 1967), 241-251.

(32) Newcomb, R. W. Active Integrated Circuit Synthesis. Englewood Cliffs, N. J.: Prentice-Hall, 1968.

(33) Orchard, H. J., and G. C. Temes. "Filter Design Using Transformed Variables." IEEE Transactions on Circuit Theory, Vol. CT-15 (December, 1968), 385-408.

(34) Rauch, D. J. "On the Realization of Time Domain Model of Real Linear Bielement Systems." (Unpub. Ph. D. Thesis, Michigan State University, 1963.)

(35) Rohrer, R. A. Circuit Theory: An Introduction to the State Variable Approach. New York: McGraw-Hill, 1970.

(36) Silverman, L. M. "Reciprocal Realization of A Matrices." IEEE Transactions on Circuit Theory, Vol. CT-16 (May, 1969), 252-253.

(37) Weinberg, L. Network Analysis and Synthesis. New York: McGraw-Hill, 1962.

(38) Wilkinson, J. H. The Algebraic Eigenvalue Problem. Oxford: Clarendon Press, 1965.

(39) Yarlagadda, R. "An Application of Tridiagonal Matrices to Network Synthesis." SIAM Journal of Applied Mathematics, Vol. 16, No. 6 (November, 1968), 1146-1162.

(40) Youla, D. C., and P. Tissi. "N-Port Synthesis via Reactance Extraction - Part I." 1966 IEEE International Convention Record, Part 7, 183-208.

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