STABILITY CONCEPTS IN DYNAMICAL SYSTEMS

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CHAPTER I

INTRODUCTION

The evolution of the theory of dynamical systems had its origins in the study of systems of ordinary differential equations. While no one individual can be credited as the founder of this theory, it is generally agreed that G. D. Birkhoff, in [15], provided the impetus that established a systematic approach to the theory. Prior to Birkhoff's work in the 1920's, the investigation of the functional and topological properties of solutions of systems of ordinary differential equations had been initiated by the French mathematician Henri Poincare and the American mathematician E. H. Moore. In the early 1930's, A. A. Markov and H. Whitney independently gave the first abstract definitions of a dynamical system. Nemytskii and Stepanov's book, "Qualitative Theory of Differential Equations," published in 1947, renewed interest in dynamical systems and provided a basis for the modern development of the theory.

In the early 1890's, Liapunov set forth a precise definition of stability, a concept that had been investigated by Lagrange and Dirichlet. During the first half of this century attention centered on developing Liapunov's methods for stability as related to systems of differential equations. Researchers then started to examine stability, in fact all of dynamical systems, in more general terms. This led to abstracting to more general topological spaces and finally to what many feel is a separate field of mathematical study.

In the 1940's, Bebutov introduced several new types of stability, the most notable one being uniform stability. In the late 1950's, Zubov, Bass, and Lefschetz all contributed to the systematic development of stability theory. It was Zubov, in [32], who gave the first complete development of the theory. During this same period, Ura introduced prolongations and characterized stability of compact sets in terms of prolongations.

Since the 1950's, stability theory has been incorporated into most of the developments in dynamical systems. These developments are surveyed in [13], [14], and [19]. In [14], Bhatia and Szego present an extensive survey of results in stability theory. Their work, as well as the work of many others, is restricted to metric spaces. A more general setting is used by Bhatia and Hajek, in [11], but the scope of their work is restricted to several particular types of dynamical systems.

The purpose of this dissertation is to investigate the concepts of stability and para-stability. Chapter II presents the basic concepts of dynamical systems theory that are necessary for studying stability theory. It is hoped that Chapter II will not only lay the groundwork for the rest of the dissertation, but will invite the reader to investigate other areas of dynamical systems theory.

In Chapter III, stability, in the sense of Liapunov, is presented. Characterizations of stability are given, including Ura's Theorem which is proven by using *-stability. Chapter III concludes with Zubov's stability criterion and relative stability.

In Chapter IV, the relationship between stability and para-stability is examined. The concept of para-stability was introduced by Hájek in 1972, see [22]. The relationship of Liapunov and para-Liapunov functions to stability and para-stability is also presented.

CHAPTER II

PRELIMINARY CONCEPTS

Dynamical Systems

Throughout this paper, R, R^+, and R^- will denote the real numbers, nonnegative real numbers, and nonpositive real numbers, respectively. The topological properties that are necessary for this chapter are elementary, and the reader is referred to [18], [24], and [25].

Definition 2.1: The pair (X,π), where X is a topological space and π is a mapping from $X \times R$ into X, is a dynamical system if and only if the following conditions hold:

(i) Identity axiom: $\pi(x,0) = x$, for all $x \in X$;

(ii) Homomorphism axiom: $\pi(\pi(x,t),s) = \pi(x,t+s)$, for all $x \in X$ and $t,s \in R$; and

(iii) Continuity axiom: the mapping π is continuous on $X \times R$.

The topological space X is called the phase space, and π is called the phase mapping. A dynamical system (X,π) is often referred to as a continuous flow or simply as a flow. Unless stated otherwise, it will always be assumed that the phase space X is a Hausdorff space. All sets and points shall be assumed to be subsets and elements of X, respectively. Unless stated otherwise, all sets will be assumed to be nonempty. For notational convenience, $\pi(x,t)$ will be denoted by xt. Thus, axiom (ii) of Definition 2.1 would be written as $(xt)s = x(t+s)$, for all $x \in X$ and $t,s \in R$. For $T \subset R$ and all $x \in X$, $\pi(x,T)$ will be denoted by xT. In particular, for $M \subset X$ and $T \subset R$, $\pi(M,T)$ will be denoted by $\bigcup_{x \in M} xT$.

Definition 2.2: For each $x \in X$, the sets $C(x) = xR = \{xt: t \in R\}$, $C^+(x) = xR^+ = \{xt: t \in R^+\}$, and $C^-(x) = xR^- = \{xt: t \in R^-\}$ are called, respectively, the trajectory of x, the positive trajectory of x, and the negative trajectory of x. The trajectory $C(x)$ is also called the orbit of x and $C^+(x)$ and $C^-(x)$ are called the semi-trajectories or semi-orbits of x.

If $M \subset X$, then $C(M) = \bigcup_{x \in M} C(x)$, $C^+(M) = \bigcup_{x \in M} C^+(x)$, and $C^-(M) = \bigcup_{x \in M} C^-(x)$.

The following two properties of $C(x)$ are immediate from Definition 2.2.

Proposition 2.3: For $x \in X$, $C(x) = C^+(x) \cup C^-(x)$.

Proposition 2.4: For $x \in X$ and any $t \in R$, $C(x) = C(xt)$.

It can be easily verified that Proposition 2.4 does not hold for $C^+(x)$ and $C^-(x)$, see Example 2.1.

Definition 2.5: If $C(x) = \{x\}$, then x is a critical or rest point. If $C(x) \neq \{x\}$ but there exists a $t \in R^+$ such that $xt = x$, then x is a periodic point.

<u>Example</u> 2.1: Let the dynamical system (R^2,π) be defined by the following system of differential equations (R^2 denotes the Euclidean plane with the usual topology):

$$\dot{x} = -x$$
$$\dot{y} = y$$

(The phase space is shown in Figure 1).

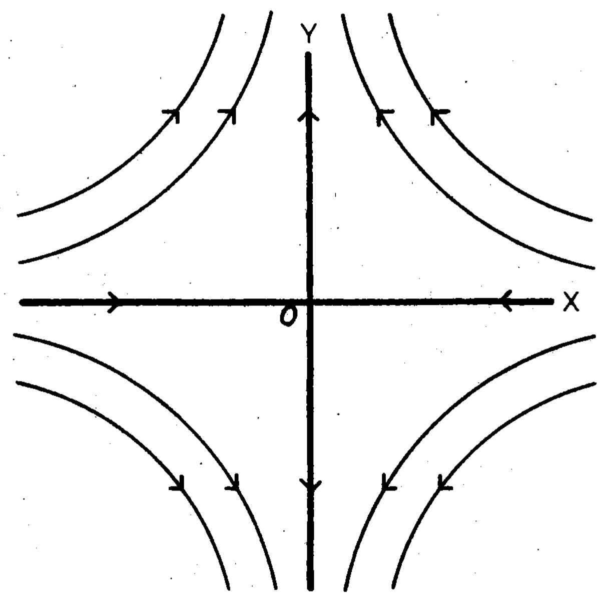

Figure 1. The Phase Space of Example 2.1.

The point (0,0) is the only critical point. For any point $P = (x,0)$, with $x \neq 0$

$$C^{+}(P) = \begin{cases} \{(x',0): \ 0 < x' \leq x\}, & \text{if } x > 0 \\ \{(x',0): \ x \leq x' < 0\}, & \text{if } x < 0. \end{cases}$$

A similar statement holds for $Q = (0,y)$, $y \neq 0$. For any point $S = (x,y)$, with $x \neq 0$, $y \neq 0$,

$$C^{+}(S) = \{(xe^{-t}, ye^{t}): \ t \in R^{+}\}.$$

Remark 2.6: In Example 2.1, the positive trajectories were the only ones given; and in Figure 1, the arrows indicate the positive direction of the trajectories. This procedure will be used throughout the paper in regard to examples. Definitions will contain both the positive and negative versions, and in most cases, the bilateral version will be given. All theorems and propositions that are true for the positive version are also true for the negative version. Since the negative version is the dual of the positive version, only the positive version will be proven. The bilateral version will generally be given for theorems and propositions, but will not be proven.

Invariance

Definition 2.7: A subset M of X is invariant if and only if $C(M) = M$. If $M \subset X$, then M is positive (negative) invariant if and only if $C^{+}(M) = M$ $(C^{-}(M) = M)$.

Proposition 2.8: If M is a subset of X, then the following are equivalent:

(a) M is invariant;

(b) $C(M) \subset M$;

(c) $C(x) \subset M$, for all $x \in M$; and

(d) $M = \bigcup_{x\in M} C(x)$.

Proof: (a) implies (b): Follows from Definition 2.7.

(b) implies (c): Since $C(M) = \bigcup_{x\in M} C(x)$ and $C(M) \subset M$, $C(x) \subset M$ for all $x \in M$. (c) implies (d): If $C(x) \subset M$ for all $x \in M$, then $\bigcup_{x\in M} C(x) \subset M$. Let $x \in M$. Then, $x \in C(x)$ and $M \subset \bigcup_{x\in M} C(x)$. Therefore, $M = \bigcup_{x\in M} C(x)$. (d) implies (a): Since $M = \bigcup_{x\in M} C(x)$ and $C(M) = \bigcup_{x\in M} C(x)$, $M = C(M)$. This implies that M is invariant. This completes the proof.

Using a similar proof, Proposition 2.8 is true for positive (negative) invariance, where $C(M)$ is replaced by $C^+(M)$ $(C^-(M))$.

Examples of invariant sets will be presented in Examples 2.5, 2.6, 2.7, and 2.8. These examples not only discuss invariance, but relate the concept of invariance to the concepts of k-invariance and d-invariance.

Proposition 2.9: For each $x \in X$, $C(x)$, $C^+(x)$, and $C^-(x)$ are invariant, positive invariant, and negative invariant, respectively.

Proof: Follows from Definition 2.7.

Proposition 2.10: A subset M of X is positive (bilaterally) invariant if and only if $X - M$ is negative (bilaterally) invariant.

Proof: First, assume that M is positive invariant and let $y \in C^-(X - M)$. Then, $y = xt$ for some $x \in X - M$ and $t \in R^-$. Suppose xt is not an element of $X - M$. Thus, $xt \in M$ and the positive invariance of M implies that $(xt)(-t) \varepsilon M$. But $(xt)(-t) = x(t + {}^{-}t) = x$. Thus, $x \varepsilon M$, and this contradicts $x \varepsilon X - M$. Therefore, $y = xt \varepsilon X - M$ and $C^{-}(X - M) \subset (X - M)$. Proposition 2.8 implies that $X - M$ is negative invariant.

Conversely, let $X - M$ be negative invariant. For any $y \varepsilon C^{+}(M)$, $y = xt$ for some $x \varepsilon M$ and $t \varepsilon R^{+}$. If $xt \notin M$, then $xt \varepsilon X - M$. This implies that $(xt)(-t) = x \varepsilon X - M$ as $X - M$ is negative invariant. This contradicts $x \varepsilon M$. Thus, $y = xt \varepsilon M$ and $C^{+}(M) \subset M$. Therefore, Proposition 2.8 implies that M is positive invariant, and the proof is complete.

<u>Proposition</u> 2.11: If (M_i) is a family of positive (bilaterally) invariant sets, then $\bigcup_i M_i$ and $\bigcap_i M_i$ are also positive (bilaterally) invariant.

<u>Proof</u>: Let $x \varepsilon C^{+}(\bigcup_i M_i)$. Then, $x = mt$ for some $m \varepsilon \bigcup_i M_i$ and $t \varepsilon R^{+}$. Thus, $m \varepsilon M_i$ for some i and $x = mt \varepsilon C^{+}(M_i)$. The positive invariance of M_i implies that $C^{+}(M_i) = M_i$. Therefore, $x \varepsilon M_i$. Thus, $x \varepsilon \bigcup_i M_i$ and $C^{+}(\bigcup_i M_i) \subset \bigcup_i M_i$. Proposition 2.8 then implies that $\bigcup_i M_i$ is positive invariant.

To prove $\bigcap_i M_i$ is positive invariant, consider $X - \bigcap_i M_i = \bigcup_i (X - M_i)$. Since M_i is positive invariant, Proposition 2.10 implies that $X - M_i$ is negative invariant. Thus, by the first part of this proof, $\bigcup_i (X - M_i) = X - \bigcap_i M_i$ is negative invariant. Proposition 2.10 implies that $\bigcap_i M_i$ is positive invariant. This completes the proof.

The boundary, interior, and closure of M will be denoted by ∂M, M°, and $\overline{M}$, respectively. The derived set of M will be denoted by M'. For the definition and basic properties of a net, the reader is referred to [25, pp. 65-66]. Nets and sequences will be denoted by (x_i). The context in which (x_i) appears will clarify if (x_i) is being used to denote a net or a sequence. A subnet (subsequence) of (x_i) will be written as $(x_{i,n})$.

Proposition 2.12: If a subset M of X is positive (bilaterally) invariant, then $\overline{M}$ and M° are positive (bilaterally) invariant.

Proof: Suppose $C^+(\overline{M})$ is not a subset of $\overline{M}$. Then, there exists $m \varepsilon \overline{M}$ and $t \varepsilon R^+$ such that $mt \notin \overline{M}$. Since $\overline{M} = M \cup M'$, $m \varepsilon M'$; for if m is in M, then $mt \varepsilon M \subset \overline{M}$ as M is positive invariant. Since $m \varepsilon M'$, there exists a net (m_i) in M such that (m_i) converges to m. Thus, $(m_i t)$ converges to mt. Since M is positive invariant, each $m_i t$ is in M, and this implies that $mt \varepsilon \overline{M}$. This contradicts mt not being an element of $\overline{M}$. Thus, $C^+(\overline{M})$ is a subset of $\overline{M}$, and Proposition 2.8 implies that $\overline{M}$ is positive invariant.

To show the interior of M is positive invariant, consider $M^\circ = X - (\overline{X - M})$. Since M is positive invariant, Proposition 2.10 implies that $X - M$ is negative invariant. Thus, by the first part of this proposition, $\overline{X - M}$ is negative invariant. Proposition 2.10 then implies that $X - (\overline{X - M})$ is positive invariant. This completes the proof.

Proposition 2.13: If M is positive (bilaterally) invariant, then the boundary of M is positive (bilaterally) invariant.

Proof: By definition, $\partial M = \overline{M} \cap (\overline{X - M})$. The positive invariance of M implies that $\overline{M}$ and $\overline{X - M}$ are positive invariant, see Proposition 2.12. Then Proposition 2.11 implies that $\overline{M} \cap (\overline{X - M}) = \partial M$ is positive invariant. This completes the proof.

The converse of Proposition 2.13 is not true as shall be shown in Example 2.2. Proposition 2.17 shows that if M is either open or closed, then the converse of Proposition 2.13 is true if ∂M is invariant. Examples 2.2 and 2.3 also show the necessity of M being either open or closed.

Definition 2.14: For each $x \in X$, the x-motion, π_x, is the mapping $\pi_x: R \to X$ defined by $\pi_x(t) = xt$, for all $t \in R$.

Proposition 2.15: For any $x \in X$, the x-motion, π_x, is continuous.

Proof: Immediate from Definition 2.15 and axiom (iii) of Definition 2.1.

Proposition 2.16: For each $x \in X$, $C(x)$ $(C^+(x), C^-(x))$ is connected.

Proof: Since $\pi_x: R \to X = \{xt: t \in R\} = C(x)$ and π_x is continuous, $C(x)$ is the continuous image of the connected set R. Thus, $C(x)$ is connected as claimed.

Proposition 2.17: If M is open or closed and ∂M is invariant, then M is invariant.

Proof: Assume that M is open, and ∂M is invariant. Suppose M is not invariant. Then, there exists $m \in M$ and $t_1 \in R$ such that $mt_1 \notin M$. Without loss of generality let t_1 be in R^+. Then

$C^+(m) \cap \partial M = \emptyset$. For if not, then there exists t in R^+ such that mt is an element of ∂M. This implies that $C(mt)$ is a subset of ∂M since ∂M is invariant. Thus, $(mt)(-t) = m(t + {}^-t) = m$ is an element of ∂M. This contradicts m being in M and M being open. Thus, $C^+(m) \cap \partial M = \emptyset$. Further, $C^+(m) \cap (X - \overline{M}) = \emptyset$. For suppose that there exists a t in R^+ such that mt is an element of $X - \overline{M}$. Since $C^+(m)$ is connected and $C^+(m) \cap \partial M = \emptyset$, C can be written as the union of the two sets $A = \{mt' : mt' \in M \text{ and } t' \in [0,t]\}$ and $B = \{mt' : mt' \in X - \overline{M} \text{ and } t' \in [0,t]\}$. Clearly, $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$. Thus, C can be written as the union of two separated sets, and this contradicts C being connected. Thus, no point of $C^+(m)$ can lie on ∂M or in $X - \overline{M}$. Therefore, $C^+(M)$ is contained in M, and Proposition 2.8 implies that M is invariant.

Assume that M is closed and ∂M is invariant. Since the interior of M is open, the proof of the first part of the proposition implies that the interior of M is invariant. Proposition 2.12 then implies that the closure of the interior of M, which is M, is also invariant. This completes the proof.

<u>Example</u> 2.2: Consider the flow defined by the system of differential equations

$$\dot{x} = y$$

$$\dot{y} = -x$$

(see Figure 2). In Figure 2, let $x \in \gamma$ and let M be the union of $\{x\}$ and the interior of the disk bounded by γ. The boundary of M is γ which is positive invariant. By the definition of M, M is neither open nor closed. Since $C^+(x) = \gamma \not\subset M$, M is not positive invariant

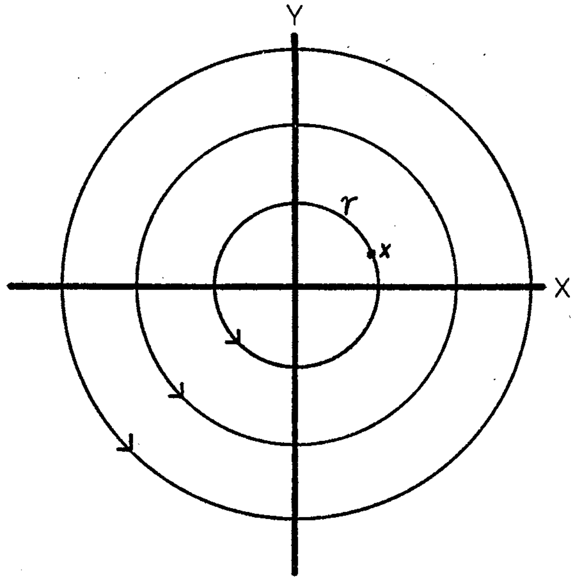

Figure 2. The Phase Space of Example 2.2.

and, hence, cannot be invariant. Thus, the condition that M is either open or closed cannot be omitted from Proposition 2.17.

Figure 3 is a portion of Figure 1 with M being the shaded region. The set M is closed, but ∂M is not invariant. For any $x \in M$ such that x is not on the x-axis, $C^+(x) \not\subset M$, and M is not invariant. By considering the interior of M, M° is a noninvariant open set with noninvariant boundary. Thus, the invariance of the boundary of M cannot be omitted from Proposition 2.17.

<u>Proposition</u> 2.18: If M is positive invariant, then each of its components is positive invariant.

<u>Proof</u>: Let A be a component of M. For $x \in A$, $C^+(x)$ is connected, see Proposition 2.16, and is a subset of M as M is positive invariant. Then, $A \cup C^+(x)$ is connected since $A \cap C^+(x) \neq \emptyset$.

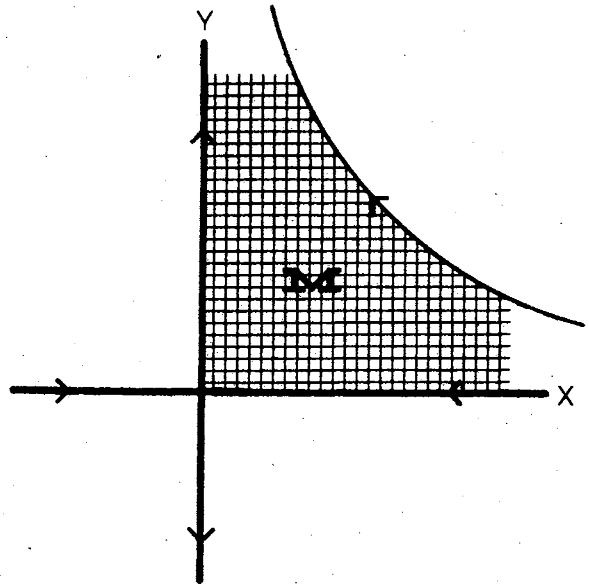

Figure 3. The Set M of Example 2.2.

Since A is a component of M, it follows that $C^+(x) \subset A$. Therefore, $\bigcup_{x \varepsilon A} C^+(x) \subset A$, and this implies that $C^+(A) \subset A$. Thus, Proposition 2.8 implies that A is positive invariant, and the proof is complete.

Proposition 2.19: If M is positive invariant, $M = M_1 \cup M_2$, and $\overline{M}_1 \cap M_2 = \emptyset = M_1 \cap \overline{M}_2$, then M_1 and M_2 are each positive invariant.

Proof: By definition

$$C^+(M_1) = \bigcup_{x \varepsilon M_1} C^+(x).$$

Proposition 2.16 implies that $C^+(x)$ is connected. Therefore, for $x \varepsilon M_1$, $C^+(x) \subset M_1$, and M_1 is positive invariant by Proposition 2.8.

An analogous proof shows that M_2 is positive invariant. This completes the proof.

Definition 2.20: A subset M of X is relatively compact if and only if M is contained in a compact set.

Proposition 2.21: The phase space X contains a compact invariant set if and only if it contains a relatively compact semi-trajectory.

Proof: Assume that X contains a compact invariant set M. For any element x of M, the invariance of M implies $C^+(x)$ is contained in M. Since M is compact and $\overline{C^+(x)}$ is a subset of $\overline{M} = M$, $\overline{C^+(x)}$ is compact. Thus, $C^+(x)$ is the desired relatively compact semi-trajectory.

Now let $C^+(x)$ be a relatively compact semi-trajectory of X. Thus, there exists a compact set A containing $C^+(x)$. The compactness of A and $\overline{C^+(x)} \subset \overline{A} = A$ implies that $\overline{C^+(x)}$ is compact. Consider the sequence (xn), $n = 1,2,3, \ldots,$ in $\overline{C^+(x)}$. Since $\overline{C^+(x)}$ is compact, there exists a subsequence (xn_k) of (xn) such that (xn_k) converges to a point y in $\overline{C^+(x)}$. The positive invariance of $\overline{C^+(x)}$ implies that $C^+(y) \subset \overline{C^+(x)}$. Let $t \in R^-$. Then $-t \in R^+$. Thus, there exists an integer N such that for all $i \geq N$, $n_{k,i} \geq -t$. consider the sequence $(x(n_{k,i} + t))$, where $(\underline{n_{k,i}} + t) \geq 0$. Thus, $(x(n_{k,i} + t)) = ((xn_{k,i})t)$ is a sequence in $\overline{C^+(x)}$ that converges to yt. Since $\overline{C^+(x)}$ is compact, yt is an element of $\overline{C^+(x)}$. Thus, $C^-(y)$ is a subset of $\overline{C^+(x)}$. Also, $C^+(y)$ is a subset of $\overline{C^+(x)}$ since y is in $\overline{C^+(x)}$. Thus, $C(y) = C^+(y) \cup C^-(y)$ is contained in $\overline{C^+(x)}$. This implies that $\overline{C(y)}$ is a subset of $\overline{C^+(x)}$, and $\overline{C(y)}$ is compact as $\overline{C^+(x)}$ is compact. Since $C(y)$ is invariant, Proposition 2.12 implies that $\overline{C(y)}$ is invariant. Thus, $\overline{C(y)}$ is the desired compact invariant subset of X. This completes the proof.

Actually, more has been proven. It has been shown that any relatively compact semi-trajectory contains a compact invariant set.

Limit Sets

Definition 2.22: For each $x \varepsilon X$, $K(x) = \overline{C(x)}$, $K^+(x) = \overline{C^+(x)}$, and $K^-(x) = \overline{C^-(x)}$.

Definition 2.23: For each $x \varepsilon X$, the positive (negative) limit set of x, $L^+(x)$ $(L^-(x))$, is given by

$$L^+(x) = \{y \varepsilon X: (xt_i) \rightarrow y \text{ for some net } (t_i) \rightarrow +\infty\}$$

$$(L^-(x) = \{y \varepsilon X: (xt_i) \rightarrow y \text{ for some net } (t_i) \rightarrow -\infty\}).$$

The limit set of x is $L(x) = L^+(x) \cup L^-(x)$.

Propositions 2.24 through 2.27 show the relationships between $C^+(x)$, $K^+(x)$, and $L^+(x)$.

Proposition 2.24: For any $x \varepsilon X$, $K^+(x) = C^+(x) \cup L^+(x)$ $(K^-(x) = C^-(x) \cup L^-(x))$.

Proof: To show that $C^+(x) \cup L^+(x) \subset K^+(x)$, first note that $C^+(x) \subset K^+(x) = \overline{C^+(x)}$. If $y \varepsilon L^+(x)$, then there exists a net (t_i) converging to $+\infty$ such that (xt_i) converges to y. For each i, $xt_i \varepsilon C^+(x)$. The positive invariance of $C^+(x)$ implies that

$(xt_i) \subset C^+(x)$. Thus, $y \in \overline{C^+(x)} = K^+(x)$ and $L^+(x) \subset K^+(x)$. Hence, $C^+(x) \cup L^+(x) \subset K^+(x)$.

To show that $K^+(x) \subset C^+(x) \cup L^+(x)$, let $y \in K^+(x)$. If $y \in C^+(x)$, then the result follows. Suppose that $y \notin C^+(x)$. Then, y cannot be written as xt' for $t' \in R^+$. But $y \in \overline{C^+(x)}$ implies that there exists a net (x_i) in $C^+(x)$ that converges to y. For each i, $x_i = xt_i$ for some $t_i \in R^+$. If the net (t_i) does not converge to $+\infty$, then y is expressible as $y = xt$ for some $t \in R^+$. But this contradicts $y \notin C^+(x)$. Thus, the net (t_i) does converge to $+\infty$ and, hence, $y \in L^+(x)$. Thus, $K^+(x) \subset C^+(x) \cup L^+(x)$, and the result follows.

<u>Proposition</u> 2.25: For any $x \in X$, $K(x) = C(x) \cup L^+(x) \cup L^-(x)$.

<u>Proof</u>: Since $C(x) = C^+(x) \cup C^-(x)$, Proposition 2.24 implies that it will suffice to show $K(x) = K^+(x) \cup K^-(x)$. This equality follows from $K(x) = \overline{C(x)} = \overline{C^+(x) \cup C^-(x)} = \overline{C^+(x)} \cup \overline{C^-(x)} = K^+(x) \cup K^-(x)$. This completes the proof.

<u>Example</u> 2.3: Consider the dynamical system given in Example 2.1. Let P and S be as in Example 2.1. Then,

$$L^+(P) = \{(0,0)\},$$

$$K^+(P) = \begin{cases} \{(x',0): \ 0 \leq x' \leq x\}, & \text{if } x > 0 \\ \{(x',0): \ x \leq x' \leq 0\}, & \text{if } x < 0, \end{cases}$$

$$L^+(S) = \emptyset,$$

and

$$K^+(S) = C^+(S).$$

Proposition 2.26: If $x \varepsilon X$, then $L^+(x) = L^+(xt)$ for all $t \varepsilon R$.

Proof: Let $y \varepsilon L^+(x)$. Then, there exists a net (t_i) converging to $+\infty$ such that (xt_i) converges to y. For each t_i there exists a t_j such that $t_i = t + t_j$ and the net (t_j) converges to $+\infty$. Thus, $(xt_i) = (x(t + t_j)) = ((xt)t_j)$ and $((xt)t_j)$ converges to y. Thus, $y \varepsilon L^+(xt)$ and $L^+(x) \subset L^+(xt)$.

To show $L^+(xt) \subset L^+(x)$, let $y \varepsilon L^+(xt)$. Then, there exists a net (t_i) converging to $+\infty$ such that $((xt)t_i)$ converges to y. But, $(xt)t_i = x(t + t_i)$, and the net $(t + t_i)$ converges to $+\infty$. This implies that $y \varepsilon L^+(x)$. Therefore, $L^+(xt) \subset L^+(x)$. Hence, $L^+(x) = L^+(xt)$, and the proof is complete.

Proposition 2.27: $L^+(x) = \bigcap_{t \varepsilon R^+} K^+(xt) = \bigcap_{t \varepsilon R} K^+(xt)$.

Proof: If $y \varepsilon L^+(x)$, then Proposition 2.26 implies that $y \varepsilon L^+(xt)$ for all $t \varepsilon R^+$. Since $L^+(xt) \subset K^+(xt)$, $y \varepsilon K^+(xt)$ for all $t \varepsilon R^+$. This implies that $y \varepsilon \cap\{K^+(xt)\colon t \varepsilon R^+\}$. Thus, $L^+(x)$ is contained in $\bigcap_{t \varepsilon R^+} K^+(xt)$.

If $y \varepsilon \bigcap_{t \varepsilon R^+} K^+(xt)$, then $y \varepsilon K^+(xt)$ for all $t \varepsilon R^+$. Since $K^+(xt) = C^+(xt) \cup L^+(xt)$ and $L^+(xt) = L^+(x)$, $K^+(xt) = C^+(xt) \cup L^+(x)$. If $y \varepsilon L^+(x)$, then $\bigcap_{t \varepsilon R^+} K^+(xt) \subset L^+(x)$. If $y \varepsilon C^+(xt)$ for all $t \varepsilon R^+$, then there exists a net (t_i) converging to $+\infty$ such that $y \varepsilon C^+(xt_i)$ for each t_i. For each i, $y = (xt_i)t_j$ for some

$t_j \geq 0$. Thus, $y = xt_i'$ where $t_i' = t_i + t_j$. Since (t_i') converges to $+\infty$ and (xt_i') converges to y, $y \in L^+(x)$. Thus, $\bigcap_{t\in R^+} K^+(xt) \subset L^+(x)$. Therefore, $\bigcap_{t\in R^+} K^+(xt) = L^+(x)$.

Now let $y \in \bigcap_{t\in R^+} K^+(xt)$. Then, $y \in K^+(xt)$ for all $t \in R^+$. Since $K^+(xt) = C^+(xt) \cup L^+(xt)$, then $y \in C^+(xt)$ or $y \in L^+(xt)$. If $y \in C^+(xt)$, then $y \in C(xt)$ for all $t \in R^+$. Since $C(x) = C(xt)$ for all $t \in R$, $y \in C(xt)$ for all $t \in R$. If $y \in L^+(xt)$ for every $t \in R^+$, then $y \in L^+(xt)$ for all $t \in R$, see Proposition 2.26. Thus, $y \in K^+(xt)$ for all $t \in R$ and $\bigcap_{t\in R^+} K^+(xt) \subset \bigcap_{t\in R} K^+(xt)$. Clearly, $\bigcap_{t\in R} K^+(xt) \subset \bigcap_{t\in R^+} K^+(xt)$. Thus, $\bigcap_{t\in R} K^+(xt) = \bigcap_{t\in R^+} K^+(xt)$. This completes the proof.

Proposition 2.28: For x in X, $K^+(x)$ is a closed positive invariant set, and $L^+(x)$ is closed and invariant.

Proof: Since $K^+(x) = \overline{C^+(x)}$, $K^+(x)$ is closed. Since $C^+(x)$ is positive invariant, Proposition 2.12 implies that $\overline{C^+(x)}$ is positive invariant. Thus, $K^+(x)$ is closed and positive invariant.

That $L^+(x)$ is closed follows from $L^+(x) = \bigcap_{t\in R^+} K^+(xt)$ and each $K^+(xt)$ being closed. To show that $L^+(x)$ is invariant, let $z \in C(L^+(x))$. Thus, $z = yt$ where $y \in L^+(x)$ and $t \in R$. Since $y \in L^+(x)$, there exists a net (t_i) converging to $+\infty$ such that (xt_i) converges to y. Thus, $(x(t_i + t)) = ((xt_i)t)$ converges to yt. Since (t_i) converges to $+\infty$, $(t_i + t)$ converges to $+\infty$. Thus, $yt \in L^+(x)$. Therefore, $C(L^+(x)) \subset L^+(x)$, and Proposition 2.8 implies that $L^+(x)$ is invariant. This completes the proof.

Prolongation

Definition 2.29: For each $x \in X$, the positive (negative) prolongation of x, $D^+(x)$ $(D^-(x))$, is given by

$$D^+(x) = \{y \in X: (x_i t_i) \to y \text{ for some net } (x_i) \to x \text{ and numbers } t_i \geq 0\}$$

$$(D^-(x) = \{y \in X: (x_i t_i) \to y \text{ for some net } (x_i) \to x \text{ and numbers } t_i \leq 0\}).$$

The prolongation of x is $D(x) = D^+(x) \cup D^-(x)$.

Definition 2.30: For each $x \in X$, the positive (negative) prolongational limit set of x, $J^+(x)$ $(J^-(x))$, is given by

$$J^+(x) = \{y \in X: (x_i t_i) \to y \text{ for some nets } (x_i) \to x \text{ and } (t_i) \to +\infty\}$$

$$(J^-(x) = \{y \in X: (x_i t_i) \to y \text{ for some nets } (x_i) \to x \text{ and } (t_i) \to -\infty\}).$$

The prolongational limit set of x is $J(x) = J^+(x) \cup J^-(x)$.

Example 2.4: Consider the flow defined in Example 2.1. Let P and S be as in that example. Then,

$$J^+(P) = \{(0,y): y \in R\},$$

$$D^+(P) = \begin{cases} \{(x',0): 0 < x' \leq x\} \cup \{(0,y): y \in R\}, & \text{if } x > 0 \\ \{(x',0): x \leq x' < 0\} \cup \{(0,y): y \in R\}, & \text{if } x < 0, \end{cases}$$

$$J^+(S) = \emptyset,$$

and

$$D^+(S) = C^+(S).$$

Propositions 2.31 through 2.37 describe some of the properties of $D^+(x)$ and $J^+(x)$ as well as showing the relationships between $C^+(x)$, $K^+(x)$, $L^+(x)$, $D^+(x)$, and $J^+(x)$.

<u>Proposition</u> 2.31: For x,y in X, $y \varepsilon D^+(x)$ if and only if $x \varepsilon D^-(y)$.

<u>Proof</u>: Let $y \varepsilon D^+(x)$. Then, there exists a net (x_i) converging to x and numbers $t_i \geq 0$ such that $(x_i t_i)$ converges to y. Thus, $-t_i \leq 0$ and $((x_i t_i)(-t_i)) = (x_i)$ converges to x. Thus, $x \varepsilon D^-(y)$. Similarly, if $x \varepsilon D^-(y)$, then $y \varepsilon D^+(x)$. Thus, the proof is complete.

<u>Proposition</u> 2.32: For x,y in X, $y \varepsilon J^+(x)$ if and only if $x \varepsilon J^-(y)$.

<u>Proof</u>: Let $y \varepsilon J^+(x)$. Then, there exist nets (x_i) and (t_i) such that (x_i) converges to x, (t_i) converges to $+\infty$, and $(x_i t_i)$ converges to y. Thus, $(-t_i)$ converges to $-\infty$. Consider $((x_i t_i)(-t_i))$. This net converges to x which implies that $x \varepsilon J^-(y)$. A similar argument shows that if $x \varepsilon J^-(y)$, then $y \varepsilon J^+(x)$. This completes the proof.

<u>Proposition</u> 2.33: For M a subset of X, $K^+(M) \subset D^+(M)$ and $L^+(M) \subset J^+(M)$ ($K^-(M) \subset D^-(M)$ and $L^-(M) \subset J^-(M)$).

<u>Proof</u>: Let $x \varepsilon K^+(M)$. Then, $x \varepsilon K^+(m) = C^+(m) \cup L^+(m)$ for some $m \varepsilon M$. If $x \varepsilon C^+(m)$, then $x = mt$ for some $t \geq 0$. Define the nets (m_i) and (t_i) by $m_i = m$ and $t_i = t$ for each i. Thus, (m_i) converges to m and $(m_i t_i)$ converges to $mt = x$. This implies that

$x \in D^+(m) \subset D^+(M)$. If $x \in L^+(m)$, then there exists a net (t_i) converging to $+\infty$ such that (mt_i) converges to x. Since (t_i) converges to $+\infty$, there exists an integer n such that for all $i > n$, $t_i \geq 0$. Define the net (m_i) by $m_i = m$ for each i. Thus, for all $i > n$, $(m_i t_i)$ converges to x. This implies that $x \in D^+(m) \subset D^+(M)$. Hence, $K^+(M) \subset D^+(M)$.

If $y \in L^+(M)$, then there exists a net (t_i) converging to $+\infty$ such that (xt_i) converges to y for some $x \in M$. Define the net (x_i) by $x_i = x$ for each i. Thus, (x_i) converges to x, (t_i) converges to $+\infty$, and $(x_i t_i)$ converges to y. Thus, $y \in J^+(M)$ and $L^+(M) \subset J^+(M)$. This completes the proof.

<u>Proposition</u> 2.34: $D^+(x) = C^+(x) \cup J^+(x)$, $(D^-(x) = C^-(x) \cup J^-(x))$.

<u>Proof</u>: If $y \in D^+(x)$, then there exists a net (x_i) converging to x and numbers $t_i \geq 0$ such that $(x_i t_i)$ converges to y. If $y \in C^+(x)$, then $D^+(x) \subset C^+(x) \cup J^+(x)$. If $y \notin C^+(x)$, then there does not exist a $t \geq 0$ such that $y = xt$. If the numbers $t_i \geq 0$ do not converge to $+\infty$, then y can be written as xt' for some $t' \geq 0$. This contradiction shows that the numbers t_i do converge to $+\infty$. Thus, (x_i) converges to x, (t_i) converges to $+\infty$, and $(x_i t_i)$ converges to y. This implies that $y \in J^+(x)$. Hence, $y \in C^+(x) \cup J^+(x)$ and $D^+(x) \subset C^+(x) \cup J^+(x)$.

Now let $y \in C^+(x) \cup J^+(x)$. If $y \in C^+(x)$, then $y = xt$ for some $t \geq 0$. Define the net (x_i) by $x_i = x$ for each i. For each i, let $t_i = t$. Hence, $t_i \geq 0$ for each i. Thus, $(x_i t_i) = (xt)$ converges to $xt = y$ and $y \in D^+(x)$. If $y \in J^+(x)$, then there exist nets (x_i) and (t_i) such that (x_i) converges to x, (t_i) converges to $+\infty$, and $(x_i t_i)$ converges to y. Since (t_i) converges to $+\infty$, there exists an integer n such that for all $i > n$, $t_i \geq 0$. Thus, $y \in D^+(x)$ and $J^+(x) \subset D^+(x)$. Therefore, $C^+(x) \cup J^+(x) \subset D^+(x)$, and the proof is complete.

<u>Proposition</u> 2.35: For each x in X, $D^+(x) = \cap\{\overline{UR^+}: U \in N(x)\}$, where $N(x)$ is the neighborhood system of x.

<u>Proof</u>: To show that $D^+(x) \subset \cap\overline{UR^+}$, let $y \in D^+(x)$. Then, there exists a net (x_i) converging to x and numbers $t_i \geq 0$ such that $(x_i t_i)$ converges to y. Let U be any neighborhood of x. Since (x_i) converges to x, there exists an integer n such that for all $i \geq n$, $x_i \in U$. Thus, for $i \geq n$, $x_i t_i \in Ut_i \subset UR^+$. Therefore, $(x_i t_i) \subset UR^+ \subset \overline{UR^+}$ for all $i \geq n$. Since $(x_i t_i)$ converges to y and $\overline{UR^+}$ is closed, $y \in \overline{UR^+}$. Since U was arbitrary, $y \in \cap\overline{UR^+}$ and, thus, $D^+(x) \subset \cap\overline{UR^+}$.

To show that $\cap\overline{UR^+} \subset D^+(x)$, let $y \in \cap\overline{UR^+}$. Order the collection $N(x) \times N(y)$ by set inclusion. Hence, $(U_i,V_i) \geq (U_j,V_j)$ if and only if $U_i \subset U_j$ and $V_i \subset V_j$, where $U_i,U_j \in N(x)$ and $V_i,V_j \in N(y)$. Thus, $N(x) \times N(y)$ is a directed set. Let A denote the ordered collection $N(x) \times N(y)$. For each (U,V) in A, $y \in \overline{UR^+}$ implies that V contains a point of UR^+. Thus, for each $(U,V) \in A$ choose $u_v \in U$ and $t_v \geq 0$ such that $u_v t_v \in V$. It is now claimed that (u_v) is a net that converges to x. Let U' be any neighborhood of x and V' a neighborhood of y. Then, for all $(U,V) \geq (U',V')$, $u_v \in U \subset U'$. Thus, (u_v) is eventually in U'. This implies that (u_v) converges to x. Similarly, the net $(u_v t_v)$ converges to y. Therefore,

$y \in D^+(x)$ and $\bigcap \overline{UR^+} \subset D^+(x)$. Thus, the desired equality has been shown.

<u>Proposition</u> 2.36: For each x in X, $J^+(x) = \bigcap_{t \in R^+} D^+(xt)$.

<u>Proof</u>: Let $y \in J^+(x)$. Thus, there exist nets (x_i) converging to x and (t_i) converging to $+\infty$ such that $(x_i t_i)$ converges to y. Let $t \in R^+$. Since (t_i) converges to $+\infty$, there exists an integer n such that for all $i \geq n$, $t_i \geq t$. Thus, $s_i = t_i - t \geq 0$ for all $i \geq n$. Also, for $i \geq n$, the net (s_i) converges to $+\infty$. The net $(x_i t)$ converges to xt and the net $(x_i t(s_i)) = (x_i t_i)$ converges to y. Thus, $y \in D^+(xt)$. Since this is true for all $t \in R^+$, $y \in \bigcap_{t \in R^+} D^+(xt)$.

Now let $y \in \bigcap_{t \in R^+} D^+(xt)$. Suppose that there exists a $t \in R^+$ such that for some net (w_i), the net $(w_i t_i)$ converges to y, where the net (w_i) converges to xt and the net (t_i) converges to $+\infty$. Then the net $(w_i(-t))$ converges to $xt(-t) = x$. Thus, $(w_i(-t)(t_i + t)) = (w_i t_i)$ converges to y. Since $(t_i + t)$ converges to $+\infty$, y is in $J^+(x)$. Suppose such a t does not exist. Since $y \in D^+(xt)$ for all $t \in R^+$, there exists a net (w_i) converging to xt and numbers $t_i \geq 0$, (t_i) not converging to $+\infty$, such that $(w_i t_i)$ converges to y. Hence, there exists a subnet $(t_{i,j})$ of (t_i) such that $(t_{i,j})$ converges to $m_t < +\infty$. Thus, $(w_{i,j})$ converges to xt and $(w_{i,j} t_{i,j})$ converges to y. But, $(w_{i,j} t_{i,j})$ converges to $(xt)m_t = x(t + m_t)$. Thus, $y = x(t + m_t)$. Define the set (x_t) by $x_t = x$ for each t. Thus, (x_t) converges to x. Also, $(t + m_t)$ converges to $+\infty$. Thus, the net $(x_t(t + m_t))$ converges to y. This implies that $y \in J^+(x)$. This completes the proof of the proposition.

<u>Proposition</u> 2.37: For x in X, $D^+(x)$ is closed and positive invariant, and $J^+(x)$ is closed and invariant.

<u>Proof</u>: Since $D^+(x) = \bigcap \overline{UR^+}$, and each $\overline{UR^+}$ is closed, $D^+(x)$ is closed. To show that $D^+(x)$ is positive invariant, let $y \varepsilon C^+(D^+(x))$. Then, $y = zt$ where z is in $D^+(x)$ and $t \varepsilon R^+$. Thus, there exists a net (x_i) converging to x and numbers $t_i \geq 0$ such that $(x_i t_i)$ converges to z. Hence, $(x_i(t_i + t)) = ((x_i t_i)t)$ converges to $zt = y$. Since $(t_i + t) \geq 0$ for all i, $y \varepsilon D^+(x)$. This implies $C^+(D^+(x)) \subset D^+(x)$. Thus, by Proposition 2.8, $D^+(x)$ is positive invariant.

Since $J^+(x) = \bigcap D^+(xt)$ and each $D^+(xt)$ is closed, $J^+(x)$ is closed. To show $J^+(x)$ is invariant, let $y \varepsilon C(J^+(x))$. This implies that $y = zt$ where $z \varepsilon J^+(x)$ and $t \varepsilon R$. Thus, there exists a net (x_i) converging to x, and a net (t_i) converging to $+\infty$ such that $(x_i t_i)$ converges to z. Since (t_i) converges to $+\infty$ there exists an integer n such that for each $j \geq n$, $(t_j + t) \geq 0$. Also, $(x_i(t_j + t)) = ((x_i t_i)t)$ converges to $zt = y$. Since $(t_j + t)$ converges to $+\infty$, $y \varepsilon J^+(x)$. Thus, $C(J^+(x)) \subset J^+(x)$. Proposition 2.8 implies that $J^+(x)$ is invariant. This completes the proof.

<u>Definition</u> 2.38: A subset M of X is <u>positive</u> (negative) <u>k-invariant</u> if and only if $K^+(M) = M$ $(K^-(M) = M)$. The set M is <u>k-invariant</u> if and only if $K(M) = M$.

If $M \subset X$, then M is <u>positive</u> (negative) <u>d-invariant</u> if and only if $D^+(M) = M$ $(D^-(M) = M)$. The set M is <u>d-invariant</u> if and only if $D(M) = M$.

<u>Example</u> 2.5: In the flow (R^2,π) defined by the system of differential equations

$$\dot{x} = y$$

$$\dot{y} = \sin^2\left(\frac{\pi}{x^2 + y^2}\right)y - x,$$

(see Figure 4), each γ_n is given by

$$\gamma_n = \left\{(x,y) : x^2 + y^2 = \frac{1}{n}, \quad n = 1,2,3, \ldots\right\}.$$

Let G_n be the disk bounded by γ_n. Each G_n° is invariant but not positive k-invariant. Each G_n is invariant and k-invariant, but not positive d-invariant. That G_n is not positive d-invariant follows from $J^+(x) \subset \gamma_{n-1}$ for any $x \in \gamma_n \subset G_n$. However, the closure of the complement of G_1 is invariant, k-invariant and positive d-invariant.

<u>Example</u> 2.6: In Example 2.2 each disk is invariant, k-invariant, and d-invariant.

<u>Example</u> 2.7: In the dynamical system (R^2,π) defined by the system of differential equations

$$\dot{r} = r(1 - r)$$

$$\dot{\theta} = 1$$

0 is the only critical point, and the unit circle γ is a periodic trajectory (see Figure 5).

The unit disk M is invariant, k-invariant, and positive d-invariant. For any x in M, $M - \{x\}$ is not positive invariant.

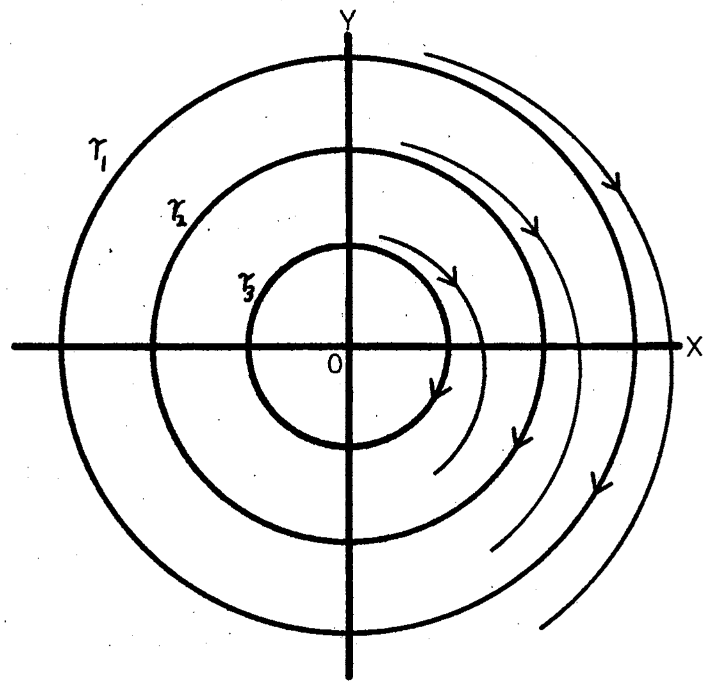

Figure 4. The Phase Space of Example 2.5.

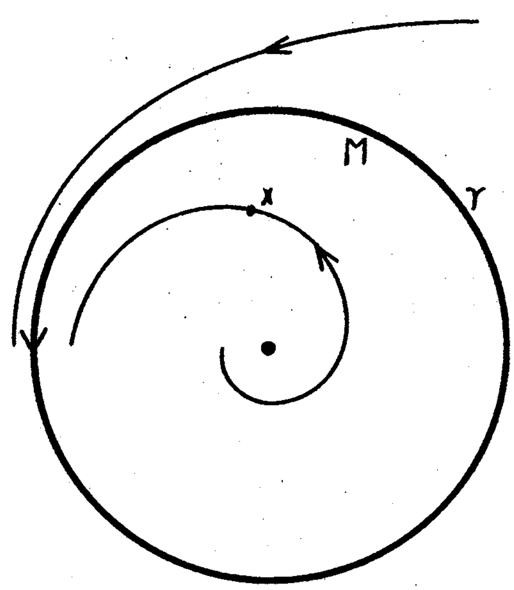

Figure 5. The Phase Space of
Example 2.7.

Example 2.8: For the flow (R^2,π) defined by the system of differential equations

$$\dot{x} = \begin{cases} x, & \text{if } x^2y^2 \geq 1 \text{ and } y > 0 \\ 2x^3y^2 - x, & \text{if } x^2y^2 < 1 \text{ and } y > 0 \\ -x, & \text{if } y \leq 0 \end{cases}$$

$$\dot{y} = \begin{cases} -y, & \text{if } y > 0 \\ 0, & \text{if } y \leq 0 \end{cases}$$

(see Figure 6), the set of critical points is

$$\{(x,y): x = 0, \quad y \in R^-\}.$$

For $x^2y^2 < 1$ and $y > 0$, all sets M, as indicated in Figure 6, are invariant, k-invariant, but not positive d-invariant.

Proposition 2.39: If M is closed and positive (bilaterally) invariant, then M is positive (bilaterally) k-invariant.

Proof: Since M is closed and positive invariant, $K^+(M) = \overline{C^+(M)} = \overline{M} = M$. This implies that M is positive k-invariant, and the proof is complete.

Proposition 2.40: Positive (bilateral) d-invariance implies positive (bilateral) k-invariance which implies positive (bilateral) invariance.

Proof: If M is positive d-invariant, then $D^+(M) = M$. Proposition 2.33 implies that $K^+(M) \subset M$. Since $M \subset K^+(M)$, $K^+(M) = M$. Thus, M is positive k-invariant. For M positive k-invariant,

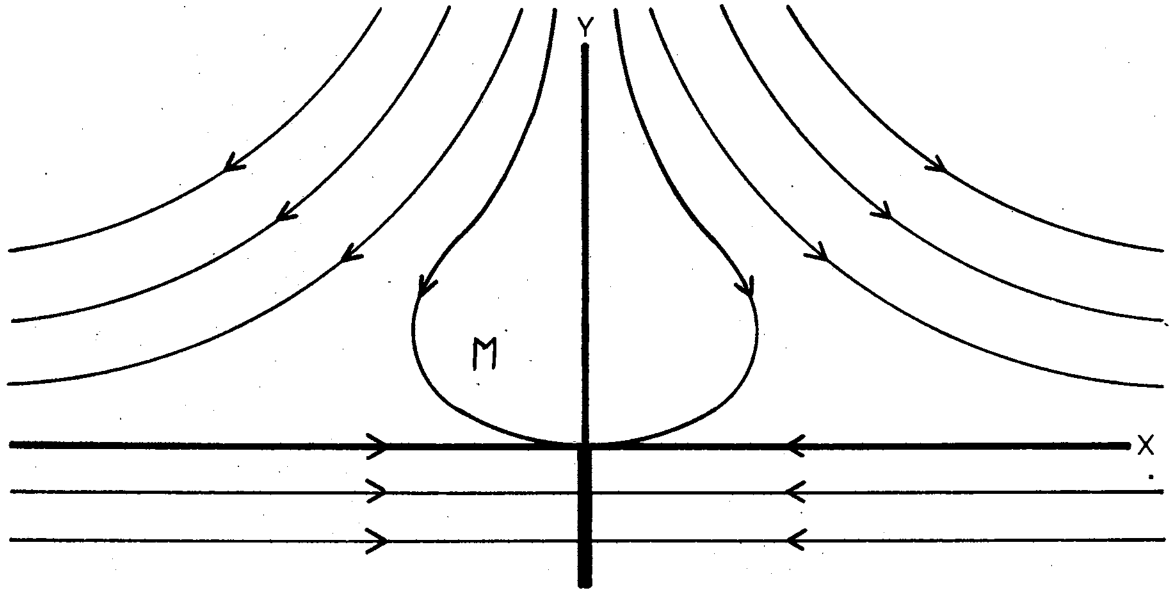

Figure 6. The Phase Space of Example 2.8.

$K^+(M) = M$. Since $C^+(M) \subset K^+(M) = M$ and $M \subset C^+(M)$, $C^+(M) = M$. This implies that M is positive invariant, and the proof is complete.

<u>Proposition</u> 2.41: If (M_i) is a family of positive k-invariant sets (negative k-invariant sets), then $\bigcup_i M_i$ and $\bigcap_i M_i$ are also positive k-invariant (negative k-invariant). A similar statement holds if the M_i are positive d-invariant or negative d-invariant.

<u>Proof</u>: To show that $\bigcup_i M_i$ is positive k-invariant, it suffices to show $K^+(\bigcup_i M_i) \subset \bigcup_i M_i$. Let $x \in K^+(\bigcup_i M_i)$. Then, $x \in C^+(\bigcup_i M_i) \cup L^+(\bigcup_i M_i)$, see Proposition 2.24. Since each M_i is positive k-invariant, Proposition 2.40 implies that each M_i is positive invariant. Proposition 2.11 implies that $\bigcup_i M_i$ is positive invariant. Hence, $C^+(\bigcup_i M_i) = \bigcup_i M_i$. Thus, if $x \in C^+(\bigcup_i M_i)$, then

$x \in \bigcup_i M_i$. This implies that $C^+(\bigcup_i M_i) \subset \bigcup_i M_i$. If $x \in L^+(\bigcup M_i)$, then there exists a net (t_i) converging to $+\infty$ such that (mt_i) converges to x for some $m \in \bigcup_i M_i$. Thus, $x \in L^+(m)$ is contained in $L^+(M_i)$ for some M_i. This implies that $x \in K^+(M_i)$ as $L^+(M_i) \subset K^+(M_i)$. Since M_i is positive k-invariant, $K^+(M_i) = M_i$. Thus, $x \in M_i \subset \bigcup_i M_i$. Therefore, $L^+(\bigcup_i M_i) \subset \bigcup_i M_i$. Therefore, $K^+(\bigcup_i M_i) \subset \bigcup_i M_i$, and $\bigcup_i M_i$ is positive k-invariant.

Now let $x \in K^+(\bigcap_i M_i) = C^+(\bigcap_i M_i) \cup L^+(\bigcap_i M_i)$. If $x \in C^+(\bigcap_i M_i)$, then $x \in \bigcap_i M_i$ as Proposition 2.1 implies that the intersection of positive invariant sets is positive invariant. If $x \in L^+(\bigcap_i M_i)$, then (mt_i) converges to x for some $m \in \bigcap_i M_i$ and some net (t_i) converging to $+\infty$. Thus, $x \in L^+(m) \subset L^+(M_i)$ for all M_i. Since $L^+(M_i) \subset K^+(M_i)$, $x \in K^+(M_i) = M_i$ for all M_i. Thus, $x \in \bigcap_i M_i$ and $K^+(\bigcap_i M_i) \subset \bigcap_i M_i$. Therefore, $\bigcap_i M_i$ is positive k-invariant.

If (M_i) is a family of positive d-invariant sets, then the proof that $\bigcup_i M_i$ and $\bigcap_i M_i$ are positive d-invariant follows as above by noting that $D^+(M_i) = C^+(M_i) \cup J^+(M_i)$ and each M_i is positive invariant. This completes the proof.

<u>Proposition</u> 2.42: A set M is positive d-invariant if and only if $X - M$ is negative d-invariant.

<u>Proof</u>: Assume that M is positive d-invariant and let $y \in M$ and $x \in X - M$. Then $x \notin M = D^+(M)$ implies that $x \notin D^+(y)$ for all $y \in M$. Proposition 2.31 implies that for each $y \in M$, $y \notin D^-(x)$. Therefore, for each $y \in M$, $y \notin D^-(X - M)$. This implies that $D^-(X - M) \subset X - M$. Hence, $X - M$ is negative d-invariant.

Now assume that $X - M$ is negative d-invariant. Let $y \in M$ and $x \in X - M$. Hence, $y \notin X - M = D^-(X - M)$. Since $y \notin D^-(x)$, Proposition 2.31 implies that $x \notin D^+(y)$. This implies that for each $x \in X - M$, $x \notin D^+(M)$. Therefore, $D^+(M) \subset M$, and M is positive d-invariant. This completes the proof.

Proposition 2.43: A set M is positive (negative) d-invariant if and only if $J^+(M) \subset M$ $(J^-(M) \subset M)$, and M is positive (negative) invariant.

Proof: First, assume that M is positive d-invariant. Proposition 2.40 implies that M is positive invariant. Proposition 2.34 implies that $D^+(M) = C^+(M) \cup J^+(M)$. Thus, $J^+(M) \subset D^+(M) = M$.

Conversely, let M be positive invariant and $J^+(M) \subset M$. Since M is positive invariant, $C^+(M) = M$. Thus, $J^+(M) \subset C^+(M)$. Proposition 2.34 implies that $D^+(M) = C^+(M) \cup J^+(M)$. Thus, $D^+(M) = C^+(M) = M$, and M is positive d-invariant. This completes the proof.

Proposition 2.44: If M is positive (bilaterally) invariant, then $\overline{M}$ is positive (bilaterally) k-invariant.

Proof: Since M is positive invariant, $\overline{M}$ is positive invariant. Thus, $\overline{M} = C^+(\overline{M})$. This implies that $\overline{M} = \overline{\overline{M}} = \overline{C^+(\overline{M})} = K^+(\overline{M})$. Thus, $\overline{M}$ is positive k-invariant, and the proof is complete.

In light of the above proposition, it would be natural to ask if $\overline{M}$ is positive d-invariant whenever M is positive k-invariant. This is not true in general as the following example shows.

Example 2.9: In Example 2.5, see Figure 4, let M be the disk G_2. Since $J^+(M) = \gamma_1$, $D^+(M) \neq M$. Hence, $\bar{M}$ is not positive d-invariant, but M is positive k-invariant.

Attraction

Definition 2.45: Let M be a subset of X and x an element of X. The point x is positively weakly attracted to M if and only if the net (xt), $t \in R^+$, is frequently in every neighborhood of M.

The point x is positively attracted to M if and only if the net (xt), $t \in R^+$, is ultimately in every neighborhood of M.

The point x is positively strongly attracted to M if and only if for any neighborhood U of M there exists a neighborhood V of x such that Vt, $t \in R^+$ is ultimately in U.

The negative versions are formed by requiring t to be in R^-. The bilateral cases are defined by taking the conjunction of the positive and negative cases.

Positive weak attraction will be denoted by weak attraction. Similarly for attraction and strong attraction. Negative and bilateral will never be omitted.

Definition 2.46: Let $M \subset X$. Then,

$$A_w^+(M) = \{x\colon x \text{ is weakly attracted to } M\},$$

$$A^+(M) = \{x\colon x \text{ is attracted to } M\},$$

and

$$A_s^+(M) = \{x\colon x \text{ is strongly attracted to } M\}.$$

The sets $A_w^+(M)$, $A^+(M)$, and $A_s^+(M)$ are called the <u>region of weak attraction</u>, the <u>region of attraction</u>, and the <u>region of strong attraction</u>, respectively.

Regions of negative attraction are defined similarly. The bilateral versions are the intersection of the regions of positive and negative attraction.

<u>Definition</u> 2.47: A subset M of X is a <u>weak attractor</u> if and only if $A_w^+(M)$ is a neighborhood of M. Similarly, M is an <u>attractor</u> if and only if $A^+(M)$ is a neighborhood of M, and a <u>strong attractor</u> if and only if $A_s^+(M)$ is a neighborhood of M. Negative and bilateral versions are defined similarly.

<u>Proposition</u> 2.48: If x is strongly attracted to M, then x is attracted to M. If x is attracted to M, then x is weakly attracted to M.

<u>Proof</u>: Let $M \subset X$ and let x be strongly attracted to M. Then, for any neighborhood U of M there exists a neighborhood V_x of x and a $t_x \leq 0$ such that $V_x t \subset U$ for all $t > t_x$. Thus, for all $t > t_x$, $xt \in U$. This implies that the net (xt) is ultimately in any neighborhood of M. This implies x is attracted to M.

If (xt) is ultimately in every neighborhood of M, then (xt) is frequently in that same neighborhood and, thus, x is weakly attracted to M. This completes the proof.

<u>Proposition</u> 2.49: Let $M \subset X$. Then $A_s^+(M) \subset A^+(M) \subset A_w^+(M)$.

Proof: Let $x \in A_s^+(M)$. Then x is strongly attracted to M. By Proposition 2.48, x is attracted to M and, hence, $x \in A^+(M)$. Since any x that is attracted to M is weakly attracted to M, $A^+(M) \subset A_w^+(M)$. This completes the proof.

Proposition 2.50: Let $M \subset X$. A point x is weakly attracted to M if and only if either the net (xt) is frequently in M or $L^+(x) \cap M \neq \emptyset$.

Proof: Assume that x is weakly attracted to M. This implies that the net (xt) is frequently in every neighborhood of M. Now suppose that the net (xt) is not frequently in M. Thus, (xt) is ultimately in $X - M$. This implies that there exists a t_k such that $xt \in X - M$ for all $t \geq t_k$. Suppose $L^+(x) \cap M = \emptyset$. Since $L^+(x) = L^+(xt_k)$, $L^+(xt_k) \subset X - M$. Also, $C^+(xt_k) \subset X - M$. Thus, $K^+(xt_k) = L^+(xt_k) \cup C^+(xt_k)$ is a subset of $X - M$. Since $K^+(xt_k)$ is closed, $X - K^+(xt_k)$ is a neighborhood of M and the net (xt), $t \geq t_k$, is not frequently in $X - K^+(xt_k)$. This contradicts x being weakly attracted to M. Thus, $L^+(x) \cap M \neq \emptyset$.

Conversely, let the net (xt) be frequently in M or $L^+(x) \cap M \neq \emptyset$. If (xt) is frequently in M, then (xt) is frequently in every neighborhood of M and is thus weakly attracted to M. If $L^+(x) \cap M \neq \emptyset$, then let $y \in L^+(x) \cap M$. This implies that there exists a net (t_i) converging to $+\infty$ such that (xt_i) converges to y. Since $y \in M$, the net (xt_i) is ultimately and, therefore, frequently in every neighborhood of M. Therefore, x is weakly attracted to M. This completes the proof.

The following three examples illustrate the regions of attraction that have been presented. They will also be used to illustrate the conditions of Theorem 3.18. Theorem 3.18 characterizes stability in terms of the regions of attraction.

<u>Example</u> 2.10: In Example 2.1, let $M = \{(0,0)\}$. Then, $A_w^+(M) = A^+(M) = \{(x,y): \; y = 0\}$ and $A_s^+(M) = \emptyset$.

If $M = \{(x,y): \; y = 0\}$, then $A_w^+(M) = \emptyset$. If $M = \{(x,y): x = 0, \; y > 0\}$, then

$$A_w^+(M) = A^+(M) = A_s^+(M) = \{(x,y): \; y > 0\}.$$

<u>Example</u> 2.11: Let M be any disk in Example 2.2. Then, $A_s^+(M) = A_s^-(M) = M$.

<u>Example</u> 2.12: As in Example 2.7, let M be the unit disk. Then, $A_s^-(M^\circ) = M^\circ$ and $A_s^+(M^\circ) = X$.

The concepts of this chapter, in particular the concepts of invariance and prolongation, will be used in Chapter III to investigate Liapunov stability, *-stability, and relative stability. For further results related to the topics presented in this chapter, as well as additional topics in the basic theory of dynamical systems, the reader is referred to [12], [13], [14], and [19].

CHAPTER III

LIAPUNOV STABILITY

In contrast to the analytical development of stability as given by Liapunov, in this chapter stability will be examined from a topological viewpoint. After giving a number of basic results, several important characterizations of stability will be proven, including Ura's characterization of stability of compact sets using prolongations. The chapter concludes by examining Zubov's stability criterion and some results on relative stability.

Stability

Definition 3.1: A set $M \subset X$ is positively (negatively) stable if and only if every neighborhood of M contains a positive (negative) invariant neighborhood of M. A set $M \subset X$ is bilaterally stable if and only if every neighborhood of M contains an invariant neighborhood of M.

The usual convention of denoting "positive stability" by "stability" will be used throughout the remainder of the dissertation. Negative and bilateral will never be omitted when referring to these types of stability.

Definition 3.2: A set $M \subset X$ is positively (negatively) k-stable if and only if every neighborhood of M contains a positive (negative) k-invariant neighborhood of M. A set M is <u>bilaterally</u> <u>k-stable</u> if and only if every neighborhood of M contains a k-invariant neighborhood of M.

<u>Positive</u> (negative) and <u>bilateral</u> <u>d-stability</u> of M are defined similarly by requiring each neighborhood of M to contain a positive (negative) d-invariant neighborhood of M or a d-invariant neighborhood of M, respectively.

Positive k-stability and positive d-stability are denoted by k-stability and d-stability. The adjectives negative and bilateral will never be suppressed.

<u>Proposition</u> 3.3: If M is an open, positive (bilaterally) invariant set, then M is (bilaterally) stable.

<u>Proof</u>: Let U be any neighborhood of M. Since M is open, M is a neighborhood of itself. Thus, U contains a positive invariant neighborhood of M. This implies that M is stable, and the proof is complete.

Proposition 3.3 remains true if M is an open, positive (bilaterally) k-invariant (d-invariant) set. The proof of this statement is analogous to the proof of Proposition 3.3.

<u>Example</u> 3.1: In Example 2.1, the set $\{0\}$ is invariant, but not stable. Thus, invariance does not imply stability.

<u>Example</u> 3.2: In Example 2.2, see Figure 2, each disk is positive d-stable and, hence, is stable. In fact, each disk is bilaterally stable.

<u>Example</u> 3.3: Example 2.5 shows that positive invariance does not imply stability. Consider the disk bounded by γ_1. This disk is positive invariant, but not stable.

The set $\{0\}$ is stable and positive k-stable, but not positive d-stable since the trajectories spiral outward.

<u>Example</u> 3.4: In Example 2.7, the unit disk is stable. The set $\{0\}$ is negatively stable. The open unit disk is stable as it is open and positive invariant, see Proposition 3.3. The open unit disk is not positive k-stable as it is a neighborhood of itself and does not contain a positive k-invariant neighborhood of itself.

<u>Example</u> 3.5: The set $\{(x,y)\colon x = 0,\ y \in R^-\}$, in Example 2.8, is stable as it is the union of the stable sets $\{(0,y)\colon y \in R^-\}$, see Proposition 3.4.

<u>Proposition</u> 3.4: The union of stable (k-stable, d-stable) sets is stable (k-stable, d-stable).

<u>Proof</u>: Let M_1 and M_2 be stable sets and U an arbitrary neighborhood of $M_1 \cup M_2$. The stability of M_1 implies that there exists a positive invariant neighborhood V of M_1 such that $V \subset U$. Similarly, there exists a positive invariant neighborhood W of M_2 such that $W \subset U$. Thus, $M_1 \cup M_2 \subset V \cup W \subset U$, and, by Proposition 2.11, $V \cup W$ is positive invariant. Thus, $M_1 \cup M_2$ is stable. Thus, for $i = 1,2$, the union of the members of the family $\mathcal{S} = \{M_i\colon M_i \text{ is stable}\}$ is stable. Proceeding by induction, assume that $\bigcup_{i=1}^{k} M_i$ is stable, where each M_i is stable. Consider $\bigcup_{i=1}^{k+1} M_i$, where each M_i is stable. Clearly, $\bigcup_{i=1}^{k+1} M_i = (\bigcup_{i=1}^{k} M_i) \cup M_{i+1}$. By the induction

hypothesis $\bigcup_{i=1}^{k} M_i$ is stable, and by the first part of the proof, $(\bigcup_{i=1}^{k} M_i) \cup M_{k+1}$ is stable. Thus, $\bigcup_{i=1}^{k+1} M_i$ is stable. Therefore, the arbitrary union of stable sets is stable. This completes the proof.

The intersection of positive invariant sets was shown to be positive invariant in Proposition 2.11. This property does not apply to stable sets as the following example shows.

<u>Example</u> 3.6: Consider the flow (R^2, π) defined by the system of differential equations

$$\dot{x} = 1$$

$$\dot{y} = 0$$

(see Figure 7).

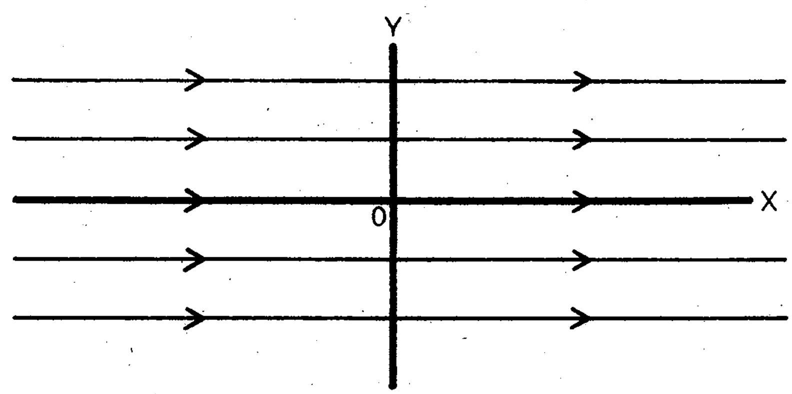

Figure 7. The Phase Space of Example 3.6.

Let $M_1 = \{(x,y): x \leq 1, \quad y = 0\} \cup \{(x,y): x \geq 1, \quad y \neq 0\}$ and let $M_2 = \{(x,y): x \leq 2, \quad y = 0\} \cup \{(x,y): x \geq 2, \quad y \neq 0\}$. Thus, M_i, $i = 1,2$, is the union of the x-axis with the half-plane $x \geq i$ minus all points on the x-axis of the form $(x,0)$, see Figure 8. That M_1 and M_2 are stable sets follows from the definition of stability. The intersection of M_1 and M_2 is

$$\{(x,y): x \leq 1, \quad y = 0\} \cup \{(x,y): x \geq 2, \quad y \neq 0\}.$$

To show that $M_1 \cap M_2$ is not stable, let

$$U = (^-\infty, \frac{5}{4}) \cup \{(x,y): x > \frac{7}{4}\}.$$

Thus, U is a neighborhood of $M_1 \cap M_2$. For any point (x,y), where $x \leq 1$, $y = 0$, $C^+(x,y) \not\subset U$ and, hence, U cannot contain a positive invariant neighborhood of $M_1 \cap M_2$. This implies that $M_1 \cap M_2$ is not stable.

By considering the definition of M_i, for $i = 1,2,3, \ldots,$ a countable collection of stable sets can be constructed such that their intersection is not stable.

<u>Proposition</u> 3.5: If M is stable (k-stable, d-stable), then M is positive invariant (positive k-invariant, positive d-invariant).

<u>Proof</u>: Let M be stable and suppose that M is not positive invariant. Then, there exists $x \in C^+(M) - M$. Since $x \in C^+(M)$, $x = mt$ where $m \in M$ and $t \in R^+$. Since $x \notin M$ and X is Hausdorff, there exists a neighborhood U of M such that $x \notin U$. But M stable

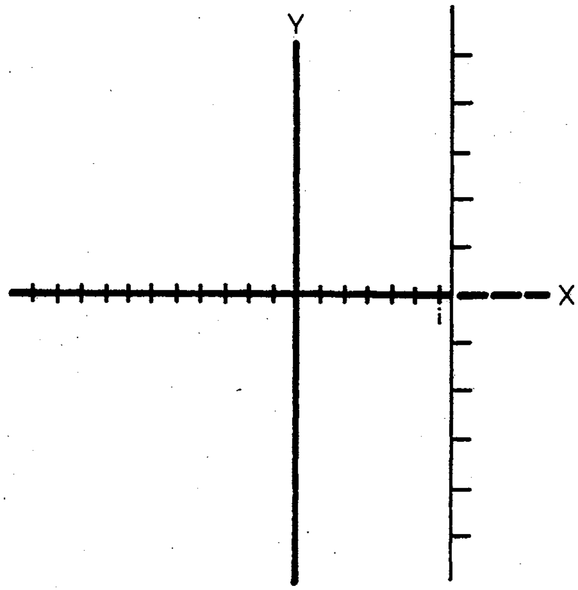

Figure 8. The Set M_i of Example 3.6.

implies there is a positive invariant neighborhood V of M contained in U. Thus, $x \in V \subset U$. This contradicts $x \notin U$. Hence, M is positive invariant, and the proof is complete.

The negative and bilateral versions of Proposition 3.5 are also true.

<u>Theorem</u> 3.6: If X is regular, M closed and stable, then M is positive d-invariant.

<u>Proof</u>: Suppose that M is not positive d-invariant. Then, there exists $x \in D^+(M) - M$. Since X is regular and M is closed, there exist disjoint neighborhoods U and V of M and x, respectively. But U is a neighborhood of M, and M stable implies that there exists a positive invariant neighborhood W of M such that $W \subset U$.

Since $x \in D^+(M)$, there exists $m \in M$ such that $x \in D^+(m)$. Thus, there is a net (m_i) converging to m and numbers $t_i \geq 0$ such that $(m_i t_i)$ converges to x. Hence, there exists an integer n such that for all $i \geq n$, $m_i \in U$. Thus, $m_i t_i \in U$ for all $i \geq n$. But this implies that $(m_i t_i)$ does not converge to x. This contradiction implies that M is positive d-invariant. This completes the proof.

Proposition 3.7: If M is d-stable, then M is k-stable. If M is k-stable, then M is stable.

Proof: Let M be d-stable and U an arbitrary neighborhood of M. Then, there exists a positive d-invariant neighborhood V of M such that $V \subset U$. Since V is positive d-invariant, Proposition 2.40 implies that V is positive k-invariant. Thus, for any neighborhood U of M there exists a positive k-invariant neighborhood of M that is contained in U. Thus, M is k-stable.

Let M be k-stable and U an arbitrary neighborhood of M. Then, there exists a positive k-invariant neighborhood V of M such that $V \subset U$. The positive k-invariance of V implies, see Proposition 2.40, that V is positive invariant. Hence, M is stable, and the proof is complete.

Theorem 3.8: If X is regular and M is closed with compact boundary, then M (negative) stable implies that M is (negative) k-stable.

Proof: Let U be a neighborhood of M. Since X is regular, there exists a neighborhood V of M such that $\overline{V} \subset U$. Since M is stable, there exists a positive invariant neighborhood W of M such that $W \subset V$. Thus, $\overline{W} \subset \overline{V} \subset U$. Proposition 2.44 implies that $\overline{W}$ is positive k-invariant. Thus, for any neighborhood U of M there exists a positive k-invariant neighborhood $\overline{W}$ of M such that $\overline{W} \subset U$. Therefore, M is k-stable, and the proof is complete.

Proposition 3.9: In a regular space X, if M is bilaterally stable and compact, then M is bilaterally k-stable.

Proof: Since M is bilaterally stable and compact, M is positive k-stable, see Theorem 3.8. By the same theorem, M is negative k-stable and, hence, M is bilaterally k-stable. This completes the proof.

Proposition 3.10: Let X be a regular space and M a closed and bilaterally stable set. Then, $X - M$ is positive and negative d-invariant.

Proof: Theorem 3.6 implies that M is positive and negative d-invariant. By Proposition 2.42, $X - M$ is then positive and negative d-invariant. This completes the proof.

Theorem 3.11: If M is bilaterally stable, then $K(x) \cap M = \emptyset$ for all $x \notin M$. Thus, if X is regular and M is closed with compact boundary, then every neighborhood U of M contains some $x \notin M$ with $K(x) \subset U - M$ if M is not open.

Proof: Suppose that $K(x) \cap M \neq \emptyset$ for some $x \notin M$. Then, there exists a y such that $y \in K(x)$ and $y \in M$. By Proposition 2.24, $K(x) = C(x) \cup L(x)$. Since M is bilaterally stable, M is invariant. Thus, $X - M$ is invariant and this implies that $y \notin C(x)$. Thus,

Theorem 3.13. If M is compact and stable, then for all $x \in M$, $D^+(x)$ is a compact and connected subset of M. Thus, $D^+(M) = M$.

Proof: To show that $D^+(x) \subset M$, suppose that there exists $y \in D^+(x) - M$. Then, there exists a net (x_i) converging to x and numbers $t_i \geq 0$ such that $(x_i t_i)$ converges to $y \notin M$. Since M is compact, there exist disjoint neighborhoods U and W of M and y, respectively. Since M is stable, there exists a positive invariant neighborhood V of M such that $V \subset U$. Since (x_i) converges to $x \in M$, there exists an integer n such that for $i > n$, $(x_i t_i) \subset V$. But, $(x_i t_i)$ converging to y implies that $(x_i t_i)$ is ultimately in W. This contradiction shows that $y \in M$. Thus, $D^+(x)$ is a closed subset of M and, hence, is compact.

Suppose that $D^+(x)$ is not connected. Then, there exists disjoint compact sets A and B such that $A \cup B = D^+(x)$. Let $x \in A$ and $y \in B$. Since $A \cap B = \emptyset$, there exist disjoint neighborhoods U and V of A and B, respectively. Since $y \in D^+(x)$, there exists a net (x_i) converging to x and numbers $t_i \geq 0$ such that $(x_i t_i)$ converges to y. Thus, there exists an integer n such that for all $i > n$, $x_i \in U$ and $x_i t_i \in V$. For each $i > n$, $C^+(x_i) \cap \partial U \neq \emptyset$. For suppose that $C^+(x_i) \cap \partial U = \emptyset$. Since (x_i) converges to $x \in A \subset U$, $C^+(x_i) \cap U \neq \emptyset$. For $i > n$, $C^+(x_i) \cap V \neq \emptyset$ and this implies that $C^+(x_i) \cap (X - U) \neq \emptyset$. Thus, $C^+(x_i)$ meets U and $X - U$ but $C^+(x_i) \cap \partial U = \emptyset$. This contradicts $C^+(x_i)$ being a connected set. Thus, $C^+(x_i) \cap \partial U \neq \emptyset$ for $i > n$. This implies that for each $i > n$ there exists a number $s_i \geq 0$ such that $x_i s_i \in \partial U$. Thus, there exists a net $(x_i s_i) \subset \partial U$. Also, $(x_i s_i) \subset M$, as M is stable and, therefore, positive invariant. Thus, there exists a subnet $(x_{i,k}s_{i,k})$ of $(x_i s_i)$ such that $(x_{i,k}s_{i,k})$ converges to a point z in M. But, ∂U closed implies that $z \in \partial U$. Thus, $z \in \partial U$ and $z \in D^+(x)$ implies that $D^+(x) \cap \partial U \neq \emptyset$. Thus, $z \in A \cup B$ and $z \in \partial U$. If $z \in A \cap \partial U$, then, since U is a neighborhood of A, there exists an open set S in U such that $A \subset S$. Thus, no element of A is an element of ∂U. Also, if $z \in B \cap \partial U$, then since V is a neighborhood of B, there exists an open set T in V such that $B \subset T$. Thus, no element of B is an element of ∂V. But, V disjoint from U and $z \in V$ implies that no element of B is an element of ∂U. These results contradict $D^+(x) \cap \partial U \neq \emptyset$. Thus, $D^+(x)$ is connected.

Since M is always contained in $D^+(M)$, and it has been proven that $D^+(M) \subset M$, $D^+(M) = M$, and the proof is complete.

Characterizations of Stability

In this section several important characterizations of stability will be established. These results not only give a greater insight into the meaning of stability, but serve as useful tools for proving further results. In the first characterization, the phase space is a locally compact metric space. The remaining results are proven in more general spaces. In Theorem 3.18, stability is characterized in terms of the regions of strong attraction. Theorem 3.19 gives a characterization in terms of nets.

<u>Theorem</u> 3.14: Let M be a closed subset of a locally compact metric phase space X. Then, M is stable if and only if $D^+(M) = M$ and every $x \in \partial M$ has a neighborhood U such that $C^+(U) - M$ has compact closure.

Proof: It will be assumed that the phase space X is σ-compact. This is possible since [13, XI, 7.3] implies that X is the direct sum of σ-compact spaces X_i which are locally compact and metric. Thus, there exists a sequence of compact subsets Q_m such that $X = \bigcup Q_m$. It is now possible to show that if P is a subset of X that does not have compact closure, then there exists a sequence (p_n) in P such that for each m and sufficiently large n, $p_n \notin Q_m$. For suppose that no such sequence exists. Then, for all sequences (p_n) in P there exists an m such that $p_n \in Q_m$ for all n. Since P is the union of all sequences in P, we have $P \subset Q_m$. Thus, since Q_m is compact and, therefore, closed, $\overline{P} \subset Q_m$. This implies that $\overline{P}$ is compact. This contradiction proves that such a sequence (p_n) does exist.

Assume that M is stable. Theorem 3.6 implies that $D^+(M) = M$. Now assume that for some $x \in \partial M$ and every neighborhood U of x, $C^+(U) - M$ does not have compact closure. Order the neighborhoods U_n of x by set inclusion where U_n is a neighborhood of radius $\frac{1}{n}$. Using the result above, if $m = n$, then for each $C^+(U_n) - M$ an element x_k can be selected such that $x_k \notin Q_n$. For each k, $x_k = x_n t_n$ where $x_n \in U_n$ and $t_n \geq 0$. Since the neighborhoods U_n of x are ordered by set inclusion, a sequence (x_n) has been constructed such that (x_n) converges to x. For each n, $x_n t_n \notin Q_n$ and $x_n t_n \notin M$, that is $x_n t_n \notin M \cup Q_n$. Therefore, the set $F = \{x_n t_n : n = 1,2,3, \ldots\}$ is closed and $F \cap M = \emptyset$. Thus, $G = X - F$ is an open neighborhood of M. Since M is stable, there exists a positive invariant neighborhood V of M such that $V \subset G$. Since V is a neighborhood of M and $x \in \partial M$, $x \in V^\circ$. Since (x_n)

converges to x, (x_n) is ultimately in V. Thus, there exists an integer k such that $x_n t_n \varepsilon Vt_n$ for all $n \geq k$. Since V is positive invariant, $Vt_n \subset V$. Hence, $V \subset G$ and $x_n t_n \varepsilon F$ imply that $x_n t_n \varepsilon U$. Since G is disjoint from F, this gives the needed contradiction and, hence, the result.

To show sufficiency, let M be closed and let $D^+(M) = M$. Further, assume that for every $x \varepsilon \partial M$ there exists a neighborhood U of x such that $C^+(U) - M$ has compact closure. Suppose that M is not stable. This implies that there exists a neighborhood G of M such that G contains no positive invariant neighborhoods of M. Thus, for all neighborhoods U of x, there exists a point x in M with $C^+(U) \not\subset G$. For suppose not. Then for all $x \varepsilon M$ there exists a neighborhood U_x of x such that $C^+(U_x) \subset G$. Let $U = \bigcup_{x \varepsilon M} U_x$. Then U is a neighborhood of M such that $C^+(U) \subset G$. But $C^+(U)$ is positive invariant and is a neighborhood of M. This implies that G contains a positive invariant neighborhood of M. This contradicts the selection of G.

The positive d-invariance of M implies that M is positive invariant. This implies that $x \varepsilon \partial M$. For if not, then $x \varepsilon M^\circ$. This implies that there exists a neighborhood U' of x such that $U' \subset M$. The positive invariance of M implies that $C^+(U') \subset M \subset G$. But this contradicts the fact that for all neighborhoods U of x, $C^+(U) \not\subset G$. Hence, $x \varepsilon \partial M$; that is, $x \varepsilon \overline{X - M}$. Thus, there exists a sequence (x_n) in $X - M$ such that (x_n) converges to x. For each $x_n \varepsilon (x_n)$, $x_n \varepsilon U_n$ and there exists a $t_n \geq 0$ with $x_n t_n \notin G$. The selection of such a sequence is possible since for all $U \varepsilon N(x)$, $C^+(U) \not\subset G$ and for each $x \varepsilon \partial M$, U must contain points of $X - M$. Since $M \subset G$,

$x_n t_n \notin M$, for each n. By assumption, there exists a neighborhood U of x such that $C^+(U) - M$ has compact closure. The sequence (x_n) is ultimately in U and, therefore, the sequence $(x_n t_n)$ is ultimately in $C^+(U)$. For each n, $x_n t_n \notin M$. This implies that $(x_n t_n)$ is ultimately in $C^+(U) - M$. Thus, there exists a convergent subsequence $(x_{n,k} t_{n,k})$ that converges to a point y in $\overline{C^+(U) - M}$. Since $(x_{n,k})$ converges to x and $t_{n,k} \geq 0$, $y \in D^+(x) \subset D^+(M) = M$. But $(x_n t_n) \not\subset G$ implies that $y \notin G^\circ \supset M$. This contradiction shows that M is stable. This completes the proof of the theorem.

<u>Proposition</u> 3.15: Let X be a locally compact Hausdorff space, M a closed invariant subset of X, $x \in \partial M$, and U a neighborhood of x. Then, $\overline{C^+(U) - M} = K^+(U - M) = K^+(\overline{U - M})$.

<u>Proof</u>: To show the equalities, it will suffice to show the following: (a) $\overline{C^+(U) - M} \subset K^+(U - M)$; (b) $K^+(U - M) \subset K^+(\overline{U - M})$; and (c) $K^+(\overline{U - M}) \subset \overline{C^+(U) - M}$. To show (a), it suffices to show $C^+(U) - M \subset K^+(U - M)$ since $K^+(U - M)$ is closed. Let $x \in C^+(U) - M$. Then, $x = ut$ where $ut \notin M$, $u \in U$, and $t \geq 0$. The positive invariance of M and $ut \notin M$ implies that $u \notin M$. Thus, $u \in U - M$. Therefore, $x = ut \in K^+(U - M)$ and $C^+(U) - M \subset K^+(U - M)$.

Statement (b) follows from the fact that $U - M \subset \overline{U - M}$. Thus, $K^+(U - M) \subset K^+(\overline{U - M})$.

To show (c), it will first be shown that $C^+(U) - M$ and, hence, $\overline{C^+(U) - M}$ is positive invariant. Let $x \in C^+(C^+(U) - M)$. Then, $x = yt$ where $y \in C^+(U) - M$ and $t \geq 0$. Since $y \notin M$, $yt \notin M$. For if $yt \in M$, then the invariance of M would imply $(yt)(-t) = y \in M$. This contradicts $y \notin M$ and, thus, $yt \notin M$. Since $y \in C^+(U)$ and

$C^+(U)$ is positive invariant, $x = yt \in C^+(U)$. Hence, $x \in C^+(U) - M$. This implies that $C^+(C^+(U) - M) \subset C^+(U) - M$. Thus, $C^+(U) - M$ is positive invariant. Since $U \subset C^+(U)$, $U - M \subset C^+(U) - M$. This implies that $\overline{U - M} \subset \overline{C^+(U) - M}$. Since $\overline{C^+(U) - M}$ is positive invariant, $C^+(\overline{U - M}) \subset \overline{C^+(U) - M}$. Thus, $K^+(\overline{U - M}) \subset \overline{C^+(U) - M}$ as $\overline{C^+(U) - M}$ is closed. This completes the proof.

The following corollaries state sufficient conditions for a point x to be positively Lagrange stable. A point x is <u>positively Lagrange stable</u> if and only if $K^+(x)$ is compact.

<u>Corollary</u> 3.16: Let X be a locally compact Hausdorff space and M a closed invariant set in X. If M is stable, then each $x \in \partial M$ is positively Lagrange stable.

<u>Proof</u>: Let M be stable and $x \in \partial M$. Theorem 3.14 implies that there exists a neighborhood U of x such that $C^+(U) - M$ has compact closure. If $x \in \partial M$, then $x \in \overline{U - M}$. Thus, $C^+(x) \subset C^+(\overline{U - M})$ and this implies that $K^+(x) \subset K^+(\overline{U - M})$. From the preceeding proposition, $K^+(\overline{U - M}) \cup \overline{C^+(U) - M}$. Thus, $K^+(x)$ is a closed subset of the compact set $\overline{C^+(U) - M}$ and, hence, is compact. This implies that x is positively Lagrange stable. This completes the proof.

<u>Corollary</u> 3.17: Let the phase space X be Hausdorff and M an invariant subset of X. If $x \notin M$ and x is attracted to M, then x is positively Lagrange stable.

<u>Proof</u>: If $K^+(x)$ is not compact, then there exists a net (xt_n) in $C^+(x)$ with no convergent subnets. Then, the set

$A = \{xt_n : xt_n \in (xt_n)\}$ is closed. Since $x \notin M$ and M is invariant, no element of A is in M. Thus, $X - A$ is a neighborhood of M. Since x is attracted to M, there exists a number $t \geq 0$ such that $C^+(xt) \subset X - A$. If $t = 0$, then $(xt)t_n = xt_n \in A$ and this contradicts $xt_n \in X - A$. If $t > 0$, then since the numbers t_n converge to $+\infty$, there exists a number m such that $t_m > t$ and $xt_m \in A$. Thus, $t' = t_m - t$ is greater than zero. Thus, $(xt)t' = x(t + t') = x(t + t_m - t) = xt_m \in A$. This again contradicts $(xt)t' \in C^+(xt) \subset X - A$. These contradictions complete the proof.

Theorem 3.18: A set M is stable if and only if it is positive invariant and $M \subset A_s^+(M)$. A set M is bilaterally stable if and only if $A_s^+(M) = M = A_s^-(M)$.

Proof: Assume that M is stable. Proposition 3.5 implies that M is positive invariant. Let $x \in M$ and let U be a neighborhood of M. The stability of M implies the existence of a positive invariant neighborhood V of M such that $V \subset U$. Since $Vt \subset C^+(V) = V \subset U$, $x \in A_s^+(M)$. Thus, $M \subset A_s^+(M)$.

Conversely, let M be positive invariant and $M \subset A_s^+(M)$. Let U be a neighborhood of M. Since $M \subset A_s^+(M)$, for each $x \in M$ there exists a neighborhood V_x of x and a number $t_x \geq 0$ such that $V_x t \subset U$ for all $t > t_x$. Now consider all numbers t such that $0 \leq t \leq t_x$. The continuity axiom implies that $x[0,t_x]$ is compact. The positive invariance of M implies that $x[0,t_x] \subset M$ and, hence, $x[0,t_x] \subset U$. Since U is a neighborhood of $x[0,t_x]$, there exists for each $xt \in x[0,t_x]$ open neighborhoods U_x of x and A_t of t such that, by the continuity axiom, $U_x A_t \subset U$. For each $xt \in x[0,t_x]$, $U_x A_t$ is an open neighborhood of xt. Thus, the collection $\{U_x A_t : xt \in x[0,t_x]\}$ is an open cover of $x[0,t_x]$. By the compactness of $x[0,t_x]$, there is a finite collection $\{U_{x,i} A_{t,i}\}$ that covers $x[0,t_x]$. Thus, $W_x = \cap U_{x,i}$ is a neighborhood of x and $W_x t \subset U$ for $0 \leq t \leq t_x$. Let $B_x = V_x \cap W_x$. Therefore, B_x is a neighborhood of x and $C^+(B_x) \subset U$. Since each $C^+(B_x)$ is positive invariant and contained in U, $B = \cup B_x$ is a positive invariant subset of U. This implies that M is stable.

To prove the second assertion, assume that M is bilaterally stable. This implies that M is stable and negative stable. Thus, M is positive invariant and negative invariant. Thus, M is invariant. Also by the above proof, $M \subset A_s^+(M)$. An analogous proof shows that $M \subset A_s^-(M)$. Since the proofs for $A_s^+(M) = M$ and $A_s^-(M) = M$ are similar, only $A_s^+(M) = M$ will be shown. Suppose that $A_s^+(M) \neq M$. Then there exists an $x \in A_s^+(M) - M$. Since $A_s^+(M) \subset A_w^+(M)$, $x \in A_w^+(M)$. Proposition 2.50 implies that either $L^+(x) \cap M \neq \emptyset$ or that $C^+(x)$ is frequently in M. Since M is invariant and $x \notin M$, $C^+(x)$ is not frequently in M. Hence, there exists a $y \in L^+(x) \cap M$. Since $y \in L^+(x)$, there exists a net (t_i) converging to $+\infty$ such that the net (xt_i) converges to y. Since X is Hausdorff and $x \notin M$, there exists a neighborhood U of M such that $x \notin U$. Let V_y be any neighborhood of y and $t \leq 0$. Since (xt_i) converges to y, there exists an integer n such that for all $i \geq n$, $xt_i \in V_y$. Let $t_k \in R^-$ such that $k \geq n$ and $t_k < t$. Then, $V_y t_k$ contains the point x. This implies that $V_y t_k \not\subset U$. Thus, $y \notin A_s^-(M)$, and this is a contradiction. Therefore, $A_s^+(M) = M$.

To show the converse, let $A_s^+(M) = M = A_s^-(M)$. Thus, $M \subset A_s^+(M)$ and $M \subset A_s^-(M)$. Since $A_s^+(M) = M$ and $A_s^+(M)$ is invariant, M is invariant. Thus, by the first statement of the theorem, M is positive and negative stable, that is, M is stable. This completes the proof.

The following examples show that it is not possible to weaken the conditions of Theorem 3.18.

Example 3.7: As in Example 2.10, let $M = \{(0,0)\}$. Then, $M \not\subset A_s^+(M)$. Since $(0,0)$ is a critical point, M is positive invariant. But M is not stable as Theorem 3.18 indicates.

In Example 3.11, see Figure 10, let M be an ϵ-disk centered at p_2. Then, $M \subset A_s^+(M) = X$. But M is not positive invariant and not stable.

Example 3.8: In Example 2.11, for any disk M, $A_s^+(M) = A_s^-(M) = M$. Thus, M is bilaterally stable.

In Example 2.7, M° denotes the open unit disk. Since $A_s^-(M^\circ) = M^\circ$ and $A_s^+(M^\circ) = X$, Theorem 3.18 implies that M° cannot be bilaterally stable. However, M° is stable.

Theorem 3.19: A set M is stable if and only if for all nets (x_i) and (y_i), $y_i \in C^+(x_i)$, the net (y_i) is ultimately in every neighborhood of M whenever the net (x_i) is ultimately in every neighborhood of M. Similar statements also hold for k-stability and d-stability.

Proof: Assume M is stable. Let (x_i) be any net that is ultimately in every neighborhood of M. Let (y_i) be a net such that for each i, $y_i \in C^+(x_i)$. If U is any neighborhood of M, then there exists a positive invariant neighborhood V of M such that $V \subset U$. Thus, (x_i) is ultimately in V. This implies that there exists an integer n such that for all $i \geq n$, $x_i \in V$. The positive invariance of V implies $C^+(x_i) \subset V$, for $i \geq n$. Thus, for $i \geq n$, y_i is in V. This implies (y_i) is ultimately in V and, thus, ultimately in U as $V \subset U$.

Conversely, suppose M is not stable. Then there exists a neighborhood U of M such that U does not contain any positive invariant neighborhoods of M. Let (U_i) be the collection of all neighborhoods of M that are contained in U. Partially order the U_i's by set inclusion and let $x \in M$. Then for each U_i there exists a $y_i = xt_i$ such that $y_i \notin U_i$ and $t_i \in R^+$. Thus, $y_i \in U_j$ for some $U_j \supset U_i$. Since U_j is not positive invariant, there exists a number $t_j \in R^+$ such that $y_j = y_i t_j = (xt_i)t_j = xt_j' \notin U_j$ where $t_j' = t_i + t_j$. Since $y_j = xt_j'$, $y_j \in C^+(x)$. Continuing in this same manner, a net (y_k) is constructed such that (y_k) is not ultimately in U_i. But, the net (x_k), defined by $x_k = x$ for all k, converges to x and is ultimately in U_i. By the hypothesis, since $y_k \in C^+(x_k)$ for each k, the net (y_k) is ultimately in U_i. This gives the desired contradiction, and thus M is stable. This completes the proof.

<u>Theorem</u> 3.20: If each component of a set M is stable, then M is stable. If M is compact and stable, then so is each component of M.

<u>Proof</u>: The first statement follows from M equaling the union of its components and the union of stable sets being stable. Now let M

be compact and stable and let C be a component of M. Since C is a closed subset of the compact set M, C is compact. For each $x \in C$, $D^+(x) \subset C$. This is true since x is in $D^+(x)$, and Theorem 3.13 implies that $D^+(x)$ is a connected subset of M. Now suppose that C is not stable. Then there exists a neighborhood U of C that contains no positive invariant neighborhoods of C. Thus, there exists $x \in C$ and $t_x \geq 0$ such that $xt_x \notin U$. It follows that there exists a net (x_i) that converges to x and numbers $t_i \geq 0$ such that $x_i t_i \notin U$. Suppose not, then for all nets (x_i) converging to x and all numbers $t_i \geq 0$, $x_i t_i \in U$. In particular, let $x_i = x$ for each x_i in (x_i). Then $xt_x \in U$. This contradicts $xt_x \notin U$. Thus, such a net exists. Since $x_i[0,t_i]$ is connected and intersects U and the complement of U, there exists a number s_i, $0 \leq s_i \leq t_i$, such that $x_i s_i \in \partial U$. Since (x_i) is ultimately in every neighborhood of C, (x_i) is ultimately in every neighborhood of M. Theorem 3.19 implies that $(x_i s_i)$ is ultimately in every neighborhood of M. Since M is compact, there exists a subnet $(x_{i,k} s_{i,k})$ of $(x_i s_i)$ that converges to a point y in M. Since $(x_{i,k})$ converges to x, $y \in D^+(x)$. But ∂U closed implies that $y \in \partial U$. This is a contradiction as $D^+(x) \subset C$ implies that $y \in C$; and, U a neighborhood of C implies that $y \notin \partial U$. Thus, C is stable. This completes the proof.

In the next section, stability of a closed set with compact boundary is characterized for a locally compact phase space. These results are given in Theorems 3.27 and 3.28.

*-stability and Ura's Theorem

The concept of *-stability will be used to prove several lemmas that culminate in the proof of Ura's Theorem. Ura's Theorem characterizes stability of closed sets with compact boundary in terms of positive d-invariance. Such restrictions are not necessary for characterizing *-stability. Examples will be given to show that stability and *-stability are not in general equivalent. Several results give sufficient conditions for a stable set to be *-stable.

Definition 3.21: A set M is *-stable if and only if for each $x \notin M$, $y \in M$, there exist neighborhoods U of x and V of y such that $U \cap C^+(V) = \emptyset$.

Example 3.9: The dynamical system (R,π) defined by the differential equation

$$\dot{x} = -x^4 \sin^2 \frac{\pi}{x}$$

(see Figure 9) yields a set, actually an infinite collection of sets, that is *-stable but not stable. Note that the only critical points are 0 and $\frac{1}{n}$, where n is a positive integer. In Figure 9, let $M = (-\frac{1}{4}, \frac{1}{4}]$ (in general, let $M_n = (-\frac{1}{n}, \frac{1}{n}]$ for $n = 1,2,3, \ldots$). To see that M is not stable, let $A = (-\frac{1}{4}, \frac{7}{24})$ be a neighborhood of M. The set A is not positive invariant and contains no positive invariant neighborhoods of M. Thus, M is not stable.

Now let x be any element not in M and y any element in M. If $x < -\frac{1}{4}$ or $x > \frac{1}{4}$, then $\rho(x,M) = \epsilon > 0$. Thus, $U = (x - \frac{1}{\epsilon}, x + \frac{1}{\epsilon})$ and $V = (y - \frac{1}{\epsilon}, y + \frac{1}{\epsilon})$ are neighborhoods of

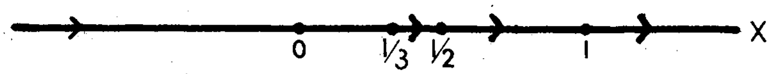

Figure 9. The Phase Space of Example 3.9.

x and y, respectively, such that $C^+(V) \cap U = \emptyset$. If $x = -\frac{1}{4}$ and $y \in M$, then $\rho(x,y) = \epsilon > 0$. Taking U and V as above, we have $C^+(V) \cap U = \emptyset$. Thus, M is *-stable.

Example 3.10: In Example 2.7, the open unit circle, M°, is stable but not *-stable. Let $x \in \gamma$ and y any element in M°. For any neighborhood V of y, $C^+(V)$ will intersect any neighborhood U of x. Thus, M° is not *-stable.

Theorem 3.22: A set M is *-stable if and only if $D^+(M) = M$.

Proof: Assume that M is *-stable. It suffices to show that $D^+(M) \subset M$. Let $y \in M$ and $x \notin M$. The *-stability of M implies that there exist neighborhoods U of x and V of y such that $U \cap C^+(V) = \emptyset$. Suppose that $x \in K^+(V)$. Since $x \notin C^+(V)$, $x \in \partial C^+(V)$. This implies, by the definition of a boundary point, that $U \cap C^+(V) \neq \emptyset$. This is a contradiction and, thus, $x \notin K^+(V)$. Therefore, $x \notin \cap\{K^+(V)\colon V$ is a neighborhood of $y\}$. Thus, $x \notin D^+(y)$ for any $y \in M$. This implies that $x \notin D^+(M) = \cup\{D^+(y)\colon y \in M\}$. Thus, $D^+(M) \subset M$ and $M = D^+(M)$.

To show the converse, let $D^+(M) = M$. Let $y \in M$ and $x \notin M$. Thus, $x \notin D^+(M)$. Since $D^+(y) \subset D^+(M)$,

$$x \notin D^+(y) = \cap\{K^+(V): V \text{ is a neighborhood of } y\}.$$

This implies that there exists a neighborhood V' of y such that $x \notin K^+(V')$. Since $K^+(V')$ is closed, $U = X - K^+(V')$ is a neighborhood of x such that $U \cap K^+(V') = \emptyset$. Hence, $U \cap C^+(V') = \emptyset$ and M is *-stable. This completes the proof.

Lemma 3.23: If M is *-stable, then M is positive invariant.

Proof: Let $x \notin M$. Then by the definition of *-stability, for each $y \in M$, there exist neighborhoods U of x and V of y such that $U \cap C^+(V) = \emptyset$. Thus, $x \notin C^+(y)$ for each $y \in M$. This implies $x \notin \bigcup_{y \in M} C^+(y) = C^+(M)$. Therefore, $C^+(M) \subset M$, and M is positive invariant. This completes the proof.

Lemma 3.24: Let M be a closed set with compact boundary. If M is stable, then M is *-stable.

Proof: Let M be stable and $x \notin M$. For each $m \in \partial M$ there exist disjoint open neighborhoods U_m of m and V_x of x. Since ∂M is compact and $\{U_m : m \in \partial M\}$ is an open cover of ∂M, there exists a finite collection $\{U_{m,i} : m_i \in \partial M\}$ that covers ∂M. Thus, $U = M \cup (\cup U_{m,i})$ is a neighborhood of M. For each $U_{m,i}$ let $V_{x,i}$ denote the corresponding neighborhood of x. Then, $V = \cap V_{x,i}$ is a neighborhood of x such that $U \cap V = \emptyset$. The stability of M implies that there exists a positive invariant neighborhood W of M such that $W \subset U$. Thus, W is a neighborhood of each $y \in M$ such that $W = C^+(W) \subset U$. This implies that $V \cap C^+(W) = \emptyset$ as $V \cap U = \emptyset$. Hence, M is *-stable, and the proof is complete.

Lemma 3.25: Let X be locally compact and M a closed subset of X. If M is stable, then M is *-stable.

Proof: Let M be stable and $x \notin M$. Since X is Hausdorff and locally compact, X is regular. Thus, there exist disjoint neighborhoods U and V of M and x, respectively. The stability of M implies the existence of apositive invariant neighborhood W of M such that $W \subset U$. Hence, $C^+(W) \cap V = \emptyset$ and M is *-stable. This completes the proof.

Lemma 3.26: Let X be locally compact and M closed with compact boundary. Then there exists a closed neighborhood U of M such that ∂U is compact.

Proof: Since X is locally compact, for each $x \in \partial M$ there exists a compact neighborhood U_x of x. The collection $\{U_x : x \in \partial M\}$ is a covering of ∂M. The compactness of ∂M implies that there exists a finite subcover $\{U_{x,i} : x_i \in \partial M\}$, of ∂M. Thus, $U = M \cup (\cup U_{x,i})$ is a neighborhood of M. Since M is closed and each $U_{x,i}$ is closed, U is a closed neighborhood of M. For each i define $V_{x,i} = U_{x,i} - M^\circ$. Then, $U = M \cup (\cup U_{x,i}) = M \cup (\cup V_{x,i})$. Thus,

$$\partial U = \partial(M \cup (\cup V_{x,i})) \subset \partial M \cup \partial(\cup V_{x,k})$$

$$\subset \partial(\cup V_{x,i}) \subset \cup(\partial V_{x,i}) \subset \cup U_{x,i}.$$

To show that ∂U is compact it suffices to show that $\cup U_{x,i}$ is compact, as ∂U is a closed subset of $\cup U_{x,i}$. Let $\mathcal{C}$ be any cover of $\cup U_{x,i}$. Then $\mathcal{C}$ is a cover of each $U_{x,i}$ and, hence, by compactness of $U_{x,i}$, there exists a finite subcover C_i of $U_{x,i}$. Thus, $\bigcup_i C_i$ is a cover of $\bigcup U_{x,i}$ and $\bigcup_i C_i$ is finite as there are only a finite number of the $U_{x,i}$. Hence, ∂U is compact, and the proof is complete.

<u>Theorem</u> 3.27: Let X be locally compact and M a closed set with compact boundary. Then M is stable if and only if M is *-stable.

<u>Proof</u>: If M is stable, then Lemma 3.25 implies that M is *-stable. To show the converse, let M be *-stable and suppose that M is not stable. Lemma 3.26 implies that there exists a closed neighborhood U of M such that ∂U is compact. It is now claimed that for any neighborhood W of M, $C^+(W) \cap \partial U \neq \emptyset$. Since M is not stable, Theorem 3.19 implies that there exist nets (x_i) and (y_i), $y_i \in C^+(x_i)$ for all i, such that (x_i) is ultimately in every neighborhood of M and (y_i) is not ultimately in every neighborhood of M. This implies that (y_i) is frequently in the complement of every neighborhood of M. Let W be a neighborhood of M. Then there exists a number $T' \geq 0$ such that for all $i \geq T'$, $x_i \in W$. Also, there exists a number $T'' \geq 0$ such that for all $i \geq T''$, $x_i \in U$. Let $T = \max\{T',T''\}$. Then, for all $i \geq T$, $x_i \in U \cap W$. Fix $i \geq T$. Then there exists a number $k \geq i \geq T \geq T''$ such that $y_k \in C^+(x_k)$ and $y_i \notin U$. Since $k \geq T$, $x_k \in U \cap W$. Thus, there exists a point x_k in $U \cap W$ such that $y_k = x_k t_k$ is not in U. This implies $C^+(x_k)$ is both in U and in the complement of U. Hence, there exists a number $s_k \geq 0$ such that $x_k s_k \in \partial U$. Thus, $C^+(W) \cap \partial U \neq \emptyset$.

The neighborhoods of M are directed by set inclusion and for each such neighborhood W it has been shown that there exists a point $x_W \in W$ such that $C^+(x_W) \cap \partial U \neq \emptyset$. Lemma 3.23 implies that M is positive invariant, and this implies that $x_w \notin M$. For suppose that $x_w \in M$. Then $C^+(x_w) \subset C^+(M) = M \subset U°$ implies that $C^+(x_w) \cap \partial U = \emptyset$. This contradiction shows $x_w \notin M$. It is now claimed that the net (x_w) has a cluster point $x \in \partial M$. For suppose not. Then for each $m \in \partial M$ there exists a neighborhood T_m of m such that (x_w) is not frequently in T_m. Thus, (x_w) is ultimately in $X - T_m$. The collection $\{T_m: m \in \partial M\}$ covers ∂M and the compactness of ∂M implies that there exists a finite subcover $\{T_{m,i}\}$, that covers ∂M. The set $A = (\bigcup T_{m,i}) \cup M°$ is a neighborhood of M such that (x_w) is not frequently in A. Thus, (x_w) is ultimately in $X - A$. But this contradicts $(x_w) \subset (x_i)$ as (x_i) is ultimately in every neighborhood of M. Thus, (x_w) has a cluster point x in ∂M. Thus, there exists a subnet $(x_{w,i})$ of (x_w) that converges to $x \in \partial M$. For each $x_{w,i}$ there is a $t_{w,i} \geq 0$ such that $x_{w,i}t_{w,i} \in \partial U$. Since ∂U is compact, there is a subnet $(x_{w,i,k}t_{w,i,k})$ of $(x_{w,i}t_{w,i})$ that converges to a point $u \in \partial U$. Also, $(x_{w,i,k})$ converges to $x \in \partial M$. This implies that $u \in D^+(x) \subset D^+(M)$. Theorem 3.22 implies that $D^+(M) = M$ and, therefore, $u \in M$. This is a contradiction as $u \in \partial U$ and $M \subset U°$. Thus, M is stable, and the proof is complete.

<u>Theorem</u> 3.28: (<u>Ura</u>) Let X be locally compact and M a closed subset of X with compact boundary. Then, M is stable if and only if $D^+(M) = M$.

<u>Proof</u>: By Theorem 3.27, M is stable if and only if M is *-stable. By Theorem 3.22, M is *-stable if and only if $D^+(M) = M$. This completes the proof.

Example 3.11: To show local compactness is necessary for Ura's Theorem, consider the flow (R^2,π) defined by the system of differential equations

$$\dot{r} = r(1 - r)$$

$$\dot{\theta} = \sin^2 \frac{\theta}{2}$$

(see Figure 10).

The unit circle is the union of γ and $\{p_2\}$. Let $X' = X - \gamma$. That X' is not locally compact follows from considering any neighborhood U of p_2. Each U will contain a net (x_i) that converges to a point $x \in \gamma$, but $x \notin U$. Thus, p_2 has no compact neighborhoods. However, $D^+(p_2) = \{p_2\}$ and $\{p_2\}$ is not stable due to the behavior of the trajectories interior and exterior to the unit circle.

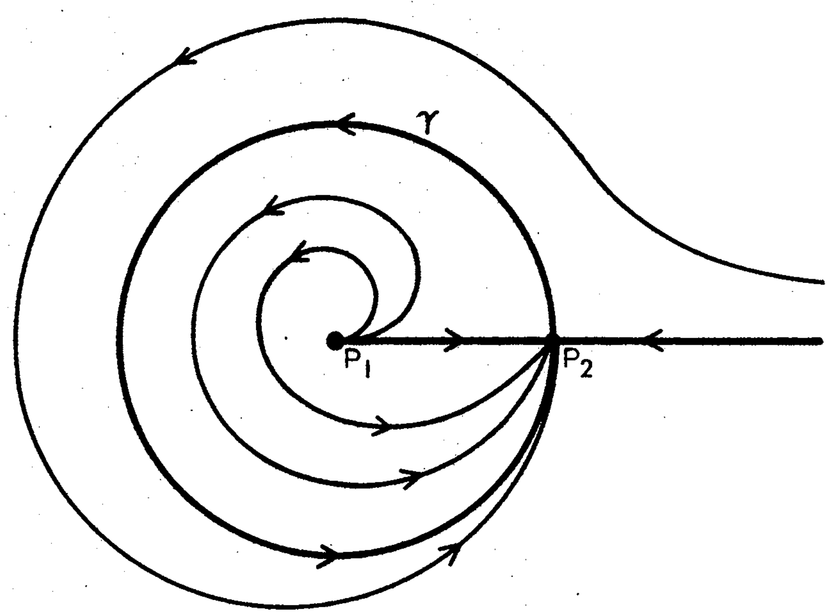

Figure 10. The Phase Space of Example 3.11.

<u>Proposition</u> 3.29: If X is locally compact, then a compact set M is positive d-invariant if and only if each of its components is positive d-invariant.

<u>Proof</u>: If M is compact and positive d-invariant, then Ura's Theorem implies that M is stable. Since M is compact and stable, every component C of M is stable, see Theorem 3.20. Since C is a closed subset of M, C is compact. Ura's Theorem implies that C is positive d-invariant.

Conversely, let each component C of M be positive d-invariant. Since M is the union of all its components, M is positive d-invariant, see Proposition 2.41. This completes the proof.

<u>Theorem</u> 3.30: Let X be locally compact and M a closed set with compact boundary. Then, the following are equivalent:

(a) M is stable;

(b) M is k-stable;

(c) $D^+(M) = M$;

(d) $J^-(X - M) \cap M = \emptyset$, and M is positive invariant; and

(e) $X - M$ is negative d-stable.

<u>Proof</u>: Equivalence will be shown by proving: 1. (a) if and only if (b); 2. (a) if and only if (c); 3. (c) if and only if (d); and 4. (c) if and only if (e).

1. Let M be stable. Since X is locally compact and Hausdorff, X is regular. Thus, Theorem 3.8 implies that M is k-stable. Conversely, if M is k-stable, then Proposition 3.7 implies that M is stable.

2. Ura's Theorem.

3. Assume that $D^+(M) = M$. Then, by Proposition 2.42, $D^-(X - M) = X - M$. Proposition 2.43 implies that $J^-(X - M) \subset X - M$, and $X - M$ is negative invariant. Thus, $J^-(X - M) \cap M = \emptyset$. Since $X - M$ is negative invariant, Proposition 2.42 implies that M is positive invariant.

Conversely, assume that $J^-(X - M) \cap M = \emptyset$ and that M is positive invariant. Thus, $J^-(X - M) \subset X - M$, and $X - M$ is negative invariant. Since, see Proposition 2.34, $D^-(X - M) = C^-(X - M) \cup J^-(X - M)$ and $C^-(X - M)$ and $J^-(X - M)$ are subsets of $X - M$, it follows that $D^-(X - M) = X - M$. Thus, M is positive d-invariant.

4. By Proposition 2.42, $D^+(M) = M$ if and only if $X - M$ is negative d-invariant. Since M is closed, $X - M$ is open. Thus, Proposition 3.3 implies that $X - M$ is negative d-stable. Conversely, if $X - M$ is negative d-stable, then $X - M$ is negative d-invariant. Thus, M is positive d-invariant. This completes the proof.

Zubov's Stability Criterion

In [32], Zubov gave the following criterion for the stability of a closed invariant set.

<u>Proposition</u> 3.31: A closed invariant set M is stable if and only if $L^-(x) \cap M = \emptyset$ for all $x \notin M$.

The necessity of the above proposition has been shown in Theorem 3.11. The criterion is not sufficient as the following example shows.

Example 3.12: Consider the flow (R^2,π) defined by the system of differential equations

$$\dot{x} = 0$$

$$\dot{y} = y^4 \sin^2 \frac{\pi}{y} + x^2$$

(see Figure 11). Let $M = \{(0,0)\}$. For each point p not in M, $(0,0) \notin L^+(P)$. Thus, $L^+(P) \cap M = \emptyset$. But clearly, M is unstable.

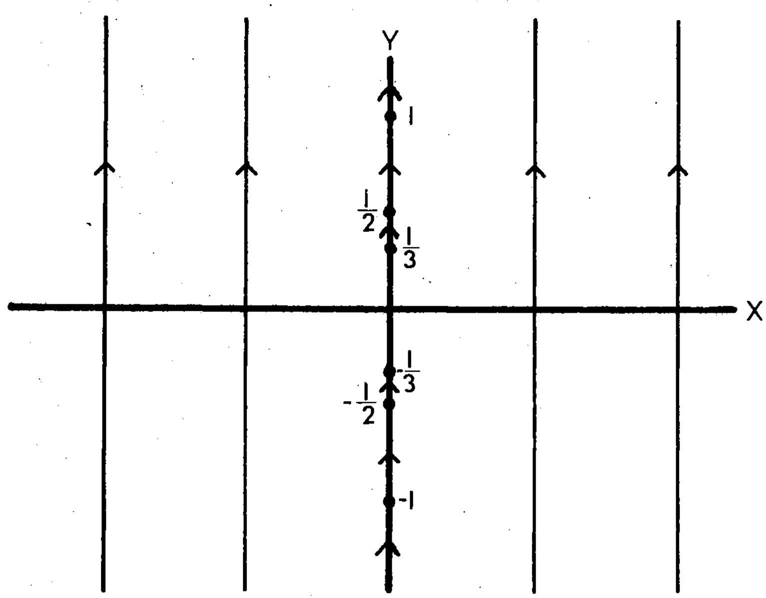

Figure 11. The Phase Space of Example 3.12.

Relative Stability

Definition 3.32: Let M be a compact set and $U \subset X$. Then $D^+(M,U)$ denotes the positive prolongation of M relative to U and is given by

$$D^+(M,U) = \bigcup_{x \in M} \{y \in X: \text{ there exist nets } (x_i) \text{ in } U, \ (t_i) \text{ in } R^+ \text{ such that } (x_i) \text{ converges to } x \text{ and } (x_it_i) \text{ converges to } y\}.$$

Proposition 3.33: If U is a neighborhood of M, then $D^+(M,U) = D^+(M)$.

Proof: If $y \in D^+(M,U)$, then there exist nets (x_i) in U and (t_i) in R^+ such that (x_i) converges to $x \in M$ and (x_it_i) converges to y. This implies that $y \in D^+(M)$ and, hence, $D^+(M,U) \subset D^+(M)$. If $y \in D^+(M)$, then there exists a net (x_i) converging to $x \in M$ and numbers $t_i \geq 0$ such that (x_it_i) converges to y. Since U is a neighborhood of M, (x_i) is ultimately in U. Thus, there exists an integer n such that $x_i \in U$ for all $i > n$. Thus, for all $i > n$, $(x_i) \subset U$. This implies that $y \in D^+(M,U)$ and, hence, $D^+(M) \subset D^+(M,U)$. This completes the proof.

In the proof of the above proposition, the fact that U is a neighborhood of M was not used to show $D^+(M,U) \subset D^+(M)$. Thus, $D^+(M,U) \subset D^+(M)$ for any $U \subset X$.

Definition 3.34: If M is compact and $U \subset X$, then M is stable relative to U, denoted by M is U-stable, if and only if for every neighborhood W of M there exists a neighborhood V of M such that $C^+(V \cap U) \subset W$.

Theorem 3.35: If M is U-stable and U is a neighborhood of M, then M is stable.

Proof: Let M be U-stable where U is a neighborhood of M. Let W be any neighborhood of M. Thus, there exists a neighborhood V of M such that $C^+(V \cap U)$ is contained in W. Since V and U are neighborhoods of M, $V \cap U$ is a neighborhood of M and $M \subset V \cap U \subset C^+(V \cap U)$. Since $C^+(V \cap U)$ is a positive invariant subset of W, M is stable and the proof is complete.

Proposition 3.36: If M is compact and stable, then M is U-stable for all $U \subset X$.

Proof: Let W be any neighborhood of M. Then the stability of M implies the existence of a positive invariant neighborhood V of M such that $V \subset W$. Thus, $V \cap U \subset V$ and $C^+(V \cap U) \subset C^+(V) = V \subset W$. This implies that M is U-stable as was to be shown.

Theorem 3.37: Let X be locally compact. A compact set M is U-stable if and only if $D^+(M,U) \subset M$. If $M \subset U$, then M is U-stable if and only if $D^+(M,U) = M$.

Proof: First, let M be U-stable and $y \in D^+(M,U)$. Since M is U-stable, for any neighborhood W of M there exists a neighborhood V of M such that $C^+(V \cap U) \subset W$. Since $y \in D^+(M,U)$, there exist nets (x_i) in U and (t_i) in R^+ such that (x_i) converges to $x \in M$ and (x_it_i) converges to y. Suppose that $y \notin M$. Then, there exist neighborhoods W of M and Y of y such that $Y \cap W = \emptyset$. The U-stability of M implies that there exists a neighborhood V of M such that $C^+(V \cap U) \subset W$. Since (x_i) converges to $x \in M$, (x_i) is ultimately in $C^+(V \cap U)$. Thus, $(x_it_i) \subset C^+(V \cap U)$ as $C^+(V \cap U)$ is positive invariant. Since (x_it_i) converges to y, (x_it_i) is ultimately in Y. But, $C^+(V \cap U) \subset W$ implies that $Y \cap C^+(V \cap U) = \emptyset$. This contradiction shows $y \in M$ and, hence, $D^+(M,U) \subset M$.

Conversely, let $D^+(M,U) \subset M$ and assume that M is not U-stable. Then there exists a neighborhood W of M such that for all neighborhoods V of M, $C^+(V \cap U) \not\subset W$. Since X is locally compact, there exists a compact neighborhood N of M such that $N \subset W$. For each neighborhood V of M such that $V \subset N$, there exists a point $x_V \in V \cap U$ and a $t_V \in R^+$ such that $x_V t_V \in \partial N$. Since (x_V) is in N and N is compact, there exists a subnet $(x_{V,k})$ of (x_V) that converges to some $x \in N$. In fact, $x \in M$. For suppose that $x \notin M$. Then, by the compactness of M, there exist disjoint neighborhoods P and Q of x and M respectively. This contradicts the construction of (x_V) and, thus, $x \in M$. For each $x_{V,k}$, $x_{V,k}t_{V,k}$ is in the compact set ∂N. Thus, the net $(x_{V,k}t_{V,k})$ has a subnet $(x_{V,k,i}t_{V,k,i})$ that converges to $y \in \partial N$. Note that $(x_{V,k,i})$ converges to $x \in M$. This implies that $y \in D^+(M,U)$. But $y \in \partial N$ implies that $y \notin M$. This contradicts $D^+(M,U) \subset M$ and, thus, M is U-stable.

To prove the second assertion, let $M \subset U$ and assume that M is U-stable. By the above proof, $D^+(M,U) \subset M$. Let $x \in M$ and define (x_i) by $x_i = x$ for each i. Let $t_i = 0$ for each i. Since $M \subset U$, (x_i) is in U and (x_i) converges to $x \in M$. Also, $(x_i t_i) = (x_i)$ and, thus, converges to x. Thus, $x \in D^+(M,U)$ and $M \subset D^+(M,U)$. Therefore, $M = D^+(M,U)$. Conversely, let $D^+(M,U) = M$. By the first part of the theorem, M is U-stable. This completes the proof.

In this chapter the concepts of stability, *-stability, and relative stability have been introduced and their properties investigated. Also, the relationships between these types of stability have been examined. The concept of stability has been extensively studied in [14]. Additional references on stability include [5], [7], [11], [12], [13], [16], [19], [21], [22], [23], [27], and [31]. The concept of relative stability is presented in [13], [14], and [31]. Additional results on stability can be found in papers that present the concept of asymptotic stability.

CHAPTER IV

PARA-STABILITY

The concepts of para-stability and para-Liapunov functions were introduced by Hájek in [22]. The definition for para-stability was motivated by several results in [21]. These results indicated that new concepts of stability should be defined by requiring a set M to be the intersection of certain neighborhoods of M. This is in contrast to the usual definitions that require a certain property to be true for all neighborhoods of M. Since stability and para-stability may be characterized in terms of Liapunov and para-Liapunov functions, the chapter begins with a presentation of these concepts. After presenting some results of para-stability theory, the relationship between stability and para-stability will be examined.

Para-Liapunov Functions

<u>Definition</u> 4.1: A function $v: X \to R$ is a <u>Liapunov function</u> on (X,π) if and only if v is continuous and $v(xt) \leq v(x)$ for all $t \geq 0$ and $x \in X$.

<u>Example</u> 4.1: In the flow (R^2,π) of Example 2.1, defined by the system of differential equations

$$\dot{x} = -x$$

$$\dot{y} = y,$$

the function $\pi\colon R^2 \times R \to R^2$ is given by $\pi((x,y),t) = (xe^{-t}, ye^{t})$. Let the continuous function $v\colon R^2 \to R$ be given by $v(x,y) = e^{-|xy|}$. That v is a Liapunov function follows from letting (x,y) be any element of R^2 and considering $v((x,y)t)$ for all $t \geq 0$. From the definitions of π and v,

$$v((x,y)t) = v(xe^{-t}, ye^{t}) = e^{-|xe^{-t}ye^{t}|} = e^{-|xy|}.$$

This implies that v is a Liapunov function on (X,π).

Definition 4.2: A mapping $v\colon X \to R^+$ on the phase space X is a para-Liapunov function for (X,π) if and only if v is continuous and for every $\epsilon > 0$ there exists a $\delta > 0$ such that $v(xt) < \epsilon$ for all $t \geq 0$ whenever $v(x) < \delta$.

Example 4.2: Let the function π be as in Example 4.1. Then the continuous function $v\colon R^2 \to R^+$ defined by $v(xt) = e^{-t}(2 + 10 \sin t)$ is a para-Liapunov function but not a Liapunov function. Let $\epsilon > 0$ be given. Since $e^{-t}(2 + 10 \sin t)$ converges to 0 as t converges to $+\infty$, there exists an integer n such that $v(xt) < \epsilon$ whenever $t \geq n$. Let $\delta = \epsilon$. If $v(x') < \delta = \epsilon$, where $x' = xt'$, then $t' \geq n$. Then, for any $t \geq 0$, $v(x't) = v(x(t + t'))$. Since $t + t' \geq t' \geq n$, $v(x(t + t')) < \epsilon$. Thus, v is a para-Liapunov function.

To show that v is not a Liapunov function, note that a Liapunov function must satisfy $v(xt) \leq v(x)$ for all x and all $t \geq 0$. Consider the point $x0$ and $t = 1$. Then, $v(x0) = e^{-0}(2 + 10 \sin 0) = 2$ while $e^{-1}(2 + 10 \sin 1) \doteq 3.83$. Thus, $v(x1) \nleq v(x)$ and v is not a Liapunov function.

Proposition 4.3: Every Liapunov function is a para-Liapunov function.

Proof: Let v be a Liapunov function and let $\epsilon > 0$. Thus, v is continuous and $v(xt) \leq v(x)$ for all $t \geq 0$ and $x \in X$. Let $\delta = \epsilon$. If $v(x) < \delta = \epsilon$, then $v(xt) \leq v(x) < \epsilon$ for all $t \geq 0$. Thus, v is a para-Liapunov function, and the proof is complete.

Proposition 4.4: If v is a para-Liapunov function on X, $x \in X$, and $v(x) = 0$, then $D^+(x) \subset v^{-1}(0)$. Thus, $D^+(v^{-1}(0)) = v^{-1}(0)$ is closed and positive invariant.

Proof: Let $y \in D^+(x)$. Then there exists a net (x_i) converging to x and numbers $t_i \geq 0$ such that (x_it_i) converges to y. From the definition of v, for any $\epsilon > 0$ there exists a $\delta > 0$ such that $v(xt) < \epsilon$ whenever $v(x) < \delta$. Since v is continuous, $v(x_i)$ converges to $v(x) = 0$. Thus, $v(x_i)$ is ultimately less than δ. Therefore, $v(x_it_i)$ is ultimately less than ϵ. Therefore, $v(y) \leq \epsilon$ and since ϵ was arbitrary, $v(y) = 0$. This implies that $y \in v^{-1}(0)$ and, hence, $D^+(x) \subset v^{-1}(0)$.

To show that $D^+(v^{-1}(0)) = v^{-1}(0)$, first note that, by the definition of $D^+(v^{-1}(0))$, $v^{-1}(0) \subset D^+(v^{-1}(0))$. If $x \in v^{-1}(0)$, then $v(x) = 0$. Thus, by the first part of the proof, $D^+(x) \subset v^{-1}(0)$. Therefore, $D^+(v^{-1}(0)) = \bigcup_{x \in v^{-1}(0)} D^+(x) \subset v^{-1}(0)$. Thus, $D^+(v^{-1}(0)) = v^{-1}(0)$. Proposition 2.37 implies that $D^+(v^{-1}(0))$ is closed and positive invariant, and the proof is complete.

Proposition 4.5: Let v be a para-Liapunov function and for $\alpha > 0$ define $V_\alpha = \{x\colon v(xt) \leq \alpha \text{ for all } t \geq 0\}$. Then V_α is closed, positive invariant, and $v^{-1}(0) = \bigcap\{V_\alpha\colon \alpha > 0\}$.

Proof: Let $x \varepsilon \overline{V}_\alpha$. Then there exists a net (x_i) in V_α such that (x_i) converges to x. Since for each i, $x_i \varepsilon V_\alpha$, $v(x_i t) \leq \alpha$ for all $t \geq 0$. Since v is continuous, $v(x_i)$ converges to $v(x)$. Suppose $v(xt') > \alpha$ for some t'. Then $v(x_i t')$ converges to $v(xt')$. Let $\epsilon' > 0$ be picked such that $v(xt') - \epsilon' > \alpha$. Thus, $v(x_i t')$ is ultimately in the ϵ'-neighborhood of $v(xt')$. Thus, there exists an integer k such that for all $i > k$, $v(x_i t') > \alpha$. This contradiction shows $v(xt) \leq \alpha$ for all $t \geq 0$ and thus $x \varepsilon V_\alpha$. Therefore, $\overline{V}_\alpha \subset V_\alpha$ and V_α is closed.

To show that V_α is positive invariant, let $y \varepsilon C^+(V_\alpha)$. Then $y = xt$ for some $x \varepsilon V_\alpha$ and some $t \varepsilon R^+$. Thus, $v(y) = v(xt)$. Since $x \varepsilon V_\alpha$, $v(xt) \leq \alpha$. Therefore, $v(y) \leq \alpha$ and $y \varepsilon V_\alpha$. This implies that $C^+(V_\alpha) \subset V_\alpha$ and V_α is positive invariant.

To show that $v^{-1}(0) = \bigcap\{V_\alpha\colon \alpha > 0\}$, let $x \varepsilon v^{-1}(0)$. By the definition of v, for any $\epsilon > 0$ there exists a $\delta > 0$ such that $v(xt) < \epsilon$, for all $t \geq 0$, whenever $v(x) < \delta$. Thus, since $v(x) = 0 < \delta$, $v(xt) < \epsilon$ for all $t \geq 0$ and for all ϵ. This implies that $x \varepsilon V_\alpha$ for each $\alpha > 0$. Thus, $x \varepsilon \bigcap\{V_\alpha\colon \alpha > 0\}$ and $v^{-1}(0) \subset \bigcap\{V_\alpha\colon \alpha > 0\}$. Now let $x \varepsilon \bigcap\{V_\alpha\colon \alpha > 0\}$. Thus, $x \varepsilon V_\alpha$ for each α and $v(x0) = v(x) \leq \alpha$ for all $\alpha > 0$. This implies that $v(x) = 0$ or that $x \varepsilon v^{-1}(0)$. Therefore, $\bigcap\{V_\alpha\colon \alpha > 0\} \subset v^{-1}(0)$, and the proof is complete.

Proposition 4.6: Let v be a para-Liapunov function and let V_α be defined as in Proposition 4.5. Then there exists a sequence (α_n), $\alpha_n > 0$ for each n, such that (α_n) converges to 0 and such that V_{α_n} is a neighborhood of $V_{\alpha_{n+1}}$ for each n.

Proof: For notational convenience V_{α_n} will be denoted by $V_{\alpha n}$. Define the sequence (α_n) inductively by letting $\alpha_1 = 1$ and choosing α_{n+1} such that $2\alpha_{n+1} \leq \alpha_n$. Thus, for each n, $\alpha_n \leq 2^{-n}$. This implies that (α_n) converges to 0. To show the second assertion, fix n and consider $V_{\alpha n}$ and $V_{\alpha(n+1)}$. Let $x \in \overline{V_{\alpha(n+1)}}$. Since, by Proposition 4.5, $V_{\alpha(n+1)}$ is closed, there exists a net (x_i) in $V_{\alpha(n+1)}$ that converges to x. Since $x \in V_{\alpha(n+1)}$, $v(x) \leq \alpha_{n+1} < 2\alpha_{n+1}$. Since (x_i) converges to x, $v(x_i)$ converges to $v(x)$; and thus, $v(x_i)$ is ultimately less than $2\alpha_{n+1}$. Let $\epsilon = \alpha_n$ and $\delta = 2\alpha_{n+1}$. Then, by definition of v being para-Liapunov, $v(x_i t) < \alpha_n$ for all $t \geq 0$. This implies that each x_i is in $V_{\alpha n}$ and, thus, $x \in V_{\alpha n}$ as $V_{\alpha n}$ is closed. Therefore, $V_{\alpha(n+1)} \subset V_{\alpha n}$. Suppose $V_{\alpha(n+1)} \not\subset V^{\circ}_{\alpha n}$. Then there exists $x \in V_{\alpha(n+1)}$ such that $x \in \partial V_{\alpha n}$. Thus, $x \in \overline{X - V_{\alpha n}}$. This implies that there exists a net (y_i) in $X - V_{\alpha n}$ such that (y_i) converges to x. Using the same argument as above, $v(y_i)$ converges to $v(x) \leq \alpha_{n+1} < 2\alpha_{n+1} \leq \alpha_n$. Thus, $v(y_i)$ is ultimately less than $2\alpha_{n+1}$. This implies, by definition of v, that $v(y_i t) < \alpha_n$ for all $t \geq 0$. Thus, $y_i \in V_{\alpha n}$. This contradicts $y_i \in X - V_{\alpha n}$. Therefore, $V_{\alpha n}$ is a neighborhood of $V_{\alpha(n+1)}$, and the proof is complete.

Corollary 4.7: Let v be a para-Liapunov function and let V_α be defined as in Proposition 4.5. Then the sets $G_n = V^{\circ}_{\alpha n}$ have the following properties: G_n is open and positive invariant; $\overline{G_{n+1}} \subset G_n$ for all $n \geq 1$; and $v^{-1}(0) = \cap G_n = \cap \overline{G_n}$.

Proof: Since $G_n = V^{\circ}_{\alpha n}$ and the interior of a set is open, G_n is open. Proposition 4.5 implies that $\overline{G_n}$ is positive invariant and

Proposition 2.12 implies that G_n is positive invariant. From Proposition 4.6, $\overline{G_n}$ is a neighborhood of $\overline{G_{n+1}}$ and, thus, $\overline{G_{n+1}} \subset G_n$ for all $n \geq 1$.

To show the last property, it is first claimed that $\cap G_n = \cap \overline{G_n}$. Since each $G_n \subset \overline{G_n}$, $\cap G_n \subset \cap \overline{G_n}$. Now let $x \in \cap \overline{G_n}$. Then, $x \in \overline{G_n}$ for each $n \geq 1$. This implies that $x \in G_{n-1}$ for all $n > 1$ and thus, $x \in \cap G_n$ for all n. Thus, $\cap \overline{G_n} \subset \cap G_n$ and equality holds. By Proposition 4.5, $v^{-1}(0) = \cap \{V_\alpha : \alpha > 0\}$. Since $\cap \overline{G_n} \supset \cap \{V_\alpha : \alpha > 0\}$, $v^{-1}(0) \subset \cap \overline{G_n}$. Let $x \in \cap \overline{G_n}$. Then, $x \in \overline{G_n}$ for each n. This implies that $x \in V_{\alpha n}$ for all α_n. Therefore, $v(x) \leq \alpha_n$ for all α_n. Since (α_n) converges to 0, $v(x) = 0$. Thus, $x \in v^{-1}(0)$. Therefore, $\cap \overline{G_n} \subset v^{-1}(0)$. Thus, $v^{-1}(0) = \cap G_n = \cap \overline{G_n}$, and the proof is complete.

<u>Theorem</u> 4.8: In a normal phase space X, let $\{G_n : n = 1,2,3, \ldots\}$ be a sequence of open positive invariant sets such that $\overline{G_{n+1}} \subset G_n$, for all n. Then there exists a para-Liapunov function $v: X \to [0,1]$ such that:

(1) $\overline{G_n} \subset \{x: v(x) \leq \frac{1}{n}\} \subset G_{n-1}$;

(2) $\cap G_n = \cap \overline{G_n} = v^{-1}(0)$; and

(3) $X - G_1 \subset v^{-1}(1)$.

<u>Proof</u>: Let $H_n = \overline{G_n} - G_{n+1}$ for each n. Since $\overline{G_n}$ is closed and G_{n+1} is open, H_n is closed for each n. For each n, $H_{n+1} \cap H_{n-1} = (\overline{G_{n+1}} - G_{n+2}) \cap (\overline{G_{n-1}} - G_n)$. Suppose that this intersection is not empty. Then there exists an x such that $x \in \overline{G_{n+1}} - G_{n+2}$ and $x \in \overline{G_{n-1}} - G_n$. Since $x \in \overline{G_{n+1}}$, $x \in G_n$. Thus,

$x \notin \overline{G_{n-1}} - G_n$, and this contradiction shows $H_{n+1} \cap H_{n-1}$ is empty for each n.

It is now claimed that for each n, $H_{n-1} \cap H_n = \partial G_n$. Let $x \in H_{n-1} \cap H_n = (\overline{G_{n-1}} - G_n) \cap (\overline{G_n} - G_{n+1})$. Then $x \in \overline{G_n}$, but $x \notin (\overline{G_n})^\circ = G_n$. Thus, $x \in \partial G_n$ and $H_{n-1} \cap H_n \subset \partial G_n$. If $x \in \partial G_n$, then $x \notin G_{n+1}$ as $\overline{G_{n+1}} \subset G_n$ and G_n is open. Thus, $x \in \overline{G_n} - G_{n+1} = H_n$. Since $x \in \partial G_n$, $x \notin G_n$. Since $\overline{G_n} \subset G_{n-1}$, $x \in \overline{G_{n-1}} - G_n = H_{n-1}$. Thus, $x \in H_n \cap H_{n-1}$ and $\partial G_n \subset H_n \cap H_{n-1}$. Thus, $\partial G_n = H_n \cap H_{n-1}$. Now consider $\partial G_n \cap \partial G_{n+1}$ for each n. Since $\overline{G_{n+1}} \subset G_n$ and G_n is open, $\overline{G_{n+1}} \cap G_n = \emptyset$. Thus, $\partial G_{n+1} \cap \partial G_n = \emptyset$.

Since H_n is a closed set in the normal space X, H_n is a normal subspace of X [18, VII, 3.3]. For each n, ∂G_n is a closed subset of H_n. Since $H_n = \overline{G_n} - G_{n+1}$ where G_{n+1} is open, ∂G_{n+1} is a closed subset of H_n. Thus, $\partial G_n \cup \partial G_{n+1}$ is a closed subset of H_n. Define the function $v'\colon \partial G_n \cup \partial G_{n+1} \to \{\frac{1}{n}, \frac{1}{n+1}\}$ by $v'(x) = \frac{1}{n}$ if $x \in \partial G_n$ and $v'(x) = \frac{1}{n+1}$ if $x \in \partial G_{n+1}$. Since v' is continuous, there exists a continuous extension of v' to a function $v''\colon H_n \to [\frac{1}{n+1}, \frac{1}{n}]$ [18, VII, 5.1]. Now define $v\colon X \to [0,1]$ by

$$v(x) = \begin{cases} 0, & \text{if } x \in \bigcap G_n \\ v''(x), & \text{if } x \in \bigcup H_n \\ 1, & \text{if } x \in X - G_1 \end{cases}$$

Let (a,b) be any open set in $[0,1]$. Then

$$\begin{aligned} v^{-1}(a,b) &= \{x \in X\colon a < v(x) < b\} \\ &= \bigcup\{G_n\colon a < v(x) < b\}. \end{aligned}$$

Thus, $v^{-1}(a,b)$ is an open set since each G_n is open. Thus, v is continuous. Let $\epsilon > 0$, choose $n > \frac{1}{\epsilon}$, and let $\delta = \frac{1}{n+1}$. For $v(x) < \delta = \frac{1}{n+1}$, $x \in G_n$. Thus, $v(xt) \leq \frac{1}{n} < \epsilon$ for all $t \geq 0$ as G_n is positive invariant. Thus, v is a para-Liapunov function.

To show (1), if $x \in \overline{G_n}$, then $v(x) \leq \frac{1}{n}$. This implies that $x \in \{x\colon v(x) \leq \frac{1}{n}\}$. Thus, $\overline{G_n} \subset \{x\colon v(x) \leq \frac{1}{n}\}$. Now let $x \in \{x\colon v(x) \leq \frac{1}{n}\}$. Since $\frac{1}{n} < \frac{1}{n-1}$, $v(x) < \frac{1}{n-1}$ and this implies that $x \in G_{n-1}$. Thus, $\{x\colon v(x) \leq \frac{1}{n}\} \subset G_{n-1}$ and (1) has been shown. The proof of statement (2) is as in the proof of Corollary 4.7. Statement (3) follows from $v(x) = 1$ if $x \in X - G_1$. Thus, if $x \in X - G_1$, $v(x) = 1$ which implies that $x \in v^{-1}(1)$ and, hence, $X - G_1 \subset v^{-1}(1)$. This completes the proof.

Para-Stability

Definition 4.9: A subset M of a phase space X is (positively) para-stable if and only if M is the intersection of sets P_i each of which has the following property: $P_i = \bigcap_{n=1}^{\infty} G_{i,n}$ for suitable open positive invariant sets $G_{i,n}$ such that $\overline{G_{i,n+1}} \subset G_{i,n}$ for all n. Thus, $M = \bigcap_i P_i = \bigcap_i \left(\bigcap_{n=1}^{\infty} G_{i,n}\right)$.

Proposition 4.10: Each para-stable set M is the intersection of its closed positive invariant neighborhoods; in particular, M is closed, positive invariant, and $D^+(M) = M$.

Proof: Let M be para-stable. Then $M = \bigcap_i \left(\bigcap_{n=1}^{\infty} G_{i,n}\right)$ where $G_{i,n}$ is an open positive invariant neighborhood of M such that for each i, $\overline{G_{i,n+1}} \subset G_{i,n}$ for all n. Since $M \subset G_{i,n} \subset \overline{G_{i,n}}$ for all

i and n, $M \subset \bigcap_i \left(\bigcap_{n=1}^{\infty} \overline{G_{i,n}} \right)$. Let $x \in \bigcap_i \left(\bigcap_{n=1}^{\infty} \overline{G_{i,n}} \right)$. Then, $x \in \overline{G_{i,n}}$ for all i and n. Since for fixed i and any $n > 1$, $\overline{G_{i,n}} \subset G_{i,n-1}$, $x \in G_{i,n}$ for all n. Hence, $x \in \bigcap_i \left(\bigcap_{n=1}^{\infty} G_{i,n} \right)$. Thus, $\bigcap_i \left(\bigcap_{n=1}^{\infty} \overline{G_{i,n}} \right) \subset \bigcap_i \left(\bigcap_{n=1}^{\infty} G_{i,n} \right) = M$. Hence, $M = \bigcap_i \left(\bigcap_{n=1}^{\infty} \overline{G_{i,n}} \right)$. Since each $G_{i,n}$ is positive invariant, Proposition 2.12 implies that $\overline{G_{i,n}}$ is positive invariant. Hence, the first statement has been shown.

That M is closed follows from each $\overline{G_{i,n}}$ being closed. Proposition 2.11 implies that M is positive invariant. Proposition 3.3 implies that each $G_{i,n}$ is stable and, thus, $D^+(G_{i,n}) = G_{i,n}$. Since $M \subset D^+(M) \subset D^+(G_{i,n}) = G_{i,n} \subset \overline{G_{i,n}}$ is true for all i and n, $D^+(M) \subset \bigcap_i \left(\bigcap_{n=1}^{\infty} \overline{G_{i,n}} \right) = M$. Thus, $D^+(M) = M$. This completes the proof.

<u>Proposition</u> 4.11: Finite unions and arbitrary intersections of para-stable sets are para-stable.

<u>Proof</u>: Let $M_1, M_2, \ldots, M_k$ be para-stable sets. Then each $M_j = \bigcap_i P_{j,i}$, $1 \leq j \leq k$, where $P_{j,i} = \bigcap_{n=1}^{\infty} G_{j,i,n}$ with the $G_{j,i,n}$ as in the definition of para-stability. Then

$$M = \bigcup_{j=1}^{k} M_j = \bigcup_{j=1}^{k} \left(\bigcap_i P_{j,i} \right).$$

It is now claimed that $\bigcup_{j=1}^{k} \left(\bigcap_i P_{j,i} \right) = \bigcap_i \left(\bigcup_{j=1}^{k} P_{j,i} \right)$. Let $x \in \bigcup_{j=1}^{k} \left(\bigcap_i P_{j,i} \right)$. Then for some $1 \leq m \leq k$, $x \in \bigcap_i P_{m,i}$. This implies that $x \in P_{m,i}$ for all i and, hence, $x \in \bigcup_{j=1}^{k} P_{j,i}$ for all i. Thus, $x \in \bigcap_i \left(\bigcup_{j=1}^{k} P_{j,i} \right)$ and set inclusion from the left has been shown. Now let $x \in \bigcap_i \left(\bigcup_{j=1}^{k} P_{j,i} \right)$. Then, $x \in \bigcup_{j=1}^{k} P_{j,i}$ for all i.

This implies that for some $1 \leq r \leq k$, $x \in P_{r,i}$ for all i. Therefore, $x \in \bigcap_i P_{r,i}$ and, hence, $x \in \bigcup_{j=1}^{k} (\bigcap_i P_{j,i})$. Thus, the equality has been established.

Thus, $M = \bigcup_{j=1}^{k} M_j$ is the intersection of sets $P_i = \bigcup_{j=1}^{k} P_{j,i}$ where, for each j, $P_{j,i} = \bigcap_{n=1}^{\infty} G_{j,i,n}$ with each $G_{j,i,n}$ being an open positive invariant set and $\overline{G_{j,i,n+1}} \subset G_{j,i,n}$ for all n. As above, $P_i = \bigcup_{j=1}^{k} (\bigcap_{n=1}^{\infty} G_{j,i,n}) = \bigcap_{n=1}^{\infty} (\bigcup_{j=1}^{k} G_{j,i,n})$. Thus, $\bigcup_{j=1}^{k} G_{j,i,n}$ is an open positive invariant set and

$$\overline{\bigcup_{j=1}^{k} G_{j,i,n+1}} \subset \bigcup_{j=1}^{k} \overline{G_{j,i,n+1}} \subset \bigcup_{j=1}^{k} G_{j,i,n}$$

for all n. Thus, M is para-stable.

Now let $\{M_j\}_{j\in\Lambda}$ be an arbitrary collection of para-stable sets and let $M = \bigcap_{j\in\Lambda} M_j$. Then

$$M = \bigcap_{j\in\Lambda} M_j = \bigcap_{j\in\Lambda} (\bigcap_i P_{j,i}) = \bigcap_{j\in\Lambda} (\bigcap_i (\bigcap_{n=1}^{\infty} G_{j,i,n})),$$

where each $G_{j,i,n}$ is an open positive invariant set such that $\overline{G_{j,i,n+1}} \subset G_{j,i,n}$ for all n. Let $P_1 = \bigcap_i P_{1,i}$, $P_2 = \bigcap_i P_{2,i}$, ... Then each P_k, $k = 1,2,3, \ldots,$ equals $\bigcap_{n,i} G_{k,i,n}$ where $\overline{G_{k,i,n+1}} \subset G_{k,i,n}$. Thus, M is para-stable, and the proof is complete.

<u>Proposition</u> 4.12: If $v: X \to R^+$ is a para-Liapunov function, then $v^{-1}(0)$ is a para-stable G_δ set.

<u>Proof</u>: Let v be a para-Liapunov function and the sets V_α and G_n be defined as in Proposition 4.5 and Corollary 4.7, respectively.

Corollary 4.7 implies that $v^{-1}(0) = \bigcap_n G_n$, where each G_n is open positive invariant and $\overline{G_{n+1}} \subset G_n$ for all $n \geq 1$. Thus, $v^{-1}(0)$ is a G_δ set. To show that $v^{-1}(0)$ is para-stable, let $P_n = G_n$ for each n. Thus, $v^{-1}(0) = \bigcap_n P_n$. This implies that $v^{-1}(0)$ is para-stable, and the proof is complete.

Theorem 4.13: Let X be a normal phase space. Then M is para-stable if and only if $M = \bigcap_i v_i^{-1}(0)$ for suitable para-Liapunov functions $v_i \colon X \to R^+$.

Proof: First, assume that $M = \bigcap_i v_i^{-1}(0)$, where for each i, v_i is a para-Liapunov function. Proposition 4.12 implies that $v_i^{-1}(0)$ is para-stable, and Proposition 4.11 implies that $M = \bigcap_i v_i^{-1}(0)$ is para-stable.

Conversely, let M be para-stable. Thus, $M = \bigcap_i \left(\bigcap_{n=1}^{\infty} G_{i,n}\right)$ where each $G_{i,n}$ is an open positive invariant set such that for all n, $\overline{G_{n+1}} \subset G_n$. Theorem 4.8 implies that for each i a para-Liapunov function v_i can be constructed such that $v_i^{-1}(0) = \bigcap_{n=1}^{\infty} G_n$. Thus, $M = \bigcap_i v_i^{-1}(0)$ as was required. This completes the proof.

Corollary 4.14: In a normal phase space X, a set M is para-stable if and only if M is the intersection of para-stable G_δ sets.

Proof: First, assume that M is para-stable. Theorem 4.13 implies that $M = \bigcap v_i^{-1}(0)$ where each v_i is a para-Liapunov function. Proposition 4.12 implies that each $v_i^{-1}(0)$ is a para-stable G_δ set. Thus, M is the intersection of para-stable G_δ sets. Conversely, let M equal the intersection of para-stable G_δ sets, G_n.

Proposition 4.11 implies that M is then para-stable. This completes the proof.

Lemma 4.15: For $n = 1,2,3,\ldots$ let $v_n: X \to [0,1]$ be para-Liapunov functions and let $\alpha_n \geq 0$ be constants such that $\sum \alpha_n < +\infty$. Then, $\sum \alpha_n v_n$ is a para-Liapunov function.

Proof: First, it will be established that if v_1 and v_2 are para-Liapunov functions, then αv_1, $\alpha \geq 0$, and $v_1 + v_2$ are para-Liapunov functions. Let $\alpha \geq 0$ and let $\epsilon > 0$ be given. Since v_1 is a para-Liapunov function, there exists a $\delta > 0$ such that if $v_1(x) < \delta$, then $v_1(xt) < \epsilon$ for all $t \geq 0$. Thus, for $\epsilon > 0$, let $\epsilon' = \frac{\epsilon}{2}$, and $\delta' = \alpha\delta$, where δ is determined by ϵ' and v_1. Then, if $v_1(x) < \delta' = \alpha\delta$, $v_1(x) < \epsilon$. Thus, $v_1(xt) < \epsilon' = \frac{\epsilon}{2}$. Thus, $\alpha v_1(xt) < \epsilon$ and αv_1 is a para-Liapunov function.

Now let $\epsilon > 0$ be given. Then for $(1/2)\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $v_1(xt) < (1/2)\epsilon$ if $v_1(x) < \delta_1$ and $v_2(xt) < (1/2)\epsilon$ if $v_2(x) < \delta_2$. Let $\delta = 2\max(\delta_1,\delta_2)$. Then for this δ, $v_1(xt) + v_2(xt) < (1/2)\epsilon + (1/2)\epsilon = \epsilon$ and $v_1 + v_2$ is a para-Liapunov function. This proof also serves as the first step for an inductive proof of $\sum \alpha_n v_n$ being a para-Liapunov function. Thus, assume $\sum \alpha_n v_n$ is a para-Liapunov function for all $n \leq k$. Consider $\sum_{n=1}^{k+1} \alpha_n v_n$. Since $\sum_{n=1}^{k+1} \alpha_n v_n = \alpha_1 v_1 + \sum_{n=2}^{k+1} \alpha_n v_n$, the induction hypothesis implies that $\sum_{n=2}^{k+1} \alpha_n v_n$ is a para-Liapunov function. By the proof for $v_1 + v_2$, $\sum_{n=1}^{k+1} \alpha_n v_n$ is a para-Liapunov function. Thus, for all n, $\sum \alpha_n v_n$ is a para-Liapunov function and this completes the proof.

Theorem 4.16: Let X be a regular Lindelof phase space and $M \subset X$. Then the following properties are equivalent:

(1) M is a para-stable G_δ set;

(2) $M = \bigcap_{n=1}^{\infty} G_n$, where each G_n is an open positive invariant set and $\overline{G_{n+1}} \subset G_n$; and

(3) $M = v^{-1}(0)$ for some para-Liapunov function v.

Proof: First, it will be shown that (3) implies (2). Using the results and notation of Proposition 4.5 and Corollary 4.7, $v^{-1}(0) = \cap G_n$ where each G_n is open, positive invariant, and $\overline{G_{n+1}} \subset G_n$. Thus, since $M = v^{-1}(0)$, $M = \cap G_n$.

To show (2) implies (3) note that X regular and Lindelof implies X is para-compact and, thus, normal [18, VIII, 6.5 and 2.2]. Thus, Theorem 4.8 implies that there exists a para-Liapunov function v such that $v^{-1}(0) = \cap G_n = M$.

That (3) implies (1) is precisely Proposition 4.12. It remains to show (1) implies (3). Let M be para-stable and $M = \cap H_n$, where each H_n is open. Fix n and let $x \varepsilon X - H_n$. Since X is normal and M is para-stable, the conditions of Theorem 4.13 are satisfied. Thus, for each $x \varepsilon X - H_n$, there exists a para-Liapunov function v_n such that $v_n: X \to [0,1]$, $M \subset v_n^{-1}(0)$, and $v_n(x) > 0$. To see that $v_n(x) > 0$, note that the sets H_n contain the sets G_n in the construction of Theorem 4.8 and $v_n(x) = 0$ only on $\cap G_n \subset H_n$. Since $x \varepsilon X - H_n$, $v_n(x) > 0$. For each $x \varepsilon X - H_n$, let $A_x = \{x: v_n(x) > 0\}$. Thus,

$$X - H_n \subset \bigcup_{x \varepsilon X - H_n} A_x.$$

Since each A_x is open,

$$\bigcup_{x\varepsilon X-H_n} A_x$$

is an open cover of $X - H_n$. Since H_n is open, $X - H_n$ is closed and, hence, is Lindelof [18, VIII, 6.6]. Thus, there exists a countable subcover for $X - H_n$. This subcover determines a family of para-Liapunov functions $\{v_{nm}: m = 1,2,3, \ldots\}$ such that $M \subset v_{nm}^{-1}(0)$ for each m. For each $x \varepsilon X - H_n$, there exists an m such that $v_{nm}(x) > 0$. Let the function v be defined by $v(x) = \sum_{n,m=1}^{\infty} 2^{-(n+m)} v_{nm}(x)$. Since for each $x \notin M = \cap H_n$ there exists a m such that $v_{nm}(x) > 0$, $v(x) > 0$ for $x \notin M$. Since $\sum_{n,m=1}^{\infty} 2^{-(n+m)} < +\infty$, Lemma 4.15 implies that v is a para-Liapunov function. Since $M \subset v_{nm}^{-1}(0)$ for all n and m, $M \subset v^{-1}(0)$. To show $v^{-1}(0) \subset M$ let $x \notin M$. Then $v(x) > 0$ which implies that $x \notin v^{-1}(0)$. Thus, $M = v^{-1}(0)$ as was to be shown. This completes the proof of the theorem.

Stability and Para-Stability

Example 4.3: In the flow (R^2,π) defined by the system of differential equations

$$\dot{x} = -xy$$

$$\dot{y} = \begin{cases} x - 1 - y^2, & \text{if } x \geq 0 \\ -x - 1 - y^2, & \text{if } x < 0 \end{cases}$$

(see Figure 12), the y-axis is para-stable but not stable. The para-stability of the y-axis follows from letting $G_{i,n}$, for all n, be the interior of the set formed by the trajectories γ_i and γ_i'.

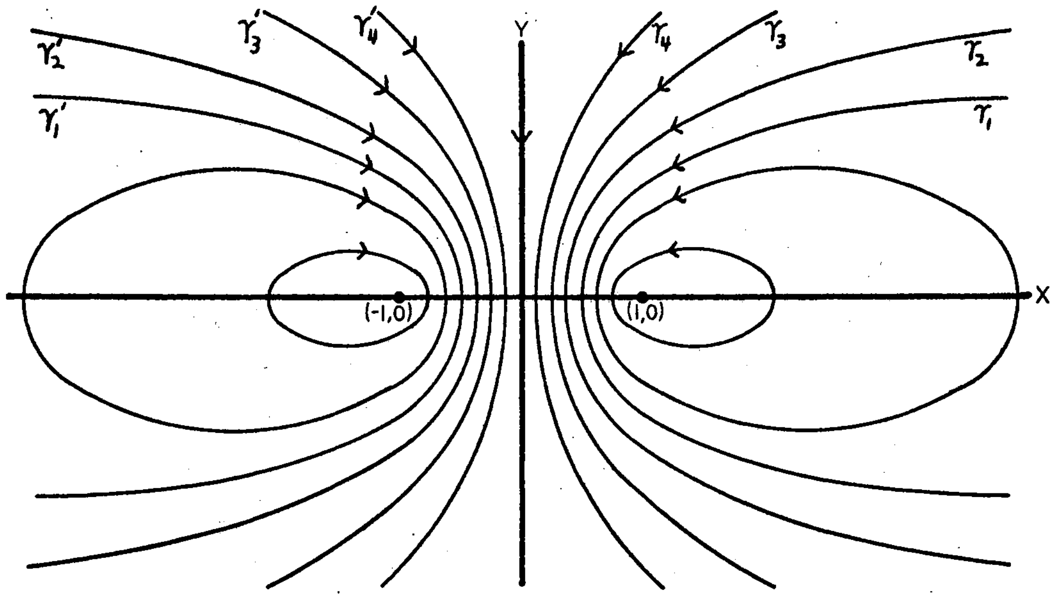

Figure 12. The Phase Space of Example 4.3.

Each $G_{i,n}$ is open and positive invariant and $P_i = \bigcap_{n=1}^{\infty} G_{i,n} = G_{i,n}$. The y-axis equals $\bigcap_i P_i$ and $\overline{G_{i,n}} \subset G_{i,n-1}$. This implies the y-axis is para-stable. Let $U = \{(x,y): -1 < x < 1\}$. Thus, U is a neighborhood of the y-axis that contains no positive invariant neighborhood of the y-axis. This implies that the y-axis is not stable.

Example 4.4: Stability does not imply para-stability since any open positive invariant set M is stable, see Proposition 3.3. But Proposition 4.10 implies that any para-stable set must be closed.

Example 4.5: In Example 2.2, every disk is both stable and para-stable.

Theorem 4.17: If X is normal and M is a closed stable set, then M is para-stable.

Proof: Let U_i be any open neighborhood of M. Thus, $X - U_i$ is closed. Since X is normal, there exist disjoint open neighborhoods V_i and W_i of M and U_i, respectively. Since M is stable, there exists an open positive invariant neighborhood $G_{i,1}$ of M such that $G_{i,1} \subset V_i \subset U_i$, and $G_{i,1} \cap W_i = \emptyset$. Thus, $\overline{G_{i,1}} \subset U_i$. Since $G_{i,1}$ is an open neighborhood of M, there exists an open positive invariant neighborhood $G_{i,2}$ of M such that $\overline{G_{i,2}} \subset G_{i,1}$. Continuing in this manner, we obtain a sequence of open positive invariant neighborhoods $G_{i,n}$ of M with $\overline{G_{i,n+1}} \subset G_{i,n}$. Let $P_i = \bigcap_{n=1}^{\infty} G_{i,n}$. Then, $M \subset P_i \subset U_i$. Since U_i was arbitrary, we have $M \subset \bigcap_i P_i$, where each $P_i = \bigcap_{n=1}^{\infty} G_{i,n}$. To show $\bigcap_i P_i \subset M$, suppose that there exists an x in $\bigcap_i P_i$ such that $x \notin M$. This implies that there exist open disjoint neighborhoods V' of M and W' of x. But since V' is an open neighborhood of M, V' is one of the U_i. Thus, $\bigcap_i P_i = \bigcap_i \left(\bigcap_{n=1}^{\infty} G_{i,n} \right) \subset V'$. This implies that $x \in V'$. This contradiction shows $x \in M$ and, thus, $\bigcap_i P_i \subset M$. Hence, $M = \bigcap_i P_i$, and M is para-stable. This completes the proof.

Theorem 4.18: Let X be locally compact and Hausdorff and let M be a closed set with the compact boundary. Then the following are equivalent:

(1) M is stable;

(2) M is para-stable; and

(3) $M = D^+(M)$.

Proof: That (1) and (3) are equivalent is Ura's Theorem. Proposition 4.10 shows that (2) implies (3). To complete the proof, it will be shown that (1) implies (2). Let M be stable and U_i any open neighborhood of M. Since X is locally compact and Hausdorff, X is regular. Thus, there exists a neighborhood V_i of M such that $\overline{V_i} \subset U_i$. The stability of M implies that there exists an open positive invariant neighborhood $G_{i,1}$ of M such that $G_{i,1} \subset \overline{V}$. Thus, $\overline{G_{i,1}} \subset U_i$. Since $G_{i,1}$ is an open neighborhood of M, there exists a neighborhood $V_{i,1}$ of M such that $\overline{V_{i,1}} \subset G_{i,1}$. Thus, there is a positive invariant neighborhood $G_{i,2}$ of M such that $\overline{G_{i,2}} \subset \overline{V_{i,1}} \subset G_{i,1}$. Continuing in this same manner, we have, as in Theorem 4.17, a sequence of open positive invariant neighborhoods $G_{i,n}$ of M such that $\overline{G_{i,n+1}} \subset G_{i,n}$. As in the proof of Theorem 4.17, $M = \bigcap_i P_i$ where $P_i = \bigcap_{n=1}^{\infty} G_{i,n}$. Thus, M is para-stable as was to be shown.

<u>Theorem</u> 4.19: Let X be para-compact, locally compact, and M a closed G_δ set with compact boundary. Then M is stable if and only if $M = v^{-1}(0)$ for some para-Liapunov function v.

<u>Proof</u>: First, assume that a para-Liapunov function v exists such that $M = v^{-1}(0)$. Proposition 4.12 implies that $v^{-1}(0)$ is para-stable and Theorem 4.18 implies that $v^{-1}(0)$ is stable. Thus, M is stable.

Conversely, let M be a stable set. Since X is para-compact and locally compact, it follows that $X = \bigcup X_i$ where each X_i is σ-compact, open, and $X_i \cap X_j = \emptyset$ for $i \neq j$ [18, XI, 7.3]. Since, for each i, X_i is σ-compact, X_i is Lindelof and locally compact [18, XI, 7.2]. Hence, each X_i is regular [18, XI, 6.4 and VII, 7]. It is now claimed that each X_i is invariant. For suppose that there exists $y \in C(X_i) - X_i$. Then there exists $x \in X_i$ and $t \in R$ such that $y = xt \notin X_i$. The openness of X_i implies that $X_i \cap \partial X_i = \emptyset$, that is,

$X_i \cap \overline{\bigcup_{j\neq i} (X_j)} = \emptyset$. Also, since each X_j, $j \neq i$, is open, $\bigcup_{j\neq i} X_j$ is open and $\bigcup_{j\neq i} X_j \cap \overline{X_i} = \emptyset$. Thus, X_i and $\bigcup_{j\neq i} X_j$ are separated sets and $C(x)$ has points in each of the sets. This is a contradiction of $C(x)$ being a connected set and, hence, there does not exist a $y \in C(X_i)$ such that $y \notin X_i$. Thus, $C(X_i) = X_i$ and X_i is invariant. Since M is stable, Proposition 2.35 implies that M is positive invariant. Proposition 2.10 implies that $X_i \cap M$ is positive invariant. Since $X_i = X - \bigcup_{j\neq i} X_j$ and $\bigcup_{j\neq i} X_j$ is open, it follows that X_i is closed. Thus, $X_i \cap M$ is closed and $X_i \cap M = X_i \cap (\bigcap G_n)$, where each G_n is open since M is a G_δ set. Since X_i is also open, $X_i \cap M$ is a G_δ set in X_i. Since X_i is open and invariant, X_i is stable according to Proposition 3.3. Thus, the stability of X_i and M implies that $X_i \cap M$ is stable in the dynamical system relativized to X_i.

Since ∂M is compact and the X_i are disjoint, ∂M intersects at most a finite number of the X_i. For if not, then the infinite open covering of ∂M by the X_i would not reduce to a finite subcovering of ∂M, which contradicts the compactness of M. Thus, $I = \{i\colon X_i \cap \partial M \neq \emptyset\}$ has cardinality less than $+\infty$. If $j \notin I$, then X_j open and $X_j \cap \partial M = \emptyset$ imply that $X_j \cap M = X_j \cap M^\circ$. Thus, $X_j \cap M$ is open in X_j. Also, $X_j \cap M$ is closed in X_j since X_j and M are closed in X_j. Since X_j is regular and Lindelof, X_j is normal [25, page 113]. Thus, there exists a continuous function $v_j\colon X_j \to R^+$ such that $v_j(x) = 0$ if $x \in X_j \cap M$ and $v_j(x) = 1$ if $x \in X_j - M$ [18, VII, 4.1]. If $i \in I$, then $X_i \cap \partial M \neq \emptyset$, that is, $X_j \cap M \neq \emptyset$. Since $X_i \cap M$ is stable, Theorem 4.18 implies that $X_i \cap M$ is para-stable. Since, from above, $X_i \cap M$ is a G_δ set, Theorem 4.16

implies the existence of a para-Liapunov function $v_i: X_i \to R^+$ such that $X_i \cap M = v_i^{-1}(0)$. Now define $v: X \to R^+$ by $v(x) = v_i(x)$ if $x \in X_i$, $i = 1,2,3, \ldots$. Since each v_i is continuous, v is continuous.

To show $M = v^{-1}(0)$, let $x \in M$. Since $M = \bigcup(X_i \cap M)$, $x \in X_i \cap M$ for some i. If $i \in I$, then $x \in v_i^{-1}(0) \subset v^{-1}(0)$ and, thus, $M \subset v^{-1}(0)$. If $i \notin I$, then $v_i(x) = 0$ and $M \subset v^{-1}(0)$. Now let $x \in v^{-1}(0)$. If $x \in X_i$, $i \in I$, then $v(x) = v_i(x) = 0$ and this implies, by the definition of v_i, that $x \in X_i \cap M \subset M$. If $x \in X_i$, $i \notin I$, then $v(x) = v_i(x) = 0$ implies $x \in X_i \cap M \subset M$. Thus, $x \in M$ and $v^{-1}(0) \subset M$. Hence, $M = v^{-1}(0)$.

It remains to show that v is a para-Liapunov function. It has been established that v is continuous and that for each $i \in I$, v_i is para-Liapunov. If $i \notin I$, then for $x \in X_i$, $x \in X_i \cap M$ or $x \in X_i - M$. If $x \in X_i \cap M$, then $v(x) = v_i(x) = 0$. For any $\epsilon > 0$, let $\delta = \epsilon$. If $v_i(x) < \delta$, then $v_i(xt) = 0 < \epsilon$. That $v_i(xt) = 0$ follows from the fact that $X_i \cap M$ is positive invariant. Thus, $xt \in X_i \cap M$ and $v_i(xt) = 0$. If $x \in X_i - M$, then $v(x) = v_i(x) = 1$. Since X_i is invariant, $xt \in X_i$ for all $t \geq 0$. Thus, $xt \in X_i \cap M$ or $xt \in X_i - M$. If for some $t' \geq 0$, $xt' \in X_i \cap M$, then $xt \in X_i \cap M$ for all $t \geq t'$ since $X_i \cap M$ is positive invariant. Let $\epsilon > 0$ and let $\delta = \epsilon$. If $v_i(x) < \delta = \epsilon$, then $v_i(xt) < \epsilon$ for all $t \geq 0$. For if $0 < \epsilon \leq 1$, then the result follows vacuously. If $1 < \epsilon$, then $v_i(x) < \delta = \epsilon$ by definition of $x \in X_i - M$. Thus, $v_i(xt)$ is either 1 or 0 and $v_i(xt) < \epsilon$ for all $t \geq 0$. Therefore, for $i \notin I$, v_i is a para-Liapunov function. Since each v_i,

$i = 1,2,3, \ldots,$ is a para-Liapunov function, v is a para-Liapunov function, and the proof is complete.

Theorem 4.20: Let X be normal and M a closed set. Then, M is stable if and only if, for every neighborhood U of M, there exists a para-Liapunov function $v: X \to [0,1]$ with $M \subset v^{-1}(0)$ and $X - U \subset v^{-1}(1)$.

Proof: First, assume that M is stable and let U be any neighborhood of M. Since X is normal, and M is closed and stable, a sequence of open positive invariant neighborhoods G_n of M can be constructed, as in Theorem 4.17, such that $\overline{G_{n+1}} \subset G_n \subset G_1 \subset U$. Theorem 4.8 implies that there exists a para-Liapunov function $v: X \to [0,1]$ such that $\cap G_n = v^{-1}(0)$ and $X - G_1 \subset v^{-1}(1)$. Since $M \subset G_n$ for all n, $M \subset \cap G_n = v^{-1}(0)$. Since $G_1 \subset U$, $X - U \subset X - G_1 \subset v^{-1}(1)$.

Conversely, let U be any neighborhood of M. Then there exists a para-Liapunov function $v: X \to [0,1]$ with $M \subset v^{-1}(0)$ and $X - U \subset v^{-1}(1)$. By definition of v, for $\epsilon = 1$ there exists a $\delta > 0$ such that if $v(x) < \delta$, then $v(xt) < \epsilon = 1$ for all $t \geq 0$. Let $V = \{x: v(x) < \delta\}$. Thus, for $x \in V$, $v(xt) < 1$, for all $t \geq 0$. This implies that $xt \notin X - U$. Thus, $xt \in U$. This implies that $C^+(V) \subset U$. Since $M \subset v^{-1}(0)$, $M \subset V \subset C^+(V)$. It remains to show $M \subset V°$. For if not, then there exists an x such that $x \in M$ and $x \in \partial V$. Thus, there exists a net (y_i) in $X - V$ such that (y_i) converges to x. Thus, $v(y_i)$ converges to $v(x) = 0$. This implies that $v(y_i)$ is ultimately less than δ and this implies $y_i \in V$. This contradiction shows $M \subset V°$. Thus, $C^+(V)$ is a positive invariant neighborhood of M such that $C^+(V) \subset U$. Since U was arbitrary, M is stable, and the proof is complete.

Theorem 4.21: Let X be normal and M a closed set. Then M is a stable G_δ set if and only if, for every neighborhood U of M, there exists a para-Liapunov function $v: X \to [0,1]$ such that $M = v^{-1}(0)$ and $X - U \subset v^{-1}(1)$.

Proof: Let M be a closed and stable G_δ set and U any neighborhood of M. Then $M = \cap H_n$ where each H_n is an open neighborhood of M. Since H_1 and $U°$ are open neighborhoods of M, $H_1 \cap U°$ is an open neighborhood of M. Since M is closed and X is normal, there exists an open set V_1 such that $M \subset V_1 \subset \overline{V_1} \subset H_1 \cap U°$. Thus, since M is stable, there exists an open positive invariant neighborhood G_1 of M such that $M \subset G_1 \subset \overline{G_1} \subset \overline{V} \subset H_1 \cap U°$. Since G_1 and H_2 are open neighborhoods of M, $H_2 \cap G_1$ is also an open neighborhood of M. Thus, there exists an open set V_2 such that $M \subset V_2 \subset \overline{V_2} \subset H_2 \cap G_1$. Hence, there exists an open positive invariant set G_2 such that $M \subset G_2 \subset \overline{G_2} \subset \overline{V_2} \subset H_2 \cap G_1$. This implies that $\overline{G_2} \subset G_1$. Continuing in this same manner, a sequence G_n of open positive invariant neighborhoods of M can be constructed such that $\overline{G_{n+1}} \subset G_n$. Since $M \subset G_n$ for all n, $M \subset \cap G_n$. To show $\cap G_n \subset M$ let $x \in \cap G_n$. Then, $x \in G_n$ for all n. By the construction of the G_n, $x \in H_n$ for all n and, hence, $x \in \cap H_n = M$. Thus, $M = \cap G_n$. Also, Theorem 4.8 implies that there exists a para-Liapunov function $v: X \to [0,1]$ such that $\cap G_n = v^{-1}(0)$ and $X - G_1 \subset v^{-1}(1)$. Since $M = \cap G_n$, $M = v^{-1}(0)$. Since $G_1 \subset U$, it follows that $X - U \subset X - G_1 \subset v^{-1}(1)$. Thus, the first assertion is complete.

Conversely, let U be any neighborhood of M. Then there exists a para-Liapunov function $v: X \to [0,1]$ such that $M = v^{-1}(0)$ and $X - U \subset v^{-1}(1)$. Proposition 4.12 implies that $v^{-1}(0)$ is a G_δ set and, hence, M is a G_δ set. The proof that U contains a positive invariant neighborhood $C^+(V)$ of M is precisely the proof used in Theorem 4.20. Thus, M is a stable G_δ set. This completes the proof.

In this chapter the relationships between stability, para-stability, and para-Liapunov functions have been examined. Due to the recent introduction of para-stability, the only reference that can be given is [22]. The relationships between stability and Liapunov functions can be found in [7], [14], [21], [27], and [28].

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