RIEMANN p-ADIC INTEGRATION

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CHAPTER I

INTRODUCTION

The concept of a p-adic number was introduced by Hensel as early as 1908, but ideas related to p-adic fields and their generalizations are still being explored. In spite of the active research that has taken place in the past decade there are still many unanswered questions. The following statement by A. F. Monna (8) illustrates this research:

> At first a theory of non-archimedean normed spaces was attempted. In more recent years a theory of locally convex spaces over non-archimedean valued fields followed.
>
> Both parts of the theory are now in development. Several problems, which have found solution in spaces over the reals, still wait solution in our case. Nevertheless, as a general conclusion it may be said, that many parts of the classical theory remain valid. It is remarkable that this is also true of parts for which one would expect the ordering of the reals to be essential. I mention, for instance, the separation theorems for convex sets; without using an ordering of the fields--even if ordering should be possible--one can define convexity of sets and prove separation theorems for convex sets. In many cases the proofs which are valid for the real spaces, cannot be given in the same way for spaces over a non-archimedean valued field K.

The purpose of this study is to provide an expository development of Riemann integration over the p-adic field by the use of Riemann sums. The structure is planned in such a manner that it will be accessible to the senior mathematics major. The role of the field in determining the properties of the integral will be probed. The major references for this study are papers by Francisco Tomás (13) and Francois Bruhat (4).

There is a great amount of substructure that is necessary in preparation for the definition of the Riemann integral. The necessary facts about p-adic numbers, algebraic systems, topology, and measure theory will be presented in this chapter. A standard notation will be established to be used with these concepts throughout the paper. The second chapter will use these properties to establish a basis for integration. The remainder of the paper will be devoted to Riemann integration over the p-adic field and its properties.

Sets

There are certain sets which are used frequently. These sets will be designated as follows: Z is the set of integers; Z^+ is the set of positive integers; Z' is the set of nonnegative integers; Z_p is the set of p-adic integers; Q_p is the set of p-adic numbers; and R' is the set of nonnegative real numbers.

If A and B are arbitrary sets, then the set

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

is the cartesian product of A and B. The elements of $A \times B$ are ordered pairs. A partition of a set A is a representation of A as the union of nonempty mutually disjoint subsets of A.

Definition 1.1. A distance function (or a metric) for a set A is a function $d: A \times A \to R'$ such that for all $x, y, z \in A$

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$, and

(3) $d(x, z) \leq d(x, y) + d(y, z)$.

Given a set A and an element x of the universe. The characteristic function χ is defined as follows:

$$\chi(x) = 1 \text{ if } x \in A$$

$$\chi(x) = 0 \text{ if } x \notin A.$$

Algebraic Systems

The algebraic systems that will be of concern are groups and fields.

Theorem 1.1. Let G be a group. For each $a, b \in G$ there exists uniquely $x \in G$ such that $a + x = b$ where + is the group operation.

The set H is a subgroup of a group G if H is a subset of G and H is a group with respect to the operation in G.

Theorem 1.2. A subset H of a group G is a subgroup of G if and only if $a - b \in H$ for each $a, b \in H$.

Let H be a subgroup of G. Then for each $x \in G$, $x + H$ is a left coset of H in G.

Theorem 1.3. If H is a subgroup of G, the left cosets of H in G form a partition of G.

Let H be a subgroup of G. Then the number of cosets of H contained in G is called the index of H in G and is denoted by $[G:H]$.

Topology

Definition 1.2. A set G is said to be a topological group if:

(1) G is a group;

(2) G is a topological space; and

(3) The group operations in G are continuous in the topological space G. In other words, the function $-x$ is continuous on G, and $x+y$ is continuous on $G \times G$.

Theorem 1.4. Let $\{U_\alpha\}$ be a basis for the open sets of the identity e of the topological group G. Then the open sets of G are unions of the sets of the form $x+U_\alpha$ where $x \in G$, and the topology of G is completely determined by the basis at e.

Theorem 1.5. If G is a topological group, and H is a subgroup which is open, then H is also closed.

The cartesian product $A \times B$ of two topological spaces A and B is a topological space with the product topology. The family $\mathfrak{B}$ of all cartesian products $U \times V$ where U is an open subset of A and V is an open subset of B is a basis for the product topology. A topological space B is compact if every open covering of B has a finite subcovering. A topological space B is locally compact if each point of B has at least one compact neighborhood.

Theorem 1.6. The cartesian product of two compact topological spaces is a compact topological space.

Theorem 1.7. The cartesian product of two locally compact topological spaces is a locally compact topological space.

Theorem 1.8. The cartesian product of two topological groups is a topological group.

Theorem 1.9. Every closed subset of a compact space is compact.

Theorem 1.10. The family of all spherical neighborhoods of points in a set A with metric d forms a basis for a topology for A.

p-Adic Numbers

The set Q_p of p-adic numbers is a field, and Z_p the set of p-adic integers is a subset of Q_p. The set Z_p is a commutative ring with unity. Both Q_p and Z_p are abelian groups with respect to addition. Any element $\alpha \in Z_p$ has a unique representation $\alpha = a_0 + a_1 p + a_2 p^2 + \ldots$ where $0 \leq a_i \leq p-1$, $i \in Z'$. This form is called the canonical form, and will be used throughout the paper assuming that the coefficients are thus restricted without specific note. Every $\alpha \in Z_p$, $\alpha \neq 0$, has a unique representation in the form $\alpha = p^m \varepsilon$ where $m \in Z'$ and ε is a unit of Z_p. Each nonzero $\alpha \in Q_p$ is uniquely expressed in the form $\alpha = p^m \varepsilon$ where $m \in Z$ and ε is a unit of Z_p.

Definition 1.3. The function $\varphi : Q_p \rightarrow R'$ is defined as follows: $\varphi(\alpha) = \frac{1}{p^k}$ if $\alpha = p^k \varepsilon$, ε a unit of Z_p, $k \in Z$; and $\varphi(0) = 0$.

Theorem 1.11. Let $\alpha, \beta \in Q_p$. The function φ has the following properties:

(1) $\varphi(\alpha) \geq 0$ with equality only if $\alpha = 0$;

(2) $\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$;

(3) $\varphi(\alpha+\beta) \leq \max(\varphi(\alpha), \varphi(\beta))$; and

(4) $\varphi(\alpha+\beta) \leq \varphi(\alpha) + \varphi(\beta)$.

Property (3) is referred to as the non-Archimedean property of φ. The function $d_p : Q_p \times Q_p \to R'$ defined by $d_p(\alpha, \beta) = \varphi(\alpha - \beta)$ is a metric on Q_p. The p-adic numbers with metric induced by φ is a totally disconnected and locally compact metric space. The p-adic integers form a compact subspace of the metric space Q_p. All discussion of Q_p and Z_p as topological spaces will be with respect to the metric induced by φ. Both topological spaces Q_p and Z_p are topological groups.

The cartesian product $Q_p \times Q_p$ denoted by Q_p^2 is a locally compact topological group. The cartesian product Z_p^2 is a compact topological group.

Theorem 1.12. The subsets $p^r Z_p$, $r \in Z$, of the topological space Q_p are open, closed and compact. Also, $p^{r_1} Z_p \subset p^{r_2} Z_p$ when $r_1, r_2 \in Z$ and $r_1 \geq r_2$.

Theorem 1.13. The subsets $p^r Z_p$, $r \in Z'$, of the topological space Z_p are open, closed and compact.

Theorem 1.14. The set $p^r Z_p$, $r \in Z$, is a subgroup of Q_p.

Proof: The set $p^r Z_p$ is a subset of Q_p. Let $p^r \varepsilon_0$, $p^r \varepsilon_1 \in p^r Z_p$ where $\varepsilon_0, \varepsilon_1 \in Z_p$. The theorem will be proved if it can be shown that $p^r \varepsilon_0 - p^r \varepsilon_1 \in p^r Z_p$. But $p^r \varepsilon_0 - p^r \varepsilon_1 = p^r(\varepsilon_0 - \varepsilon_1)$. Also $\varepsilon_0 - \varepsilon_1 \in Z_p$ as

Z_p is a ring. Thus $p^r(\varepsilon_0 - \varepsilon_1) \in p^r Z_p$. Then $p^r\varepsilon_0 - p^r\varepsilon_1 \in p^r Z_p$. Therefore $p^r Z_p$ is a subgroup of Q_p.

Theorem 1.15. The set $p^r Z_p$, $r \in Z^+$, is a subgroup of Z_p.

Proof: Let $p^r\varepsilon_0, p^r\varepsilon_1 \in p^r Z_p$ where $\varepsilon_0, \varepsilon_1 \in Z_p$. But $p^r\varepsilon_0 - p^r\varepsilon_1 = p^r(\varepsilon_0 - \varepsilon_1)$. And $\varepsilon_0 - \varepsilon_1 \in Z_p$ as Z_p is a ring. So $p^r(\varepsilon_0 - \varepsilon_1) \in p^r Z_p$. Thus $p^r\varepsilon_0 - p^r\varepsilon_1 \in p^r Z_p$. Therefore $p^r Z_p$ is a subgroup of Z_p.

The cosets of a subgroup K of a group G constitute a partition of the group G. This partition will be used in the same manner as the partition of an interval is used in the development of Riemann integration over the field of real numbers. It will be of value to know the number of distinct cosets in a particular partition.

Theorem 1.16. If $G = Z_p$, and $K = p^i Z_p$ where $i \in Z'$ then $[Z_p : K] = p^i$.

Proof: Let $x \in Z_p$, then $x + K = x + p^i Z_p$. Let x be of the form $x = a_0 + a_1 p + \ldots + a_{i-1}p^{i-1}$ where $a_j \in Z$, $0 \leq a_j \leq p-1$ for each $0 \leq j \leq i-1$. These choices for x will give the distinct cosets of K in Z_p. There are p choices for each a_j and i choices for each j. So there are a total of p^i choices for x. Therefore $[Z_p : K] = p^i$.

Theorem 1.17. If $G = Z_p^2$, and $K = p^i Z_p \times p^j Z_p$ where $i, j \in Z'$ then $[Z_p^2 : K] = p^{i+j}$.

Proof: Let $x \in Z_p^2$, then $x + K = x + p^i Z_p \times p^j Z_p$. Let x be of the form

$$x = (a_0 + a_1 p + \dots + a_{i-1} p^{i-1}, b_0 + b_1 p + \dots + b_{j-1} p^{j-1})$$

where $a_n \in Z$, $0 \leq a_n \leq p-1$ for $0 \leq n \leq i-1$ and $b_m \in Z$, $0 \leq b_m \leq p-1$ for $0 \leq m \leq j-1$. These choices for x will give the distinct cosets of K in Z_p^2. There are p choices for each a_n and i choices for n. There are p choices for each b_m and j choices for m. So there are $p^i \cdot p^j = p^{i+j}$ choices for x. Therefore $[Z_p^2 : K] = p^{i+j}$.

Example 1.1. Let $G = Z_p$, and $K = p^7 Z_p$. By Theorem 1.16, $[Z_p : p^7 Z_p] = p^7$.

Example 1.2. Let $G = Z_p^2$, and $K = p^3 Z_p \times p^5 Z_p$. By Theorem 1.17, $[G:K] = p^{3+5} = p^8$.

Haar Measure

Let G be a disconnected and locally compact topological group with its topology originating from a metric $d(a, b)$. Haar proved the existence of a left invariant Lebesque measure in G.

Definition 1.4. A function $m(H)$ is a left invariant Haar measure if it satisfies the following properties:

(1) The function $m(H)$ is defined for all sets $H \subset G$, and its values are real numbers such that $0 \leq m(H) \leq +\infty$.

(2) For every nonempty open set $K \neq \{e\}$, $m(K) > 0$; for every compact set K, $m(K) < +\infty$.

(3) If $H_1, H_2, \ldots$ is a finite or infinite sequence of sets, then $m(H_1) + m(H_2) + \ldots \geq m(H_1 \cup H_2 \cup \ldots)$.

(4) If H, K are two sets with $D(H,K) > 0$

$$\left(D(H,K) = \inf\{D(x,y) : x \in H,\ y \in K\}\right),$$

then $m(H) + m(K) = m(H \cup K)$.

(5) The function $m(H) = \inf\{m(K) : K$ is an open set, $K \supset H\}$.

(6) The function $m(H) = m(a+H)$ for all $a \in G$, $H \subset G$,

(7) The function $m(H) = m(-H)$.

The only disconnected and locally compact groups G of concern in this paper are those which are subsets of Q_p or subsets of Q_p^2.

A normalized Haar measure will be given for Q_p. Let $m(Z_p) = 1$. Then as $[Z_p : p^r Z_p]_{r \in Z^+} = p^r$, define $m(p^r Z_p) = \frac{1}{p^r} = p^{-r}$. By the left invariant property (6), $m(\alpha + p^r Z_p) = p^{-r}$ for all $\alpha \in Q_p$. Similarly, as $[p^{-r} Z_p : Z_p]_{r \in Z^+} = p^r$, define $m(p^{-r} Z_p) = p^r$. By the left invariant property (6), $m(\alpha + p^{-r} Z_p) = p^r$. This definition will lead to a left invariant Haar measure, but all that will be needed in this paper is the measure of the sets that have been given.

A normalized Haar measure for Q_p^2 is derived from the measure on Q_p as follows: $m(A \times B) = m(A) \cdot m(B)$ for each $A \times B \in Q_p^2$. For example: $m(Z_p \times Z_p) = m(Z_p) \cdot m(Z_p) = 1$; and

$$m(p^r Z_p \times p^s Z_p)_{r,s \in Z} = m(p^r Z_p)_{r \in Z} \cdot m(p^s Z_p)_{s \in Z} = p^{-r} \cdot p^{-s} = p^{-r-s},$$

For each $\alpha \in Q_p^2$, $m(\alpha + p^r Z_p \times p^s Z_p) = p^{-r-s}$.

CHAPTER II

PRELIMINARY CONCEPTS

Functions

The functions that will be of interest in developing Riemann p-adic integration are mappings from G to Q_p.

Definition 2.1. A function f is a constant function on a set $S \subset G$ if $f(x) = c$ for all $x \in S$.

Example 2.1. Let $f(x) = 3$ for all $x \in G$, then f is a constant function on G.

Definition 2.2. A function f is a locally constant function if for each $x \in G$ there exists a neighborhood N of x such that f is a constant function on N.

Example 2.2. Let $G = Z_p$, and

$$f(x) = \begin{cases} 0 & \text{if } x \in p^2 Z_p - p^5 Z_p \\ 1 & \text{otherwise} . \end{cases}$$

Since $p^5 Z_p$ is an open set any element of $p^5 Z_p$ has a neighborhood, $p^5 Z_p$, over which the function is constant. Also, $Z_p - p^2 Z_p$ is an open set as $p^2 Z_p$ is a closed set. Thus for each element of $Z_p - p^2 Z_p$, $Z_p - p^2 Z_p$ is a neighborhood over which f is a constant.

Similarly $p^2Z_p - p^5Z_p$ is an open set, so for each element of $p^2Z_p - p^5Z_p$, $p^2Z_p - p^5Z_p$ is a neighborhood over which f is a constant. Thus f is a locally constant function.

Example 2.3. Let $G = Z_p^2$, and $f = \sum_{i=0}^{\infty} p^{3i} \chi(K_i)$ where $K_i = p^iZ_p \times p^{2i}Z_p$, $i \in Z'$, and χ is the characteristic function. Let $x = (p^4, p^2)$. Then $x \in K_0$, $x \in K_1$ and $x \in K_2$. But $x \notin K_i$ for $i \geq 3$. So $\chi(K_0) = 1$, $\chi(K_1) = 1$, $\chi(K_2) = 1$ and $\chi(K_i) = 0$ for $i \geq 3$. Therefore $f(p^4, p^2) = p^0 + p^3 + p^6$.

Let N be any neighborhood of $(0,0)$. Then there exists a $j \in Z^+$ such that $p^jZ_p \times p^{2j}Z_p \subset N$ as $\cap \{K | K \in \mathcal{C}\} = \{0\}$. Also, $p^{j+1}Z_p \times p^{2(j+1)}Z_p \subset p^jZ_p \times p^{2j}Z_p$. Let $x \in p^{j+1}Z_p \times p^{2(j+1)}Z_p$, and $y \in (p^jZ_p \times p^{2j}Z_p) - (p^{j+1}Z_p \times p^{2(j+1)}Z_p$. So $f(x) - f(y) = p^{3(j+1)}$ and x and $y \in N$. Therefore f is not locally constant.

Limit of a Function

In the present setting the limit of a function is defined in the usual way, making use of the metric in G and in Q_p.

Definition 2.3. Let $x, \alpha \in G$, $\beta \in Q_p$, and f be a function, $f: G \to Q_p$. The limit of the function f as x approaches α is β, denoted by $\lim_{x \to \alpha} f(x) = \beta$, if for each real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that $\varphi(f(x) - \beta) < \epsilon$ whenever $\varphi(x - \alpha) < \delta$.

Example 2.4. Let $G = Z_p$, and

$$f(x) = \begin{cases} 0 & \text{if } x \in p^2Z_p - p^5Z_p \\ 1 & \text{otherwise}. \end{cases}$$

Show $\lim_{x \to p^2} f(x) = 0$.

In order to satisfy the definition it must be shown that for each $\epsilon > 0$ there exists $\delta > 0$ such that $\varphi(f(x) - 0) < \epsilon$ whenever $\varphi(x - p^2) < \delta$. Let $\delta = p^{-2}$. If $\varphi(x - p^2) < p^{-2}$ then $x = p^2 + a_0 p^3 + a_1 p^4 + \dots$. Thus $f(x) = 0$ so $\varphi(f(x)) = \varphi(0) = 0 < \epsilon$. Therefore $\lim_{x \to p^2} f(x) = 0$.

In order to be able to apply the definition of limit to a function whose domain is a subset of Q_p^2, it will be necessary to define a function φ that induces a metric for the product space.

<u>Definition 2.4.</u> Let $(a, b) \epsilon Q_p^2$, then $\varphi((a, b)) = \max\{\varphi(a), \varphi(b)\}$.

The fact that $\varphi((a, b))$ induces a metric follows immediately from the properties for φ in the space Q_p.

<u>Theorem 2.1.</u> The function φ for Q_p^2 space is non-Archimedean.

Proof: Let $(x_1, y_1), (x_2, y_2) \epsilon Q_p^2$. Then

$$\begin{aligned} \varphi((x_1, y_1) + (x_2, y_2)) &= \varphi(x_1 + x_2, y_1 + y_2) \\ &= \max[\varphi(x_1 + x_2), \varphi(y_1 + y_2)] \end{aligned}$$

by definition. This implies that

$$\begin{aligned} \varphi[(x_1, y_1) + (x_2, y_2)] &\leq \max\Big[\max[\varphi(x_1), \varphi(y_1)], \max[\varphi(x_2), \varphi(y_2)]\Big] \\ &= \max[\varphi(x_1, y_1), \varphi(x_2, y_2)] \end{aligned}$$

as

$$\varphi(x_1, y_1) = \max[\varphi(x_1), \varphi(y_1)] \quad \text{and} \quad \varphi(x_2, y_2) = \max[\varphi(x_2), \varphi(y_2)].$$

Therefore the metric for Q_p^2 is non-Archimedean.

Example 2.5. Let $G = Z_p^2$, and $f = \sum_{i=0}^{\infty} p^{3i} \chi(K_i)$ where $K_i = p^i Z_p \times p^i Z_p$. Show

$$\lim_{(x,y) \to (p^2, p)} f(x, y) = 1 + p^3 \quad \text{and} \quad \lim_{(x,y) \to (0,0)} f(x, y) = \sum_{i=0}^{\infty} p^{3i}.$$

For each $\epsilon > 0$, a $\delta > 0$ must be found such that $\varphi(f(x, y) - (1 + p^3)) < \epsilon$ whenever $\varphi((x, y) - (p^2, p)) < \delta$. Let $\delta = p^{-1}$. But $\varphi((x, y) - (p^2, p)) = \varphi(x - p^2, y - p)$. So

$$\varphi((x, y) - (p^2, p)) = \max\{\varphi(x - p^2), \varphi(y - p)\}.$$

Thus $\varphi(x - p^2) < p^{-1}$ and $\varphi(y - p) < p^{-1}$. This implies that $y = p + a_0 p^2 + a_1 p^3 + \ldots$ and $x = b_0 p^2 + b_1 p^3 + \ldots$. Therefore $(x, y) \in K_1 \subset K_0$ and $(x, y) \notin K_i$ for $i \geq 2$. Consequently $f(x, y) = p^0 + p^3$. Thus $\varphi(f(x, y) - (1 + p^3)) = 0 < \epsilon$. Therefore

$$\lim_{(x,y) \to (p^2, p)} f(x, y) = 1 + p^3.$$

In order to prove that

$$\lim_{(x,y) \to (0,0)} f(x, y) = \sum_{i=0}^{\infty} p^{3i},$$

for each $\epsilon > 0$, a $\delta > 0$ must be found such that $\varphi\left(f(x, y) - \sum_{i=0}^{\infty} p^{3i}\right) < \epsilon$ whenever $\varphi((x, y)) < \delta$. There exists an $r \in Z'$ such that $p^{-r} \leq \epsilon < p^{-r+1}$. Let $\delta = p^{-r}$. But $\varphi((x, y)) = \max(\varphi(x), \varphi(y))$. Since $\varphi(x) \leq p^{-r}$ and $\varphi(y) \leq p^{-r}$,

$$x = a_{r+1} p^{r+1} + a_{r+2} p^{r+2} + \ldots$$

and

$$y = b_{r+1}p^{r+1} + a_{r+2}p^{r+2} + \dots \ .$$

Then $f(x, y) = p^0 + p^3 + \dots + p^{3r} + R$ where $R = \sum_{i=r+1}^{a} p^{3i}$ and $a \in Z'$, $a \geq r+1$. So

$$\varphi\left(f(x, y) - \sum_{i=0}^{\infty} p^{3i}\right) \leq p^{-3(r+2)} < \epsilon \ .$$

Therefore

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \sum_{i=0}^{\infty} p^{3i} .$$

Continuity of a Function

Continuity plays a surprising role in the space of integrable functions.

Definition 2.5. A function $f : G \to Q_p$ is continuous at a if $\lim_{x \to a} f(x) = f(a)$. A function f is continuous on G if it is continuous at each point of G.

Example 2.6. Let $G = Z_p$, and

$$f(x) = \begin{cases} 0 & \text{if } x \in p^2 Z_p - p^5 Z_p \\ 1 & \text{otherwise} . \end{cases}$$

By Example 2.4, the $\lim_{x \to p^2} f(x) = 0$. Also $f(p^2) = 0$. Thus $\lim_{x \to p^2} f(x) = f(p^2)$. Therefore f is continuous at p^2. It can easily be shown that this function is continuous on Z_p. If $x_0 \in p^2 Z_p - p^5 Z_p$ $\lim_{x \to x_0} f(x) = 0$. If $x_0 \notin p^2 Z_p - p^5 Z_p$ $\lim_{x \to x_0} f(x) = 1$.

Example 2.7. Let $G = Z_p^2$, and $f = \sum_{i=0}^{\infty} p^{3i} \chi(K_i)$ where $K_i = p^i Z_p \times p^i Z_p$. By Example 2.5, $\lim_{(x,y) \to (p^2, p)} f(x, y) = 1 + p^3$. Also, $f(p^2, p) = 1 + p^3$. Thus f is continuous at (p^2, p).

Also by Example 2.5,

$$\lim_{(x,y) \to (0,0)} f(x, y) = \sum_{i=0}^{\infty} p^{3i}.$$

But $f(0, 0) = \sum_{i=0}^{\infty} p^{3i}$. So f is continuous at $(0, 0)$.

Derivative of a Function

The definition of a derivative will be the same as for real numbers. The non-Archimedean metric will cause some results to vary from that which was expected in the real numbers.

Definition 2.6. Let $f: G \to Q_p$. The derivative of f at b denoted by $f'(b)$ is

$$f'(b) = \lim_{x \to b} \frac{f(x) - f(b)}{x - b}.$$

Example 2.8. Let $G = Z_p$, and $f(x) = p$. Let $y \in Z_p$. Then

$$f'(y) = \lim_{x \to y} \frac{p - p}{x - y} = \lim_{x \to y} 0 = 0.$$

Therefore $f'(y) = 0$ for all $y \in Z_p$.

Example 2.9. Let $G = Z_p$,

$$f_i(x) = \begin{cases} p^{2i} & \text{if } x \in p^i + p^{3i} Z_p \\ 0 & \text{otherwise}, \end{cases}$$

and f be defined by $f = \sum_{i=1}^{\infty} f_i$. Show $f'(y) = 0$ for all $y \in Z_p$.

Now $f'(y) = 0$ if and only if

$$\lim_{x \to y} \frac{f(x) - f(y)}{x - y} = 0, \quad \text{and} \quad \lim_{x \to y} \frac{f(x) - f(y)}{x - y} = 0$$

if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\varphi\left(\frac{f(x) - f(y)}{x - y}\right) < \epsilon$ whenever $\varphi(x - y) < \delta$. Let $y \in Z_p$. Either there exists uniquely i_0 such that $y \in p^{i_0} + p^{3i_0} Z_p$ or $y \notin p^i + p^{3i} Z_p$ for all $i \in Z^+$.

Consider the case where there exists uniquely i_0 such that $y \in p^{i_0} + p^{3i_0} Z_p$. Then $y = p^{i_0} + a_0 p^{3i_0} + a_1 p^{3i_0+1} + \ldots$. Let $\delta = p^{-3i_0}$. Then $\varphi(x - y) < p^{-3i_0}$ which implies that

$$x = p^{i_0} + a_0 p^{3i_0} + b_0 p^{3i_0+1} + b_1 p^{3i_0+2} + \ldots .$$

This means that $f(y) = p^{2i_0}$ and $f(x) = p^{2i_0}$. So

$$\varphi\left(\frac{f(x) - f(y)}{x - y}\right) = \varphi\left(\frac{p^{2i_0} - p^{2i_0}}{x - y}\right) = \varphi(0) = 0 < \epsilon .$$

Therefore $f'(y) = 0$ when $y \in p^{i_0} + p^{3i_0} Z_p$ for some i_0.

Consider the case where $y \notin p^i + p^{3i} Z_p$ for all $i \in Z^+$. This implies $y = 0$ or $y = p^m(a_0 + a_1 p + a_2 p^2 + \ldots)$ where one of the following occur:

(1) $m = 0$, and $1 \leq a_0 \leq p-1$;

(2) $m \in Z^+$, and $2 \leq a_0 \leq p-1$; or

(3) $m \in Z^+$, $a_0 = 1$, and there exists an i such that $m < i < 3m$ and $a_i \neq 0$.

If $y = 0$ let $\delta = p^{-r}$ where $p^{-r} \leq \epsilon < p^{-r+1}$. Then $\varphi(x - y) < p^{-r}$ which implies that $x = p^k \varepsilon_0$ where $k \geq r + 1$ and ε_0 is a unit of Z_p. Either $f(x) = 0$ or $f(x) = p^{2k}$, and $f(y) = 0$. If $f(x) = 0$ then

$$\varphi\left(\frac{f(x) - f(y)}{x - y}\right) = \varphi(0) = 0 < \varepsilon .$$

If $f(x) = p^{2k}$ then

$$\varphi\left(\frac{f(x) - f(y)}{x - y}\right) = \varphi\left(\frac{f(x)}{x}\right) = \varphi\left(\frac{p^{2k}}{p^k}\right) = \varphi(p^k) .$$

Thus $\varphi\left(\frac{f(x) - f(y)}{x - y}\right) = p^{-k} < p^{-r} \leq \epsilon$ since $k \geq r + 1$. Therefore $f'(0) = 0$.

If

$$y = p^m(a_0 + a_1 p^1 + a_2 p^2 + \dots)$$

where $m = 0$ and $1 \leq a_0 \leq p-1$ let $\delta = p^{-1}$. Then $\varphi(x - y) < p^{-1}$ which implies that

$$x = a_0 + a_1 p^1 + b_0 p^2 + b_1 p^3 + \dots \quad .$$

Thus $f(x) = 0$ and $f(y) = 0$. So $\varphi\left(\frac{f(x) - f(y)}{x - y}\right) = 0 < \epsilon$. Therefore $f'(y) = 0$ when $y = a_0 + a_1 p^1 + a_2 p^2 + \dots$ and $1 \leq a_0 \leq p-1$.

If $y = p^m(a_0 + a_1 p + a_2 p^2 + \dots)$ where $m \epsilon Z^+$ and $2 \leq a_0 \leq p-1$ then let $\delta = p^{-m}$. Then $\varphi(x - y) < p^{-m}$ which implies that $x = p^m(a_0 + b_0 p + b_1 p^2 + \dots)$. Thus $f(x) = 0$ and $f(y) = 0$. So $\varphi\left(\frac{f(x) - f(y)}{x - y}\right) = 0 < \epsilon$. Therefore $f'(y) = 0$ when $y = p^m(a_0 + a_1 p + a_2 p^2 + \dots)$, $m \epsilon Z^+$ and $2 \leq a_0 \leq p-1$.

If $y = p^m(a_0 + a_1 p + a_2 p^2 + \dots)$ where $m \in Z^+$, $a_0 = 1$ and there exists i such that $m < i < 3m$ and $a_i \neq 0$ then let $\delta = p^{-3m}$. Then $\varphi(x - y) < p^{-3m}$ which implies that

$$x = p^m(a_0 + a_1 p + \dots + a_{3m} p^{3m} + b_0 p^{3m+1} + b_1 p^{3m+2} + \dots) .$$

Thus $f(y) = 0$ and $f(x) = 0$. Again, $\varphi\left(\frac{f(x) - f(y)}{x - y}\right) = 0 < \epsilon$. Therefore $f'(y) = 0$ when $y = p^m(a_0 + a_1 p + a_2 p^2 + \dots)$, $m \in Z^+$, $a_0 = 1$ and there exists i such that $m < i < 3m$ and $a_i \neq 0$.

Therefore $f'(y) = 0$ for all $y \in Z_p$.

A Family $\mathfrak{C}$

Let $\mathfrak{C}$ represent a family of open compact subgroups of the group G with the properties:

(1) $\bigcap \{K : K \in \mathfrak{C}\} = \{e\}$ where e is the identity element of G, and

(2) $A \in \mathfrak{C}$ and $B \in \mathfrak{C}$ implies that $A \cap B \in \mathfrak{C}$.

Since such a family plays a central role in Riemann p-adic integration it is advantageous to exhibit such a family.

<u>Example 2.10.</u> Let $G = Q_p$, then

$$\mathfrak{D} = \{p^{kr} Z_p : r \in Z \text{ and } k \text{ is a constant, } k \in Z^+\}$$

is a family $\mathfrak{C}$.

By Theorem 1.14, $p^{kr} Z_p$ is a subgroup of Q_p for each r. Also, $p^{kr} Z_p$ is an open compact subset of Q_p for each $kr \in Z$. Let $p^{kr_1} Z_p$, $p^{kr_2} Z_p \in \mathfrak{D}$ where $r_1 < r_2$. Then

$$p^{kr_1}Z_p \cap p^{kr_2}Z_p = p^{kr_2}Z_p ,$$

and $p^{kr_2}Z_p \in \mathfrak{D}$. The identity of Q_p is 0.

Finally it must be shown that $\cap \{D : D \in \mathfrak{D}\} = \{0\}$. It is clear that $\{0\} \subset \cap \{D : D \in \mathfrak{D}\}$. Assume $\cap \{D : D \in \mathfrak{D}\} \not\subset \{0\}$. Then there exists $\alpha \in Q_p$, $\alpha \neq 0$, such that $\alpha \in \cap \{D : D \ \mathfrak{D}\}$. But $\alpha = p^{\ell}\varepsilon_0$ where $\ell \in Z$ and ε_0 is a unit of Z_p. There exists $r_1 \in Z$ such that $kr_1 < \ell$. So $\alpha \notin p^{kr_1}Z_p$. This implies that $\alpha \notin \cap \{D : D \in \mathfrak{D}\}$. Thus $\cap \{D : D \in \mathfrak{D}\} \subset \{0\}$. Therefore $\mathfrak{D}$ is a family $\mathfrak{C}$.

<u>Example 2.11</u>. Let $G = Z_p$, then

$$\mathfrak{D} = \{p^{kr}Z_p : r \in Z', \ k \text{ is a constant}, \ k \in Z^+\}$$

is a family $\mathfrak{C}$. The demonstration is similar to the demonstration for Example 2.10.

<u>Example 2.12</u>. Let $G = Q_p^2$, then

$$\mathfrak{D} = \{p^{kr}Z_p \times p^{\ell s}Z_p : r, s \in Z \ \text{ and } \ k, \ell \ \text{ constants}, \ k, \ell \in Z^+\}$$

is a family $\mathfrak{C}$.

Q_p is a locally compact topological group. Then Q_p^2 is a locally compact topological group. By Theorem 1.14, $p^{kr}Z_p$ and $p^{\ell s}Z_p$ are compact subgroups of Q_p. Thus $p^{kr}Z_p \times p^{\ell s}Z_p$ is a compact subgroup of $Q_p \times Q_p$. Also as $p^{kr}Z_p$ and $p^{\ell s}Z_p$ are open in Q_p, then $p^{kr}Z_p \times p^{\ell s}Z_p$ is open in $Q_p \times Q_p$. Let

$$p^{kr_1}Z_p \times p^{\ell s_1}Z_p \quad \text{and} \quad p^{kr_2}Z_p \times p^{\ell s_2}Z_p$$

be elements of $\mathfrak{D}$. Then

$$p^{kr_1}Z_p \times p^{\ell s_1}Z_p \cap p^{kr_2}Z_p \times p^{\ell s_2}Z_p$$
$$= p^{k\max\{r_1, r_2\}} \times p^{\ell \max\{s_1, s_2\}} \in \mathfrak{D}.$$

Show $\cap\{D : D \in \mathfrak{D}\} = \{0\}$. It is clear that $\{0\} \subset \cap\{D : D \in \mathfrak{D}\}$. Assume that $\cap\{D : D \in \mathfrak{D}\} \not\subset \{0\}$. Then there exists $\alpha \in Q_p^2$, $\alpha \neq 0$, such that $\alpha \in \cap\{D : D \in \mathfrak{D}\}$. Thus $\alpha = p^n Z_p \times p^m Z_p$ where $n, m \in Z^+$ as $\alpha \in p^k Z_p \times p^\ell Z_p$. There exists $r_1 \in Z$ and $s_1 \in Z$ such that $r_1 k > n$ and $s_1 \ell > m$. Therefore $\alpha \notin p^{r_1 k} Z_p \times p^{s_1 \ell} Z_p$. So $\alpha \notin \cap\{D : D \in \mathfrak{D}\}$. So $\cap\{D : D \in \mathfrak{D}\} = \{0\}$. Therefore $\mathfrak{D}$ is a family $\mathcal{C}$.

Example 2.13. Let $G = Z_p^2$, then

$$\mathfrak{D} = \{p^{kr}Z_p \times p^{\ell s}Z_p : r, s \in Z', \ k, \ell \ \text{constants}, \ k, \ell \in Z^+\}$$

is a family $\mathcal{C}$. The demonstration is similar to the demonstration for Example 2.12.

The families $\mathcal{C}$ will be restricted to subfamilies of $\{p^r Z_p : r \in Z\}$ or $\{p^r Z_p \times p^s Z_p : r, s \in Z\}$.

Riemann Sums

Riemann sums will be developed with respect to the family $\mathcal{C}$.

Definition 2.7. Let G be a compact group with $\varphi(m(G)) = \gamma$, and f be a function from G to Q_p. The Riemann sum of f relative to K and $\{\xi_i\}$ is the sum $\sum_{i=1}^{s} m(K) f(\xi_i)$ where $K \in \mathcal{C}$, $s = [G : K]$, and

$\{\xi_i\}$ is a set consisting of one and only one element from each of the cosets of K with respect to G. The notation $S(f, K, \xi)$ is used to represent this sum. (Note $m(K) = m(x+K)$ for all $x \in G$.)

Example 2.14. Let $G = Z_p$, $\mathcal{C} = \{K_r : K_r = p^{2r} Z_p, \ r \in Z'\}$, and $f(x) = p$ for all $x \in Z_p$. Consider $K_0 \in \mathcal{C}$. By Theorem 1.14, $[Z_p : K_0] = p^0 = 1$. So $S(f, K_0, \xi) = m(K_0) f(\xi_1)$ where $\xi_1 \in Z_p$. Or $S(f, K_0, \xi) = 1 \cdot p = p$.

Consider $K_1 \in \mathcal{C}$. By Theorem 1.14, $[Z_p : K_1] = p^2$. So

$$S(f, K_1, \xi) = \sum_{i=1}^{p^2} m(K_1) f(\xi_i) = \sum_{i=1}^{p^2} p^{-2} \cdot p$$

Thus $S(f, K_1, \xi) = p^2 \cdot p^{-2} \cdot p$. Finally $S(f, K_1, \xi) = p$.

Consider $K_2 \in \mathcal{C}$. By Theorem 1.14, $[Z_p : K_2] = p^4$. So

$$S(f, K_2, \xi) = \sum_{i=1}^{p^4} m(K_2) f(\xi_i) = p^4 \cdot p^{-4} \cdot p$$

Thus $S(f, K_2, \xi) = p$.

Consider $K_r \in \mathcal{C}$. Then $[Z_p : K_r] = p^{2r}$, and

$$S(f, K_r, \xi) = \sum_{i=1}^{p^{2r}} m(K_r) f(\xi_i) .$$

Or $S(f, K_r, \xi) = p^{2r} \cdot p^{-2r} \cdot p$. Thus $S(f, K_r, \xi) = p$. In this case $S(f, K_r, \xi) = p$ for every r and for every admissible $\{\xi_i\}$.

Example 2.15. Let $G = Z_p^2$,

$$\mathcal{C} = \{K_r : K_r = p^r Z_p \times p^r Z_p, \ r \in Z'\} ,$$

and

$$f = \sum_{r=0}^{\infty} p^{3r} \chi(K_r) .$$

Consider $K_0 \in \mathcal{C}$. By Theorem 1.17, $[Z_p^2 : K_0] = 1$. Thus $S(f, K_0, \xi) = m(K_0) f(\xi_1)$ where $\xi_1 \in Z_p^2$. Either $\xi_1 = (0,0)$ or $\xi_1 = \left(p^{k_1}\varepsilon_1, p^{k_2}\varepsilon_2\right)$ where $k_1, k_2 \in Z'$ and $\varepsilon_1, \varepsilon_2$ are units of Z_p. If $\xi_1 = (0,0)$ then $f(\xi_1) = \sum_{r=0}^{\infty} p^{3r}$. If $\xi_1 = \left(p^{k_1}\varepsilon_1, p^{k_2}\varepsilon_2\right)$ then

$$f(\xi_1) = \sum_{r=0}^{\min\{k_1, k_2\}} p^{3r} .$$

So

$$S(f, K_0, \xi) = \begin{cases} p^0 \sum_{r=0}^{\infty} p^{3r} & \text{if } \xi_1 = (0,0) \\ p^0 \sum_{r=0}^{\min\{k_1,k_2\}} p^{3r} & \text{if } \xi_1 = \left(p^{k_1}\varepsilon_1, p^{k_2}\varepsilon_2\right) \end{cases}$$

$$= p^0 + R_0$$

where

$$R_0 = \begin{cases} 0 \\ \sum_{r=1}^{a} p^{3r} \quad \text{where } a \in Z, a \geq 1 \text{ or } a = \infty . \end{cases}$$

Consider $K_1 \in \mathcal{C}$. So $[Z_p^2 : K_1] = p^2$. Then

$$S(f, K_1, \xi) = \sum_{i=1}^{p^2} m(K_1) f(\xi_i) = m(K_1) \sum_{i=1}^{p^2} f(\xi_i) .$$

For all cosets of K_1 except K_1 itself any element chosen from the coset will have a function value of p^0. Let ξ_1 be the element chosen from K_1. Then $\xi_1 = (0,0)$ or $\xi_1 = \left(p^{k_1}\varepsilon_1, p^{k_2}\varepsilon_2\right)$ where $\varepsilon_1, \varepsilon_2$ are units of Z_p and $k_1, k_2 \in Z$, $k_1, k_2 \geq 1$. Thus

$$S(f,K,\xi) = p^{-2}\left[p^0(p^2-1) + \begin{cases} \sum\limits_{r=0}^{\infty} p^{3r} & \text{if } \xi_1 = (0,0) \\ \sum\limits_{r=0}^{\min\{k_1,k_2\}} p^{3r} & \text{if } \xi_1 = \left(p^{k_1}\varepsilon_1, p^{k_2}\varepsilon_2\right) \end{cases}\right]$$

$$= p^{-2}[p^2 - p^0 + p^0 + p^3 + R_1]$$

where

$$R_1 = \begin{cases} 0 \\ \sum\limits_{r=2}^{a} p^{3r} \quad \text{where } a \in Z, a \geq 2 \text{ or } a = \infty . \end{cases}$$

So $S(f,K,\xi) = p^0 + p^1 + p^{-2}R_1$.

Consider $K_2 \in \mathbf{C}$. Then $[Z_p^2 : K_2] = p^4$. Thus

$$S(f,K_2,\xi) = \sum_{i=1}^{p^4} m(K_2) f(\xi_i) \quad \text{or} \quad S(f,K_2,\xi) = p^{-4} \sum_{i=1}^{p^4} f(\xi_i) .$$

The distinct cosets of K_2 can be represented by $x + K_2$ where $x = (a_0 + a_1 p, b_0 + b_1 p)$. If $a_0 \neq 0$ and $b_0 \neq 0$ there are $p^2(p^2-1)$ cosets and the function value of any element in these cosets is p^0. If $a_0 = 0$, $b_0 = 0$ and a_1, b_1 not both zero there are $p^2 - 1$ cosets and the function value for any element in these cosets is $p^0 + p^3$. If $a_0, a_1, b_0, b_1 = 0$ the coset is K_2. Let ξ_1 be the element chosen from K_2. Then $\xi_1 = (0,0)$ or $\xi_1 = p^{k_1}\varepsilon_1, p^{k_2}\varepsilon_2$ where $\varepsilon_1, \varepsilon_2$ are units of Z_p and $k_1, k_2 \epsilon Z$, $k, k_2 \geq 2$. So

$S(f, K_2, \xi)$

$$= p^{-4}\left[p^2(p^2-1)p^0 + (p^2-1)(p^0+p^3) + \begin{cases} \sum_{r=0}^{\infty} p^{3r} & \text{if } \xi_1 = (0,0) \\ \sum_{r=0}^{\min\{k_1,k_2\}} p^{3r} & \text{if } \xi_1 = \left(p^{k_1}\varepsilon_1, p^{k_2}\varepsilon_2\right) \end{cases}\right]$$

$$= p^{-4}[p^4 - p^2 + p^2 - p^0 + p^5 - p^3 + p^0 + p^3 + p^6 + R_2]$$

$$= p^{-4}[p^4 + p^5 + p^6 + R_2]$$

where

$$R_2 = \begin{cases} 0 \\ \sum_{r=3}^{a} p^{3r} & \text{where } a \epsilon Z, a \geq 3 \text{ or } a = \infty . \end{cases}$$

Thus $S(f, K_2, \xi) = p^0 + p + p^2 + p^{-4}R_2$.

By the use of mathematical induction it can be shown that $S(f, K_n, \xi) = p^0 + p^1 + \ldots + p^n + p^{-2n}R_n$ where $n \epsilon Z'$ and

$$R_n = \begin{cases} 0 \\ \sum_{r=n+1}^{a} p^{3r} & \text{where } a \epsilon Z, a \geq n+1 \text{ or } a = \infty , \end{cases}$$

Pseudo Distance

In order to discuss the relationship of Riemann integrability and continuity, it is necessary to have a function with special properties that is related to the family $\mathcal{C}$.

Definition 2.8. The function $w_{\mathcal{C}}$ associated with the family $\mathcal{C}$ is defined as $w_{\mathcal{C}}(x) = \inf\{m(K) : K \in \mathcal{C}_x\}$ where $\mathcal{C}_x = \{K : K \in \mathcal{C}$ and $x \in K\}$.

This function $w_{\mathcal{C}}$ will be used to induce a function ρ such that $\rho(x, y) = w_{\mathcal{C}}(x-y)$. The function ρ fails to satisfy property (1) of a distance function, and is called a pseudo distance function.

Example 2.16. Let $G = Z_p$, $\mathcal{C} = \{p^r Z_p, r \in Z'\}$, and $x = p^3 \varepsilon_0$ where ε_0 is a unit of Z_p. Then $\mathcal{C}_x = \{p^r Z_p : r \in Z,\ 0 \leq r \leq 3\}$. So

$$w_{\mathcal{C}}(x) = \inf\{m(p^r Z_p) : r \in Z,\ 0 \leq r \leq 3\}$$
$$= \inf\{p^{-r} : r \in Z,\ 0 \leq r \leq 3\}$$

Therefore $w_{\mathcal{C}}(x) = p^{-3}$.

Note that the function $w_{\mathcal{C}}$ is p^{-r}, where $p^r Z_p$ is the smallest disc in $\mathcal{C}$ that contains x.

Example 2.17. Let $G = Q_p^2$, $\mathcal{C} = \{p^{2r} Z_p \times p^r Z_p : r \in Z\}$, and $x = (p^{-4}\varepsilon_0, p\varepsilon_1)$ where $\varepsilon_0, \varepsilon_1$ are units of Z_p. Then

$$\mathcal{C}_x = \{p^{2r} Z_p \times p^r Z_p : r \in Z,\ r \leq -2\}.$$

So

$$w_{\mathcal{C}}(x) = \inf\{m(p^{2r} Z_p \times p^r Z_p) : r \in Z,\ r \leq -2\}$$
$$= \inf\{p^{-3r} : r \in Z,\ r \leq -2\}$$

Thus $w_{\mathcal{C}}(x) = p^6$.

Theorem 2.2. The function $w_{\mathcal{C}}$ has the following properties:

(1) The function $w_{\mathcal{C}}(e) = 0$ where e is the identity of G.

(2) The function $\rho(x, y) = w_{\mathcal{C}}(x-y)$ is not a distance function.

(3) If $\mathcal{C}$ and $\mathcal{C}'$ are two appropriate families such that $\mathcal{C} \subset \mathcal{C}'$ then $w_{\mathcal{C}}(x) \geq w_{\mathcal{C}'}(x)$ for all $x \in G$.

(4) Zero is the greatest lower bound of $w_{\mathcal{C}}(x)$.

(5) if $G = Q_p$, $\mathcal{C} = \{p^r Z_p : r \in Z\}$ then $w_{\mathcal{C}}(x) = \varphi(x)$ for all $x \in Q_p$.

Proof: (1) Assume $G \subset Q_p$. Then since $0 \in K$ for every $K \in \mathcal{C}$, $w_{\mathcal{C}}(0) = \inf\{m(K) : K \in \mathcal{C}\}$. Suppose $\inf\{m(K) : K \in \mathcal{C}\} \neq 0$, then $\inf\{m(K) : K \in \mathcal{C}\} = \epsilon > 0$, as $m(K) > 0$ for each $K \in \mathcal{C}$. There exists $k \in Z$ such that $p^{-k} \leq \epsilon < p^{-k+1}$. The family $\mathcal{C}$ is a subfamily of $\{p^r Z_p : r \in Z\}$. Let $r > k, r \in Z$. If $p^r \in K$ for all $K \in \mathcal{C}$ then $\bigcap \{K : K \in \mathcal{C}\} \neq \{0\}$ which is a contradiction. Therefore there exists $K \in \mathcal{C}$ such that $p^r \notin K$. This implies that $m(K) \leq p^{-r} < p^{-k} \leq \epsilon$ which is a contradiction. Thus it follows that $\inf\{m(K) : K \in \mathcal{C}\} = 0$, and that $w_{\mathcal{C}}(0) = 0$.

Assume $G \subset Q_p^2$. Then $\mathcal{C}$ is a subfamily of

$$\{p^r Z_p \times p^s Z_p : r, s \in Z\} .$$

Since $(0, 0) \in K$ for every $K \in \mathcal{C}$, $w_{\mathcal{C}}(0, 0) = \inf\{m(K) : K \in \mathcal{C}\}$. Suppose the $\inf\{m(K) : K \in \mathcal{C}\} \neq 0$, then $\inf\{m(K) : K \in \mathcal{C}\} = \epsilon > 0$, as $m(K) > 0$ for each $K \in \mathcal{C}$. There exists $k \in Z$ such that

$p^{-k} \leq \epsilon < p^{-k+1}$. Let $t > k, t \in Z$. There exists $K' \in \mathcal{C}$ such that $(p^t, 0) \notin K'$ for if not $(p^t, 0) \in \cap \{K : K \in \mathcal{C}\}$ which is a contradiction. Thus $K' = p^{r_0}Z_p \times p^s Z_p$ where $r_0, s \in Z, r_0 > t$. Let $u > k, u \in Z$. There exists $K'' \in \mathcal{C}$ such that $(0, p^u) \notin K''$ for if not $(0, p^u) \in \cap \{K : K \in \mathcal{C}\}$ which is a contradiction. Thus $K'' = p^r Z_p \times p^{s_0} Z_p$ where $r, s_0 \in Z$, $s_0 > u$. But

$$K' \cap K'' = p^{\max\{r_0, r\}} Z_p \times p^{\max\{s_0, s\}} Z_p \in \mathcal{C} .$$

It follows that

$$m(K' \cap K'') = p^{-m\{r_0, r\} - \max\{s_0, s\}} \leq p^{-r_0 - s_0}$$
$$< p^{-t-u} < p^{-2k} < p^{-k} \leq \epsilon .$$

Which contradicts that $\inf\{m(K) : K \in \mathcal{C}\} = \epsilon$. Therefore $\inf\{m(K) : K \in \mathcal{C}\} = 0$, and $w_{\mathcal{C}}(0, 0) = 0$.

(2) Let $G = Z_p^2$, $\mathcal{C} = \{p^r Z_p \times p^s Z_p : r, s \in Z'\}$, and $x = (1, 0)$. Then $\mathcal{C}_x = \{Z_p \times p^s Z_p : s \in Z'\}$. So

$$w_{\mathcal{C}}(x) = \inf\{m(Z_p \times p^s Z_p) : s \in Z'\} = \inf\{p^{-s} : s \in Z'\} = 0 .$$

But $(1, 0) \neq (0, 0)$ so $w_{\mathcal{C}}$ fails to satisfy property (1) of a distance function.

(3) Let $x \in G$. For each $K \in \mathcal{C}_x, K \in \mathcal{C}'_x$ as $\mathcal{C}_x \subset \mathcal{C}'_x$. So $w_{\mathcal{C}}(x) = \inf\{m(K) : K \in \mathcal{C}_x\}$. But

$$\inf\{m(K) : K \in \mathcal{C}_x\} \geq \inf\{m(K) : K \in \mathcal{C}'_x\} = w_{\mathcal{C}'}(x) .$$

Therefore $w_{\mathcal{C}}(x) \geq w_{\mathcal{C}'}(x)$ for all $x \in G$.

(4) For each $K \in \mathcal{C}$, $m(K) > 0$. This implies the $w_{\mathcal{C}}(x) \geq 0$ for each $x \in G$. But $w_{\mathcal{C}}(e) = 0$. Therefore the greatest lower bound of $w_{\mathcal{C}}(x)$ is 0.

(5) Let $x \in Q_p$. Then $x = 0$ or $x = p^k \varepsilon_0$ for some $k \in Z$ and ε_0 a unit in Z_p. If $x = 0$ then $\mathcal{C}_x = \mathcal{C}$. So $w_{\mathcal{C}}(0) = \inf\{p^{-r} : r \in Z\} = 0$. Also $\varphi(0) = 0$.

If $x = p^k \varepsilon_0$ then $\mathcal{C}_x = \{p^r Z_p : r \leq k\}$. So

$$w_{\mathcal{C}}(p^k \varepsilon_0) = \inf\{p^{-r} : r \leq k\} = \inf\{p^{-r} : r \leq k\} = p^{-k}.$$

Also $\varphi(p^k \varepsilon_0) = p^{-k}$. Therefore $w_{\mathcal{C}}(x) = \varphi(x)$ for all $x \in G$.

In a similar manner it can be shown that if $G = Z_p$, $\mathcal{C} = \{p^r Z_p : r \in Z'\}$ then $w_{\mathcal{C}}(x) = \varphi(x)$.

<u>Example 2.18</u>. Let $G = Z_p^2$, $\mathcal{C}' = \{p^r Z_p \times p^s Z_p : r, s \in Z'\}$, and $\mathcal{C} = \{p^{2r} Z_p \times p^{3r} Z_p : r \in Z'\}$. Thus $\mathcal{C} \subset \mathcal{C}'$. So by property (2) of Theorem 2.2 $w_{\mathcal{C}}(x) \geq w_{\mathcal{C}'}(x)$ for each $x \in G$. In particular, $w_{\mathcal{C}}(p^2, p^3) = p^{-5} = w_{\mathcal{C}'}(p^2, p^3)$. Also $w_{\mathcal{C}}(p, p^2) = 1$ but $w_{\mathcal{C}'}(p, p^2) = p^{-3}$. So $w_{\mathcal{C}}(p, p^2) > w_{\mathcal{C}'}(p, p^2)$.

<u>Example 2.19</u>. Let $G = Z_p$, and $\mathcal{C} = \{p^r Z_p : r \in Z'\}$. Then $w_{\mathcal{C}}(x) = \varphi(x)$ for each $x \in Z_p$. But $\varphi(x) \leq 1$ for each $x \in Z_p$. Therefore $w_{\mathcal{C}}(x) \leq 1$. So for this particular situation $w_{\mathcal{C}}(x)$ is bounded above.

<u>Example 2.20</u>. Let $G = Q_p$, and $\mathcal{C} = \{p^r Z_p : r \in Z\}$. Then $w_{\mathcal{C}}(x) = \varphi(x)$ for each $x \in Q_p$. But $\varphi(x)$ is not bounded above for $x \in Q_p$. Therefore $w_{\mathcal{C}}(x)$ is not bounded above for this situation.

The two previous examples illustrate that $w_{\mathcal{C}}$ may or may not be bounded above.

Theorem 2.3. Let $G \subset Q_p$, and $\mathcal{C} \subset \{p^r Z_p : r \in Z\}$. Then $\rho(x, y) = w_{\mathcal{C}}(x-y)$ is a non-Archimedean distance function.

Proof: By Theorem 2.2, $w_{\mathcal{C}}(0) = 0$, and $w_{\mathcal{C}}(x-y) \geq 0$. Suppose $x \neq y$. Then $x - y \neq 0$, and $x - y = p^m \varepsilon_0$ where $m \in Z$ and ε_0 is a unit of Z_p. There exists a $K' \in \mathcal{C}$ such that $p^m \varepsilon_0 \notin K'$ for if not $p^m \varepsilon_0 \in \bigcap_{K \in \mathcal{C}} K$ which contradicts $\bigcap_{K \in \mathcal{C}} K = \{e\}$. Therefore $w_{\mathcal{C}}(x-y) = \inf\{m(K) : K \in \mathcal{C}_{x-y}\} \geq m(K) > 0$. Thus ρ has property 1 of a distance function.

Also, ρ has property 2 of a distance function. For each $K \in \mathcal{C}$ such that $x - y \in K$, it follows that $y - x \in K$ as K is a subgroup of G. Therefore $w_{\mathcal{C}}(x-y) = w_{\mathcal{C}}(y-x)$.

Consider $w_{\mathcal{C}}(\alpha + \beta) \leq w_{\mathcal{C}}(\alpha) + w_{\mathcal{C}}(\beta)$ where $\alpha, \beta \in G$. Either $\varphi(\alpha) \geq \varphi(\beta)$ or $\varphi(\alpha) \leq \varphi(\beta)$. Without loss of generality assume that $\varphi(\alpha) \geq \varphi(\beta)$. If $\varphi(\alpha) > \varphi(\beta)$ then $\varphi(\alpha + \beta) = \varphi(\alpha)$ as $\varphi(\alpha + \beta) \leq \max\{\varphi(\alpha), \varphi(\beta)\}$ with equality when $\varphi(\alpha) \neq \varphi(\beta)$. This implies that $\mathcal{C}_{\alpha+\beta} = \mathcal{C}_{\alpha}$, and the latter implies that $w_{\mathcal{C}}(\alpha + \beta) = w_{\mathcal{C}}(\alpha)$. Since $\varphi(\alpha) > \varphi(\beta)$, $\mathcal{C}_{\alpha} \subset \mathcal{C}_{\beta}$, and $w_{\mathcal{C}}(\alpha) \geq w_{\mathcal{C}}(\beta)$. Thus $w_{\mathcal{C}}(\alpha + \beta) = \max\{w_{\mathcal{C}}(\alpha), w_{\mathcal{C}}(\beta)\}$. If $\varphi(\alpha) = \varphi(\beta)$ then $\varphi(\alpha + \beta) \leq \varphi(\alpha)$. This implies that $\mathcal{C}_{\alpha+\beta} \supset \mathcal{C}_{\alpha}$, and the latter implies that $w_{\mathcal{C}}(\alpha + \beta) \leq w_{\mathcal{C}}(\alpha)$. Thus $w_{\mathcal{C}}(\alpha + \beta) \leq \max\{w_{\mathcal{C}}(\alpha), w_{\mathcal{C}}(\beta)\}$ as $w_{\mathcal{C}}(\alpha) = w_{\mathcal{C}}(\beta)$. Let $\alpha = x - y$, and $\beta = y - z$. Then $w_{\mathcal{C}}(\alpha + \beta) \leq \max\{w_{\mathcal{C}}(\alpha), w_{\mathcal{C}}(\beta)\}$ becomes $w_{\mathcal{C}}(x-z) \leq \max\{w_{\mathcal{C}}(x-y), w_{\mathcal{C}}(y-z)\}$. This implies that $w_{\mathcal{C}}(x-z) \leq w_{\mathcal{C}}(x-y) + w_{\mathcal{C}}(y-z)$. Therefore ρ is a non-Archimedean distance function.

CHAPTER III

RIEMANN p-ADIC INTEGRATION

General Riemann p-Adic Integration

The general Riemann p-adic integration of a function will depend upon the existence of the limit of the Riemann sums in much the same way as it does in Riemann integration over the real numbers. This limit must be independent of the choices of an element in each of the cosets, and is thus rather complicated. The Riemann sums in this case are derived from the partition of G by the left cosets of a subgroup of G and will be similar to what is called a regular partition in the study of integration over the real numbers.

Definition 3.1. The $\lim_{K \in \mathcal{C}} S(f, K, \xi) = A$ if for each $\epsilon > 0$ there exists a $K_0 \in \mathcal{C}$ such that $\varphi(A - S(f, K, \xi)) < \epsilon$ for each $K \subset K_0$, $K \in \mathcal{C}$, and for each choice of $\{\xi_i\}$.

Definition 3.2. A function $f: G \to Q_p$ is $R - \mathcal{C}$ integrable if $\lim_{K \in \mathcal{C}} S(f, K, \xi)$ exists. The space of $R - \mathcal{C}$ integrable functions is represented by $L^{R,\mathcal{C}}(G)$. If $f \in L^{R,\mathcal{C}}(G)$, then denote $\lim_{K \in \mathcal{C}} S(f, K, \xi)$ by $\int^{R,\mathcal{C}} f$.

Example 3.1. Let $G = Z_p$, $\mathcal{C} = \{K_r : K_r = p^{2r} Z_p, r \in Z'\}$, and $f(x) = p$ for all $x \in Z_p$. By Example 2.14, $S(f, K_r, \xi) = p$ for all $r \in Z'$ and all choices of $\{\xi_i\}$. So

$$\int^{R,\mathcal{C}} f = \lim_{r\to\infty} p \,.$$

Thus $\int^{R,\mathcal{C}} f = p$.

Example 3.2. Let $G = Z_p^2$, $\mathcal{C} = \{K_r : K_r = p^r Z_p \times p^r Z_p,\ r \in Z'\}$, and $f = \sum_{r=0}^{\infty} p^{3r} \chi(K_r)$. By Example 2.15,

$$S(f, K_r, \xi) = p^0 + p^1 + \ldots + p^r + p^{-2r} R_r$$

where

$$R_r = \begin{cases} 0 \\ \sum_{i=r+1}^{a} p^{3r} \quad \text{where } a \in Z',\ a \geq r+1 \text{ or } a = \infty, \end{cases}$$

Thus

$$\int^{R,\mathcal{C}} f = \lim_{r\to\infty} (p^0 + p^1 + \ldots + p^r + p^{-2r} R_r) = \sum_{i=0}^{\infty} p^i .$$

Example 3.3. Let $G = Z_p^2$, $K_r' = p^r Z_p \times p^{2r} Z_p$, $r \in Z'$,

$$\mathcal{C} = \{K_r : K_r = p^{2r} Z_p \times p^r Z_p,\ r \in Z'\}, \quad \text{and} \quad f = \sum_{r=0}^{\infty} p^{4r} \chi(K_r').$$

Show $f \notin L^{R,\mathcal{C}}(Z_p^2)$.

Consider the Riemann sums formed by using different sets $\{\xi_{ri}\}$. Let

$$S_1 = \sum_{i=1}^{s} m(K_r) f\{\xi_{ri}\} , \quad \text{and} \quad S_2 = \sum_{i=1}^{s} m(K_r) f(\xi_{ri}')$$

where $\xi_{r1} = (0,0)$, $\xi_{r1}' = (0, p^r)$ and $\xi_{ri} = \xi_{ri}'$ for all i such that

$i \neq 1$. Then

$$S_1 - S_2 = f(\xi_{r1})m(K_r) - f(\xi'_{r1})m(K_r) = [f(\xi_{r1}) - f(\xi'_{r1})]m(K_r).$$

Thus

$$S_1 - S_2 = [f(0,0) - f(0,p^r)]p^{-3r} = \left[\sum_{i=0}^{\infty} p^{4i} - \sum_{i=0}^{[\frac{r}{2}]} p^{4i}\right] p^{-3r}$$

where $[\frac{r}{2}]$ is the greatest integer less than $\frac{r}{2}$. Now

$$S_1 - S_2 = p^{-3r} \sum_{i=[\frac{r}{2}]+1}^{\infty} p^{4i} = \sum_{i=[\frac{r}{2}]+1}^{\infty} p^{4i-3r}$$

Thus

$$\varphi(S_1 - S_2) = \varphi\left(\sum_{i=[\frac{r}{2}]+1}^{\infty} p^{4i-3r}\right) = p^{3r-4[\frac{r}{2}]-4}.$$

When $[\frac{r}{2}]$ is removed, this becomes

$$\varphi(S_1 - S_2) = \begin{cases} p^{6k-4k-4} & \text{if } r = 2k,\ k \in Z'. \\ p^{6k+3-4k-4} & \text{if } r = 2k+1,\ k \in Z'. \end{cases}$$

$$= \begin{cases} p^{2k-4} & \text{if } r = 2k. \\ p^{2k-1} & \text{if } r = 2k+1. \end{cases}$$

As $r \to \infty$, $k \to \infty$ also, and as $k \to \infty$, $\varphi(S_1 - S_2) \to \infty$. Therefore $\lim\limits_{K \in \mathcal{C}} S(f, K, \xi)$ does not exist. Thus $f \notin L^{R,\mathcal{C}}(Z_p^2)$.

Some properties of $L^{R,\mathcal{C}}(G)$ will be developed.

<u>Theorem 3.1.</u> Let $f, g \in L^{R,\mathcal{C}}(G)$, and $a \in Q_p$. Then $f+g \in L^{R,\mathcal{C}}(G)$, $af \in L^{R,\mathcal{C}}(G)$, $\int^{R,\mathcal{C}} (f+g) = \int^{R,\mathcal{C}} f + \int^{R,\mathcal{C}} g$, and $\int^{R,\mathcal{C}} (af) = a \int^{R,\mathcal{C}} f$.

Proof: Let $f, g \in L^{R,\mathcal{C}}$. Consider $f+g$. First by the definition it can be shown that $\lim\limits_{K \in \mathcal{C}} S(f+g, K, \xi)$ exists as follows:

$$S(f+g, K, \xi) = \sum_{i=1}^{s} m(K)(f+g)(\xi_i) = \sum_{i=1}^{s} m(K)[f(\xi_i) + g(\xi_i)]$$

$$= \sum_{i=1}^{s} [m(K) f(\xi_i) + m(K) g(\xi_i)]$$

$$= \sum_{i=1}^{s} m(K) f(\xi_i) + \sum_{i=1}^{s} m(K) g(\xi_i).$$

Consequently

$$\lim_{K \in \mathcal{C}} S(f+g, K, \xi) = \lim_{K \in \mathcal{C}} \sum_{i=1}^{s} m(K) f(\xi_i) + \lim_{K \in \mathcal{C}} \sum_{i=1}^{s} m(K) g(\xi_i),$$

since both of the limits on the right exist, It follows that $\lim\limits_{K \in \mathcal{C}} S(f+g, K, \xi)$ exists. Therefore $f+g \in L^{R,\mathcal{C}}(G)$, and

$$\int^{R,\mathcal{C}} f+g = \int^{R,\mathcal{C}} f + \int^{R,\mathcal{C}} g .$$

Let $f \in L^{R,\mathcal{C}}(G)$ and $a \in Q_p$. Consider af. Then

$$\lim_{K \in \mathcal{C}} \sum_{i=1}^{s} m(K)(af)(\xi_i) = \lim_{K \in \mathcal{C}} \left[a \sum_{i=1}^{s} m(K) f(\xi_i) \right]$$

$$= a \lim_{K \in \mathcal{C}} \sum_{i=1}^{s} m(K) f(\xi_i).$$

This last limit exists as $f \in L^{R,\mathcal{C}}(G)$. Therefore

$$af \in L^{R,\mathcal{C}}(G), \quad \text{and} \quad \int^{R,\mathcal{C}}(af) = a\int^{R,\mathcal{C}} f.$$

The previous theorem leads to the conclusion that $L^{R,\mathcal{C}}(G)$ is a vector space.

Theorem 3.2. If $f, g \in L^{R,\mathcal{C}}(G)$ and $a, b \in Q_p$ then

$$\int^{R,\mathcal{C}}(af+bg) = a\int^{R,\mathcal{C}} f + b\int^{R,\mathcal{C}} g.$$

Proof: By Theorem 3.1,

$$\int^{R,\mathcal{C}}(af+bg) = \int^{R,\mathcal{C}} af + \int^{R,\mathcal{C}} bg = a\int^{R,\mathcal{C}} f + b\int^{R,\mathcal{C}} g.$$

Example 3.4. Let $G = Z_p^2$, $\mathcal{C} = \{K_r : K_r = p^r Z_p \times p^r Z_p, \ r \in Z^+\}$, and $f = p^2 \sum_{r=0}^{\infty} p^{3r} \chi(K_r)$. Find $\int^{R,\mathcal{C}} f$.

Note that $f = p^2 f_1$ where $f_1 = \sum_{r=0}^{\infty} p^{3r} \chi(K_r)$. By Example 3.2 $\int^{R,\mathcal{C}} f_1 = \sum_{i=0}^{\infty} p^i$. Then

$$\int^{R,\mathcal{C}} f = \int^{R,\mathcal{C}} p^2 f_1 = p^2 \int^{R,\mathcal{C}} f_1 \qquad \text{by Theorem 3.2.}$$

Thus $\int^{R,\mathcal{C}} f = p^2 \sum_{i=0}^{\infty} p^i = \sum_{i=0}^{\infty} p^{i+2}$,

Theorem 3.3. Let σ_x be the left translation of a function $f \in L^{R,\mathcal{C}}(G)$, that is, $\sigma_x f(k) = f(x+k)$, then $\int^{R,\mathcal{C}} \sigma_x f = \int^{R,\mathcal{C}} f$.

Proof: By definition $\int^{R,\mathcal{C}} f = \lim_{K \in \mathcal{C}} \sum_{\ell=1}^{s} m(K) f(\xi_\ell)$ and the limit exists as $f \in L^{R,\mathcal{C}}(G)$. Also,

$$\int^{R,\mathcal{C}} \sigma_x f = \lim_{K \in \mathcal{C}} \sum_{i=1}^{s} m(K) \sigma_x f(\xi_i) = \lim_{K \in \mathcal{C}} \sum_{i=1}^{s} m(K) f(x+\xi_i)$$

if this limit exists. These two integrals will be shown equal by showing that each Riemann sum of one is also a Riemann sum of the other one. This will also show that $\sigma_x f \in L^{R,C}(G)$. Now $\sum_{i=1}^{s} m(K) f(x+\xi_i)$ is a Riemann sum of the type $S(f, K, \xi)$ if and only if $\{x+\xi_i\}$ is a collection of one and only one element from each coset of K. Suppose this is not the case, that is, $\xi_j, \xi_k \in \{\xi_i\}$ such that $x+\xi_k, x+\xi_j$ belong to the same coset for some $y \in G$. Then there exists $k_1, k_2 \in K$ such that $x+\xi_j = y+k_1$ and $x+\xi_k = y+k_2$. Thus

$$(x+\xi_j) - (x+\xi_k) = (y+k_1) - (y+k_2)$$

which implies that $\xi_j - \xi_k = k_1 - k_2$. But K is a subgroup of G so $k_1 - k_2 \in K$. So there exists $k_3 \in K$ such that $k_3 = k_1 - k_2 = \xi_j - \xi_k$. Then $\xi_j = \xi_k + k_3$ which implies that $\xi_j \in \xi_k + K$. But $\xi_k \in \xi_k + K$. Thus ξ_j and ξ_k are elements of the same coset of K. This is a contradiction. Thus $\{x+\xi_i\}$ is a collection of one and only one element from each coset of K. Therefore $\sum_{i=1}^{s} m(K) f(x+\xi_i)$ is a Riemann sum of the type $S(f, K, \xi)$.

Now it will be shown that $\sum_{\ell=1}^{s} m(K) f(\xi_\ell)$ is a Riemann sum of the type $S(\sigma_x f, K, \xi)$. For each $\xi_m \in \{\xi_\ell\}$ there exists uniquely $\alpha_m \in G$ such that $\xi_m = x+\alpha_m$. Therefore

$$\sum_{\ell=1}^{s} m(K) f(\xi_\ell) = \sum_{\ell=1}^{s} m(K) f(x+\alpha_m) = \sum_{\ell=1}^{s} m(K) \sigma_x f(\alpha_m) .$$

This will be a Riemann sum of the type $S(\sigma_x f, K, \xi)$ if $\{\alpha_m\}$ is a collection of one and only one element from each coset of K. Again assume that $\alpha_i, \alpha_j \in \{\alpha_m\}$ such that $\alpha_i, \alpha_j \in y+K$, $y \in G$. There exist

$k_1, k_2 \in K$ such that $\alpha_i = y + k_1$ and $\alpha_j = y + k_2$. Therefore $\xi_i - x = y + k_1$ and $\xi_j - x = y + k_2$. Thus $\xi_i - \xi_j = k_1 - k_2$. From here the argument follows as before. So $\{\alpha_m\}$ is a collection of one and only one element from each coset of K. Therefore $\sum_{\ell=1}^{s} m(K) f(\xi_\ell)$ is a Riemann sum of the type $S(\sigma_x f, K, \xi)$. Consequently $\lim_{K \in \mathcal{C}} S(\sigma_x f, K, \xi)$ exists if and only if $\lim_{K \in \mathcal{C}} S(f, K, \xi)$ exists, and

$$\int^{R,\mathcal{C}} \sigma_x f = \int^{R,\mathcal{C}} f.$$

<u>Theorem 3.4.</u> If $\mathcal{C} \subset \mathcal{C}'$ then $L^{R,\mathcal{C}'} \subset L^{R,\mathcal{C}}$ and $\int^{R,\mathcal{C}'} f = \int^{R,\mathcal{C}} f$.

Proof: Let $f \in L^{R,\mathcal{C}'}$. Then for each $\epsilon > 0$ there exists $K_0' \in \mathcal{C}'$ such that

$$\varphi\left(S(f, K', \xi) - \int^{R,\mathcal{C}'} f\right) < \epsilon$$

whenever $K' \subset K_0'$, $K' \in \mathcal{C}'$. There exists $K_0 \in \mathcal{C}$ such that $K_0 \subset K_0'$ as $\cap \{K | K \in \mathcal{C}\} = \{e\}$ and $\mathcal{C} \subset \mathcal{C}'$. Thus for each $K \subset K_0$ it follows that $K \subset K_0 \subset K_0'$ and $K \in \mathcal{C} \subset \mathcal{C}'$. Therefore

$$\varphi\left(S(f, K, \xi) - \int^{R,\mathcal{C}'} f\right) < \epsilon$$

for each $K \subset K_0$. So $f \in L^{R,\mathcal{C}}$ and $\int^{R,\mathcal{C}'} f = \int^{R,\mathcal{C}} f$.

The largest family $\mathcal{C}$ that is possible would be composed of all open compact subgroups of G. This family would give the smallest space of integrable functions. The question may arise as to whether $L^{R,\mathcal{C}'}$ is a proper subset of $L^{R,\mathcal{C}}$ when $\mathcal{C}$ is a proper subset of $\mathcal{C}'$.

Two appropriate families and a special function are chosen in the next example to show that the subspace is a proper subspace.

Example 3.5. Let $G = Z_p^2$, $\mathcal{C} = \{K_r : K_r = p^r Z_p \times p^r Z_p,\ r \in Z'\}$, $\mathcal{C}' = \{K_{rs} : K_{rs} = p^r Z_p \times p^s Z_p,\ r \in Z',\ s \in Z'\}$, and $f = \sum_{r=0}^{\infty} p^{3r} \chi(K_r)$. By Example 3.2, $f \in L^{R,\mathcal{C}}(Z_p^2)$. But $f \notin L^{R,\mathcal{C}'}(Z_p^2)$ as will now be demonstrated.

The demonstration will be complete if it can be shown that $\lim_{K \in \mathcal{C}} S(f, K, \xi)$ does not exist. That is, there exists $\epsilon > 0$ with the property that for each $K \subset \mathcal{C}$ there exists $K', K'' \subset K$ such that $\varphi\ S(f, K', \xi) - S(f, K'', \xi) \geq \epsilon$. These elements of $\mathcal{C}$ must be found, and the right choices for the element in each coset must be made.

Let N be any positive integer. There exists $k \in Z$ such that $p^k > N$. There also exists $r, s \in Z'$ such that $r + s \geq k$. So $\varphi\Big(m(K_{rs})\Big) = \varphi(p^{-r-s}) = p^{r+s} \geq p^k > N$. Let $y = (p^{\ell}, p^{m+3})$, $z = (p^{\ell}, p^m)$ where ℓ and m are constants, $\ell, m \in Z'$, $m \geq s$, and $\ell \geq \max\{r, m+3\}$. Thus y and z are in the same coset of K_{rs} namely K_{rs} itself.

Two Riemann sums, S_1 and S_2, of f with respect to K_{rs} will be exhibited. Let the choices of the elements in each coset be the same except for the choices in the coset K_{rs}. Choose $y \in K_{rs}$ for S_1, and choose $z \in K_{rs}$ for S_2. Thus

$$\varphi(S_1 - S_2) = \varphi\Big[\Big(f(y) - f(z)\Big) m(K_{rs})\Big] = \varphi\Big(f(y) - f(z)\Big)\varphi\Big(m(K_{rs})\Big)$$

Then

$$\varphi(S_1 - S_2) = \varphi\left(\sum_{i=0}^{m+3} p^{3i} - \sum_{i=0}^{m} p^{3i}\right)\varphi\Big(p^{-r-s}\Big)$$

$$= \varphi\left(p^{3m+3} + p^{3m+6} + p^{3m+9}\right) p^{r+s}$$

$$= p^{-3m-3} p^{r+s}$$

$$= p^{r+s-3m-3} .$$

Let r increase without bound. Then y and z remain in the same coset of K_{rs} as they have the same first coordinate and s is fixed. Thus $\varphi(S_1 - S_2)$ increases without bound. Therefore $f \notin L^{R,\mathcal{C}'}(Z_p^2)$.

Note that in the example above $\mathcal{C} \subset \mathcal{C}'$. Thus $L^{R,\mathcal{C}'}(Z_p^2) \subset L^{R,\mathcal{C}}(Z_p^2)$ by Theorem 3.4. But $f \in L^{R,\mathcal{C}}(Z_p^2)$, and $f \notin L^{R,\mathcal{C}'}(Z_p^2)$. Therefore $L^{R,\mathcal{C}'}(Z_p^2)$ is a proper subset of $L^{R,\mathcal{C}}(Z_p^2)$.

The following theorem is a generalization of the previous example.

Theorem 3.5. Let $G = Z_p^2$, and

$$\mathcal{C} = \{K_{rs} : K_{rs} = p^r Z_p \times p^s Z_p ,\ r \in Z' ,\ s \in Z'\} .$$

Then $f \in L^{R,\mathcal{C}}$ if and only if there exists $K \in \mathcal{C}$ such that f is a constant function on each coset of K.

Proof: The "only if" statement will be proved first. This will be proved by contradiction. Suppose that $f \in L^{R,\mathcal{C}}$ and no such K exists. Then for each $K \in \mathcal{C}$ there exists $x, y, z \in G$ such that $y, z \in x + K$ and $f(y) \neq f(z)$. Also for each positive integer N there exists $K' \in \mathcal{C}$ such that $\varphi(m(K')) > N$ as there exists a $k \in Z$ such that $p^k > N$. Now choose $K' = K_{rs}$ such that $r + s \geq k$, and then

$$\varphi\left(m(K_{rs})\right) = \varphi\left(p^{-r-s}\right) = p^{r+s} \geq p^k > N .$$

Consequently there exists $x, y, z \in G$ such that $y \in x+K'$ and $z \in x+K'$ and $f(y) \neq f(z)$ where $y = (y_1, y_2)$ and $z = (z_1, z_2)$. This y and z could be chosen such that they have a coordinate in common. For suppose a y and a z were chosen such that $y_1 \neq z_1$ and $y_2 \neq z_2$. Consider $u = (y_1, z_2)$. Then $u \in x+K$ and $f(u) \neq f(y)$ or $f(u) \neq f(z)$. For suppose $f(u) = f(y)$ and $f(u) = f(z)$ then $f(y) = f(z)$ which is a contradiction. Thus a pair has been found satisfying the given conditions and having a coordinate in common. Let y and z be such a pair. Without loss of generality it can be assumed that the first coordinate is the common one, that is, $y_1 = z_1$.

Let S_1 and S_2 be two Riemann sums of f with respect to K'. Let the choices of the elements in each coset be the same except for the choices in the coset K'. Choose $y \in K'$ for S_1 and choose $z \in K'$ for S_2. Now $S_1 - S_2 = [f(y) - f(z)]\, m(K')$. Then

$$\varphi(S_1 - S_2) = \varphi\{[f(y) - f(z)]\, m(K')\} = \varphi[f(y) - f(z)]\varphi(m(K')) .$$

Let r increase without bound. Then $\varphi(m(K'))$ increases without bound. Also, y and z remain in the same coset of $K' = K_{rs}$ as they have the same first coordinate. So $\varphi[f(y) - f(z)]$ is a nonzero constant. Thus $\varphi(S_1 - S_2)$ increases without bound. Therefore $f \notin L^{R,\mathbb{C}}$.

The "if" statement will now be proved. Assume there exists $K \in \mathbb{C}$ such that f is a constant on each coset of K. The value of $S(f, K', \xi)$ is independent of the choice of $\{\xi_i\}$, and independent of the subset K' of K. Therefore

$$f \in L^{R,\mathbb{C}} \quad \text{and} \quad \int^{R,\mathbb{C}} f = S(f, K', \xi) .$$

Strong Riemann p-Adic Integration

Strong Riemann p-adic integration allows more types of partitions than the general Riemann p-adic integration. The general Riemann p-adic integration was compared to regular partitions, and strong Riemann p-adic integration could be compared to a general partition in the study of the integration over the real numbers. The same type of limit will be used here as was used in general Riemann integration but there will be more potential Riemann sums.

Definition 3.3. Let G be compact. A partition, denoted by δ, is a decomposition of G with $G = \bigcup_{i=1}^{t} x_i + K_i$ where $x_j + K_j \cap x_\ell + K_\ell = \varphi$ if $j \neq k$, $k_i \in \mathbf{C}$ for all i and where there exists $K \in \mathbf{C}$ such that $K_i \subset K$ for all i. The smallest set $K \in \mathbf{C}$ that contains all K_i is called the norm of the partition and is represented by $N(\delta)$.

Definition 3.4. The Riemann sums corresponding to the partition δ are defined as $S(f, \delta, \xi) = \sum_{i=1}^{t} m(K_i) f(\xi_i)$ where $\xi_i \in x_i + K_i$.

Example 3.6. Let $G = Z_p$, $\mathbf{C} = \{K_r : K_r = p^r Z_p, r \in Z'\}$, and f be a function.

Then $\{x + p Z_p\}$ where $x \in \{a_0 : a_0 \in Z', 0 \leq a_0 \leq p-1\}$ is a partition of G. The set is just the distinct cosets of $p Z_p$ so they are pairwise disjoint and their union is G. In this case $K_i = p Z_p$ for all i. Thus $N(\delta) = p Z_p$ and

$$S(f, \delta, \xi) = \sum_{i=1}^{p} m(p Z_p) f(\xi_i) = S(f, p Z_p, \xi) .$$

Similar results could be obtained for any $K \in \mathbf{C}$.

An example of a partition that does not result in a Riemann sum of the form $S(f, K_1, \xi)$ is as follows. Consider $\{x + pZ_p\} \cup \{y + p^2 Z_p\}$ where

$$x \in \{a_0 \mid a_0 \in Z', \ 1 \le a_0 \le p-1\} \quad \text{and} \quad y \in \{a_1 p \mid a_1 \in Z', \ 0 \le a_1 \le p-1\}$$

as a partition of G. The $\{y + p^2 Z_p\}$ are the distinct cosets of $p^2 Z_p$ in pZ_p, and thus $\cup \{y + p^2 Z_p\} = pZ_p$, which are pairwise disjoint. The $\{x + pZ_p\}$ are all of the distinct cosets of pZ_p except pZ_p. So the union of all of these sets is G, and they are pairwise disjoint. Then $N(\delta) = pZ_p$, and

$$S(f, \delta, \xi) = \sum_{i=1}^{p+(p-1)} m(K_i) f(\xi_i) .$$

The latter is not a Riemann sum of the form

$$S(f, pZ_p, \xi) = \sum_{i=1}^{p} m(pZ_p) f(\xi_i) .$$

Definition 3.5. A function f is strong $R - \mathfrak{C}$ integrable if $\lim_{N(\delta) \in \mathfrak{C}} S(f, \delta, \xi)$ exists. A strong $R - \mathfrak{C}$ integrable function is said to be $R' - \mathfrak{C}$ integrable. The space of the $R' - \mathfrak{C}$ - integrable functions will be denoted by $L^{R', \mathfrak{C}}(G)$.

The space $L^{R', \mathfrak{C}}(G)$ will be probed in the following theorems.

Theorem 3.6. Let G be compact. If f is a locally constant function then $f \in L^{R', \mathfrak{C}}(G)$.

Proof: Let f be locally constant. Then there exists $\{x_i + K_i\}_{1 \le i \le n}$ such that $x_j + K_j \cap x_\ell + K_\ell = \varphi$, $\cup \{x_i + K_i\}_{1 \le i \le n} = G$ and f is constant in each $x_i + K_i$. The value of the Riemann sums corresponding to partitions δ such that $N(\delta) \subset \bigcap_{i=1}^{n} K_i$ is always the same. Therefore $f \in L^{R',\mathcal{C}}(G)$.

Example 3.7. Let $G = Z_p$, $\mathcal{C} = \{K_r : K_r = p^r Z_p, r \in Z'\}$, and

$$f = \begin{cases} 0 & \text{if } x \in p^2 Z_p - p^5 Z_p \\ 1 & \text{otherwise} . \end{cases}$$

Consider the partition

$$\{x + p^5 Z_p : x = a_0 + a_1 p + a_2 p^2 + a_3 p^3 + a_4 p^4\}$$

of Z_p. If $a_i = 0$ for $i \in Z'$, $0 \leq i \leq r$, then $x + p^5 Z_p = p^5 Z_p$. For each $\alpha \in p^5 Z_p$, $f(\alpha) = 1$. If $a_0 = 0$, $a_1 = 0$, and there exists $i \in Z'$, $2 \leq i \leq 4$ such that $a_i \neq 0$, then $x + p^5 Z_p \subset p^2 Z_p - p^5 Z_p$. There are $p^3 - 1$ of these sets, and for any element chosen in any one of them the function value is 0. If $a_0 \neq 0$ or $a_1 \neq 0$ then $x + p^5 Z_p \subset Z_p - p^5 Z_p$. There are $p^3(p^2 - 1)$ of these sets, and for any element chosen in any of these sets the function value is 1. For this partition $N(\delta) = p^5 Z_p$. Thus

$$S(f, \delta, \xi) = [(p^3 - 1) \cdot 0 + p^3(p^2 - 1) + 1] p^{-5} = (p^5 - p^3 + 1) p^{-5} = 1 - p^{-2} + p^{-5}$$

regardless of the choices of the element in each set of the partition.

Thus if $N(\delta) \subset p^5 Z_p$ then the Riemann sum would be equal to $1 - p^{-2} + p^{-5}$. Therefore $\lim_{N(\delta) \in \mathcal{C}} S(f, \delta, \xi) = 1 - p^{-2} + p^{-5}$ and $f \in L^{R',\mathcal{C}}(Z_p)$.

Recall that $f(x) = 1$ for all $x \in Z_p - p^2Z_p$, $x \in p^5Z_p$, and $f(x) = 0$ for all other x. Note that

$$m(Z_p - p^2Z_p \cup p^5Z_p) = m(Z_p - p^2Z_p) + m(p^5Z_p)$$
$$= 1 - p^2 + p^{-5} = \int^{R,\mathcal{C}} f.$$

So the integral of a function whose range is 1 on some set and 0 elsewhere is equal to the measure of the set.

Theorem 3.7. The space $L^{R',\mathcal{C}} \subset L^{R,\mathcal{C}}$.

Proof: Let $f \in L^{R',\mathcal{C}}$. Thus $\lim_{N(\delta) \in \mathcal{C}} S(f, \delta, \xi) = A$. This implies that for each $\epsilon > 0$ there exists $K_0 \in \mathcal{C}$ such that $\varphi(S(f, \delta, \xi) - A) < \epsilon$ for each $N(\delta) \subset K_0$.

Assume $f \notin L^{R,\mathcal{C}}$. Therefore $\lim_{K \in \mathcal{C}} S(f, K, \xi) \neq A$. So there exists $\epsilon > 0$ such that for any $K_1 \in \mathcal{C}$ there exists $K \subset K_1$, $K \in \mathcal{C}$ and $\varphi(S(f, K, \xi) - A) \geq \epsilon$. But for each $K \in \mathcal{C}$ the cosets of K form a partition for G and $N(\delta) = K$. Thus $S(f, K, \xi)$ is of the form $S(f, \delta, \xi)$ where $N(\delta) = K$. Therefore $f \notin L^{R',\mathcal{C}}$. This is a contradiction so it must be concluded that $f \in L^{R,\mathcal{C}}$. So $L^{R',\mathcal{C}} \subset L^{R,\mathcal{C}}$.

It will be shown later that $L^{R',\mathcal{C}} = L^{R,\mathcal{C}}$ for certain families $\mathcal{C}$. An example will also be given to show that in certain cases $L^{R',\mathcal{C}}$ is a proper subset of $L^{R,\mathcal{C}}$.

CHAPTER IV

SOME SPECIAL PROPERTIES OF THE SPACES OF INTEGRABLE FUNCTIONS

The discussions in this chapter will be limited to G as a subset of Q_p. This eliminates the use of product spaces and makes $w_{\mathcal{C}}(x-y)$ greater than 0 if $x \neq y$.

It has been shown that $L^{R',\mathcal{C}} \subset L^{R,\mathcal{C}}$ for each family $\mathcal{C}$. Each of these spaces is dependent on the specific family $\mathcal{C}$ that is chosen. In particular, Theorem 3.4 showed that if $\mathcal{C} \subset \mathcal{C}'$ then $L^{R,\mathcal{C}'} \subset L^{R,\mathcal{C}}$. Is it possible to choose a family $\mathcal{C}$ such that $L^{R,\mathcal{C}} = L^{R',\mathcal{C}}$? If so what are the characteristics of the family $\mathcal{C}$? Is it possible to choose a family $\mathcal{C}$ such that a function f exists where $f \in L^{R,\mathcal{C}}$ and $f \notin L^{R',\mathcal{C}}$? These questions will be answered, and as they are, other interesting questions will arise. The requirement that $f \in L^{R,\mathcal{C}}$ places some limitations on the behavior of f relative to the family $\mathcal{C}$. A reasonable starting point for this inquiry is a study of these limitations.

Riemann Integration and Derivative

The first tool that is needed is a relationship between an element of $L^{R,\mathcal{C}}$ and a special limit.

<u>Theorem 4.1.</u> If $f \in L^{R,\mathcal{C}}(G)$ then $\lim_{x \to y} \dfrac{\varphi(f(x) - f(y))}{w_{\mathcal{C}}(x-y)} = 0$ uniformly.

Proof: Let $f \in L^{R,\mathcal{C}}(G)$. Then $\lim_{K \in \mathcal{C}} S(f, K, \xi)$ exists. This implies that for each $\epsilon > 0$ there exists $K_0 \in \mathcal{C}$ such that $\varphi[S(f, K, \xi) - S(f, K, \eta)] < \epsilon$ where K is any element of $\mathcal{C}$ that is also a subset of K_0, and $\{\xi_i\}$ and $\{\eta_i\}$ are arbitrary sets of representatives from each coset of K. Let $x - y \in K \subset K_0$, which implies that x and y are in the same coset of K. Choose a particular pair of sets, $\{\xi_i\}, \{\eta_i\}$, where $\xi_i = \eta_i$ for $2 \le i \le s$, $\xi_1 = x$, and $\eta_1 = y$. Then

$$\begin{aligned}\varphi[S(f, K, \xi) - S(f, K, \eta)] &= \varphi[m(K) f(x) - m(K) f(y)] \\ &= \varphi[(f(x) - f(y)) m(K)] .\end{aligned}$$

The properties of φ make

$$\varphi[S(f, K, \xi) - S(f, K, \eta)] = \varphi(f(x) - f(y)) \varphi(m(K)) < \epsilon$$

for each $K \in \mathcal{C}_{x-y}$, $K \subset K_0$. It now follows that

$$\varphi(f(x) - f(y)) < \frac{\epsilon}{\varphi(m(K))} = \epsilon\, m(K)$$

as $\varphi(m(K)) > 0$ and $\varphi(m(K)) = \frac{1}{m(K)}$. Take the infimum of each side of the inequality. Then

$$\varphi(f(x) - f(y)) \le \epsilon \inf\{m(K) : K \in \mathcal{C}_{x-y}, K \subset K_0\} .$$

Therefore $\varphi(f(x) - f(y)) \le \epsilon\, w_{\mathcal{C}}(x-y)$ which implies $\frac{\varphi(f(x) - f(y))}{w_{\mathcal{C}}(x-y)} < \epsilon$ as in the present setting $w_{\mathcal{C}}(x-y) > 0$ for all $x, y \in G$ such that $x \ne y$.

<u>Corollary 4.1.</u> If $f \in L^{R,\mathcal{C}}(G)$ then f is continuous.

Proof: Let $f \in L^{R,\mathcal{C}}(G)$. Then by Theorem 4.1, $\lim_{x \to y} \frac{\varphi(f(x) - f(y))}{w_{\mathcal{C}}(x-y)} = 0$ uniformly. Remembering that $w_{\mathcal{C}}(x-y) = \inf\{m(K) : K \in \mathcal{C}_{x-y}\}$, then

$w_{\mathcal{C}}(x-y) \leq m(K)$, $K \in \mathcal{C}_{x-y}$. Consequently $w_{\mathcal{C}}(x-y) \leq m(G)$ as $K \subset G$ and $m(K) \leq m(G)$. Now it follows that $\lim\limits_{x \to y} \frac{\varphi(f(x) - f(y))}{m(G)} = 0$. But as $m(G)$ is a constant $\frac{1}{m(G)} \lim\limits_{x \to y} \varphi(f(x) - f(y)) = 0$. Thus $\lim\limits_{x \to y} \varphi(f(x) - f(y)) = 0$ which implies that $\lim\limits_{x \to y} f(x) = f(y)$. Therefore f is continuous.

This is interesting in its contrast to the situation of the Riemann integral over the real numbers. A function may be discontinuous and still be Riemann integrable over the real numbers. Recall, however, that locally constant functions are continuous in this setting but not in the Euclidean spaces. Although a function must be continuous to be integrable in the p-adic setting, not all continuous functions are integrable as is illustrated in the following example.

Example 4.1. Let $G = Z_p$, $\mathcal{C} = \{K_r : K_r = p^r Z_p,\ r \in Z'\}$, and $f(x) = x$ for all $x \in Z_p$. The function f is continuous on Z_p, and $f \notin L^{R,\mathcal{C}}$.

The function f is continuous on Z_p if $\lim\limits_{x \to y} x = y$ for each $y \in Z_p$. That is, for each $\epsilon > 0$ there exists $\delta > 0$ such that $\varphi(x-y) < \epsilon$ whenever $\varphi(x-y) < \delta$. In this case, all that is needed is to let $\delta = \epsilon$. Therefore f is continuous on Z_p.

The function $f \notin L^{R,\mathcal{C}}$ if $\lim\limits_{K \in \mathcal{C}} S(f, K, \xi)$ does not exist. That is, there exists $\epsilon > 0$ with the property that for all $K \in \mathcal{C}$ there exists $K_r \subset K$, $K_r \in \mathcal{C}$, such that $\varphi(S(f, K_r, \xi) - S(f, K_r, \eta)) \geq \epsilon$. Let $\xi_i = \eta_i$ if $i \neq 1$, $\xi_1 = p^r$, and $\eta_1 = 0$. Then

$$S(f, K_r, \xi) - S(f, K_r, \eta) = (f(p^r) - f(0))\, m(p^r Z_p) = p^r \cdot p^{-r} = 1,$$

Consider $\epsilon = \frac{1}{2}$. For any $K \in \mathcal{C}$ there exists $r \in Z'$ such that $K_r \subset K$, and

$$\varphi(S(f, K_r, \xi) - S(f, K_r, \eta)) = \varphi(1) \geq \frac{1}{2}.$$

Therefore $f \notin L^{R,\mathcal{C}}$.

Also, note that

$$f'(y) = \lim_{x \to y} \frac{f(x) - f(y)}{x - y} = \lim_{x \to y} \frac{x - y}{x - y} = \lim_{x \to y} 1 = 1$$

for all $y \in Z_p$.

<u>Theorem 4.2.</u> Let $G = Z_p$, and $\mathcal{C} = \{p^r Z_p : r \in Z'\}$. If $f \in L^{R,\mathcal{C}}(G)$ then $f' = 0$.

Proof: Let $f \in L^{R,\mathcal{C}}(Z_p)$. Then

$$\lim_{x \to y} \varphi\left(\frac{f(x) - f(y)}{x - y}\right) = \lim_{x \to y} \frac{\varphi(f(x) - f(y))}{\varphi(x - y)} = \lim_{x \to y} \frac{\varphi(f(x) - f(y))}{w_{\mathcal{C}}(x - y)} = 0$$

as $\varphi(x-y) = w_{\mathcal{C}}(x-y)$ by Theorem 2.2 (5). Since $\lim_{x \to y} \varphi\left(\frac{f(x) - f(y)}{x - y}\right) = 0$ implies $\lim_{x \to y} \frac{f(x) - f(y)}{x - y} = 0$, it is seen that $f'(y) = 0$ for all $y \in Z_p$.

The converse of Theorem 4.2 is not true as the following example shows.

<u>Example 4.2.</u> Let $G = Z_p$, $\mathcal{C} = \{K_r : K_r = p^r Z_p, r \in Z'\}$, and

$$f_i(x) = \begin{cases} p^{2i} & \text{if } x \in p^i + p^{3i} Z_p \\ 0 & \text{otherwise}, \end{cases}$$

Let f be defined by $f = \sum_{i=1}^{\infty} f_i$. By Example 2.9, $f'(y) = 0$ for all $y \in Z_p$. But $f \notin L^{R,\mathcal{C}}(Z_p)$ as will be shown.

This can be shown by finding two sets of elements $\{\xi_i\}$ and $\{\eta_i\}$ such that the difference of the corresponding Riemann sums remains large. Let

$$S_1 = \sum_{i=1}^{s} m(p^{3\ell-1}Z_p)\, f(\xi_i)\,, \quad \text{and} \quad S_2 = \sum_{i=1}^{s} m(p^{3\ell-1}Z_p)\, f(\eta_i)\,.$$

Choose $\xi_i = \eta_i$ for each $2 \le i \le s$, $\xi_1 = p^\ell$ and $\eta_1 = p^\ell + p^{3\ell-1}$. Note that ξ_1 and η_1 belong to $p^\ell + p^{3\ell-1}Z_p$. Then

$$S_1 - S_2 = \left[f(p^\ell) - f(p^\ell + p^{3\ell-1}) \right] m\left(p^{3\ell-1}Z_p\right) = \left(p^{2\ell} - 0\right) p^{1-3\ell} = p^{1-\ell}\,.$$

Thus $\varphi(S_1 - S_2) = p^{\ell-1}$, and $\varphi(S_1 - S_2)$ increases as ℓ increases. The limit of $S(f, K, \xi)$ cannot exist. Therefore $f \notin L^{R,\mathcal{C}}(Z_p)$.

By the previous example it is seen that the fact that the derivative of a function is the zero function does not imply that the function is an element of $L^{R,\mathcal{C}}$. It is reasonable therefore to turn our attention to the properties of $\mathcal{C}$ which might ensure that $L^{R,\mathcal{C}} = L^{R',\mathcal{C}}$.

Ample Families

Definition 4.1. The family $\mathcal{C}$ is ample if there exists a real number $\beta > 0$ possessing the property: given any K and K' elements of $\mathcal{C}$ with $K' \subset K$, there exists $\{K_i : K_i \in \mathcal{C},\ i \in Z,\ 0 \le i \le n\}$ such that $K' = K_n \subset K_{n-1} \subset \dots \subset K_1 \subset K_0 = K$ and $\varphi[K_j : K_{j+1}] \ge p^{-\beta}$ where $0 \le j \le n-1$.

To emphasize what this definition implies, an ample family and a family that is not ample will be exhibited.

Example 4.3.

(a) Let $\mathcal{C} = \{K_r : K_r = p^{2r} Z_p,\ r \in Z\}$. $\mathcal{C}$ is an ample family with $\beta = 2$. To see this consider any $r_1 \in Z$. Then

$$K_{r_1+1} = p^{2(r_1+1)} Z_p, \quad \text{and} \quad K_{r_1} = p^{2r_1} Z_p .$$

Therefore $[K_{r_1+1} : K_{r_1}] = p^2$. Thus $\varphi[K_{r_1+1} : K_{r_1}] = p^{-2} \geq p^{-2}$.

(b) Let $\mathcal{C} = \{K_r : K_r = p^{2^r} Z_p,\ r \in Z\}$. Assume there exists β that satisfies the conditions of the definition of an ample family. Thus for each $r \in Z$, $\varphi[K_r : K_{r+1}] \geq p^{-\beta}$. Now there exists $r_1 \in Z$ such that $2^{r_1 - 1} \leq \beta < 2^{r_1}$. Since

$$K_{r_1} = p^{2^{r_1}} Z_p \quad \text{and} \quad K_{r_1+1} = p^{2^{r_1+1}} Z_p ,$$

then

$$[K_{r_1} : K_{r_1+1}] = p^{2^{r_1+1} - 2^{r_1}} = p^{2 \cdot 2^{r_1} - 2^{r_1}} = p^{2^{r_1}} .$$

Consequently $\varphi[K_{r_1} : K_{r_1+1}] = p^{-2^{r_1}}$. But $-2^{r_1} < -\beta \leq -2^{r_1 - 1}$. So $\varphi[K_{r_1} : K_{r_1+1}] < p^{-\beta}$. This implies that $\mathcal{C}$ is not an ample family.

The ample family $\mathcal{C}$ is proposed as the family that will cause $L^{R,\mathcal{C}}$ to be equal to $L^{R',\mathcal{C}}$.

Theorem 4.3. Let $\mathcal{C}$ be ample. Then $L^{R,\mathcal{C}} = L^{R',\mathcal{C}}$, and $f \in L^{R,\mathcal{C}}$ if and only if $\lim_{x \to y} \dfrac{\varphi(f(x) - f(y))}{w_{\mathcal{C}}(x-y)} = 0$ uniformly, Proof: The "only if" part of the proof follows directly from Theorem 4.1. The "if" part of the proof will be established if $f \in L^{R',\mathcal{C}}$. For if $f \in L^{R',\mathcal{C}}$ then $f \in L^{R,\mathcal{C}}$ as $L^{R',\mathcal{C}} \subset L^{R,\mathcal{C}}$. It has been shown that if $f \in L^{R,\mathcal{C}}$ then $\lim\limits_{x \to y} \frac{\varphi(f(x) - f(y))}{w_{\mathcal{C}}(x-y)} = 0$ uniformly. If $\lim\limits_{x \to y} \frac{\varphi(f(x) - f(y))}{w_{\mathcal{C}}(x-y)} = 0$ uniformly implies that $f \in L^{R',\mathcal{C}}$ then $L^{R,\mathcal{C}} \subset L^{R',\mathcal{C}}$. The conclusion is that $L^{R,\mathcal{C}} = L^{R',\mathcal{C}}$.

So assuming that $\lim\limits_{x \to y} \frac{\varphi(f(x) - f(y))}{w_{\mathcal{C}}(x-y)} = 0$ uniformly, it will be shown that $f \in L^{R',\mathcal{C}}$. As $\mathcal{C}$ is an ample family there exists β satisfying the conditions of an ample family. Then for each $\epsilon > 0$ there exists $K_0 \in \mathcal{C}$ such that

$$(4.1) \qquad \frac{\varphi(f(x) - f(y))}{w_{\mathcal{C}}(x-y)} < \epsilon p^{-\beta} \text{ for each } K \subset K_0 \text{ and } x - y \in K.$$

Let $S(f, \delta, \xi)$ and $S(f, \delta', \xi')$ be any two Riemann sums such that $N(\delta) \subset K_0$ and $N(\delta') \subset K_0$. Then $f \in L^{R',\mathcal{C}}$ if and only if $\varphi(S(f, \delta, \xi) - S(f, \delta', \xi') < \epsilon p^{-\beta}$. Let δ be the partition $G = \bigcup\limits_{i=1}^{s} (x_i + K_i)$, and δ' be the partition $G = \bigcup\limits_{i=1}^{t} (x_i' + K_i')$. Let

$$K = \left(\bigcap_{i=1}^{s} K_i\right) \cap \left(\bigcap_{i=1}^{t} K_i'\right).$$

It follows that

$$\begin{aligned} &\varphi(S(f, \delta, \xi) - S(f, \delta', \xi')) \\ &\qquad = \varphi(S(f, \delta, \xi) - S(f, K, \eta) + S(f, K, \eta) - S(f, \delta', \xi')) \\ &\qquad \leq \sup\{\varphi(S(f, \delta, \xi) - S(f, K, \eta)), \varphi(S(f, K, \eta) - S(f, \delta', \xi'))\}. \end{aligned}$$

It is seen that

$$\varphi(S(f, \delta, \xi) - S(f, \delta', \xi')) < \epsilon p^{-\beta}$$

if

$$\sup \{\varphi(S(f, \delta, \xi) - S(f, K, \eta)), \varphi(S(f, K, \eta) - S(f, \delta', \xi'))\} < \epsilon p^{-\beta}$$

Without loss of generality assume that the maximum is represented by $\varphi(S(f, \delta, \xi) - S(f, K, \eta))$; then it is sufficient to show that

$$\varphi(S(f, \delta, \xi) - S(f, K, \eta)) < \epsilon p^{-\beta}.$$

Consider the terms of $S(f, \delta, \xi) - S(f, K, \eta)$ which involve $x_i + K_i$. They are $f(\xi_i) m(K_i) - \sum_{j=1}^{t} f(\eta_{ij}) m(K)$ where $\{\eta_{ij}\}$ is a complete collection of representatives of the cosets of K contained in $x_i + K_i$ and ξ_i is any element of $x_i + K_i$. If

$$\varphi(f(\xi_i) m(K_i) - \sum_{j=1}^{t} f(\eta_{ij}) m(K)) < \epsilon p^{-\beta}$$

for each $1 \leq i \leq s$ then $\varphi(S(f, \delta, \xi) - S(f, K, \eta)) < \epsilon p^{-\beta}$. The problem is equivalent to the following statement: if $K \subset K' \subset K_0$ then

$$\varphi(f(\xi) m(K') - \sum_{j=1}^{t} f(\eta_j) m(K)) < \epsilon p^{-\beta}$$

where $\{\eta_j\}$ is a complete collection of representatives of the cosets of K contained in $\xi + K'$. As $\mathfrak{C}$ is an ample family there exists $\{K_i'\}$ such that $K = K_n' \subset K_{n-1}' \subset \ldots \subset K_2' \subset K_1' \subset K_0' = K'$ and $\varphi[K_i' : K_{i+1}'] \geq p^{-\beta}$. Let $\{y_{s,i}\}$ be a complete set of representatives of the cosets of K_s contained in $\xi + K'$. There is only one representative in K_0 as $K_0 = K'$ so $y_0 = \xi$. Thus

$$\varphi\left(f(\xi)\,m(K') - \sum_{j=1}^{t} f(\eta_j)\,m(K)\right) = \varphi\left(\sum_{i=1}^{1} f(y_0)\,m(K_0') - \sum_i f(y_{1,\ })\,m(K_1')\right.$$

$$+ \sum_i f(y_{1,i})\,m(K_1') - \sum_i f(y_{2,i})\,m(K_2')$$

$$+ \sum_i f(y_{2,i})\,m(K_2') - \dots - \sum_i f(y_{n-1,i})\,m(K_{n-1}')$$

$$\left.+ \sum_i f(y_{n-1,i})\,m(K_{n-1}') - \sum_i f(y_{n,i})\,m(K_n')\right)$$

$$\leq \sup\left\{\varphi\left(\sum_i f(y_{s,i})\,m(K_s) - \sum_i f(y_{s+1,i})\,m(K_{s+1})\right)\right\}$$

Thus it will be sufficient if it is proved that

$$\varphi\left(\sum_i f(y_{s,i})\,m(K_s) - \sum_i f(y_{s+1,i})\,m(K_{s+1})\right) < \epsilon\, p^{-\beta}$$

for each $0 \leq s \leq n-1$. This can be reduced as before by just considering the terms of

$$\sum_i f(y_{s,i})\,m(K_s) - \sum_i f(y_{s+1,i})\,m(K_{s+1})$$

that involve a particular coset of K_s, say $y + k_s$. They are

$$f(y_{s,1})\,m(K_s) - \sum_{j=1}^{t} f(y_{s+1,j})\,m(K_{s+1})$$

where $y_{s,1} \in y + K_s$, $[K_s : K_{s+1}] = t \leq p^{\beta}$, and $\{y_{s+1,j}\}$ is a complete collection of representatives of the cosets of K_{s+1} contained in $y + K_s$. If

$$\varphi\left(f(y_{s,1})\,m(K_s) - \sum_{j=1}^{t} f(y_{s+1,j})\,m(K_{s+1})\right) < \epsilon\, p^{-\beta}$$

for each possible coset of K_s then

$$\varphi\left(\sum_i f(y_{s,i}) m(K_s) - \sum_i f(y_{s+1,i}) m(K_{s+1})\right) < \epsilon p^{-\beta}$$

Let

$$\alpha = f(y_{s,1}) m(K_s) - \sum_{j=1}^{t} f(y_{s+1,j}) m(K_{s+1}) .$$

As $m(K_s) = t\, m(K_{s+1})$, it follows that

$$\begin{aligned} \varphi(\alpha) &= \varphi\left(t\, m(K_{s+1}) f(y_{s,1}) - m(K_{s+1}) \sum_{j=1}^{t} f(y_{s+1,j})\right) \\ &= \varphi\left[\left(t f(y_{s,1}) - \sum_{j=1}^{t} f(y_{s+1,j})\right) m(K_{s+1})\right] \\ &= \varphi\left(t f(y_{s,1}) - \sum_{j=1}^{t} f(y_{s+1,j})\right) \varphi\left(m(K_{s+1})\right) \\ &= \varphi\left[\sum_{j=1}^{t} \left(f(y_{s,1}) - f(y_{s+1,j})\right)\right] \varphi\left(m(K_{s+1})\right) \\ &\leq \sup\left\{\varphi\left(f(y_{s,1}) - f(y_{s+1,j})\right) ; 1 \leq j \leq t\right\} \varphi\left(m(K_{s+1})\right) . \end{aligned}$$

By equation 4. 1 ,

$$\varphi\left(f(y_{s,1}) - f(y_{s+1,j})\right) \leq \epsilon p^{-\beta} w_{\mathfrak{C}}\left(f(y_{s,1}) - f(y_{s+1,j})\right) .$$

Consequently

$$\varphi(\alpha) < \epsilon p^{-\beta} \varphi\left(m(K_{s+1})\right)\left(\sup\{w_{\mathfrak{C}} f(y_{s,1}) - f(y_{s+1,j}) : 1 \leq j \leq t\}\right) .$$

As $w_{\mathfrak{C}}\left(f(y_{s,1}) - f(y_{s+1,j})\right) \leq m(K_s)$, it follows that

$$\varphi(\alpha) < \epsilon p^{-\beta} m(K_s)\, \varphi\big(m(K_{s+1})\big) .$$

Remembering that $\varphi\big(m(K_{s+1})\big) = \dfrac{1}{m(K_{s+1})}$,

$$\varphi(\alpha) < \epsilon p^{-\beta} \frac{m(K_s)}{m(K_{s+1})} = \epsilon p^{-\beta} \frac{t\, m(K_{s+1})}{m(K_{s+1})} = \epsilon p^{-\beta} t .$$

But $t = [K_s : K_{s+1}] \le p^{\beta}$ as $\varphi[K_s : K_{s+1}] \ge p^{-\beta}$. Therefore $\varphi(\alpha) < \epsilon p^{-\beta} p^{\beta} = \epsilon$. This implies that $f \in L^{R',\mathcal{C}}$.

The previous theorem gives the conditions for $L^{R,\mathcal{C}} = L^{R',\mathcal{C}}$. If these conditions are not satisfied it is possible to find a function f such that $f \in L^{R,\mathcal{C}}$ and $f \notin L^{R',\mathcal{C}}$.

<u>Example 4.4.</u> Let $G = Z_p$,

$\mathcal{C} = \{K_r : K_r = p[r]Z_p,\ r \in Z'$ and $p[r]$ represents $p^{2^r}\}$,

$$f_i = p[i]\ \chi\left(\bigcup_{j=0}^{p[i-1]-1} (j + p[i]Z_p) \right) ,$$

and

$$f = \sum_{i=1}^{\infty} f_i .$$

By Example 4.3 , it is known that $\mathcal{C}$ is not an ample family. The investigation will be conducted in four parts. (a) The f_i and f will be explored. (b) Evaluation of Riemann sums of f for $R - \mathcal{C}$ integration will be discussed. (c) It will be shown that $f \in L^{R,\mathcal{C}}$. (d) It will be shown that $f \notin L^{R',\mathcal{C}}$.

(a) For $i = 1$,

$$f_1 = p^2 \chi\left(\bigcup_{j=0}^{p-1} (j + p^2 Z_p)\right)$$

$$= p^2 \chi\left((0 + p^2 Z_p) \cup (1 + p^2 Z_p) \cup \ldots \cup ((p-1) + p^2 Z_p)\right)$$

$$= p^2 \chi(a_0 + p^2 Z_p)$$

where $0 \le a_0 \le p-1$. Thus it is seen that

$$f_1(x) = \begin{cases} p^2 & \text{if } x \in a_0 + p^2 Z_p \\ 0 & \text{if } x \notin a_0 + p^2 Z_p . \end{cases}$$

For $i = 2$,

$$f_2 = p^4 \chi\left(\bigcup_{j=0}^{p^2-1} (j + p^4 Z_p)\right) = p^4 \chi(a_0 + a_1 p + p^4 Z_p)$$

where $0 \le a_0 \le p-1$ and $0 \le a_1 \le p-1$. It follows that

$$f_2(x) = \begin{cases} p^4 & \text{if } x \in a_0 + a_1 p + p^4 Z_p \\ 0 & \text{otherwise} . \end{cases}$$

For $i = 3$,

$$f_3 = p^8 \chi\left(\bigcup_{j=0}^{p^4-1} (j + p^8 Z_o)\right) = p^8 \chi(a_0 + a_1 p + a_2 p^2 + a_3 p^3 + p^8 Z_p)$$

where $a_i \in Z'$, $0 \le a_i \le p-1$. The evaluation of f_3 gives

$$f_3(x) = \begin{cases} p^8 & \text{if } x \in a_0 + a_1 p + a_2 p^2 + a_3 p^3 + p^8 Z_p \\ 0 & \text{otherwise} , \end{cases}$$

For $i = n$,

$$f_n = p^{2^n} \chi\left(\bigcup_{j=0}^{p^{2^{n-1}}} (j + p^{2^n} Z_p)\right)$$

$$= p^{2^n} \chi\left(a_0 + a_1 p + \ldots + a_{2^{n-1}-1} p^{2^{n-1}-1} + p^{2^n} Z_p\right) .$$

Consequently,

$$f(0) = \sum_{i=1}^{\infty} p^{2^i} , \qquad f(1) = \sum_{i=1}^{\infty} p^{2^i} , \qquad f(p) = \sum_{i=2}^{\infty} p^{2^i} ,$$

$$f(p^2) = p^2 + \sum_{i=3}^{\infty} p^{2^i} , \qquad f(p^3) = p^2 + \sum_{i=3}^{\infty} p^{2^i} ,$$

and

$$f(p^4) = p^2 + p^4 + \sum_{i=4}^{\infty} p^{2^i} .$$

(b) The general Riemann sum will be approached by splitting the sum into two sums using

$$f = \sum_{j=1}^{i} f_j + \sum_{j=i+1}^{\infty} f_j .$$

This gives

$$S(f, p[i]Z_p, \xi_i) = S\left(\sum_{j=1}^{\infty} f_j, p[i]Z_p, \xi_i\right)$$

$$= S\left(\sum_{j=1}^{i} f_j + \sum_{j=i+1}^{\infty} f_j, p[i]Z_p, \xi_i\right)$$

$$= S\left(\sum_{j=1}^{i} f_j, p[i]Z_p, \xi_i\right) + S\left(\sum_{j=i+1}^{\infty} f_j, p[i]Z_p, \xi_i\right) .$$

Each of these terms will now be examined.

It is desired to show that

$$S\left(\sum_{j=1}^{i} f_j, p[i]Z_p, \xi_i\right) = \sum_{j=0}^{i-1} p[j]$$

which will be accomplished by mathematical induction. Let

$$M = \{i : \text{the previous statement is valid}\} ,$$

and show that $M = Z^+$. Is 1 an element of M ? It is if

$$\sum_{\ell=1}^{p^2} \left[\left(\sum_{j=1}^{1} f_j(\xi_{1\ell})\right) m(p^2 Z_p)\right] = \sum_{j=0}^{0} p[j] = p$$

where $[Z_p : p^2 Z_p] = p^2$. Evaluation of the left hand side yields

$$\sum_{\ell=1}^{p^2} \left[\left(\sum_{j=1}^{1} f_j(\xi_{1\ell})\right) m(p^2 Z_p)\right] = \sum_{\ell=1}^{p^2} f_1(\xi_{1\ell}) p^{-2} = p(p^2)p^{-2} = p$$

as there are only p of the cosets of $p^2 Z_p$ that are subsets of a set of the form $a_0 + p^2 Z_p$ and the rest of the cosets have an empty intersection with the sets of that form. Thus $1 \in M$. It will be assumed that the statement is true for $i = k$ and be shown true for $i = k+1$. From the induction hypothesis it follows that

$$\sum_{j=0}^{k-1} p[j] = \sum_{\ell=1}^{p^{2^k}} \sum_{j=1}^{k} f_j(\xi_{k\ell}) m\left(p^{2^k} Z_p\right) = \sum_{\ell=1}^{p^{2^k}} \sum_{j=1}^{k} \sum_{h=1}^{p^{2^k}} f_j(\xi_{k\ell h}) m\left(p^{2^{k+1}} Z_p\right)$$

where

$$\left[Z_p : p^{2^k} Z_p\right] = p^{2^k}, \left[p^{2^k} Z_p : p^{2^{k+1}} Z_p\right] = p^{2^{k+1}-2^k} = p^{2^k(2-1)} = p^{2^k},$$

and $\{\xi_{k\ell h}\}$ is a collection of one representative of each coset of $p^{2^{k+1}} Z_p$ contained in the coset of $p^{2^k} Z_p$ of which $\xi_{k\ell}$ is a representative. Note that each coset of $p^{2^k} Z_p$ is either contained in a set of the form

$$a_0 + a_1 p + \dots + a_{2^{k-1}-1} p^{2^{k-1}-1} + p^{2^k} Z_p$$

or it is disjoint from any such set. Therefore

$$\sum_{j=0}^{k-1} p[j] = \sum_{g=1}^{p^{2^{k+1}}} \sum_{j=1}^{k} f_j(\xi_{kg})\, m\left(p^{2^{k+1}} Z_p\right)$$

where

$$\left[Z_p : p^{2^{k+1}} Z_p\right] = p^{2^{k+1}} = p^{2^k \cdot 2} = p^{2^k + 2^k} = p^{2^k} \cdot p^{2^k},$$

and $\{\xi_{kg}\}$ is a collection of representatives of the cosets of $p^{2^{k+1}} Z_p$ contained in Z_p. (The set $\{\xi_{kg}\}$ is simply the set $\{\xi_{k\ell h}\}$ indexed in a different way.) Add

$$\sum_{g=1}^{p^{2^{k+1}}} f_{k+1}\left(\xi_{k+1,g}\right) m\left(p^{2^{k+1}} Z_p\right)$$

to each side of the equation, and

$$\sum_{g=1}^{p^{2^{k+1}}} f_{k+1}\left(\xi_{k+1,g}\right) m\left(p^{2^{k+1}} Z_p\right) + \sum_{j=0}^{k-1} p[j] =$$

$$= \sum_{g=1}^{p^{2^{k+1}}} \sum_{j=1}^{k+1} f_j\left(\xi_{k+1,g}\right) m\left(p^{2^{k+1}} Z_p\right) .$$

Note that the number of cosets of $p^{2^{k+1}} Z_p$ contained in a set of the form

$$a_0 + a_1 p + \ldots + a_{2^k-1} p^{2^k-1} + p^{2^{k+1}} Z_p$$

is p^{2^k}, and all other cosets are disjoint from a set of this form. This implies that

$$S\left(\sum_{j=0}^{k+1} f_j, p^{2^{k+1}} Z_p, \xi_{k+1}\right) = p^{2^k} \cdot p^{2^{k+1}} \cdot p^{-2^{k+1}} + \sum_{j=0}^{k-1} p[j]$$

$$= p^{2^k} + \sum_{j=0}^{k-1} p[j] = \sum_{j=0}^{k} p[j] = \sum_{j=0}^{(k+1)-1} p[j] .$$

Thus $k+1 \in M$, and $M = Z^+$.

Now $S\left(\sum_{j=i+1}^{\infty} f_j, p[i] Z_p, \xi_i\right)$ will be considered. First let $i = 1$; it is necessary to evaluate $\sum_{j=2}^{\infty} f(\xi_{1\ell})$ where $\{\xi_{1\ell}\}$ is a complete collection of representatives of the cosets of $p^2 Z_p$ contained in Z_p. For every set $\{\xi_{1\ell}\}$, $\sum_{j=2}^{\infty} f_j(\xi_{1\ell}) = p^{r_1}\alpha_1$ where $r_1 = 2^{m_1}$, $m_1 \geq i+1$ and $\alpha_1 \in Z_p$. Note that if an appropriate choice of $\{\xi_{1\ell}\}$ is made, then $\alpha = 0$. Such a choice is, for example, if $\xi_{1\ell} \in a_0 + a_1 p + p^2 Z_p$ then choose $\xi_{1\ell} = a_0 + a_1 p + p^2 + p^3 + \ldots + p^n + \ldots$.

Next let $i = 2$; it is necessary to evaluate $\sum_{j=3}^{\infty} f(\xi_{2\ell})$ where $\{\xi_{2\ell}\}$ is a complete collection of representatives of the cosets of $p^4 Z_p$ contained in Z_p. For every set $\{\xi_{2\ell}\}$, $\sum_{j=3}^{\infty} f_j(\xi_{2\ell}) = p^{r_2}\alpha_2$ where $r_2 = 2^{m_2}$, $m_2 \geq i+1$ and $\alpha_1 \in Z_p$. Note that if an appropriate choice of $\{\xi_{2\ell}\}$ is made, then $\alpha_2 = 0$. Such a choice is, for example, if

$$\xi_{2\ell} \in a_0 + a_1 p + a_2 p^2 + a_3 p^3 + p^4 Z_p$$

then choose

$$\xi_{2\ell} = a_0 + a_1 p + a_2 p^2 + a_3 p^3 + p^4 + p^5 + \dots + p^n + \dots \quad .$$

By mathematical induction it can be shown that $\sum_{j=i+1}^{\infty} f_j(\xi_{i\ell}) = p^{r_i}\alpha_i$ where $r_i = 2^{m_i}$, $m_i \geq i+1$, $\alpha_i \in Z_p$, and $\{\xi_{i\ell}\}$ is a complete collection of the representatives of the cosets of $p^{2^i} Z_p$ contained in Z_p. Choosing $\{\xi_{i\ell}\}$ such that

$$\xi_{i\ell} = a_0 + a_1 p + \dots + a_{2^i - 1} p^{2^i - 1} + p^{2^i} + \dots + p^n + \dots \quad ,$$

then $\sum_{j=i+1}^{\infty} f_j(\xi_{i\ell}) = 0$.

It now follows that

$$\varphi\left[\sum_{\ell=1}^{p^{2^i}} \left(\sum_{j=i+1}^{\infty} f_j(\xi_{i\ell})\, m(p^{2^i} Z_p)\right)\right]$$

$$\leq \sup\left\{\varphi\left(\sum_{j=i+1}^{\infty} f_j(\xi_{i\ell})\right) \cdot \varphi\left(m(p^{2^i} Z_p)\right) : 1 \leq \ell \leq p^{2^i}\right\} \leq p^{-2^i}$$

as

$$\varphi\left(\sum_{j=i+1}^{\infty} f_j(\xi_{i\ell})\right) \leq p^{-2^{i+1}} , \ \varphi\left(m(p^{2^i} Z_p)\right) = p^{2^i} ,$$

and

$$\varphi\left(\sum_{j=i+1}^{\infty} f_j(\xi_{ij})\right) \cdot \varphi\left(m(p^{2^i} Z_p)\right) \leq p^{2^i} p^{-2^{i+1}} = p^{2^i - 2^{i+1}} = p^{2^i(1-2)} = p^{-2^i} .$$

The two terms have now be investigated. It has been found that

$$S(f, p[i]Z_p, \xi_i) = \sum_{j=0}^{i-1} p[j] + \alpha_i$$

where $\varphi(\alpha_i) \leq p^{-2^i}$.

(c) In order to show that $f \in L^{R,\mathcal{C}}$, it must be shown that $\lim_{K \in \mathcal{C}} S(f, K, \xi)$ exists. It will be shown that $\lim_{K \in \mathcal{C}} S(f, K, \xi) = \sum_{j=0}^{\infty} p[j]$. That is, for each $\epsilon > 0$ there exists $K' \in \mathcal{C}$ such that

$$\varphi\left(S(f, K, \xi) - \sum_{j=0}^{\infty} p[j]\right) < \epsilon$$

whenever $K \subset K'$. Let any $0 < \epsilon < 1$ be given. There exists $\ell \in Z'$ such that $p^{-\ell} \leq \epsilon < p^{-\ell+1}$. Choose $K' = p[h]Z_p$ where $2^h > \ell$. Then for any $K \subset K'$, $K = p[i]Z_p$ where $i \geq h$. This implies

$$\begin{aligned}\varphi\left(S(f, K_i, \xi) - \sum_{j=0}^{\infty} p[j]\right) &= \varphi\left(\sum_{j=0}^{i-1} p[j] + \alpha_i - \sum_{j=0}^{\infty} p[j]\right) \\ &= \varphi\left(\alpha_i - \sum_{j=i}^{\infty} p[j]\right) \leq \max\left\{\varphi(\alpha_i), \varphi\left(\sum_{j=i}^{\infty} p[j]\right)\right\} \\ &= p^{-2^i}\end{aligned}$$

as $\varphi(\alpha_i) \leq p^{-2^{i+1}}$ and $\varphi\left(\sum_{j=i}^{\infty} p[j]\right) = p^{-2^i}$. Thus

$$\varphi\, S(f, K_i, \xi) - \sum_{j=0}^{\infty} p[j] \leq p^{-2^h} < p^{-\ell} \leq \epsilon$$

as $i \geq h$, $2^h > \ell$, and $p^{-\ell} \leq \epsilon$. Therefore $f \in L^{R,\mathcal{C}}$.

(d) In order to show that $f \notin L^{R',\mathbf{C}}$, it will be necessary to show that $\lim_{N(\delta)\in\mathbf{C}} S(f, \delta, \eta)$ does not exist. That is, there exists an $\epsilon > 0$ with the property that for each $K' \in \mathbf{C}$ there exists $N(\delta') \subset K'$ and $N(\delta'') \subset K'$ such that $\varphi(S(f, \delta', \eta) - S(f, \delta'', \eta)) \geq \epsilon$. Two such Riemann sums will now be found.

The sum $S(f, p[i]Z_p, \xi_i) = S(f, \delta_i', \xi_i)$ where $N(\delta_i') = p[i]Z_p$. It has been shown in part (b) that with a suitable choice of $\{\xi_{ij}\}$ this sum is equal to $\sum_{j=0}^{i-1} p[j]$. Let δ_i'' be the partition formed by the cosets of $p[i]Z_p$ with the exception of $p[i]Z_p$ itself and the cosets of $p[i+1]Z_p$ contained in $p[i]Z_p$. By the appropriate choice of $\{\eta_i\}$, it can be shown that $S(f, \delta_i'', \eta_i) = 1 + \sum_{j=0}^{i-1} p[j]$ for each i, where $N(\delta_i'') = p[i]Z_p$. This will be illustrated for $i = 1$. In this case

$$S(f, \delta_1'', \eta_1) = \sum_{j=1}^{p^2+(p^2-1)} f(\eta_{1j})\, m(K_i)$$

where $[Z_p : p^2 Z_p] = p^2$, $[p^2 Z_p : p^4 Z_p] = p^2$, and $\{\eta_{ij}\}$ is a complete collection of representatives of the sets of the partition. The appropriate cosets of $p^2 Z_p$ are $a_0 + a_1 p + p^2 Z_p$ where $a_0 \neq 0$ or $a_1 \neq 0$. Consider the cosets of this collection of the form $a_0 + p^2 Z_p$, $a_0 \neq 0$. Choose the element in this case as $\alpha = a_0 + p^2 + p^3 + \ldots + p^n + \ldots$, and $f(\alpha) = f_1(\alpha) = p^2$. There are $p - 1$ of these cosets. Next consider the cosets of the form $a_0 + a_1 p + p^2 Z_p$ where $a_1 \neq 0$. Choose the element in this case to be $\alpha = a_0 + a_1 p + p^2 + p^3 + \ldots + p^n + \ldots$, and $f(\alpha) = 0$. The appropriate cosets of $p^4 Z_p$ are $a_2 p^2 + a_3 p^3 + p^4 Z_p$. If a_2 and a_3 are not both equal to zero, then choose the element of these sets as $\alpha = a_2 p^2 + a_3 p^3 + p^4 + p^5 + \ldots + p^n + \ldots$. Consequently $f(\alpha) = f_1(\alpha) = p^2$, and there are $p^2 - 1$ of these cosets. If $a_2 = 0$ and

$a_3 = 0$, then choose the element of $p^4 Z_p$ as

$$p^4 + p^5 + p^6 + \dots + p^n + \dots \ .$$

It follows that

$$f(p^4 + p^5 + \dots) = f_1(p^4 + p^5 + \dots) + f_2(p^4 + p^5 + \dots) = p^2 + p^4 \ .$$

Therefore

$$\begin{aligned} S(f, \delta_1'', \eta_1) &= m(p^2 Z_p)(p-1)p^2 + m(p^4 Z_p)\Big((p^2-1)p^2 + \ p^2 + p^4)\Big) \\ &= p^{-2}(p-1)p^2 + p^{-4}(p^4 + p^4) \\ &= p-1+2 = p+1 = 1 + \sum_{j=0}^{i-1} p[j] \ , \ i = 1 \ . \end{aligned}$$

It could be proven that

$$S(f, \delta_i'', \eta_i) = 1 + \sum_{j=0}^{i-1} p[j] \ \text{ for all } i \ .$$

Consequently $S(f, \delta_i', \xi_i) - S(f, \delta_i'', \eta_i) = 1$ for all i.

Let $\epsilon = \frac{1}{2}$ and K' be any element of $\mathcal{C}$. There exists $i \in Z'$ such that $K_i \subset K'$. But $\varphi(S(f, \delta_i', \xi_i) - S(f, \delta_i'', \eta_i)) = \varphi(1) = 1 \geq \frac{1}{2}$.

Integration of Products

Theorem 4.4. Let G be compact, and $\mathcal{C}$ be an ample family. If $f, g \in L^{R,\mathcal{C}}$ then $f g \in L^{R,\mathcal{C}}$.

Proof: According to Theorem 4.3 $f g \in L^{R,\mathcal{C}}$ if $\lim\limits_{x \to y} \dfrac{f g(x) - f g(y)}{w_{\mathcal{C}}(x-y)} = 0$ uniformly. That is, if for each $\epsilon > 0$ there exists $K_0 \in \mathcal{C}$ such that

$\frac{\varphi(fg(x) - fg(y))}{w_{\mathcal{C}}(x-y)} < \epsilon$ whenever $K \subset K_0$ and $K \in \mathcal{C}_{x-y}$. The following computation gives the appropriate inequality:

$$\begin{aligned}
\varphi(f(x)\,g(x) - f(y)\,g(y)) &= \varphi(f(x)\,g(x) - f(x)\,g(y) + f(x)\,g(y) - f(y)\,g(y)) \\
&= \varphi\Big(f(x)\,(g(x) - g(y)) + (f(x) - f(y))\,g(y)\Big) \\
&\leq \sup\{\varphi(f(x)\,(g(x) - g(y))),\ \varphi(g(y)\,(f(x) - f(y)))\} \\
&= \sup\{\varphi(f(x))\,\varphi(g(x) - g(y)),\ \varphi(g(y))\,\varphi(f(x) - f(y))\} \\
&\leq \sup\{\varphi(f(x)),\varphi(f(y))\}\sup\{\varphi(g(x)-g(y)),\varphi(f(x)-f(y))\}.
\end{aligned}$$

The hypothesis that f and g belong to $L^{R,\mathcal{C}}$ implies that

$$\frac{\varphi(f(x) - f(y))}{w_{\mathcal{C}}(x-y)} < \epsilon, \quad \text{and} \quad \frac{\varphi(g(x) - g(y))}{w_{\mathcal{C}}(x-y)} < \epsilon .$$

That is, $\varphi(f(x) - f(y)) < \epsilon\, w_{\mathcal{C}}(x-y)$, and $\varphi(g(x) - g(y)) < \epsilon\, w_{\mathcal{C}}(x-y)$. Then

$$\varphi\ f(x)\,g(x) - f(y)\,g(y) \leq \sup\{\varphi(f(x)),\ \varphi(f(y))\}\epsilon\ w_{\mathcal{C}}(x-y) .$$

Thus

$$\frac{\varphi(f(x)\,g(x) - f(y)\,g(y))}{w_{\mathcal{C}}(x-y)} \leq \sup\{\varphi(f(x)),\ \varphi(f(y))\}\ \epsilon ,$$

and $\sup\{\varphi(f(x)),\ \varphi(f(y))\}$ is a constant. Therefore $fg \in L^{R,\mathcal{C}}$.

The basic ideas of Riemann integration have now been developed. Comparisons have been made between integration over the real number field and integration over the p-adic number field. The investigation has shown how the cosets of an element of a family $\mathcal{C}$ partition the group and play a role comparable to that of subintervals for Riemann integration over the real numbers. The reader who is interested in integration over the p-adic field will find an area of active research. The bibliography will give him an introduction to this research.

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Education: Graduated from White Pine County High School, Ely, Nevada in 1947; received the Bachelor of Science degree from Utah State University in June, 1954; received the Master of Science degree from Utah State University in June, 1963; completed the requirements for the Doctor of Education degree at Oklahoma State University in May, 1972.

Professional Experience: Graduate Assistant, Utah State University, Logan, Utah, 1957-1960; Instructor of Mathematics, Weber State College, Ogden, Utah, 1960-1964; Assistant Professor of Mathematics, Weber State College, Ogden, Utah, 1964-1967 and 1970; Graduate Assistant, Oklahoma State University, Stillwater, Oklahoma, 1967-1970.

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