THE ASYMPTOTIC DENSITY OF CERTAIN INTEGER SEQUENCES

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CHAPTER I

INTRODUCTION

The study of subsets of the set of integers has intrigued mathematical minds since the threshold of mathematics. Through the investigation of the sequence of squares, Pythagoras (569?- 500? B.C.) was led to realize a huge gap in the mathematical theory of his day, a gap that was filled later by the introduction of irrational numbers. Interest in sets of integers did not diminish as mathematics became more sophisticated. Fermat (1601?- 1665) in studying the set of primes became interested in the set of numbers of the form $2^{2^n} + 1$, where n is a positive integer. These numbers were later called Fermat numbers. Fermat conjectured that each of the Fermat numbers belongs to the set of primes. Although Euler (1707-1783) was able to disprove this conjecture, the investigation of Fermat numbers led to the consideration of other interesting sequences, and to the invention of tests for primality.

Many famous mathematicians, among them Gauss (1777-1855) and Dirichlet (1805-1859), investigated integer sequences called arithmetical progressions. This investigation was not confined to a particular arithmetical progression, but rather to the properties of this entire class of integer sequences. It is, of course, impossible to list all the sequences of integers that have attracted attention throughout the centuries. A few others that should at least be mentioned are the amicable numbers, the perfect numbers, the square-free numbers, and the abundant numbers.

It was not until the twentieth century that considerable attention was devoted to distribution characteristics of integer sequences in general. The intuitive concept of density of a sequence was formalized in a number of ways. It is the distinguished Soviet scholar Lev Genrichovitch Schnirelmann who is most responsible for stimulating interest in the general theory of density of integer sequences. Schnirelmann introduced a formal measure of density that was later called Schnirelmann density. Although the primary concern of this paper will not be the measure of density proposed by Schnirelmann, his work must be credited as a basis for that which followed.

In the early 1930's the prolific Hungarian mathematician Paul Erdős, who had been deeply involved in the study of abundant numbers, turned his attention to the study of the asymptotic density of various classes of integer sequences. Erdős was joined in this endeavor by H. Davenport, A. Besicovitch, F. Berhend, S. Chowla, and others. The scope of their studies was extensive. The classes of integer sequences that were of primary concern with regard to asymptotic density were sets of multiples, primitive sequences, and sequences of k-free integers.

Halberstam and Roth in Sequences [14] present a detailed account of the accomplishments of these men concerning sets of multiples and primitive sequences. This book reveals a similarity between the original proofs of results concerning sets of multiples and those of results concerning primitive sequences. In 1967 Ralph Alexander [1] introduced a technique that has been shown to be useful in proving results concerning both of these classes of integer sequences.

The purpose of this dissertation is to study the concept of asymptotic density and its application to certain classes of integer sequences. A modification of techniques introduced by Alexander will be employed whenever possible to unify the efforts of previous authors. Throughout ths paper particular examples of integer sequences will be introduced to illustrate density properties of the classes of integer sequences to which they belong.

Chapter II will serve as an introduction to the general theory of asymptotic density. The definitions of various measures of density will be presented, and relationships concerning these measures will be proved.

In Chapter III a method of analyzing an integer sequence will be investigated. This method, a decomposition into particular subsequences, is a modification of Alexander's method. This decomposition will be utilized many times throughout the remainder of the dissertation.

Chapters IV and V will investigate particular classes of integer sequences. In Chapter IV results concerning sets of multiples and sequences of k-free integers will be stated and proved. In Chapter V primitive sequences will be investigated with particular attention devoted to relationships among primitive sequences, sets of multiples, and the concepts of Chapter III.

CHAPTER II

DENSITY OF INTEGER SEQUENCES

Density, defined intuitively, is any quantity designed to describe the manner in which the elements of one set are distributed in some reference set. We are concerned here with the density of sets of positive integers relative to the set of all positive integers.

Various measures of density of positive integer sequences have been defined. Among them are asymptotic density, Schnirelmann density, logarithmic density, and a generalized asymptotic density. Although our primary concern is with asymptotic density, other measures of density will be introduced and referred to for purposes of comparison and to ensure an awareness of parallel avenues of study.

Certain nomenclature and notations are used consistently throughout the paper. The words "set" and "sequence" are used interchangeably to mean a strictly increasing sequence of positive integers (i.e., a set of distinct positive integers arranged in increasing order of magnitude). Sets (sequences) are often denoted by capital letters, with the capital letters N and P reserved for the set of natural numbers and the set of primes, respectively.

By the complement of a set A, denoted by A^c, we mean the set of positive integers that are not elements of A. That is $A^c = N \sim A$. For any integer n, the notation A(n) is used to denote the number of elements in the sequence A that do not exceed n. Other terms and notations related to set theory are used with their customary meanings.

Asymptotic Density

If A is a sequence of positive integers, then the ratio

$$\frac{A(n)}{n}$$

provides a natural measure of density for A relative to the subset $\{1, 2, 3, \ldots, n\}$ of N. A consideration of this ratio for large n would be an appropriate measure of density for A relative to N. The following definitions arise spontaneously.

Definition 2.1. Let A be a sequence of positive integers. If the limit of the sequence $\frac{A(n)}{n}$ exists, then A is said to possess asymptotic density, and the asymptotic density of A is defined by

$$\delta A = \lim_{n\to\infty} \frac{A(n)}{n} .$$

Definition 2.2. If A is a sequence of positive integers, then

$$\underline{\delta} A = \liminf_{n\to\infty} \frac{A(n)}{n}$$

is called the lower asymptotic density of A.

Definition 2.3. If A is a sequence of positive integers, then

$$\overline{\delta} A = \limsup_{n\to\infty} \frac{A(n)}{n}$$

is called the upper asymptotic density of A.

The reader who is not familiar with the concepts of limit, limit inferior, and limit superior is directed to the definitions and theorems appearing in the appendix.

Since the ratio $\frac{A(n)}{n}$ is bounded by $0 \leq \frac{A(n)}{n} \leq 1$, the definition of limit superior and limit inferior imply that the measures $\underline{\delta}$ and $\overline{\delta}$ exist for all integer sequences. On the other hand, the definition of limit indicates that certain sequences may not possess asymptotic density.

The following theorem, which follows immediately from Theorems A.4 and A.5 of the appendix, exhibits relationships among the measures $\underline{\delta}$, $\overline{\delta}$, and δ.

<u>Theorem 2.1.</u> If A is a sequence of positive integers, then

(i) $\underline{\delta}A \leq \overline{\delta}A$,

(ii) if $\underline{\delta}A = \overline{\delta}A = v$, then δA exists and is equal to v.

<u>Example 2.1.</u> Let A be the set of even integers. Then $\{A(n)\} = 0, 1, 1, 2, 2, 3, 3, \ldots$. That is $A(n) = \frac{n}{2}$ if n is even; and $A(n) = \frac{n-1}{2}$ if n is odd. Thus

$$\frac{1}{2} - \frac{1}{2n} \leq \frac{A(n)}{n} \leq \frac{1}{2}$$

for all n. Since $\lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2n}\right) = \frac{1}{2}$, $\delta A = \lim_{n\to\infty}\frac{A(n)}{n} = \frac{1}{2}$. In general, the set M_a of positive multiples of the integer a possesses asymptotic density $\frac{1}{a}$.

Generalizing further, consider the arithmetical progression

$$T = \{t_k\} = \{ak+b\},$$

where a and b are fixed positive integers. We may write $T(n) = k$, where k satisfies the inequalities

$$ak + b \leq n < ak + b + a .$$

Thus

$$\frac{n - b - a}{a} < k \leq \frac{n - b}{a} ,$$

$$\frac{n - b - a}{an} < \frac{T(n)}{n} \leq \frac{n - b}{an}$$

$$\frac{1}{a} - \frac{b + a}{an} < \frac{T(n)}{n} \leq \frac{1}{a} - \frac{b}{an} .$$

Since $\lim_{n\to\infty} \left(\frac{1}{a} - \frac{b+a}{an}\right) = \lim_{n\to\infty} \frac{1}{a} - \frac{b}{an} = \frac{1}{a}$, we have

$$\delta T = \lim_{n\to\infty} \frac{T(n)}{n} = \frac{1}{a} .$$

Consider the ratio $\frac{k}{a_k}$ where a_k is the k^{th} term in the infinite sequence A. How is this ratio related to the ratio $\frac{A(n)}{n}$? If we refer to the sequence $T = \{t_k\}$ in Example 2.1, we see that

$$\frac{k}{t_k} = \frac{k}{ak + b} .$$

Hence,

$$\lim_{k\to\infty} \frac{k}{t_k} = \frac{1}{a} = \delta T .$$

Is it true in general that if the asymptotic density of an infinite sequence A exists, then $\lim_{k\to\infty} \frac{k}{a_k}$ exists and equals δA ? Niven and

Zuckerman [21] have demonstrated that the answer to this question is yes.

Theorem 2.2. If A is an infinite sequence, then

$$\underline{\delta} A = \liminf_{k\to\infty} \frac{k}{a_k} ;$$

and if δA exists, then $\delta A = \lim_{k\to\infty} \frac{k}{a_k}$.

Proof. Since there are k elements $a_1, a_2, \ldots, a_k$ in A that do not exceed a_k, $A(a_k) = k$. Thus for every integer k there exists an integer n (namely $n = a_k$), such that $\frac{k}{a_k} = \frac{A(n)}{n}$. Hence the sequence $\{\frac{k}{a_k}\}$ is a subsequence of the sequence $\{\frac{A(n)}{n}\}$, and (by Theorem A.7)

$$\liminf_{k\to\infty} \frac{k}{a_k} \geq \liminf_{n\to\infty} \frac{A(n)}{n} , \tag{2.1}$$

$$\limsup_{k\to\infty} \frac{k}{a_k} \leq \limsup_{n\to\infty} \frac{A(n)}{n} . \tag{2.2}$$

If n is an integer $\geq a_1$ and a_k is the smallest integer in A that exceeds n, then $a_{k-1} \leq n < a_k$. Thus $A(n) = k-1$ and

$$\frac{k}{a_k} - \frac{A(n)}{n} = \frac{k}{a_k} - \frac{k-1}{n} < \frac{k}{n} - \frac{k-1}{n} = \frac{1}{n} ;$$

and given $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$, and the corresponding values of k,

$$0 \leq \frac{k}{a_k} - \frac{A(n)}{n} < \varepsilon . \tag{2.3}$$

Suppose that $\liminf_{n\to\infty} \frac{A(n)}{n} = v$. Then (by Theorem A. 10) there exist an infinite number of terms $\frac{A(n)}{n}$ such that $\frac{A(n)}{n} < v + \varepsilon$. Thus by (2.3) there exist an infinite number of terms $\frac{k}{a_k}$ such that

$$\frac{k}{a_k} < \frac{A(n)}{n} + \varepsilon < v + 2\varepsilon .$$

Hence (by Theorem A. 10)

$$\liminf_{k\to\infty} \frac{k}{a_k} \le v = \liminf_{n\to\infty} \frac{A(n)}{n} . \tag{2.4}$$

By (2. 1) and (2. 4) we have that $\liminf_{k\to\infty} \frac{k}{a_k} = \liminf_{n\to\infty} \frac{A(n)}{n}$, and from (2. 2) we have

$$\limsup_{n\to\infty} \frac{A(n)}{n} \ge \limsup_{k\to\infty} \frac{k}{a_k} \ge \liminf_{k\to\infty} \frac{k}{a_k} = \liminf_{n\to\infty} \frac{A(n)}{n} . \tag{2.5}$$

Thus if $\delta A = \lim_{n\to\infty} \frac{A(n)}{n}$ exists, then

$$\delta A \ge \limsup_{k\to\infty} \frac{k}{a_k} \ge \liminf_{n\to\infty} \frac{k}{a_k} = \delta A ,$$

and so (by Theorem A. 5) $\lim_{k\to\infty} \frac{k}{a_k}$ exists and is equal to δA. This completes the proof of the theorem. △

Theorem 2. 2 yields a procedure for finding the asymptotic density of a sequence. However, this procedure (according to the theorem) relies on the assumption that it is known that the sequence possesses asymptotic density. Thus the following theorem may prove to be of more use.

<u>Theorem 2.3.</u> Let $A = \{a_k\}$ be an infinite sequence of positive integers. If $\lim_{k\to\infty} \frac{k}{a_k}$ exists, then so does $\delta A = \lim_{n\to\infty} \frac{A(n)}{n}$ and they are equal.

<u>Proof.</u> If is sufficient (by Theorem A.5) to show that $\liminf_{k\to\infty} \frac{k}{a_k} = \liminf_{n\to\infty} \frac{A(n)}{n}$ and $\limsup_{k\to\infty} \frac{k}{a_k} = \limsup_{n\to\infty} \frac{A(n)}{n}$. Since we have (2.5), we need only show that $\limsup_{n\to\infty} \frac{A(n)}{n}$ does not exceed $\limsup_{k\to\infty} \frac{k}{a_k}$.

Let $\limsup_{n\to\infty} \frac{A(n)}{n} = v$. Then (by Theorem A.10) given $\varepsilon > 0$, there are an infinite number of values of n for which

$$\frac{A(n)}{n} > v - \varepsilon \,. \tag{2.6}$$

Suppose a particular n satisfies (2.6). Let $k = A(n)$; so that $a_k \leq n < a_{k+1}$, and

$$\frac{k}{a_k} = \frac{A(n)}{a_k} \geq \frac{A(n)}{n} > v - \varepsilon \,.$$

Thus there exists at least one integer k such that

$$\frac{k}{a_k} > v - \varepsilon \,. \tag{2.7}$$

For any such k, there exists an integer $n' \geq a_{k+1}$ such that n' satisfies (2.6). Now let $k' = A(n')$. Then $a_{k'} \leq n' < a_{k'+1}$ and

$$\frac{k'}{a_{k'}} = \frac{A(n')}{a_{k'}} \geq \frac{A(n')}{n'} > v - \varepsilon \,.$$

Therefore, k' satisfies (2.7); also, since $n' \geq a_{k+1}$, $k' = A(n') \geq k+1 > k$. Thus there exist an infinite number of values for k that satisfy (2.7), and the limit superior of $\frac{k}{a_k}$ must be at least as great as v. Hence, the theorem is proved. $\triangle$

Example 2.2. Theorem 2.3 may be used to determine the asymptotic density, if it exists, of the geometric progression $G = \{ar^k\}$, where a and r are fixed integers, $a \geq 1$, $r > 1$. Since it is difficult to determine the value $G(n)$ for any integer n, direct application of the definition of asymptotic density is not appealing.

Due to the restrictions on the fixed integers a and r, we have

$$\frac{k}{a_k} = \frac{k}{ar^k} \leq \frac{k}{2^k} .$$

Since $k/2^k$ tends to zero as k becomes large, by comparison $\lim_{k\to\infty} \frac{k}{r^k} = 0$. Thus by Theorem 2.3, the geometric progression G possesses zero asymptotic density.

Example 2.3. Suppose that the sequence $A = \{a_k\}$ possesses asymptotic density δA. Theorems 2.2 and 2.3 can be used to show that the sequence $mA = \{m a_k\}$, $m \in N$, possesses asymptotic density $m^{-1}\,\delta A$. By Theorem 2.2, $\delta A = \lim_{k\to\infty} \frac{k}{a_k}$. Hence

$$\lim_{k\to\infty} \frac{k}{m a_k} = m^{-1} \lim_{k\to\infty} \frac{k}{a_k} = m^{-1}\,\delta A .$$

Since $m a_k$ is the k^{th} term of the sequence mA, by Theorem 2.3 the sequence mA possesses asymptotic density $m^{-1}\,\delta A$.

The sequences that have been presented as examples have all possessed asymptotic density. Now, we shall consider a sequence that does not possess asymptotic density.

Example 2.4. Let A be the set of integers n that satisfy

$$2^m \leq n < 2^m + 2^{m-1} ,$$

for some positive integer m. We can represent A as the countable union of the sets $A_i = \{2^i, 2^i+1, \ldots, 2^i+2^{i-1}-1\}$, $i = 1, 2, \ldots$.

Consider the first element 2^m in the set A_m. Since there are exactly 2^{i-1} elements in each set A_i, $i < m$, we have

$$A(2^m) = 1 + \sum_{i=1}^{m-1} 2^{i-1} = 1 + (2^{m-1} - 1) = 2^{m-1} ,$$

and $\frac{A(2^m)}{2^m} = \frac{2^{m-1}}{2^m} = \frac{1}{2}$. Thus for an infinite number of integers n, $\frac{A(n)}{n} = \frac{1}{2}$. Hence (by Theorem A. 10)

$$\underline{\delta} A = \liminf_{n\to\infty} \frac{A(n)}{n} \leq \frac{1}{2} . \tag{2.8}$$

On the other hand, consider the last element $2^m + 2^{m-1} - 1$ in the set A_m. We have

$$A(2^m+2^{m-1}-1) = A(2^m) + 2^{m-1} - 1 = 2^{m-1} + 2^{m-1} - 1 = 2^m - 1 ,$$

and

$$\frac{A(2^m + 2^{m-1} - 1)}{2^m + 2^{m-1} - 1} = \frac{2^m - 1}{3 \cdot 2^{m-1} - 1} > \frac{2^m - 1}{3 \cdot 2^{m-1}} = \frac{2}{3} - \frac{1}{3 \cdot 2^{m-1}} .$$

Thus if $\varepsilon > 0$, then there exist an infinite number of values for n such that $\frac{A(n)}{n} > \frac{2}{3} - \varepsilon$. Hence (by Theorem A. 10)

$$\overline{\delta} A = \limsup_{n\to\infty} \frac{A(n)}{n} \geq \frac{2}{3} . \tag{2.9}$$

Thus by (2.8) and (2.9) δA does not exist.

This same result can be obtained by considering the ratio $\frac{k}{a_k}$. For $k' = 2^i$, $i = 1, 2, \ldots$, we have $a_{k'} = 2^{i+1}$; and for $k'' = 2^{i+1} - 1$, $a_{k''} = 2^{i+1} + 2^i - 1$. It follows that $\frac{k'}{a_{k'}} = \frac{1}{2}$ and $\frac{k''}{a_{k''}} > \frac{2}{3} - \frac{1}{k''}$; so that $\liminf_{k\to\infty} \frac{k}{a_k} \neq \limsup_{k\to\infty} \frac{k}{a_k}$.

Set theoretic relationships among sequences of integers suggest plausible relationships concerning their densities.

Theorem 2.4. Let A and B be sequences of integers. If $A \cup B = C$ and $A \cap B = \emptyset$, then

$$\overline{\delta} C \leq \overline{\delta} A + \overline{\delta} B ,$$

$$\underline{\delta} C \geq \underline{\delta} A + \underline{\delta} B .$$

Proof. Since A and B are disjoint, $C(n) = A(n) + B(n)$. Thus

$$\frac{C(n)}{n} = \frac{A(n)}{n} + \frac{B(n)}{n} ,$$

and the desired results follow from Theorem A.6 of the appendix. △

Theorem 2.5. Let A and B be disjoint sequences of integers, and let $C = A \cup B$. If two of the sequences A, B, C, possess asymptotic density, then the third sequence also possesses asymptotic density and

$$\delta C = \delta A + \delta B .$$

Proof. Suppose that δA and δB exist. Since

$$\frac{C(n)}{n} = \frac{A(n)}{n} + \frac{B(n)}{n} ,$$

and since $\lim \frac{A(n)}{n}$ and $\lim \frac{B(n)}{n}$ exist, we have

$$\lim_{n\to\infty} \frac{C(n)}{n} = \lim_{n\to\infty}\left(\frac{A(n)}{n} + \frac{B(n)}{n}\right)$$

$$= \lim_{n\to\infty} \frac{A(n)}{n} + \lim_{n\to\infty} \frac{B(n)}{n} .$$

Thus δC exists and $\delta C = \delta A + \delta B$. Likewise, the theorem is true if δB and δC exist, or if δA and δC exist. $\triangle$

<u>Corollary</u>. If δA exist, then δA^c exists and is equal to $1 - \delta A$.

<u>Proof.</u> In Theorem 2.5, replace C by N and B by A^c. Since $N = A \cup A^c$ and $A \cap A^c = \emptyset$, we have that δA^c exists and

$$\delta A^c = \delta N - \delta A = 1 - \delta A , \qquad \triangle$$

The Corollary to Theorem 2.5 may be used to uncover a necessary and sufficient condition for a sequence to possess asymptotic density.

<u>Theorem 2.6</u>. The sequence A possesses asymptotic density if, and only if, $\underline{\delta} A + \underline{\delta} A^c = 1$.

<u>Proof</u>. Suppose δA exists. Then by the Corollary to Theorem 2.5 $\underline{\delta} A + \underline{\delta} A^c = 1$.

Suppose $\underline{\delta} A + \underline{\delta} A^c = 1$. Then

$$\liminf_{n\to\infty} \frac{A(n)}{n} + \liminf_{n\to\infty} \frac{A^c(n)}{n} = 1 .$$

Since $A^c(n) = n - A(n)$, we have $\frac{A^c(n)}{n} = 1 - \frac{A(n)}{n}$. Thus

$$\liminf_{n\to\infty} \frac{A(n)}{n} + \liminf_{n\to\infty} \left(1 - \frac{A(n)}{n}\right) = 1 ,$$

and (by Theorems A. 8 and A. 9)

$$\liminf_{n\to\infty} \frac{A(n)}{n} + 1 + \liminf_{n\to\infty} \left(- \frac{A(n)}{n}\right)$$

$$= \liminf_{n\to\infty} \frac{A(n)}{n} + 1 - \limsup_{n\to\infty} \frac{A(n)}{n} = 1 .$$

Thus the limit inferior and the limit superior of $\frac{A(n)}{n}$ are equal, and (by Theorem A. 5) A possesses asymptotic density. $\triangle$

Theorem 2. 5 can be used to show that a sequence can be altered in certain ways without affecting its asymptotic density. If the sequence B in Theorem 2. 5 possess zero asymptotic density, then $\delta(A \cup B) = \delta A$. Since A and B are disjoint and $C = A \cup B$, $A = C \setminus B$; and so $\delta(C \setminus B) = \delta C$. Thus we have the following theorem.

Theorem 2. 7. Let A be a sequence of integers with asymptotic density δA . If A is altered by including or deleting a set of integers possessing zero asymptotic density, then the resulting sequence possesses asymptotic density δA .

Since the asymptotic density of a finite set of integers is zero, we have the following corollary.

<u>Corollary</u>. Let A be a sequence of integers with asymptotic density δA. If A is altered by including or deleting a finite number of integers, then the asymptotic density of the resulting sequence exists and is equal to δA.

A sequence can be altered in yet another manner without affecting its asymptotic density. If $A = \{a_1, a_2, \ldots\}$ is a sequence of integers and t is a fixed integer, then the sequence $A+t = \{a_1+t, a_2+t, a_3+t, \ldots\}$ is called a translation of A. The following theorem indicates that the asymptotic densities of a sequence and its translation are equal.

<u>Theorem 2.8</u>. Let $A = \{a_1, a_2, \ldots\}$ be a sequence of integers. If $A+t = \{a_1+t, a_2+t, \ldots\}$, where t is a fixed integer, then

(i) $\underline{\delta}(A+t) = \underline{\delta}A$,

(ii) $\overline{\delta}(A+t) = \overline{\delta}A$,

(iii) $\delta(A+t) = \delta A$ if δA exists.

In other words, the measures $\underline{\delta}$, $\overline{\delta}$, and δ are translation invariant.

<u>Proof</u>. By Theorems 2.2 and 2.3, it is appropriate to investigate the ratio $\frac{k}{a_k+t}$. We have

$$\frac{k}{a_k+t} = \frac{k}{a_k+t} + \frac{k}{a_k} - \frac{k}{a_k} = \frac{k}{a_k} - \frac{kt}{a_k(a_k+t)} . \tag{2.10}$$

Since the ratio $\frac{k}{a_k}$ is bounded $(0 \leq \frac{k}{a_k} \leq 1)$ and $\lim_{k\to\infty} \frac{t}{(a_k+t)} = 0$,

$$\lim_{k\to\infty} \frac{kt}{a_k(a_k+t)} = 0 . \tag{2.11}$$

Thus taking the limit inferior and limit superior respectively of (2.10) and applying (2.11), we have

$$\underline{\delta}(A+t) = \liminf_{k\to\infty} \frac{k}{a_k+t} = \liminf_{k\to\infty} \frac{k}{a_k} = \underline{\delta} A ,$$

$$\overline{\delta}(A+t) = \limsup_{k\to\infty} \frac{k}{a_k+t} = \limsup_{k\to\infty} \frac{k}{a_k} = \overline{\delta} A .$$

If δA exists, then the limit of (2.10) shows that $\delta(A+t)$ exists and

$$\delta(A+t) = \lim_{k\to\infty} \frac{k}{a_k+t} = \lim_{k\to\infty} \frac{k}{a_k} = \delta A .$$

Thus the theorem is proved. $\triangle$

Schnirelmann Density

As mentioned earlier, it seems appropriate to consider the ratio $\frac{A(n)}{n}$ when defining a measure of density for integer sequences. Asymptotic, lower asymptotic, and upper asymptotic densities were defined as the limit, limit inferior, and limit superior, respectively, of this ratio. Now we shall define a density by considering the greatest lower bound of the ratio $\frac{A(n)}{n}$ over all natural numbers n. As might be suspected, this density possesses properties that differ from those of asymptotic density.

<u>Definition 2.4</u>. Let A be a sequence of integers. The Schnirelmann density of A, denoted by $\delta_S A$, is defined as

$$\delta_S A = \underset{n \in N}{\text{g.l.b.}} \frac{A(n)}{n}$$

By its definition, the Schnirelmann density exists for all integer sequences, and equals zero for finite integer sequences. Since the inequality

$$0 \leq \frac{A(n)}{n} \leq 1$$

is true for all sequences A, we have

$$0 \leq \delta_S A \leq 1 .$$

Furthermore, by comparing the definitions of greatest lower bound and limit inferior, we have

$$0 \leq \delta_S A \leq \underline{\delta} A \leq 1 .$$

A major difference between Schnirelmann density and asymptotic density is the manner in which the first few terms in a sequence influence the measure of density of the sequence. In the Corollary to Theorem 2.7, we observe that the inclusion or exclusion of a finite number of terms does not change the asymptotic density of a sequence. On the other hand, Schnirelmann density depends heavily of the first few terms of a sequence. For instance, if the first term in a sequence A is not 1, then $\frac{A(1)}{1} = 0$; and so $\delta_S A = \text{g.l.b.} \frac{A(n)}{n} = 0$. Other differences between these measures of density are illustrated in the following example.

<u>Example 2.5.</u> Let A be the sequence of positive integers that are congruent to 1, mod 3; that is $A = \{3k+1\}$. Then

$$\left\{\frac{A(n)}{n}\right\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{4}, \frac{2}{5}, \frac{2}{6}, \frac{3}{7}, \ldots\right\} ,$$

and $\delta_S A = \text{g.l.b.}\ \frac{A(n)}{n} = \frac{1}{3}$. As mentioned in the previous paragraph, if we alter the sequence A by removing the first term 1, we have

$$\delta_S(A \setminus \{1\}) = 0 \neq \delta_S A, \qquad (2.12)$$

whereas $\delta(A \setminus \{1\}) = \delta A = \frac{1}{3}$.

We can also alter the sequence A by adding the integer 2 to every term, obtaining $A+2 = \{3k\}$. Since the first term in the sequence $A+2$ is not 1, we have

$$\delta_S(A+2) = 0 \neq \delta_S A, \qquad (2.13)$$

whereas $\delta(A+2) = \delta A = \frac{1}{3}$. Now, if we include the integer 1 in the sequence $A+2$, we obtain the sequence

$$A^* = \{1, 3, 6, \ldots\},$$

and

$$\left\{\frac{A^*(n)}{n}\right\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \frac{3}{6}, \ldots\right\}.$$

Thus for $n \geq 2$, $\frac{A(n)}{n}$ is of the form $\frac{k}{(3k-1)}$ or $\frac{k}{(3k-2)}$ or $\frac{k}{(3k-3)}$, so that $\text{g.l.b.}\ \frac{A(n)}{n} = \frac{1}{3}$. Therefore

$$\delta_S A^* = \delta_S\Big((A+2) \cup \{1\}\Big) = \frac{1}{3} \neq \delta_S(A+2) + \delta_S 1 = 0, \qquad (2.14)$$

whereas $\delta A^* = \delta(A+2) = \frac{1}{3}$.

Statements (2.12), (2.13), and (2.14) in the previous example indicate that the properties of asymptotic density described in Theorems 2.5 and 2.8 are not true for Schnirelmann density.

The differences that exist between asymptotic and Schnirelmann density do not infer that either measure is not significant, but rather that the applications of Schnirelmann density differ from those of asymptotic density.

In 1942 Henry B. Mann [18] proved a theorem that was conjectured by Schnirelmann and Landau in 1931. The theorem may be stated as follows:

> Let A and B be sequences of positive integers, and let $A + B$ denote the sequence of integers $\{0\} \cup A \cup B \cup \{a + b : a \in A, b \in B\}$. Then $\delta_S(A+B) \geq \delta_S A + \delta_S B$.

Through this theorem, Schnirelmann density can be applied to problems concerning the representation of integers. Among these problems is the theorem by Vinogradov, that all sufficiently large odd numbers can be represented as the sum of at most three primes. Lagranges' theorem, that every positive integer can be represented as the sum of at most four squares, can be proved using Mann's theorem. Also, Schnirelmann density led to the first elementary proof of Waring's conjecture, that every positive integer can be represented as the sum of k n^{th} powers, where k is related to n. An interesting account of applications of Schnirelmann density appears in Three Pearls of Number Theory by A. Y. Khinchin [17].

Logarithmic Density

Our primary concern is not with Schirelmann density, but rather with asymptotic density. We now consider a measure of density that is closely related to asymptotic density.

<u>Definition 2.5</u>. Let A be a sequence of positive integers. If the quantity

$$\delta_L A = \lim_{n\to\infty} \frac{1}{\log n} \sum_{a_i \le n} \frac{1}{a_i}$$

exists, we define it to be the logarithmic density of A .

We also define measures of density analogous to lower and upper asymptotic density.

<u>Definition 2.6</u>. We define

$$\underline{\delta}_L A = \liminf_{n\to\infty} \frac{1}{\log n} \sum_{a_i \le n} \frac{1}{a_i} ,$$

$$\overline{\delta}_L A = \limsup_{n\to\infty} \frac{1}{\log n} \sum_{a_i \le n} \frac{1}{a_i}$$

to be respectively the lower and upper logarithmic densities of A .

If a sequence A consists of a finite number of positive integers, then by Definition 2.5 the sequence A possesses zero logarithmic density. The following theorem presents a sufficient condition for an infinite sequence of positive integers to possess zero logarithmic density.

<u>Theorem 2.9</u>. Let $A = \{a_1, a_2, \ldots\}$ be an infinite sequence of positive integers. If the series $\Sigma_{i=1}^{\infty} a_i^{-1}$ converges, then A possess zero logarithmic density.

Proof. Let S denote the sum of the series $\Sigma_{i=1}^{\infty} \frac{1}{a_i}$. Since the a_i's are positive, the partial sum $\sum_{a_i \le n} \frac{1}{a_i}$ does not exceed S, for any $n \ge 1$. Thus

$$\delta_L A = \lim_{n\to\infty} \frac{1}{\log n} \sum_{a_i \le n} \frac{1}{a_i} \le \lim_{n\to\infty} \frac{S}{\log n} = 0 ,$$

and the theorem is proved. △

Applying Theorem 2.9, we see that the set of squares, $A = \{1, 4, 9, 16, \dots\}$, possesses zero logarithmic density since the series $\Sigma_{k=1}^{\infty} \frac{1}{k^2}$ converges. More generally, since the series $\Sigma_{k=1}^{\infty} \frac{1}{k^r}$ converges whenever $r > 1$, the sequence of positive r^{th} powers $\{1^r, 2^r, \dots\}$ possess zero logarithmic density. Also, since the geometric series $\Sigma_{k=1}^{\infty} (\frac{1}{r})^k$ converges for $r > 1$, the geometric progression $G = \{ar^k\}$ in Example 2.2 possess zero logarithmic density.

If the logarithmic density is to be consistent with the intuitive concept of density, we should expect that the logarithmic density of the sequence of natural numbers N is 1. This property is a direct result of the following lemma.

Lemma 2.1.

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} = 1 .$$

Proof. Let n be a positive integer. From calculus we have

$$\int_1^n \frac{1}{x}\, dx = \log n .$$

Thus the area under the curve $y = \frac{1}{x}$, $1 \leq x \leq n$, is $\log n$. We may then consider the sums $\Sigma_{k=2}^{n} \frac{1}{k}$ and $\Sigma_{k=1}^{n-1} \frac{1}{k}$ as being lower and upper Riemann sums, respectively, of the function $f(x) = \frac{1}{x}$, $1 \leq x \leq n$. Therefore, we have

$$\sum_{k=2}^{n} \frac{1}{k} \leq \log n \leq \sum_{k=1}^{n-1} \frac{1}{k} ,$$

so that

$$\frac{1}{\log n} \sum_{k=2}^{n} \frac{1}{k} < 1 < \frac{1}{\log n} \sum_{k=1}^{n-1} \frac{1}{k} ,$$

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{\log n} < 1 < \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} ,$$

$$-\frac{1}{\log n} < 1 - \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} < 0 < \frac{1}{\log n} .$$

Therefore,

$$\left| 1 - \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \right| < \frac{1}{\log n} ,$$

and the lemma is proved. △

Unlike Schnirelmann density, logarithmic density possesses properties analogous to the properties of asymptotic density that are demonstrated in Theorems 2.4 and 2.5.

Theorem 2.4'. Let A and B be sequences of integers. If $C = A \cup B$ and $A \cap B = \emptyset$, then

$$\overline{\delta}_L C \leq \overline{\delta}_L A + \overline{\delta}_L B ,$$

$$\underline{\delta}_L C \geq \underline{\delta}_L A + \underline{\delta}_L B .$$

Proof. Since $C = A \cup B$ and $A \cap B = \emptyset$, we have

$$\{\frac{1}{c_i} : c_i \in C,\ 1 \leq c_i \leq n\} = \{\frac{1}{a_i} : a_i \in A,\ 1 \leq a_i \leq n\}$$
$$\cup \{\frac{1}{b_i} : b_i \in B,\ 1 \leq b_i \leq n\},$$

and this union is disjoint. Thus

$$\frac{1}{\log n} \sum_{c_i \leq n} \frac{1}{c_i} = \frac{1}{\log n} \sum_{a_i \leq n} \frac{1}{a_i} + \frac{1}{\log n} \sum_{b_i \leq n} \frac{1}{b_i} ,$$

for all n. The desired result follows from Theorem A.6 of the appendix. $\triangle$

Theorem 2.5'. Let A and B be disjoint sequences, and let $C = A \cup B$. If two of the sequences A, B, C, possess logarithmic density, then the third sequence possesses logarithmic density and $\delta_L C = \delta_L A + \delta_L B$.

Proof. The proof parallels the proof of Theorem 2.5. $\triangle$

The following example illustrates that the evaluation of the logarithmic density of a sequence can be laborious.

Example 2.6. Let A be the sequence of even integers. Consider the quantity

$$L_n = \frac{1}{\log n} \sum_{a_i \leq n} \frac{1}{a_i} .$$

We wish to find the limit of L_n as $n \to \infty$, if this limit exists. Since the integers in A are of the form $2k$, where k is an integer, we have

$$L_n = \frac{1}{\log n} \sum_{k=1}^{[n/2]} \frac{1}{2k} ,$$

where the brackets denote the greatest integer function. Thus we have

$$\begin{aligned} L_n &= \frac{1}{2} \cdot \frac{1}{\log n} \sum_{k=1}^{[n/2]} \frac{1}{k} \\ &= \frac{1}{2} \cdot \frac{1}{\log \frac{n}{[n/2]} + \log [n/2]} \sum_{k=1}^{[n/2]} \frac{1}{k} . \end{aligned}$$

The quantity $\frac{n}{[n/2]}$ is bounded for $n > 2$ by

$$2 \leq \frac{n}{[n/2]} \leq \frac{n}{\frac{n}{2} - 1} \leq 6 .$$

Thus

$$\begin{aligned} \lim_{n\to\infty} L_n &= \frac{1}{2} \lim_{n\to\infty} \frac{1}{\log \frac{n}{[n/2]} + \log [n/2]} \sum_{k=1}^{[n/2]} \frac{1}{k} \\ &= \frac{1}{2} \lim_{n\to\infty} \frac{1}{\log [n/2]} \sum_{k=1}^{[n/2]} \frac{1}{k} , \end{aligned}$$

and so by Lemma 2.1,

$$\delta_L A = \lim_{n\to\infty} L_n = \frac{1}{2} .$$

The asymptotic density of the sequence of even integers is $\frac{1}{2}$. Thus we see that, at least for this sequence, the logarithmic and asymptotic densities coincide. The following theorem illustrates, among other things, that if both the asymptotic and logarithmic densities exist for some sequence, then they must coincide. As in Example 2.6, Lemma 2.1 will play an important part in the proof of this theorem.

Theorem 2.10. If A is an integer sequence, then

$$0 \leq \underline{\delta} A \leq \underline{\delta}_L A \leq \overline{\delta}_L A \leq \overline{\delta} A \leq 1 .$$

Proof. Since $0 \leq \frac{A(n)}{n} \leq 1$, we have

$$0 \leq \underline{\delta} A \leq \overline{\delta} A \leq 1 . \tag{2.15}$$

By the definitions of the measures $\underline{\delta}$, $\overline{\delta}$, $\underline{\delta}_L$, and $\overline{\delta}_L$, it is appropriate that we compare the quantities

$$\frac{A(n)}{n} \quad \text{and} \quad \frac{1}{\log n} \sum_{a_i \leq n} \frac{1}{a_i}$$

as n becomes large. For any integers n and N, such that $n > N \geq 1$, we have

$$\frac{1}{\log n} \sum_{a_i \leq n} \frac{1}{a_i} = \frac{1}{\log n} \sum_{a_i \leq N} \frac{1}{a_i} + \frac{1}{\log n} \sum_{N < a_i \leq n} \frac{1}{a_i} . \tag{2.16}$$

By the definition of $A(n)$,

$$A(m) - A(m-1) = \begin{cases} 0, & \text{if } m \notin A, \\ 1, & \text{if } m \in A. \end{cases}$$

Thus for the right most sum in (2.16), we have

$$\sum_{N<a_i\leq n} \frac{1}{a_i} = \sum_{m=N+1}^{n} \frac{A(m)-A(m-1)}{m} = \sum_{m=N+1}^{n} \frac{A(m)}{m} - \sum_{m=N+1}^{n} \frac{A(m-1)}{m}$$

$$= \sum_{m=N+1}^{n} \frac{A(m)}{m} - \sum_{m=N}^{n-1} \frac{A(m)}{m+1}$$

$$= \sum_{m=N+1}^{n-1} A(m)\left(\frac{1}{m} - \frac{1}{m+1}\right) + \frac{A(n)}{n} - \frac{A(N)}{N+1}$$

$$= \sum_{m=N+1}^{n-1} \frac{A(m)}{m}\left(\frac{1}{m+1}\right) + \frac{A(n)}{n} - \frac{A(N)}{N+1} .$$

Thus

$$\sum_{N<a_i\leq n} \frac{1}{a_i} \geq \left(\inf_{n>N} \frac{A(n)}{n}\right) \sum_{m=N+1}^{n-1} \frac{1}{m+1} + \frac{A(n)}{n} - \frac{A(N)}{N+1}, \quad (2.17)$$

$$\sum_{N<a_i\leq n} \frac{1}{a_i} \leq \left(\sup_{n>N} \frac{A(n)}{n}\right) \sum_{m=N+1}^{n-1} \frac{1}{m+1} + \frac{A(n)}{n} - \frac{A(N)}{N+1} \quad (2.18)$$

Substituting (2.17) and (2.18) into (2.16)., we have

$$\frac{1}{\log n} \sum_{a_i\leq n} \frac{1}{a_i} \geq \frac{1}{\log n} \sum_{a_i\leq N} \frac{1}{a_i} + \left(\inf_{n>N} \frac{A(n)}{n}\right) \frac{1}{\log n} \sum_{m=N+1}^{n-1} \frac{1}{m+1}$$

$$+ \frac{1}{\log n} \frac{A(n)}{n} - \frac{1}{\log n} \frac{A(N)}{N+1}, \quad (2.19)$$

$$\frac{1}{\log n} \sum_{a_i \leq n} \frac{1}{a_i} \leq \frac{1}{\log n} \sum_{a_i \leq N} \frac{1}{a_i} + \left(\sup_{n > N} \frac{A(n)}{n} \right) \frac{1}{\log n} \sum_{m=N+1}^{n-1} \frac{1}{m+1}$$

$$+ \frac{1}{\log n} \frac{A(n)}{n} - \frac{1}{\log n} \frac{A(N)}{N+1} \qquad (2.20)$$

From Lemma 2.1, we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{m=N+1}^{n-1} \frac{1}{m+1} = \lim_{n\to\infty} \frac{1}{\log n} \sum_{m=0}^{n} \frac{1}{m+1} - \lim_{n\to\infty} \frac{1}{\log n} \sum_{m=0}^{N} \frac{1}{m+1}$$

$$= 1 - 0 = 1 .$$

Also,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{a_i \leq N} \frac{1}{a_i} = \lim_{n\to\infty} \frac{1}{\log n} \frac{A(n)}{n} = \lim_{n\to\infty} \left(- \frac{1}{\log n} \frac{A(n)}{n} \right) = 0 .$$

Thus upon taking the $\liminf\limits_{n\to\infty}$ and $\limsup\limits_{n\to\infty}$ of (2.19) and (2.20), respectively we have

$$\underline{\delta}_L A = \liminf_{n\to\infty} \frac{1}{\log n} \sum_{a_i \leq n} \frac{1}{a_i} \geq \inf_{n > N} \frac{A(n)}{n} , \qquad (2.21)$$

$$\overline{\delta}_L A = \limsup_{n\to\infty} \frac{1}{\log n} \sum_{a_i \leq n} \frac{1}{a_i} \leq \sup_{n > N} \frac{A(n)}{n} . \qquad (2.22)$$

By taking the limit as $N \to \infty$ of (2.21) and (2.22), we have

$$\underline{\delta} A \leq \underline{\delta}_L A \leq \overline{\delta}_L A \leq \overline{\delta} A ,$$

which is the desired result. $\triangle$

Theorem 2.10 indicates that, for a given sequence, the existence of asymptotic density implies the existence of logarithmic density. However, the existence of logarithmic density need not imply the existence of asymptotic density. In Chapter V a sequence demonstrating this will be constructed.

Generalized Asymptotic Density

The relationship between asymptotic density and logarithmic density can be examined from another viewpoint. In 1967, Ralph Alexander [1] defined what he called μ-density. In essence, Alexander defined a class of densities which includes asymptotic density and logarithmic density.

Definition 2.7. Let $A = \{a_1, a_2, \ldots\}$ be a sequence of positive integers, and let $\{c_1, c_2, \ldots\}$ be a sequence of positive real numbers, not necessarily monotone. We define the quantity

$$\mu A = \lim_{n\to\infty} \sum_{a_i \leq n} c_{a_i} \Big/ \sum_{i=1}^{n} c_i$$

to be the μ-density of A, if it exists.

Definition 2.8. For A and $\{c_i\}$ as above, we define the quantities

$$\underline{\mu} A = \liminf_{n\to\infty} \sum_{a_i \leq n} c_{a_i} \Big/ \sum_{i=1}^{n} c_i ,$$

$$\overline{\mu} A = \limsup_{n\to\infty} \sum_{a_i \leq n} c_{a_i} \Big/ \sum_{i=1}^{n} c_i$$

to be respectively the lower μ-density and the upper μ-density of A .

It is obvious that these definitions encompass an extensive class of densities, and that each μ-density is determined by the sequence of real numbers $\{c_i\}$ that is selected. We refer to this sequence as the index sequence for μ .

Consider the index sequence $\{1, 1, 1, \ldots\}$. Then

$$\sum_{a_i \leq n} c_{a_i} = \sum_{a_i \leq n} 1 = A(n) ,$$

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} 1 = n .$$

Therefore, asymptotic density is the μ-density that has the sequence $\{1, 1, \ldots\}$ as its index sequence.

Now consider the index sequence $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\}$. In this case

$$\sum_{a_i \leq n} c_{a_i} = \sum_{a_i \leq n} \frac{1}{a_i} ,$$

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} \frac{1}{i} .$$

Thus if the μ-density, with index sequence $\frac{1}{k}$, exists for a sequence of integers A , then

$$\mu A = \lim_{n\to\infty} \sum_{a_i \leq n} \frac{1}{a_i} \Big/ \sum_{i=1}^{n} \frac{1}{i} ,$$

and it follows from Lemma 2.1 that

$$\mu A = \lim_{n\to\infty} \sum_{a_i \leq n} \frac{1}{a_i} \Big/ \log n = \delta_L A ,$$

the logarithmic density of A .

A number of results concerning asymptotic density remain true for generalized asymptotic density. In particular, Theorems 2.4 , 2.6 , and 2.7 are valid if "asymptotic density" is replaced by "μ-density".

CHAPTER III

THE DECOMPOSITION THEOREM

In this chapter we shall investigate an interesting and useful decomposition of sequences of positive integers. This decomposition is a modification of a method introduced by Ralph Alexander [1]. The prime divisors of an integer will be the central theme of the decomposition.

<u>Definition 3.1</u>. Let $g(a)$ represent the greatest prime divisor of the integer a, and let P_a represent the set of natural numbers n such that the prime divisors of n are greater than $g(a)$. The set $aP_a = \{ax : x \in P_a\}$ is called the set of higher multiples of a.

Certain properties of the sets P_a and aP_a are immediate consequences of Definition 3.1. For example, the positive integers x and y are members of P_a if, and only if xy is a member of P_a. Another obvious, but useful, result is that if a divides b, then P_b is a subset of P_a.

<u>Example 3.1</u>. Consider the set P_a for $a = 21$. Since the greatest prime divisor of 21 is 7, P_{21} is the set of positive integers that are not divisible by any of the primes 2, 3, 5, and 7. This set may also be described as the set of positive integers that are relatively prime to 2, 3, 5, and 7. An integer is relatively prime to 2, 3, 5,

and 7 if, and only if, it is relatively prime to the product $2 \cdot 3 \cdot 5 \cdot 7 = 210$. Hence, we may write $P_{21} = \{k : (k, 210) = 1\}$; and the set of higher multiples of 21 can be expressed as $21 P_{21} = \{21k : (k, 210) = 1\}$.

We can generalize what is shown in Example 3.1. Let a be any positive integer. If $p_1, p_2, \ldots, p_r$ are the primes that do not exceed $g(a)$, and $m = p_1 p_2 \cdots p_r$, then P_a is composed of those integers that are relatively prime to m. That is

$$P_a = \{n : (n, m) = 1\} .$$

Thus it may be suspected that a process of counting the elements of P_a can be related to the Euler ϕ-function. Recall that $\phi(n)$ is defined to be the number of natural numbers $\leq n$ that are relatively prime to n.

<u>Theorem 3.1.</u> The set P_a possesses asymptotic density, and

$$\delta P_a = \frac{\phi(m)}{m} ,$$

where m is the product $p_1 p_2 \cdots p_r$ of the primes that do not exceed $g(a)$.

<u>Proof.</u> Since $P_a = \{n : (n, m) = 1\}$, there are exactly $\phi(m)$ elements of P_a in the interval $(0, m]$. In fact, in any interval $((k-1)m, km]$ there are exactly $\phi(m)$ elements of P_a. Therefore, for a given integer n, there exist integers k and t such that

$$P_a(n) = k\,\phi(m) + t ,$$

where

$$km < n \leq (k+1)m, \quad 0 \leq t < \phi(m).$$

Consider the ratio $P_a(n)/n$. This ratio may be expressed as

$$\frac{P_a(n)}{n} = \frac{k\,\phi(m)+t}{n}$$

From the inequalities involving n and t we obtain

$$\frac{k\,\phi(m)}{(k+1)m} \leq \frac{P_a(n)}{n} < \frac{(k+1)\,\phi(m)}{km} .$$

By the manner in which k was chosen, $k \to \infty$ as $n \to \infty$, and

$$\lim_{n\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{k}{k+1} = 1 ,$$

$$\lim_{n\to\infty} \frac{k+1}{k} = \lim_{k\to\infty} \frac{k+1}{k} = 1 .$$

Thus,

$$\lim_{n\to\infty} \frac{k\,\phi(m)}{(k+1)m} = \lim_{n\to\infty} \frac{(k+1)\,\phi(m)}{km} = \frac{\phi(m)}{m} ;$$

and so $\delta P_a = \lim_{n\to\infty} P_a(n)/n$ exists and is equal to $\frac{\phi(m)}{m}$. △

<u>Corollary</u>. The asymptotic densities of the sets P_a and aP_a are

$$\delta P_a = \prod_{p \leq g(a)} (1 - \frac{1}{p}) ,$$

$$\delta aP_a = a^{-1} \prod_{p \leq g(a)} (1 - \frac{1}{p}) .$$

Proof. The formula for the asymptotic density of P_a is a direct result of Theorem 3.1 and the formula for $\phi(m)$,

$$\phi(m) = \prod_{p|m} (p - 1) .$$

The formula for the asymptotic density of aP_a follows from Example 2.3. △

Definition 3.2. Let $A = \{a_i\}$ be a sequence of natural numbers such that $1 \notin A$. The primary part of A is defined to be

$$A' = \{a_i \in A : a_i \notin a_j P_{a_j} , \text{ for any } j\} .$$

That is, A' is the set of members of A that are not higher multiples of other members of A.

The concept of a primary part will form the nucleus of our decomposition. Thus it is worthwhile to consider a few examples.

Example 3.2. Let M_{10} denote the set consisting of the positive multiples of 10. An integer m in M_{10} can be expressed in the form $m = 2^i \cdot 3^j \cdot 5^k \cdot n$, where $i \geq 1$, $j \geq 0$, $k \geq 1$, $n \geq 1$, and $(n, 2 \cdot 3 \cdot 5) = 1$. Since n is composed of primes greater than 5, m is a higher multiple of another integer in M_{10} whenever $n \neq 1$. Therefore, the integers m' in the primary part of M_{10} are of the form $2^i \cdot 3^j \cdot 5^k$, where $i \geq 1$, $j \geq 0$, and $k \geq 1$. In other words,

$$M'_{10} = \{m' = 2^i 3^j 5^k : i \geq 1,\ j \geq 0,\ k \geq 1\}$$

$$= \{10, 20, 30, 40, 50, 60, 80, 90, 100, 120, 140, \ldots\}.$$

Example 3.3. Let S be the set of square-free integers (natural numbers that are not divisible by the square of a prime); then each member of S is a product of distinct primes. Thus the primary part of S is precisely the set of all primes.

Example 3.4. A sequence of integers in which no member is divisible by another member is called a primitive sequence. We shall investigate these sequences in detail, later. For the present, we observe that the primary part of a primitive sequence is the entire sequence. It is not true, however, that the primary part of any sequence is necessarily primitive. This is demonstrated in Example 3.2 since $M'_{10} = \{2^i 3^j 5^k : i \geq 1,\ j \geq 0,\ k \geq 1\}$ is not primitive.

Theorem 3.2 (Decomposition Theorem). Let A be any sequence of natural numbers with $1 \notin A$. If a is any member of A, then either a belongs to the primary part of A or a is a higher multiple of a unique member of the primary part of A.

Proof. Suppose that $a \notin A'$, the primary part of A. Then by definition, there is at least one a_i in A such that

$$a = a_i x_i, \quad \text{where } x_i \notin P_{a_i}. \tag{3.1}$$

Let a_i be the least member of A that satisfies (3.1). We wish to show that $a_i \in A'$. Assume that $a_i \notin A'$, then there exists at least one $a_j \in A$ such that $a_i = a_j x_j$, where $x_j \in P_{a_j}$; and so

$$a = a_j x_j x_i \, . \tag{3.2}$$

Since a_j divides a_i, P_{a_i} is a subset of P_{a_j}. Thus both x_i and x_j are members of P_{a_j}; and so we have

$$x_j x_i \in P_{a_j} \, . \tag{3.3}$$

From (3.2) and (3.3) we have that a_j satisfies (3.1). However, $a_j < a_i$, since $a_i = a_j x_i$ and $x_j > 1$. This is a contradiction since a_i was chosen to be the least element of A for which (3.1) is satisfied. Thus $a_i \in A'$ and a has at least one representation in the desired form.

To prove the uniqueness of the representation, it will suffice to show that the sets of higher multiples of distinct integers in A' are disjoint. Assume that a_i and a_j are distinct members of A' and that $a_i x_i = a_j x_j$, where x_i and x_j belong to P_{a_i} and P_{a_j}, respectively. We assume without loss of generality that the greatest prime divisor of a_i does not exceed the greatest prime divisor of a_j. Then a_i and x_j are relatively prime; and since $a_i | a_j x_j$, we have that $a_i | a_j$. Thus there is an x such that $a_i x = a_j$; so that $a_i x x_j = a_j x_j = a_i x_i$, which implies that $x | x_i$. Since $x_i \in P_{a_i}$ and $x | x_i$, $x \in P_{a_i}$, which is a contradiction since $a_j \in A'$. Therefore, the representation is unique.

$\triangle$

Let us now illustrate the Decomposition Theorem using the previous three examples. In example (3.2) we saw that the integers in the primary part of the set M_{10} were of the form $2^i 3^j 5^k$, where i, j, and k are integers, $i \geq 1$, $j \geq 0$, $k \geq 1$. Then any integer x not in the primary part is of the form $2^i 3^j 5^k n$, where n is composed of powers of primes that exceed 5. Thus x is a higher multiple of an integer in the primary part of M_{10}.

In Example 3.3 we saw that the primary part of the set of square-free integers S is the set of all primes. Then a member x of S, that does not belong to the primary part, is a product of two or more primes. Thus x is a higher multiple of the least of its prime divisors.

Since the primary part of a primitive sequence is the entire sequence, Example 3.4 illustrates the Decomposition Theorem in a trivial manner.

At times we shall find it essential to refer to known results concerning the distribution of primes. In particular, we shall utilize the asymptotic formula due to Mertens,

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim e^{-C} (\log x)^{-1} ,$$

where C is Euler's constant, (see A. E. Ingham [16], p. 22). We modify this formula for our own purpose in the following manner.

<u>Lemma 3.1</u>. Let p denote a prime number. We have the inequality.

$$\log x < \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} < M \log x, \quad \text{for } x \geq 2 ,$$

where M is a positive constant.

<u>Proof</u>. We first show that $\log x < \prod_{p \leq x} (1 - \frac{1}{p})^{-1}$. Let m be any positive integer. Since

$$\frac{1}{(1-u)} > \frac{(1-u^{m+1})}{(1-u)} = 1 + u + u^2 + \ldots + u^m ,$$

for $0 < u < 1$, we have

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} > \prod_{p \leq x} (1 + p^{-1} + p^{-2} + \ldots + p^{-m}) .$$

We may express the product on the right as the sum $\Sigma' \frac{1}{n}$, where the integers n are of the form

$$n = \prod_{p_i \leq x} p_i^{\alpha_i} , \quad 0 \leq \alpha_i \leq m .$$

All positive integers less than $\min(x, 2^{m+1})$ are of this form. If m is chosen so that $2^{m+1} > x$, then every n, $1 \leq n \leq x$, is included in $\Sigma' \frac{1}{n}$. Thus

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} > \sum_{n=1}^{[x]} \frac{1}{n} > \int_1^{[x]+1} \frac{1}{t} dt = \log([x]+1) > \log x .$$

The proof that $\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} < M \log x$ follows from the asymptotic formula by Mertens. From

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim e^{-C} (\log x)^{-1} ,$$

we have

$$\lim_{x \to \infty} \frac{\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1}}{e^{C} \log x} = 1 ;$$

and so

$$\prod_{p \leq x} (1 - \frac{1}{p})^{-1} < (1 + \frac{1}{2}) e^C \log x ,$$

for all x larger than some N. Let

$$M' = \max_{2 \leq x \leq N} \left(\frac{\prod_{p \leq x} (1 - \frac{1}{p})^{-1}}{\log x} \right) ,$$

and let $M = \max(M', \frac{3}{2} e^C)$. Then

$$\prod_{p \leq x} (1 - \frac{1}{p})^{-1} < M \log x ,$$

for all $x \geq 2$. △

We first employ this lemma in the proof of the following theorem.

Theorem 3.3. Let A be an arbitrary sequence of natural numbers such that $1 \notin A$, let A' be the primary part of A, and let $g(a')$ denote the greatest prime divisor of the integer a' in A'. Then

$$\sum_{a' \in A'} (a' \log g(a'))^{-1} \leq M ,$$

where M is the constant in Lemma 3.1.

Proof. In proving "uniqueness" in the Decomposition Theorem, we demonstrated that sets of higher multiples of distinct members of A' are disjoint. That is, if $a_i' \neq a_j'$, then $a_i' P_{a_i'} \cap a_j' P_{a_j'} = \emptyset$. By the corollary to Theorem 3.1, we have that

$$\delta(a' P_{a'}) = \frac{1}{a'} \prod_{p \leq g(a')} (1 - \frac{1}{p}) .$$

Hence by Lemma 3.1, we have the following inequality:

$$\delta(a' P_{a'}) > (a' M \log g(a'))^{-1} ,$$

for all a' in A'. Since the sum of the densities of a collection of disjoint sets does not exceed 1, we conclude that for any n,

$$\sum_{i=1}^{n} (a_i' \log g(a_i'))^{-1} < M \sum_{i=1}^{n} \delta(a_i' P_{a_i'}) \leq M .$$

This is the desired result if A' has a finite number of elements. If A' is infinite, taking the limit as $n \to \infty$, we have

$$\sum_{a' \in A'} (a' \log g(a'))^{-1} = \lim_{n \to \infty} \sum_{i=1}^{n} (a_i' \log g(a_i'))^{-1} \leq M . \qquad \Delta$$

Theorem 3.4. If A' is the primary part of a sequence of integers, then A' possesses zero logarithmic density.

Proof. Since we are concerned with the logarithmic density of A', we consider the quantity

$$(\log n)^{-1} \sum_{a' \leq n} \frac{1}{a'} ,$$

where $a' \in A'$ and n is any positive integer. For any integer k such that $1 \leq k < n$,

$$(\log n)^{-1} \sum_{a' \leq n} \frac{1}{a'} = (\log n)^{-1} \sum_{a' \leq k} \frac{1}{a'} + (\log n)^{-1} \sum_{k < a' \leq n} \frac{1}{a'} .$$

Given $\varepsilon > 0$, we shall show that each quantity on the right is $< \frac{\varepsilon}{2}$ for a particular k and large enough n.

Since $\log n \geq \log a'$ when $a' \leq n$, we have

$$(\log n)^{-1} \sum_{k < a' \leq n} \frac{1}{a'} \leq \sum_{k < a' \leq n} (a' \log a')^{-1} \leq \sum_{k < a'} (a' \log a')^{-1} .$$

From Theorem 3.3,

$$\sum_{a'} (a' \log g(a'))^{-1} \leq M ,$$

where $g(a')$ denotes the greatest prime divisor of a', and M is a positive constant. Since $\log g(a') \leq \log a'$,

$$\sum_{a'} (a' \log a')^{-1} \leq \sum_{a'} (a' \log g(a'))^{-1} \leq M .$$

Thus the series on the left converges, and there exists a k such that

$$\sum_{k < a'} (a' \log a')^{-1} < \frac{\varepsilon}{2} .$$

Hence,

$$(\log n)^{-1} \sum_{k < a' \leq n} \frac{1}{a'} < \frac{\varepsilon}{2} ,$$

for large enough k, and for every $n > k$.

For a fixed k satisfying the above inequality, there exists $N > k$ such that for all $n > N$,

$$(\log n)^{-1} \sum_{a' \leq k} \frac{1}{a'} < \frac{\varepsilon}{2} .$$

Thus for $n > N$,

$$(\log n)^{-1} \sum_{a' \leq n} \frac{1}{a'} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

and so $\delta_L A'$ exists and equals zero. $\triangle$

The following corollary is an immediate consequence of Theorem 2.5'.

Corollary. $\delta_L A = \delta_L (A \smallsetminus A')$.

We wish to formulate a method of determining the logarithmic density of a sequence by utilizing the preceding two theorems. The following two lemmas are essential.

Lemma 3.2. If $p_1, p_2, \ldots, p_r$ are distinct prime numbers (not necessarily the first r primes), then

$$\prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)^{-1} = 1 + \Sigma' \frac{1}{d} ,$$

where the sum is over all integers d that have no prime divisors other than $p_1, p_2, \ldots, p_r$.

Proof. The geometric series $\sum_{k=0}^{\infty} \frac{1}{p^k}$ has as its sum $(1 - \frac{1}{p})^{-1}$ for each p. Since every d is the product of powers of the primes $p_1, p_2, \ldots, p_r$, we have

$$1 + \Sigma' \frac{1}{d} = \prod_{i=1}^{r} \left(\sum_{k=0}^{\infty} \frac{1}{p_i^k} \right) = \prod_{i=1}^{r} (1 - \frac{1}{p_i})^{-1},$$

the desired result. Δ

The following Lemma introduces a method for determining the logarithmic density of a countable union of integer sequences.

Lemma 3.3. Let $\{A_i\}$ be a sequence of sets of natural numbers such that each A_i and each finite union $\bigcup_{i=1}^{j} A_i$ possess logarithmic density. Let $\{C_i\}$ be a sequence of real numbers such that $\Sigma\, C_i$ converges. If for all i and n,

$$(\log n)^{-1} \sum_{\substack{a \in A_i \\ a \leq n}} \frac{1}{a} \leq C_i ,$$

then $\delta_L(\bigcup_{i=1}^{\infty} A_i)$ exists and is equal to $\lim_{j\to\infty} \delta_L(\bigcup_{i=1}^{j} A_i)$.

Proof. By the properties of logarithmic density,

$$0 \leq \delta_L(\bigcup_{i=1}^{j} A_i) \leq \delta_L(\bigcup_{i=1}^{j+1} A_i) \leq 1 ,$$

which implies that the sequence $\{\delta_L(\bigcup_{i=1}^{j} A_i)\}$ is bounded and increasing. Thus $\lim_{j\to\infty} \delta_L(\bigcup_{i=1}^{j} A_i)$ exists. Since for each j, $\delta_L(\bigcup_{i=1}^{j} A_i) \leq \underline{\delta}_L(\bigcup_{i=1}^{\infty} A_i)$, we have

$$\lim_{j\to\infty} \delta_L(\bigcup_{i=1}^{j} A_i) \leq \underline{\delta}_L(\bigcup_{i=1}^{\infty} A_i) .$$

For the remainder of the proof we utilize the following notation:

$$L(n\,;b,c) = (\log n)^{-1} \sum \frac{1}{a} ,$$

where the sum is over all integers a such that $a \in \bigcup_{i=b}^{c} A_i$ and $a \leq n$. In this notation, $\overline{\delta}_L(\bigcup_{i=1}^{b} A_i) = \limsup_{n\to\infty} L(n\,;1,b)$. For any j and n, we have

$$L(n\,;j+1,\infty) \leq \sum_{i=j+1}^{\infty} (\log n)^{-1} \sum_{\substack{a \in A_i \\ a \leq n}} \frac{1}{a} \leq \sum_{i=j+1}^{\infty} C_i .$$

Since $L(n\,;1,\infty) = L(n\,;j+1,\infty) + L(n\,;1,j)$, we have

$$L(n\,;1,\infty) \leq \sum_{i=j+1}^{\infty} C_i + L(n\,;1,j) ;$$

so that taking the limit superior as $n \to \infty$, we have, for each j,

$$\overline{\delta}_L(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=j+1}^{\infty} C_i + \delta_L(\bigcup_{i=1}^{j} A_i) .$$

Since $\Sigma\, C_i$ converges, $\lim_{j\to\infty} \sum_{i=j+1}^{\infty} C_i = 0$; and it follows that

$$\overline{\delta}_L(\bigcup_{i=1}^{\infty} A_i) \leq \lim_{j\to\infty} \delta_L(\bigcup_{i=1}^{j} A_i) . \qquad (3.5)$$

From (3.4) and (3.5) we conclude that the logarithmic density of the countable union of A_i's exists and

$$\delta_L(\bigcup_{i=1}^{\infty} A_i) = \lim_{j\to\infty} \delta_L(\bigcup_{i=1}^{j} A_i) . \qquad \triangle$$

Theorem 3.5. Let A be any set of natural numbers with $1 \notin A$. Suppose that $\delta_L(A \cap a'P_{a'})$ exists for all a' in A'. Then $\delta_L(A)$ exists and is equal to $\sum_{a' \in A'} \delta_L(A \cap a'P_{a'})$.

Proof. By the Decomposition Theorem, the integers in $A \setminus A'$ are those members of A which are higher multiples of integers in A'. Thus, we have

$$\delta_L(A \setminus A') = \delta_L\left[A \cap \left(\bigcup_{a' \in A'} a'P_{a'}\right)\right] = \delta_L\left[\bigcup_{a' \in A'} (A \cap a'P_{a'})\right]. \quad (3.6)$$

If A' is finite, then the theorem is a result of (3.6) and the finite additivity of δ_L.

If A' is infinite an additional argument is needed. Let a' be a fixed member of A' and let n be a positive integer. Let b denote any integer satisfying $b \in A \cap a'P_{a'}$ and $b \leq n$. Then we have

$$\sum \frac{1}{b} \leq \sum_{\substack{x \in P_{a'} \\ a'x \leq n}} \frac{1}{a'x} \leq \left(\frac{1}{a'}\right) \sum_{\substack{x \in P_{a'} \\ x \leq n}} \frac{1}{x}. \quad (3.7)$$

If $g(a')$ denotes the greatest prime divisor of a', then all of the x's counted in the sum on the far right in (3.7) are composed of primes satisfying $g(a') < p \leq n$. By Lemma 3.2 this sum does not exceed

$$\prod_{g(a') < p \leq n} \left(1 - \frac{1}{p}\right)^{-1}.$$

Thus, we have

$$\sum \frac{1}{b} \leq \frac{1}{a'} \left(\prod_{g(a') < p \leq n} \left(1 - \frac{1}{p}\right)^{-1}\right).$$

The product on the right can be rewritten as the quotient

$$\frac{\prod_{p \leq n} (1 - \frac{1}{p})^{-1}}{a' \left(\prod_{p \leq g(a')} (1 - \frac{1}{p})^{-1} \right)} .$$

By Lemma 3.1, the numerator of this quotient is less than $M \log n$, and the denominator exceeds $a' \log g(a')$. Thus

$$\Sigma \frac{1}{b} < \frac{M \log n}{a' \log g(a')} .$$

Thus for any a' and n, we have

$$(\log n)^{-1} \Sigma \frac{1}{b} \leq M(a' \log g(a'))^{-1} = C_{a'} . \tag{3.8}$$

By Theorem 3.3,

$$\sum_{a' \in A'} (a' \log g(a'))^{-1} \leq M ;$$

and so,

$$\sum_{a' \in A'} C_{a'} \leq M^2 . \tag{3.9}$$

By statements (3.3) and (3.9) the sets $A \cap a' P_{a'}$ satisfy the hypotheses of Lemma 3.3. Thus

$$\delta_L(A \setminus A') = \delta_L \left[\bigcup_{a' \in A'} (A \cap a' P_{a'}) \right]$$

exists and is equal to

$$\lim_{m\to\infty} \delta_L \left[\bigcup_{\substack{a' \in A' \\ a' \leq m}} (A \cap a' P_{a'}) \right] .$$

From the proof of "uniqueness" in the Decomposition Theorem, the sets $A \cap a' P_{a'}$ are disjoint for distinct a'. Thus

$$\delta_L(A \smallsetminus A') = \lim_{m\to\infty} \sum_{\substack{a' \in A' \\ a' \leq m}} \delta_L(A \cap a' P_{a'}) = \sum_{a' \in A'} \delta_L(A \cap a' P_{a'}) .$$

Since $\delta_L A = \delta_L(A \smallsetminus A')$ by the corollary to Theorem 3.4, the theorem is proved. Δ

CHAPTER IV

THE SET OF MULTIPLES OF A SEQUENCE

In this chapter we shall be concerned with the density of the set of positive multiples of a sequence. Throughout the chapter we shall assume that $A = \{a_1, a_2, \ldots\}$ is an increasing sequence of natural numbers such that $a_1 \neq 1$. We exclude the integer 1 since the set of positive multiples of 1 is identically the set N of all natural numbers, which presents an uninteresting case.

The Asymptotic Density of a Set of Multiples

Definition 4.1. Let A be a set of natural numbers. The set of multiples of A, denoted by $B(A)$, is the set of all positive multiples of the integers in A. When A is understood, we simply replace $B(A)$ by B.

Example 4.1. If the set A is composed of only one integer a, then by Example 2.3 $\delta B(A) = \frac{1}{a}$. Suppose $A = \{a', a''\}$. Then $B(A) = M_{a'} \cup M_{a''}$, where $M_{a'}$ and $M_{a''}$ denote the sets of positive multiples of a' and a'', respectively. The set $B = B(A)$ may be represented as the union of disjoint sets by $B = M_{a'} \cup (M_{a''} \setminus M_{a'})$. Since $M_{a''} \setminus M_{a'} = M_{a'} \setminus (M_{a'} \cap M_{a''})$ and $M_{a'} \cap M_{a''} = M_{[a', a'']}$, where $[a', a'']$ denotes the least common multiple of a' and a'', we have $B = M_{a'} \cup (M_{a''} \setminus M_{[a', a'']})$. By Theorem 2.6 and Theorem 2.4

we have

$$\delta B = \delta M_{a'} + \delta(M_{a''} \smallsetminus M_{[a',a'']}) = \delta M_{a'} + \delta M_{a''} - \delta M_{[a',a'']} .$$

Thus

$$\delta B = \frac{1}{a'} + \frac{1}{a''} - \frac{1}{[a', a'']} .$$

Example 4.2. Let P be the set of primes. Since every integer > 1 is a multiple of primes, $B = B(P) = N \smallsetminus \{1\}$; and so $\delta B(P) = 1$. Now, let P' denote the set of all primes except the prime p_i. Does $B' = B(P')$ also possess asymptotic density 1 ? It is obvious that $B \supset B'$, and the only integers in B that are not in B' are the powers of p_i. That is $B' = B \smallsetminus \{p_i^{\alpha} \mid \alpha = 1, 2, \ldots\}$. Since $B(n)$, the number or integers in B not exceeding n, is equal to $n - 1$, we have $B'(n) = n - 1 - \alpha(n)$, where $\alpha(n)$ is the greatest integer such that $p_i^{\alpha(n)} \leq n$. Since $\alpha(n) = \log_{p_i} p_i^{\alpha(n)} \leq \log_{p_i} n$, we have

$$\delta B' \geq \lim_{n\to\infty} \frac{(n - 1 - \log_{p_i} n)}{n} = 1 .$$

We may make the stronger statement, that if P'' is the set of all primes except $p_1, p_2, \ldots, p_k$ (not necessarily the first k primes), then $B'' = B(P'')$ also possesses asymptotic density 1. The sets $\{p_i^{\alpha} \mid \alpha = 1, 2, \ldots\}$, $i = 1, 2, \ldots, k$, are pairwise disjoint and

$$\delta B'' = \lim_{n\to\infty} \frac{B''(n)}{n} \geq \lim_{n\to\infty} \left(\frac{n-1}{n} - \sum_{i=1}^{k} n^{-1} \log_{p_i} n \right) = 1 .$$

Definition 4.2. Let $A = \{a_1, a_2, \ldots\}$ be a sequence of natural numbers. By $B_m(A)$, or simply B_m, we mean the set of positive multiples of the first m terms of A.

Definition 4.3. Let $A = \{a_1, a_2, \ldots\}$ be a sequence of natural numbers. We denote by $B^{(m)}(A)$, or simply $B^{(m)}$, the set of all natural numbers which are multiples of a_m but not multiples of a_1 or a_2 or $\cdots$ or a_{m-1}.

There are obvious relationships among the sets B_m, $B^{(m)}$, and M_a. For instance,

$$B_m(A) = B(\{a_1, a_2, \ldots, a_m\}) = \bigcup_{i=1}^{m} M_{a_i} = \bigcup_{i=1}^{m} B^{(i)}(A),$$

and

$$B^{(m)}(A) = B_m(A) \smallsetminus B_{m-1}(A) = M_{a_m} \smallsetminus \bigcup_{j=1}^{m-1} M_{a_j}.$$

These relationships, as well as the following counting principle, will be utilized to determine the existence of and a formula for $\delta B_m(A)$.

Lemma 4.1. Let A be any set, and let $c(A)$ denote the number of elements in A. If $A = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_m$, then

$$c(A) = \sum_{i=1}^{m} c(A_i) - \sum_{i<j} c(A_i \cap A_j) + \sum_{i<j<k} c(A_i \cap A_j \cap A_k)$$
$$- \ldots + (-1)^{m-1} c(A_1 \cap A_2 \cap \cdots \cap A_m).$$

The validity of Lemma 4.1 can be realized intuitively and proved inductively. If the reader desires to see a proof, consult Finite

Mathematics with Applications by A. W. Goodman and J. S. Ratti; Macmillan, 1971, pages 118-119.

Theorem 4.1. The set B_m possesses asymptotic density and

$$\delta B_m = \sum_{i=1}^{m} \frac{1}{a_i} - \sum_{i<j} \frac{1}{[a_i, a_j]} + \sum_{i<j<k} \frac{1}{[a_i, a_j, a_k]} - \dots + \frac{(-1)^{m-1}}{[a_1, a_2, \dots, a_m]} .$$

Proof. As usual, we let $B_m(n)$ denote the number of integers in B_m that do not exceed n. Since $B_m = \bigcup_{i=1}^{m} M_{a_i}$, $B_m(n) = \left(\bigcup_{i=1}^{m} M_{a_i}\right)(n)$. Thus by Lemma 4.1, we have

$$B_m(n) = \sum_{i=1}^{m} M_{a_i}(n) - \sum_{i<j} \left(M_{a_i} \cap M_{a_j}\right)(n) + \sum_{i<j<k} \left(M_{a_i} \cap M_{a_j} \cap M_{a_k}\right)(n)$$

$$+ \dots + (-1)^{m-1} \left(\bigcap_{i=1}^{m} M_{a_i}\right)(n)$$

$$= \sum_{i=1}^{m} \left[\frac{n}{a_i}\right] - \sum_{i<j} \left[\frac{n}{[a_i, a_j]}\right] + \sum_{i<j<k} \left[\frac{n}{[a_i, a_j, a_k]}\right]$$

$$+ \dots + (-1)^{m-1} \left[\frac{n}{[a_1, a_2, \dots, a_m]}\right]$$

where the smaller brackets indicate the least common multiple and the larger brackets indicate the greatest integer function. Now dividing by n, we have

$$B_m \frac{(n)}{n} = \sum_{i=1}^{m} n^{-1} \left[\frac{n}{a_i}\right] - \sum_{j<j} n^{-1} \left[\frac{n}{[a_i, a_j]}\right]$$

$$+ \dots + (-1)^{m-1} n^{-1} \left[\frac{n}{[a_1, a_2, \dots, a_m]}\right] .$$

Let $n \to \infty$. By Theorem A. 3 of the appendix, we have

$$\delta B_m = \sum_{i=1}^{m} \frac{1}{a_i} - \sum_{i<j} \frac{1}{[a_i, a_j]} + \sum_{i<j<k} \frac{1}{[a_i, a_j, a_k]}$$

$$+ \ldots + \frac{(-1)^{m-1}}{[a_1, a_2, \ldots, a_m]} \; . \qquad \triangle$$

The following corollary is a direct consequence of Theorem 4. 1 and the identity $B^{(m)}(A) = B_m(A) \smallsetminus B_{m-1}(A)$.

<u>Corollary.</u> The set $B^{(m)}$ possesses asymptotic density and

$$\delta B^{(m)} = \frac{1}{a_m} - \sum_{i\;1}^{m} \frac{1}{[a_i, a_m]} + \sum_{i<j<m} \frac{1}{[a_i, a_j, a_m]}$$

$$- \ldots + \frac{(-1)^{m-1}}{[a_1, a_2, \ldots, a_m]}$$

<u>Corollary.</u> If A is a set of pairwise relatively prime positive integers, then the set $B_m(A)$ possesses asymptotic density

$$\delta B_m(A) = 1 - \prod_{i=1}^{m} \left(1 - \frac{1}{a_i}\right) .$$

<u>Proof.</u> Since the integers in A are pairwise relatively prime, the least common multiple of any subset of A is the product of the elements in the subset. Thus, utilizing Theorem 4. 1 , we have

$$\delta B_m(A) = \sum_{i=1}^{m} \frac{1}{a_i} - \sum_{i<j} (a_i a_j)^{-1} + \sum_{i<j<k} (a_i a_j a_k)^{-1}$$

$$- \ldots + (-1)^{m-1} (a_1 a_2 \cdots a_m)^{-1} .$$

Observe that the terms are composed of products of the rational numbers $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_m}$. Consider the product $\prod_{i=1}^{m}\left(1 - \frac{1}{a_i}\right)$.

$$\prod_{i=1}^{m}\left(1 - \frac{1}{a_i}\right) = 1 - \sum_{i=1}^{m} \frac{1}{a_i} + \sum_{i<j} (a_i a_j)^{-1} - \sum_{i<j<k} (a_i a_j a_k)^{-1}$$

$$- \ldots + (-1)^m (a_1 a_2 \cdots a_m)^{-1}$$

$$= 1 - \delta B_m(A) .$$

Thus we have

$$\delta B_m(A) = 1 - \prod_{i=1}^{m}\left(1 - \frac{1}{a_i}\right) ,$$

the desired result. $\triangle$

<u>Example 4.3.</u> We can utilize the preceding corollary to determine the asymptotic density of the set of multiples of the first m primes, $B_m(P)$. We have

$$\delta B_m(P) = 1 - \prod_{i=1}^{m}\left(1 - \frac{1}{p_i}\right) .$$

Since the value of the Euler ϕ-function for the product $p_1 p_2 \cdots p_m$ is

$$\phi(p_1 p_2 \cdots p_m) = \prod_{i=1}^{m} (p_i - 1) ,$$

we can express $\delta B_m(P)$ in terms of ϕ as follows:

$$\delta B_m(P) = 1 - \prod_{i=1}^{m} \frac{(p_i - 1)}{p_i}$$

$$= 1 - (p_1 p_2 \cdots p_m)^{-1} \prod_{i=1}^{m} (p_i - 1)$$

$$= 1 - \frac{\phi(p_1 p_2 \cdots p_m)}{p_1 p_2 \cdots p_m} .$$

If we compare this result with Theorem 3.1, we see that

$$\delta B_m(P) = 1 - \delta P_{p_m} , \; P_{p_m} = \{n : (n, p_1 p_2 \cdots p_m) = 1\} .$$

This is understandable since the only integers not relatively prime to $p_1 p_2 \cdots p_m$ are multiples of at least one of these primes. Thus $B_m(P) = P^c_{p_m}$.

We now consider the density of $B = B(A)$ for any A. By the definition of B_m, we have the following relationship,

$$B_1 \subset B_2 \subset \cdots \subset B .$$

Thus

$$0 \leq \delta B_1 \leq \delta B_2 \leq \ldots \leq \underline{\delta} B \leq 1 .$$

The sequence $\{\delta B_m\}$ is bounded and monotone increasing. Hence, $\lim_{m\to\infty} \delta B_m$ exists and

$$\lim_{m\to\infty} \delta B_m \leq \underline{\delta} B .$$

It is natural to expect that the density of B is $\lim_{m\to\infty} \delta B_m$. However, this is not true in general. Later, we shall consider a theorem by A. S. Besicovitch [5] that there exists a set of multiples that does not possess asymptotic density. The following theorem illustrates a sufficient condition on A so that $\delta B(A) = \lim_{m\to\infty} \delta B_m(A)$.

Theorem 4.2. Let $A = \{a_1, a_2, \ldots\}$ be a sequence of natural numbers. If the series $\Sigma \frac{1}{a_i}$ converges, then the set of multiples, $B = B(A)$, of A possesses asymptotic density and $\delta B(A) = \lim_{m\to\infty} \delta B_m(A)$.

Proof. We have seen that in general $\lim_{m\to\infty} \delta B_m \leq \underline{\delta} B$. Thus we need only show that $\lim_{m\to\infty} \delta B_m \geq \overline{\delta} B$ whenever $\Sigma \frac{1}{a_i}$ converges.

We can express B as a disjoint union of sets,

$$B = B_m \cup (B \setminus B_m),$$

so that the number of integers in B that do not exceed n is

$$B(n) = B_m(n) + (B \setminus B_m)(n).$$

Since $(B \setminus B_m) \subset \left(\bigcup_{i=m+1}^{\infty} M_{a_i} \right)$, we have

$$B(n) \leq B_m(n) + \sum_{i=m+1}^{\infty} M_{a_i}(n).$$

Since $M_{a_i}(n) = [\frac{n}{a_i}]$ and $[\frac{n}{a_i}] \leq \frac{n}{a_i}$,

$$B(n) \leq B_m(n) + n \sum_{i=m+1}^{\infty} \frac{1}{a_i}.$$

Dividing by n, we have

$$\frac{B(n)}{n} \leq \frac{B_m(n)}{n} + \sum_{i=m+1}^{\infty} \frac{1}{a_i};$$

and taking the lim sup, we obtain

$$\overline{\delta} B \leq \delta B_m + \sum_{i=m+1}^{\infty} \frac{1}{a_i} .$$

Since $\Sigma \frac{1}{a_i}$ converges, taking the limit as $m \to \infty$, we have

$$\overline{\delta} B \leq \lim_{m \to \infty} \delta B_m ;$$

and the theorem is proved. △

Theorem 4.2 presented a sufficient condition on A for B(A) to possess asymptotic density. The following theorem will illustrate that this condition is not a necessary condition.

<u>Theorem 4.3</u>. There exists a sequence A of natural numbers such that B(A) possesses asymptotic density and the series $\sum_{a \in A} \frac{1}{a}$ is divergent.

<u>Proof</u>. Let $A = P$, the set of primes. Since $B(P) = N$, $\delta B(P) = 1$. Thus we need only show that the series $\sum_{p \in P} \frac{1}{p}$ is divergent.

Let $S(x) = \sum_{p \leq x} \frac{1}{p}$ and $P(x) = \prod_{p \leq x} (1 - \frac{1}{p})^{-1}$. From Lemma 3.1, we have

$$P(x) > \log x .$$

Also,

$$\log P(x) = \sum_{p \leq x} \log (1 - \frac{1}{p})^{-1} = - \sum_{p \leq x} \log (1 - \frac{1}{p}) .$$

The series expansion of $\log (1 - \frac{1}{p})$ gives

$$-\log\left(1-\frac{1}{p}\right) = p^{-1} + 2^{-1}p^{-2} + 3^{-1}p^{-3} + \dots$$

$$< p^{-1} + 2^{-1}p^{-2}(1+p^{-1}+p^{-2}+\dots)$$

$$= p^{-1} + 2^{-1}p^{-2}\left(1-\frac{1}{p}\right)^{-1}$$

Thus

$$-\log\left(1-\frac{1}{p}\right) - \frac{1}{p} < 2^{-1}p^{-2}\left(1-\frac{1}{p}\right)^{-1} ,$$

and

$$\log P(x) - S(x) < \sum_{p\leq x} 2^{-1}p^{-2}\left(1-\frac{1}{p}\right)^{-1}$$

$$= \sum_{p\leq x} \frac{1}{2p(p-1)} < \sum_{n=2}^{\infty} \frac{1}{2n(n-1)} = \frac{1}{2} .$$

Therefore, by Lemma 3.1,

$$S(x) > \log P(x) - \frac{1}{2} > \log\log x - \frac{1}{2} .$$

Since $S(x)$ is the partial sum of the series $\Sigma\frac{1}{p}$, the series is divergent, and the theorem is established. △

The Set of k-Free Integers

In order to illustrate the application of Theorem 4.1 and Theorem 4.2, we shall investigate an important class of integral sequences, the sequences of k-free integers. A positive integer is said to be square-free if it is not divisible by the square of a prime. In a general sense, if k is an integer greater than 1, then a positive integer is said to be k-free if it is not divisible by the k^{th} power of a prime. We denote the set of k-free integers by $S^{(k)}$. A formula for the asymptotic distribution of $S^{(k)}$ was developed in 1931 by C. J. Evelyn and E. H. Linfort [12]. Their proof relies on extensive use of series involving the Möbius function. In January, 1972 J. E. Nymann [22] presented a new proof using a Möbius inversion formula. It is possible to avoid either approach by utilizing the theory developed in the first section of this chapter.

It is obvious that the sets $S^{(k)}$, $k = 2, 3, \ldots$, are not sets of multiples. However, it is possible to express a set of k-free integers as the complement of a set of multiples. A positive integer x is k-free if, and only if, x is not a multiple of p^k, for any prime p. Thus, if $P^k = \{p_1^k, p_2^k, \ldots\}$ is the set of k^{th} powers of all primes and $B(P^k)$ is the set of multiples of P^k, then $S^{(k)} = N \setminus B(P^k)$, where N is the set of natural numbers. Thus $\delta S^{(k)} = 1 - \delta B(P^k)$.

The following lemma is essential for developing an expression for $\delta B(P^k)$. A generalization of the lemma is presented in The Distribution of Prime Numbers by A. E. Ingham [16].

Lemma 4.2. Let $\sum_{n=1}^{\infty} \frac{1}{n^k} = \xi(k)$. Then

$$\xi(k) = \prod_p (1 - p^{-k})^{-1}$$

for all integers k, such that $k \geq 2$.

Proof. Consider the product

$$Q(x) = \prod_{p \leq x} \left(1 + \frac{1}{p^k} + \frac{1}{p^{2k}} + \ldots\right), \quad x > 2 .$$

The number of factors in this product is finite, and each factor is a convergent series. Therefore, we may take the product in any order. In particular, we have

$$Q(x) = \Sigma' \frac{1}{n^k} ,$$

where the summation is over all positive integers n whose prime factors do not exceed x. Since we may write

$$\sum_{n=1}^{\infty} \frac{1}{n^k} = \Sigma' \frac{1}{b^k} + \Sigma'' \frac{1}{n^k} ,$$

where the summation farthest to the right is over all positive integers n with at least one prime factor greater than x, we have

$$\xi(k) = Q(x) + \Sigma'' \frac{1}{n^k} .$$

Thus

$$|\xi(k) - Q(x)| \leq \Sigma'' \frac{1}{n^k} .$$

Since every n counted in the summation on the right is greater than x, we have

$$|\xi(k) - Q(x)| \leq \sum_{n>x} \frac{1}{n^k} .$$

The series $\Sigma \frac{1}{n^k}$ converges for $k > 2$, therefore, $\sum_{n>x} \frac{1}{n^k}$ approaches zero as $x \to \infty$. Thus

$$\lim_{x \to \infty} Q(x) = \xi(k) .$$

For each prime p the series $1 + \frac{1}{p^k} + \frac{1}{p^{2k}} + \ldots$ is a geometric series with sum $\left(1 - \frac{1}{p^k}\right)^{-1}$. Thus

$$\xi(k) = \prod_p \left(1 + \frac{1}{p^k} + \frac{1}{p^{2k}} + \ldots\right) = \prod_p \left(1 - \frac{1}{p^k}\right)^{-1} . \qquad \Delta$$

Theorem 4.4. The asymptotic density of the set of k-free integers exists and is equal to $\frac{1}{\xi(k)}$.

Proof. We have seen that $\delta S^{(k)} = 1 - \delta B(P^k)$, where $P^k = \{p^k : p \text{ is a prime}\}$. Thus it is sufficient to determine the density of $B(P^k)$.

Since the hyper-harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges for $k > 1$, the subseries $\sum_p \frac{1}{p^k}$ converges. Therefore by Theorem 4.2,

$$\delta B(P^k) = \lim_{m\to\infty} \delta B_m(P^k) .$$

From a corollary to Theorem 4.1 we have

$$\delta B_m(P^k) = 1 - \prod_{i=1}^{m} \left(1 - \frac{1}{p_i^k}\right) .$$

Thus

$$\delta B(P^k) = 1 - \prod_p \left(1 - \frac{1}{p^k}\right) ,$$

$$\delta S^{(k)} = \prod_p \left(1 - \frac{1}{p^k}\right) .$$

From Lemma 4.1 we have

$$\delta S^{(k)} = \frac{1}{\xi(k)} ,$$

the desired result. △

Using complex analysis, in particular the calculus of residues (see Hille [15], pages 258-264), or using Fourier analysis (see Rainville [23], pages 187-196), the function $\xi(k)$ can be evaluated for $k = 2$,

$$\xi(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} .$$

The following corollary is immediate.

Corollary. The asymptotic density of the set of square-free integers exists and is equal to $\frac{6}{\pi^2}$.

The Logarithmic Density of a Set of Multiples

In 1936 Davenport and Erdős [6] proved that the set of multiples of any sequence of integers possesses logarithmic density and that the logarithmic density is equal to the lower asymptotic density. Their original proof involved a previous theorem by Hardy and Littlewood and utilized the Dirichlet series. In 1951 Davenport and Erdős [7] presented a more direct and elementary proof. This later proof involved a decomposition similar to that of the Decomposition Theorem. For this reason, it is no surprise that the Davenport-Erdős Theorem can be treated as an application of the Decomposition Theorem.

Theorem 4.5 (Davenport-Erdős Theorem). If $B = B(A)$ is the set of multiples of a sequence A of natural numbers, then the logarithmic density of B exists and is equal to the lower asymptotic density of B.

Proof. Since $\delta_L B' = 0$ by Theorem 3.4, $\delta_L B$ is equal to $\delta_L(B \setminus B')$, if these densities exist. We will use Theorem 3.5 to show that $\delta_L(B \setminus B')$ does indeed exist.

If b' is a member of B', then b' is a multiple of some integer in A; and so $b' P_{b'}$, the set of higher multiples of b', is a subset of B. Thus

$$B \cap b' P_{b'} = b' P_{b'} . \tag{4.1}$$

for each b'. By the corollary to Theorem 3.1, $\delta_L b' P_{b'}$ exists. Thus $\delta_L(B \cap b' P_{b'})$ exists, and

$$\delta_L(B \cap b' P_{b'}) = \delta_L b' P_{b'} .$$

By Theorem 3.5, we have that $\delta_L B$ exists and

$$\delta_L B = \sum_{b' \in B'} \delta_L(B \cap b' P_{b'}) = \sum_{b' \in B'} \delta_L(b' P_{b'}) . \tag{4.2}$$

Now, from Theorem 2.10, we have

$$\underline{\delta} B \leq \delta_L B . \tag{4.3}$$

By the Decomposition Theorem and (4.1),

$$B \setminus B' = \bigcup_{b' \in B'} (B \cap b' P_{b'}) = \bigcup_{b' \in B'} b' P_{b'} . \tag{4.4}$$

Now, for each j,

$$0 \le \delta\left(\bigcup_{i=1}^{j} b_i' P_{b_i'}\right) \le \delta\left(\bigcup_{i=1}^{j+1} b_i' P_{b_i'}\right) \le 1 ;$$

so that $\lim_{j\to\infty} \delta\left(\bigcup_{i=1}^{j} b_i' P_{b_i'}\right)$ exists. By set inclusion and (4.4),

$$\delta\left(\bigcup_{i=1}^{j} b_i' P_{b_i'}\right) \le \underline{\delta}\left(\bigcup_{i=1}^{\infty} b_i' P_{b_i'}\right) = \underline{\delta}(B \smallsetminus B') .$$

Since $B \smallsetminus B' \subset B$, taking the limit as $j \to \infty$, we obtain

$$\lim_{j\to\infty} \delta\left(\bigcup_{i=1}^{j} b_i' P_{b_i'}\right) \le \underline{\delta}(B \smallsetminus B') \le \underline{\delta} B . \tag{4.5}$$

By Theorem 2.10,

$$\lim_{j\to\infty} \delta\left(\bigcup_{i=1}^{j} b_i' P_{b_i'}\right) = \lim_{j\to\infty} \delta_L\left(\bigcup_{i=1}^{j} b_i' P_{b_i'}\right) .$$

Since $b_i' P_{b_i'}$ is a subset of B, the uniqueness property of the Decomposition Theorem (Theorem 3.2) implies that the sets $b_i' P_{b_i'}$ are disjoint. Therefore,

$$\lim_{j\to\infty} \delta_L\left(\bigcup_{i=1}^{j} b_i P_{b_i}\right) = \sum_{i=1}^{\infty} \delta_L(b_i P_{b_i}) . \tag{4.6}$$

By (4.2),

$$\sum_{i=1}^{\infty} \delta_L(b_i P_{b_i}) = \delta_L B .$$

This, together with (4.5) and (4.6), implies that $\underline{\delta} B \ge \delta_L B$. By this inequality and (4.3), $\underline{\delta} B = \delta_L B$: and the theorem is proved. Δ

An Example by A. S. Besicovitch

We have seen that the set of multiples of any sequence possesses logarithmic density. It was conjectured by S. Chowla that these sets also possess asymptotic density. In 1934 A. S. Besicovitch [5] disproved this conjecture by exhibiting a sequence whose set of multiples does not possess asymptotic density. The sequence that Besicovitch constructed is closely related to the sequence of integers in the interval $(a, 2a]$. We shall denote this sequence by I_a. An investigation of the distribution of $B(I_a)$ will better prepare us for Besicovitch's result.

If, as before, we let M_i denote the set of positive multiples of the integer i, then

$$B(I_a) = M_{a+1} \cup M_{a+2} \cup \cdots \cup M_{2a} .$$

The integer $(2a)!$ is a member of each set M_i above. Let t be an integer in $B(I_a)$, that is, t is in M_j for some j, $j = a+1, a+2, \ldots, 2a$. Since $(2a)!$ is also in M_j, the integer $t + (2a)!$ is in M_j. Thus $t + (2a)!$ is in $B(I_a)$; and the distribution of $B(I_a)$ is of a periodic nature. Conversely, if $t + (2a)!$ is in $B(I_a)$, then so is t.

Let $[B(I_a)](n)$ denote the number of elements of $B(I_a)$ that do not exceed n. By the periodic nature of $B(I_a)$, there are exactly $[B(I_a)]((2a)!)$ elements of $B(I_a)$ that are in any interval $(k(2a)!, (k+1)(2a)!]$. Suppose that m is an integer such that m is in the interval $(k(2a)!, (k+1)(2a)!]$, $k \in N$. Then $[B(I_a)](m)$ satisfies the inequalities

$$k[B(I_a)]((2a)!) \leq [B(I_a)](m) \leq (k+1)[B(I_a)]((2a)!) ;$$

and so dividing by m, we have

$$\frac{[B(I_a)](m)}{m} \leq \frac{(k+1)[B(I_a)]((2a)!)}{m} < \frac{(k+1)[B(I_a)]((2a)!)}{k(2a)!}$$

$$\frac{[B(I_a)](m)}{m} \geq \frac{k[B(I_a)]((2a)!)}{m} \geq \frac{k[B(I_a)]((2a)!)}{(k+1)(2a)!} .$$

Since $k \to \infty$ as $m \to \infty$, and since

$$\lim_{k\to\infty} \frac{(k+1)[B(I_a)]((2a)!)}{k(2a)!} = \lim_{k\to\infty} \frac{k[B(I_a)]((2a)!)}{(k+1)(2a)!} = \frac{[B(I_a)]((2a)!)}{(2a)!} ,$$

the asymptotic density of $B(I_a)$ exists and

$$\delta B(I_a) = \frac{[B(I_a)]((2a)!)}{(2a)!} .$$

We can use the inequalities of the previous paragraph to determine an upper bound on $[B(I_a)](m)$ that will be used in Besicovitch's example. Since $[B(I_a)](m) \leq (k+1)[B(I_a)]((2a)!)$, and $2m > 2k(2a)! \geq (k+1)(2a)!$, we have

$$[B(I_a)](m) < 2m \frac{[B(I_a)]((2a)!)}{(2a)!} = 2m\ \delta B(I_a) .$$

Thus we have proved the following lemma.

<u>Lemma 4.3</u>. Let I_a denote the set of integers in the interval $(a, 2a]$. Then $\delta B(I_a)$ exists. If $m > (2a)!$, then the number of elements of $B(I_a)$ that do not exceed m is less than $2m\ \delta B(I_a)$.

We shall now investigate the limit of $\delta B(I_a)$ as a approaches infinity. Observing the values of the first few terms in the sequence

$\{\delta B(I_a)\}$, $a = 1, 2, \ldots,$ offers little help in finding a possible limit. For instance, for $a = 1, 2, 3, 4,$ and 5 we have,

$$\delta B(I_1) = 0.5$$

$$\delta B(I_2) = 0.5$$

$$\delta B(I_3) \doteq 0.4667$$

$$\delta B(I_4) \doteq 0.4869$$

$$\delta B(I_5) \doteq 0.4476$$

There seems to be no intuitive way of finding the limit. In 1935 P. Erdős [11] proved that the limit exists and is equal to zero. His proof utilizes a theorem by Hardy and Ramanujan, which had been proved in 1920. In 1934 Turan [24] presented a more elementary proof of the same result.

<u>The Theorem of Hardy and Ramanujan.</u> If $\delta > 0$, $r > 0$, and $n \geq a$, then the number of integers not exceeding n, having fewer than

$$\log(\log a) - r(\log(\log a))^{\frac{1}{2}+\delta}$$

or more than

$$\log(\log a) + r(\log(\log a))^{\frac{1}{2}+\delta}$$

prime divisors (counting multiplicities) less than a, is $o(n)$.

A more casual interpretation of this theorem is that the normal number of prime divisors less than a of an integer is $\log\log a$. We will modify the theorem to fit our needs by letting $\delta = \frac{1}{2}$ and $r = \frac{1}{3}$, thus obtaining the following corollary.

Corollary. Let $\varepsilon > 0$ and $n \geq a$. The number of natural numbers not exceeding n, having fewer than $\frac{2}{3} \log\log a$ or more than $\frac{4}{3} \log\log a$ prime divisors less than a, is less than εn for large enough a.

We are now prepared to state and prove Erdős' result formally.

Theorem 4.6. The limit of $\delta B(I_a)$, as a approaches infinity, is zero.

Proof. We begin by dividing the integers in I_a into two classes. In the first class we place the integers $b_1, b_2, \ldots, b_y$ having at most $\frac{2}{3} \log\log a$ prime divisors. Thus the second class consists of integers $c_1, c_2, \ldots, c_z$ having more than $\frac{2}{3} \log\log a$ prime divisors. Let B_b and B_c represent, respectively, the sets of multiples of the b_i's and the c_i's. We have $B(I_a) = B_b \cup B_c$, and the number of integers in $B(I_a)$ not exceeding n is not greater than $B_b(n) + B_c(n)$.

We concentrate first on the set B_b. Since for any i there are at most $\frac{n}{b_i}$ multiples of B_i that do not exceed n,

$$B_b(n) \leq \sum_{i=1}^{y} \frac{n}{b_i} < \sum_{i=1}^{y} \frac{n}{a} = \frac{ny}{a},$$

for any n. Now, replacing both n and a in the preceeding corollary by $2a$ and realizing that $\log\log a < \log\log 2a$, we have that $y < \varepsilon 2a$. Thus

$$B_b(n) < 2n\varepsilon . \tag{4.7}$$

Centering our attention on B_c, we let

$$n > 2a^2 . \tag{4.8}$$

The integers counted in $B_c(n)$ are of the form $c_i x$, where $1 \leq x \leq \frac{n}{c_i}$. We may arrange these integers into two distinct subsets. In the first we place those $c_i x$ for which x has at most $\frac{2}{3} \log\log a$ prime factors less than a. From (4.8) we have $a < \frac{n}{2a}$, and since for each i, $c_i \leq 2a$, we have

$$a < \frac{n}{c_i} , \quad \text{for each} \quad i = 1, 2, \ldots, z .$$

Thus if we replace n by $\frac{n}{c_i}$ in the corollary, then there are less than $\frac{\varepsilon n}{c_i}$ values of x that have at most $\frac{2}{3} \log\log a$ prime factors less than a, for each c_i. Therefore, the number of integers $c_i x$ in the first subset is less than $\sum_{i=1}^{z} \frac{\varepsilon n}{c_i}$. Since $a < c_i \leq 2a$ and $z \leq a$,

$$\sum_{i=1}^{z} \frac{\varepsilon n}{c_i} < \frac{\varepsilon n z}{a} \leq \varepsilon n .$$

Thus the number of integers $c_i x$ in the first subset is less than εn.

The second subset of integers counted in $B_c(n)$ consists of integers of the form $c_i x$, where each c_i and each x have more than $\frac{2}{3} \log\log a$ prime factors less than a. Thus each $c_i x$ has more than $\frac{4}{3} \log\log a$ prime factors less than a. Thus the number of these integers is less than εn by the preceeding corollary.

Therefore, by (4.7) the number of positive integers not exceeding n in $B(I_a)$ is less than $2\varepsilon n + \varepsilon n + \varepsilon n = 4\varepsilon n$, for large enough a and $n > 2a^2$. Hence,

$$\limsup_{n\to\infty} \frac{\{B(I_a)\}(n)}{n} < 4\varepsilon ,$$

for large enough a. Thus $\lim_{a\to\infty} \delta B(I_a) = 0$. $\triangle$

We have presented a sufficient foundation from which to encounter and verify Besicovitch's example.

Theorem 4.7 (Besicovitch). There exists a sequence of natural numbers whose set of multiples does not possess asymptotic density.

Proof. Let ε, $0 < \varepsilon < \frac{1}{2}$, be given. We define the sequence $\varepsilon_1, \varepsilon_2, \ldots$ by $\varepsilon_k = (\frac{1}{2})^{k+1}\varepsilon$. Then

$$\sum_{i=1}^{\infty} \varepsilon_k = \varepsilon \sum_{k=1}^{\infty} (\frac{1}{2})^{k+1} = \frac{\varepsilon}{2} . \tag{4.9}$$

By Theorem 4.6 there exist a sequence of integers $\{a_1, a_2, \ldots\}$ satisfying both

$$\delta B(I_{a_k}) < \varepsilon_k , \tag{4.10}$$

$$a_{k+1} > (2a_k)! , \tag{4.11}$$

for $k = 1, 2, \ldots$. Let G be the union of the sets I_{a_k}, $k = 1, 2, \ldots$. Since $I_{a_k} \subset B(G)$ and I_{a_k} contains exactly a_k integers, the number of integers not exceeding $2a_k$ in $B(G)$, $[B(G)](2a_k)$, is at least a_k.

Thus $[B(G)](n) \geq \frac{n}{2}$ for infinitely many favlues of n, and

$$\overline{\delta} B(G) = \limsup_{n\to\infty} \frac{[B(G)](n)}{n} \geq \frac{1}{2} . \qquad (4.12)$$

By (4.11) and Lemma 4.3 (replacing m by a_k), we have

$$[B(I_{a_i})](a_k) \leq 2a_k \, \delta B(I_{a_i}) ,$$

For $i = 1, 2, \ldots, k-1$. Thus

$$[B(G)](a_k) \leq \sum_{i=1}^{k-1} [B(I_{a_i})](a_k) \leq 2a_k \sum_{i=1}^{k-1} \delta B(I_{a_i}) ,$$

and by (4.9) and (4.10),

$$2a_k \sum_{i=1}^{k-1} \delta B(I_{a_i}) < 2a_k \sum_{i=1}^{k-1} \varepsilon_i < 2a_k \left(\frac{\varepsilon}{2}\right) = a_k \varepsilon .$$

Therefore,

$$\underline{\delta} B(G) = \liminf_{n\to\infty} \frac{[B(G)](n)}{n} \leq \varepsilon .$$

From (4.12) and (4.13), we have $\underline{\delta} B(G) \neq \overline{\delta} B(G)$. Hence, $B(G)$ does not possess asymptotic density. $\triangle$

CHAPTER V

PRIMITIVE SEQUENCES

In this chapter we shall investigate properties of a collection of positive integer sequences called primitive sequences. The concept of a primitive sequence was introduced briefly in Chapter III as an example following the Decomposition Theorem. One purpose of the present chapter is to examine the relationships that exist among primitive sequences, sets of multiples, and the concept of "primary part" (Decomposition Theorem).

Definition 5.1. A sequence of positive integers is called a primitive sequence if no member of the sequence divides another member.

Perhaps the most obvious and well-studied example of a primitive sequence is the set of primes. In fact, any sequence of pairwise relatively prime integers is an example of a primitive sequence. Of course it is not essential that the elements of a primitive sequence be relatively prime. An important example of a finite primitive sequence is the set I_a, the set of integers in the interval $(a, 2a]$. The set I_a was utilized in the construction of Besicovitch's example (Theorem 4.7).

Primitive Generating Sequences

In Chapter IV, a set of multiples B was defined by first presenting a sequence A, such that $B = B(A)$. However, the sequence A

may not be unique. For instance, the sequence $\{2, 3, 4, \ldots\}$ may be expressed as the set of multiples of the set of primes, or it may be expressed as the set of multiples of the set of square-free integers.

A set of multiples B can be characterized without refering to a particular sequence A.

Theorem 5.1. A sequence B is a set of multiples if, and only if, for every b in B, kb is also in B, $k = 1, 2, \ldots$.

Proof. Assume that B is a set of multiples. Then there exists some sequence A, such that $B = B(A)$. If b is any integer in B, then there exists an integer a in A and an integer n in N, such that $b = na$. Let kb be any positive multiple of b. Then $kb = kna$, and kb is in B.

Assume that for every b in B, every multiple of b is also in B. Then B is the set of multiples of itself. That is, $B = B(B)$, and the theorem is proved. △

Definition 5.2. If $B = B(A)$ is the set of multiples of the sequence A, then A is said to generate B and A is called a generating sequence for B. If A is also a primitive sequence, A is called a primitive generating sequence for B.

We see that the set of primes is a primitive generating sequence for the sequence $\{2, 3, 4, \ldots\}$. The set of square-free integers generates this same sequence but is not primitive since, for example, 15 and 105 are square-free and 15 divides 105. It is important to notice that the set of primes must be a subset of any generating sequence for $\{2, 3, 4, \ldots\}$, since a prime is a multiple of no integer other than itself and 1. Thus the set of primes is the intersection of all the generating sequences for $\{2, 3, 4, \ldots\}$.

<u>Definition 5.3.</u> The intersection of the generating sequences of a set of multiples B is called the minimal generating sequence for B, and is denoted by $\hat{B}$.

In order that Definition 5.3 be meaningful, it is necessary to demonstrate that the intersection of the generating sequences for a set of multiples B is itself a generating sequence for B. Let b be any integer in B. If b is not divisible by any other integer in B, then b must be in every generating sequence for B, and hence b must be in the intersection of the generating sequences. On the other hand, suppose that b is divisible by some other integer in B. Since there are finitely many integers in B that are less than b, there exists a smallest integer b' in B such that b' divides b. Then b' is not divisible by any other integer in B, and b' must be in every generating sequence for B. Thus b is a multiple of b', and b' is in the intersection of the generating sequences. Hence, each integer in B is generated by an integer in the intersection of generating sequences, and so this intersection is a generating sequence.

Referring again to the generating sequences of $\{2, 3, 4, \ldots\}$, we see that the set of primes is both primitive and minimal, on the other hand the set of square-free integers is neither primitive nor minimal. The fact that the minimal generating sequence is a primitive sequence is not indigenous to this particular example.

Theorem 5.2. Let B be any set of multiples. A generating sequence for B is primitive if, and only if, it is the minimal generating sequence for B.

Proof. Let A be a primitive generating sequence for B, and let a be an integer in A. Suppose that a is divisible by b in B, where $a \neq b$. Since A is primitive, b is not in A, and b is not a multiple of any integer in A. This contradicts the fact that A is a generating sequence for B. Thus a is not divisible by another integer in B. Since a is a member of B, a must be a member of every generating sequence for B. Therefore, A is the minimal generating sequence for B.

On the other hand, let A be the minimal generating sequence for B. Suppose that A is not a primitive sequence. Then there exist integers a and a' in A such that a divides a'. Since every integer in B that is generated by a' is also generated by a, $A \setminus \{a'\}$ is a generating sequence for B. This contradicts the fact that A is the minimal generating sequence for B. Thus A is a primitive sequence, and the theorem is proved. △

By definition, the minimal generating sequence for a set of multiples is unique. Thus by Theorem 5.2 every set of multiples has a unique primitive generating sequence. Also, given any primitive sequence, it is the primitive generating sequence for one, and only one, set of multiples. Thus we have the following theorem.

Theorem 5.3. There exists a one-to-one correspondence between the collection of sets of multiples and the collection of primitive sequences.

Important relationships also exist between the primitive sequences and the Decomposition Theorem (Theorem 3.2) with its subsequent results. We recall that a higher multiple of an integer a is defined to be an integer of the form ka, where the prime divisors of k exceed the greatest prime divisor of a. Also, the primary part, A', of A is defined to be the set of members of A that are not higher multiples of other members of A. Since no member of a primitive sequence is a multiple of another member, we have the following theorem.

Theorem 5.4. The primary part of a primitive sequence is the entire sequence.

We now consider a relationship that involves primitive sequences, sets of multiples, and the Decomposition Theorem.

Example 5.1. Let A be the set of square-free integers. Let B be the set of multiples of A; that is, $B = B(A) = \{2, 3, 4, \ldots\}$. In terms of the Decomposition Theorem, B', the primary part of B, is the set consisting of all positive integer powers of the primes. The primary part of A, is the set of primes. Since the set of primes generates B, the set of primes is also the primitive (minimal) generating sequence for B. Thus we have the relationship

$$B' \supset A' = \hat{B}, \quad B' \neq A' .$$

We may ask the question, is it true in general that the primitive generating sequence for a set of multiples B is equal to the primary part of any generating sequence A? We see that this is the case when

$B = \{2,3,4,\ldots\}$ and A is the set of square-free integers. It is also true if A is the set of primes, which also generates B. However, it is not true if $A = B$ - in this case B is a generating sequence for itself - for we have

$$B' = A' \supset \hat{B}, \quad A' \neq \hat{B} .$$

Thus it seems that the relationship that exists in general could at best be

$$B' \supset A^{\nu} \supset \hat{B} . \tag{5.1}$$

We shall see that (5.1) is indeed true in general. To prove this we shall utilize the following lemma. The lemma is important, in itself, because it illustrates an interesting relationship that exists between a generating sequence and its primary part.

Lemma 5.1. If the sequence A is a generating sequence for the sequence B, then A', the primary part of A, also generates B. That is $B(A') = B(A) = B$.

Proof. Since $A' \subset A$, it is obvious that $B(A') \subset B(A)$. Thus we need only show that $B(A') \supset B(A)$. If b is an integer in B, then there exists an integer a in A and an integer k such that $b = ka$. By the Decomposition Theorem (Theorem 3.2) either a belongs to A' or there exists a unique integer a' in A' such that $a = k'a'$ for some integer k'. If a belongs to A', then $b = ka$ belongs to $B(A')$. If $a = k'a'$, then $b = kk'a'$ belongs to $B(A')$. Thus $B(A') \supset B(A)$, and the lemma is proved. $\triangle$

Theorem 5.5. Let $B = B(A)$ be the set of multiples of a sequence A, let A' and B' denote the primary parts of A and B, respectively, and let $\hat{B}$ denote the primitive generating sequence for B. Then we have

$$B' \supset A' \supset \hat{B} .$$

Proof. The mere fact that A is a subset of B is not sufficient for A' to be a subset of B'. If a' is in the primary part of A, then a' is not a higher multiple of any integer in A. Thus a' is not a higher multiple of any integer in B, the set of multiples of A. Therefore, a' is a member of B', the primary part of B. Thus $A' \supset B'$.

By Lemma 5.1, A' generates B since A' is the primary part of A and A is a generating sequence for B. Since $\hat{B}$ is the primitive generating sequence for B, by Theorem 5.2 it is the minimal generating sequence. Thus $A' \supset \hat{B}$, and the proof is complete. $\triangle$

The Asymptotic Density of Primitive Sequences

We shall now concentrate on the asymptotic density properties of primitive sequences. Relationships that we have encountered concerning primitive sequences, sets of multiples, and the concept of "primary part" will be utilized. We shall see that the upper asymptotic density of a primitive sequence does not exceed $\frac{1}{2}$, and that there exists a primitive sequence that does not possess asymptotic density. However, every primitive sequence possesses zero logarithmic density.

P. Erdős [11] credits the proof of the following theorem to M. Wachsberger and E. Weissfeld.

<u>Theorem 5.6.</u> Let A be an infinite primitive sequence and let $A(n)$ denote the number of integers in A that do not exceed n. Then

$$A(2n) \leq n \,. \tag{5.2}$$

<u>Proof.</u> Suppose that for some integer n, $A(2n) > n$. In other words, there are at least n elements of A that do not exceed 2n. Thus for each of the first $n+1$ elements $a_1, a_2, \ldots, a_{n+1}$ of A, $a_i < 2n$. Let b_i denote the greatest odd divisor of a_i, for $i = 1, 2, \ldots, n+1$. In other words,

$$a_i = 2^{\alpha_i} b_i \,. \tag{5.3}$$

Since there are no more than n distinct odd integers that do not exceed 2n, two of the integers $b_1, b_2, \ldots, b_{n+1}$ must be equal, say $b_i = b_j$, $1 \leq i < j \leq n+1$. By (5.3) we have that either a_i divides a_j or a_j divides a_i. This is a contradiction since A is a primitive sequence. Thus (5.2) holds. $\triangle$

<u>Theorem 5.7.</u> If A is a primitive sequence, then the upper asymptotic density of A does not exceed $\frac{1}{2}$.

<u>Proof.</u> From Theorem 5.6 we have that if x is an even integer then

$$\frac{A(x)}{x} \leq \frac{1}{2} \,. \tag{5.4}$$

Any odd integer may be expressed as $x+1$, where x is even. Since $A(x+1)$ does not exceed $A(x)+1$, we have

$$A(x+1) \le \frac{x}{2} + 1 = \frac{x+1}{2} + \frac{1}{2} .$$

Thus

$$\frac{A(x+1)}{x+1} \le \frac{1}{2} + \frac{1}{2(x+1)} . \tag{5.5}$$

From (5.4) and (5.5) we have that for any natural number n,

$$\frac{A(n)}{n} \le \frac{1}{2} + \frac{1}{2n} .$$

Hence,

$$\begin{aligned} \overline{\delta}A &= \limsup_{n\to\infty} \frac{A(n)}{n} \\ &\le \limsup_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right) \\ &\le \limsup_{n\to\infty} \frac{1}{2} + \limsup_{n\to\infty} \frac{1}{2n} \\ &= \frac{1}{2} . \end{aligned}$$

$\triangle$

This theorem establishes an upper bound for the upper asymptotic density of a primitive sequence. Besicovitch constructed an example that illustrates that $\frac{1}{2}$ is the least upper bound for the upper asymptotic densities of primitive sequences. We shall consider this example at the end of this section. First, we consider a result concerning the lower asymptotic density of a primitive sequence.

In 1935 F. Behrend [4] and P. Erdős [11], working independently, established that primitive sequences possess zero logarithmic density, and hence zero lower asymptotic density. Their proofs were lengthy and involved. Although we shall prove this important result, we shall employ neither Behrend's nor Erdős' proof. Instead, we shall use the fact that the logarithmic density of the primary part of any sequence is zero (Theorem 3.4), which is a consequence of the Decomposition Theorem.

Theorem 5.8. The lower asymptotic density and logarithmic density of a primitive sequence are zero.

Proof. Let A be a primitive sequence. From Theorem 2.10 we have that

$$0 \leq \underline{\delta} A \leq \delta_L A .$$

Thus it suffices to show that $\delta_L A = 0$. From Theorem 5.4 we have that the primary part of A is all of A. Thus from Theorem 3.4 we have $\delta_L A = 0$, and the theorem is proved. △

It was conjectured by H. Davenport and others that the asymptotic density of any primitive sequence exists and is equal to zero. Indeed this is the case with the examples of primitive sequences that we have encountered.

In 1934 A. S. Besicovitch [5] presented a counter example to Davenport's conjecture. Recall that it was Besicovitch (Theorem 4.7) who disproved Chowla's conjecture that any set of multiples possesses asymptotic density. This, together with the relationships that exist between sets of multiples and primitive sequences, indicates that the construction used in Theorem 4.7 may again be useful. In fact, the example described here is the primitive generating sequence of the set of multiples that was constructed in the proof of Theorem 4.7.

Theorem 5.9. There exists a primitive sequence that does not possess asymptotic density.

Proof. Let ε, $0 < \varepsilon < \frac{1}{4}$, be given. As in Theorem 4.7, we define the sequence of real numbers $\{\varepsilon_1, \varepsilon_2, \dots\}$ by $\varepsilon_k = (\frac{1}{2})^{k+1}\varepsilon$ so that

$$\sum_{k=1}^{\infty} \varepsilon_k = \frac{\varepsilon}{2} .$$

Also, as before, we choose a sequence of integers $\{a_1, a_2, \dots\}$ satisfying both the following inequalities

$$\delta B(I_{a_k}) < \varepsilon_k , \quad a_{k+1} > (2a_k)! \tag{5.6}$$

for $k = 1, 2, \dots$. We again define G to be the union of the sets I_{a_k}, $k = 1, 2, \dots$. Whereas in Theorem 4.7 the set of multiples $B = B(G)$ was the desired example, in the present situation we consider the primitive (minimal) generating sequence $\hat{B}$ for B. Since G is a generating sequence for B, $\hat{B}$ is a subset of G. We claim that B can be obtained from $G = I_{a_1} \cup I_{a_2} \cup \dots$ by removing from each set I_{a_k} those elements that belong to any of the sets $B(I_{a_1}), B(I_{a_2}), \dots, B(I_{a_{k-1}})$. In other words,

$$B = I_{a_1} \cup \{I_{a_2} \smallsetminus B(I_{a_1})\} \cup \{I_{a_3} \smallsetminus \bigcup_{i=1}^{2} B(I_{a_u})\} \cup \{I_{a_4} \smallsetminus \bigcup_{i=1}^{3} B(I_{a_i})\} \cup \dots .$$

It should be noted that the sets $\{I_{a_k} \smallsetminus \bigcup_{i=1}^{k-1} B(I_{a_i})\}$ are disjoint since $I_{a_k} \cap I_{a_{k-1}} = \emptyset$ for $a_k > (2a_{k-1})!$.

It is necessary to show that $\hat{B}$ is indeed the primitive generating sequence for B. Suppose b belongs to $\hat{B}$, and x divides b, $x \neq b$. For some j we have

$$b \in I_{a_j} \smallsetminus \bigcup_{i=1}^{j-1} B(I_{a_i}) .$$

Thus b is not a multiple of any integer in I_{a_i}, $i = 1, 2, \ldots, j-1$. Hence, x is not an element of I_{a_i} for $i = 1, 2, \ldots, j-1$. Since the set I_{a_j} is primitive, x is not an element of this set. Therefore, x is not in $\hat{B}$, and $\hat{B}$ is primitive. That $\hat{B}$ generates $B = B(G)$ is a result of the fact that each element in G is a multiple of some element in $\hat{B}$ and G generates B.

Since $\hat{B}$ is primitive, by Theorem 5.8 we have $\underline{\delta}\hat{B} = 0$. Thus it suffices to show that $\overline{\delta}\hat{B} > 0$.

Consider the subset $(\hat{B} \cap I_{a_k})$ of $\hat{B}$. Let $[B \cap I_{a_k}](2a_k)$ denote the number of integers $\leq 2a_k$ that are in $(B \cap I_{a_k})$. Since

$$\hat{B} \cap I_{a_k} = I_{a_k} \smallsetminus \bigcup_{i=1}^{k-1} B(I_{a_i}) ,$$

we have

$$\begin{aligned}
[\hat{B} \cap I_{a_k}](2a_k) &= \left[I_{a_k} \smallsetminus \bigcup_{i=1}^{k-1} B(I_{a_i}) \right](2a_k) \\
&\geq I_{a_k}(2a_k) - \left[\bigcup_{i=1}^{k-1} B(I_{a_i}) \right](2a_k) \\
&= a_k - \left[\bigcup_{i=1}^{k-1} B(I_{a_i}) \right](2a_k) \\
&\geq a_k - \sum_{i=1}^{k-1} [B(I_{a_i})](2a_k) .
\end{aligned}$$

By (5.6), $2a_k > (2a_i)!$, for $i = 1, 2, \ldots, k-1$; so by Lemma 4.3

$$[B(I_{a_i})](2a_k) \leq 2(2a_k)\,\delta B(I_{a_i}), \quad i = 1, 2, \ldots, k-1 .$$

Since by (5.6) $\delta B(I_{a_i}) < \varepsilon_i$, $[B(I_{a_i})](2a_k) < 4a_k\varepsilon_i$. Because of this inequality we can determine a lower bound for $[B \cap I_{a_k}](2a_k)$. We have

$$[\hat{B} \cap I_{a_k}](2a_k) > a_k - 4a_k \sum_{i=1}^{k-1} \varepsilon_i > a_k - 4a_k \sum_{i=1}^{\infty} \varepsilon_i$$

$$= a_k - 4a_k\left(\frac{\varepsilon}{2}\right) = a_k(1 - 2\varepsilon) .$$

Since $[\hat{B} \cap I_{a_k}]$ is a subset of $\hat{B}$, this inequality implies $\hat{B}(2a_k) > a_k(1 - 2\varepsilon)$. Hence,

$$\frac{\hat{B}(2a_k)}{2a_k} > \frac{a_k(1-2\varepsilon)}{2a_k} = \frac{1}{2} - \varepsilon , \tag{5.7}$$

and so the ratio $\frac{\hat{B}(n)}{n}$ exceeds $\frac{1}{2} - \varepsilon$ for infinitely many values of n. Thus $\overline{\delta}\hat{B} = \limsup_{n\to\infty} \frac{\hat{B}(n)}{n} \geq \frac{1}{2} - \varepsilon > \frac{1}{4}$; and the theorem is proved. Δ

From (5.7) we have the following corollary which, together with Theorem 5.7, establishes that $\frac{1}{2}$ is the least upper bound of the upper asymptotic densities of primitive sequences.

<u>Corollary</u>. Given $\varepsilon > 0$, there exists an infinite primitive sequence A such that $\overline{\delta}A > \frac{1}{2} - \varepsilon$.

Sequences That Do Not Possess Zero Logarithmic Density

Theorem 5.8 established that the logarithmic density of any primitive sequence exists and is equal to zero. In this section we consider properties of sequences of integers that do not possess zero logarithmic density. These properties are contained in a chain of theorems, each more detailed than the one preceding it. Whenever possible, results of the Decomposition Theorem are utilized.

Theorem 5.8 can be restated in the following manner.

Theorem 5.8'. If the infinite sequence A does not possess zero logarithmic density (i.e., the upper logarithmic density of A is positive), then A is not a primitive sequence.

Example 5.2. We continue to exploit the characteristics of the set S of square-free integers. It has been shown in the corollary to Theorem 4.4 that the set of square-free integers possesses positive asymptotic density, and thus positive logarithmic density. Consider the following subsequences of S:

$$S_1 = \{2, 6, 10, 14, \ldots\},$$

$$S_2 = \{2, 6, 30, 210, \ldots\}.$$

Subsequence S_1 consists of the integer 2 and the integers of the form $2p$, where p is an odd prime; S_2 is the set of integers of the form $p_1 p_2 \cdots p_i$, where p_i denotes the i-th prime. Both of these subsequences consist of higher multiples of the integer 2. The fact that S contains subsequences that consist of higher multiples of an element of S is a property of all sequences that do not possess zero logarithmic density.

Theorem 5.10. If the infinite integer sequence A does not possess zero logarithmic density, then A contains an infinite subsequence consisting of higher multiples of some integer in A.

Proof. As in Chapter III, let aP_a denote the set of higher multiples of the integer a. Recall from Theorem 3.5 that if $\delta_L(A \cap a'P_{a'})$ exists for all a' in A', the primary part of A, then

$$\delta_L A = \sum_{a' \epsilon A'} \delta_L(A \cap a'P_{a'}) .$$

If the set $(A \cap a'P_{a'})$ contains finitely many integers, for each a' in A', then $\delta(A \cap a'P_{a'}) = 0$, and consequently $\delta_L A = 0$. Thus if A does not possess zero logarithmic density, then there exists at least one element a' in A', such that infinitely many higher multiples of a' are in A. This completes the proof. $\triangle$

Referring to Example 5.2, the subsequences S_1 and S_2 illustrate Theorem 5.10, in that both consist of higher multiples of the integer 2. Subsequence S_2 has an additional property. Each term in S_2 is a divisor of the next term.

Definition 5.4. Let $C = \{c_1, c_2, \ldots\}$ be an infinite set of natural numbers. If c_i divides c_{i+1} for each i, then C is called a division chain.

In 1936 Davenport and Erdős [6] demonstrated that any sequence that does not possess zero logarithmic density contains a division chain. This is a qualitative refinement of Theorem 5.10.

Theorem 5.11. If the sequence A does not possess zero logarithmic density, then A contains a division chain.

Halberstam and Roth [14] present a proof of this result that utilizes Theorem 4.4. We shall prove a more detailed result by utilizing consequences of the Decomposition Theorem in a more direct and constructive manner.

Consider the subsequence S_2 in Example 5.2, again. By its definition, $S_2 = \{p_1 p_2 \cdots p_k\}$, is obviously a division chain. Also, each term in S_2 is a higher multiple of the preceding term.

Theorem 5.12. If the sequence A does not possess zero logarithmic density, then A contains a division chain $\{c_1, c_2, \ldots\}$ such that c_{i+1} is a higher multiple of c_i for all i.

Proof. Consider the set $A \smallsetminus A'$, where A' is the primary part of A. By the Decomposition Theorem this set consists of higher multiples of elements in A'. Hence,

$$A \smallsetminus A' = A \cap \Big(\bigcup_{a' \in A'} a' P_{a'} \Big) = \bigcup_{a' \in A'} (A \cap a' P_{a'}) . \qquad (5.8)$$

Since A' possess zero logarithmic density by Theorem 3.4 and the upper logarithmic density of A is positive, the upper logarithmic density of $A \smallsetminus A'$ is also positive. Thus from (5.8)

$$0 < \overline{\delta}_L(A \smallsetminus A') \leq \sum_{a' \in A'} \overline{\delta}_L(A \cap a' P_{a'}) .$$

Therefore there exists an integer c_1 in A' such that

$\overline{\delta}_L(A \cap c_1 P_{c_1}) > 0$. Let $A_1 = A \cap c_1 P_{c_1}$. Following the same reasoning, since the upper logarithmic density of A_1 is positive, there exists an integer c_2 in A_1' such that $\overline{\delta}_L(A_1 \cap c_2 P_{c_2}) > 0$. In this manner we may construct a sequence of sets $A_1, A_2, \ldots$ and a subsequence $c_1, c_2, \ldots$ of A such that $A_{i+1} = A \cap c_{i+1} P_{c_{i+1}}$ and c_{i+1} is an element of A_i'. Thus c_{i+1} is a higher multiple of c_i for all i, and the theorem is proved. $\triangle$

Theorems 5.8', 5.10, 5.11, and 5.12 may be viewed as a chain of sequentially more detailed statements concerning sets of natural numbers which do not possess zero logarithmic density. Ralph Alexander [1] has proved an additional refinement by utilizing a generalization of the Decomposition Theorem. We conclude this chapter by stating Alexander's result, without proof.

Theorem 5.13. Let $K_1, K_2, \ldots$ be a sequence of positive numbers, not necessarily in increasing order. If the sequence A of positive integers does not possess zero logarithmic density, then A contains a division chain of the form $\{q_1, q_1 q_2, q_1 q_2 q_3, \ldots\}$, where q_{i+1} is composed entirely of primes greater than $(q_1 q_2 \cdots q_i)^{K_i}$.

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APPENDIX

This appendix is intended for those readers with a limited background in analysis. A reader whose experience includes the use of a textbook of parity with Mathematical Analysis by T. M. Apostol [3] should have little need to refer to this appendix.

Throughout the appendix the notation $\{a_n\}$, or $\{b_n\}$, is used to denote a sequence of real numbers and capital letters are used to denote constant real numbers.

Limit of a Sequence of Real Numbers

Definition A. 1. If for every $\varepsilon > 0$, there exists an integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$, then we say that L is the limit (as n approaches infinity) of $\{a_n\}$; and we write $\lim_{n\to\infty} a_n = L$.

The proofs of Theorems A. 1 and A. 2 are not provided here since they can be found in most mathematical analysis textbooks. A proof of Theorem A. 3 is provided since it is not easily accessible.

Theorem A. 1. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

$$\text{(i)} \quad \lim_{n\to\infty} (a_n \pm b_n) = A \pm B ,$$

$$\text{(ii)} \quad \lim_{n\to\infty} a_n b_n = AB ,$$

$$\text{(iii)} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B} \quad \text{if} \quad B \neq 0 .$$

Theorem A.2. The limit of $\{a_n\}$ is L if, and only if, the limit of every subsequence of $\{a_n\}$ is L.

Theorem A.3. If n and K are integers, K being constant and not zero, then

$$\lim_{n\to\infty} \frac{[\frac{n}{K}]}{n} = \frac{1}{K},$$

where the brackets indicate the greatest integer function.

Proof. By the division algorithm, for a given n there exists integers q and r such that

$$n = Ka + r, \quad 0 \leq r < K.$$

Therefore,

$$\frac{[\frac{n}{K}]}{n} = \frac{[\frac{(Kq+r)}{K}]}{Kq+r} = \frac{q}{Kq+r},$$

$$\left| \frac{[\frac{n}{K}]}{n} - \frac{1}{K} \right| = \left| \frac{q}{Kq+r} - \frac{1}{K} \right| = \left| \frac{Kq - Kq - r}{K(Kq+r)} \right|$$

$$= \frac{r}{K(Kq+r)} = \frac{r}{Kn} < \frac{K}{Kn} = \frac{1}{n}.$$

The theorem follows. △

Limit Superior and Limit Inferior of a Sequence of Real Numbers

Definition A.2. Let $\{a_n\}$ be a sequence of real numbers that is bounded above, and let $M_n = \text{l.u.b.}\ \{a_n, a_{n+1}, \ldots\}$. If $\lim_{n\to\infty} M_n = M$, then we say that M is the limit superior of $\{a_n\}$; and we write $\limsup_{n\to\infty} a_n = M$.

Definition A.3. Let $\{a_n\}$ be a sequence of real numbers that is bounded below, and let $K_n = \text{g.l.b.}\,\{a_n, a_{n+1}, \ldots\}$. If $\lim_{n\to\infty} K_n = K$, then we say that K is the limit inferior of $\{a_n\}$; and we write $\liminf_{n\to\infty} a_n = K$.

Proofs of the remaining theorems can be found in Methods of Real Analysis by R. R. Goldberg [13].

Theorem A.4. For any sequence $\{a_n\}$, $\liminf_{n\to\infty} a_n \leq \inf \sup_{n\to\infty} a_n$.

Theorem A.5. If $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_n = L$.

Theorem A.6. If $\{a_n\}$ and $\{b_n\}$ are bounded sequences, then

$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n ,$$

$$\liminf_{n\to\infty} (a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n .$$

Theorem A.7. If $\{a_n\}$ is bounded and $\{b_n\}$ is a subsequence of $\{a_n\}$, then

$$\limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty} a_n ,$$

$$\liminf_{n\to\infty} b_n \geq \liminf_{n\to\infty} a_n .$$

Theorem A.8. If $\{a_n\}$ is bounded and M is a constant real number, then

$$\limsup_{n\to\infty} (M + a_n) = M + \limsup_{n\to\infty} a_n ,$$

$$\liminf_{n\to\infty} (M + a_n) = M + \liminf_{n\to\infty} a_n .$$

Theorem A.9. If $\{a_n\}$ is bounded, then

$$\liminf_{n\to\infty} (-a_n) = -\limsup_{n\to\infty} a_n ,$$

$$\limsup_{n\to\infty} (-a_n) = -\liminf_{n\to\infty} a_n .$$

Theorem A.10. Let $\{a_n\}$ be bounded. Then

(i) $\limsup_{n\to\infty} a_n = M$ if, and only if, for any $\varepsilon > 0$, $a_n > M - \varepsilon$ for infinitely many values of n, and $a_n > M + \varepsilon$ for a finite number of values of n.

(ii) $\liminf_{n\to\infty} a_n = K$ if, and only if, for any $\varepsilon > 0$, $a_n < M + \varepsilon$ for infinitely many values of n, and $a_n < M - \varepsilon$ for a finite number of values of n.

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