LINEAR SPACES OVER NON-ARCHIMEDEAN VALUED FIELDS

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CHAPTER I

INTRODUCTION

In recent years increasing attention has been given to the study of the field Q_p of p-adic numbers. The simplest way of describing this field is that it is the completion of the field Q of rational numbers using the p-adic valuation $| \ |_p$ in place of the usual absolute value for establishing convergence criteria. Previous theses of an expository nature have covered in considerable detail the development of the p-adic number field and many of its properties. Valuations have been discussed at length. It has been observed that a valuation on a field induces a metric on the field. In particular, the p-adic valuation induces the p-adic metric d_p, a non-archimedean metric. It has been shown that the metric space (Q_p, d_p) is totally-disconnected. The space is not compact but the subset 0_p of p-adic integers is compact.

The field Q_p may be considered as a linear (or vector) space over itself. The valuation on Q_p as a field is a norm on Q_p as a linear space over itself. We then have a normed linear space over a non-archimedean valued field, in this case a non-archimedean normed space. This leads to the consideration of arbitrary normed linear spaces E over non-archimedean valued fields K and hence to non-archimedean normed linear spaces.

Normed linear spaces over the real or complex number fields have played an important role in many areas of mathematics. The question arises as to the implications on the linear space E when the field K has a non-archimedean valuation. Of particular interest is the situation in which the norm on E is also non-archimedean.

This study begins in Chapter II with a discussion of the properties of the norm. Several examples of non-archimedean normed linear spaces are given. The relation between the valuation on K and the norm on E is studied.

The emphasis in Chapter III is on properties of a topological nature. The metric induced by a norm is discussed, as well as some of the properties of spheres when the metric is non-archimedean. Compactness and connectedness are studied. The most basic consideration in this context is the fact that any non-archimedean metric space is 0-dimensional. Finally, the concepts of completeness and spherical completeness are introduced and compared. These concepts are important in analysis concerning non-archimedean normed linear spaces.

In Chapter IV interest is centered on whether or not the non-archimedean property of the metric is the determining factor for the properties discussed in Chapter III. It is demonstrated that it is possible to define on a set S metrics of various types, for example archimedean, non-archimedean, and locally non-archimedean metrics, which are equivalent under certain conditions. The theorems in this chapter show that to insure the fundamental topological properties exhibited in Chapter III it is sufficient but not necessary that the metric be non-archimedean.

The topic of convexity is frequently studied in connection with linear spaces. The last chapter is a brief introduction to the concept of convexity in a linear space over a non-archimedean valued field. Of special interest is the fact that the definition is of necessity independent of order.

Many articles written in the area of non-archimedean normed linear spaces eventually lead into the study of locally convex spaces and the results in this paper are of interest in this connection. However, the discussion of locally convex spaces is beyond the scope of this study.

Most of the literature in the area of non-archimedean normed linear spaces has appeared since 1946. At the present time there is no convenient single source for this material as the numerous articles appear in a variety of journals. Of these articles a high percentage are in European journals and have not been translated into English. One of the major contributors has been A. F. Monna who has produced a steady stream of articles from 1946 to the present. In addition to being somewhat inaccessible, many of these articles are written at a level of difficulty and require such an extensive background that many of the interesting properties of non-archimedean normed linear spaces are lost to the reader. A French-language book by Monna was published in 1970, see [10]. However, this book is a summary of the results of his articles and not a definitive study of the area.

This paper is aimed at a level which requires an understanding of the basic properties of the p-adic numbers and of elementary topology. Thus anyone, who has had an elementary topology course and a number theory course or seminar in which the p-adic numbers have been discussed, should be able to read it with understanding.

Before proceeding with the study of non-archimedean normed linear spaces, background material concerning valuations, the p-adic number field, and pertinent topology will be briefly reviewed. In addition, some notation will be introduced. Readers familiar with this background material can proceed directly to Chapter II. However, Chapter I will serve as a convenient source for those definitions and theorems essential to the remaining chapters.

Algebra

The basic algebraic system with which we will be concerned is the linear space.

Definition 1.1. A nonempty set E is said to be a linear space (or vector space) over a field K if E is an abelian group under an operation which we denote by +, and if for every $\alpha \in K$, $x \in E$ there is an element, written αx in E subject to

(1) $\alpha(x+y) = \alpha x + \alpha y$,

(2) $(\alpha+\beta)x = \alpha x + \beta x$,

(3) $\alpha(\beta x) = (\alpha\beta)x$,

(4) $1x = x$,

for all $\alpha, \beta \in K$ and $x, y \in E$ (where the 1 represents the unit element of K under multiplication).

It will be assumed that the reader is familiar with other basic algebraic systems such as groups, rings, and fields. A limited use will be made of modules and submodules.

Definition 1.2. Let R be a ring; a nonempty set M is said to be an R-module (or, a module over R) if M is an abelian group under an operation $+$ such that for every $r \in R$ and $m \in M$ there exists an element rm in M subject to:

(1) $r(a+b) = ra + rb$,

(2) $r(sa) = (rs)a$, and

(3) $(r+s)a = ra + sa$,

for all $a, b \in M$ and $r, s \in R$.

Definition 1.3. An additive subgroup A of the R-module M is called a submodule (or an R-submodule) of M if whenever $r \in R$ and $a \in A$, then $ra \in A$.

Order

Definition 1.4. A set S is partially ordered by a binary relation $\leq$ on S if

(1) $a \leq a$ for $a \in S$, (Reflexive)

(2) $a \leq b$ and $a \neq b$ implies $b \nleq a$, (Anti-symmetric)

(3) $a \leq b$ and $b \leq c$ implies $a \leq c$. (Transitive)

The relation $\subset$ is a common partial order relation. We say that a collection of subsets of a space E is partially ordered by set inclusion. Note that under a partial ordering not every pair of elements of S are related. For example, if $E = \{1, 2, 3\}$ then neither $A = \{1, 2\}$ or $B = \{2, 3\}$ is a subset of the other.

<u>Definition 1.5</u>. A set S is linearly ordered by the relation $\leq$ on S if

(1) S is partially ordered by $\leq$,

(2) $a, b \in S$ implies that either $a \leq b$ or $b \leq a$.

Thus a linearly ordered set is a set S which is partially ordered by a relation $\leq$ relatively to which each pair of elements of S are related. For example, if $S_1 \supset S_2 \supset \ldots$ is a monotonic decreasing sequence of subsets, then the collection $\{S_n\}$ of sets is linearly ordered by set inclusion.

<u>Definition 1.6</u>. A linearly ordered set such that every non-void subset has a least element is <u>well ordered</u>.

The set of positive integers with the natural ordering is well ordered.

Topology

Included in this section are those topological concepts which are especially appropriate to this study. For further references see [5], [7], or [20].

<u>Definition 1.7</u>. A <u>topological space</u> is a pair (X, τ) consisting of a set X and a collection τ of subsets of X, called <u>open sets</u>, satisfying the following axioms:

(i) The union of open sets is an open set.

(ii) The finite intersection of open sets is an open set.

(iii) The set X and the empty set $\emptyset$ are open sets.

The collection τ is called a <u>topology</u> for X.

When it is clear which topology X has, we sometimes refer to the space X. If the topology on X is induced by the metric d, we will write (X, d).

<u>Definition 1.8</u>. A family β of sets is a <u>base</u> for a topology if and only if β is a subfamily of τ and for each point x of the space and each open set U containing x, there is a member V of β such that $x \in V \subset U$.

Two topologies which can be assigned to any set are the trivial topology and the discrete topology.

<u>Definition 1.9</u>. Let E be any set.

(1) The <u>trivial topology</u> on the set E is the topology whose only elements are E and $\emptyset$.

(2) The <u>discrete topology</u> on the set E is the topology containing every subset of E; that is, every subset of E is open with respect to the discrete topology.

In this study we will be primarily interested in metric spaces.

<u>Definition 1.10</u>. A <u>metric</u> on a set E is a function d from $E \times E$ into R such that

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$, and

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for each $x, y, z \in E$.

The set S with metric d is a <u>metric space</u> and is denoted (E, d). If the metric d also satisfies the strong inequality

(iv) $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ for each $x, y, z \in E$,

then d is called a <u>non-archimedean metric</u> and the space (E, d) is called a non-archimedean metric space.

For our purposes a <u>neighborhood</u> of a point p of a topological space E will mean any open set containing the point p. A point p is a <u>limit point</u> of a set A if every neighborhood of p contains at least one point of A distinct from p. The <u>closure</u> of a set A is the set together with its limit points and is denoted $\overline{A}$. The closure of a set is a closed set. It is sometimes defined to be the intersection of all closed sets containing A.

The <u>interior</u> of a set A, denoted $\overset{o}{A}$, is the largest open set contained in A or equivalently the union of all open sets contained in A. It might be noted that the interior of A is the complement of the closure of the complement of A.

<u>Definition 1.11.</u> The <u>boundary</u> of A, denoted bdry A, is the set of all points which are in the closure of A but not in the interior of A. If we denote the complement of A by $E \sim A$, then we have

$$\text{bdry } A = \overline{A} \cap (\overline{E \sim A}) .$$

A situation which will be of special interest to us is the one in which the set A is both open and closed. In this case the boundary of A is empty.

<u>Theorem 1.1</u>. The set A has an empty boundary if and only if A is both open and closed.

Proof: Suppose A is both open and closed. Since A is open, $E \sim A$ is closed so that $\overline{E \sim A} = E \sim A$. Since A is closed $\overline{A} = A$. Thus $\text{bdry}\, A = \overline{A} \cap \overline{E \sim A} = A \cap (E \sim A) = \emptyset$. Conversely, if $\text{bdry}\, A = \emptyset$, that is $\overline{A} \cap \overline{E \sim A} = \emptyset$, since $E \sim A \subset \overline{E \sim A}$ we have $\overline{A} \cap (E \sim A) = \emptyset$. Thus A contains all its limit points and so A is closed. Similarly, $A \subset \overline{A}$ implies $A \cap \overline{E \sim A} = \emptyset$ so that the set $E \sim A$ contains all its limit points. Thus $E \sim A$ is closed which implies A is open.

The set A is <u>dense</u> in E if $\overline{A} = E$, that is, every point of E is a point or a limit point of A. A space E is separable if it has a countable dense subset. Thus the rationals are dense in the reals. A subset A of E is said to be nowhere dense in E if no nonempty open set of E is contained in $\overline{A}$. In other words, the interior of the closure of a nowhere dense set is empty.

Since we will be dealing almost exclusively with metric spaces, the separation properties are not of much interest. This results from the fact that every metric space is completely normal and hence normal, regular and Hausdorff.

A subset A of a space E is <u>compact</u> if every open cover contains a finite subcover. A subset of E is <u>countably compact</u> if every infinite subset of A has at least one limit point in A. Every compact subset A of E is countably compact. However, in a metric space we have the stronger theorem.

Theorem 1.2. In a metric space E, compactness and countable compactness are equivalent.

Other important compactness properties are contained in the following theorem.

Theorem 1.3. (1) Every closed subset of a compact set is compact.

(2) Every compact subset of a Hausdorff space is closed.

A subset A of a space E is connected if it is not the union of two disjoint non-empty sets each of which is open in A. An equivalent statement is that A is connected if and only if no proper subset of A is both open and closed in A. A subset is non-degenerate if it contains at least two distinct points. A space E is totally disconnected if its only connected subsets are points, that is, if no non-degenerate subset is connected.

Valuations

In this section the definition of a valuation will be given and some of the properties of interest to this study will be listed.

Definition 1.12. A valuation on a field K is a function $|\ |$ from K into the reals such that

(i) $|\alpha| \geq 0$ and $|\alpha| = 0$ if and only if $\alpha = 0$,

(ii) $|\alpha\beta| = |\alpha|\,|\beta|$ for all $\alpha, \beta \varepsilon K$,

(iii) $|\alpha + \beta| \leq |\alpha| + |\beta|$ for all $a, \beta \varepsilon K$.

If $|\ |$ satisfies the additional property

(iv) $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$ for all $\alpha, \beta \in K$,

then $|\ |$ is said to be a <u>non-archimedean valuation</u>.

The valuation $|\ |$ induces a metric d on K by defining $d(x, y) = |x - y|$.

<u>Notation</u>: Since we will be using the term non-archimedean numerous times, we will abbreviate it to n.a. For grammatical purposes n.a. should be read non-archimedean.

In this paper the notation $|\ |$ will be used for an arbitrary valuation on the field K, whether the valuation is archimedean or non-archimedean. However, in a few cases the usual absolute value, which is a valuation, is used. Whenever the symbol $|\ |$ is used for the absolute value, this will be pointed out. The other special valuation is the p-adic valuation, denoted $|\ |_p$, where p is a prime integer. We will have more to say about this valuation in the section on the p-adic number fields.

Properties of valuations are given in this theorem.

<u>Theorem 1.4</u>. If $|\ |$ is a valuation on K, then

(1) $|1| = 1$

(2) $|-\alpha| = |\alpha|$

(3) $|\alpha^{-1}| = |\alpha|^{-1}$

(4) If $|\ |$ is n.a. then $|\alpha| > |\beta|$ implies $|\alpha + \beta| = |\alpha|$.

<u>Theorem 1.5</u>. A valuation $|\ |$ on the rational numbers is non-archimedean if and only if $|n| \leq 1$ for every n in Z, the set of integers.

In the following discussion the term rational integers will be used for the set Z of integers to distinguish them from the ring $\mathfrak{O}$ of integers given in the following definition.

Definition 1.13. Let K be a field with n.a. valuation $| \ |$. Denote by $\mathfrak{O}$ the set $\mathfrak{O} = \{x \in K \,\big|\, |\alpha| \leq 1\}$. The set $\mathfrak{O}$ is called the valuation ring or the ring of integers of the field K with respect to the valuation $| \ |$. As the name indicates $\mathfrak{O}$ is a ring. The set $\mathfrak{O}$ contains the ring Z of rational integers as a subset.

Definition 1.14. The trivial valuation is that valuation defined by

$$|\alpha| = 1, \quad \text{if } \alpha \neq 0,$$

$$|\alpha| = 0, \quad \text{if } \alpha = 0.$$

Definition 1.15. The non-trivial valuations $| \ |_a$ and $| \ |_b$ on a field K are equivalent if for each $\alpha \in K$, $|\alpha|_a < 1$ if and only if $|\alpha|_b < 1$.

The following definitions concern convergence with respect to a valuation $| \ |$ on a field K.

Definition 1.16. Let $| \ |$ be a valuation on a field K.

(1) A sequence $\{\alpha_n\}$ of K converges, with respect to the valuation $| \ |$, to the point α if for each $\epsilon > 0$ there exists an N such that $|\alpha_n - \alpha| < \epsilon$ whenever $n \geq N$. We write $\lim \alpha_n = \alpha$.

(2) A sequence $\{x_n\}$ is Cauchy if for each $\epsilon > 0$ there exists an N such that $|\alpha_n - \alpha_m| < \epsilon$ whenever $m, n \geq N$.

(3) A valued field K is <u>complete</u>, with respect to the valuation $| \ |$, if every Cauchy sequence of K converges to a point of K.

(4) The sequence $\{\alpha_n\}$ is a <u>null sequence</u>, with respect to the valuation $| \ |$, provided that for each $\epsilon > 0$ there exists an N such that $|\alpha_n| < \epsilon$ whenever $n \geq N$.

(5) The set A of elements of a field K with valuation $| \ |$, is <u>bounded</u> with respect to $| \ |$, if the set of norm values $|\alpha|$, $\alpha \in A$ is bounded above.

The following theorem gives necessary and sufficient conditions for two valuations to be equivalent.

<u>Theorem 1.6.</u> Two non-trivial valuations $| \ |_a$ and $| \ |_b$ are equivalent if and only if they determine the same convergence criteria. That is, if for each sequence $\{\alpha_n\}$ there exists a point α such that $\lim |\alpha_n - \alpha|_a = 0$ if and only if $\lim |\alpha_n - \alpha|_b = 0$.

<u>Theorem 1.7.</u> Ostrowski's Theorem. The only non-trivial valuations on the field Q of rational numbers are those equivalent to $| \ |_p$, the p-adic valuation for some prime p, or to $| \ |$, the absolute value.

As with the absolute value we have the following theorem which holds for any valuation.

<u>Theorem 1.8.</u> Let K be a field with valuation $| \ |$. If $\lim \alpha_n = \alpha$, then $\lim |\alpha_n| = |\alpha|$.

The p-Adic Number Fields

The simplest non-trivial example of a non-archimedean valued field is the p-adic number field Q_p. It is assumed that the reader has had some experience with Q_p. However, there are certain properties which are basic to the discussion in the remaining chapters. This section contains a brief review of these properties and the necessary definitions. For a development of the p-adic number field and other background material of this kind see [1], [2] or [3].

The set of p-adic integers 0_p is the set $0_p = \{\alpha \in Q_p \mid |\alpha|_p \leq 1\}$. Referring to our general discussion of valuations the set 0_p is the ring of integers of the field Q_p with respect to the valuation $|\ |_p$. Thus 0_p contains the set of rational integers as a subset.

The following theorem contains statements concerning the representation of p-adic numbers. The term unit is used as in algebra. The element x is a unit in a ring R if it has an inverse; that is, if there exists an element $y \in R$ such that $xy = yx = 1$.

Theorem 1.9. (1) Every non-zero p-adic number α has a unique series representation

$$\alpha = p^m \sum_{n=0}^{\infty} a_n p^n = \sum_{n=0}^{\infty} a_n p^{m+n},$$

where $0 \leq a_n \leq p-1$ and $a_0 \neq 0$.

(2) If $m \geq 0$, then $\alpha \in 0_p$.

(3) A p-adic integer $\alpha = \sum_{n=0}^{\infty} a_n p^n$ is a unit in 0_p if and only if $a_0 \neq 0$.

(4) Every p-adic number $\alpha \varepsilon Q_p$ has a unique representation in the form $\alpha = p^n \mathcal{E}$ where $\mathcal{E}$ is a unit in 0_p, and $n \varepsilon Z$.

With this brief background we can now define the p-adic valuation for future reference.

Definition 1.17. The p-adic valuation $| \; |_p$ is the valuation on Q_p defined as follows. Let $\alpha = p^n \mathcal{E}$ where $\mathcal{E}$ is a unit in 0_p. Then $|\alpha|_p = \frac{1}{p^n}$, $\alpha \neq 0$, and $|0|_p = 0$.

The p-adic valuation is non-archimedean, that is $|\alpha + \beta|_p \leq \max\{|\alpha|_p, |\beta|_p\}$ for all $\alpha, \beta \varepsilon Q_p$. It has some interesting convergence properties which are contained in the following theorem.

Theorem 1.10: The following are properties of sequences of p-adic numbers:

(1) A sequence $\{\alpha_n\}$ of p-adic numbers is Cauchy if and only if for each $\epsilon > 0$ there exists an N such that $|\alpha_{n+1} - \alpha_n| < \epsilon$ whenever $n \geq N$.

(2) Every Cauchy sequence of p-adic numbers is bounded.

(3) From any bounded sequence of p-adic numbers, it is possible to select a convergent subsequence.

(4) A sequence $\{\alpha_n\}$ of p-adic numbers converges to a p-adic number α if and only if $\{\alpha_n\}$ is Cauchy.

(5) The field $(Q_p, | \; |_p)$ is complete.

(6) The field Q of rational numbers is dense in Q_p.

(7) Let $\{\alpha_n\}$ be a non-null Cauchy sequence of p-adic numbers. Then the sequence $\{|\alpha_n|_p\}$ of real numbers is eventually constant.

In his thesis, Snook [19] proved several properties of the p-adic metric space (Q_p, d_p) where d_p is the metric induced on Q_p by the p-adic valuation. That is, $d_p(x, y) = |x - y|_p$. The fact that d_p is a n.a. metric follows from the corresponding properties of the valuation $|\ |_p$. For example,

$$d_p(x, z) = |x - z|_p = |x-y + y-z|_p \leq \max\{|x-y|_p, |y-z|_p\}$$

$$= \max\{d(x, y), d(y, z)\}.$$

The following theorem gives some of the topological properties of the space (Q_p, d_p).

<u>Theorem 1.11</u>. Let (Q_p, d_p) be the space of p-adic numbers with the p-adic metric d_p. Then

(1) The set 0_p is a compact subset of (Q_p, d_p).

(2) The set 0_p of p-adic integers is a closed and bounded subset of (Q_p, d_p).

(3) Any closed and bounded subset of (Q_p, d_p) is compact.

(4) The space (Q_p, d_p) is totally disconnected.

In general we denote by $G = \{|\alpha| \mid \alpha \in K, \alpha \neq 0\}$ the set of non-zero values. In the p-adic case this set is a cyclic group generated by $\frac{1}{p}$. That is $G = \bigcup_{n \in Z} \{\frac{1}{p^n}\}$.

CHAPTER II

EXAMPLES AND PROPERTIES OF NORMS

When most students first encounter a linear space over a field, the scalar field is the rational, real, or complex number field. The valuation on the field is the absolute value, an archimedean valuation. However, one can consider linear spaces over n.a. valued fields, for example, over Q_p. When a norm is introduced on the linear space, the space is referred to as a normed linear space. If the norm also satisfies the non-archimedean property, the space is called a non-archimedean normed space.

In this chapter, E is a normed linear space over a field with non-archimedean valuation. The first section contains the basic properties of the norm and the set of values of the norm. Some examples of n.a. normed linear spaces are given. In addition, the relation between the value on K and the norm on E is studied.

Definition 2.1. A <u>normed linear space E over a field K</u> is a linear space E for which there is a mapping, $\| \ \|: E \to R$, called a norm such that:

(i) $\|x\| \geq 0$ for all $x \in E$ and $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for all $x \in E$ and $\alpha \in K$ where $| \ |$ is the valuation on K, and

(iii) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \varepsilon E$.

If in addition the normed linear space E satisfies property

(iv) $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \varepsilon E$, then E is called a <u>non-archimedean normed linear space</u>.

It is the case in which the valuation on K is non-archimedean with which we are concerned. Some properties of the norm will now be observed.

First, it should be noted that property (iv) implies property (iii) of a normed space. That is, if $\|x+y\| \leq \max\{\|x\|, \|y\|\}$, then $\|x+y\| \leq \|x\| + \|y\|$. In addition, a norm has some properties analogous to those of a valuation.

<u>Theorem 2.1</u>. If $\| \ \|$ is a norm on the space E, then

(1) $\|-x\| = \|x\|$.

(2) Let $\| \ \|$ be n.a. If $\|y\| < \|x\|$, then $\|x+y\| = \|x\|$.

Proof: (1) $\|-x\| = \|(-1)x\| = |-1| \, \|x\| = \|x\|$.

(2) By property (iv) $\|x+y\| \leq \max\{\|x\|, \|y\|\} = \|x\|$.

But $\|x\| = \|x+y-y\| \leq \max\{\|x+y\|, \|y\|\} = \|x+y\|$. Therefore $\|x+y\| = \|x\|$.

The following is an example of a n.a. normed linear space which will be used later.

<u>Example 2.1</u>. Let $E = K^n = \{x \mid x = (x_1, \ldots, x_n), x_i \varepsilon K\}$. Consider E as a linear space over K, a field with n.a. valuation $| \ |$. Define $\|x\| = \max_{1 \leq i \leq n} |x_i|$. The norm $\| \ \|$ is a non-archimedean norm.

Proof: Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, $x, y \varepsilon K^n$.

(i) $\|x\| \geq 0$, since $|x_i| \geq 0$ by definition of $|\ |$, which implies that $\max_{1 \leq i \leq n} |x_i| = \|x\| \geq 0$. $\|x\| = 0$ if and only if $x = 0$, since $\|x\| = \max_{1 \leq i \leq n} |x_i| = 0$ if and only if $|x_i| = 0$, $i = 1, 2, \ldots, n$ which is true if and only if $x_i = 0$, $i = 1, 2, \ldots, n$, or $x = 0$.

(ii) $\|\alpha x\| = |\alpha| \|x\|$. Let $\alpha \varepsilon K$, $x \varepsilon K^n$, then $\alpha x = (\alpha x_1, \ldots, \alpha x_n)$. Thus $\|\alpha x\| = \max_{1 \leq i \leq n} |\alpha x_i| = |\alpha| \max_{1 \leq i \leq n} |x_i| = |\alpha| \|x\|$.

(iii) $\|x + y\| \leq \max(\|x\|, \|y\|)$. We have $\|x + y\| = \max_{1 \leq i \leq n} |x_i + y_i|$. But $|x_i + y_i| \leq \max\{|x_i|, |y_i|\}$ for each $i = 1, 2, \ldots, n$, since $|\ |$ is a n.a. valuation. Therefore

$$\|x + y\| \leq \max_{1 \leq i \leq n} [\max\{|x_i|, |y_i|\}] = \max\{\max_{1 \leq i \leq n} |x_i|, \max_{1 \leq i \leq n} |y_i|\}$$

$$= \max\{\|x\|, \|y\|\}.$$

Thus K is a non-archimedean normed linear space.

Examples of normed linear spaces which are archimedean (not n.a.) include the following:

(1) $E = K = Q$, with the usual absolute value on Q as the valuation and as the norm.

(2) $E = C[a, b]$, the set of all real valued continuous functions on the closed interval $[a, b]$. Let $K = R$, with the usual absolute value and the norm defined by $\|x\| = \max_{t \varepsilon [a, b]} |x(t)|$.

(3) $E = C^n$, the complex Euclidean n-space. Let $K = C$ with the

usual absolute value. The norm for $x = (\alpha_1, \ldots, \alpha_n) \varepsilon C^n$, is given by $\|x\| = (|\alpha_1|^2 + \ldots + |\alpha_n|^2)^{1/2}$.

As another example of a n.a. normed linear space we have the following.

Example 2.2. Let $E = Q_p$ and $K = Q$ with the p-adic valuation as the valuation on Q and its extension to Q_p as the norm on E. Thus $\|x\| = |x|_p$ for $x \varepsilon E = Q_p$ and $|\alpha| = |\alpha|_p$ for $\alpha \varepsilon K = Q$. Since $\|\alpha x\| = |\alpha x|_p = |\alpha|_p |x|_p = |\alpha| \, \|x\|$ we have property (ii) of a n.a. normed linear space. The remaining properties (i) and (iv) are immediate consequences of the corresponding properties for $|\ |_p$. Thus $E = Q_p$ as a linear space over $K = Q$ is a n.a. normed linear space.

Consider the case $E = Q_5$ and $K = Q$ and in addition let the absolute value be the valuation on Q and the 5-adic value be the norm on Q_5. Let $\alpha = 2 \varepsilon Q$ and $x = 3 \varepsilon Q_5$. Then $\|\alpha x\| = \|2 \cdot 3\| = \|6\| = |6|_5 = 1$. However $|\alpha| \, \|x\| = |2| \, |3|_5 = 2 \cdot 1 = 2$. Thus $\|\alpha x\| \neq |\alpha| \, \|x\|$ so $\|\ \|$, as defined, is not a norm on Q_5 as a linear space over Q with the absolute value.

The preceding discussion indicates that an archimedean valuation on K and a n.a. norm on E may not be compatible. This leads one to ask what conditions on K are necessary and/or sufficient for a norm, $\|\ \|$, on E to be n.a. It turns out that the condition that K be n.a. is necessary but not sufficient.

Theorem 2.2. If E is n.a. then K is n.a.

Proof: Suppose there exist $\alpha, \beta \in K$ such that $|\alpha+\beta| > \max\{|\alpha|, |\beta|\}$. Let $x \neq 0$ be in E. Then

$$\begin{aligned}\|\alpha x+\beta x\| = \|(\alpha+\beta)x\| = |\alpha+\beta|\ \|x\| &> \max\{|\alpha|, |\beta|\}\|x\| \\ &= \max\{|\alpha|\ \|x\|, |\beta|\ \|x\|\} \\ &= \max\{\|\alpha x\|, \|\beta x\|\}.\end{aligned}$$

Thus $\| \ \|$ is not n.a. which implies E is not n.a. This is a contradiction, so if E is n.a. K is n.a.

While K being n.a. is a necessary condition for E to be n.a., the following example shows it is not a sufficient condition. In particular, an archimedean normed linear space E over a n.a. valued field K will be constructed.

<u>Example 2.3.</u> Let K be a n.a. field. Let E be the linear space of all sequences $x = (x_1, x_2, \ldots)$, $x_i \in K$, such that $\sum_{i=1}^{\infty} |x_i| < \infty$, with the norm $\|x\| = \sum_{i=1}^{\infty} |x_i|$. Then it is not true that

$$\|x+y\| \leq \max\{\|x\|, \|y\|\} \quad \text{for all} \quad x, y \in E,$$

as will be shown. First, $\| \ \|$ is a norm since $\sum_{i=1}^{\infty} |x_i| \geq 0$ and $\sum_{i=1}^{\infty} |x_i| = 0$ if and only if $x_i = 0$ for $i = 1, 2, \ldots$ so that $x = 0$. And

$$\|\alpha x\| = \sum_{i=1}^{\infty} |\alpha x_i| = |\alpha| \sum_{i=1}^{\infty} |x_i| = |\alpha|\ \|x\|.$$

Finally,

$$\|x+y\| = \sum_{i=1}^{\infty} |x_i + y_i| \leq \sum_{i=1}^{\infty} (|x_i| + |y_i|) = \sum_{i=1}^{\infty} |x_i| + \sum_{i=1}^{\infty} |y_i|$$

$$= \|x\| + \|y\| .$$

Now let $x = (1, 0, 0, \ldots)$ and $y = (0, 1, 0, 0, \ldots)$. Then $\|x\| = \|y\| = 1$ and $\|x+y\| = 2$. Thus $\|x+y\| > \max\{\|x\|, \|y\|\}$. Thus E is archimedean even though K is n. a.

Suppose K is a n. a. valued field. It has been shown that E need not be n. a. Is it possible in this case for the inequality $\|x+y\| > \max\{\|x\|, \|y\|\}$ to hold for all non-zero x and y in E? The answer is negative as the following argument shows. Recall the theorem that a valuation $|\ |$ is n. a. if and only if $|n| \leq 1$ for every integer n. Let x be a non-zero element of E. Then $\|x+x\| = \|2x\| = |2|\ \|x\| \leq \|x\| = \max\{\|x\|, \|x\|\}$. Thus $\|x+x\| \leq \max\{\|x\|, \|x\|\}$. Thus it is never possible to have $\|x+y\| > \max\{\|x\|, \|y\|\}$ for all non-zero elements.

To summarize, two situations may occur:

A. $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in E$, in which case E is n. a.

B. There exist $x, y \neq 0$ in E such that $\|x+y\| > \max\{\|x\|, \|y\|\}$. In this case E is said to be archimedean. However, as observed in the preceding paragraph, this inequality cannot hold for all non-zero elements of E.

Norm Values

Let E be a n. a. normed linear space over a n. a. valued field K and define the sets $G = \{|\alpha| \mid \alpha \in K, \alpha \neq 0\}$ and

$H = \{\|x\| \mid x \in E, x \neq 0\}$. In the examples of normed linear spaces considered so far, it has been the case that $H \subset G$, in fact $H = G$. However, in general H is not a subset of G. The following example illustrates this situation.

Example 2.4. Let $K = Q_p$, $p \neq 2$, then the value group G is generated by $\frac{1}{p}$. Thus $G = \{\frac{1}{p^n} \mid n \in Z\}$. Consider the set of sequences $x = \{a_i\}$ where $a_i \in Q_p$ and $\lim a_i = 0$. These sequences form a linear space E over Q_p. Let $\{C_i\}$ be an increasing sequence of positive numbers such that $\lim_i C_i = C > 0$. Then $\lim |a_i|_p C_i = 0$ since $\lim a_i = 0$ and the C_i are bounded above by C. Define $\|x\| = \max_i |a_i|_p C_i$. This maximum exists since $\lim_i |a_i|_p C_i = 0$ where $\{|a_i|_p C_i\}$ is a bounded sequence of real numbers.

As defined above, $\| \ \|$ is a n.a. norm on E as the following argument shows. Clearly $|a_i|_p C_i \geq 0$ so that $\|x\| = \max_i |a_i|_p C_i \geq 0$. Further, $\|x\| = 0$ if and only if $|a_i|_p = 0$ which is equivalent to $a_i = 0$, $i = 1, 2, \ldots$ or $x = 0$. For $a \in Q_p$, $ax = \{a a_i\}$. Thus $\|ax\| = \max_i |a a_i|_p C_i = |a|_p \max_i |a_i|_p C_i = |a|_p \|x\|$. Finally, let $x = \{a_i\}$ and $y = \{b_i\}$. Then

$$\begin{aligned}
\|x+y\| &= \max_i |a_i + b_i|_p C_i \leq \max_i [\max\{|a_i|_p, |b_i|_p\} C_i] \\
&= \max_i [\max\{|a_i|_p C_i, |b_i|_p C_i\}] \\
&= \max[\max_i |a_i|_p C_i, \max_i |b_i|_p C_i] \\
&= \max\{\|x\|, \|y\|\}.
\end{aligned}$$

It can now be shown that the set $H = \{\|x\| \mid x \in E, x \neq 0\}$ is not the same as the set $G = \{\frac{1}{p^n} \mid n \in Z\}$. In fact

$H = \{\frac{1}{p^n} C_i \mid n \varepsilon Z, \ i \varepsilon Z^+\}$. To see this fix n and k and define $x = \{a_i\}$ where $a_i = 0$ for $i \neq k$, and $a_k = p^n$. Then $|a_i|_p C_i = 0$, $i \neq k$ and $|a_k|_p C_k = |p^n|_p C_k = \frac{1}{p^n} C_k$. Thus $\|x\| = \max_i |a_i|_p C_i = \frac{1}{p^n} C_k$. Therefore for any fixed increasing sequence $\{C_i\}$ of positive numbers with $\lim_i C_i = C > 0$, for which at least one C_k is not an integral power of p, we have $H = \{\|x\|\} = \{\frac{1}{p^n} C_k\} \not\subset G = \{\frac{1}{p^n}\}$ for $n \varepsilon Z$, $k \varepsilon Z^+$.

In the preceding example, the set G of non-zero values $|\alpha|$, $\alpha \varepsilon K$, had no limit point but 0. Such a valuation is said to be discrete. The following definitions identify properties of the sets G and H where $G = \{|\alpha| \mid \alpha \varepsilon K, \alpha \neq 0\}$ and $H = \{\|x\| \mid x \varepsilon E, x \neq 0\}$.

<u>Definition 2.2</u>. The valuation on K is said to be <u>discrete</u> if the set G has no limit point but 0. If the valuation on K is discrete we say that <u>K is discrete</u>.

Similarly, discreteness is defined for E.

<u>Definition 2.3</u>. The norm on E is said to be <u>discrete</u> if the set H has no limit point but 0. If the norm on E is discrete, we say that <u>E is discrete</u>.

The p-adic valuation $|\ |_p$ on the field Q_p is a familiar discrete valuation. The absolute value on R is a valuation which is not discrete. A valuation which is not discrete is sometimes said to be dense. If Q_p is considered as a linear space over itself with both the valuation and the norm being the p-adic valuation $|\ |_p$, then the norm is a discrete norm. A norm which is not discrete will appear in Example 2.5.

A natural question one might ask is whether or not there is any relation between K being discrete and E being discrete. The following theorem and example show that K being discrete is a necessary condition but not a sufficient condition for E to be discrete.

Theorem 2.3. If E is discrete, then K is discrete.

Proof: Let E be discrete. Suppose K is not discrete. Then the set of values G has a limit $L > 0$. Thus for any n there exists a point $\alpha_n \varepsilon K$ such that $\big|\,|\alpha_n| - L\big| < \frac{1}{n}$. (Note that the outer symbol $\big|\ \big|$ represents the ordinary absolute value on the reals). Thus the sequence $\{|\alpha_n|\}$ has the limit L. Pick $x \varepsilon E$ such that $\|x\| = a > 0$. Consider the sequence $\{\|\alpha_n x\|\}$, $\alpha_n x \varepsilon E$. We have $\lim\limits_{n\to\infty}\|\alpha_n x\| = \lim\limits_{n\to\infty} |\alpha_n|\,\|x\| = \|x\| \lim\limits_{n\to\infty} |\alpha_n| = a \cdot L > 0$. Thus $\{\|\alpha_n x\|\}$, $\alpha_n x \varepsilon E$ is a sequence of norm values with limit point $aL > 0$. This is a contradiction since E is discrete by hypothesis. Therefore if E is discrete, K is discrete.

The following example shows that the converse of the preceding theorem does not hold. A linear space E over a field K will be given in which K is discrete but E is not discrete.

Example 2.5. Let $E = S^Q$ be the set of all power series $x = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \ldots$ where $\alpha_1, \alpha_2, \ldots$ is a set of rational numbers well-ordered in the natural order, that is, a strictly increasing well-ordered set of rational numbers, and the a_i are contained in some given field Γ. Define addition and multiplication in the usual way. For example, if

$$x = t^{\frac{1}{2}} - 3t^{\frac{5}{3}} + 2t^2 + \frac{3}{8}t^{\frac{5}{2}} + \ldots \quad \text{and} \quad y = 2t^{\frac{3}{4}} + 3t^{\frac{5}{3}} - \frac{7}{8}t^{\frac{15}{8}} + 7t^2 + \ldots$$

then

$$x + y = t^{\frac{1}{2}} + 2t^{\frac{3}{4}} - \frac{7}{8}t^{\frac{15}{8}} + 9t^2 + \ldots$$

and

$$x \cdot y = 2t^{\frac{5}{4}} + 3t^{\frac{13}{6}} - 6t^{\frac{29}{12}} - 9t^{\frac{10}{3}} + 6t^{\frac{11}{3}} + \ldots .$$

The set S^Q with addition and multiplication thus defined is a field.

Define for $x \varepsilon E$, $\|x\| = e^{-\alpha_1}$, $x \neq 0$ and $\|0\| = 0$. Here $\| \ \|$ is actually a n.a. valuation on field E. Clearly $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$. Let

$$x = \sum_{i=1}^{\infty} a_i x^{\alpha_i} \quad \text{and} \quad y = \sum_{i=1}^{\infty} b_i x^{\beta_i} .$$

Then the first term of xy is $a_1 b_1 x^{\alpha_1 + \beta_1}$ so that

$$\|x \cdot y\| = e^{-(\alpha_1 + \beta_1)} = e^{-\alpha_1} \cdot e^{-\beta_1} = \|x\| \cdot \|y\| .$$

Finally, the first term of $x + y$ would be determined by $\min(\alpha_1, \beta_1)$ so that

$$\|x + y\| \leq e^{-\min(\alpha_1, \beta_1)} = \max\{e^{-\alpha_1}, e^{-\beta_1}\} = \max\{\|x\|, \|y\|\} .$$

Thus $\| \ \|$ is a n.a. valuation on S^Q.

Consider the subset $K = S^Z$ of S^Q consisting of all series $\sum_{i=1}^{\infty} a_i t^{n_i}$ where $\{n_i\}$ is an ordered subset of Z, the set of integers. The set S^Z is a subfield of S^Q.

Now consider S^Q as a linear space over the subfield S^Z. Let the valuation on S^Z be the valuation induced on S^Z by the valuation $\| \ \|$ on S^Q. Then for $a \varepsilon S^Z$ and $x \varepsilon S^Q$ we have $\|ax\| = \|a\| \, \|x\|$ since a and x are both in S^Q. The other properties of a n.a. norm were verified when it was shown that $\| \ \|$ was a valuation on S^Q. Thus $\| \ \|$ is a n.a. norm on S^Q as a linear space over S^Z.

The following arguments demonstrate that S^Z is discrete but that S^Q is not discrete. The sets G and H in this case are $G = \{e^{-n}\}_{n \varepsilon Z}$ and $H = \{e^{-\alpha}\}_{\alpha \varepsilon Q}$. To show that S^Z is discrete let $L > 0$. There exists an N such that $\frac{1}{e^{N+1}} \leq L < \frac{1}{e^N}$. If $L = \frac{1}{e^{N+1}}$, then there exists an $a > \frac{1}{e^{N+2}}$ and $b < \frac{1}{e^{N+1}}$ with $L \varepsilon (a,b)$. Thus $(a,b) \cap G = \{L\}$ and L is not a limit point of G. If $L \neq \frac{1}{e^{N+1}}$, there exists an $a > \frac{1}{e^N}$ and $b < \frac{1}{e^{N+1}}$ such that $L \varepsilon (a,b)$ and $(a,b) \cap G = \{L\}$. Thus L is not a limit point of G, so G has no limit point but 0 and S^Z is discrete.

However S^Q is not discrete; in fact, it will be shown that H is dense in the non-negative reals and hence certainly has limit points other than 0. Let $L > 0$ and $L \varepsilon (a,b)$, $a > 0$. Then $\log a < \log b$ and there exists a real number α such that $\log a < -\alpha < \log b$. Thus $a < e^{-\alpha} < b$ so that $e^{-\alpha} \varepsilon (a,b)$ and L is a limit point of H. Similarly for $\epsilon > 0$ there is an α such that $0 < e^{-\alpha} < \epsilon$ so 0 is a limit point of H and H is dense in the non-negative reals.

We have then an example of a n.a. linear space $E = S^Q$ over a field $K = S^Z$ where K is discrete but E is not discrete. Thus, as indicated before, K being discrete is a necessary condition but not a sufficient condition for E to be discrete.

CHAPTER III

TOPOLOGICAL PROPERTIES OF NON-ARCHIMEDEAN NORMED LINEAR SPACES

In Chapter II properties of a linear space were discussed which are independent of a topology on E. The discreteness of E and K, as defined in Chapter II, are independent of any topology on E or K. However, given any normed linear space E over a field K with norm, $\| \ \|$, there is always an associated metric. This metric then induces a topology on E. This chapter is concerned with properties of E which result from this topology.

After defining the metric on E, properties of spheres when the metric is non-archimedean are examined. Conditions under which the topologies on E and K are discrete are explored. This is followed by a discussion of connectedness and compactness. It is noted that a key to the topological properties is the fact that every n.a. metric space is 0-dimensional. The chapter ends with a discussion of completeness and spherical completeness and the basic difference between them.

The Metric Induced by a Norm

Given a normed linear space E over K with norm, $\| \ \|$, the norm induces a metric on E as follows. Define for all $x, y \in E$,

$d(x, y) = \|x - y\|$. That d is a metric is shown in the next theorem.

Theorem 3.1. The function $d : E \times E \to R$ is a metric on E.

Proof: (i) $d(x, y) \geq 0$ since $\|x - y\| \geq 0$. Further, $d(x, y) = 0$ if and only if $x = y$ since $\|x - y\| = 0$ if and only if $x - y = 0$ which is true if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ since $\|x - y\| = \|-(y - x)\| = \|y - x\|$.

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ since

$$\|x - z\| = \|x-y + y-z\| \leq \|x - y\| + \|y - z\| .$$

Furthermore, if the norm is n.a.,

(iv) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ since

$$\|x - z\| = \|(x - y) + (y - z)\| \leq \max\{\|x - y\|, \|y - z\|\} .$$

Thus d is a metric on E and if $\| \ \|$ is n.a. then d satisfies the strong inequality.

If the metric d satisfies the strong triangle inequality,

(1) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$, for every $x, y, z \in E$, in addition to the usual properties of a metric, then d is called a non-archimedean metric. If a metric is not n.a., then it is called archimedean. Thus, if the strong triangle inequality (1) fails to hold for even one triple of points, the metric is archimedean.

One indirect consequence of Theorem 2.1 (2) is that every triangle in a n.a. metric space is at least isosceles. To see this suppose that $d(y, x) < d(x, z)$. Then $\|y - x\| < \|x - z\|$ but then by Theorem 2.1 (2) $\|y - z\| = \|y-x + x-z\| = \max\{\|y - x\|, \|x - z\|\} = \|x - z\|$.

Thus $d(y, z) = \|y - z\| = \|x - z\| = d(x, z)$. Thus at least two sides of any triangle must be equal in length.

Properties of Spheres

As usual the following definitions are made for any metric d.

<u>Definition 3.1</u>. Let $S(x_0, r) = \{x \varepsilon E \mid d(x_0, x) < r\}$. The set $S(x_0, r)$ is called an <u>open sphere</u> with center x_0 and radius $r > 0$.

<u>Definition 3.2</u>. Let $S[x_0, r] = \{x \varepsilon E \mid d(x_0, x) \leq r\}$. The set $S[x_0, r]$ is called a <u>closed sphere</u> with center x_0 and radius $r > 0$.

If there is any question as to what metric is being used, then the notation $S_d(x_0, r)$ will be adopted when the metric is d.

It is known that the collection of all open spheres, as defined above, is a base for a topology on E and this topology is said to be induced by the metric d. That is, the open sets are those which are unions of open spheres. Thus $U \subset E$ is open, with respect to the topology induced by d, if and only if for any $x \varepsilon U$ there exists a positive number ϵ such that the open sphere $S(x, \epsilon) \subset U$. As one would hope, the open spheres are open sets and the closed spheres are closed sets. However, if d is a n.a. metric, the spheres have some unusual properties.

<u>Theorem 3.2</u>. Every open sphere in a n.a. metric space (E, d) is a closed set.

Proof: Let y be a limit point of the open sphere $S(x, r)$. Since E is a metric space there exists a sequence $\{x_n\}$, $x_n \varepsilon S(x, r)$, such that $d(x_n, y) < \frac{1}{n}$. Also there exists an N such that $\frac{1}{N} < r$, so that $d(x_N, y) < \frac{1}{N} < r$. It follows that, $d(y,x) \leq \max\{d(y, x_N), d(x_N, x)\} < r$ so $y \varepsilon S(x, r)$. Therefore, $S(x, r)$ is closed, since it contains all its limit points.

Theorem 3.3. Every closed sphere in a n.a. metric space (E, d) is an open set in E.

Proof: Let $y \varepsilon S[x, r]$. To show that $S(y, r) \subset S[x, r]$, let $z \varepsilon S(y, r)$, then $d(y, z) < r$. Thus, $d(x, z) \leq \max\{d(x, y), d(y, z)\} \leq r$, so $z \varepsilon S[x, r]$. Since $S(y, r) \subset S[x, r]$, $S[x, r]$ is an open set.

In addition to the results contained in the preceding two theorems, there are some even more surprising properties of spheres in a n.a. metric space which are given in the following theorems. The theorems are stated for closed spheres but are also valid for open spheres.

Theorem 3.4. In a n.a. metric space E, if the intersection of two closed spheres is non-empty, then one sphere contains the other.

Proof: Given two spheres $S[x, r]$ and $S[y, \rho]$, we may assume $r \leq \rho$. If the intersection is non-empty it will be shown that $S[x, r] \subset S[y, \rho]$. Since the intersection is assumed to be non-empty there exists a point w contained in both spheres. Thus for $z \varepsilon S[x, r]$ $d(y, z) \leq \max\{d(y, w), d(w, z)\} \leq \max\{d(y, w), \max[d(w, x), d(x, z)]\}$. But $w \varepsilon S[y, \rho]$ implies $d(w, y) \leq \rho$, $w \varepsilon S[x, r]$ implies $d(x, w) \leq r$, and $z \varepsilon S[x, r]$ implies $d(x, z) \leq r$. Therefore, $d(y, z) \leq \rho$ so $z \varepsilon S[y, \rho]$. Thus if $S[x, r] \cap S[y, \rho] \neq \emptyset$ and $r \leq \rho$, then $S[x, r] \subset S[y, \rho]$.

Theorem 3.5. In a n.a. metric space any point of a closed sphere may be taken as its center.

Proof: Given $S[x_0, r]$, let $y \varepsilon S[x_0, r]$ and consider the sphere $S[y, r]$. Since $y \varepsilon S[x_0, r] \cap S[y, r]$, by the previous theorem one sphere is a subset of the other. But from the proof of the previous theorem, since the radii are equal, each sphere is a subset of the other. This implies that $S[x_0, r] = S[y, r]$ where y was any point in $S[x_0, r]$. Thus any point of a sphere may be taken as its center.

If one contemplates the preceding theorems it is not surprising that many results for a n.a. metric space will be different from what we have come to expect from the study of the reals with the usual metric. This section concludes with a converse of Theorem 3.4. Theorem 3.4 and its converse give a characterization of n.a. metric spaces.

Theorem 3.6. Let E be a metric space. Suppose that any two spheres in E are either disjoint or one is a subset of the other. Then E is a n.a. metric space.

Proof: It must be shown that for any $a, b, c \varepsilon E$, $d(a, c) \leq \max\{d(a, b), d(b, c)\}$. Suppose this is not true, then there exist three points a, b, and c such that $d(a, c) > \max\{d(a, b), d(b, c)\}$. Let $d(a, b) = \delta$ and suppose that $d(c, b) = \delta' \leq \delta$. There are two cases, (1) $\delta' < \delta$. Consider $S[a, \delta]$ and $S[b, \delta']$. Clearly $b \varepsilon S[b, \delta']$ and since $d(a, b) = \delta$, $b \varepsilon S[a, \delta]$. Thus b is contained in both spheres so by hypothesis one sphere is a subset of the other. Since $d(c, a) > \max\{d(a, b), d(b, c)\} = \delta$, we have $c \not\varepsilon S[a, \delta]$. And since one of the spheres is a subset of the other it must be that $S[a, \delta]$ is a proper subset of $S[b, \delta']$ since $c \in S[b, \delta']$. However, since $d(a, b) = \delta > \delta'$, $a \notin S[b, \delta']$. This is a contradiction.

(2) $\delta = \delta'$. Since $d(c, b) = \delta' = \delta$, $b \in S[c, \delta]$, and since $d(a, b) = \delta$, $b \in S[a, \delta]$. Thus the two spheres intersect and so by hypothesis one must be a subset of the other. Since $d(c, a) > \delta = \max\{d(a, b), d(b, c)\}$, then $c \notin S[a, \delta]$ and $a \notin S[c, \delta]$. Thus each sphere must be a proper subset of the other, a contradiction.

Since both cases lead to contradictions, E must be a n.a. metric space.

Discrete Topology

One possible topology which any set may be assigned is the discrete topology, that is the topology in which every subset is an open set. It could be that the topology induced by a valuation or norm is the discrete topology. It turns out that this can happen only when the valuation on K is the trivial valuation. Recall that the trivial valuation defines $|x| = 1$ for $x \neq 0$.

A word of caution is appropriate here. The term discrete has been used here in relation to the topology on K. Earlier a valuation on K was called discrete if the set G of values had no limit point but 0. There is no connection between these two concepts so one must be careful to distinguish between them.

In keeping with our earlier terminology, the statement K is discrete means that the valuation on K is discrete. If we are referring to the topology on K we will always say that K has the discrete topology. Thus there should be no reason for confusion.

Theorem 3.7. In order that K have a discrete topology, it is necessary and sufficient that the valuation on K be trivial.

Proof: Assume that the valuation, $|\ |$, on K is the trivial valuation. Let $x_0 \varepsilon K$. Consider $S(x_0, \frac{1}{2}) = \{x \varepsilon K \mid d(x_0, x) < \frac{1}{2}\}$. If $d(x_0, x) = |x_0 - x| < \frac{1}{2}$, then $|x_0 - x| = 0$ so that $x = x_0$. Thus $S(x_0, \frac{1}{2}) = \{x_0\}$ so $\{x_0\}$ is open. Since x_0 was an arbitrary point in K, the topology on K induced by d is the discrete topology.

Conversely, assume that K has the discrete topology. Then $\{0\}$ is open. Thus there exists an $r_0 > 0$ such that $S(0, r_0) \subset \{0\}$. Since $1 \varepsilon K$, $|1| = 1$. Suppose there is a $y \varepsilon K$ such that $|y| = \delta = 0$ and $y \neq 1$. This assumption implies that there is an element $z \varepsilon K$ such that $0 < |z| < 1$. In particular, if $0 < \delta < 1$, let $z = y$. If $\delta > 1$, then $y^{-1} \varepsilon K$ and $|y^{-1}| = |y|^{-1} < 1$ so that $z = y^{-1}$ will suffice. In any case there is a $z \neq 1$ such that $0 < |z| < 1$. But then $z^n \varepsilon K$ for any n and $|z^n| = |z|^n \to 0$ as $n \to \infty$. Thus the sequence $\{z^n\}$ converges to 0. This is a contradiction since there exists an $r_0 > 0$ with $S(0, r_0) \subset \{0\}$. Therefore $|x| = 1$ for any $x \neq 0$ and $|\ |$ is the trivial valuation.

In considering a linear space E over a field K, the valuation on K induces a topology on K and the norm on E induces a topology on E. One might suspect that there is some relationship between these two topologies. The following theorem and example show that K having the discrete topology is a necessary condition but not a sufficient condition for E to have the discrete topology.

Theorem 3.8. If E has the discrete topology, then K has the discrete topology.

Proof: Assume E has the discrete topology. Then $\{0\}$ is an open set in E so there exists a number $r_0 > 0$ such that $S(0, r_0) \subset \{0\}$. Thus if $x \neq 0$ we have $\|x\| \geq r_0 > 0$. If K does not have the discrete topology there exists an $\alpha \varepsilon K$ such that $\{\alpha\}$ is not open in K. Thus for any $n > 0$, $S(\alpha, \frac{1}{n})$ contains a point $\beta_n \neq \alpha$ with $0 < |\beta_n - \alpha| < \frac{1}{n}$. Let x be a non-zero element of E. Then $\|(\beta_n - \alpha)x\| = |\beta_n - \alpha| \; \|x\| \to 0$ as $n \to \infty$ so there exists an N such that $\|(\beta_N - \alpha)x\| < r_0$ and $(\beta_N - \alpha)x \neq 0$. This is a contradiction since it was shown earlier that the norm of any non-zero element of E must be at least r_0. Therefore, if E has the discrete topology then K has the discrete topology.

Corollary 3.9. If E has the discrete topology, then the valuation on K is the trivial valuation.

Proof: This follows immediately from Theorems 3.8 and 3.7.

The converse of Theorem 3.8 is not true as the following example shows. In this example K is a field with the discrete topology. However, it will be shown that E, considered as a linear space over K, does not have the discrete topology.

Example 3.1. Let K be a field with a trivial valuation. By Theorem 3.7, K has the discrete topology. Let E be the linear space over K consisting of the set of all power series $x = a_1 t^{n_1} + a_2 t^{n_2} + \dots$, where $n_1, n_2, \dots$ is an ordered subset of Z, the set of integers. Define $\|x\| = e^{-n_1}$, $x \neq 0$, and $\|0\| = 0$. E is a n.a. normed space over K as in Example 2.5.

Also, as in Example 2.5, 0 is the only limit point of the set of norm values $\|x\|$, $x \varepsilon E$ so that E is discrete. However, E does not have the discrete topology since $\{0\}$ is not an open set in E. This follows since for any $\epsilon > 0$, there exists an integer n such that $e^{-n} < \epsilon$. Thus no sphere $S(0, \epsilon) \subset \{0\}$ so $\{0\}$ is not an open set. The space E is then a linear space over a field K, where K has the discrete topology but E does not have the discrete topology. Thus K having the discrete topology is not a sufficient condition for the topology on E to be discrete.

Dimension 0

It has been observed that the collection of open spheres in a metric space is a base for the topology. In the case of a n.a. metric, Theorem 3.2 tells us that each open sphere is also a closed set. Thus in a n.a. metric space there is always a base consisting of sets which are both open and closed. Recall that by Theorem 1.1, a set has empty boundary if and only if it is both open and closed. Thus any n.a. metric space has a base consisting of sets with empty boundaries. The following definition identifies the property.

Definition 3.3. A topological space E has dimension 0 at a point x if x has arbitrarily small neighborhoods with empty boundaries. That is, given any neighborhood V of x there is a neighborhood U of x with empty boundary such that $x \varepsilon U \subset V$. The space E is called 0-dimensional if it has dimension 0 at each point of E.

Since any n.a. metric space has a base consisting of sets with empty boundaries it is clear that the definition of 0-dimensional is satisfied. Thus we have:

Theorem 3.10. Every n.a. metric space is 0-dimensional. In particular (Q_p, d_p) is 0-dimensional.

Connectedness

Another immediate result of the existence of proper subsets which are both open and closed is that no n.a. metric space is connected. Thus in the case of the p-adic numbers Q_p, with the topology induced by the p-adic metric d_p, (Q_p, d_p) is not connected. In this case a much stronger result holds. In Snook [19], p. 78, it was proven that (Q_p, d_p) is totally disconnected; that is, the only connected sets are singleton sets.

One might ask if this is always the case in 0-dimensional spaces. However, any set E with the trivial topology, that is with the only open sets being the empty set and the set E, serves as an example of a space which is 0-dimensional but not totally disconnected. The space is 0-dimensional since the only neighborhood of any point $x \in E$ is the set E which is both open and closed. E is actually connected, since there is certainly no proper subset of E which is both open and closed. Thus E cannot be totally disconnected.

A more restrictive question and one of more interest to this study might be, is every n.a. metric space totally disconnected? The affirmative answer to this question results from the following theorem and the fact that every n.a. metric space is 0-dimensional.

Theorem 3.11. Every 0-dimensional metric space is totally disconnected.

Proof: Let E be a 0-dimensional metric space. Then the topology for E has a base β consisting of sets which are both open and closed in E. Let H be any non-degenerate subset of E and suppose H is connected. Since H is non-degenerate, there exist distinct points x and y in H. But since E is a metric space, there exists an open set U such that $x \varepsilon U$ and $y \not\varepsilon U$. Without loss of generality, $U \varepsilon \beta$. But $U \varepsilon \beta$ implies U is both open and closed in E and hence $E \sim U$ is open in E. But then $x \varepsilon H \cap U$ and $y \varepsilon H \cap (E \sim U)$ and both are open sets in H. Clearly their intersection is empty. Thus H is the union of disjoint nonempty sets each of which is open in H. Therefore H is not connected. Since H was an arbitrary non-degenerate subset of E, we have that E is totally disconnected.

Since every n.a. metric space is 0-dimensional by Theorem 3.10, we have the following theorem.

Theorem 3.12. Every n.a. metric space is totally disconnected.

Compactness

Compactness in (Q_p, d_p) was discussed by Snook [19]. It was shown that the set 0_p is a compact subset of (Q_p, d_p) but that the space (Q_p, d_p) is not compact since it is not bounded. Recall that in a metric space every compact subset is closed and bounded. It was shown that, as in the reals, every closed and bounded subset of (Q_p, d_p) is compact. It follows that the spheres, being closed and bounded, are compact. But every point in Q_p is contained in a sphere. Hence (Q_p, d_p) has the property that every point of Q_p has a compact neighborhood. This property is called local compactness.

Definition 3.4. A subset of a space E is locally compact if every point of E is contained in a compact neighborhood.

The above argument proves the following theorem.

Theorem 3.13. The space (Q_p, d_p) is locally compact.

Thus Q_p is a n.a. metric space which is locally compact. One might ask if every n.a. metric space is locally compact. However, the space S^∞, which we have previously encountered, furnishes an example of a n.a. metric space which is not locally compact.

Example 3.2. As in Example 2.8, let S^∞ be the subset of S^Q consisting of all formal series $x = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \ldots$ where $\alpha_1, \alpha_2, \ldots$ is a finite sequence or a simple sequence of rational numbers tending to infinity. Recall that since S^∞ is a subset of S^Q the sequence of exponents is strictly increasing.

The norm on S^∞ is defined by $\|x\| = e^{-\alpha_1}$, if $a_1 \neq 0$, $\|0\| = 0$. Let d be the induced metric on S^∞. To show that (S^∞, d) is not locally compact we will show that 0 is not contained in any compact neighborhood. To do this consider the sphere $S[0, \epsilon]$, $\epsilon > 0$. Then there exists an $\alpha \in Q$ such that $\alpha \geq -\log \epsilon$, that is, $-\alpha \leq \log \epsilon$ or equivalently $e^{-\alpha} \leq \epsilon$. Consider the set $A = \{x \in S^\infty \mid x = at^\alpha, a \in R, a \neq 0\}$. Then $x \in A$ implies that $\|x - 0\| = \|x\| = e^{-\alpha} \leq \epsilon$ so that $x \in S[0, \epsilon]$. Thus A is an infinite subset of $S[0, \epsilon]$. Let $x, y \in A$, $x \neq y$. Then $x = bt^\alpha$ and $y = ct^\alpha$

where $b \neq c$. Therefore $\|x - y\| = \|bt^{\alpha} - ct^{\alpha}\| = \|(b-c)t^{\alpha}\| = e^{-\alpha}$ since $b - c \neq 0$. Clearly A has no limit point since any two distinct points are a distance $e^{-\alpha}$ apart where α is fixed. Thus A is an infinite subset of $S[0, \epsilon]$ which has no limit point. It follows that none of the spheres $S[0, \epsilon]$ is compact, since in a metric space compactness and countable compactness are equivalent.

But this implies no neighborhood of 0 is compact since suppose U were a compact neighborhood of 0. There exists an $\epsilon > 0$ such that $0 \in S[0, \epsilon] \subset U$. But then $S[0, \epsilon]$ is a closed subset of the compact set U and hence is compact. This contradiction implies that no neighborhood of 0 is compact and hence S is not locally compact.

We have then that (S^{∞}, d) is an example of a n.a. metric space in which not all closed and bounded sets are compact, not all spheres are compact, and which is not locally compact.

Completeness

Two examples of complete valued fields which have been encountered are the set of real numbers with the absolute value and the set of p-adic numbers with the p-adic valuation. The concept of completeness extends to normed linear spaces. In the following definition some terms which have previously been applied only to valued fields will be extended to normed linear spaces.

Definition 3.5. Let E be a normed linear space with norm $\| \ \|$.

(a) A sequence $\{x_n\}$ in E is Cauchy if and only if for any $\epsilon > 0$ there exists an integer N such that $m, n \geq N$ implies that $\|x_n - x_m\| < \epsilon$.

(b) A sequence $\{x_n\}$ in E <u>converges</u> to x in E if and only if for any $\epsilon > 0$ there exists an N such that $n \geq N$ implies that $\|x_n - x\| < \epsilon$.

(c) A normed linear space E is <u>complete</u> if and only if every Cauchy sequence in E converges to an element of E.

(d) A complete normed linear space is called a <u>Banach space</u>.

Several theorems carry over as well and they will be stated without proof.

<u>Theorem 3.14.</u> Let E be a n.a. normed linear space.

(a) A sequence $\{x_n\}$ in E is Cauchy if and only if for each $\epsilon > 0$ there exists an N such that $\|x_{n+1} - x_n\| < \epsilon$ whenever $n \geq N$.

(b) If $\{x_n\}$ is a non-null Cauchy sequence in a n.a. normed linear space then the sequence $\{\|x_n\|\}$ is eventually constant.

<u>Example 3.3.</u> The space K^n of Example 2.1, where K is a complete valued field, is a complete normed linear space. To see that K^n is complete, let $\{x_m\} = \{(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}\}$ be a Cauchy sequence in K^n. Recall that $\|x\| = \|(x_1, \ldots, x_n)\| = \max_{1 \leq i \leq n} |x_i|$ where $|\ |$ is the n.a. valuation on K. Then for each $\epsilon > 0$ there exists an M such that $\|x_{m+1} - x_m\| = \max_{1 \leq i \leq n} |x_i^{(m+1)} - x_i^{(m)}| < \epsilon$ whenever $m \geq M$. Hence for each i, $1 \leq i \leq n$, $|x_i^{(m+1)} - x_i^{(m)}| < \epsilon$ whenever $m \geq M$, so that the sequence $\{x_i^{(m)}\}$ is Cauchy, with respect to the valuation $|\ |$, for $i = 1, 2, \ldots, n$. Since K is complete, there exists an $\hat{x}_i$ such that $\lim_{m \to \infty} x_i^{(m)} = \hat{x}_i$. Thus for any $\epsilon > 0$ there exists an N_i such that

$m \geq N_i$ implies that $|x_i^{(m)} - \hat{x}_i| < \epsilon$. Let $N = \max_{1 \leq i \leq n} N_i$ and $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$. Then $\hat{x} \varepsilon K$ and $\|x_m - \hat{x}\| = \max_{1 \leq i \leq n} |x_i^{(m)} - \hat{x}_i| < \epsilon$ for $m \geq N$. Thus the Cauchy sequence $\{x_m\}$ converges to $\hat{x} \varepsilon K^n$ so K^n is complete.

The following classical theorem of Cantor gives a characterization of metric spaces which are complete.

Theorem 3.15. Among the metric spaces the complete spaces are characterized by the following property; every sequence of non-void closed sets $A_1 \supset A_2 \supset \ldots$ of which the diameters tend to 0, has a non-empty intersection.

Spherical Completenss

The notion of spherically complete spaces was introduced by Ingleton [9], for the study of the Hahn-Banach theorem in n.a. normed spaces. He showed that a n.a. valued field has the Hahn-Banach property if and only if it is spherically complete. Monna [13] has since generalized the concept to locally convex spaces. Recall the K is said to have the Hahn-Banach property if, for any n.a. space F over K, every linear functional defined on a subspace of F possesses an extension of the same norm defined on the whole space F. This study will not be concerned with the Hahn-Banach property. However, because of the importance of spherical completeness in relation to the Hahn-Banach property and other areas of n.a. analysis, spherical completeness will be discussed and compared to completeness.

Definition 3.6. A n.a. normed linear space E over a n.a. valued field K is called spherically complete if every family of closed spheres which is linearly ordered by set inclusion has a non-empty intersection.

Recall that if two spheres intersect in a n.a. normed linear space, then one is a subset of the other. Thus any family $\mathfrak{F}$ of spheres such that any two intersect is linearly ordered by set inclusion. For practical purposes, in proofs of spherical completeness it is sufficient to show that any decreasing sequence $S_1 \supset S_2 \supset \ldots$ of closed spheres has a non-empty intersection. This results from the fact that from any family $\mathfrak{F}$ of closed spheres that is linearly ordered by set inclusion one may extract a decreasing sequence $S_1 \supset S_2 \supset \ldots$ such that each sphere in the original family $\mathfrak{F}$ contains one of the spheres S_n. Thus any point in common with each of the S_i, $i = 1, 2, \ldots$ will be a common point of the spheres in $\mathfrak{F}$.

An important relationship between completeness and spherical completeness is given by the following theorem.

<u>Theorem 3.16</u>. Each spherically complete space E is complete with respect to the topology induced on E by the norm on E.

Proof: Note that spherical completeness is by definition a property of n.a. normed linear spaces. Thus E is n.a. by hypothesis. The proof follows from the characterization of complete metric spaces given by Theorem 3.15. Let $A_1 \supset A_2 \supset \ldots$ be a sequence of non-void closed subsets of E whose diameters tend to 0. Let d_n be the diameter of the set A_n. For each $n = 1, 2, \ldots$ pick $x_n \in A_n$, then $A_n \subset S[x_n, d_n]$ and $\{d_n\} \to 0$. We then have $A_{n+1} \subset A_n \subset S[x_n, d_n]$. Since $A_{n+1} \subset S[x_{n+1}, d_{n+1}]$,

$S[x_{n+1}, d_{n+1}] \cap S[x_n, d_n] \neq \emptyset$. But without loss of generality $d_{n+1} \leq d_n$ so $S[x_{n+1}, d_{n+1}] \subset S[x_n, d_n]$. Thus $S[x_1, d_1] \supset S[x_2, d_2] \supset \ldots$. But E is spherically complete so there exists

$$x_0 \in \bigcap_{n=1}^{\infty} S[x_n, d_n] .$$

Suppose $x_0 \notin \bigcap_{n=1}^{\infty} A_n$. Then there exists an N such that $x_0 \notin A_N$ and since $A_n \subset A_N$ for $n \geq N$, $x_0 \notin A_n$ for $n \geq N$. Since A_N is closed and $x_0 \in E \sim A_N$, there exists a sphere $S(x_0, \epsilon) \subset E \sim A_N$. But $\{d_n\} \to 0$ implies there exists an N_0 such that $d_{N_0} < \epsilon$. Let $N_1 = \max(N_0, N)$. Then $S(x_0, d_{N_1}) \subset E \sim A_{N_1}$, that is $S(x_0, d_{N_1}) \cap A_{N_1} = \emptyset$. However $x_{N_1} \in A_{N_1}$ so $d(x_0, x_{N_1}) > d_{N_1}$. But $x_0 \in S[x_{N_1}, d_{N_1}]$ implies that $d(x_0, x_{N_1}) \leq d_{N_1}$. This contradiction implies that $x_0 \in \bigcap_{n=1}^{\infty} A_n$ and hence that E is complete.

Thus spherical completeness implies completeness. The converse is not true in general as will be demonstrated in Example 3.4. However, by means of additional restrictions on E, sufficient conditions can be stated for a complete space to be spherically complete. For example, consider the following theorem.

<u>Theorem 3.17</u>. If E is a complete n.a. normed linear space whose norm is discrete, then E is spherically complete.

Proof: It suffices to show that the intersection of any sequence of closed spheres $S_1 \supset S_2 \supset \ldots$ is non-empty. Two cases must be considered.

(a) Suppose that the diameters of the spheres tend to 0. Then by Theorem 3.15, since E is complete, the intersection is non-empty.

(b) Suppose that the diameters do not tend to 0. In this case the diameters converge to a positive number and hence are constant from some point on since E is discrete. Thus the intersection is non-empty.

<u>Corollary 3.18.</u> Each complete field with a discrete n.a. valuation is spherically complete. In particular, the p-adic field Q_p with the p-adic valuation $| \;|_p$ is spherically complete.

It has previously been shown that if K is complete then K^n is complete. A similar result holds for K^n with respect to spherical completeness.

<u>Theorem 3.19.</u> The space K^n is spherically complete if K is spherically complete.

Proof: Recall that the norm on K^n was $\|x\| = \max\limits_{1 \le i \le n} |x_i|$. It suffices to consider a sequence of closed spheres in K^n, $S_1 \supset S_2 \supset \ldots$ and show that their intersection is non-empty. Let $S_i = [a^{(i)}, d_i]$. Then if $x = (x_1, x_2, \ldots, x_n) \varepsilon S_i$, $\|x - a^{(i)}\| = \max\limits_{1 \le j \le n} |x_j - a_j^{(i)}| \le d_i$. Let $P_k : K^n \to E_k$ be the projection from K^n into the k^{th} coordinate space E_k; that is, $P_k(x) = (0, \ldots, x_k, 0, \ldots, 0)$. Let $P_k(S_i) = S_i^{(k)}$. Then

$$S_i^{(k)} = \{x^{(k)} \varepsilon E_k \mid |x^{(k)} - a_k^{(i)}| \le d_i\} = S^{(k)}[a_k^{(i)}, d_i].$$

Thus $S_1^{(k)} \supset S_2^{(k)} \supset \ldots$ is a sequence of closed spheres in E_k. But E_k is isomorphic to K by the isomorphism φ defined by $\varphi(0,\ldots,x_k,0,\ldots,0) = x_k$. Since K is spherically complete, E is spherically complete and $\bigcap_{i=1}^{\infty} S_i^{(k)}$ is non-empty for each $k = 1,2,\ldots,n$. Choose $\alpha_k \varepsilon \bigcap_{i=1}^{\infty} S_i^{(k)}$ for $k = 1,2,\ldots,n$. Then $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a point of K^n, contained in each of the spheres $S_1, S_2, \ldots$ so that the intersection is non-empty. Therefore K^n is spherically complete.

From the study of metric spaces it is known that any closed subset of a complete metric space is complete. The situation with regard to spherical completeness is not the same. In fact the following example provides a spherically complete space containing a closed subset which is not spherically complete.

Example 3.4. Let E be the space of Example 2.5. That is, $E = S^Q$ is the set of formal power series $x = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \ldots$ where $\alpha_1, \alpha_2, \ldots$ is a strictly increasing well-ordered set of rational numbers and the a_i are contained in some field Γ. As before, define $\|x\| = e^{-\alpha_1}$, $x \neq 0$, $\|0\| = 0$. It was observed that S^Q is a field and $\|\ \|$ is a n.a. valuation on the field. Let $K = S^\infty$ be the subfield of S^Q consisting of power series $\sum_{i=1}^{\infty} a_i t^{\alpha_i}$ where $\alpha_1, \alpha_2, \ldots$ is a finite sequence or a simple sequence tending to ∞. By simple we mean the set $\bigcup_{i=1}^{\infty} \{\alpha_i\}$ has no limit point but ∞.

Since S^∞ is a subfield of S^Q, S^Q may be considered as a linear space over S^∞. The valuation $\|\ \|$ on field S^Q becomes a norm on the n.a. linear space S^Q over S^∞ since for $a \varepsilon S^\infty$ and $x \varepsilon S^Q$, $\|ax\| = \|a\|\,\|x\|$. It will be shown that the space S^Q is spherically complete but that S^∞ is a closed subspace of S^Q that is not spherically complete.

S^∞ is a Closed Subspace of S^Q

Let $x_0 = c_1 t^{r_1} + c_2 t^{r_2} + \ldots$ be a limit point of S^∞. Suppose $x_0 \notin S^\infty$. Then it must be the case that the set $r_1, r_2, \ldots$ is not a simple sequence and hence has a finite limit point. Since the set $r_1, r_2, \ldots$ is well-ordered in the natural (increasing) order, it must have a smallest limit point L.

Since x_0 is a limit point of S^∞ in S^Q and S^Q is a metric space, there exists a sequence $x_1, x_2, \ldots$ where $x_n \varepsilon S^\infty$, such that $\|x_0 - x_n\| < e^{-n}$. Let $N > L$ and

$$x_N = \sum_{i=1}^{\infty} a_{i,N} t^{\alpha_{i,N}}$$

Then $\|x_0 - x_N\| < e^{-N} < e^{-L}$. However, since L is the smallest limit point of the set $r_1, r_2, \ldots$, which is well-ordered so that the r_β's are increasing, this requires that the <u>sequence</u> $r_1, r_2, \ldots$ have L as its limit point. Thus $r_i \leq L$ so that $e^{-r_i} \geq e^{-L}$ for $i = 1, 2, \ldots$.

Furthermore, suppose that $\alpha_{i,N} = r_i$ for $i = 1, 2, \ldots$ and $a_{i,N} = c_i$ for $i = 1, 2, \ldots$. Then the sequence $\alpha_{1,N}, \alpha_{2,N}, \ldots$ would have L as a limit point. This is a contradiction since this sequence is a simple sequence whose only limit point is ∞. Thus there exists an M such that $\alpha_{M,N} \neq r_M$ or $a_{M,N} \neq c_M$. In either case we would have $\|x_0 - x_N\| \geq e^{-r_M} \geq e^{-L} > e^{-N}$. But earlier we had that $\|x_0 - x_N\| < e^{-N}$. This contradiction implies that $x_0 \varepsilon S^\infty$

and since x was an arbitrary limit point of S, it follows that S is a closed subspace of S^{Ω}.

S^{Ω} is Spherically Complete

It suffices to consider a sequence $S_1 \supset S_2 \supset \ldots$ of closed spheres in S^{Ω} and show that their intersection is non-empty. Let

$$x_n = a_{1,n} t^{\alpha_{1,n}} + a_{2,n} t^{\alpha_{2,n}} + \ldots$$

be a fixed point in S_n for $n = 1, 2, \ldots$. Let r_n be the radius of S_n and denote by $i(n)$ the ordinal of the set of all i such that $\exp[-\alpha_{i,n}] > r_n$. Thus $i > i(n)$ implies $\exp[-\alpha_{i,n}] \leq r_n$. Let $x_{n+1} \varepsilon S_{n+1}$, then $x_{n+1} \varepsilon S_n$ so $\|x_{n+1} - x_n\| \leq r_n$. Thus

$$x_{n+1} - x_n = b_1 t^{\alpha_1} + b_2 t^{\alpha_2} + \ldots$$

where $\alpha_1 > \alpha_{i(n)}$. This can only happen if $\alpha_{i,n+1} = \alpha_{i,n}$ and $a_{i,n+1} = a_{i,n}$ for $i < i(n)$. Further, if $\alpha_{i(n),n+1}$ exists, then $\exp[-\alpha_{i(n),n+1}] \leq r_n$. Since $S_{n+1} \subset S_n$, $r_{n+1} \leq r_n$, so that $i(n+1) \geq i(n)$. A common point of the spheres S_n can now be constructed. Let

$$x_0 = \sum_{i < i(1)} a_{i,1} t^{\alpha_{i,1}} + \sum_{i(1) \leq i < i(2)} a_{i,2} t^{\alpha_{i,2}} + \ldots$$

$$+ \sum_{i(n-1) \leq i < i(n)} a_{i,n} t^{\alpha_{i,n}} + \ldots .$$

By construction, x_0 agrees with x_1 for $i < i(1)$ so that $x_0 \varepsilon S_1$.

But $x_2 \varepsilon S_2 \subset S_1$ implies that x_2 agrees with x_1 and hence x_0 for $i < i(1)$. Again, by construction, x_0 agrees with x_2 for $i(1) \leq i < i(2)$ and hence for $i < i(2)$ so $x_0 \varepsilon S_2$. This process may be continued so that for any n, $x_0 \varepsilon S_n$ and hence $x_0 \varepsilon \bigcap_{n=1}^{\infty} S_n$. Thus S^Q is spherically complete.

S^∞ is not Spherically Complete

Let $x_n = \sum_{i=1}^{\infty} t^{\alpha_i}$ where $\alpha_i = \frac{i}{i+1}$ for $i = 1, 2, \ldots, n$ but $\alpha_i = i$ for $i > n$. Then $x_n \varepsilon S^\infty$ since the sequence $\alpha_1, \alpha_2, \ldots$ is a simple sequence which tends to ∞. Let $r_n = \exp[-\frac{n+1}{n+2}]$. Define $S_n = S[x_n, r_n] = \{x \varepsilon S^\infty \mid \|x - x_n\| \leq r_n\}$. We will show (i) $S_{n+1} \subset S_n$ and (ii) $\bigcap_{n=1}^{\infty} S_n = \emptyset$.

(i) Let $x \varepsilon S_{n+1}$. Then $\|x - x_n\| \leq \max\{\|x - x_{n+1}\|, \|x_{n+1} - x_n\|\}$.

But $\|x - x_{n+1}\| \leq r_{n+1} < r_n$ and

$$\|x_{n+1} - x_n\| = \|\sum_{i=1}^{n+1} t^{\frac{i}{i+1}} + \sum_{i=n+2}^{\infty} t^i - \sum_{i=1}^{n} t^{\frac{i}{i+1}} - \sum_{i=n+1}^{\infty} t^i\|$$

$$= \|-t^{\frac{n+1}{n+2}} - t^{n+1}\| = \exp[-\frac{n+1}{n+2}] = r_n .$$

Thus $\|x - x_n\| \leq r_n$ so $x \varepsilon S_n$. Therefore $S_{n+1} \subset S_n$.

(ii) Suppose there exists an $x \varepsilon \bigcap_{n=1}^{\infty} S_n$, then $x \varepsilon S[x_n, r_n]$ so $\|x - x_n\| \leq r_n$ for $n = 1, 2, \ldots$. Thus x agrees with x_n for $1 \leq i \leq n$; that is, if $x = \sum_{i=1}^{\infty} b_i t^{\beta_i}$, then $b_i = 1$ and $\beta_i = \frac{i}{i+1}$ for $i = 1, 2, \ldots, n$. Since this is true for every $n = 1, 2, \ldots$, then $x = \sum_{i=1}^{\infty} t^{\frac{i}{i+1}}$. But the sequence $\{\frac{i}{i+1}\}_{i=1}^{\infty}$ has the limit 1.

Thus $x \notin S^\infty$. This is a contradiction since $x \in \bigcap_{n=1}^{\infty} S_n$ implies that $x \in S^\infty$. Therefore $\bigcap_{n=1}^{\infty} S_n = \emptyset$ and S^∞ is not spherically complete.

We have then an example of a spherically complete space S^Q, containing a closed subspace S^∞ which is not spherically complete. Of course since spherical completeness implies completeness S^Q is complete. Thus S^∞ is complete since any closed subspace of a complete space is complete.

The example can be taken one step further and consider the subfield S^Z of S^∞, consisting of all formal series of the form $x = a_1 t^{n_1} + a_2 t^{n_2} + \ldots$ where the sequence $n_1, n_2, \ldots$ is a simple sequence of integers increasing to ∞ and where $a_1, a_2, \ldots$ belong to Γ. Define $\|x\| = e^{-n_1}$ if $a_1 \neq 0$. We can then consider S^∞ as a linear space over S^Z. The norm on S^Z is discrete since the set $\{e^{-n}\}_{n \in Z}$ of norm values has no limit point but 0.

The space S^Z is complete. This results from the fact that S^Z is a closed subspace of the complete space S^∞ and hence is complete. The set S^Z is a closed subset of S^∞ as follows. Let x_0 be a limit point of S^Z. Since S^∞ is a metric space there exists a sequence $x_1, x_2, \ldots$ of elements in S^Z such that $\|x_0 - x_n\| < e^{-n}$. Suppose $x_0 \notin S^Z$. Let $x_0 = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \ldots$ and

$$x_n = a_{1,n} t^{\alpha_{1,n}} + a_{2,n} t^{\alpha_{2,n}} + \ldots .$$

Since $x_0 \in S^\infty \sim S^Z$ it must be that there exists α_K such that α_K is not an integer. But then the term $a_K t^K$ would necessarily appear in the difference $x_0 - x_K$. Thus $\|x_0 - x_K\| \geq e^{-K}$. But we had

$\|x_0 - x_K\| < e^{-K}$. This contradiction implies that $x_0 \varepsilon S^Z$ and hence S^Z is closed since it contains all its limit points.

It has been shown that S^Z is a complete normed linear space whose norm is discrete. By Theorem 2.16, S^Z is spherically complete.

To summarize, S^Q is a spherically complete space containing a closed subspace S^∞ which is not spherically complete. And the space S^∞ which is not spherically complete contains a closed subspace S^Z which is spherically complete.

CHAPTER IV

EQUIVALENT METRICS

Given a set E one can define various metrics on the set. Suppose that two metrics d and d' are defined on the set E. With each of the metrics is associated a collection of spheres. Each collection of spheres $B = \{S_d(x, r) \mid x \varepsilon E, \ r > 0\}$ and $B' = \{S_{d'}(x, r) \mid x \varepsilon E, \ r > 0\}$ is a base for a topology on E. From a topological standpoint it is of interest to see if the topologies induced on E by d and d' are the same. In particular, is a set which is open in (E, d) also open in (E, d') and conversely. If this is the case, d and d' are called equivalent or topologically equivalent metrics.

Definition 4.1. Let E be a set. Let d and d' be two metrics on E. The metrics d and d' are called topologically equivalent metrics for E if they determine the same topology on E.

In this paper, the term equivalent metric will be used. To prove that two metrics d and d' are equivalent it is sufficient to show that given any point $x \varepsilon E$ and any sphere $S_d(x, r)$ there exists an $r' > 0$ such that $S_{d'}(x, r') \subset S_d(x, r)$ and for any $x \varepsilon E$ and sphere $S_{d'}(x, r)$ there exists an $r' > 0$ such that $S_d(x, r') \subset S_{d'}(x, r)$. If this is true, since each of the two sets of spheres is a base for the respective topologies, any set open in one topological space will necessarily be open in the other. Thus the open sets are the same which means the topologies on E are the same.

In the preceding chapter it was noted that a norm on a linear space E induces a metric d on E. To illustrate the concept of equivalent metrics the first three examples of this chapter involve metrics induced on the same linear space by three different norms. Although the metrics are of different types, non-archimedean and archimedean, they are shown to be equivalent. The linear space involved is the space Q_p^2 where Q_p^2 denotes the set of all ordered pairs of p-adic numbers.

Example 4.1. Define $\|x\|_1 = \max\{|x_1|_p, |x_2|_p\}$ for $x = (x_1, x_2) \in Q_p^2$. This example is a special case of the space K^n given in Example 2.1 where $K = Q_p$ and $n = 2$. Recall that if K is a n.a. valued field then K^n with the norm $\|x\|_1 = \max_{1 \leq i \leq n} |x_i|$ is a n.a. normed linear space. Since Q_p is a n.a. valued field it follows that Q_p^2 is a n.a. normed linear space. Thus the induced metric d_1 on Q_p^2 is a n.a. metric.

In searching for an example of an archimedean metric on Q_p^2 a first step is to analyze the situation for R^2. The standard metric on R^2, considered as a linear space over R, is induced by the norm $\|x\| = \sqrt{x_1^2 + x_2^2}$ for $x = (x_1, x_2) \in R^2$. Thus $\|\ \|$ is indeed a mapping from R^2 into R, which it must be to be a norm on R^2. However if the same definition were used in Q_p^2, that is, $\|x\| = \sqrt{x_1^2 + x_2^2}$ for $x = (x_1, x_2) \in Q_p^2$ problems would be encountered immediately since this mapping is not defined for all $x \in Q_p^2$ and when it is defined its values are not necessarily in R.

In R^2 it is known that $\sqrt{x_1^2 + x_2^2} = \sqrt{|x_1|^2 + |x_2|^2}$ so this suggests that one might try the mapping $\|x\| = \sqrt{|x_1|_p^2 + |x_2|_p^2}$ which is certainly well-defined and is a mapping of Q_p^2 into R. However, it is easily shown that in R^2 the metric induced by the norm $\|x\| = |x_1| + |x_2|$ is equivalent to that induced by the norm $\|x\| = \sqrt{|x_1|^2 + |x_2|^2}$. The simpler mapping $\|x\| = |x_1|_p + |x_2|_p$ has been chosen as an example of an archimedean norm on Q_p^2.

Example 4.2. Define $\|x\|_2 = |x_1|_p + |x_2|_p$, $x = (x_1, x_2) \varepsilon Q_p^2$. First, $\| \ \|_2$ is a norm on Q_p^2 as follows. Clearly $\|x\|_2 \geq 0$ and $\|x\|_2 = 0$ if and only if $x = 0$. Let $\alpha \varepsilon Q_p$ and $x = (x_1, x_2) \varepsilon Q_p^2$. Then

$$\|\alpha x\|_2 = \|(\alpha x_1, \alpha x_2)\|_2 = |\alpha x_1|_p + |\alpha x_2|_p = |\alpha|_p(|x_1|_p + |x_2|_p)$$

$$= |\alpha|_p \|x\|_2 .$$

Finally, let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then

$$\|x+y\|_2 = \|(x_1 + y_1, x_2 + y_2)\|_2 = |x_1 + y_1|_p + |x_2 + y_2|_p$$

$$\leq (|x_1|_p + |y_1|_p) + (|x_2|_p + |y_2|_p)$$

$$= (|x_1|_p + |x_2|_p) + (|y_1|_p + |y_2|_p)$$

$$= \|x\|_2 + \|y\|_2 .$$

Next it is shown that the norm $\| \ \|_2$ is archimedean. To see this let $x = (1, 0)$ and $y = (0, 1)$. Then

$$\|x+y\|_2 = |x_1 + y_1|_p + |x_2 + y_2|_p = |1|_p + |1|_p = 2 .$$

But

$$\max(\|x\|_2, \|y\|_2) = \max(|1|_p + |0|_p, |0|_p + |1|_p) = \max(1,1) = 1.$$

Therefore, $\|x+y\|_2 > \max(\|x\|_2, \|y\|_2)$. Thus $\| \ \|_2$ is archimedean and the metric $d_2(x, y) = \|x - y\|_2$ is archimedean.

Before moving on to the next example let us make one other observation which will be of interest later in the chapter. Consider the sphere $S_{d_2}(0, \epsilon) = \{x \varepsilon Q_p^2 \mid d_2(x, 0) < \epsilon\}$ where ϵ is an arbitrary positive number. The metric d_2 of Example 4.2 is not n.a. in any such sphere, that is, for any $\epsilon > 0$. This can be seen as follows. For any $\epsilon > 0$ there exists an N such that $\frac{1}{p^N} < \epsilon$. Let $x = (p^N, 0)$ and $y = (0, p^N)$. Then $d_2(x, 0) = \|x - 0\| = \|x\|$ $= |p^N|_p + |0|_p = \frac{1}{p^N} < \epsilon$ and similarly $d_2(y, 0) = \frac{1}{p^N} < \epsilon$. Thus $0, x, y \varepsilon S_{d_2}(0, \epsilon)$ and yet

$$d_2(x, y) = \|x - y\| = \|(p^N, -p^N)\| = |p^N|_p + |-p^N|_p = \frac{1}{p^N} + \frac{1}{p^N}$$

$$= \frac{2}{p^N} > \frac{1}{p^N} = \max\{d_2(x, 0), d_2(0, y)\}.$$

Thus d_2 is not n.a. on $S_{d_2}(0, \epsilon)$ for any $\epsilon > 0$.

Example 4.3. Define $\|x\|_3 = \min\{|x_1|_p + |x_2|_p, 1\}$. This is a norm which has the same properties as the one in Example 4.2 if the sum $|x_1|_p + |x_2|_p$ is at most 1. Thus the norm $\| \ \|_3$ is archimedean so the induced metric d_3 is also archimedean. Let x, y, and z be three points of Q_p^2 such that the distance, with respect to d_2, between two pairs of points is ≥ 1. Then the distance, with respect to d_3, between the two pairs of points is 1. Therefore, the triangle determined by the three points is isosceles and the base is the shortest side. Thus the strong inequality is satisfied for these three points. This metric is then n.a. "in a large sense".

Although the metrics in Example 4.1 to 4.3 are of different types, they will now be shown to be equivalent. To do this it will be shown that the metrics d_1 and d_3 are each equivalent to d_2. First consider d_1 and d_2. Let $S_{d_1}[y, r]$ be given where $y = (y_1, y_2)$. Let $x = (x_1, x_2) \varepsilon S_{d_2}[y, r]$. Then

$$\|x - y\|_2 = \|(x_1 - y_1, x_2 - y_2)\|_2 = |x_1 - y_1|_p + |x_2 - y_2|_p \leq r .$$

But then

$$\|x - y\|_1 = \max\{|x_1 - y_1|_p, |x_2 - y_2|_p\} \leq |x_1 - y_1|_p + |x_2 - y_2|_p \leq r$$

so $x \varepsilon S_{d_1}[y, r]$. Thus $S_{d_2}[y, r] \subset S_{d_1}[y, r]$ so any d_1-open set is d_2-open. Now let $S_{d_2}[y, r]$ be given. Let $x = (x_1, x_2) \varepsilon S_{d_1}[y, \frac{r}{2}]$. Then $\|x - y\|_1 = \max\{|x_1 - y_1|_p, |x_2 - y_2|_p\} \leq \frac{r}{2}$ so $|x_1 - y_1|_p \leq \frac{r}{2}$ and $|x_2 - y_2|_p \leq \frac{r}{2}$. But then

$$\|x - y\|_2 = |x_1 - y_1|_p + |x_2 - y_2|_p \leq \frac{r}{2} + \frac{r}{2} = r$$

so $x \varepsilon S_{d_2}[y, r]$. Thus $S_{d_1}[y, \frac{r}{2}] \subset S_{d_2}[y, r]$ so any d_2-open set is d_1-open. Hence d_1 and d_2 are equivalent.

Now it will be shown that d_3 and d_2 are equivalent. Let $S_{d_3}[y, r]$ be given where $y = (y_1, y_2)$. Let $x \varepsilon S_{d_2}[y, r]$. Then

$$\|x - y\|_3 = \min\{|x_1 - y_1|_p + |x_2 - y_2|_p, 1\}$$

$$\leq |x_1 - y_1|_p + |x_2 - y_2|_p \leq r$$

so that $x \varepsilon S_{d_3}[y, r]$. Thus $S_{d_2}[y, r] \subset S_{d_3}[y, r]$ so any d_3-open set is d_2-open. Now let $S_{d_2}[y, r]$ be given. Let $r' < \min\{1, r\}$. Let $x \varepsilon S_{d_3}[y, r']$. Then

$$\|x - y\|_3 = \min\{|x_1 - y_1|_p + |x_2 - y_2|_p, 1\}$$

$$= |x_1 - y_1|_p + |x_2 - y_2|_p$$

$$= \|x - y\|_2 .$$

But $x \varepsilon S_{d_3}[y, r']$ implies that $\|x - y\|_3 \leq r' < r$. Thus $\|x - y\|_2 < r$ so $x \varepsilon S_{d_2}[y, r]$. Hence $S_{d_3}[y, r'] \subset S_{d_2}[y, r]$ so every d_2-open set is d_3-open. It follows that d_2 and d_3 are equivalent.

It has been shown that d_1 and d_3 are each equivalent to d_2. Since equivalence of metrics is an equivalence relation this implies d_1 is also equivalent to d_3.

Locally Non-Archimedean Metrics

In Example 4.2 (and 4.3) there exists at least one point which had no neighborhood in which the metric was n.a. However, it is possible for a metric d to be archimedean and yet have the property that every point has a neighborhood in which d is non-archimedean.

Definition 4.2. The metric d is called locally non-archimedean on the space E, if for each point $a \varepsilon E$ there exists a neighborhood U of a such that for any $x, y, z \varepsilon U$, $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

The following example gives a construction of an archimedean metric which is locally n. a.

<u>Example 4.4.</u> Let $E = 0_3$, the set of 3-adic integers, with the n. a. metric $d_3(x, y) = |x - y|_3$. Consider the three disjoint spheres of radius $\frac{1}{3}$, $X = S[0, \frac{1}{3}]$, $Y = S[1, \frac{1}{3}]$ and $Z = S[2, \frac{1}{3}]$. Let $w \varepsilon 0_3$, where $w = \sum_{i=0}^{\infty} a_i 3^i$. Then $a_0 = 0, 1,$ or 2 and $w \varepsilon X$ if and only if $a_0 = 0$, $w \varepsilon Y$ if and only if $a_0 = 1$, and $w \varepsilon Z$ if and only if $a_0 = 2$. Thus $X \cup Y \cup Z = 0_3$ and they are disjoint. Furthermore, X, Y, and Z are each open and closed. For example, $S[1, \frac{1}{3}] = S(1, 1)$.

Now define a new metric d by

$$d(x_1, x_2) = \frac{d_3(x_1, x_2)}{1 + d_3(x_1, x_2)} \quad \text{for} \quad x_1, x_2 \varepsilon X$$

and similarly for Y and Z. Also define:

$$d(x, y) = 1, \quad x \varepsilon X, \ y \varepsilon Y,$$

$$d(y, z) = 1, \quad y \varepsilon Y, \ z \varepsilon Z, \ \text{and}$$

$$d(x, z) = 2, \quad x \varepsilon X, \ z \varepsilon Z.$$

One can show that d is a metric. However, if $x \varepsilon X$, $y \varepsilon Y$, and $z \varepsilon Z$, we have $2 = d(x, z) > \max\{d(x, y), d(y, z)\} = 1$ so that d is archimedean. Even though d is not n. a. it is still locally n. a. as the following argument shows.

Let $x_1, x_2, x_3 \varepsilon X$. (An analogous proof holds for Y and Z). Then,

$$\max\{d(x_1, x_3), d(x_3, x_2)\} = \max\left\{\frac{d_3(x_1, x_3)}{1+d_3(x_1, x_3)}, \frac{d_3(x_3, x_2)}{1+d_3(x_3, x_2)}\right\}$$

$$\geq \max\left\{\frac{d_3(x_1, x_3)}{1+\max[d_3(x_1, x_3), d_3(x_3, x_2)]}, \frac{d_3(x_3, x_2)}{1+\max[d_3(x_3, x_2), d_3(x_1, x_3)]}\right\}$$

$$= \frac{\max\{d_3(x_1, x_3), d_3(x_3, x_2)\}}{1+\max\{d_3(x_1, x_3), d_3(x_3, x_2)\}}$$

$$= \frac{1}{1+\dfrac{1}{\max\{d_3(x_1, x_3), d_3(x_3, x_2)\}}} .$$

Since $d_3(x_1, x_2) \leq \max\{d_3(x_1, x_3), d_3(x_3, x_2)\}$,

$$\frac{1}{1+d_3(x_1, x_2)} \geq \frac{1}{1+\max\{d_3(x_1, x_3), d_3(x_3, x_2)\}}$$

and so

$$\max\{d(x_1, x_3), d(x_3, x_2)\} \geq \frac{1}{1+\dfrac{1}{\max\{d_3(x_1, x_3), d_3(x_3, x_2)\}}}$$

$$\geq \frac{1}{1+\dfrac{1}{d_3(x_1, x_2)}}$$

$$= \frac{d_3(x_1, x_2)}{1+d_3(x_1, x_2)}$$

$$= d(x_1, x_2) .$$

Thus for $x_1, x_2, x_3 \in X$ we have $d(x_1, x_2) \leq \max\{d(x_1, x_3), d(x_3, x_2)\}$.

Any point $p \in E$ must be in one of the sets X, Y, or Z, say X, and there exists a neighborhood U of p such that $p \in U \subset X$, since X is open. Since d is then n.a. on U, it follows that d is locally n.a. Thus d is an example of an archimedean metric which is locally n.a.

The question arises, if d is archimedean (n.a.) on two subsets is d archimedean (n.a.) on their intersection? In this example, if d is n.a. in a neighborhood U of a point p, say $p \in X$, then $U \subset X$. Two such neighborhoods intersect only if both are contained in the same subset X, Y, or Z on which d is n.a. Given neighborhoods U and V of p and q respectively on which d is n.a., if they intersect, then one is a subset of the other. Of course d will also be n.a. on the intersection or union since either is a subset of X (or Y or Z).

On the other hand, for $x \in X = S_d[0, \frac{1}{3}]$ and $z \in Z = S_d[2, \frac{1}{3}]$, $S_d(x, 1\frac{1}{2}) = X \cup Y$ and $S_d(z, 1\frac{1}{2}) = Z \cup Y$, where $Y = S_d[1, \frac{1}{3}]$ so that d is archimedean on both spheres. But $S_d(x, 1\frac{1}{2}) \cap S_d(z, 1\frac{1}{2}) = Y$ on which d is n.a. Thus d may be n.a. on the intersection of two spheres on which d is archimedean.

Existence of Equivalent Metrics

The non-archimedean metric d_1 and the archimedean metric d_2 of Examples 4.1 and 4.2 were shown to be equivalent. In these examples the metrics were induced on the linear space Q_p^2 by norms on Q_p^2. For the remainder of Chapter IV the metric d, unless otherwise specified, will be an arbitrary metric defined on the set under consideration. In the following paragraphs several interesting theorems concerning the equivalence of various types of metrics will be proven. A rather startling result is found in the following theorem.

Theorem 4.1. For each space E of at least three points with a n.a. metric there exists an equivalent archimedean metric for E.

Proof: A metric on a space E with less than three points is necessarily n.a. and hence cannot be archimedean. Therefore, let E be a space consisting of at least three points and let d be a n.a. metric on E. Let x, y, and z be distinct points of E. Let $r < \min\{d(x,y), d(y,z), d(x,z)\}$. Then $X = S[x,r]$ and $Y = S[y,r]$ are disjoint open and closed subsets which do not contain z. Thus $Z = E \sim (X \cup Y)$ is a neighborhood of z which is both open and closed and $E = X \cup Y \cup Z$. Moreover, X, Y, and Z are disjoint.

A new metric d' is then introduced on E by a method illustrated in Example 4.4. Define

$$d'(x_1,x_2) = \frac{d(x_1,x_2)}{1+d(x_1,x_2)} \quad \text{for} \quad x_1, x_2 \in X.$$

Similarly, define d' on Y and Z respectively. Also define

$$d'(x,y) = 1 \quad \text{for} \quad x \in X,\ y \in Y,$$

$$d'(y,z) = 1 \quad \text{for} \quad y \in Y,\ z \in Z \quad \text{and}$$

$$d'(x,z) = 2 \quad \text{for} \quad x \in X,\ z \in Z.$$

Just as in the proof in Example 4.4, the metric d' can be shown to be archimedean but locally n.a. Moreover, d' and d are equivalent as the following argument shows. Let $x \in E$, and without loss of generality assume $x \in X$. Since X is open, there exists a neighborhood U of x contained in X. Thus, for any points x_1 and x_2 in U, $d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}$ and this metric is known to be equivalent to d. The equivalence results from the fact that $S_d[x, r] = S_{d'}[x, \frac{r}{1+r}]$. Thus, given any n. a. metric on a set E, there exists an equivalent archimedean metric for E.

Before proceeding with the discussion of other equivalences, it would be well to ponder for a moment the significance of this theorem with respect to the discussion in Chapter III of fundamental topological properties of n. a. normed linear spaces. This theorem tells us that given any n. a. normed linear space E, with the metric d induced by the norm, an equivalent archimedean metric d' exists on E. This means that the topologies on the spaces (E, d) and (E, d') are the same. Thus the fundamental topological properties, depending only on the open sets, must be the same. It then becomes apparent that the archimedean metric space (E, d') is an example of a topological space which shares the same fundamental topological properties as the n. a. normed linear space (E, d). Thus the condition that the space be a n. a. normed linear space or n. a. metric space is sufficient to insure the fundamental topological properties exhibited in Chapter III but it is not a necessary condition. However, it should be noted that since the two spaces have the same open sets they also have the same closed sets and hence each has a base consisting of sets that are both open and closed; that is, each space is a 0-dimensional metric space. This points up again the fact that basic to the fundamental topological properties of a n.a. normed linear space E is the 0-dimensionality of E, along with the fact that E is a metric space.

The following theorem gives another equivalence between two different types of metrics. In order to prove this theorem, the following lemma is needed.

Lemma 4.2. If d is n.a. on $S(a, r)$ and $r_1 < r$, then $S[a, r_1]$ is both an open and a closed set.

Proof: Let $y \in S[a, r_1]$ and $r_2 < \min\{r_1, r-r_1\}$. It will be shown that $S(y, r_2) \subset S[a, r_1]$. Let $x \in S(y, r_2)$ then $d(a, x) \leq d(a, y) + d(y, x) \leq r_1 + r_2$. Since $r_2 < \min\{r_1, r-r_1\}$ we have $r_2 < r - r_1$ so $d(a, x) \leq r_1 + r_2 < r_1 + r - r_1 = r$. Thus $x \in S(a, r)$. Therefore x, y and a are all contained in $S(a, r)$ on which d is n.a. The strong inequality then applies and $d(a, x) \leq \max\{d(a, y), d(y, x)\} \leq \max\{r_1, r_2\} = r_1$ since $r_2 < \min\{r_1, r-r_1\}$. Since $d(a, x) \leq r_1$, $x \in S[a, r_1]$. We have then $S(y, r_2) \subset S[a, r_1]$, where y is an arbitrary point of $S[a, r_1]$, so $S[a, r_1]$ is an open set. Since any closed sphere is a closed set the proof is complete.

Theorem 4.3. Let d be a locally n.a. metric on the separable space E. Then there exists an equivalent n.a. metric d' on E.

Proof: Let $\epsilon > 0$. Let $D = \{x_i\}_{i=1}^{\infty}$ be a countable dense subset of E. With each $x_i \in D$, associate the following collection of spheres: $A_i = \{S[x_i, r] \mid r \leq \frac{\epsilon}{2}$, r is rational, d is n.a. on $S[x_i, r]$ and $S[x_i, r]$ is both open and closed$\}$, for $i = 1, 2, \ldots$. Since d is locally n.a. there exists an r such that d is n.a. on $S(x_i, r)$.

Moreover, Lemma 4.2 shows that for $r_1 < r$, $S[x_i, r_1]$ is both open and closed. Thus the collection A_i is non-empty for each $n = 1, 2, \ldots$. Since the set of rational numbers is countable, each A_i is countable. Let $A = \bigcup_{i=1}^{\infty} A_i$. The set A is countable being the union of a countable collection of countable sets and hence we can rename its elements $V_1, V_2, \ldots$. Since each V_n is one of the spheres $S[x_i, r]$ in A, each set V_n is both open and closed, the diameter of each is $\leq \epsilon$ and d is n.a. on each V_n. Now define the sequence $U_1, U_2, \ldots$ as follows.

$$U_1 = V_1, \; U_2 = V_2 - V_1, \ldots, U_n = V_n - \bigcup_{i=1}^{n-1} V_i, \ldots .$$

Then $C = \{U_i\}_{i=1}^{\infty}$ is a collection of disjoint open and closed sets. Since each $U_n \subset V_n$, each U_n is also of diameter not more than ϵ and d is n.a. on each U_n.

The family C is a cover of E as the following argument shows. Let $x \varepsilon E$ and select $r \leq \frac{\epsilon}{2}$ such that r is rational and d is n.a. on $S[x, r]$. Since D is dense in E, there exists an $x_1 \varepsilon S[x, \frac{r}{2}]$. Thus $d(x, x_1) \leq \frac{r}{2}$ so that $x \varepsilon S[x_1, \frac{r}{2}] \subset S[x, r]$. The latter containment holds since if $y \varepsilon S[x_1, \frac{r}{2}]$, then $d(x, y) \leq d(x, x_1) + d(x_1, y) \leq \frac{r}{2} + \frac{r}{2} = r$, so that $y \varepsilon S[x, r]$. Thus d is n.a. on $S[x_1, \frac{r}{2}]$ and $S[x_1, \frac{r}{2}] = V_j$ for some $j = 1, 2, \ldots$. We have then $x \varepsilon V_j$ for some j which implies $x \varepsilon U_k$ for some $k \leq j$. Thus C is a cover of E.

Now define a new metric d' by:

$d'(x, y) = d(x, y)$ if x and y are contained in the same U_i;

$d'(x, y) = \epsilon$, if x and y are contained in different U_i's.

The following discussion proves that d' is n.a.

For any three points x, y and z in E there are three possibilities.

(a) All three points are contained in the same U_i. In this case, since $d = d'$ on U_i and d is n.a. the strong inequality holds.

(b) Two points, say x and y, are in U_i and $z \varepsilon U_j$, $i \neq j$. Thus $d'(x, z) = d'(y, z) = \epsilon$ and $d'(x, y) = d(x, y) \leq \epsilon$. Hence the maximum of any pair of distances is ϵ and the third distance is certainly $\leq \epsilon$.

(c) No two points are contained in the same U_i. In this case $d'(x, y) = d'(y, z) = d'(x, z) = \epsilon$ and the desired result follows. Thus d' is a n.a. metric on E.

That metrics d and d' are equivalent can be demonstrated as follows. Let $x \varepsilon E$ and $S_d[x, r]$ be given. As a result of Lemma 4.2, since d is locally n.a. it may be assumed $S_d[x, r]$ is d-open. We have $x \varepsilon U_i$ for some i and hence $x \varepsilon U_i \cap S_d[x, r]$ which is d-open. Thus there exists a sphere $S_d[x, r'] \subset U_i \cap S_d[x, r]$. But then $d = d'$ on this sphere so that $S_{d'}[x, r'] \subset S_d[x, r]$. On the other hand given $x \varepsilon E$ and sphere $S_{d'}[x, r]$, let $r' < \max\{r, \epsilon\}$. To show that $S_d[x, r'] \subset S_{d'}[x, r]$, let $y \varepsilon S_d[x, r']$. Then $d(x, y) \leq r' < \epsilon$ so that x and y are in the same U_i. Thus $S_d[x, r'] \subset U_i$ and since $d' = d$ on U_i and $r' < r$, we have that $S_d[x, r'] = S_{d'}[x, r'] \subset S_{d'}[x, r]$.

Thus it has been shown that d' is a n.a. metric equivalent to the locally n.a. metric d.

As was pointed out in Chapter III, every n.a. metric space is 0-dimensional. Using Lemma 4.2 it can now be shown that every locally n.a. metric space is 0-dimensional. Although this theorem is really a corollary to the proof of the previous theorem, the short argument is repeated here.

Theorem 4.4. Every locally n.a. metric space E is 0-dimensional.

Proof: Let $a \in E$ and $r > 0$. The sphere $S(a, r)$ is then an arbitrary open sphere in E. Since E is locally n.a. there exists an $r_1 < r$ such that d is n.a. on $S(a, r_1)$. Then for $r_2 < r_1$, by Lemma 4.2, $S[a, r_2]$ is an open and closed set. But $r_2 < r_1 < r$ implies that $S[a, r_2] \subset S(a, r)$. Since $S(a, r)$ was an arbitrary base element, E is 0-dimensional.

A locally n.a. metric satisfies the strong inequality (1), if the triangles are "sufficiently small". Example 4.3 demonstrated a metric which is n.a. "in a large sense", that is, which satisfies the strong inequality if the points are sufficiently far apart. In the following example the metric is shown to have the property that it is not n.a. in any neighborhood of any point and yet is n.a. "in a large sense".

Example 4.5. Consider the Euclidean space R^2 with the usual metric $d(x, y) = \{(x_1 - y_1)^2 + (x_2 - y_2)^2\}^{1/2}$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Define a new metric d' as follows:

$$d'(x, y) = d(x, y), \quad \text{if } d(x, y) \leq 1;$$

$$d'(x, y) = 1, \quad \text{if } d(x, y) > 1.$$

That d' is a metric can be verified as follows. The properties $d'(x, y) \geq 0$, $d'(x, y) = 0$ if and only if $x = y$, and $d'(x, y) = d'(y, x)$ follow immediately from the definition of d' and the corresponding properties of d. To prove the triangle inequality there are four cases.

(i) If $d(x, y)$, $d(y, z)$, and $d(x, z) \leq 1$, then $d'(x, y) = d(x, y)$, $d'(y, z) = d(y, z)$, and $d'(x, z) = d(x, z)$ so the triangle inequality holds.

(ii) If $d(x, y) > 1$, $d(x, z) \leq 1$, and $d(y, z) \leq 1$, then $1 < d(x, y) \leq d(x, z) + d(y, z)$ so $d'(x, y) = 1 < d(x, z) + d(y, z) = d'(x, z) + d'(y, z)$. Clearly $d'(x, z) \leq 1 = d'(x, y) \leq d'(x, y) + d'(x, z)$ and similarly for $d'(y, z)$.

(iii) If $d(x, y) > 1$, $d(y, z) > 1$, and $d(x, z) \leq 1$, then $d'(x, y) = 1 = d'(y, z)$ and $d'(x, z) = d(x, z) \leq 1$. This case is clear.

(iv) If $d(x, y) > 1$, $d(y, z) > 1$, and $d(x, z) > 1$, then $d'(x, y) = d'(y, z) = d'(x, z) = 1$.

Thus the triangle inequality holds in any case and d' is a metric.

Let $x \varepsilon R^2$, then $S_d(x, \frac{1}{2})$ is a neighborhood of x in which d and d' are identical. Thus d and d' are equivalent. But the metric d is not n. a. in any neighborhood of any point. This follows since given any open set containing x there exists a sphere $S_d(x, r)$ containing x and this sphere contains a right triangle for which, in R^2, the hypotenuse is longer than either leg. Thus d is not n.a. in any neighborhood of any point and since d and d' are identical in $S_d(x, \frac{1}{2})$ for any x, then d' also has this property.

However, suppose three points x, y, and z determine a triangle in which two of the sides have lengths ≥ 1 with respect to d. Then these sides have length 1 with respect to d' and it is clear that $d'(x, y) \leq \max\{d'(x, z), d'(y, z)\} = 1$. Thus any triangle that is "sufficiently large" satisfies the strong inequality so that d', like the metric in Example 4.3, is n.a. "in a large sense".

In the proof of Theorem 4.1 and in Example 4.4 the archimedean metric d', which is equivalent to the given n.a. metric d, is locally n.a. This suggests the following question: if d is n.a. metric on a space E, does there exist an equivalent archimedean metric that is <u>not</u> locally n.a.? This question is answered affirmatively if E is a separable space as the following example and theorem show.

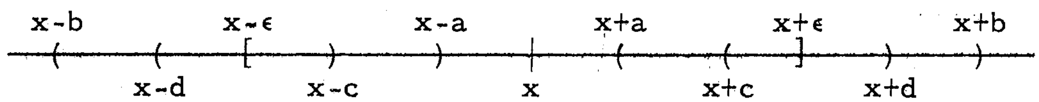
The Cantor ternary set C furnishes an example of a separable, 0-dimensional space which is familiar to most graduate students in mathematics. Of special interest here is the fact that one can define on C equivalent metrics, one n.a. and the other archimedean. Moreover, the archimedean metric has the property that <u>no</u> point has a neighborhood in which the metric is n.a. This fact then leads to the proof of the next theorem. Before stating this theorem, let us consider the Cantor set in some detail and verify the properties which it possesses.

Example 4.6. The Cantor ternary set. Recall that the Cantor ternary set is defined to be the subset C of the interval $[0, 1]$ consisting of all numbers x whose ternary expansion contains no 1's. That is, $x \in C$ implies $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} = .a_1 a_2 \ldots$ where $a_i = 0$ or 2. With the ordinary metric, $d(x, y) = |x - y|$, that is the absolute value, the set C with the relative topology is separable and 0-dimensional.

(C, d) is separable. To prove that (C, d) is separable it will be shown that the set $D = \{ \sum_{i=1}^{n} \frac{C_i}{3^i} \mid C_i = 0 \text{ or } 2,\ n \geq 1 \text{ an integer}\}$ is dense in C. To see this let $x \in C$. Then $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$, $a_i = 0$ or 2. Considering C as a subspace of $[0, 1]$, let (a, b) be any open interval containing x. Then there exists an integer k such that $\left(x - \frac{1}{3^k}, x + \frac{1}{3^k}\right) \subset (a, b)$. Let $y = \sum_{i=1}^{k} \frac{a_i}{3^i}$, then $y \in D$ and $|x - y| = \sum_{i=k+1}^{\infty} \frac{a_i}{3^i} < \frac{1}{3^k}$ so that $y \in \left(x - \frac{1}{3^k}, x + \frac{1}{3^k}\right) \subset (a, b)$. Thus x is a limit point of D and D is dense in C.

(C, d) is 0-dimensional. Let $x \in C$. It is known that C is nowhere dense in R, that is, between any two distinct points of C there is an open subinterval of $[0, 1] \sim C$. Let U be any open set of (C, d) containing x. It will be shown that there is a subset of U containing x which is both open and closed. Since U is open, there exists an r_1 such that $S(x, r_1) \subset U$. Moreover, in the reals there exists a subinterval $(x + a, x + b)$ of $(x, x + r_1)$ containing no points of C, since C is nowhere dense in R. Consider $(x-a, x-b)$ which is contained in $(x-r_1, x)$. Since C is nowhere dense in R, there is a subinterval $(x-d, x-c)$ of $(x-b, x-a)$ containing no points of C. It follows that $(x+c, x+d) \subset (x+a, x+b)$ and neither $(x+c, x+d)$ nor $(x-d, x-c)$ contains any point of C. Pick ϵ such that $c < \epsilon < d$.

Then $x+\epsilon \in (x+c, x+d)$ and $x-\epsilon \in (x-d, x-c)$ so that neither $x+\epsilon$ nor $x-\epsilon$ is contained in C.



Now consider $S(x, \epsilon)$. The preceding argument shows that $S(x, \epsilon) \subset U$ and the boundary of $S(x, \epsilon)$, with respect to C, is empty since $S[x, \epsilon] = S(x, \epsilon)$ in C. Thus C is 0-dimensional since for any point $x \in C$ and, for any neighborhood U of x, $V = S(x, \epsilon)$ is a neighborhood of x, with empty boundary, such that $V \subset U$.

<u>There is no point of (C, d) having a neighborhood U in which d is n.a.</u> Let $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$ be an arbitrary <u>fixed</u> point of C. It will first be shown that the collection $\{U_n\}_{n=1}^{\infty}$ where $U_n = \{ \sum_{i=1}^{n} \frac{a_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{\epsilon_i}{3^i} \mid \epsilon_i = 0, 2\}$ is a neighborhood base at x. To do this it will be shown that for each $n = 1, 2, \ldots, U_n = S_d(x, \frac{1}{3^n})$. The set of all such spheres is clearly a neighborhood base at x. To prove $U_n = S_d(x, \frac{1}{3^n})$, let $y \in U_n$, then $y = \sum_{i=1}^{n} \frac{a_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{\epsilon_i}{3^i}$, $\epsilon_i = 0, 2$. Thus

$$|x - y| = \left| \sum_{i=1}^{\infty} \frac{a_i}{3^i} - \left\{ \sum_{i=1}^{n} \frac{a_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{\epsilon_i}{3^i} \right\} \right| = \left| \sum_{i=n+1}^{\infty} \frac{a_i}{3^i} - \sum_{i=n+1}^{\infty} \frac{\epsilon_i}{3^i} \right|$$

$$= \left| \sum_{i=n+1}^{\infty} \frac{a_i - \epsilon_i}{3^i} \right| \leq \sum_{i=n+1}^{\infty} \frac{|a_i - \epsilon_i|}{3^i} \leq \sum_{i=n+1}^{\infty} \frac{2}{3^i} < \frac{1}{3^n} .$$

Note that a_i, $\epsilon_i = 0, 2$ implies $|a_i - \epsilon_i| = 0$ or 2. We have shown that $U_n \subset S_n(x, \frac{1}{3^n})$. Now let $y \varepsilon S_d(x, \frac{1}{3^n})$. Then $y = \sum_{i=1}^{\infty} \frac{b_i}{3^i}$, $b_i = 0, 2$. Suppose $y \notin U_n$. Then $a_i \neq b_i$ for some $i, 1 \leq i \leq n$. Let k be the smallest such index. Then $a_k - b_k = \pm 2$ so that $\frac{a_k - b_k}{3^k} = \frac{2}{3^k} \geq \frac{2}{3^n}$ or $\frac{a_k - b_k}{3^k} = -\frac{2}{3^k} \leq -\frac{2}{3^n}$. But since $-2 \leq a_i - b_i \leq 2$, $i = k+1, k+2, \ldots,$ we have

$$-\frac{1}{3^n} \leq -\frac{1}{3^k} < \sum_{i=k+1}^{\infty} -\frac{2}{3^i} \leq \sum_{i=k+1}^{\infty} \frac{a_i - b_i}{3^i} \leq \sum_{i=k+1}^{\infty} \frac{2}{3^i} < \frac{1}{3^k} \leq \frac{1}{3^n}.$$

Therefore

$$\frac{a_k - b_k}{3^k} + \sum_{i=k+1}^{\infty} \frac{a_i - b_i}{3^i} > \frac{2}{3^n} - \frac{1}{3^n} = \frac{1}{3^n}$$

or

$$\frac{a_k - b_k}{3^k} + \sum_{i=k+1}^{\infty} \frac{a_i - b_i}{3^i} < -\frac{2}{3^n} + \frac{1}{3^n} = -\frac{1}{3^n}.$$

But then

$$|x - y| = \left| \frac{a_k - b_k}{3^k} + \sum_{i=k+1}^{\infty} \frac{a_i - b_i}{3^i} \right| > \frac{1}{3^n}$$

which is a contradiction since it was assumed $y \varepsilon S_d(x, \frac{1}{3^n})$. Therefore, $a_i = b_i$, $i = 1, 2, \ldots$ so $y \varepsilon U_n$. This implies $S_d(x, \frac{1}{3^n}) \subset U_n$ so $U_n = S_d(x, \frac{1}{3^n})$. The collection $\{U_n\}_{n=1}^{\infty}$ is then a neighborhood base at x.

Suppose there exists a neighborhood U of $x \varepsilon C$ such that d is n.a. on U. Then there is an N such that d is n.a. on $U_N \subset U$.

Let $y = \sum_{i=1}^{N} \frac{a_i}{3^i}$, $z = \sum_{i=1}^{N} \frac{a_i}{3^i} + \frac{2}{3^{N+1}}$ and $w = \sum_{i=1}^{N} \frac{a_i}{3^i} + \frac{2}{3^{N+2}}$. Then y, z, and w are in U_N so,

$$\frac{2}{3^{N+1}} = |y - z| \leq \max\{|y - w|, |z - w|\}$$

$$= \max\{\frac{2}{3^{N+2}}, \frac{2}{3^{N+1}} - \frac{2}{3^{N+2}}\} = \frac{2}{3^{N+2}} .$$

This contradiction implies that d is not n.a. in U. Hence there is no point $x \varepsilon C$ having a neighborhood U in which d is n.a.

<u>There is a n.a. metric d' which is equivalent to d</u>. Define the new metric d' on C as follows. Let $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$ and $y = \sum_{i=1}^{\infty} \frac{b_i}{3^i}$ be points of C. Then define

$d'(x, y) = 0$ if $x = y$, and

$d'(x, y) = \frac{1}{n}$, if $x \neq y$ and n is the first index for which $a_i \neq b_i$.

Clearly $d'(x, y) \geq 0$, $d'(x, y) = 0$ if and only if $x = y$, and $d'(x, y) = d'(y, x)$. To prove that the strong inequality is satisfied, let $z = \sum_{i=1}^{\infty} \frac{C_i}{3^i}$ and x and y as above. Let $d'(x, y) = \frac{1}{n_1}$, $d'(y, z) = \frac{1}{n_2}$ and $d'(x, z) = \frac{1}{n_3}$. Suppose $\frac{1}{n_1} > \max\{\frac{1}{n_2}, \frac{1}{n_3}\}$, then $n_1 < n_2$ and $n_1 < n_3$. Also $a_{n_1} \neq b_{n_1}$ since $d'(x, y) = \frac{1}{n_1}$. But $d'(y, z) = \frac{1}{n_2}$ and $n_1 < n_2$ implies $b_{n_1} = c_{n_1}$. And $d'(x, z) = \frac{1}{n_3}$ and $n_1 < n_3$ implies $a_{n_1} = c_{n_1}$ so that $a_{n_1} = b_{n_1}$. This contradiction implies that $d'(x, y) \leq \max\{d'(y, z), d'(x, z)\}$ so d' is a n.a. metric.

As before define

$$U_n = \left\{ \sum_{i=1}^{n} \frac{a_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{\epsilon_i}{3^i} \mid \epsilon_i = 0, 2 \right\} .$$

Then, if $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$, U_n is a neighborhood of x. In fact, $U_n = S_{d'}(x, \frac{1}{n})$. This latter follows since $y = \sum_{i=1}^{n} \frac{a_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{\epsilon_i}{3^i}$ if and only if $d'(x, y) < \frac{1}{n}$. Thus for any $x \varepsilon E$, $S_d(x, \frac{1}{3^n}) = S_{d'}(x, \frac{1}{n})$ so that the metrics d and d' are equivalent. The following theorem summarizes the preceding discussion.

Theorem 4.5. There exist equivalent metrics d and d' on the Cantor ternary set C such that d' is n.a. and d is an archimedean metric for which no point of C has a neighborhood in which d is n.a.

This theorem is a special case of the more general theorem which is stated without proof.

Theorem 4.6. On any separable, 0-dimensional space, equivalent metrics d and d' can be defined such that d' is n.a. and d is an archimedean metric for which no point has a neighborhood in which d is n.a.

It should be noted that even though a metric is not locally n.a. there may still be neighborhoods in which the metric is n.a. At most one can say that there is at least one point which has no neighborhood in which the metric is n.a. In the proof of Theorem 4.5 we actually saw an archimedean metric d for which no point had a neighborhood in which d was n.a. and yet d was equivalent to a n.a. metric d'.

Metrics on a Field

It has been demonstrated that a n.a. metric may be equivalent to an archimedean metric. In fact, by Theorem 4.1, given any space E with a n.a. metric, there exists an equivalent archimedean metric.

Suppose that the space is a field $E = K$ with a topology induced by a valuation. Recall that if $|x|$, $x \varepsilon K$, denotes a valuation on K, the relation $d(x, y) = |x - y|$ defines a metric on K that is n.a. if and only if the valuation is n.a. While it is possible for two metrics on K, one archimedean and one n.a., to be equivalent, the following theorem shows that if they are equivalent they cannot both be induced by valuations on K.

Theorem 4.7. Let K be a metric field with d' and d'' equivalent metrics on K. Suppose that d' is n.a. and d'' is archimedean. Then d' and d'' are not both induced by valuations on K.

Proof: Every field K contains a subfield Q isomorphic to the field of rational numbers. The metrics d' and d'' induce metrics on Q. If a metric is induced by a valuation on K, then it induces a valuation on Q. Suppose d' is induced by n.a. valuation on K. Then Q has an induced n.a. valuation. However, it is known that every non-trivial n.a. valuation on Q is equivalent to one of the p-adic valuations. See Palmer [17, p. 46]. Ostrowski's theorem states that the only non-trivial valuations on Q are those equivalent to a p-adic valuation $|\ |_p$ or the ordinary absolute value $|\ |$. Thus the valuation induced on Q by d' is equivalent to $|\ |_p$ and the valuation induced on Q by d'' is equivalent to $|\ |$. However, the valuations $|\ |_p$ and $|\ |$ are not equivalent. Thue d' and d'' cannot both have been induced by a valuation on K.

As an illustration of the previous theorem, the p-adic valuation, $|\ |_p$, on the field Q_p of p-adic numbers induces a n.a. metric d_p on the set Q_p. It has been observed that this space (Q_p, d_p) is separable and 0-dimensional and thus by Theorem 4.6 an equivalent archimedean metric d' can be defined on Q_p. Since d_p was induced by the n.a. valuation $| \ |_p$, it is impossible for d' to have been induced by a valuation on Q_p.

CHAPTER V

CONVEXITY

The discussion of linear spaces often leads to the topic of convexity. In the preceding chapters some of the properties of linear spaces over n.a. valued fields have been discussed. At this time convexity in a non-archimedean setting will be investigated.

Most of the published work on convexity in the n.a. case has been done by Monna. Convexity is a starting point for the study of locally convex spaces over K. This study will not be pursued but convexity in the n.a. case is of sufficient interest to warrant some attention on its own merit.

In this chapter convexity in E will be defined and several resulting properties of convex sets will be observed. A characterization of convex sets in a n.a. valued field K, considered as a linear space over itself, will be given. This chapter is intended only to introduce the concept of convexity in the n.a. setting, examine a few of its properties, and remark briefly on some of the problems involved with convexity in linear spaces over n.a. valued fields. For further discussion on convexity the articles by Monna are the best current source. See [11] and [13].

Convexity is usually defined for linear spaces over R, the field of real numbers. In this situation a set A is said to be convex if for any $x, y \in A$ and $a \geq 0$, $b \geq 0$ in R, such that $a + b = 1$, the point $ax + by \in A$. A similar definition cannot be used for a linear space E over a n.a. valued field K since the field K is not ordered, so that statements such as $a \geq 0$ are meaningless. Thus our definition of convexity must be independent of order on K.

Throughout this chapter, $\mathfrak{G}$ will denote the ring of integers of K; that is, $\mathfrak{G} = \{\lambda \in K \mid |\lambda| \leq 1\}$. In the case $K = Q_p$, then $\mathfrak{G} = 0_p$. The proof that $\mathfrak{G}$ is an integral domain appears in Palmer's thesis [17, p. 39]. The definition of convexity used is the following. It should be noted that this definition does not require E to have a topology and, in keeping with earlier remarks, it is independent of any order on K.

Definition 5.1. A subset of E is called convex if $\lambda x + \mu y + \nu z \in A$ for every $x, y, z \in A$ and $\lambda, \mu, \nu \in \mathfrak{G}$ for which $\lambda + \mu + \nu = 1$.

Sometimes this notion is called K-convex to emphasize that the convexity is with respect to K. However we will write simply convex, referring to a linear space E over a n.a. valued field K. First, there are several basic theorems which result from the given definition of convexity.

Properties of Convex Sets

Theorem 5.1. The intersection of a family of convex sets is convex.

Proof: Let A_α, $\alpha \in \Lambda$ be convex and $A = \bigcap_{\alpha \in \Lambda} A_\alpha$. Let $x, y, z \in A$ and $\lambda, \mu, \nu \in \mathfrak{G}$ with $\lambda + \mu + \nu = 1$. Since $x, y, z \in A$, then $x, y, z \in A_\alpha$ for each $\alpha \in \Lambda$. Thus $\lambda x + \mu y + \nu z \in A_\alpha$ for each $\alpha \in \Lambda$ since A is convex. Hence $\lambda x + \mu y + \nu z \in A$ and A is convex.

The following theorem states that any translation of a convex set is convex.

Theorem 5.2. If $A \subset E$ is convex, $w \in E$, and $\eta \in K$, then $w+A$ and ηA are convex.

Proof: Let $x, y, z \in w + A$ and $\lambda, \mu, \nu \in \mathfrak{O}$ with $\lambda + \mu + \nu = 1$. Then $x = w + x_0$, $y = w + y_0$ and $z = w + z_0$, where $x_0, y_0, z_0 \in A$. Since A is convex, then $\lambda x_0 + \mu y_0 + \nu z_0 \in A$. Thus

$$\begin{aligned}\lambda x + \mu y + \nu z &= \lambda(w + x_0) + \mu(w + y_0) + \nu(w + z_0)\\ &= (\lambda + \mu + \nu)w + \lambda x_0 + \mu y_0 + \nu z_0\\ &= w + (\lambda x_0 + \mu y_0 + \nu z_0) \in w + A\end{aligned}$$

and therefore $w+A$ is convex.

The proof that ηA is convex is similar.

The convex subsets of E containing 0 have an interesting algebraic structure as the following theorem shows. Here E is considered as an $\mathfrak{O}$-module; that is, a module over $\mathfrak{O}$, the ring of integers of K.

Theorem 5.3. Let $A \subset E$ and $0 \in A$. Then A is convex if and only if A is an $\mathfrak{O}$-submodule of E.

Proof: Suppose A is convex. Let $x, y \in A$, then since $0 \in A$, $x - y = 1 \cdot x + (-1)y + 1 \cdot 0 \in A$. Thus A is an additive subgroup of the $\mathfrak{O}$-module E. Let $\lambda, \mu \in \mathfrak{O}$ and $x \in A$. Then $\lambda x = \lambda x + \mu 0 + (1 - \lambda - \mu)0 \in A$ so that A is an $\mathfrak{O}$-submodule.

Conversely, if A is a submodule of the $\mathfrak{O}$-module E, $x, y, z \in A$, and $\lambda, \mu, \nu \in \mathfrak{O}$ with $\lambda + \mu + \nu = 1$, then $\lambda x + \mu y + \nu z \in A$ so that A is convex.

Corollary 5.4. If Q_p is considered as a linear space over itself then 0_p is convex.

Proof: The set $\mathfrak{O} = 0_p$ is clearly an 0_p-submodule.

Theorem 5.3 gives a characterization of the convex subsets of E. That is, a subset is convex if and only if it is an $\mathfrak{O}$-submodule or a translation of an $\mathfrak{O}$-submodule of E.

Theorem 5.5. Let A be any subset of E containing 0. Then A is convex if and only if A satisfies the following condition (C). (C) If x and y are contained in set S and λ and μ are elements of $\mathfrak{O}$ then $\lambda x + \mu y$ is contained in set S.

Proof: If A is convex and contains 0, then $\lambda x + \mu y + (1 - \lambda - \mu) \cdot 0 = \lambda x + \mu y$ so $\lambda x + \mu y$ is in A and A satisfies condition (C). Conversely, assume A satisfies condition (C). Then for any x, y and z in A and λ, μ and ν in $\mathfrak{O}$, $\lambda x + \mu y$ is an element of A, say w, so that $\lambda x + \mu y + \nu z = w + \nu z$ which is an element of A. Thus A is convex.

Another characterization of convex sets in E results from the preceding theorem. A subset A of E is convex if and only if A satisfies condition (C) or A is a translation of a set satisfying condition (C).

Definition 5.2. A subset A of E is symmetric if $A = -A$.

Theorem 5.6. If A is a convex subset of E containing 0, then A is symmetric.

Proof: If $x \in A$, then $-x = (-1)x + 0 \cdot x \in A$ by Theorem 5.5. Thus $A = -A$, so A is symmetric.

Definition 5.3. Let V and W be two subsets of E. Then V absorbs W if there exists an $a > 0$ such that $W \subset \lambda V$ for every $\lambda \in K$, $|\lambda| \geq a$. A subset A of E is called absorbing if it absorbs every point of E.

Theorem 5.7. If a field K is considered as a linear space over itself, then $\mathcal{O}$ is an absorbing set.

Proof: Let y be a non-zero element of K. Let $a = |y|$. Let λ be any element of K such that $|\lambda| \geq a$. Thus $|\lambda| \geq |y| > 0$. Since K is a field, there exists an element $z \in K$ such that $y = \lambda z$. Now $|y| = |\lambda z| = |\lambda|\,|z|$ so that $|z| = \dfrac{|y|}{|\lambda|}$. But $|\lambda| \geq |y|$ so $|z| \leq 1$. Thus z is an element of $\mathcal{O}$. We have shown that for any $y \in K$, there exists an $a > 0$ such that for any λ with $|\lambda| \geq a$, $\{y\} \subset \lambda\mathcal{O}$. Thus $\mathcal{O}$ absorbs $\{y\}$ and since y was an arbitrary element of K, $\mathcal{O}$ absorbs every point of K. That is, $\mathcal{O}$ is an absorbing set.

In particular, if Q_p is considered as a linear space over itself then 0_p is an absorbing set.

Theorem 5.8. Each absorbing set contains 0.

Proof: If A is an absorbing set then in particular it absorbs 0. Thus there exists a number $a > 0$ such that $0 \varepsilon \lambda A$ for any $|\lambda| \geq a$. If $a < 1$, then $0 \varepsilon \lambda A$ for any $|\lambda| = 1$ and in particular for $\lambda = 1$. Thus $0 \varepsilon 1 \cdot A = A$. If $a \geq 1$, let $|\lambda_0| \geq a$ so that $0 \varepsilon \lambda_0 A$. Let $\lambda_1 \varepsilon K$ such that $|\lambda_1| < |\lambda_0|$. Let $\mu = \lambda_1 \lambda_0^{-1}$. Then $0 \varepsilon \lambda_0 A$ implies that $\mu \cdot 0 \varepsilon \mu \cdot \lambda_0 A$, that is, $0 \varepsilon \lambda_1 A$ since $\mu\lambda_0 = \lambda_1$. Since this is true for any λ_1 such that $|\lambda_1| < |\lambda_0|$, it is true for $\lambda_1 = 1$ and hence $0 \varepsilon 1 \cdot A = A$. The case $a = \lambda_0 = 1$ is trivial. Thus any absorbing set contains 0.

The definition of convexity leads to a very simple characterization of convex sets in a n.a. valued field K. With this in mind, the case where $E = K$, considered as a linear space over itself, will now be studied.

Theorem 5.9. Let A be a non-degenerate convex subset of K. Then $A = K$ or A is a sphere; that is, $A = K$, $A = \{x \varepsilon K \mid |x - x_0| \leq r_0\}$ or $A = \{x \varepsilon K \mid |x - x_0| < r_0\}$ for some $x_0 \varepsilon K$ and $r_0 > 0$. In particular, the conclusion is valid for $K = Q_p$.

Proof: Consider first the case where $0 \varepsilon A$. By Theorem 5.3, A is an $\mathfrak{G}$-submodule. Thus for any $\lambda \varepsilon \mathfrak{G}$ and $x \varepsilon A$, $\lambda x \varepsilon A$. In particular, $1 \varepsilon \mathfrak{G}$ so that $\mathfrak{G} \cdot A = A$. Let $r_0 = \sup_{x \varepsilon A} |x|$. If the set $\{|x| \mid x \varepsilon A\}$ has no upper bound we will say $r_0 = \infty$. First suppose that $r_0 = \infty$. In this case $A = K$. To see this let $y \varepsilon K$. Since $\mathfrak{G}$ is an absorbing set by Theorem 5.7, there exists an $a > 0$ such that $|\lambda| \geq a$ implies $y \varepsilon \lambda \mathfrak{G}$. Since $r_0 = \infty$, the set $\{|x| \mid x \varepsilon A\}$ is not bounded above and hence there exists an $x \varepsilon A$ such that $|x| \geq a$.

Thus $y \varepsilon x\mathfrak{O}$, where $x \varepsilon A$ so that $y \varepsilon \mathfrak{O} \cdot A = A$. Thus $A = K$ since y was an arbitrary element of K.

If r_0 is finite, there are two possibilities. One possibility is that there exists an $x_0 \varepsilon A$ such that $|x_0| = r_0$. In this case the claim is that $A = S[0, r_0] = \{x \mid |x| \leq r_0\}$. Clearly $A \subset S[0, r_0]$ by the definition of r_0. Moreover, for any $y \varepsilon S[0, r_0]$, since K is a field, there exists an element $z \varepsilon K$ such that $y = x_0 z$. Thus $|y| = |x_0| \, |z|$ or equivalently $|z| = \dfrac{|y|}{|x_0|}$. But $|x_0| = r_0$ and $|y| \leq r_0$, since $y \varepsilon S[0, r]$, so that $|z| \leq 1$. Therefore, z is an element of $\mathfrak{O}$. Since $y = x_0 z$ with $x_0 \varepsilon A$ and $z \varepsilon \mathfrak{O}$, it follows that $y \varepsilon A \cdot \mathfrak{O} = A$. It has been shown that if $y \varepsilon S[0, r_0]$, then $y \varepsilon A$ so that $S[0, r_0] \subset A$. Hence $A = S[0, r_0]$.

The other possibility is that there is no $x_0 \varepsilon A$ such that $|x_0| = r_0$. In this case $A = S(0, r_0)$. The proof is similar to the proof for the other case.

It has been shown that if 0 is contained in the non-degenerate convex set A, then $A = K$ or A is a sphere. The case $0 \notin A$ can be handled by a translation. If $0 \notin A$, let $x_0 \varepsilon A$. Consider the set $A' = -x_0 + A$. By Theorem 5.2, A' is convex and since $x_0 \varepsilon A$, we have $0 \varepsilon A'$. Then, by the first part of the proof, there are three possibilities. If $A' = K$, then $A = K$. If $A' = \{x \mid |x| \leq r_0\}$, then $A = x_0 + A' = x_0 + \{x \mid |x| \leq r_0\}$. But $x_0 + \{x \mid |x| \leq r_0\} = \{x \mid |x - x_0| \leq r_0\} = S[x_0, r_0]$. Thus $A = S[x_0, r_0]$. Finally, if $A' = \{x \mid |x| < r_0\}$, then $A = \{x \mid |x - x_0| < r_0\} = S(x_0, r_0)$. In any case, if A is a non-degenerate convex subset of K, then $A = K$ or A is a sphere.

The following theorem completes the characterization of convex sets in K.

Theorem 5.10. Every sphere in K is a convex set.

Proof: Let $S[x_0, r_0]$ be a sphere in K. Then $S[x_0, r_0] = \{x \in K \mid |x - x_0| \le r_0\}$. Consider the sphere $S[0, r_0] = \{x \in K \mid |x| \le r_0\}$. Let $x, y, z \in S[0, r_0]$ and $\lambda, \mu, \nu \in \Theta$, with $\lambda + \mu + \nu = 1$. Then

$$\begin{aligned} |\lambda x + \mu y + \nu z| &\le \max\{|\lambda|\,|x|, |\mu|\,|y|, |\nu|\,|z|\} \\ &\le \max\{|x|, |y|, |z|\} \\ &\le r_0 . \end{aligned}$$

Thus $\lambda x + \mu y + \nu z \in S[0, r_0]$ and $S[0, r_0]$ is convex. Thus $S[x_0, r_0] = x_0 + S[0, r_0]$ is convex by Theorem 5.2. Similarly the sphere $S(x_0, r_0)$ is convex.

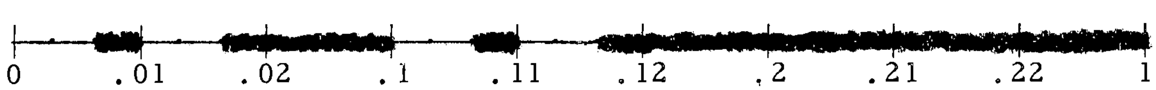
As a result of the preceding two theorems K, considered as a normed linear space over itself, is convex and moreover is locally convex since the (convex) spheres are a base for K. In this case, the only non-degenerate convex sets are the spheres which are both open and closed and hence have no boundary. This fact has important implications which will be discussed further in the conclusion.

A Geometric Model for 0_2 and 0_2^2

In this section a geometric representation is given for 0_2 and 0_2^2, using the n.a. norm of Example 4.1 for 0_2^2. A similar interpretation can be done for 0_p and 0_p^2 where p is any prime. In Chapter I, it was noted that any element $\alpha \varepsilon 0_p$, has a unique representation in the form $a_0 + a_1 p + a_2 p^2 + \ldots$ where $0 \leq a_i \leq p-1$. Thus, in 0_2, any element can be represented in series form where $a_i = 0$ or 1. It is also known that any real number r, $0 \leq r < 1$ can be represented, using base 3, in the form $r = \frac{a_0}{3} + \frac{a_1}{3^2} + \frac{a_2}{3^3} + \ldots$ which can be written $r = .a_0 a_1 a_2 \ldots$. If one identifies the 2-adic integer $\alpha = a_0 + a_1 p + a_2 p^2 + \ldots$ with the real number $.a_0 a_1 a_2 \ldots$ in base 3, then a one-to-one correspondence is established between the set 0_2 and the set

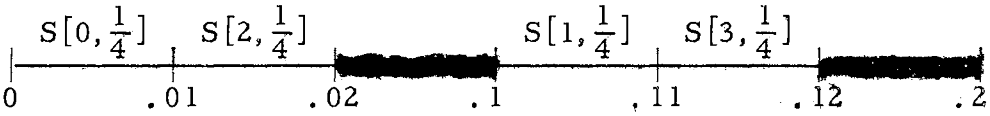
$$H = \{r \varepsilon [0, 1) \mid r = .a_0 a_1 a_2 \ldots, \text{ base } 3, \text{ and } a_i = 0 \text{ or } 1\}.$$

By this identification, one obtains a geometric interpretation of the set 0_2. In the graph below, the shaded portion represents some of the points which do not correspond to points of 0_2 since $a_i = 2$ for at least one $i = 0, 1, 2, \ldots$. As an example of the correspondence, the point $5 = 1 + 0 \cdot 2 + 1 \cdot 2^2$ in 0_2 is represented by the number $.101$. For a more detailed discussion of this geometric interpretation of 0_2 see Agnew [1].

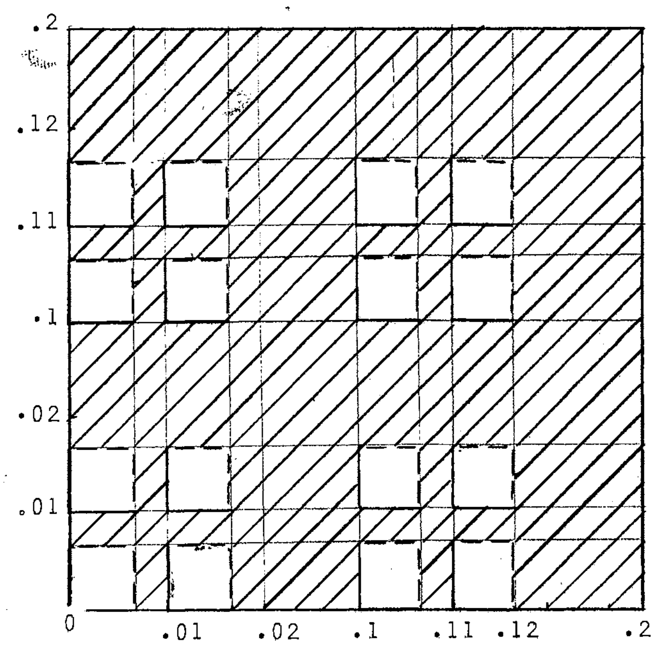


Of special interest to this study is the fact that spheres in 0_2 are represented as subintervals of $[0, 1) \cap H$. For example, there are four spheres of radius $\frac{1}{4}$ in 0_2; the spheres $S[0, \frac{1}{4}]$, $S[1, \frac{1}{4}]$, $S[2, \frac{1}{4}]$, and $S[3, \frac{1}{4}]$. These spheres are subsets of the subintervals

$[0,.01)$, $[.1,.11)$, $[.01,.02)$ and $[.11,.12)$ respectively, as indicated below.



The geometric model for 0_2 lends itself to a natural interpretation of 0_2^2 as indicated in the following graph. The shaded areas represent some of the points not in 0_2^2. The graph also indicates the 16 spheres of radius $\frac{1}{4}$ in 0_2^2.



If A and B are convex subsets of 0_2, then $A \times B$ is a convex subset of 0_2^2. This follows since for $x = (x_1, x_2)$, $y = (y_1, y_2)$ and

$z = (z_1, z_2)$ in $A \times B$ and λ, μ, and ν in 0_2, with $\lambda + \mu + \nu = 1$,

$$\lambda x + \mu y + \nu z = \lambda(x_1, x_2) + \mu(y_1, y_2) + \nu(z_1, z_2)$$

$$= (\lambda x_1 + \mu y_1 + \nu z_1, \lambda x_2 + \mu y_2 + \nu z_2)$$

The first and second components are in A and B respectively since A and B are convex. Thus $\lambda x + \mu y + \nu z$ is in $A \times B$ and $A \times B$ is convex.

In 0_2 the only convex sets are points and spheres. It follows that the collection of convex subsets of 0_2^2 includes the cartesian product of the spheres and points in 0_2. However, it includes other sets as well. For example, consider the set A defined as follows. Let $x = (1, 1)$ and $y = (0, 0)$. Then $\lambda x + (1 - \lambda)y = \lambda(1, 1) + (1 - \lambda)(0, 0) = (\lambda, \lambda)$. Define $A = \{(\lambda, \lambda) | \lambda \varepsilon 0_2\}$. Then for $a = (a_0, a_0)$, $b = (b_0, b_0)$ and $c = (c_0, c_0)$ it is clear that $\lambda x + \mu y + \nu z = (\lambda a_0 + \mu b_0 + \nu c_0, \lambda a_0 + \mu b_0 + \nu c_0) \varepsilon A$. Thus A is convex. It is also clear that A is not the cartesian product of any two subsets of 0_2. The set A defined above, relative to the points $x = (1, 1)$ and $y = (0, 0)$, is actually the smallest convex set containing x and y. This is verified in the next section.

Convex Hull

Definition 5.4. Let $S \subset E$. The convex hull of S is the intersection of the convex subsets containing S: it is denoted $C_0(S)$.

Since, by Theorem 5.1, the intersection of any family of convex sets is convex, it follows that the convex hull of any subset S of E is a convex set. Consider now the special case in which $S = \{x, y\}$.

<u>Theorem 5.11.</u> The convex hull of the set $\{x, y\}$, $x, y \in E$ is the set $C_0(\{x, y\}) = \{\lambda x + (1-\lambda)y \mid |\lambda| \leq 1\}$.

Proof: Let $z \in C_0(\{x, y\})$. Then z is contained in every convex set containing x and y. With $\lambda = 1$ and $\lambda = 0$ one sees that x and y respectively are contained in the set on the right. Thus we must show it is convex. Let $z_1, z_2, z_2 \in \{\lambda x + (1-\lambda)y \mid |\lambda| \leq 1\}$ and $\lambda, \mu, \nu \in \mathfrak{G}$, $\lambda + \mu + \nu = 1$. Then $z_i = \lambda_i x + (1-\lambda_i)y$, for $i = 1, 2, 3$. Hence

$$\begin{aligned}
\lambda z_1 + \mu z_2 + \nu z_3 &= \lambda[\lambda_1 x + (1-\lambda_1)y] + \mu[\lambda_2 x + (1-\lambda_2)y] + \nu[\lambda_3 x + (1-\lambda_3)y] \\
&= (\lambda\lambda_1 + \mu\lambda_2 + \nu\lambda_3)x + [\lambda(1-\lambda_1) + \mu(1-\lambda_2) + \nu(1-\lambda_3)]y \\
&= (\lambda\lambda_1 + \mu\lambda_2 + \nu\lambda_3)x + [(\lambda + \mu + \nu) - (\lambda\lambda_1 + \mu\lambda_2 + \nu\lambda_3)]y \\
&= \lambda' x + (1-\lambda')y
\end{aligned}$$

where $\lambda' = \lambda\lambda_1 + \mu\lambda_2 + \nu\lambda_3$. Furthermore,

$$|\lambda'| = |\lambda\lambda_1 + \mu\lambda_2 + \nu\lambda_3| \leq \max\{|\lambda|\,|\lambda_1|, |\mu|\,|\lambda_2|, |\nu|\,|\lambda_3|\} \leq 1.$$

Thus $\{\lambda x + (1-\lambda)y \mid |\lambda| \leq 1\}$ is a convex set containing x and y so that $C_0(\{x, y\}) \subset \{\lambda x + (1-\lambda)y \mid |\lambda| \leq 1\}$.

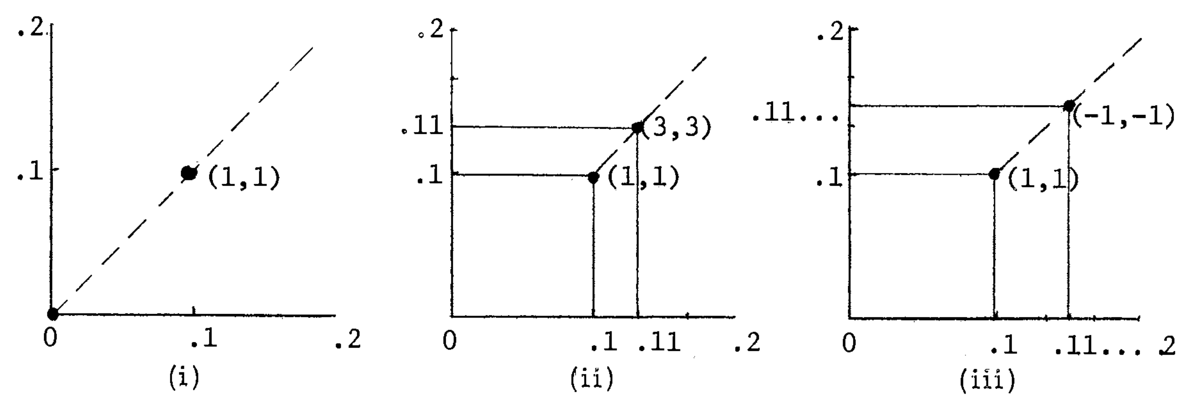
Now let $z \in \{\lambda x + (1-\lambda)y \mid |\lambda| \leq 1\}$. Let C be any convex set containing x and y, then $z = \lambda x + (1-\lambda)y$ for some λ, $|\lambda| \leq 1$. Since x and y are in C and C is convex we have that $z = \lambda x + (1-\lambda)y = \lambda x + 1 \cdot y - \lambda y \in C$. Thus z is in every convex set containing x and y so that $\{\lambda x + (1-\lambda)y \mid |\lambda| \leq 1\} \subset C_0(\{x, y\})$. The conclusion then follows.

Following are three examples of the convex hull of two points in $0_2^{\,2}$.

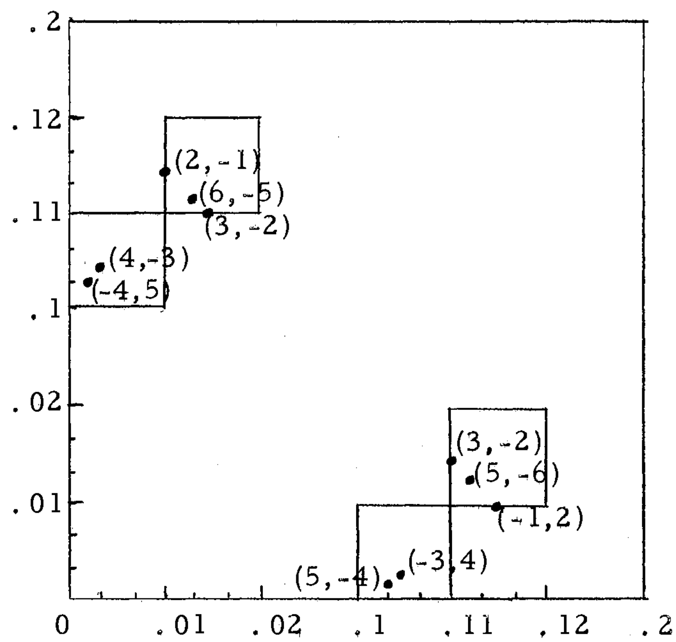
(i) $x = (1,1)$, $y = (0,0)$, $C_0(\{x,y\}) = \{(\lambda,\lambda) \mid \lambda \varepsilon 0_2\}$.

(ii) $x = (3,3)$, $y = (1,1)$, $C_0(\{x,y\}) = \{(2\lambda + 1, 2\lambda + 1) \mid \lambda \varepsilon 0_2\}$.

(iii) $x = (1,1)$, $y = (-1,-1)$, $C_0(\{x,y\}) = \{(2\lambda - 1, 2\lambda - 1) \mid \lambda \varepsilon 0_2\}$.



One generally expects the convex hull of two points to be the line segment joining the two points. In these examples, the convex hull is a "segment" but not all the points are "between" the two points x and y. In the final example, with graph following, the graph of the convex hull does not even resemble a segment. Let $x = (1,0)$ and $y = (0,1)$. Then the convex hull is given by $C_0(\{x,y\}) = \{(\lambda, 1-\lambda) \mid \lambda \varepsilon 0_2\}$. A few of the points contained in the convex hull are indicated on the graph. It can be argued that since the sum of the coordinates must be 1, the points must all lie in the four spheres indicated.



Quasi-convexity

Monna has introduced the term quasi-convex related to Theorem 5.11.

Definition 5.5. A subset A of E is called quasi-convex if for any x and $y \in S$, $\lambda x + (1-\lambda)y \in S$ for every $\lambda \in K$ such that $|\lambda| \leq 1$.

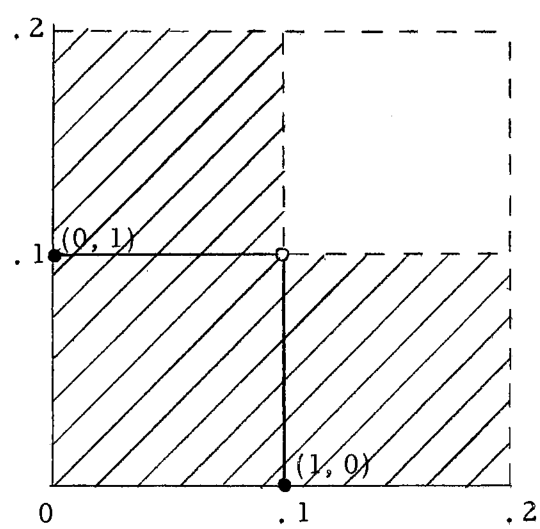
As a result of Theorem 5.11 it follows that a quasi-convex set A contains the convex hull of each pair of points in A.

Convexity and quasi-convexity are not equivalent. However, if one notes that $\lambda x + (1-\lambda)y = \lambda x + 1 \cdot y - \lambda y$ then whenever S is convex, we have $\lambda x + (1-\lambda)y$ is a member of S. This proves the following theorem.

Theorem 5.12. Every convex set is quasi-convex.

The converse of this theorem is not true as the following example shows, that is, a set is defined which is quasi-convex, but not convex. However, in most cases, the two concepts are equivalent. Theorem 5.13 gives one general case in which they are equivalent.

<u>Example 5.1.</u> Let $E = Q_2^2$; that is, the set of ordered pairs (x_1, x_2) such that x_1 and x_2 are elements of Q_2. Let $x = (1, 0)$ and $y = (0, 1)$. Define the set S by $S = \{(\alpha\beta, (1 - \alpha)\mu) \mid \alpha, \beta, \mu \varepsilon 0_2\}$. The set S will be shown to be quasi-convex but not convex. The set S is the union of three spheres indicated in the following graph.



Let z_1 and z_2 be elements of S. To prove that S is quasi-convex, it must be shown that $\lambda z_1 + (1 - \lambda)z_2$ is an element of S for any λ in 0_2. Since z_1 and z_2 are elements of S, they can be represented as $z_1 = (\alpha_1\beta_1, (1 - \alpha_1)\mu_1)$ and $z_2 = (\alpha_2\beta_2, (1 - \alpha_2)\mu_2)$ where the elements $\alpha_1, \beta_1, \mu_1, \alpha_2, \beta_2$, and μ_2 are in 0_2.

Therefore,

$$\lambda z_1 + (1-\lambda)z_2 = \lambda(\alpha_1\beta_1,\ (1-\alpha_1)\mu_1) + (1-\lambda)(\alpha_2\beta_2,\ (1-\alpha_2)\mu_2)$$

$$= (\lambda\alpha_1\beta_1 + (1-\lambda)\alpha_2\beta_2,\ \lambda(1-\alpha_1)\mu_1$$

$$+ (1-\lambda)(1-\alpha_2)\mu_2)$$

$$= (\alpha_3, \beta_3)$$

where $\alpha_3 = \lambda\alpha_1\beta_1 + (1-\lambda)\alpha_2\beta_2$ and $\beta_3 = \lambda(1-\alpha_1)\mu_1 + (1-\lambda)(1-\alpha_2)\mu_2$. To show that (α_3, β_3) is an element of S, it must be demonstrated that $(\alpha_3, \beta_3) = (\alpha\beta, (1-\alpha)\mu)$ for some $\alpha, \beta,$ and μ in 0_2. In this example, the symbol $|\ \ |$ is used for the 2-adic valuation.

To begin with, the following argument shows that the statements $|\alpha_3| = 1$ and $|\beta_3| = 1$ cannot both be true. Suppose that $|\alpha_3| = 1$, that is, $|\lambda\alpha_1\beta_1 + (1-\lambda)\alpha_2\beta_2| = 1$. By the n.a. property, $|\lambda\alpha_1\beta_1 + (1-\lambda)\alpha_2\beta_2| \leq \max\{|\lambda\alpha_1\beta_1|, |(1-\lambda)\alpha_2\beta_2|\}$, so that either $|\lambda\alpha_1\beta_1| = 1$ or $|(1-\lambda)\alpha_2\beta_2| = 1$. If $|\lambda\alpha_1\beta_1| = 1$, then $|\lambda| = 1$ and $|\alpha_1| = 1$. In 0_2 this implies that $|1-\lambda| < 1$ and $|1-\alpha_1| < 1$. By the n.a. property, $|\beta_3| \leq \max\{|\lambda(1-\alpha_1)\mu_1|, |(1-\lambda)(1-\alpha_2)\mu_2|\}$. But $|1-\lambda| < 1$ implies that $|(1-\lambda)(1-\alpha_2)\mu_2| < 1$ and $|1-\alpha_1| < 1$ implies that $|\lambda(1-\alpha_1)\mu_1| < 1$. Thus $|\beta_3| < 1$. Likewise, if $|(1-\lambda)\alpha_2\beta_2| = 1$, then $|1-\lambda| = 1$ and $|\alpha_2| = 1$ so that $|\lambda| < 1$ and $|1-\alpha_2| < 1$. Then, as before, $|\beta_3| < 1$. Thus either $|\alpha_3| < 1$ or $|\beta_3| < 1$.

To show that there exist elements $\alpha, \beta,$ and μ in 0_2 such that $(\alpha_3, \beta_3) = (\alpha\beta, (1-\alpha)\mu)$, the two cases $|\alpha_3| < 1$ and $|\beta_3| < 1$ must be considered. If $|\alpha_3| < 1$, then $|1-\alpha_3| = 1$. In this case, let $\alpha = \alpha_3$ and $\beta = 1$. Then we must have $\beta_3 = (1 - \alpha)\mu$, so with $\alpha = \alpha_3$ let $\mu = \beta_3(1 - \alpha_3)^{-1}$. It remains to show that α, β, and μ are elements of 0_2. Clearly α and β are in 0_2 and μ is in 0_2 since $|1 - \alpha_3| = 1$. Thus, in the case $|\alpha_3| < 1$, it has been shown that $(\alpha_3, \beta_3) \varepsilon S$.

If $|\beta_3| < 1$, then $|1 - \beta_3| = 1$. In this case, let $\mu = 1$ and $1 - \alpha = \beta_3$, that is, $\alpha = 1 - \beta_3$. This requires $\alpha\beta = \alpha_3$ or equivalently $\beta = \alpha_3 \cdot \alpha^{-1} = \alpha_3(1 - \beta_3)^{-1}$. But then α and μ are clearly in 0_2 and β is in 0_2 since $|1 - \beta_3| = 1$. Thus, in either case, (α_3, β_3) is in S.

It has been shown that for any z_1 and z_2 in S and $\lambda \varepsilon 0_2$, the element $\lambda z_1 + (1 - \lambda)z_2$ is in S. Thus S is quasi-convex.

It is easily demonstrated that the set S is not convex. Since the set S is given by $\{(\alpha\beta, (1 - \alpha)\mu) \mid \alpha, \beta, \mu \varepsilon 0_2\}$, it is clear that the points $(1, 0)$, $(0, 1)$ and $(0, 0)$ are all contained in S. However, the point $(1, 0) + (0, 1) - (0, 0) = (1, 1)$ is not in S. This follows since for $\alpha \varepsilon 0_2$ either $|\alpha| < 1$ or $|1 - \alpha| < 1$. Thus either $|\alpha\beta| < 1$ or $|(1 - \alpha)\mu| < 1$ so it is impossible that both statements $\alpha\beta = 1$ and $(1 - \alpha)\mu = 1$ be true. Since the point $(1, 1)$ is not in S, S is not convex.

The set S has been shown to be quasi-convex but not convex. Thus in Q_2^2 convexity and quasi-convexity are not equivalent. However, the following theorem states that in any n.a. valued field K, considered as a linear space over itself, convexity and quasi-convexity are equivalent.

Theorem 5.13. Let K be any n.a. valued field considered as a linear space over itself. Then a set A in K is convex if and only if it is quasi-convex.

Proof: In view of Theorem 5.12, it is only necessary to show that if the set is quasi-convex it is convex. In fact, it is sufficient to show that every quasi-convex set containing 0 is convex. The case where $0 \notin A$ can be handled by a translation. The proof that any translation of a quasi-convex set is quasi-convex is analogous to the proof of Theorem 5.2 for convex sets.

Therefore, let A be any quasi-convex subset of K containing 0. Then for any x in A and $\lambda \varepsilon \mathfrak{G}$, λx is in A since $\lambda x = \lambda x + (1 - \lambda) \cdot 0$.

Moreover, if x and y are non-zero elements of the field K and $|y| \leq |x|$, then $y = \alpha x$ for some α in $\mathfrak{G}$. Thus for λ and μ in $\mathfrak{G}$, $\lambda x + \mu y = \lambda x + \mu(\alpha x) = (\lambda + \mu\alpha)x = \beta x$ and $|\beta| = |\lambda + \mu\alpha| \leq \max\{|\lambda|, |\mu\alpha|\} \leq 1$. Let x and y be in A and, without loss of generality, let $|y| \leq |x|$. Then $\lambda x + \mu y = \beta x$ for some $\beta \varepsilon \mathfrak{G}$. By the preceding paragraph, since A is a quasi-convex set containing 0, βx is in A. Thus A is convex.

In Theorem 5.11 it was shown that the convex hull of the set $\{x, y\}$, $x, y \varepsilon E$ is the set $C_0(\{x, y\}) = \{\lambda x + (1 - \lambda)y \mid |\lambda| \leq 1\}$. In particular, if $E = K$, since the convex hull is convex and contains at least two points, it must be a sphere by Theorem 5.9. Thus it must be the smallest sphere containing x and y. Let $|x - y| = r$. Then it is clear that $S[x, r]$ is the smallest sphere containing x and y. This argument proves the following theorem.

Theorem 5.14. Let x and y be in K. Then $C_0(\{x, y\})$, the convex hull of $\{x, y\}$, is the sphere $S[x, r]$ where $r = |x - y|$.

In the case that $K = Q_p$, one can also make the following observations.

Theorem 5.15. In Q_p, $p \neq 2$, every symmetric convex set contains 0.

Proof: Since every non-degenerate convex subset of Q_p, is a sphere, suppose $S[x_0, r] = -S[x_0, r]$. Then $x_0 \varepsilon S[x_0, r]$ implies that $-x_0 \varepsilon S[x_0, r]$ so that

$$r \geq |x_0 - (-x_0)|_p = |2x_0|_p = |2|_p |x_0|_p = |x_0|_p = |x_0 - 0|_p .$$

Thus $0 \varepsilon S[x_0, r]$.

One can see from the proof that $p \neq 2$ was necessary. In fact in Q_2 consider

$$S[1, \tfrac{1}{2}] = \{x \mid |x - 1|_2 \leq \tfrac{1}{2}\} = \{x \mid |x - 1|_2 < 1\} = \{x \mid |x|_2 = 1\} .$$

That is $S[1, \frac{1}{2}] = \{x \mid |x|_2 = 1\}$. From the lefthand side of the last equation one sees that this set is closed and convex and from the right-hand side, that it is symmetric but does not contain 0. Thus the conclusion of Theorem 5.15 is false in Q_2.

Another way of observing the result in the preceding paragraph is as follows. The elements of the set $A = \{x \varepsilon Q_2 \mid |x|_2 = 1\}$ are of the form $1 + \sum_{n=1}^{\infty} a_n 2^n$ where $a_n = 0$ or 1. Thus

$A = 1 + \{x \varepsilon Q_2 \mid |x|_2 < 1\} = 1 + \{x \varepsilon Q_2 \mid |x|_2 \leq \frac{1}{2}\}$ so that A is convex being of the form $x_0 + A'$ where A' is convex.

If $p \neq 2$, $A = \{x \varepsilon Q_p \mid |x|_p = 1\}$ is not a sphere, as it is in Q_2, since if it were a sphere it would be convex. For example in Q_3, let $x = 1$, $y = 2$, $z = 1$. Then $x, y, z \varepsilon A$ but $3x - y - z = 0 \notin A$, even though $3 + (-1) + (-1) = 1 \varepsilon 0_3$. Thus A is not convex and hence is not a sphere.

Conclusion

One area of study in convexity involves the concept of extreme points in convex sets. A point x of a convex set A is an <u>extreme point</u> of A if and only if x is not an interior point of any line segment whose end points belong to A. Theorems such as the Krein-Milman Theorem are concerned with the existence of extreme points in convex sets. But in K it was found that the only non-degenerate sets are spheres which are both open and closed. Thus every point of a sphere S is an interior point. It appears that a different definition of boundary or extreme point, or possibly a different definition of convexity, is needed if theorems such as the Krein-Milman Theorem are to have analogues in the non-archimedean setting.

The articles [12] and [13] by Monna contain a more detailed discussion of the problems involved. Convexity in linear spaces over non-archimedean valued fields appears to be an area for additional study and research.

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