ASYMPTOTIC INNER-PRODUCT FEEDBACK CONTROL

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CHAPTER I

INTRODUCTION

Control theory is a relatively new area of engineering technology, but has played an increasingly prominent role in the advancement of modern civilization. Although little organized theory existed prior to 1940, the rapid growth of control theory since that time has lead to widespread applications of automatic control systems. Today, automatic control systems are employed in space vehicle and missile guidance, weapon fire-control systems, aircraft-piloting systems, as well as in numerous applications of domestic and industrial import.

The early developments in control theory are characterized by the use of frequency domain methods (1, 2, 3). Such methods determine the stability of closed-loop systems on the basis of the open-loop response to steady-state sinusoidal inputs. These early methods, such as the Bode plot, Nyquist plot, and Nichols chart, are chiefly graphical techniques which rely on the use of frequency domain plots. Although basically trial-and-error techniques, the frequency response methods were used until the late 1940's to design feedback control systems on the basis of satisfying design specifications such as bandwidth, gain and phase margin, peak resonance, and cutoff rate.

The introduction of the root locus technique by Evans (4) in 1948 improved on the frequency response methods by providing information on the transient response of systems as well as the frequency response.

The frequency response techniques and the root locus methods comprise what is commonly classified as classical control theory.

The classical approach leads to feedback control systems which are stable and satisfy a set of performance requirements. In general, the resultant controls are not optimal in any sense, but simply represent one of many control systems that work.

The principal disadvantage of the classical approach is its inapplicability to multiple-input multiple-output systems. With few exceptions, it is applicable only to linear time-invariant systems which are also single-input single-output. Since modern engineering systems are often quite complex, and are often time-varying and nonlinear, as well as multiple-input multiple-output, the classical approach does not apply to such systems.

Modern control theory, based on the concept of state, was developed about 1960 to cope with the weaknesses encountered in the classical approach (5, 6). Aided by the advent of the electronic computer and spurred on by the coming of the space age, a new approach to control theory was derived. The objective of modern control theory is to design the control for which the overall system behavior is optimal in some prescribed sense.

To measure the relative "goodness" of control systems, a performance measure is prescribed. The objective of the performance measure is to incorporate in a single number a quantitative measure of the performance of the system. Modern control theory is concerned with the determination of the control signal vector $\underline{U}(t)$ for which a given performance measure is minimized or maximized. The performance measure often assumes the form

$$J = \int_{t_o}^{t_1} L(\underline{X}(t), \underline{U}(t))\, dt \tag{1.1}$$

where L is a functional relationship, $\underline{X}(t)$ is the state vector and $\underline{U}(t)$ is the control vector. A control vector which minimizes or maximizes equation (1.1) is an optimal control relative to the given performance measure.

The modern approach to control theory is superior to the classical approach in many aspects. Besides being applicable to nonlinear time-varying multiple-input multiple-output systems, modern control theory is based on a time-domain approach rather than the complex frequency domain approach. Modern control theory can be carried out for a class of inputs instead of a specific input function, and it allows the inclusion of initial conditions in the control system design.

Despite the advantages of modern control theory, it is not without its shortcomings relative to the classical approach. Paramount among the disadvantages of the modern approach is that a feedback control is possible only in special cases. For linear systems with quadratic performance measures, the optimal control is the well known linear regulator (7). However, if the system is nonlinear or the performance measure is nonquadratic, an analytic solution for the control is normally impossible.

Recent results by Mulholland and Rhoten (8, 9, 10) have shown that the use of inner-product performance measures lead to optimal feedback control laws for a wide class of problems. This dissertation reports on research undertaken to characterize and extend the use of inner-product performance measures for asymptotic control systems.

History of Inner-Product Control

The inner-product formulation of optimal control problems originated in 1968 from the work of Mulholland (11) in the consideration of limit cycles in nonlinear feedback control systems. An optimal nonlinear feedback control scheme is considered with a limit set configuration as the control objective. If the limit set is the surface of a hypersphere of radius a, the control policy depends upon the distance of the state from the limit set. If the norm of the state vector $\underline{X}(t)$ is denoted by

$$\rho = \underline{X}^T \underline{X} \,, \tag{1.2}$$

where $\underline{X}^T$ denotes the transpose of the state vector, then the control objective is accomplished if

$$\rho = a^2.$$

The norm of the state vector is an inner-product which provides a measure of how close the system is to the control objective. The inner-product formulation of control utilizes this norm of the state vector as the basis for the formulation of an optimal system control.

The control system considered is a modification of the direct control problem of Lur'e. A comparison of the Lur'e formulation and the inner-product formulation is given by Mulholland and Rhoten (12). The system is illustrated in Figure 1, and is described by the equations:

$$\dot{\underline{X}} = \underline{A}\,\underline{X} + \xi\,\underline{B}\,\underline{X} \tag{1.3}$$

$$\xi = f(\rho) \tag{1.4}$$

$$\rho = \underline{X}^T \underline{X}$$

where $\underline{A}$ is the n x n matrix describing the plant, $\underline{B}$ is the n x m input matrix and ξ is a scalar feedback control signal.

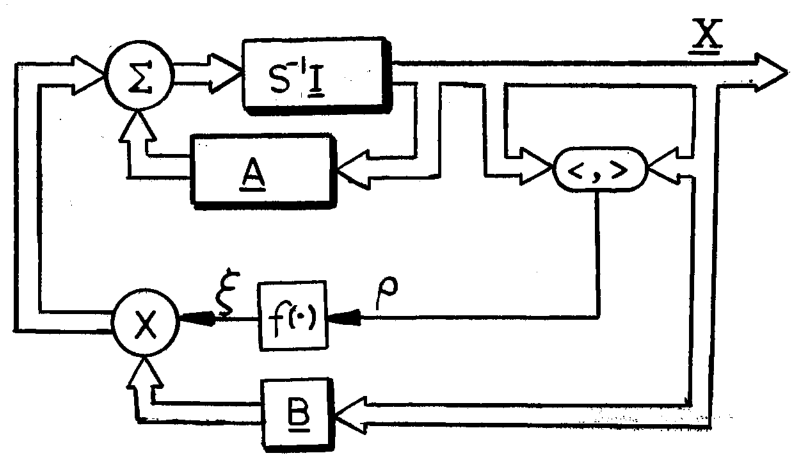

Figure 1. An Inner-Product Control Structure

The system is a closed loop formulation in which the function $f(\rho)$ is arbitrary and is selected to drive the system to the limit set in an optimal manner. The primary consideration in the design of $f(\rho)$ is to drive the state vector to the limit set commensurate with a reasonable expenditure of control energy. Failure of the system to reach the limit set is measured by a non-negative function of the state distance from the limit set. An indirect measure of the control input energy at any instant of time is provided by a non-negative function of $d\rho/dt$. A rather general form for the performance measure is therefore given by

$$J = \int_{t_0}^{\infty} \left[h(\rho) + (d\rho/dt)^2 \right] dt \qquad (1.5)$$

where $h(\rho)$ is a positive real-valued function of ρ with $h(a^2) = 0$. Motivation for this performance measure is drawn from the simplest problem of the calculus of variations in which $h(\rho)$ is related to the potential energy and $(d\rho/dt)^2$ is related to the kinetic energy of a mass.

The formulation of an inner-product performance measure leads to globally optimal control laws if $\underline{A}$ is skew symmetric and $\underline{B}$ is positive definite. The control laws are explicitly realizable by the single nonlinear transducer $f(\rho)$ as a closed form function of the system error signal.

The system defined in equation (1.3) is equivalent to the linear system

$$\dot{\underline{X}} = \underline{A}\,\underline{X} + \underline{B}\,\underline{U}$$

in which the control vector $\underline{U}$ is given by

$$\underline{U} = \xi\,\underline{X}. \tag{1.6}$$

The use of equation (1.3) is therefore equivalent to a linear system with a fixed configuration control structure. The inner-product control of fixed configuration linear systems has received extensive attention from Rhoten and Mulholland (8, 13, 14) for cases in which the control objective is the origin of the state space and $\underline{B}$ is nonsingular. Nonlinear and bilinear applications have also been considered by Mulholland and Rhoten (9) and Sridhar and Rhoten (14) respectively. Stochastic extensions of the problem have been provided by Sims, et al. (15).

The principal limiting feature of these papers is the inability to obtain optimal bounded controls for problems in which the control input matrix $\underline{B}$ is singular. A major emphasis of this dissertation is the extension of the inner-product approach to general systems in which the control input matrix is singular or nonsingular.

Problem Formulation

The problem considered is the asymptotic inner-product control of dynamic systems. The state equations describing the systems are assumed to be of the form

$$\dot{\underline{X}}(t) = \underline{F}(\underline{X}(t),t) + \underline{B}(t)\,\underline{U}(t) \tag{1.7}$$

where $\underline{X}(t)$ is the n x 1 state vector and $\underline{U}(t)$ is the m x 1 control vector. The control portion of the system is separable and appears linearly in the system equations. $\underline{F}(\underline{X}(t),t)$ is an n x 1 functional vector describing the unforced system, and $\underline{B}$ is an n x m matrix describing the distribution of the control vector to the system.

This formulation encompasses linear and nonlinear plants for which the control is linearly separable. Since feedback control is sought, it is desired to determine the control vector as an instantaneous function of the state vector components, i.e.,

$$\underline{U} = \underline{U}(\underline{X}(t)).$$

The control input matrix $\underline{B}$ is singular or nonsingular. The control structure includes the fixed configuration controller of equation (1.6) but does not limit the control to this form.

It is assumed that the control objective is accomplished by driving certain linear combinations of components of the state vector to zero, while limiting the control energy necessary to accomplish this. A general system error signal is therefore defined by

$$\rho(t) = \underline{X}^T(t)\ \underline{Q}(t)\ \underline{X}(t) \tag{1.8}$$

where $\underline{Q}$ is a symmetric matrix which is positive definite or positive semidefinite. This definition of the system error was suggested by Sims, et al. (15) as a generalization of the inner-product definition of equation (1.2). The inner-product of equation (1.8) is a measure of distance in a subspace of the state space, and is invariant for some changes in the state vector if $\underline{Q}$ is semidefinite.

The inner-product formulation of optimal control is based upon the use of the inner-product error signal in the performance measure. Since ρ is a measure of the system error, the time derivative of ρ provides an indirect measure of the control input energy. Using a non-negative function of $d\rho/dt$ in the performance measure penalizes the power input by penalizing any rapid changes in the distance of the state from the origin. A rather general form for the performance measure is therefore given by equation (1.5)

$$J = \int_{t_0}^{\infty} \left[h(\rho) + (d\rho/dt)^2\right]\ dt$$

where $h(\rho)$ is a positive real-valued function of ρ with $h(0) = 0$.

While the form of the inner-product performance index differs somewhat from more conventional performance indices, it does penalize system error directly through $h(\rho)$ and control cost by weighting rapid changes in the error signal. The selection of a performance measure of this form allows an elegant and direct solution to the problem in which the resultant control laws are globally optimal feedback controls.

Scope of Study

The principal objectives of this study are:

(a) to define the underlying structure of systems for which the inner-product approach is suitable; and

(b) to formulate these requirements in the form of an acceptable theory and procedure for determining the optimal inner-product control when applicable.

The solution procedure follows a somewhat different course than is customary. The trajectory minimizing the inner-product performance measure is first determined disregarding the system equations. The minimizing trajectory is developed in Chapter II, and provides the fundamental inner-product law. The trajectory is a differential equation involving the inner-product error signal and its time derivative, and prescribes in norm the trajectory the system must follow in order to minimize the performance measure.

While the inner-product approach to optimal control is applicable to most systems, those systems which yield a feedback control are of the utmost interest because of the implementation advantages offered. Chapter III and Chapter IV consider the necessary and sufficient conditions under which the inner-product approach yields optimal feedback controls. The control systems considered in Chapter III are true closed loop feedback control systems.

The inner-product approach also yields optimal feedback controls which are not true closed loop systems. Although the terms "feedback" and "closed loop" are often used interchangeably in modern control literature, the systems considered in Chapter IV are feedback controls which possess open loop characteristics. The control systems are closed loop from the feedback nature of the controls, and open loop from the behavior characteristics of the controls. The extension of the inner-product approach to these open loop type of feedback systems greatly broadens the class of systems to which the inner-product approach is applicable. A summary of the dissertation is given in Chapter V together with some conclusions and suggestions for further study.

Appendix A considers the solution of a vector equation required in the development of Chapter II. The detailed nature of the solution development is not suitable for inclusion in the main text of the dissertation and is included as an appendix in order to preserve the train of thought in Chapter II. The solution and its development represent a significant step in the general development.

Two equivalent forms of the solution are indicated in Appendix A. The two forms are quite different in structure, but are shown to be equivalent. The main body of the thesis utilizes the simpler of the two forms in the development of the inner-product theory. A summary of equivalent results for the alternate form is given in Appendix B.

CHAPTER II

THE FUNDAMENTAL INNER-PRODUCT LAW

The Problem Defined

The problem to be considered is the asymptotic control of a dynamic system described by a set of n differential equations,

$$\dot{\underline{X}}(t) = \underline{F}(\underline{X}(t),t) + \underline{B}(t)\,\underline{U}(t) \tag{2.1}$$

where $\underline{X}(t)$ is the n x 1 state vector and $\underline{U}(t)$ is the m x 1 control vector. The control portion of the system is assumed to be separable, and to appear linearly in the state equations. $\underline{F}(\underline{X}(t),t)$ is an n x 1 functional vector describing the unforced system, and $\underline{B}(t)$ is an n x m matrix describing the distribution of the control vector to the system.

The asymptotic control of the system is to be accomplished by the specification of a control vector $\underline{U}(t)$ which minimizes the performance measure

$$J = \int_{t_o}^{\infty} \left[h(\rho) + (d\rho/dt)^2 \right] dt \tag{2.2}$$

where ρ is the system error signal and is defined by the scalar

$$\rho(t) = \underline{X}^T(t)\,\underline{Q}(t)\,\underline{X}(t). \tag{2.3}$$

The matrix $\underline{Q}(t)$ is a symmetric matrix which is positive definite or positive semidefinite, and $\underline{X}^T(t)$ denotes the transpose of the state vector $\underline{X}(t)$. The scalar function $h(\rho)$ in the performance measure is restricted to real-valued positive definite functions of the error signal.

The system error signal is a quadratic form, and the rank of the form is that of the matrix $\underline{Q}(t)$. Let r be the rank of $\underline{Q}(t)$. A quadratic form in n variables and of rank r is equivalent to a quadratic form in r variables. That is, there exists a nonsingular transformation $\underline{T}$ such that

$$\underline{Y}(t) = \underline{T}(t)\,\underline{X}(t) \tag{2.4}$$

and the system error signal is given by

$$\rho = y_1^{\,2} + y_2^{\,2} + \dots + y_r^{\,2} \tag{2.5}$$

or

$$\rho(t) = \underline{Y}_r^T(t)\,\underline{Y}_r(t) \tag{2.6}$$

where

$$\underline{Y}_r(t) = \underline{I}_r\,\underline{Y}(t) \tag{2.7}$$

and

$$\underline{I}_r = \left[\underline{I} \mid \underline{0}\right]. \tag{2.8}$$

$\underline{I}_r$ is an r x n matrix, $\underline{I}$ is an r x r identity matrix and $\underline{0}$ is an r x (n-r) zero submatrix (16).

The transformation $\underline{T}$ can also be applied to the state equations to obtain a transformed set of state equations

$$\dot{\underline{Y}}(t) = \underline{F}'(\underline{Y}(t),t) + \underline{B}'(t)\,\underline{U}(t) \tag{2.9}$$

where

$$\underline{F}'(\underline{Y}(t),t) = \dot{\underline{T}}(t)\,\underline{T}^{-1}(t)\,\underline{Y}(t) + \underline{T}(t)\,\underline{F}(\underline{T}^{-1}(t)\,\underline{Y}(t),\ t)$$

and

$$\underline{B}'(t) = \underline{T}(t)\,\underline{B}(t).$$

Since the state equations of (2.9) are identical in structure to those of equation (2.1) it can be assumed that the state equations are given by (2.1) and the system error signal is defined by

$$\rho(t) = \underline{X}_r^T(t)\, \underline{X}_r(t) \tag{2.10}$$

where

$$\underline{X}_r(t) = \underline{I}_r\, \underline{X}(t). \tag{2.11}$$

Since the transformation $\underline{T}$ is nonsingular, no generality is lost by this simplified form.

The time derivative of the system error signal in the performance measure is given by

$$d\rho/dt = 2\, \underline{X}_r^T(t)\, \dot{\underline{X}}_r(t) \tag{2.12}$$

which depends implicitly upon the control $\underline{U}$.

Control Problem Solution

A conventional formulation of many asymptotic optimal control problems is the following. Determine the control vector $\underline{U}(t)$ which will minimize the performance measure

$$I = \int_{t_o}^{\infty} L(\underline{X}(t), \underline{U}(t))\, dt$$

where the state $\underline{X}(t)$ and control $\underline{U}(t)$ are related through the vector differential equation

$$\dot{\underline{X}}(t) = \underline{F}(\underline{X}(t), \underline{U}(t), t).$$

The definition of the system error signal and its derivative can be used to convert the original problem to the above formulation. Standard dynamic optimization techniques applicable to the above problem are therefore also applicable to the original problem. These techniques all depend to some extent upon classical variational calculus methods, and rely therefore upon satisfying a set of necessary conditions to obtain the optimum control. For linear systems with performance measures which are quadratic in the state and control, it is well known that the standard techniques lead to a linear feedback control. A feedback control system is desirable from an engineering point of view because of the general nature of the solution and the ease of implementation, relative to open loop solutions. It would be desirable if a closed loop solution could be obtained in all cases.

Unfortunately, if the system is nonlinear or the performance measure is nonquadratic, the use of standard optimization techniques leads to a set of simultaneous first order differential equations which lack a complete set of boundary conditions. An analytical solution of such two-point boundary-value problems is possible only in special cases. Consequently, numerical trial-and-error techniques usually must be used to solve the problem, which thereupon lead to numerical open loop control solutions rather than the desired closed loop control laws.

The form of the performance measure in equation (2.2) allows the solution of the problem to follow a somewhat different course than is customary. This procedure leads to a closed loop control law in many problems. The conventional approach is to select the control such that the performance measure is minimized when evaluated along the solution trajectory of the system. The alternate approach herein considered is to select the control such that the system trajectory follows a minimal-trajectory of the performance measure. Although the two procedures appear almost identical in statement, the latter approach leads to a new design technique for optimal control.

Following the procedure outlined above, the system equations are ignored for the present and the performance measure is minimized independent of the system equations. The performance measure of equation (2.2) is in the form of the simplest problem of the calculus of variations: Determine the function y(t) which will minimize the integral

$$\int_{t_o}^{\infty} H(\dot{y}(t), y(t), t)\, dt$$

where the function H is a known continuous function. If the extremum of the integral is assumed to occur along a curve y(t) which is twice differentiable, then a necessary condition for minimization of the integral is that y(t) must satisfy the Euler-Lagrange equation:

$$\frac{\partial H}{\partial y} - \frac{d}{dt}\frac{\partial H}{\partial \dot{y}} = 0. \tag{2.13}$$

This basic result of the calculus of variations, together with the associated boundary conditions, must be solved to obtain the optimal trajectory y(t) of the performance measure.

For the performance measure of equation (2.2) the Euler-Lagranage equation and associated boundary conditions are given by

$$2\, d^2\rho/dt^2 = dh(\rho)/d\rho \tag{2.14}$$

$$\rho(t_o) = \underline{X}_r^T(t_o)\, \underline{X}_r(t_o) \tag{2.15}$$

$$\lim_{t\to\infty} d\rho/dt = 0. \tag{2.16}$$

Multiplying equation (2.14) by $d\rho/dt$ and integrating once gives

$$(d\rho/dt)^2 = h(\rho) \tag{2.17}$$

where the initial condition still applies and the constant of integration has been found to be zero from the final condition of (2.16).

Equation (2.17) is equivalent to

$$d\rho/dt = -\sqrt{h(\rho)} \tag{2.18}$$

where the negative sign of the square root is selected in order to minimize the performance measure for the desired objective of

$$\lim_{t \to \infty} \rho(t) = 0. \tag{2.19}$$

Equation (2.18) is the <u>fundamental</u> <u>inner-product</u> <u>law</u> of the performance measure since it describes in norm the trajectory which minimizes the performance measure of equation (2.2). Since the system error and its derivative depend implicitly upon the state $\underline{X}(t)$ and the control $\underline{U}(t)$, the fundamental inner-prodcut law of equation (2.18) is implicitly a scalar equation involving the components of the state vector and the control vector. Selecting the control vector so that the fundamental inner-product law is satisfied will therefore minimize the performance measure and accomplish the desired objective.

Utilizing the definitions of the error signal and its derivative, from (2.10) and (2.12), the fundamental inner-product law reduces to

$$2\, \underline{X}_r^T(t)\, \dot{\underline{X}}_r(t) = -\sqrt{h(\rho)} \tag{2.20}$$

or equivalently

$$2\, \underline{X}_r^T(t)\, \dot{\underline{X}}_r(t) = 2\, \phi(\rho)\, \underline{X}_r^T(t)\, \underline{X}_r(t) \tag{2.21}$$

where

$$\phi(\rho) = -\sqrt{h(\rho)}/2\rho. \tag{2.22}$$

Equation (2.21) is equivalent to the vector equation

$$K_1 \underline{A}^T \underline{B} = K_2 \underline{A}^T \underline{A} \tag{2.23}$$

where $K_1 = 2$, $K_2 = 2\,\phi(\rho)$, $\underline{A} = \underline{X}_r(t)$ and $\underline{B} = \dot{\underline{X}}_r(t)$.

The solution of this vector equation is considered in Appendix A. If $\underline{A}$, K_1 and K_2 are assumed given, then the solution vector $\underline{B}$ can be represented in the form

$$\underline{B} = (K_2/K_1)\,\underline{A} + \underline{S}\,\underline{A} \tag{2.24}$$

where $\underline{S}$ is any r x r skew symmetric matrix. The solution to equation (2.21) can therefore be represented in the form

$$\dot{\underline{X}}_r(t) = \phi(\rho)\,\underline{X}_r(t) + \underline{S}\,\underline{X}_r(t) \tag{2.25}$$

where $\underline{S}$ is any r x r skew symmetric matrix. Appendices A and B consider the solution for the equivalent representation

$$\dot{\underline{X}}_r(t) = \zeta(\underline{X})\,\underline{M}\,\underline{X}_r(t) + \underline{S}\,\underline{X}_r(t) \tag{2.26}$$

where $\underline{S}$ is any r x r skew symmetric matrix, $\underline{M}$ is any r x r definite matrix and

$$\zeta(\underline{X}) = -\sqrt{h(\rho)}/2(\underline{X}_r^T\,\underline{M}\,\underline{X}_r). \tag{2.27}$$

Substitution of the state equations into equation (2.25) gives the <u>fundamental inner-product law</u> in terms of the control and state:

$$\underline{F}_r(\underline{X}(t),t) + \underline{B}_r(t)\,\underline{U}(t) = \phi(\rho)\,\underline{X}_r(t) + \underline{S}\,\underline{X}_r(t) \tag{2.28}$$

where $$\underline{B}_r(t) = \underline{I}_r\, \underline{B}(t) \tag{2.29}$$

and

$$\underline{F}_r(\underline{X}(t),t) = \underline{I}_r\, \underline{F}(\underline{X}(t),t). \tag{2.30}$$

The solution of the set of r algebraic equations of (2.28) for the control vector $\underline{U}(t)$ has therefore replaced the normal two-point boundary value problem. The solution of this set of equations essentially prescribes the solution for the state equations (2.1) which in norm track the optimal trajectory of the performance measure. Further implications and applications of this result are presented in Chapter III and Chapter IV. The following example illustrates the development and use of the fundamental inner-product law of equation (2.28).

An Introductory Example

Consider the linear system

$$d/dt \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{2.31}$$

The control objective is to drive the components of the state vector to the origin commensurate with a reasonable expenditure of control energy. A reasonable system error signal is therefore given by

$$\rho(t) = x_1{}^2(t) + x_2{}^2(t) + x_3{}^2(t) \tag{2.32}$$

which is in the form prescribed by equation (2.10).

The performance measure to be minimized is

$$J = \int_{t_0}^{\infty} \left[4\rho^2(t) + (d\rho/dt)^2\right] \; dt. \tag{2.33}$$

From equation (2.18), the minimizing trajectory of the performance measure is given by

$$d\rho/dt = -2\ \rho(t). \tag{2.34}$$

Utilizing the definition of the system error signal, the minimizing trajectory reduces to the single algebraic equation

$$x_1\,\dot{x}_1 + x_2\,\dot{x}_2 + x_3\,\dot{x}_3 = -\,(x_1{}^2 + x_2{}^2 + x_3{}^2). \tag{2.35}$$

This equation is analogous to equation (2.21) in the development, and can be expressed in the form of equation (2.23).

$$K_1\,\underline{A}^T\,\underline{B} = K_2\,\underline{A}^T\,\underline{A}$$

where $K_1 = 1$, $K_2 = -1$, $\underline{A}^T = (x_1 \quad x_2 \quad x_3)$ and $\underline{B}^T = (\dot{x}_1 \quad \dot{x}_2 \quad \dot{x}_3)$. The solution of this equation, from (2.25), can be expressed in the form

$$d/dt \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} \\ -\alpha_{12} & 0 & \alpha_{23} \\ -\alpha_{13} & -\alpha_{23} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{2.36}$$

where α_{12}, α_{13}, and α_{23} are the arbitrary components of the skew symmetric matrix $\underline{S}$ in equation (2.25). They may be constants, functions of the state vector or general time-varying functions. The only restriction is that the α coefficients must be bounded.

Decomposing the vector equation into its three component equations and substituting the state equations of (2.31) gives

$$x_2 + x_3 + u_1 = -\,x_1 + \alpha_{12}\,x_2 + \alpha_{13}\,x_3 \tag{2.37}$$

$$-\,x_2 + 2\,x_3 = -\,x_2 - \alpha_{12}\,x_1 + \alpha_{23}\,x_3 \tag{2.38}$$

$$x_1 + x_3 + u_2 = -\,x_3 - \alpha_{13}\,x_1 - \alpha_{23}\,x_2\,. \tag{2.39}$$

Equation (2.37) and (2.39) can be solved directly for the controls u_1 and u_2, and equation (2.38) can be satisfied by selecting $\alpha_{12} = 0$ and $\alpha_{23} = 2$. The resultant controls are then given by

$$u_1 = - x_1 - x_2 + (\alpha_{13} - 1) x_3$$
$$u_2 = - (\alpha_{13} + 1) x_1 - 2 x_2 - 2 x_3.$$

These controls prescribe the general class of one parameter controls for which the state equations of (2.31) track in norm the optimal trajectory of the performance measure of equation (2.33). The controls represent a class of solutions because the coefficient α_{13} is arbitrary. Different choices for the coefficient will obviously result in different system trajectories. The different choices are in reality only describing various optimal trajectories on the optimal manifold of equation (2.34). All such trajectories satisfy in norm the minimizing trajectory of the performance measure of equation (2.34).

Inner-Product Controllable Systems

For the example just considered, the inner-product approach resulted in control laws which were closed form highly flexible solutions. However desirable results of this nature may be, the applicability of the inner-product procedure is not without limitations. The succeeding chapters consider those systems which are suitable to the inner-product approach.

Definition 2.1

A system is inner-product controllable if and only if it satisfies the fundamental inner-product law

$$d\rho/dt = -\sqrt{h(\rho)}$$

associated with the performance measure

$$J = \int_{t_0}^{\infty} \left[h(\rho) + (d\rho/dt)^2\right] dt.$$

Theorem 2.1

A system described by a set of n differential equations of the form

$$\dot{\underline{X}}(t) = \underline{F}(\underline{X}(t),t) + \underline{B}(t)\ \underline{U}(t)$$

is inner-product controllable if the r algebraic equations equivalent to the fundamental inner-product law are satisfied

$$\underline{F}_r(\underline{X}(t),t) + \underline{B}_r(t)\ \underline{U}(t) = \phi(\rho)\ \underline{X}_r(t) + \underline{S}\ \underline{X}_r(t)$$

where $\underline{S}$ is an r x r skew symmetric matrix, $\phi(\rho)$ is defined in (2.22), and $\underline{X}_r$, $\underline{B}_r$ and $\underline{F}_r(\underline{X}(t),t)$ are defined in (2.11), (2.29) and (2.30).

Theorem 2.1 is a direct consequence of the equivalence of the algebraic equations of (2.28) and the fundamental inner-product law of equation (2.18). Either Definition 2.1 or Theorem 2.1 may be used to consider the inner-product controllability of a system.

The determination of the inner-product controllability of a system is a function of the performance measure specified since both the fundamental inner-product law and the algebraic equations contain functions of $h(\rho)$, the error penalty function. A system may therefore be inner-product controllable for one performance measure while being not inner-product controllable for another performance measure.

While the inner-product approach is aimed at obtaining closed form feedback control laws, the resultant controls may not be true closed loop control laws. In some problems the closed form control is essentially a feedback control modeling an open loop control solution. These controls result from the utilization of unspecified initial conditions of the state variables. In order to distinguish the two types of control solution, the following definitions are presented.

Definition 2.2

A system is ρ-controllable if and only if it is inner-product controllable for all $\underline{X}(t_o)$.

Definition 2.3

A system is δ-controllable if and only if it is inner-product controllable for some $\underline{X}(t_o)$ but not all $\underline{X}(t_o)$.

For ρ-controllable systems the control solution consists of the control vector and the skew symmetric matrix for which the fundamental inner-product control or the equivalent algebraic equations are satisfied. For δ-controllable systems the control solution consists of the control vector, the skew symmetric matrix, and any necessary initial condition constraints of the state vector components. In general some of the state vector initial conditions are given and the remainder are arbitrary. The control solution for a δ-controllable system would then specify only the arbitrary initial conditions or constraints on the arbitrary initial conditions, together with the associated control vector and skew symmetric matrix.

Chapter III considers ρ-controllable systems, while Chapter IV considers δ-controllable systems.

CHAPTER III

CLOSED LOOP INNER-PRODUCT FEEDBACK CONTROL

Introduction

Implicit in the inner-product approach is the heretofore tacit assumption that the absolute minimum of the performance measure can be attained by the state equations. The fundamental inner-product law expresses the minimizing trajectory of the performance measure while ignoring the interaction implied by the state equations, and therefore represents the absolute or unconstrained minimum of the performance measure. When the system equations relating the state and control variables are considered, they introduce constraints into the procedure which may nullify the use of the fundamental inner-product law.

The assumption that a solution is possible for a problem can sometimes be a dangerous one, as is illustrated by Perron's Paradox (17). A statement of Perron's paradox is the following: Let N be the largest positive integer. Then for $N \neq 1$, $N^2 > N$ which is contrary to the definition of N as the largest integer. Therefore, $N = 1$.

The implication of this seemingly trivial paradox is that in seeking a solution to a problem, it can not always be assumed that a solution does indeed exist. It does not imply the nonexistence of a solution, but cautions against the assumption that a solution must exist. Chapters III and IV will characterize those systems for which the assumption of a solution is valid for inner-product control problems.

Assuming the applicability of the fundamental inner-product law, the original asymptotic control problem can be restated in the following form. For a dynamic system described by n differential equations,

$$\dot{\underline{X}}(t) = \underline{F}(\underline{X}(t),t) + \underline{B}(t)\,\underline{U}(t) \tag{3.1}$$

determine a control vector $\underline{U}(t)$ and a skew symmetric matrix $\underline{S}$ for which the r algebraic equations equivalent to the fundamental inner-product law are satisfied:

$$\underline{F}_r(\underline{X}(t),t) + \underline{B}_r(t)\,\underline{U}(t) = \phi(\rho)\,\underline{X}_r(t) + \underline{S}\,\underline{X}_r(t) \tag{3.2}$$

where

$$\underline{X}_r(t) = \underline{I}_r\,\underline{X}(t), \tag{3.3}$$

$$\underline{B}_r(t) = \underline{I}_r\,\underline{B}(t), \tag{3.4}$$

$$\underline{F}_r(\underline{X}(t),t) = \underline{I}_r\,\underline{F}(\underline{X}(t),t), \tag{3.5}$$

$$\underline{I}_r = \left[\underline{I} \;\middle|\; \underline{0}\right]. \tag{3.6}$$

$$\phi(\rho) = -\sqrt{h(\rho)}/2\rho, \tag{3.7}$$

and

$$\rho(t) = \underline{X}_r^T(t)\,\underline{X}_r(t). \tag{3.8}$$

Chapter III considers the possibility of satisfying equation (3.2) for all $\underline{X}(t_0)$, while Chapter IV considers the possibility of satisfying equation (3.2) for restricted regions of the state space.

Incomplete Control Solutions

A closed loop control solution for a given dynamic system and inner-product performance measure is defined by the specification of a control vector $\underline{U}$ and a skew symmetric matrix $\underline{S}$ which satisfy the fundamental inner-product law or the equivalent algebraic equations. A control solution may, however, leave several control components unspecified, either in the $\underline{S}$ matrix or in the control vector. Such solutions are <u>incomplete control solutions</u> and must be completed on the basis of factors other than the direct optimization of the performance measure.

The example concluding Chapter II illustrates an incomplete control solution in which a component of the $\underline{S}$ matrix remained arbitrary. The control solution for the example is given by

$$u_1 = - x_1 - x_2 + (\alpha_{13} - 1)\, x_3$$
$$u_2 = - (\alpha_{13} + 1)\, x_1 - 2\, x_2 - 2\, x_3$$

$$\underline{S} = \begin{bmatrix} 0 & 0 & \alpha_{13} \\ 0 & 0 & 2 \\ -\alpha_{13} & -2 & 0 \end{bmatrix}.$$

with the α_{13} component of the $\underline{S}$ matrix arbitrary.

Unspecified components of the $\underline{S}$ matrix are not a disadvantage, but may present an advantage by introducing flexibility into the control solution. The arbitrary α_{13} component of the $\underline{S}$ matrix provides the ability to select a "best" optimal control, based on such factors as implementation simplification. For the above example a possible consideration might be to select α_{13} in order to minimize the hardware necessary to implement the control solution. If the hardware is limited to multipliers and two-input summers, Figure 2 illustrates a minimal implementation for $\alpha_{13} = 1$. This implementation is equivalent to the implementation in Figure 3 in terms of minimizing the performance measure, but is superior in terms of implementation simplicity.

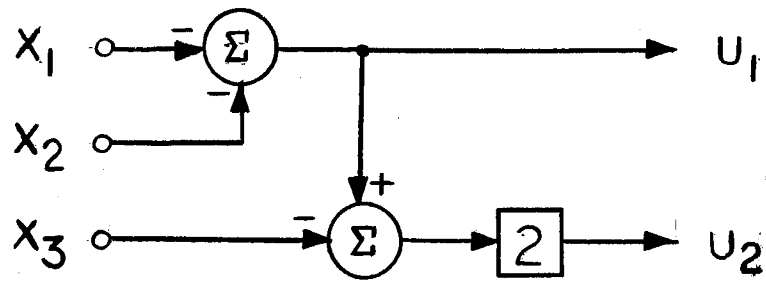

Figure 2. An Optimal Control Structure for $\alpha_{13} = 1$

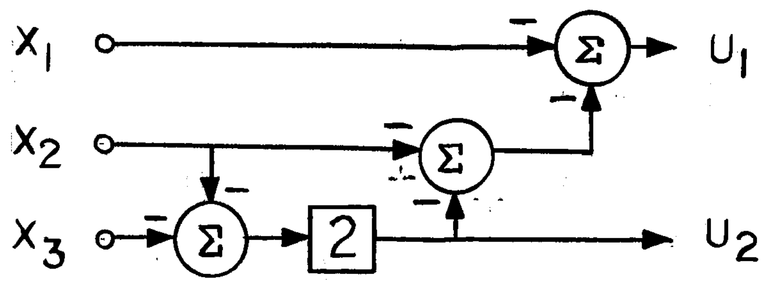

Figure 3. An Optimal Control Structure for $\alpha_{13} = -1$

All choices of α_{13} constitute optimal solutions, for the choices all specify an optimal trajectory on the optimal manifold prescribed by the fundamental inner-product law

$$d\rho/dt = -\sqrt{h(\rho)} \, . \tag{3.9}$$

For the example concluding Chapter II, $h(\rho) = 4\rho^2$, so the solution to the differential equation given by equation (3.9) is

$$\rho(t) = \rho(t_o)\, e^{-2(t - t_o)}. \tag{3.10}$$

After specifying the system error at an initial time t_o, the above equation prescribes precisely the value of the system error for all time after t_o. Since the system error is defined as $\rho = x_1^2 + x_2^2 + x_3^2$, the system error also prescribes the sphere in the three dimensional state space, upon which the system state must lie at any time $t \geq t_o$. Different choices of a_{13} will cause the system to follow different trajectories in state space, but at any time after t_o all such trajectories will lie on the appropriate sphere, which characterizes the optimal manifold in the state space.

If components of the control vector are unspecified, the control solution is incomplete and the arbitrary controls must be selected on the basis of such factors as subsystem stability. The specification of a secondary error signal and performance measure may be utilized to complete the control vector. An illustration of an incomplete control solution with an unspecified control component is provided by the following example.

Example 3.1

Consider the following linear system,

$$d/dt \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{3.11}$$

with a system error signal of

$$\rho = x_1^2 + x_2^2. \tag{3.12}$$

The performance measure to be minimized is given by

$$J = \int_{t_o}^{\infty} (4\rho^2 + (d\rho/dt)^2) \quad dt, \tag{3.13}$$

and is minimized by the fundamental inner-product law,

$$d\rho/dt = - 2 \rho .$$

Substitution of the error signal definition reduces the minimizing trajectory to

$$2 x_1 (dx_1/dt) + 2 x_2 (dx_2/dt) = - 2 (x_1^2 + x_2^2).$$

The algebraic equations equivalent to the above are

$$dx_1/dt = - x_1 + \alpha x_2$$
$$dx_2/dt = - x_2 - \alpha x_1$$

where α denotes the nonzero entry of the $\underline{S}$ matrix.

Removing derivatives by use of the state equations yields

$$- x_1 + 2 x_2 = - x_1 + \alpha x_2$$
$$x_1 + x_3 + u_1 = - x_2 - \alpha x_1.$$

The first equation is satisfied if the nonzero entry of the $\underline{S}$ matrix is selected to be 2. The second equation is satisfied if u_1 is selected in feedback form to satisfy the equation. The control solution is therefore incomplete for it fails to specify the control component u_2.

$$\underline{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad u_1 = - 3 x_1 - x_2 - x_3 \tag{3.14}$$

$$\underline{S} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

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The utilization of a secondary error signal therefore completes the control solution by specifying the arbitrary control vector component. A complete control solution for Example 3.1 is given by

$$\underline{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -3 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\underline{S} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

In general, a secondary error signal can be defined as the inner-product of those states whose derivatives are controlled by the missing controls. In the above example, the unspecified control u_2 drives the derivative of x_3, hence the secondary error signal of x_3^2 was selected. If u_1 and u_2 had been unspecified in the above example, a secondary error signal of $x_2^2 + x_3^2$ would be defined since u_1 and u_2 appear in the state equations of x_2 and x_3. The utilization of a secondary error signal as described results in a secondary problem which is classed as a directly ρ-controllable system. The solution of directly ρ-controllable systems is presented in the following section.

Incomplete control solutions present no major drawback to inner-product optimal control problems. The unspecified components may be determined on the basis of secondary considerations. In the above, sub-system stability and implementation simplification were used to illustrate possible approaches. Other factors could be utilized to complete the control solution, and result in valid optimal control systems since the fundamental inner-product law remains satisfied. The only restriction is that solutions must be tractable and feasible.

An amplification and extension of the incomplete control concept includes those control solutions which utilize only the control vector or only the $\underline{S}$ matrix to satisfy the fundamental inner-product law. If a control solution uses only the control vector the system is termed a directly ρ-controllable system. If a control solution uses only the $\underline{S}$ matrix the system is called an α-controllable system. The following two sections will consider directly ρ-controllable systems and α-controllable systems.

Directly ρ-Controllable Systems

When the rank of the control distribution matrix $\underline{B}$ is of rank n, the system is termed directly controllable (18). The controls of such a system can affect all components of the time derivative of the state vector directly, hence the name directly controllable systems. The ability to directly control the state vector derivative reduces the control of the system to the control of n first order differential equations. In order to extend the directly controllable concept, define the system error signal as a quadratic form of rank r as in equation (3.8),

$$\rho = x_1^2 + x_2^2 + \dots + x_r^2 .$$

Definition 3.1

If the controls of the system can affect independently and directly the time derivative of the states $x_1, x_2, \dots, x_r$, then the system is termed directly ρ-controllable.

The ability to control directly the first r state vector derivatives results in control solutions which use only control vector components to satisfy the fundamental inner-product law.

Since directly controllable systems control independently and directly all state vector derivatives, they are included in the class of systems which are directly ρ-controllable.

Without loss of generality, the state equations of a directly controllable system can be assumed to be of the form:

$$\dot{\underline{X}}(t) = \underline{F}(\underline{X}(t),t) + \underline{V}(t) \tag{3.17}$$

where $\underline{V}(t) = \underline{B}(t)\,\underline{U}(t)$. If the $\underline{B}$ matrix is m x n and of rank n, then n = m or m $>$ n. Either case is represented by the control vector $\underline{V}(t)$.

If the $\underline{B}$ matrix is n x n and of rank n, then it is nonsingular. Once a solution for $\underline{V}(t)$ is obtained, a solution for the control $\underline{U}(t)$ can be determined uniquely by

$$\underline{U}(t) = \underline{B}^{-1}(t)\,\underline{V}(t). \tag{3.18}$$

If the $\underline{B}$ matrix is m x n and m $>$ n, the control portion of the system is analogous to a consistent system of n equations with m unknowns. Such a system can be solved for n unknowns in terms of the remaining m-n unknowns. The control vector can therefore be decomposed into two parts and equated to the control vector $\underline{V}(t)$,

$$\underline{V}(t) = \underline{B}_1(t)\,\underline{U}_1(t) + \underline{B}_2(t)\,\underline{U}_2(t), \tag{3.19}$$

where $\underline{B}_1$ is n x n and nonsingular and $\underline{B}_2$ is (m - n) x n. Once a solution for $\underline{V}(t)$ is obtained for the system in equation (3.17), a solution for $\underline{U}_1(t)$ can be determined in terms of the solution $\underline{V}(t)$ and the remaining terms of the control.

$$\underline{U}_1(t) = \underline{B}_1^{-1}(t)\left[\underline{V}(t) - \underline{B}_2(t)\,\underline{U}_2(t)\right] \tag{3.20}$$

The specification of a $\underline{U}_2$ vector defines the remaining control vector $\underline{U}_1$ and therefore defines the total control vector $\underline{U}(t)$. Very few control problems involve a $\underline{B}$ matrix for which $m > n$; however, if such is the case, the state equations of such a system are represented by the directly controllable system of equation (3.17) if $\underline{B}$ is of rank n.

Theorem 3.1

All directly controllable systems are ρ-controllable.

Proof: For directly controllable systems, the algebraic equations of (3.2) are given by

$$\underline{I}_r\, \underline{V}(t) = \phi(\rho)\, \underline{X}_r(t) + \underline{S}\, \underline{X}_r(t) - \underline{F}_r(\underline{X}(t),t). \qquad (3.21)$$

If the control vector components v_1, v_2, ..., v_r are defined in feedback form as specified by the above equations, then the r algebraic equations equivalent to the fundamental inner-product law are satisfied for all $\underline{X}(t)$. If the equations are satisfied for all $\underline{X}(t)$ they are satisfied for all $\underline{X}(t_o)$ and the system is ρ-controllable from Definition 2.2.

Equation (3.21) defines the first r components of the n-dimensional control vector $\underline{V}(t)$, and leaves n - r control vector components and the $\underline{S}$ matrix components unspecified. The techniques of the previous section can be employed to complete the control solution. The following example by Leeper and Mulholland (19) illustrates the control solution for a directly controllable system.

Example 3.2

Consider the problem of a body spinning in free space, in which the control objective is to stop the tumbling of the body and stabilize the attitude of the system. The state equations of the system are given by

$$dx_1/dt = \beta_1\ x_2\ x_3 + v_1$$
$$dx_2/dt = \beta_2\ x_1\ x_3 + v_2$$
$$dx_3/dt = \beta_3\ x_1\ x_2 + v_3$$

where x_1, x_2 and x_3 are the components of the angular momentum vector and v_1, v_2 and v_3 are the control torques. The β coefficients are constants defined by

$$\beta_1 = (I_2 - I_3)/I_2\ I_3$$
$$\beta_2 = (I_3 - I_1)/I_1\ I_3$$
$$\beta_3 = (I_1 - I_2)/I_1\ I_2$$

where I_1, I_2 and I_3 are the moments of inertia about the principal axis.

If the angular momentum vector is zero then the angular velociites are zero and the body is not tumbling. The objective can therefore be stated simply as driving the angular momentum vector to zero, and a natural and reasonable error signal is given by

$$\rho = {x_1}^2 + {x_2}^2 + {x_3}^2.$$

Using the general performance measure of equation (2.2),

$$J = \int_{t_0}^{\infty} \left[h(\rho) + (d\rho/dt)^2\right] dt,$$

the minimizing trajectory from the fundamental inner-product law is given by

$$x_1v_1 + x_2v_2 + x_3v_3 = \phi(\rho)\ (x_1^2 + x_2^2 + x_3^2)$$

where

$$\phi(\rho) = -\sqrt{h(\rho)}/\ 2\rho.$$

The algebraic equations equivalent to the fundamental inner-product law are

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \phi(\rho) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} \\ -\alpha_{12} & 0 & \alpha_{23} \\ -\alpha_{13} & -\alpha_{23} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

All optimal controllers satisfying the algebraic equations are defined uniquely by the three parameters

$$(\alpha_{12} \quad \alpha_{13} \quad \alpha_{23}).$$

Specification of the parameters will define the control structure for the above algebraic equations.

Mulholland and Rhoten (9) have solved this problem using the above formulation under the assumption of a fixed control configuration. Although the resultant control is a closed form controller minimizing the performance measure, the solution is a member of the general class of solutions obtained by utilizing the skew symmetric matrix.

The fixed configuration controller used by Mulholland and Rhoten is given by

$$v_i = \phi(\rho)\ x_i \qquad i = 1,\ 2,\ 3.$$

Clearly this control is a special solution of the algebraic equations obtained when $\alpha_{12} = \alpha_{13} = \alpha_{23} = 0.$

With $\beta_1 = -.25$, $\beta_2 = .75$, $\beta_3 = -.50$ and $x_1(t_o)=x_2(t_o)=x_3(t_o)=5$, and a performance measure of

$$J = \int_{t_o}^{\infty} \left[4\,\rho^4 + (d\rho/dt)^2\right] dt,$$

the optimal controls are given by

$$\begin{aligned} v_1 &= -\rho\, x_1 + \alpha_{12}\, x_2 + \alpha_{13}\, x_3 \\ v_2 &= -\rho\, x_2 - \alpha_{12}\, x_1 + \alpha_{23}\, x_3 \\ v_3 &= -\rho\, x_3 - \alpha_{13}\, x_1 - \alpha_{23}\, x_2 \quad . \end{aligned}$$

A set of trajectories for $x_1(t)$ is given in Figure 4, where each trajectory represents the response of $x_1(t)$ to a particular control. The three parameters identified with each trajectory are α_{12}, α_{13}, and α_{23} which define the control structure. The state trajectories of x_2 and x_3 are similar in form.

The example illustrates the control structure flexibility introduced by the skew symmetric matrix for directly controllable systems. The generality is not without limits. If the fundamental inner-product law is solved for this example, the minimizing trajectory of the system in norm is

$$\rho(t) = 75 \;/\; (150(t - t_o) + 1).$$

For all time the solution trajectory of x_1, x_2, and x_3 satisfy this equation, and are bounded by the fixed values of $\pm\sqrt{\rho(t)}$. The skew symmetric matrix in reality introduces damped oscillations which may yield improved system trajectories.

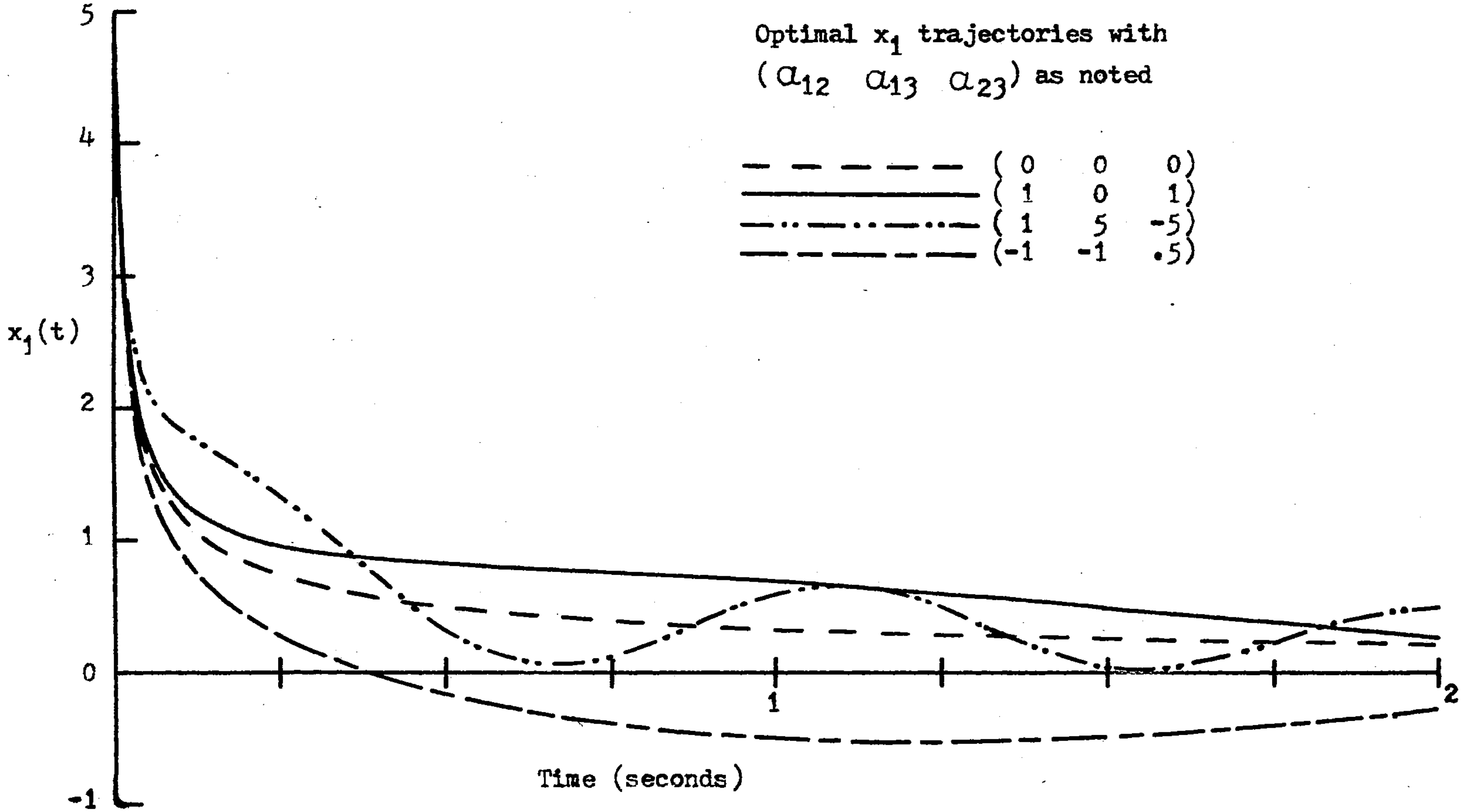

Figure 4. Optimal x_1 Trajectories for Example 3.2

Theorem 3.2

All directly ρ-controllable systems are ρ-controllable.

Proof: The algebraic equations of (3.2) are

$$\underline{B}_r(t)\ \underline{U}(t) + \underline{F}_r(\underline{X}(t),t) = \phi(\rho)\ \underline{X}_r(t) + \underline{S}\ \underline{X}_r(t). \tag{3.22}$$

A system is directly ρ-controllable if the controls of the system can affect dx_1/dt, $dx_2/dt, \ldots, dx_r/dt$ directly and independently. The $\underline{B}_r$ matrix is therefore an r x m matrix of rank r.

If m = r, then $\underline{B}_r$ is nonsingular and the control is given by

$$\underline{U}(t) = \underline{B}_r^{-1}(t)\ \left[\left(\phi(\rho)\ \underline{I} + \underline{S}\right)\ \underline{X}_r - \underline{F}_r(\underline{X}(t),t)\right]. \tag{3.23}$$

If m $>$ r, the control signal $\underline{B}_r\ \underline{U}$ can be decomposed into two parts,

$$\underline{B}_r\ \underline{U}(t) = \underline{B}_{r1}\ \underline{U}_1(t) + \underline{B}_{r2}\ \underline{U}_2(t)$$

where $\underline{B}_{r1}$ is an r x r nonsingular matrix, and $\underline{B}_{r2}$ is an r x (m - r) matrix. The control solution for m $>$ r is given by

$$\underline{U}_1(t) = \underline{B}_{r1}^{-1}\ \left[\left(\phi(\rho)\ \underline{I} + \underline{S}\right)\ \underline{X}_r(t) - \underline{F}_r(\underline{X}(t),t) - \underline{B}_{r2}\ \underline{U}_2(t)\right]. \tag{3.24}$$

Since the control vector $\underline{U}$ is composed of the components of $\underline{U}_1$ and $\underline{U}_2$, the above control solution is an incomplete control solution. The skew symmetric matrix $\underline{S}$ and the control vector $\underline{U}_2$ must be determined on the basis of secondary considerations as presented in the previous section. Once $\underline{U}_2$ and $\underline{S}$ are defined, the control vector $\underline{U}_1$ will be defined. Since the fundamental inner-product law is satisfied for all $\underline{X}(t)$, the system is ρ-controllable.

Corollary 3.1

If the rank $\underline{B} \geq$ rank $\underline{I}_r\ \underline{B} = r$, then the system is ρ-controllable.

Proof: If rank $\underline{B} \geq$ rank $\underline{I}_r\ \underline{B} = r$ then the system is directly ρ-controllable and from Theorem 3.2 is then ρ-controllable.

α-Controllable Systems

While some control solutions utilize only the control vector in the control solution, the control solution may also only specify components of the skew symmetric matrix $\underline{S}$. The elements of the $\underline{S}$ matrix are denoted by the $n(n-1)/2$ elements α_{ij} where $i = 1, 2, \ldots, n-1$ and $j = i+1, i+2, \ldots, n$.

Definition 3.2

A system is called an α-controllable system is it is ρ-controllable and utilizes only the $\underline{S}$ matrix in the control solution.

Lemma 3.1

If a system is α-controllable then $\underline{I}_r\ \underline{B} = \underline{0}$.

Proof: If a system is α-controllable then the algebraic equations of (3.2) are satisfied for a given skew symmetric matrix. This control solution is valid for all bounded control vectors, so assume $\underline{U} = \underline{0}$. The $\underline{S}$ matrix of the control solution satisfies

$$\phi(\rho)\ \underline{X}_r + \underline{S}\ \underline{X}_r - \underline{F}_r(\underline{X}(t),t) = \underline{0}. \qquad (3.25)$$

Assume $\underline{I}_r\ \underline{B} \neq \underline{0}$. At least one of the algebraic equations of (3.2) contains a control vector component. Suppose the kth component of $\underline{I}_r\ \underline{B}\ \underline{U}$ is a nonzero component, and denote it by u_1. The kth algebraic equation is given by

$$u_1 = 0$$

since the kth row of the remainder of the equation is zero from (3.25).

This contradicts the assumption that u_1 is nonzero, so the assumption that a system can be α-controllable with $\underline{B}_r \neq \underline{0}$ is false and the theorem is proven.

Theorem 3.3

A system is α-controllable if and only if

i) $\underline{B}_r = \underline{0}$ and (3.26)

ii) $\underline{F}_r(\underline{X}(t),t) = \left[\phi(\rho)\,\underline{I} + \underline{S}\right]\,\underline{X}_r(t)$ (3.27)

for some r x r skew symmetric matrix $\underline{S}$.

Proof: Substitution of equations (3.26) and (3.27) into the algebraic equations of (3.2) verifies the sufficiency of the conditions. Equation (3.26) is a necessary condition from Lemma 3.1, so the theorem is proven if equation (3.27) is shown to be a necessary condition for satisfying the algebraic equations of (3.2). Substitution of (3.26) into (3.2) reduces the equations to the form of equation (3.27), proving the theorem.

Example 3.3

Consider the nonlinear system

$$d/dt \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2\, x_3 - x_1 \\ -\, x_2 - x_1\, x_3 \\ x_1{}^2 + x_2 \\ x_1\, x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

with a system error signal of

$$\rho = x_1^{\,2} + x_2^{\,2}$$

and a performance measure of

$$J = \int_{t_0}^{\infty} \left[4\rho^2 + (d\rho/dt)^2 \right] dt.$$

It is readily verfied that the conditions of Theorem 3.3 are satisfied for the above formulation. For the error signal specified,

$$\underline{I}_r = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

and

$$\underline{B}_r = \underline{I}_r \, \underline{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which verifies equation (3.26).

For the performance measure specified, $h(\rho) = 4\rho^2$, so

$$\phi(\rho) = -1.$$

The algebraic equations of (3.27) are given by

$$x_2 x_3 - x_1 = -x_1 + \alpha_{12} x_2$$
$$-x_2 - x_1 x_3 = -\alpha_{12} x_1 - x_2$$

and are satisfied if $\alpha_{12} = x_3$. The above formulation is therefore an α-controllable system in which the control vector components are selected from secondary considerations.

In the general system formulation of equation (3.1), the condition of equation (3.26) requires the r algebraic equations of (3.2) to contain no control terms. If a control appears in an equation the control is then specified by the equation and the system is no longer an α-controllable system. The condition of equation (3.27) specifies the form which the plant structure must possess in order to be an α-controllable system.

In order to better visualize the implications of Theorem 3.3, the theorem can be applied to linear systems. An n-dimensional linear system given by

$$\dot{\underline{X}} = \underline{A}\,\underline{X} + \underline{B}\,\underline{U} \tag{3.28}$$

can be represented in the partitioned form

$$d/dt \begin{bmatrix} \underline{X}_r \\ \hline \underline{X}_s \end{bmatrix} = \left[\begin{array}{c|c} \underline{A}_{rr} & \underline{A}_{rs} \\ \hline \underline{A}_{sr} & \underline{A}_{ss} \end{array}\right] \begin{bmatrix} \underline{X}_r \\ \hline \underline{X}_s \end{bmatrix} + \begin{bmatrix} \underline{B}_r \\ \hline \underline{B}_s \end{bmatrix} \underline{U} \tag{3.29}$$

where $\underline{X}_r = \underline{I}_r\,\underline{X}$, $\underline{B}_r = \underline{I}_r\,\underline{B}$ and the remaining matrices and vectors are appropriately dimensioned submatrices and subvectors of $\underline{A}$, $\underline{B}$ and $\underline{X}$. Let $s = (n - r)$. $\underline{X}_r$ is r x 1, $\underline{X}_s$ is s x 1, $\underline{A}_{rr}$ is r x r, $\underline{A}_{rs}$ is r x s, $\underline{A}_{sr}$ is s x r, $\underline{A}_{ss}$ is s x s, $\underline{B}_r$ is r x m, and $\underline{B}_s$ is s x m.

<u>Theorem 3.4</u>

A linear system in the form of equation (3.29) is α-controllable if and only if

i) $\underline{B}_r = \underline{0}$, (3.30)

ii) $\underline{A}_{rs} = \underline{0}$, (3.31)

and iii) $\underline{A}_{rr} = \phi(\rho)\,\underline{I} + \underline{S}$ (3.32)

for some r x r skew symmetric matrix $\underline{S}$.

Proof: Equation (3.30) of Theorem 3.4 and equation (3.26) of Theorem 3.3 are equivalent. If equation (3.27) is equivalent to equations (3.31) and (3.32) the theorem prrof is complete.

Substituting the state equations of the linear system of (3.29) into equation (3.27) yields

$$\underline{A}_{rs}\,\underline{X}_s + \left[\,\underline{A}_{rr} - \phi(\rho)\,\underline{I} - \underline{S}\,\right]\,\underline{X}_r = \underline{0}. \qquad (3.33)$$

The sufficiency of equations (3.31) and (3.32) follows from the substitution of the equations into (3.33).

A system which is α-controllable is ρ-controllable, and from the definition of a ρ-controllable system equation (3.33) must be valid for all $\underline{X}(t_0)$. If equation (3.33) must be valid for all $\underline{X}(t_0)$, it must be valid for all $\underline{X}(t)$.

The necessity of equation (3.31) is verified from equation (3.33) if $\underline{X}_r = \underline{0}$ and the components of $\underline{X}_s$ are 1. The necessity of equation (3.32) is verified from equation (3.33) if $\underline{X}_s = \underline{0}$ and the components of $\underline{X}_r$ are 1. Therefore equation (3.27) is equivalent to (3.31) and (3.32) for the linear system of (3.29), and the proof is completed.

The conditions of Theorem 3.4 reduce the linear system of (3.29) to the following form:

$$d/dt \begin{bmatrix} \underline{X}_r \\ \hline \underline{X}_s \end{bmatrix} = \left[\begin{array}{c|c} \underline{A}_{rr} & \underline{0} \\ \hline \underline{A}_{sr} & \underline{A}_{ss} \end{array}\right] \begin{bmatrix} \underline{X}_r \\ \hline \underline{X}_s \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \hline \underline{B}_s \end{bmatrix} \underline{U}. \qquad (3.34)$$

In addition, equation (3.32) requires the diagonal elements of $\underline{A}_{rr}$ to be equal to the function $\phi(\rho)$, and the off diagonal elements of $\underline{A}_{rr}$ to be skew symmetric. If the principal diagonal elements of $\underline{A}_{rr}$ are constant and equal, then the function $\phi(\rho)$ must also be a constant. Since

$$\phi(\rho) = -\sqrt{h(\rho)}/2\rho \ , \tag{3.35}$$

$\phi(\rho)$ is a constant if and only if $h(\rho)$ can be expressed in the form

$$h(\rho) = 4\ k^2\ \rho^2$$

where k is a constant real number. This leads directly to the following corollary of Theorem 3.4.

Corollary 3.2

A linear system in the form of equation (3.29), for which the diagonal elements of $\underline{A}_{rr}$ are constant, is α-controllable if and only if

i) $\underline{B}_r = \underline{0}$, (3.36)

ii) $\underline{A}_{rs} = \underline{0}$, (3.37)

iii) $h(\rho) = 4\ k^2\ \rho^2$ where k is a real number, and (3.38)

iv) $\underline{A}_{rr} = k\ \underline{I} + \underline{S}$ (3.39)

for some r x r skew symmetric matrix $\underline{S}$.

The proof of Corollary 3.2 follows directly from Theorem 3.4 and the discussion preceeding the corollary.

The results of Corollary 3.2 are not restricted to constant coefficient linear systems, but only to those with constant coefficients in the diagonal entries of the matrix $\underline{A}_{rr}$.

The extension of Theorem 3.4 to linear systems for which the diagonal elements of $\underline{A}_{rr}$ are time-varying is more difficult since the diagonal elements must equal $\phi(\rho)$ which is not an explicit time function. However, if $\phi(\rho)$ can be expressed as a time function then Theorem 3.4 can be extended to this situation.

If the diagonal elements of $\underline{A}_{rr}$ are time-varying and equal, the system may be α-controllable if $\rho(t)$ can be determined as a closed form time function. If the fundamental inner-product law can be solved directly this requirement can be satisfied.

For example, if $h(\rho) = \rho^4$ then the fundamental inner-product law is given by

$$d\rho/dt = - \rho^2.$$

The solution of this differential equation is given by

$$\rho(t) = \rho(t_o)/(\rho(t_o)(t - t_o) + 1)$$

and $\phi(\rho)$ can then be expressed as a time function

$$\phi(t) = - 1/(t + a)$$

where a is the constant given by

$$a = - t_o + (1/\rho(t_o)).$$

A linear system with time-varying coefficients can then satisfy the conditions of Theorem 3.4 if the diagonal elements of $\underline{A}_{rr}$ are equal to the time function $\phi(t)$.

<u>Corollary 3.3</u>

A linear system in the form of equation (3.29), for which the diagonal elements of $\underline{A}_{rr}$ are time-varying is α-controllable if and only if

i) the fundamental inner-product law can be solved for $\rho(t)$ as a time function,

ii) $\underline{B}_r = \underline{0}$,

iii) $\underline{A}_{rs} = \underline{0}$, and

iv) $\underline{A}_{rr}(t) = (\phi(t)\,\underline{I} + \underline{S}(t))$

for some r x r skew symmetric matrix $\underline{S}(t)$.

Proof: The sufficiency of the conditions follows directly from equation (3.2) and the necessity of the conditions follows from Theorem 3.4 and the remarks preceding the corollary.

To illustrate the conditions of Corollary 3.2 and Corollary 3.3 the following example is given.

Example 3.4

Consider the general fourth order linear system

$$d/dt \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

with an error signal of

$$\rho = {x_1}^2 + {x_2}^2 .$$

The condition that $\underline{B}_r = \underline{0}$ for Corollary 3.2 and Corollary 3.3 requires

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \qquad (3.40)$$

The condition that $\underline{A}_{rs} = \underline{0}$ requires

$$\begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \qquad (3.41)$$

The off diagonal elements of $\underline{A}_{rr}$ must be skew symmetric, which requires

$$a_{12} = -a_{21} . \tag{3.42}$$

The diagonal elements of $\underline{A}_{rr}$ must equal $\phi(\rho)$, which requires

$$a_{11} = a_{22} = \phi(\rho). \tag{3.43}$$

The general fourth order linear system is α-controllable for a system error of

$$\rho = {x_1}^2 + {x_2}^2$$

if the system is of the form

$$d/dt \left[\begin{array}{c} x_1 \\ x_2 \\ \hline x_3 \\ x_4 \end{array}\right] = \left[\begin{array}{cc|cc} a_{11} & -a_{21} & 0 & 0 \\ a_{21} & a_{11} & 0 & 0 \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \hline x_3 \\ x_4 \end{array}\right] + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline b_{31} & b_{32} \\ b_{41} & b_{42} \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where

$$a_{11} = \phi(\rho).$$

If a_{11} and a_{22} are constants, then the system is α-controllable only if the error penalty function is of the form

$$h(\rho) = 4\ k^2\ \rho^2$$

and

$$a_{11} = k .$$

If a_{11} and a_{22} are time-varying, then the system is α-controllable only if the fundamental inner-product law can be solved for $\rho(t)$ as a time function. For example, if $h(\rho) = \rho(t)^{2k}$ with $k > 1$, then the fundamental inner-product law can be solved for $\rho(t)$, and the function $\phi(\rho)$ can then be expressed in the form

$$\phi(t) = -\tfrac{1}{2}\, a^{k-1} / \left[(t - t_o) + (a/\rho(t_o))^{k-1}\right],$$

where

$$a = 1/(k-1)^{1/(k-1)}.$$

The development of this result follows from the results of Rhoten and Mulholland (8).

ρ-Controllable Systems

If a system is ρ-controllable it satisfies the r algebraic equations

$$\underline{B}_r\ \underline{U}(t) = \phi(\rho)\ \underline{X}_r + \underline{S}\ \underline{X}_r - \underline{F}_r(\underline{X}(t),t)$$

for all $\underline{X}(t)$. If the rank of $\underline{B}_r$ is denoted by b then there exists a nonsingular transformation matrix $\underline{T}$ such that the above equations can be transformed and partitioned into the form

$$\begin{bmatrix} \underline{U}'(t) \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{T}_1 \\ \underline{T}_2 \end{bmatrix} \left(\left[\phi(\rho)\ \underline{I} + \underline{S}\right] \quad \underline{X}_r - \underline{F}_r(\underline{X}(t),t) \right)$$

where

$$\underline{T} = \begin{bmatrix} \underline{T}_1 \\ \hline \underline{T}_2 \end{bmatrix}$$

and

$$\underline{U}'(t) = \underline{T}_1\ \underline{B}_r\ \underline{U}(t).$$

The transformation matrix $\underline{T}_1$ is b x r and the transformation matrix $\underline{T}_2$ is (r - b) x r. The transformation decomposes the r algebraic equations into two sets of equations. The first r algebraic equations constitute a directly ρ-controllable subsystem where $\underline{U}'(t)$ is specified in feedback form as dictated by the first r equations:

$$\underline{U}'(t) = \underline{T}_1(\phi(\rho) \underline{X}_r + \underline{S} \underline{X}_r - \underline{F}_r(\underline{X}(t),t) \qquad (3.44)$$

for any r x r skew symmetric matrix $\underline{S}$. If the remaining (r - b) algebraic equations, given by

$$\underline{T}_2 \underline{S} \underline{X}_r = \underline{T}_2 (\underline{F}_r(\underline{X}(t),t) - \phi(\rho) \underline{X}_r), \qquad (3.45)$$

are satisfied by specification of the $\underline{S}$ matrix components, then the system is ρ-controllable. Likewise, if the system is ρ-controllable then the algebraic equations of (3.44) and (3.45) must be satisfied. Since the equations of (3.44) are satisfied by specifying the controls as indicated, the general requirement for a system to be ρ-controllable is that the equations of (3.45) be satisfied.

Theorem 3.5

Let b denote the rank of $\underline{B}_r$. A system is ρ-controllable if and only if

$$\underline{T}_2 \underline{S} \underline{X}_r = \underline{T}_2 (\underline{F}_r(\underline{X}(t),t) - \phi(\rho) \underline{X}_r)$$

for some r x r skew symmetric matrix $\underline{S}$. $\underline{T}_2$ is an (r - b) x r matrix of rank (r - b) and

$$\underline{T}_2 \underline{B}_r = \underline{0} \, .$$

Theorem 3.5 simply expresses the requirement of equation (3.45) in a formal theorem. The proof is a direct result of the development.

As in the section on α-controllable systems, the import of the general theorem is best illustrated when applied to linear systems. An n-dimensional linear system can be partitioned in the form

$$d/dt \begin{bmatrix} \underline{X}_r \\ \hline \underline{X}_s \end{bmatrix} = \begin{bmatrix} \underline{A}_r & | & \underline{A}_s \end{bmatrix} \begin{bmatrix} \underline{X}_r \\ \hline \underline{X}_s \end{bmatrix} + \begin{bmatrix} \underline{B}_r \\ \hline \underline{B}_s \end{bmatrix} \underline{U} \tag{3.46}$$

where (3.46) is equivalent to the partitioned linear system of (3.29) and $\underline{A}_r$ and $\underline{A}_s$ are given by

$$\underline{A}_r = \begin{bmatrix} \underline{A}_{rr} \\ \hline \underline{A}_{sr} \end{bmatrix} \qquad \underline{A}_s = \begin{bmatrix} \underline{A}_{rs} \\ \hline \underline{A}_{ss} \end{bmatrix} .$$

Theorem 3.6

A linear system in the form of equation (3.46) is ρ-controllable if and only if

i) $\underline{T}_2 \underline{A}_s = \underline{0}$, and

ii) $\underline{T}_2 \underline{A}_r = \underline{T}_2 \underline{S} + \phi(\rho) \underline{T}_2$

for some r x r skew symmetric matrix $\underline{S}$. The matrix $\underline{T}_2$ is an (r - b) x r matrix of rank (r - b) for which

iii) $\underline{T}_2 \underline{B}_r = \underline{0}$.

Proof: The sufficiency of the conditions is verified by substitution into the algebraic equations of (3.45). From Theorem 3.5, the linear system is ρ-controllable if

$$\left[\underline{T}_2 \underline{A}_r - \phi(\rho) \underline{T}_2 - \underline{T}_2 \underline{S} \right] \underline{X}_r + (\underline{T}_2 \underline{A}_s) \underline{X}_s = \underline{0} \tag{3.47}$$

where $\underline{T}_2$ is an (r - b) x r matrix of rank (r - b) and

$$\underline{T}_2 \underline{B}_r = \underline{0}.$$

Since equation (3.47) must be valid for all $\underline{X}(t)$, the necessity of condition i is verified from equation (3.47) if $\underline{X}_r(t) = \underline{0}$ and the components of $\underline{X}_s$ are 1. Condition ii is verified from equation (3.47) if $\underline{X}_s = \underline{0}$ and the components of $\underline{X}_r$ are 1, completing the proof.

The use of Theorem 3.6 is illustrated by the following example.

Example 3.5

Consider the linear system

$$d/dt \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & -2 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

with a system error signal of

$$\rho = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

and a performance measure of

$$J = \int_{t_o}^{\infty} \left[4\,\rho^2 + (d\rho/dt)^2\right] dt.$$

The algebraic equations which minimize the performance measure are

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 & a_{12} & a_{13} & a_{14} \\ -a_{12} & -1 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & -1 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & -2 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

For this example, r = n, so condition i of Theorem 3.6 is trivially satisfied, The rank of $\underline{B}_r$ is 2, so a suitable transformation matrix is given by

$$\underline{T} = \begin{bmatrix} \underline{T}_1 \\ \hline \underline{T}_2 \end{bmatrix} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ \hline 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

and the transformed algebraic equations are given by

$$\left[\begin{array}{c} u_1 \\ u_2 \\ \hline 0 \\ 0 \end{array}\right] = \left[\begin{array}{cccc} -1 & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ (-\alpha_{13}+1) & (-\alpha_{23}-\alpha_{12}) & (-1-\alpha_{13}) & (\alpha_{34}-\alpha_{14}) \\ \hline (\alpha_{12}-\alpha_{13}-2) & (2\alpha_{12}+1-\alpha_{23}) & (2\alpha_{13}-\alpha_{23}-1) & (2\alpha_{14}-\alpha_{24}+\alpha_{34}) \\ -\alpha_{14} & -\alpha_{24} & -\alpha_{34} & -1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & -1 & 2 & 3 \\ -2 & 1 & -1 & 2 \\ -2 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

The first two algebraic equations correspond to equation (3.44) and are satisfied by specifying u_1 and u_2 as dictated by the equations. The last two equations correspond to equation (3.45) and are satisfied by specification of the α coefficients. Equating the coefficients in the last two equations to zero yields the necessary conditions:

$$(-2 + \alpha_{12} - \alpha_{13}) + 2 = 0, \qquad \alpha_{14} - 2 = 0,$$
$$(2\alpha_{12} + 1 - \alpha_{23}) - 1 = 0, \qquad \alpha_{24} + 1 = 0,$$
$$(2\alpha_{13} - \alpha_{23} + 1) + 1 = 0, \qquad \alpha_{34} - 1 = 0,$$
$$(2\alpha_{14} - \alpha_{24} + \alpha_{34}) - 2 = 0 .$$

The necessary conditions therefore reduce to

$$\alpha_{14} = 2, \qquad \alpha_{24} = 1, \qquad \alpha_{34} = -1,$$

$$\alpha_{23} = 2\,\alpha_{12} = 2\,\alpha_{13}.$$

The control solution of the example is then given by

$$\underline{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 & (\alpha_{12} - 1) & (\alpha_{12} + 1) & 3 \\ (2 - \alpha_{12}) & (1 - 3\alpha_{12}) & (-1 - \alpha_{12}) & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\underline{S} = \begin{bmatrix} 0 & \alpha_{12} & \alpha_{12} & 2 \\ -\alpha_{12} & 0 & 2\alpha_{12} & 1 \\ -\alpha_{12} & -2\alpha_{12} & 0 & -1 \\ -2 & -1 & 1 & 0 \end{bmatrix}$$

where α_{12} is selected from secondary considerations.

The results of Corollary 3.2 and Corollary 3.3 can be extended to general ρ-controllable systems.

<u>Corollary 3.4</u>

A linear system with constant coefficients is ρ-controllable if and only if

i) $\underline{T}_2\, \underline{A}_s = \underline{0}$,

ii) $h(\rho) = 4\, k^2 \rho^2$ where k is a real number,

iii) $\underline{T}_2\, \underline{A}_r = \underline{T}_2\, \underline{S} + k\, \underline{T}_2$ for some skew symmetric matrix $\underline{S}$.

The matrix $\underline{T}_2$ is an $(r - b) \times r$ matrix of rank $(r - b)$ and

iv) $\underline{T}_2\, \underline{B}_r = \underline{0}$.

Corollary 3.4 follows directly from Theorem 3.4, Theorem 3.6 and their proofs.

Corollary 3.5

If the fundamental inner-product law can be solved for $\rho(t)$ as a time function, then a linear system with time-varying coefficients is ρ-controllable if and only if

i) $\underline{T}_2 \underline{A}_s = \underline{0}$, and

ii) $\underline{T}_2 \underline{A}_r = \underline{T}_2 \underline{S} + \phi(t) \underline{T}_2$ for some r x r skew symmetric matrix $\underline{S}$. The matrix $\underline{T}_2$ is an (r - b) x r matrix of rank (r - b) and

iii) $\underline{T}_2 \underline{B}_r = \underline{0}$.

In Corollary 3.3 the fundamental inner-product law had to be solved as a time function if the diagonal elements of $\underline{A}_{rr}$ were time-varying. In Corollary 3.5 this requirement is stated as an assumption since the $\underline{T}_2$ matrix has destroyed the properties of $\underline{A}_r$. Therefore, if a linear system has both constant and time-varying coefficients, then the conditions of Corollary 3.4 and Corollary 3.5 present possible necessary and sufficient conditions.

Fixed Configuration Control

The fixed configuration control structure considered is the system introduced by Mulholland and Rhoten (10). The system is described by the equations

$$\dot{\underline{X}} = \underline{A}\,\underline{X} + \xi \underline{B}\,\underline{X}$$
$$\xi = f(\rho)$$
$$\rho = \underline{X}_r^T \underline{X}_r$$

where ξ is a scalar feedback control signal. The system is equivalent to a linear system in which the control vector is given by

$$\underline{U} = \xi \, \underline{X}.$$

The procedure presented in the previous section reflects the philosophy of satisfying the scalar fundamental inner-product law indirectly through the equivalent algebraic equations. By fixing the control structure as indicated, the scalar fundamental inner-product law can be satisfied by specifying the scalar controller $f(\rho)$ in some problems.

Substituting the system equation into the fundamental inner-product law yields

$$\underline{X}_r^T \, \underline{A} \, \underline{X} + \xi \, \underline{X}_r^T \, \underline{B} \, \underline{X} = -\tfrac{1}{2}\sqrt{h(\rho)}\ .$$

Solving directly for the control signal ξ gives

$$\xi = -\left(\tfrac{1}{2}\sqrt{h(\rho)} + \underline{X}_r^T \, \underline{A} \, \underline{X}\right)/\left(\underline{X}_r^T \, \underline{B} \, \underline{X}\right)$$

where

$$\underline{X}_r^T \, \underline{B} \, \underline{X} = \underline{X}^T \, \underline{I}_r^T \, \underline{B} \, \underline{X}\ .$$

If $\underline{I}_r^T \, \underline{B}$ is semidefinite or indefinite, then there exist regions of the state space for which the control signal is unbounded. Since unbounded countrols are physically unrealizable, the above controller is applicable only if the denominator is nonzero for all $\underline{X}(t) \neq 0$. In order to insure this requirement for all $\underline{X}(t) \neq 0$, the symmetric portion of $\underline{I}_r \, \underline{B}$ must be positive definite or negative definite. This requires $\underline{I}_r$ to be the n x n identity matrix and the symmetric portion of $\underline{B}$ to be a definite matrix. If these conditions are satisfied, then the above control signal prescribes the optimal feedback control signal for the fixed configuration. This restriction of the applicability of the fixed configuration was a principal factor in selecting a general control structure and a general error signal for this research.

Under the condition that $\underline{I}_r = \underline{I}$ the control signal reduces to

$$\xi = -(\tfrac{1}{2}\sqrt{h(\rho)} + \underline{X}^T \underline{A}\, \underline{X})/(\underline{X}^T \underline{B}\, \underline{X}).$$

If the plant matrix $\underline{A}$ is skew symmetric and the system is directly controllable with $\underline{B} = \underline{I}$, then the feedback control signal becomes a true inner-product controller,

$$f(\rho) = -\sqrt{h(\rho)}/2\rho.$$

The control structure in this case consists of a single nonlinear transducer whose input is the system error signal. A complete summary of the results for fixed configuration controllers with $\underline{B}$ nonsingular are presented by Mulholland and Rhoten (10).

Summary

The theorems presented in Chapter III provide the necessary and sufficient conditions for systems to be ρ-controllable. Theorem 3.1, Theorem 3.2 and Corollary 3.1 apply to directly controllable systems and directly ρ-controllable systems. Theorem 3.3 gives the general necessary and sufficient conditions for α-controllable systems, and Theorem 3.4 and Corollaries 3.2 and 3.3 extend the results to linear systems. Theorem 3.5 considers the necessary and sufficient conditions for systems to be ρ-controllable, and Theorem 3.6, and Corollaries 3.4 and 3.5 extend the result to linear systems.

Together these theorems provide a firm foundation for determining the applicability of the inner-product approach to any given system, if a globally optimal feedback control is desired. Chapter IV considers non-globally optimal feedback controls.

CHAPTER IV

OPEN LOOP INNER-PRODUCT FEEDBACK CONTROL

Introduction

A control system is usually considered to be a closed loop system if it is a feedback system, and an open loop system if it is not a feedback system. In most modern control literature the terms closed loop and feedback are used interchangeably, as are the terms open loop and nonfeedback. However, in the inner-product formulation, the resultant control system can exhibit properties of an open loop control system and a closed loop control system. From the conventional point of view such controls are neither open loop nor closed loop, since the classification is normally a mutually exclusive one. The term <u>open loop inner-product feedback control</u> has therefore been adopted to indicate this particular class of control solutions. The following example and preparatory remarks illustrate the open loop and feedback characteristics of such solutions.

The open loop inner-product feedback controls are based upon satisfying the algebraic equations equivalent to the fundamental inner-product law by use of the following theorem. The theorem is a direct consequence of elementary calculus, and although it is quite simple in concept it results in a significant extension in the applicability of the inner-product control theory.

<u>Theorem 4.1</u>

Let $g(t)$ be a continuous real-valued function. The function $g(t) = a$ where a is a constant for all $t \geq t_o$ if

i) $g(t_o) = a$, and (4.1)

ii) $dg(t)/dt = 0$ for all $t \geq t_o$. (4.2)

Proof: The theorem follows from the mean value theorem. For any time $t_1 \geq t_o$, the mean value theorem states that there exists a time t_2 such that

$$t_1 \geq t_2 \geq t_o$$

and

$$g(t_1) - g(t_o) = dg(t_2)/dt \quad (t_1 - t_o). \tag{4.3}$$

If $g(t_o) = a$ and $dg(t)/dt = 0$ for all $t \geq t_o$, then equation (4.3) reduces to

$$g(t_1) = a$$

which proves the theorem.

The idea expressed in the theorem is that a time-varying function is equal to a constant value for all time if is is set at the desired value and not allowed to vary. If $dg(t)/dt = 0$ then the function will stay in place and $g(t) = a$ for $t \geq t_o$ as desired.

In some control problems the initial conditions of some of the state variables are unspecified. In the inner-product formulation the added degree of freedom may provide the flexibility needed to solve an optimal control problem. Theorem 4.1 can also be used to obtain control solutions which require no adjustment of the initial conditions, but specify constraints on the initial conditions such as $x_3(t_0) \neq 0$.

The algebraic equations equivalent to the fundamental inner-product law are time-varying functions analogous to the $g(t)$ function in Theorem 4.1. If the control vector and skew symmetric matrix components can not satisfy the equations, then Theorem 4.1 provides an alternate approach. The control vector and skew symmetric matrix components may be able to satisfy the derivative of the equation. If this is the case, and the equation can be satisfied at the initial time t_0, then the algebraic equation is satisfied. The following example illustrates the use of Theorem 4.1, as well as the open loop and feedback characteristics of the controls resulting from the use of Theorem 4.1.

Example 4.1

Consider the problem of a moving unit mass which is to be controlled through the acceleration component of the state vector. It is assumed that the initial position of the mass is given at time t_0, and the control is to be selected to drive the position of the mass to the origin, and maintain this position. The system equations are

$$dx_1/dt = x_2$$

$$dx_2/dt = u$$

where x_1 represents the distance and x_2 denotes the velocity.

To place the problem in an inner-product formulation, a system error signal is given by

$$\rho = x_1^2$$

and the performance measure is selected as

$$J = \int_{t_0}^{\infty} \left[4\, \rho^2 + (d\,\rho/dt)^2 \right] dt. \qquad (4.4)$$

The algebraic equation minimizing the performance measure is

$$x_2 = -\tfrac{1}{2}\, x_1.$$

From Theorem 4.1, the minimizing equation is satisfied if

$$x_2(t_o) = -\tfrac{1}{2}\, x_1(t_o) \qquad (4.5)$$

and

$$dx_2(t)/dt = -\tfrac{1}{2}\, dx_1(t)/dt.$$

The state equations reduce the latter condition to the form

$$u = -\tfrac{1}{2}\, x_2. \qquad (4.6)$$

The performance measure is therefore minimized if the initial velocity of the moving mass is selected according to equation (4.5) and the acceleration control is selected in feedback form as indicated by equation (4.6). The specification of the initial velocity places the system trajectory on the minimizing trajectory of the performance measure, and the feedback control maintains the system on the trajectory.

In contrasting the control solution of equation (4.5) and (4.6) with conventional open loop and closed loop controls, the feedback nature of the solution is readily apparent. The control of equation (4.6) is in feedback form for it specifies the control input as a function of the observed system output. However, the control of equation (4.6) specifies only one half of the total control solution. The remaining one half of the solution given by (4.5) is not a feedback control, but characterizes the open loop portion of the control.

Since an open loop control does not utilize the system output to determine the control, the deviation of the system from the assumed trajectory is not accounted for in the control. The control must therefore be carefully calibrated and must maintain that calibration in order to be useful. If external or internal disturbances are present in the system, an open loop control will not perform the task for which it was designed. Instead, it will continue to apply the precomputed control designed for the assumed trajectory. This results in the system following a trajectory which is no longer necessarily optimal.

If equation (4.5) is satisfied then the system will begin initially on the optimal trajectory and the feedback control will maintain this trajectory. However, if the initial conditions are inaccurately adjusted or if the system is disturbed at some later time, then the feedback control will not sense the error but will continue to apply the feedback control on the assumption that the system is following the original optimal trajectory. If a true feedback control is disturbed from the assumed trajectory, it essentially assumes a new optimal trajectory originating at the present observed state of the system. In a sense then, the feedback control of (4.6) is modeling the open loop solution of the problem.

The necessity of tracking precisely the minimizing trajectory of the performance measure is easily illustrated if the system is solved assuming the control of equation (4.6) without satisfying equation (4.5). The solution for the state variable $x_1(t)$ is given by

$$x_1(t) = - 2\, x_2(t_o)\, e^{-(t - t_o)/2} + x_1(t_o) + 2\, x_2(t_o).$$

If the initial velocity is adjusted to comply with equation (4.5), then the equation for $x_1(t)$ reduces to an exponential which decays to zero. If the initial velocity is inaccurately specified or if the system is perturbed at some later time, then the steady-state value of x_1 will not be zero and the performance measure will become infinite.

To illustrate the insensitive nature of the control to system variations, assume that at some time t_1 the moving mass encounters unexpected friction. If the friction ceases at time t_2, then the state equation for the velocity component is given by

$$dx_2/dt = \begin{cases} -\frac{1}{2} x_2 & t < t_1, \quad t \geq t_2 \\ -\frac{1}{2} x_2 - K^2 & t_2 > t \geq t_1 \end{cases} \tag{4.7}$$

where K^2 denotes the effect of the friction. During the time interval from t_1 to t_2 the acceleration provided by the control will be insufficient to maintain the required velocity relationship with the system position. As a result the relationship between x_1 and x_2 will be altered from the desired form of

$$\tfrac{1}{2} x_1 + x_2 = 0 \tag{4.8}$$

to

$$\tfrac{1}{2} x_1 + x_2 = k$$

where $k > 0$. The control will maintain the latter relationship for all time $t > t_2$ in the absence of any further disturbances. The resultant behavior of the system position component under these assumptions is illustrated in Figure 5, where the steady-state value of x_1 is now k.

The feedback control of equation (4.6) is therefore not a globally optimal feedback control. Since the control measures and uses only the velocity component of the state vector, it does not sense the deviation of the system from the desired trajectory of equation (4.8). This

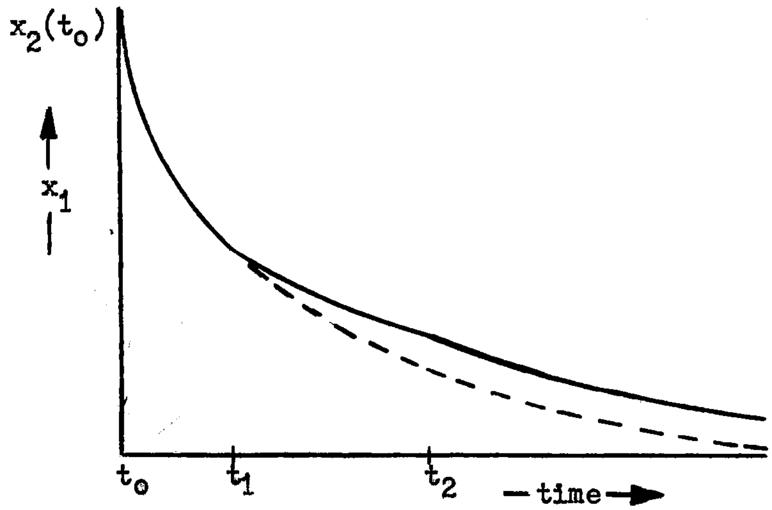

Figure 5. x_1 Trajectory for Example 4.1 with Friction in System

failure to sense the deviation of the system from the desired performance is characteristic of open loop control systems. The total control solution is therefore a feedback control which exhibits the properties of an open loop control; hence the control is designated an open loop inner-product feedback control.

The form of the control solution of Example 4.1 differs from the control solutions of Chapter III in that it specifies a required relation for the initial conditions of the components of the state vector. Since $x_1(t_o)$ is fixed, the control solution of Example 4.1 is applicable only if $x_2(t_o)$ can satisfy equation (4.5). In order to distinguish the globally optimal controls of Chapter III from the control solutions of Chapter IV, the following definition is restated from Chapter II.

Definition 4.2

A system is δ-controllable if and only if it is inner-product controllable for some $\underline{X}(t_0)$ but not all $\underline{X}(t_0)$.

The use of the term δ-controllable stems from the fact that the initial condition requirement of equation (4.5) can be included in the feedback control of equation (4.6) if an impluse function is added. The resultant control is given by

$$u = -\tfrac{1}{2} x_2 - \left[x_2(t_0) + \tfrac{1}{2} x_1(t_0)\right] \delta(t - t_0) \qquad (4.9)$$

where $\delta(t - t_0)$ is a unit impulse at time t_0. The combination of the conditions of equation (4.5) and equation (4.6) in the form of equation (4.9) is a compact notation of the total control solution.

A distinct disadvantage of the use of singularity functions is that the performance measure of equation (4.4) no longer includes an indirect measure of the control input energy. However, once the system reaches the trajectory of equation (4.8), for $t > t_0$, the control of (4.9) reduces to the control of equation (4.6) and $(d\rho/dt)^2$ again provides a measure of the control input. In order to avoid the difficulty of the singularity functions, the controls will remain in the form of equations (4.5) and (4.6), and the general performance measure considered for the δ-controllable systems will be

$$J = \int_{t_0^+}^{\infty} \left[h(\rho) + (d\rho/dt)^2\right] dt. \qquad (4.10)$$

The performance measure of equation (4.10) is equivalent to the previous general performance measure, except the control effect of the initial condition constraints is not included in the performance measure.

A recently published result by Leeper and Mulholland (20) for the δ-controllable systems has been for a class of single input systems.

Example 4.1 illustrates the class of single input systems, and the following section generalizes the results of the paper.

Single Input Systems

The nth order single input process

$$d^n x/dt^n - f(x, dx/dt, \ldots, d^{n-1}x/dt^{n-1}) = u$$

can be expressed in the equivalent state formulation given by the n state equations

$$\begin{aligned} dx_i/dt &= x_{i+1} \qquad i = 1,2,\ldots,n-1 \\ dx_n/dt &= f(x_1, x_2, \ldots, x_n) + u \,. \end{aligned} \tag{4.11}$$

If the control objective is to drive the x_j component of the state vector to the origin and maintain that state, then a reasonable error signal is given by

$$\rho = x_j^{\,2}. \tag{4.12}$$

If $j = n$, then the above formulation is a directly ρ-controllable system from Corollary 3.1, and the system is no longer a δ-controllable system. It is therefore assumed that $1 \geq j > n$.

From Chapter II the algebraic equation minimizing the performance measure is given by

$$dx_j/dt = \phi(\rho)\, x_j \tag{4.13}$$

where

$$\phi(\rho) = -\sqrt{h(\rho)}/2\rho . \tag{4.14}$$

If the error penalty function is of the form

$$h(\rho) = 4\, k^2\, \rho^2, \tag{4.15}$$

then the minimizing trajectory of equation (4.13) reduces to

$$x_{j+1} = - k\, x_j. \tag{4.16}$$

Differentiating equation (4.16) n - j times, and substituting for the derivatives from the state equations gives

$$x_n = - k \left[f(x_1, x_2, \ldots, x_n) + u \right].$$

Solving for the control u, the feedback control is given by

$$u = -(1/k)\, x_n - f(x_1, x_2, \ldots, x_n)\ . \tag{4.17}$$

Equation (4.16) is the equation describing the optimal trajectory of the performance measure. The development of equation (4.17) from (4.16) produces n - 1 - j similar equations from the first n - 1 - j differentiations. The optimal trajectory is therefore defined by

$$x_{i+1} = - k\, x_i \qquad i = j,\ j+1, \ldots, n-1.$$

For j = 1 this denotes a line in the n-dimensional state space, and for j = 2 it denotes a plane.

Since the control objective is to drive the x_j state to the origin, $x_j(t_o)$ is assumed given and not open to selection as part of the control solution. From Theorem 4.1 the control of equation (4.17) is applicable only if the n - j initial conditions $x_{j+1}(t_o)$, $x_{j+2}(t_o)$, ..., $x_n(t_o)$ can be specified to satisfy the equations

$$x_i(t_o) = (-1/k)^{i-j}\, x_j(t_o) \qquad i = j+1, j+2, \ldots, n. \tag{4.18}$$

The complete control solution is therefore given by equation (4.17) and (4.18). The above development is summarized by the following theorem.

Theorem 4.2

The single input system of equation (4.11) is δ-controllable for a system error of $\rho = x_j^2$ if

i) $h(\rho) = 4k^2\rho^2$ where k is a real number

and ii) $x_i(t_o) = (-1/k)^{i-j}\, x_j(t_o)$ $i = j+1, j+2, \ldots, n.$

The resultant feedback control is given by

$$u = -(1/k)\, x_n - f(x_1, x_2, \ldots, x_n).$$

Since the state equations of $x_1, x_2, \ldots, x_{j-1}$ are not used in the development of Theorem 4.2, the first j - 1 equations in the single input system do not need to be in the form of equation (4.11). This allows the extension of Theorem 4.2 to multiple input systems if only one control term appears in the state equations of $x_j, \ldots, x_n$. The single input system of equation (4.11) can therefore be generalized to

$$dx_i/dt = \begin{cases} g_i(x_1, x_2, \ldots, x_n, u_1, \ldots, u_m) & 1 \leq i \leq j-1 \\ x_{i+1} & j \leq i \leq n-1 \\ f(x_1, x_2, \ldots, x_n) + u_\sigma & i = n \end{cases} \qquad (4.19)$$

where $1 \leq \sigma \leq m$.

Corollary 4.1

The system of equation (4.19) is δ-controllable for a system error of $\rho = x_j^2$ if

i) $h(\rho) = 4k^2\rho^2$ where k is a real number, and

ii) $x_i(t_o) = (-1/k)^{i-j}\, x_j(t_o)$ $i = j+1, j+2, \ldots, n.$

The resultant feedback control is given by

$$u_\sigma = -(1/k)\, x_n - f(x_1, x_2, \ldots, x_n). \qquad (4.20)$$

Although systems in the form of equations (4.11) or (4.19) may be δ-controllable for

$$h(\rho) \neq 4 k^2 \rho^2,$$

the general formulation of the solution for such systems is difficult to express. If $h(\rho)$ is not in the assumed form of equation (4.15), the solution proceeds as follows.

Since $h(\rho)$ is a function of ρ, and $\rho = x_j^2$, the error penalty function can be expressed as a function of x_j and

$$\phi(\rho) = \phi(x_j). \tag{4.21}$$

Equation (4.13) becomes

$$x_{j+1} = g(x_j) \tag{4.22}$$

where

$$g(x_j) = \phi(x_j)\, x_j. \tag{4.23}$$

While equation (4.16) can be differentiated n - j times directly, equation (4.22) is not linear and therefore does not lend itself to such a direct approach. Define

$$g^{(n)} = d^n g(x_j)/dx_j^n.$$

If $j > n - 4$, then at least four differentiations of equation (4.22) are necessary. The first four differentiations are given below.

$$x_{j+2} = g^{(1)}\, x_{j+1} \tag{4.24}$$

$$x_{j+3} = g^{(1)}\, x_{j+2} + g^{(2)}\, x_{j+1}^2 \tag{4.25}$$

$$x_{j+4} = g^{(1)}\, x_{j+3} + 3\, g^{(2)}\, x_{j+1}\, x_{j+2} + g^{(3)}\, x_{j+1}^3 \tag{4.26}$$

$$x_{j+5} = g^{(1)}\, x_{j+4} + g^{(2)}\left[4\, x_{j+1}\, x_{j+3} + 3\, x_{j+2}^2\right] + 6\, g^{(3)}\, x_{j+1}^2\, x_{j+2} + g^{(4)}\, x_{j+1}^4 \qquad (4.27)$$

It is readily apparent that as $n - j$ becomes larger, the equations corresponding to (4.24) - (4.27) become more and more complex. These equations, together with the original equation (4.22) define the opeimal trajectory in state space upon which the system must lie at time t_o. The initial condition requirements of the system therefore come from these equations, and the feedback control comes from the (n-j)th differentiation of equation (4.22). To illustrate the extension of Theorem 4.2 to systems with other error penalty functions, the following example is given.

Example 4.2

Consider the linear system

$$dx_1/dt = x_2$$
$$dx_2/dt = x_3$$
$$dx_3/dt = x_4$$
$$dx_4/dt = x_1 + 2x_2 + 3x_3 + 4x_4 + u$$

with a system error signal of

$$\rho = x_1{}^2$$

and an error penalty function of

$$h(\rho) = 4\,\rho^4.$$

Defining the error penalty function defines the performance measure.

For the error penalty function specified,

$$\phi(\rho) = -\rho = -x_1^2$$

and the minimizing trajectory of the system from equation (4.13) is

$$dx_1/dt = (-x_1^2)\,x_1$$

or

$$x_2 = -x_1^3. \tag{4.28}$$

Equation (4.28) describes the optimal trajectory for the performance measure specified. In contrast to the optimal trajectory when $h(\rho) = 4\,k^2\,\rho^2$, this equation is nonlinear while the corresponding equation (4.16) is linear. Differentiating equation (4.28) three times and substituting for the derivative from the state equations gives

$$x_3 = -3\,x_1^2\,x_2 \tag{4.29}$$

$$x_4 = -6\,x_1\,x_2^2 - 3\,x_1^2\,x_3 \tag{4.30}$$

$$x_1 + 2\,x_2 + 3x_3 + 4x_4 + u = -6\,x_2^3 - 18\,x_1x_2x_3 - 3\,x_1^2\,x_4. \tag{4.31}$$

The feedback control is obtained from equation (4.31) by solving for u, and the initial condition requirements are given by equations (4.28), (4.29) and (4.30) at time t_o. The initial condition requirements reduce to

$$x_2(t_o) = -x_1^3(t_o),$$

$$x_3(t_o) = 3\,x_1^5(t_o),$$

and

$$x_5(t_o) = -15\,x_1^7(t_o).$$

Therefore the single input systems are δ-controllable for error penalty functions other than the function of equation (4.15). Theorem

4.2 and Corollary 4.1 are intended to indicate the sufficient conditions for a class of error functions. While the general case is solvable, the continuous application of the chain rule to nonlinear equations results in an optimal trajectory which is difficult to express.

Multiple Single Input Systems

A natural extension of the single input systems are systems which can be formulated as multiple single input systems. The dynamics of a multiple single input system are assumed to be described by k single input processes in the form of equation (4.19). The state formulation of such systems is given by

$$dx_i/dt = x_{i+1} \qquad 1 \leq i < n_k; \; i \neq n_1, n_2, \ldots, n_k, \tag{4.32}$$

$$dx_{n_i}/dt = f_i(x_1, \ldots, x_n) + u_i \qquad i = 1, 2, \ldots, k \tag{4.33}$$

$$dx_i/dt = g_i(x_1, x_2, \ldots, x_n, u_1, \ldots, u_m) \quad n_k < i \leq n, \tag{4.34}$$

and the system error signal is given by

$$\rho = x_1^2 + x_{n_1+1}^2 + x_{n_2+1}^2 + \cdots + x_{n_{k-1}+1}^2 \,. \tag{4.35}$$

The following system illustrates a multiple single input system

$$\begin{aligned}
dx_1/dt &= x_2 \\
dx_2/dt &= x_3 \\
dx_3/dt &= x_1 + x_4 + u_1 \\
\hline
dx_4/dt &= x_5 \\
dx_5/dt &= x_2 + x_3 + u_2 \\
\hline
dx_6/dt &= x_1 + x_5 + u_3 \,.
\end{aligned}$$

If the system error signal is given by

$$\rho = x_1^2 + x_4^2$$

then the state equations for x_1, x_2 and x_3 constitute a single input system, as do the equations for x_4 and x_5.

The state equations for x_1, x_2 and x_4 correspond to equation (4.32) while the state equations for x_3 and x_5 correspond to equation (4.33). The single state equation for x_6 corresponds to (4.34) and illustrates that additional equations can be included in the formulation, as long as the error signal and first n_k state equations are in the form indicated.

The example system fits the formulation for a multiple single input system for many system error signals. If the system error signal is given by

$$\rho = x_2^2 + x_4^2 + x_6^2$$

the state equation for x_6 becomes a degenerate single input system which is a directly ρ-controllable subsystem. Although directly ρ-controllable systems were omitted from consideration in the previous section, they are acceptable in the multiple single input system formulation if at least one of the k single input systems is not directly ρ-controllable.

Since ρ is of rank k, the algebraic equations minimizing the performance measure are specified by the following k equations

$$\underline{\dot{X}} = \phi(\rho)\, \underline{X} + \underline{S}\, \underline{X} \tag{4.36}$$

where

$$\underline{X}^T = (x_1 \quad x_{n_1+1} \quad x_{n_2+1} \quad \cdots \quad x_{n_{k-1}+1})$$

and $\underline{S}$ is any r x r skew symmetric matrix.

The k algebraic equations of (4.36) are considered separately as k single input systems. The significant difference between the multiple single input systems and the single input systems is that in satisfying the initial condition constraints, the components of the $\underline{S}$ matrix are available in the former. This can reduce the apparent number of initial conditions which are to be adjusted, and in some cases eliminate the need for adjusting any initial conditions.

The solution procedure for multiple single input systems is illustrated by the following example by Leeper and Mulholland (19).

Example 4.3

Consider the problem of a body spinning in free space. The state equations for the system are given by

$$dx_1/dt = x_2, \quad dx_3/dt = x_4, \quad dx_5/dt = x_6$$
$$dx_2/dt = \beta_1 \, x_4 \, x_6 + u_1$$
$$dx_4/dt = \beta_2 \, x_2 \, x_6 + u_2$$
$$dx_6/dt = \beta_3 \, x_2 \, x_4 + u_3 \; .$$

For the problem of attitude control, the control objective is to fix the position of the spinning body. It is assumed that the state equations have been defined so that the desired attitude is achieved if the angular displacement components are driven to zero. Since x_1, x_3 and x_5 correspond to the angular displacements of the body, a natural error signal is

$$\rho = x_1^2 + x_3^2 + x_5^4 \; .$$

Using the general performance measure of equation (4.10), the minimizing trajectory is given by the fundamental inner-product law

$$x_1 x_2 + x_3 x_4 + x_5 x_6 = - \tfrac{1}{2}\sqrt{h(\rho)}.$$

The algebraic equations corresponding to the fundamental inner-product law are given by

$$\begin{bmatrix} x_2 \\ x_4 \\ x_6 \end{bmatrix} = \phi(\rho) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} \\ -\alpha_{12} & 0 & \alpha_{23} \\ -\alpha_{13} & -\alpha_{23} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} \qquad (4.37)$$

where $\phi(\rho)$ is defined in equation (4.14).

In general the components of the $\underline{S}$ matrix may be time-varying, but in this example they are restricted to constants.

Differentiating the above equations with respect to time and substituting for the derivatives from the state equations gives the feedback controls

$$\begin{aligned} u_1 &= -\beta_1 x_4 x_6 + \phi(\rho) x_2 + \alpha_{12} x_4 + \alpha_{13} x_6 + \psi(\rho) x_1 \\ u_2 &= -\beta_2 x_2 x_6 + \phi(\rho) x_4 - \alpha_{12} x_2 + \alpha_{23} x_6 + \psi(\rho) x_3 \\ u_3 &= -\beta_3 x_2 x_4 + \phi(\rho) x_6 - \alpha_{13} x_2 - \alpha_{23} x_4 + \psi(\rho) x_5 \end{aligned}$$

where

$$\psi(\rho) = d\phi(\rho)/d\rho \left[d\rho/dt\right].$$

The control solution is given by the feedback controls defined above and by the initial condition constraints of (4.37) which must be satisfied at time t_0. For the nth order single input process, n - 1 initial conditions had to be satisfied, however only one initial condition of (4.37) need be altered to satisfy the equations. This is due to the fact that the skew symmetric components of $\underline{S}$ can be utilized to satisfy two of the three equations. Therefore the feedback controls are valid and the system is δ-controllable if $x_2(t_o)$, $x_4(t_o)$ or $x_6(t_o)$ can be adjusted.

In Example 4.3 each of the three single input systems is of order two, so the system is solved simultaneously. When the order of the single input systems differs, the systems must be solved separately, and the results combined for the total control solution. In such situations the lower order single input subsystems must be solved first, for these results are required in the higher order single input systems. The following example illustrates the solution for a multiple single input system with different orders.

<u>Example 4.4</u>

Consider the linear system

$$\begin{aligned} dx_1/dt &= x_2 \\ dx_2/dt &= x_1 + x_4 + u_1 \\ dx_3/dt &= x_4 \\ dx_4/dt &= x_5 \\ dx_5/dt &= x_3 + x_4 + u_2 \end{aligned}$$

with a system error signal of

$$\rho = x_1^{\,2} + x_3^{\,2} .$$

The x_1 and x_2 states form a second order single input system, while the x_3, x_4 and x_5 states constitute a third order single input subsystem.

If the error penalty function is selected as

$$h(\rho) = 4\,\rho^2$$

then the algebraic equations minimizing the performance measure are

$$x_2 = - x_1 + \alpha_{12} x_3$$
$$x_4 = - x_3 - \alpha_{12} x_1 .$$

The first equation is solved as a single input system by differentiating it once. The resultant equation for α_{12} constant is

$$u_1 = - x_1 - x_2 + (\alpha_{12} - 1) x_4. \tag{4.38}$$

The second equation is differentiated twice and the resultant equations are

$$x_5 = - x_4 - \alpha_{12} x_2$$

and

$$(x_3 + x_4 + u_2) = - x_5 - \alpha_{12} (x_1 + x_4 + u_1).$$

Using the solution for u_1 in equation (4.38), the above equation reduces to

$$u_2 = \alpha_{12} x_2 - x_3 - (1 - \alpha_{12}^2) x_4 - x_5. \tag{4.39}$$

The feedback controls are defined by equations (4.38) and (4.39), and the initial condition constraints are

$$x_2(t_o) = - x_1(t_o) + \alpha_{12} x_3(t_o)$$
$$x_4(t_o) = - x_3(t_o) - \alpha_{12} x_1(t_o)$$
$$x_5(t_o) = - x_4(t_o) - \alpha_{12} x_2(t_o) .$$

If a system can be represented in the multiple single input formulation of equations (4.32)-(4.34), with a system error signal in the form of equation (4.35), then the system is δ-controllable if the initial condition constraints can be satisfied.

δ-Controllable Systems

For a general dynamic system described by n differential equations

$$\dot{\underline{X}}(t) = \underline{F}(\underline{X}(t),t) + \underline{B}(t)\,\underline{U}(t)$$

the techniques of Chapter IV can often be combined with the techniques of Chapter III to yield open loop inner-product feedback controls.

The necessary conditions of inner-product controllability are given by the r algebraic equations

$$\underline{B}_r\,\underline{U} = \phi(\rho)\,\underline{X}_r + \underline{S}\,\underline{X}_r - \underline{F}_r(\underline{X}(t),t)$$

where $\underline{X}_r$, $\underline{B}_r$ and $\underline{F}_r(\underline{X}(t),t)$ are defined by equations (3.3), (3.4) and (3.5), and $\underline{S}$ is any r x r skew symmetric matrix. The above system can be transformed into two subsystems by a nonsingular transformation $\underline{T}$ as in Chapter III. The first b algebraic equations constitute a directly ρ-controllable subsystem which defines b components of the control vector. The remaining r - b algebraic equations are given by

$$\underline{T}_2\,\underline{S}\,\underline{X}_r = \underline{T}_2\;\underline{F}_r(\underline{X}(t),t) - \phi(\rho)\,\underline{X}_r$$

where $\underline{T}_2$ is an (r-b) x r matrix of rank (r-b) and

$$\underline{T}_2\,\underline{B}_r = \underline{0}\,.$$

If k of the (r-b) equations can be solved by specification of the $\underline{S}$ matrix components, the remaining (r-b-k) equations can be considered as possible δ-controllable systems. In such cases, the open loop nature of the δ-controllable subsystem will render the total system δ-controllable.

The previous examples in this chapter have required the adjustment of initial conditions of the state vector in the control solution. Although the adjustment of initial conditions is meaningful in some physical systems, the majority of control solutions do not allow this added freedom. The control solution for δ-controllable systems can lead to open loop feedback controls which do not require the adjustment of the initial conditions of the state variables. The initial condition requirements of such systems are satisfied by specification of the skew symmetric matrix components. The following example illustrates the procedure for δ-controllable systems, and yields a control solution that requires no adjustment of the initial conditions.

Example 4.5

Consider the linear system

$$dx_1/dt = x_2 + x_4 + u_1$$
$$dx_2/dt = x_1 - x_2$$
$$dx_3/dt = x_3 + x_4$$
$$dx_4/dt = x_1 - x_3 + u_2$$

with a system error signal of

$$\rho = x_1^2 + x_2^2 + x_3^2.$$

If the error penalty function is selected as

$$h(\rho) = 4\,\rho^2$$

then the algebraic equations minimizing the performance measure can be expressed in the form

$$x_2 + x_4 + u_1 = - x_1 + \alpha_{12} x_2 + \alpha_{13} x_3$$
$$x_1 - x_2 = - x_2 - \alpha_{12} x_1 + \alpha_{23} x_3$$
$$x_3 + x_4 = - x_3 - \alpha_{13} x_1 - \alpha_{23} x_2 \ .$$

The first equation is directly solvable for the control u_1. The second equation is satisfied if $\alpha_{12} = - 1$ and $\alpha_{23} = 0$. The third equation must be satisfied by Theorem 4.1. If the equation can be satisfied by Theorem 4.1, then the algebraic equations of Example 4.5 can be decomposed into a directly ρ-controllable system, an α-controllable system and a δ-controllable system.

Differentiating the third algebraic equation with α_{13} assumed constant, and simplifying the result yields the control u_2

$$u_2 = (3\alpha_{13} - 1)\ x_1 + \alpha_{13}\ x_2 + (3 - \alpha_{13}^2)\ x_3 + 2\ x_4.$$

The complete control solution is then given by

$$u_1 = - x_1 - 2\ x_2 + \alpha_{13}\ x_3 - x_4$$
$$u_2 = (3\alpha_{13} - 1)\ x_1 + \alpha_{13}\ x_2 + (3 - \alpha_{13}^2)\ x_3 + 2\ x_4$$

$$\alpha_{13}\ x_1(t_o) + 2\ x_3(t_o) + x_4(t_o) = 0.$$

If $x_1(t_o) \neq 0$, then the initial condition constraint can be satisfied by specification of α_{13}, and hence no initial conditions need be adjusted in order to satisfy the equations. The solution is still an open loop type of inner-product control and the system is therefore a δ-controllable system.

Summary

The extension of the inner-product approach to δ-controllable systems leads to feedback controls which possess open loop characteristics. Although few theorems are given, the procedures have been outlined and illustrated by examples. The formulation of necessary and sufficient conditions for δ-controllable systems is deemed to be not mathematically tractable although the procedures are well defined.

CHAPTER V

SUMMARY AND CONCLUSIONS

Summary

This dissertation summarizes the results of the development of a theory of asymptotic inner-product control for general nonlinear systems. The problem is limited to those systems in which the control is linearly separable, and seeks to determine the optimal control in feedback form.

The inner-product formulation of optimal control is based upon the use of an inner-product performance criteria. The performance criteria selected is an integral function of an inner-product error signal and its derivative. The specification of a performance measure of this form allows a direct solution to the problem which avoids the conventional two-point boundary value problem. The philosophy of the approach is to select the control such that the system trajectory follows a minimizing trajectory of the performance measure. The minimizing trajectory of the performance measure is called the fundamental inner-product law and is developed in Chapter II.

The inner-product approach is aimed entirely at obtaining closed form feedback control laws for optimal control problems. The control solutions are of two types: true closed loop feedback controls, and feedback controls with open loop response characteristics. The former represent globally optimal feedback control laws. The feedback controls in Chapter IV are closed form feedback controls, but also possess open loop characteristics relative to the error correction ability of the controls.

Chapter III presents the necessary and sufficient conditions for a system to yield closed loop inner-product feedback controls. The theorems provide a firm mathematical foundation for determining the applicability of the inner-product approach to a given system, if a closed loop globally optimal control is desired. The solution procedures and solution feasibility are illustrated by several examples.

The extension of the inner-product approach to feedback controls with open loop characteristics is considered in Chapter IV. The controls of Chapter IV are non-globally optimal. Sufficient conditions are given for special system configurations and the general procedures are illustrated by examples.

The results of Chapter III and Chapter IV depend upon the equivalence of the fundamental inner-product law and a set of algebraic equations. This equivalence is based on the results of Appendix A, and represents a major step in the general development.

Two sets of algebraic equations are indicated by the results of Appendix A. The two forms are quite different in structure but are proven equivalent. Appendix B presents a summary of results for the alternate form which are analogous to those of Chapters III and IV.

Conclusions

This thesis has accomplished the objectives set forth in Chapter I. For the closed loop inner-product control systems, the underlying structure of systems for which the inner-product approach is suitable are well defined in the theorems of Chapter III. For the open loop controls the general necessary conditions are discussed, although an explicit formulation as in Chapter III is deemed to be not mathematically tractable. Within the framework specified, the procedures for determining the optimal inner-product control are well defined and are illustrated by several examples.

The principal limiting feature of the original work in the inner-product control theory is the inability to obtain optimal bounded controls for problems in which the control input matrix is singular. A major emphasis of this dissertation is the extension of the inner-product approach to systems in which the control input matrix is singular.

Original efforts in this direction assumed the fixed configuration illustrated in Figure 1, and attempted to extend the results for a nonsingular $\underline{B}$ matrix to a corresponding singular $\underline{B}$ matrix. The suboptimal approximations were based on the use of generalized matrix inverses (21). The resultant solutions were necessarily suboptimal, and proved to be unstable for several problems. The unstable nature of the solutions made this approach impractical.

The removal of the constraint of a fixed configuration controller is essential to the extension of the original work to singular control input matrices. The use of a general feedback control structure also allows the system error signal to be generalized in the resulting inner-product theory.

In retrospect, it is noted that the inner-product theory development is based upon the use of two simple mathematical properties and the use of linear algebra. The equivalence of the fundamental inner-product law and the r algebraic equations of (2.27) is basic to the development of Chapter III and Chapter IV. This equivalence evolved from the incorporation of a general skew symmetric matrix in the solution of the vector equation. It is well known that

$$\underline{A}^T \underline{S} \underline{A} = 0$$

for any vector $\underline{A}$ if $\underline{S}$ is a skew symmetric matrix, but the realization that this equality was the key to decomposing the fundamental inner-product law was a major step in the development.

The second mathematical property used was the initial condition theorem of Chapter IV. Theorem 4.1 is also quite simple in concept, but it results in a significant extension of the inner-product approach. The realization that this theorem could be used on the algebraic equations of (2.27) provided the key to the open loop type of feedback controls in Chapter IV.

Suggestions for Further Study

The inner-product approach to optimal control offers an unlimited number of areas in which further research would be useful and fruitful. The areas indicated in this section are considered to be reasonable extensions of the results of this thesis.

Within the framework of this study, further study of the necessary and sufficient conditions of the controls of Chapter IV is desirable. As noted in the conclusions, the general conditions are not considered mathematically tractable; however, for special system configurations it is probable that some closed form definite results are possible.

It would also be desirable to conduct a sensitivity study into the control solutions of Chapter IV. It is possible that a sensitivity analysis of special system configurations would provide a key to the general conditions of Chapter IV.

On a more general scale, it is noted that the inner-product approach is dependent upon the fundamental inner-product law. While this dissertation was restricted to the asymptotic control of systems using the performance measure of equations (1.5), similar results could be obtained from similar fundamental inner-product laws. The results of this study could therefore be extended to finite time control problems and to other general inner-product performance measures, as long as an expression similar to the fundamental inner-product law is obtained from the Euler-Lagrange equation.

In investigating more general inner-product performance measures, it would be desirable if a better measure of the control input energy could be incorporated in the performance measure, without destroying the fundamental inner-product law results. Some effort has been focused in this direction, but no significant results were obtained.

Finally, it is obvious that some effort is necessary in considering the inner-product formulation in the presence of system constraints. In most modern control problems there are constraints on the range of values which the system components and control components can assume. Consideration of systems with constraints will be necessary before the inner-product approach becomes a realistic engineering tool.

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APPENDIX A

SOLUTION OF THE VECTOR EQUATION $K_1 \underline{A}^T\underline{B} = K_2 \underline{A}^T\underline{A}$

Let $\underline{A}$ and $\underline{B}$ be n-dimensional vectors in the Euclidean space E^n. The inner-product of $\underline{A}$ and $\underline{B}$ is the real number

$$\underline{A}^T \underline{B} = a_1b_1 + a_2b_2 + \dots + a_nb_n \tag{A.1}$$

where

$$\underline{A}^T = (a_1 \quad a_2 \quad \dots \quad a_n)$$
$$\underline{B}^T = (b_1 \quad b_2 \quad \dots \quad b_n).$$

<u>Lemma A.1</u>

Let $\underline{A}$ be a given nonzero vector in E^n. Any vector $\underline{B}$ satisfying the equation

$$\underline{A}^T \underline{B} = 0 \tag{A.2}$$

can be expressed in the form

$$\underline{B} = \underline{S}\,\underline{A} \tag{A.3}$$

where $\underline{S}$ is an n x n skew symmetric matrix.

Proof: The theorem is proven by constructing a skew symmetric matrix $\underline{S}$ which satisfies equation (A.3) for a given solution vector $\underline{B}$.

Let $\underline{B}$ be a solution vector of (A.2). Since $\underline{A}$ is a nonzero vector it contains at least one nonzero component, a_r. The $\underline{S}$ matrix is constructed in the following manner:

$$s_{ir} = b_i/a_r \qquad i = 1, 2, \ldots, n; \quad i \neq r$$
$$s_{ri} = - s_{ir} \tag{A.4}$$
$$s_{jk} = 0 \text{ otherwise}$$

where s_{jk} denotes the element in the jth row and kth column of the $\underline{S}$ matrix.

The vector equation of (A.3) is satisfied if the n equations equivalent to it are satisfied. Using the skew symmetric matrix of equation (A.4), the n equations are given by

$$b_i = s_{ir}\, a_r \qquad i = 1, 2, \ldots, n; \quad i \neq r \tag{A.5}$$

$$b_r = s_{r1}\, a_1 + s_{r2}\, a_2 + \ldots + s_{rn}\, a_n \,. \tag{A.6}$$

The n - 1 equations of (A.5) are automatically valid due to the definition of the s_{ir}. Substituting the s_{ri} in equation (A.6) and multiplying by a_r gives

$$a_1 b_1 + a_2 b_2 + \ldots + a_n b_n = 0 \tag{A.7}$$

which is an expansion of the vector equation of (A.2).

Since the equations of (A.5) and (A.6) are satisfied, the equivalent vector equation (A.3) is satisfied. Therefore any vector $\underline{B}$ which satisfies equation (A.2) can be expressed in the form of equation (A.3) if the skew symmetric matrix $\underline{S}$ is defined as in equation (A.4). Although the matrix $\underline{S}$ is not unique, equation (A.4) verifies that there exists at least one matrix $\underline{S}$ satisfying equation (A.3), thus proving the lemma.

Theorem A.1

Let $\underline{A}$ be a given nonzero vector in E^n. The solution of the scalar equation

$$\underline{A}^T \underline{B} = \underline{A}^T \underline{A} \tag{A.8}$$

is

$$\underline{B} = \underline{A} + \underline{S}\,\underline{A} \tag{A.9}$$

where $\underline{S}$ is any n x n skew symmetric matrix.

Proof: Equation (A.9) is a solution of (A.8) since

$$\underline{A}^T \underline{S}\,\underline{A} = 0 \tag{A.10}$$

for any vector $\underline{A}$ if $\underline{S}$ is skew symmetric. Therefore it is necessary and sufficient to prove that any solution of (A.8) can be expressed in the form of equation (A.9).

Let $\underline{B}$ be a solution vector of (A.8) and define

$$\underline{C} = \underline{B} - \underline{A}\,. \tag{A.11}$$

Multiplying equation (A.11) on the left by $\underline{A}^T$ reduces the equation to

$$\underline{A}^T \underline{C} = 0\,. \tag{A.12}$$

From Lemma A.1, any vector $\underline{C}$ satisfying equation (A.12) can be expressed in the form

$$\underline{C} = \underline{S}\,\underline{A} \tag{A.13}$$

where $\underline{S}$ is an n x n skew symmetric matrix.

Solving equation (A.11) for $\underline{B}$ and using the result of equation (A.13), any vector $\underline{B}$ satisfying equation (A.8) can readily be expressed in the form of equation (A.9). Therefore all solutions of equation (A.8) are contained in the class of solutions characterized by equation (A.9), thus proving the theorem.

The dot product of $\underline{A}$ and $\underline{B}$ is equivalent to the inner product of $\underline{A}$ and $\underline{B}$ and offers insight into the results of Theorem A.1. Considering equation (A.8) as a dot product gives

$$|\underline{A}||\underline{B}| \cos \theta = |\underline{A}||\underline{A}| \tag{A.14}$$

where θ is the angle between $\underline{A}$ and $\underline{B}$, and $|\underline{A}|$ is the norm of the vector $\underline{A}$.

$$|\underline{A}| = (a_1{}^2 + a_2{}^2 + \dots + a_n{}^2)^{\frac{1}{2}} \tag{A.15}$$

Since $\underline{A}$ is assumed to be a nonzero vector, equation (A.14) reduces to

$$|\underline{B}| \cos \theta = |\underline{A}|. \tag{A.16}$$

For a vector to satisfy equation (A.9) its norm must therefore equal the norm of $\underline{A}$ when projected along the vector $\underline{A}$. In the Euclidean space E^2 this requires the end point of the vector $\underline{B}$ to lie on the line perpendicular to $\underline{A}$ and passing through the endpoint of the vector $\underline{A}$. In E^3 the end point of the solution vectors must lie in the plane normal to $\underline{A}$ and passing through the end point of $\underline{A}$. In general, the end point of $\underline{B}$ must lie in the hyperplane in E^n which is normal to $\underline{A}$ and passes through the endpoint of the vector $\underline{A}$.

Figure 6 illustrates the solution for the two-dimensional case. Any vector $\underline{B}$ whose end point lies on the line L will satisfy the equation (A.8). The skew symmetric matrix $\underline{S}$ is simply a method of describing the vector $\underline{C}$. For this two-dimensional case, the $\underline{S}$ matrix

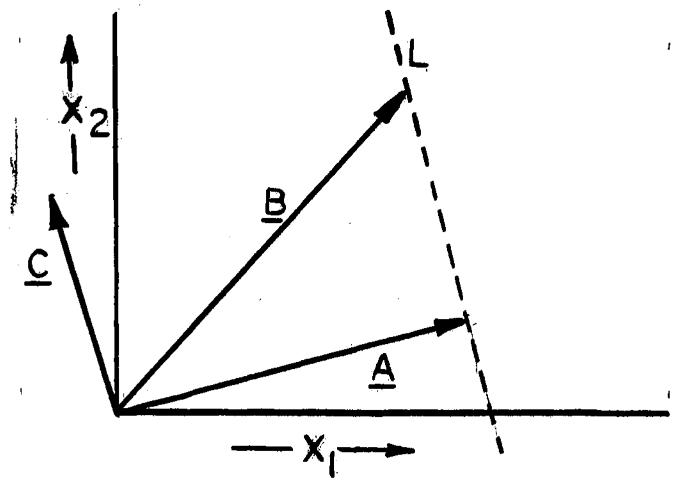

Figure 6. E^2 Illustration of $\underline{A}^T\underline{B} = \underline{A}^T\underline{A}$

describes the $\underline{C}$ vector indirectly by describing the length and direction of $\underline{C}$ relative to $\underline{A}$. In particular,

$$\underline{S} = \begin{bmatrix} 0 & -(|\underline{C}|/|\underline{A}|) \\ (|\underline{C}|/|\underline{A}|) & 0 \end{bmatrix}. \qquad \text{(A.17)}$$

The absolute value of s_{12} corresponds to the normalized length of $\underline{C}$, and the sign of s_{12} determines if the $\underline{B}$ vector is above or below $\underline{A}$.

Corollary A.1

Let $\underline{A}$ be a given nonzero vector in E^n, and K_1 and K_2 be real numbers with $K_1 \neq 0$. The solution of the vector equation

$$K_1 \underline{A}^T \underline{B} = K_2 \underline{A}^T \underline{A} \qquad \text{(A.18)}$$

is given by

$$\underline{B} = (K_2/K_1) \underline{A} + \underline{S} \underline{A} \qquad \text{(A.19)}$$

where $\underline{S}$ is any n x n skew symmetric matrix.

The proof of Corollary A.1 is identical in structure to that of Theorem A.1.

Another general form of solution for equation (A.18), proposed by Sridhar (22), is given by

$$\underline{B} = (K_2/K_1)(\underline{A}^T\underline{A}/\underline{A}^T\underline{M}\ \underline{A})\ \underline{M}\ \underline{A} + \underline{S}\ \underline{A} \tag{A.20}$$

where $\underline{M}$ is an n x n matrix. In order to guarantee that the coefficient $(\underline{A}^T\underline{A}/\underline{A}^T\underline{M}\ \underline{A})$ is bounded, $\underline{M}$ is assumed to be a positive definite or negative definite matrix. Substitution of (A.20) into equation (A.18) readily verifies that it is a solution, and although (A.19) and (A.20) are quite different in structure, the two solutions are equivalent.

Theorem A.2

Equations (A.19) and (A.20) are equivalent solutions for any vector $\underline{A}$, if $K_1 \neq 0$ and $\underline{M}$ is a definite matrix.

Proof: The two vectors are equivalent if and only one can be derived from the other. Equation (A.19) can be derived directly from (A.20) by selecting the $\underline{M}$ matrix to be the identity matrix. The remaining problem is to show that any solution in the form of equation (A.20) can be expressed in the form of equation (A.19). Let

$$\underline{R} = \underline{S} + (\underline{A}^T\underline{A}/\underline{A}^T\underline{M}\ \underline{A})\ \underline{M} - (K_1/K_2)\ \underline{I} \tag{A.21}$$

and express equation (A.20) in the form

$$\underline{B} = (K_2/K_1)\ \underline{A} + \underline{R}\ \underline{A}\ . \tag{A.22}$$

If the $\underline{R}$ matrix is skew symmetric, then equation (A.20) is in the form of equation (A.19) and the theorem proof is complete.

A square matrix can be expressed as the sum of a symmetric matrix and a skew symmetric matrix. Assume that the matrix $\underline{R}$ has a symmetric part, and let

$$\underline{R} = \underline{R}_s + \underline{R}_{ss} \tag{A.23}$$

where $\underline{R}_s$ is an n x n symmetric matrix and $\underline{R}_{ss}$ is an n x n skew symmetric matrix. Equation (A.22) then reduces to

$$\underline{B} = (K_2/K_1)\,\underline{A} + \underline{R}_s\,\underline{A} + \underline{R}_{ss}\,\underline{A}\,. \tag{A.24}$$

Since equation (A.24) is equivalent to equation (A.20), it must satisfy equation (A.18). Substituting (A.24) into (A.18) reduces to

$$\underline{A}^T\,\underline{R}_s\,\underline{A} = 0 \tag{A.25}$$

which is true for a symmetric matrix $\underline{R}_s$ only if $\underline{R}_s$ is a zero matrix. Therefore, equation (A.20) can be expressed in the form of (A.19) and the theorem is proven.

Although a solution in the form of (A.20) can be expressed in the form of (A.19) for any given $\underline{A}$, the representation may require the use of switching functions if $\underline{A}$ is time-varying. An illustration of the correspondence between (A.19) and (A.20) is given by the following example.

<u>Example A.1</u>

Consider the solution of (A.18) with $K_1 = K_2 = 1$ and $n = 2$.

$$a_1b_1 + a_2b_2 = a_1^{\,2} + a_2^{\,2}.$$

The solution corresponding to (A.19) is

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + \alpha_{12}\, a_2 \\ a_2 - \alpha_{12}\, a_1 \end{bmatrix} \qquad \text{(A.26)}$$

where α_{12} is the arbitrary entry of the 2 x 2 skew symmetric matrix $\underline{S}$.

The solution corresponding to (A.20) depends upon the $\underline{M}$ matrix specified. For this example, if

$$\underline{M} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

then the solution is given by

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \Gamma a_1 + (\Gamma + \gamma)\, a_2 \\ (\Gamma - \gamma)\, a_1 + 2\,\Gamma a_2 \end{bmatrix} \qquad \text{(A.27)}$$

where γ is the arbitrary entry of the skew symmetric matrix, and

$$\Gamma = (a_1^{\,2} + a_2^{\,2})/(a_1^{\,2} + 2a_1a_2 + 2a_2^{\,2}).$$

If the two solutions are equivalent, then there exists an α_{12} such that the first formulation corresponding to (A.19) can represent the solution corresponding to (A.20). Equating the two solutions yields the equations

$$a_1 + \alpha_{12}\, a_2 = \Gamma a_1 + (\Gamma + \gamma)\, a_2$$
$$a_2 - \alpha_{12}\, a_1 = (\Gamma - \gamma)\, a_1 + 2\,\Gamma\, a_2.$$

If $a_1 \neq 0$, the equations are satisfied if α_{12} is selected as

$$\alpha_{12} = \gamma - \Gamma - (2\Gamma - 1)(a_2/a_1).$$

If $a_2 \neq 0$, the equations are satisfied if α_{12} is selected as

$$\alpha_{12} = \gamma + \Gamma + (\Gamma - 1)(a_1/a_2).$$

If both $a_1 = a_2 = 0$, the two formulations are both 0 and are then equivalent.

If the vector $\underline{A}$ is time-varying as illustrated in Figure 7, then a_1 and a_2 may periodically be zero. In order to represent the solution of (A.27) in the form of (A.26), α_{12} must be a switching function. An example of such a function is given by

$$\alpha_{12} = \begin{cases} \gamma + \Gamma + (\Gamma - 1)(a_1/a_2) & \text{if} \quad |a_1| \leq |a_2| \\ \gamma - \Gamma - (2\Gamma - 1)(a_2/a_1) & \text{if} \quad |a_2| > |a_1| \\ 0 & \text{if} \quad a_1 = a_2 = 0. \end{cases}$$

The solution formulation of equation (A.19) is utilized in this thesis whenever the vector equation (A.18) is encountered. The corresponding results for the solution formulation of (A.20) are presented in Appendix B.

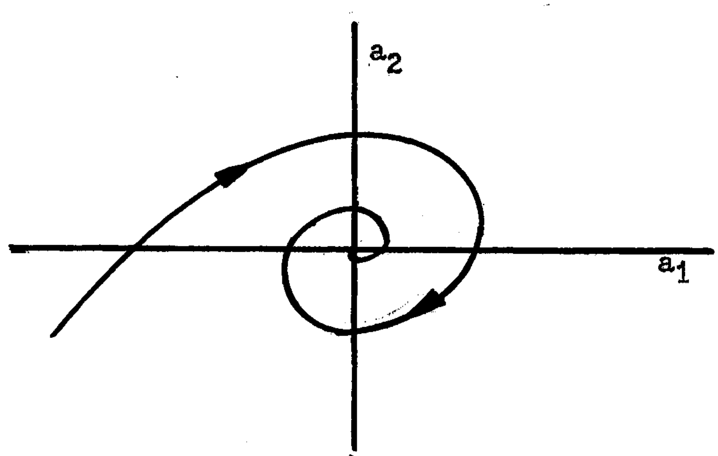

Figure 7. Example State Trajectory of $\underline{A}$

APPENDIX B

ALTERNATE REPRESENTATION RESULTS

The solution of the fundamental inner-product law can be represented in the form

$$\dot{\underline{X}}_r(t) = \left[\phi(\rho)\,\underline{I} + \underline{S}'\right]\,\underline{X}_r(t) \tag{B.1}$$

where $\underline{X}_r = \underline{I}_r\,\underline{X}$, $\underline{S}'$ is any r x r skew symmetric matrix and

$$\phi(\rho) = -\sqrt{h(\rho)}/2\rho. \tag{B.2}$$

An equivalent representation is given by

$$\dot{\underline{X}}_r(t) = \left[\zeta(\underline{X})\,\underline{M} + \underline{S}\right]\,\underline{X}_r(t) \tag{B.3}$$

where $\underline{S}$ is any r x r skew symmetric matrix, $\underline{M}$ is any r x r definite matrix and

$$\zeta(\underline{X}) = -\sqrt{h(\rho)}/(2\,\underline{X}_r^T\,\underline{M}\,\underline{X}_r). \tag{B.4}$$

The representation of equation (B.1) is utilized in the development of Chapter III and Chapter IV of the dissertation. This appendix notes those results which change if the representation of equation (B.3) were used in Chapters III and IV. The actual results of the dissertation do not change, but the representation of those results differs for the representation of (B.3).

The $\underline{S}'$ matrix of (B.1) incorporates the effects of the $\underline{S}$ matrix and $\underline{M}$ matrix of (B.3) by the use of switching functions. Therefore when Chapter III considers the use of the $\underline{S}'$ matrix in the control solution, the use of the $\underline{S}$ and $\underline{M}$ matrix of (B.3) are implied. This observation requires a revision of Definition 3.2 for the representation of (B.3).

Definition B.1

A system is called an <u>α-controllable system</u> if it is ρ-controllable and utilizes only the $\underline{S}$ matrix and $\underline{M}$ matrix in the control solution.

The revision of Definition 3.2 necessitates a revision of the statement of Theorem 3.3.

Theorem B.1

A system is α-controllable if and only if

i) $\underline{B}_r = \underline{0}$, (B.5)

and ii) $\underline{F}_r(\underline{X}(t),t) = \left[\zeta(\underline{X})\, \underline{M} + \underline{S} \right] \underline{X}_r(t)$ (B.6)

for some r x r skew symmetric matrix $\underline{S}$ and some r x r definite matrix $\underline{M}$.

The application of Theorem 3.3 to linear systems is not changed by the representation of (B.3). For the linear system of equation (3.29), (B.6) reduces to

$$\left[\underline{A}_{rr} - \zeta(\underline{X})\, \underline{M} - \underline{S} \right] \underline{X}_r + \underline{A}_{rs}\, \underline{X}_s = \underline{0}. \tag{B.7}$$

This equation must be valid for all $\underline{X}(t)$. If $\underline{X}_s = \underline{0}$ and the components of $\underline{X}_r$ are 1, then equation (B.7) reduces to

$$\underline{A}_{rr} = \zeta(\underline{X})\, \underline{M} + \underline{S} . \tag{B.8}$$

Since the components of $\underline{A}_{rr}$ are the coefficients of a linear system they must be constant or time-varying terms. Since $\underline{X}(t)$ is not known explicitly as a function of time, $\zeta(\underline{X})$ can not be constant or time-varying explicitly unless $\underline{M} = k\,\underline{I}$ for some real number k. If $\underline{M} = k\,\underline{I}$ the representation of (B.3) reduces to the representation of (B.1), so the results of Chapter III for linear systems are directly applicable to the representation of (B.3).

The only remaining theorem of Chapter III which is revised by the representation of (B.3) is Theorem 3.5.

Theorem B.2

Let b denote the rank of $\underline{B}_r$. A system is ρ-controllable if and only if

$$\underline{T}_2\,\underline{E}_r(\underline{X}(t),t) = \underline{T}_2\,\underline{S}\,\underline{X}_r + \zeta(\underline{X})\,\underline{T}_2\,\underline{M}\,\underline{X}_r \tag{B.9}$$

for some r x r skew symmetric matrix $\underline{S}$ and some r x r definite matrix $\underline{M}$, where $\underline{T}_2$ is an (r-b) x r matrix of rank (r-b) and

$$\underline{T}_2\,\underline{B}_r = \underline{0}. \tag{B.10}$$

The results of Chapter IV for single input systems are identical for the representation of (B.3) since the two representations are the same if the rank of the system error signal is one.

For δ-controllable systems, the equations of (B.9) are applicable. If k of the (r-b) equations can be satisfied by specification of the $\underline{S}$ and $\underline{M}$ matrix components, the remaining (r-b-k) equations are considered as possible δ-controllable systems. The (r-b-k) equations are differentiated until the equation can be satisfied by specification of the

$\underline{S}$ and $\underline{M}$ matrix components, or by a control vector component. The initial condition requirements are then satisfied by $\underline{S}$ and $\underline{M}$ matrix components or by adjusting the free initial conditions of the state variables.

Since $\zeta(\underline{X})$ is an explicit function of the state variables, but is dependent upon the $\underline{M}$ matrix specified, the use of the representation of (B.3) is quite complex for the simpliest of problems. The following example illustrates the complexity which is encountered with the use of (B.3).

<u>Example B.1</u>

Consider the linear system

$$dx_1/dt = x_3$$
$$dx_2/dt = u_1$$
$$dx_3/dt = u_2$$

with a system error signal of

$$\rho = x_1^2 + x_2^2$$

and an error penalty function of

$$h(\rho) = 4\rho^2.$$

The algebraic equations minimizing the performance measure are

$$x_3 = \zeta(\underline{X})\ (m_{11}\ x_1 + m_{12}\ x_2) + a_{12}\ x_2$$
$$u_1 = \zeta(\underline{X})\ (m_{21}\ x_1 + m_{22}\ x_2) - a_{12}\ x_1$$

where

$$\zeta(\underline{X}) = -(x_1^2 + x_2^2)/(m_{11}\ x_1^2 + (m_{12}+m_{21})x_1x_2 + m_{22}x_2^2).$$

The second equation is satisfied if u_1 is specified in feedback form as indicated by the equation. The first equation must be satisfied by use of Theorem 4.1. In addition, the solution must insure that $\underline{M}$ is positive definite or negative definite.

If the components of the $\underline{S}$ and $\underline{M}$ matrix are assumed constant and the first equation is differentiated and solved for u_2, the control vector is complete.

$$u_1 = \zeta(\underline{X})\ (m_{21}\ x_1 + m_{22}\ x_2) - \alpha_{12}\ x_1$$

$$u_2 = \eta(\underline{X})\ (m_{11}x_1 + m_{12}x_2) + \zeta(\underline{X})\ m_{11}x_3$$

$$+ (\alpha_{12} + m_{12}\ \zeta(\underline{X}))(\zeta(\underline{X})(m_{21}x_1 + m_{22}x_2) - \alpha_{12}x_1)$$

$$\underline{S} = \begin{matrix} 0 & -\alpha_{12} \\ \alpha_{12} & 0 \end{matrix} \qquad \underline{M} = \begin{matrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{matrix}$$

$$x_3(t_o) = \zeta(\underline{X}(t_o))(m_{11}x_1(t_o) + m_{12}x_2(t_o)) + \quad {}_{12}\ x_2(t_o)$$

where

$$\eta(\underline{X}) = d/dt \quad \zeta(\underline{X}).$$

The generality of the solution is obvious, but equally obvious is the fact that for complex system the calculations become quite cumbersome. The characterization of the general necessary conditions becomes more remote with the representation of (B.3). For this reason, the representation of (B.1) was adopted for the main body of the thesis.

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