THE DUNFORD-PETTIS PROPERTY

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PREFACE

This dissertation deals with real Banach spaces and linear maps from one to another. All mappings are continuous unless specified otherwise. It is the purpose of this paper to give various characterizations of the Dunford-Pettis property as well as some of the applications of this property. Two tables are provided in Chapter II of some of the most common Banach spaces and whether each space has the Dunford-Pettis property. Since the concept of the Dunford-Pettis property has been involved in a recent flurry of activity by some of the leading students of functional analysis, it is my hope that students in analysis with interest in the Dunford-Pettis property will find this dissertation of benefit.

The desired audience for this paper is the student who has completed a six hour course in functional analysis. The reader should have an understanding of Chapters II, IV, V and VI of Dunford-Schwartz's (11) book, since it will be referred to often. The notation and terminology in this paper will rather closely follow that in (11). The references used in the form (11, IV.3.7) will refer to number 11 in the bibliography while IV is the chapter number, 3 is the section and 7 is the theorem number in section 3.

Chapter I is concerned with the relationships among the following operators: unconditionally converging, completely continuous, weakly compact, compact and weak Cauchy. Some of the characterizations of the Dunford-Pettis property will be given in terms of these operators.

Chapter II contains basic theorems pertaining to the Dunford-Pettis property. Some of the permanence properties of the Dunford-Pettis property are also examined. It is determined whether or not the most common Banach spaces studied in an introductory course of functional analysis have the Dunford-Pettis property. Some generalizations are considered of the Banach space $C(S)$, where S is a compact Hausdorff space, and $L_1(\mu)$. These generalizations are involved in various open questions involving the Dunford-Pettis property.

Chapter III deals with some properties that are similar to the Dunford-Pettis property. A very brief treatment is given to the Dieudonné, V, and weak Cauchy V properties, and some of their applications will be applied in the study of linear operators on Banach spaces. There are two tables at the end of this chapter that give some sufficient conditions on a Banach space to determine whether it will possess the Dunford-Pettis property.

Finally, Chapter IV contains a summary and a list of some open questions involving the Dunford-Pettis property. A generalization of the idea of the Dunford-Pettis property is considered in this chapter. A table of Banach spaces is provided and it is determined whether these spaces have various properties. Also included in the open questions are some partial results.

I wish to express my appreciation to all who have helped me in the preparation and writing of this dissertation. Particular gratitude is due to Dr. Joe Howard, my dissertation advisor, who not only made valuable suggestions concerning this paper, but is also responsible for my interest in this topic. A special thanks also goes to Dr. E. K. McLachlan, Dr. John Jobe, Dr. Robert Alciatore and Dr. Vernon Troxel for serving as members of my advisory committee.

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CHAPTER I

OPERATORS AND THE DUNFORD-PETTIS PROPERTY

In addition to the most common operators, compact and weakly compact, several other operators have been studied in recent years. Lacy and Whitley (31) studied the completely continuous operators; Howard (22) studied the weak Cauchy operators and both Pelczynski (36) and Howard (24) studied the unconditionally converging operators.

The purpose of this chapter is to define five different types of operators and to exhibit a Venn diagram that will describe their interrelation. These operators will then be used in giving some of the characterizations of the Dunford-Pettis property. With these objectives in mind, we shall proceed in giving the necessary definitions and facts for defining these operators.

Definition 1.1 Let X be a Banach space (B-space). A series Σx_i in X is unconditionally convergent (u.c.) if for each subseries of Σx_i, there exists an element x in X such that the subseries converges to x, the convergence being relative to the topology on X.

Each of the following conditions is proved equivalent to Definition 1.1 in (35).

(a) A series Σx_i in X is subseries convergent relative to the weak topology on X.

Let $\quad S = \left\{ \Sigma x_i : i \text{ belongs to } F \text{ and } F \text{ a finite set} \right\}.$

(b) The weak closure of S is compact in the weak topology for X.

Definition 1.2 Let X be a B-space. A series Σx_i of elements in X is weakly unconditionally convergent (w.u.c.) if for every continuous linear functional, x' in the conjugate space of X, X', the series $\Sigma|x'(x_i)|$ is finite.

Operators

In this section of Chapter I the different operators will be defined and the relationship given in Figure 1 will be developed.

Definition 1.3 Let X and Y be B-spaces. A linear operator T from X to Y is unconditionally converging (u.c. operator) if T maps every w.u.c. series in X into a u.c. series in Y.

Definition 1.4 Let X and Y be B-spaces. A linear operator T from X to Y is completely continuous (c.c. operator) if T maps weak Cauchy sequences in X into norm convergent sequences in Y.

Definition 1.5 Let X and Y be B-spaces and S the closed unit sphere in X. A linear operator T from X to Y is a compact operator (cpt. operator) if the strong closure of T(S) is compact in the strong topology for Y.

A useful characterization of a compact operator is that it takes bounded sequences into sequences which have a convergent subsequence. It will also be noticed that if X is a reflexive space, then the c.c. and cpt. operators agree. Since much of the early work in functional analysis was done in the setting of a Hilbert space, which is reflexive, no distinction was made between these two operators. Later it will be shown that the class of completely continuous and compact operators do not agree in general for Banach spaces.

Definition 1.6 Let X and Y be B-spaces and S the closed unit sphere in X. A linear operator T from X to Y is weakly compact (w.c. operator) if the weak closure of T(S) is compact in the weak topology of Y.

The Eberlein-Smulian Theorem gives a very useful characterization of weakly compact operators, which is as follows: a linear operator is weakly compact if and only if it maps bounded sets into weakly sequentially compact sets.

The last operator to be defined was originated and studied by Howard (22).

Definition 1.7 Let X and Y be B-spaces. A linear operator T from X to Y is weak Cauchy (w.Cy. operator) if T maps bounded sequences of X into sequences in Y which have a weak Cauchy subsequence.

The following theorems and examples will enable us to see how the above defined operators are related to each other.

Theorem 1.8 Every compact operator is also weakly compact.

Proof. Let T be a compact operator from X to Y and S the closed unit sphere in X. The strong closure of T(S) is compact in the norm topology. From the duality between X and X' the closed convex sets in the weak and norm topology are the same. The norm topology being stronger than the weak topology implies that the weak closure of T(S) is compact in the weak topology. Therefore, T is a weakly compact operator.

Not every weakly compact operator is a compact operator. Let X be a infinite dimensional reflexive B-space and T the identity on X. Since X is infinite dimensional, the closed unit sphere, S, is not compact (7, page 40). Since X is reflexive, it follows that S is weakly compact (47, XIII.4.1). Therefore, T is a weakly compact operator which is not a compact operator.

Definition 1.9 A B-space X is almost reflexive if every bounded sequence in X contains a weak Cauchy subsequence.

The above definition is a generalization of a reflexive space. Let X be a reflexive space and (x_n) a bounded sequence in X, the sequence (x_n) is weakly sequentially compact (11, II.3.28). Therefore, X is almost reflexive. The B-space of null sequences, c_o , is not reflexive, but c_o is almost reflexive (31).

It should be noted that by the Eberlein-Smulian Theorem, a weakly complete space which is almost reflexive is reflexive.

Theorem 1.10 Every weakly compact operator is a weak Cauchy operator.

Proof. Let T be a weakly compact operator from X to Y and (x_n) a bounded sequence in X. The hypothesis implies that (Tx_n) is weakly sequentially compact in Y. The sequence (Tx_n) has a subsequence which is weakly convergent to a point in Y and thus, is a weak Cauchy subsequence of (Tx_n). Therefore, T is a weak Cauchy operator.

The converse of the previous theorem is false. Let T be the identity on the B-space of null sequences, c_o. Let S be the closed unit sphere in c_o. Since c_o is not reflexive, it follows that S is not weakly compact. Therefore, T is not a weakly compact operator. It follows that T is a weak Cauchy operator from the fact that c_0 is almost reflexive.

<u>Theorem 1.11</u> Every compact operator is completely continuous.

<u>Proof.</u> Let T be a compact operator from X to Y. Let (x_n) be a weak Cauchy sequence in X. Let (m) and (k) be subsequences of the indices of (x_n). The sequence $(x_m - x_k)$ converges weakly to zero from (16, page 138). Since T is compact and weakly continuous, it follows that $T(x_m - x_k)$ is norm convergent to zero. Hence, (Tx_n) is a Cauchy sequence in Y and, therefore, a norm convergent sequence.

From the following example, it can be seen that the converse of the above theorem is false. Let T be the identity map on the B-space of all absolutely convergent sequences, ℓ. Any map with domain or range ℓ is completely continuous (31). Since ℓ is infinite dimensional, the closed unit sphere in ℓ is not compact. Therefore, T is a completely continuous operator, but not a compact operator.

The following theorem gives a very useful characterization of a u.c. operator. This theorem enables us to represent some of the known results in functional analysis in terms of u.c. operators.

<u>Theorem 1.12</u> The linear map T from X to Y is a u.c. operator if and only if T has no bounded inverse on a subspace E of X which is linearly homeomorphic to c_0.

<u>Proof.</u> Assume T is not a u.c. operator. Thus, T has a bounded inverse on a subspace isomorphic to c_0 by (37).

Conversely, let T be a u.c. operator and assume T has a bounded

Conversely, let T be a u.c. operator and assume T has a bounded inverse on a subspace E, that is isomorphic to c_o. Let Σx_n be a w.u.c. series in E, then it is also a w.u.c. series in X. Since T is a u.c. operator, ΣTx_n is a u.c. series in T(E). But $(T_{|E})^{-1} = T^{-1}_E$ is bounded and therefore continuous. Also, $T^{-1}_E(\Sigma x_n) = \Sigma x_n$ is a u.c. series since continuous maps preserve w.u.c. and u.c. series. Therefore in E, every w.u.c. series is a u.c. series. Thus, E has no subspace isomorphic to c_o by (5). This is a contradiction since E is itself isomorphic to c_o. Therefore, T has no bounded inverse on a subspace isomorphic to c_o.

The above characterization for u.c. operators will be useful in showing that every completely continuous operator is also a u.c. operator.

Theorem 1.13 If T is a completely continuous operator from X to Y, then T is also a u.c. operator.

Proof. Assume T is not a u.c. operator. Thus by Theorem 1.12, T has a bounded inverse on a subspace E of X, that is isomorphic to c_o. Let $x_1, x_2, \ldots$ be elements of E which correspond to the unit base vectors of c_o under this isomorphism. For any f in $(c_o)' = \ell$, then $f(a) = \Sigma t_k a_k$ with t belonging to ℓ and $t_k = f(e_k)$ where the e_k's are the unit base vectors of c_o (47, page 91). Hence, $(f(e_k)) = (t_k)$ and since t is in ℓ, we have (x_n) converges weakly to zero.

By hypothesis and continuity of T the sequence (Tx_n) converges in norm to $0 = T(0)$. Now T^{-1} is continuous on T(E); therefore, the sequence $(T^{-1}(Tx_n)) = (x_n)$ converges in norm to zero. Hence, the unit base vectors of c_o converge to zero in norm. This is a contradiction since the unit base vectors of c_o do not converge to zero in norm. Therefore, T is a u.c. operator.

In order to show that the converse of the above theorem is not true we shall need the following two theorems.

Theorem 1.14 Let X be an almost reflexive B-space and T a map from X to the B-space Y. If T is completely continuous, then T is compact.

Proof. Let S be the closed unit sphere in X. Let (y_n) be a sequence in T(S). Hence, there exists a sequence (x_n) in S such that $Tx_n = y_n$. Since X is almost reflexive, there is a weak Cauchy subsequence of (x_n). From the fact that T is completely continuous and Y is a B-space, we have (y_n) has a convergent subsequence. Hence, T(S) is compact and T is a compact operator.

Using Theorems 1.11 and 1.14 it is seen that the completely continuous and compact operators agree on almost reflexive B-spaces.

Theorem 1.15 Every weakly compact operator is an unconditionally converging operator.

Proof. Let T be a weakly compact operator from X to Y. Let Σx_i be a w.u.c. series. For any x' in X' we have that the set

$$\left\{ \Sigma x'(x_i) : i \text{ belongs to a finite subset of the indices} \right\}$$

is bounded. Hence, the set

$$H = \left\{ \Sigma x_i : i \text{ belongs to a finite subset of indices} \right\}$$

is weakly bounded. Hence, H is bounded in the norm topology by (28, page 409). Since T is a weakly compact operator, the weak closure of T(S) is compact in the weak topology. From the equivalent condition (b) in Definition 1.1 it follows that ΣTx_i is a u.c. series in Y. Therefore, T is a u.c. operator.

Now we can readily give an example of a u.c. operator that is not completely continuous. Let X be an infinite dimensional reflexive B-space and T be the identity on X. Thus, T is weakly compact since the closed unit sphere is compact in the weak topology. From Theorem 1.15 we have T is a u.c. operator. Now we need to show T is not completely continuous. Suppose T is completely continuous. Then by Theorem 1.14 T is a compact operator. This implies that the closed unit sphere of X is compact and X must be a finite dimensional B-space. This is a contradiction, so T is not completely continuous.

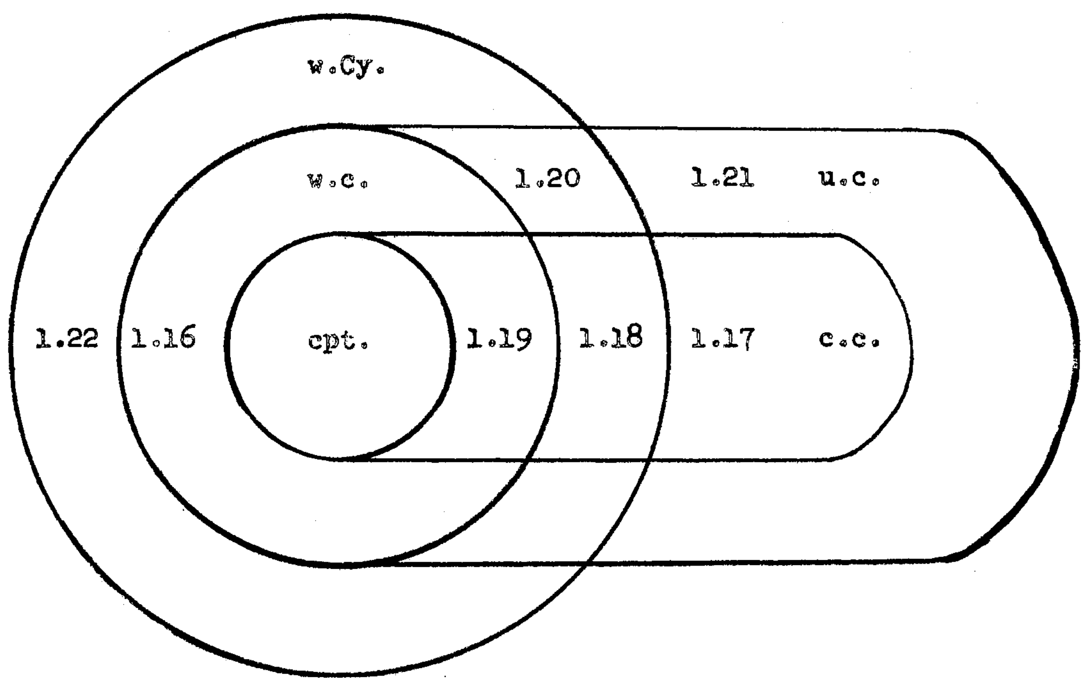


Figure 1. Interrelation of Five Operators

In order to show that the Venn diagram is drawn correctly, it is necessary to find operators that will satisfy the specific properties of each region of Figure 1. It will now be shown that the set of compact operators is non-void. Let X be a Banach space, z a fixed vector in X and f a continuous linear functional on X. Define the map T of X into X as follows $T(x) = f(x)z$. The linearity of T follows from the linearity of f. Let S be a bounded set in X. Let (Tx_n) be a sequence in T(S). Since S was assumed to be bounded, there exists some M such that $|x_n| < M$ for every n. Hence, $|f(x_n)| \leq |f||x_n| \leq |f|\ M$ for every n and it is seen that $(f(x_n))$ is a uniformly bounded sequence of real numbers. By the Bolzano-Weierstrass Theorem, there is a convergent subsequence of $(f(x_n))$, say $(f(x_{nk}))$. Let a be the limit of this convergent subsequence. Therefore, $(Tx_{nk}) = (f(x_{nk})z)$ converges to a z and a convergent subsequence of (Tx_n) has been exhibited. The linear operator is a compact operator and, thus, the set of compact operators is non-void.

The following numbered examples will indicate that the corresponding numbered regions of the Venn diagram in Figure 1 are non-void.

Example 1.16 There exists an operator which is weakly compact but not completely continuous. Such an example was given after the proof of Theorem 1.15.

Example 1.17 There exists an operator which is completely continuous but not weak Cauchy. Consider the identity map T on the B-space, ℓ. Since any map with domain or range ℓ is completely continuous, it follows from Theorem 1.13 that T is a u.c. operator. The space ℓ is not almost reflexive, otherwise it would contradict Theorem

1.14. Thus the identity on ℓ is not a weak Cauchy operator.

Example 1.18 There exists a T that is completely continuous and weak Cauchy but not weakly compact. Let (a_n) be in ℓ and define $T((a_n)) = (\sum_{k=n}^{\infty} a_k)$. Thus, T maps ℓ into c_o. Since c_o is almost reflexive, it follows that T is weak Cauchy. Also, T is completely continuous since ℓ is the domain of T. Let e_n be the unit base vectors in ℓ. Thus, $(T(e_n))$ is a weak Cauchy sequence in c_o since $(\sum_{i=1}^{n} t_i)$ is convergent (47, page 91). But since not all the t_k can be zero, we have that $(T(e_n))$ does not have a weak limit in c_o. Hence, T is not a weakly compact operator.

Example 1.19 There exists an operator that is weakly compact and completely continuous but not compact. Let T be the injection of ℓ into ℓ_2. Let (e_i) be the unit base vectors in ℓ. Assume T is compact. Thus, $T(e_i) = (e_i)$ is sequentially compact. But for i different from j, $|e_i - e_j| = \sqrt{2}$ in ℓ_2. Hence, there is no subsequence of (e_i) that is convergent, which contradicts (e_i) being sequentially compact. Therefore, T is not compact. Since the domain of T is ℓ, it follows that T is completely continuous. Since ℓ_2 is reflexive it follows that T is a weakly compact operator.

Example 1.20 There exists an operator that is weak Cauchy and u.c. but not weakly compact nor completely continuous. Let $X = \ell$, then there exists a separable space E such that E'' is the direct sum of J(E) and ℓ, where J is the natural embedding of E into E'', from Theorem 1 of (25). Let T be the identity map on E'. Since E'' is separable, then E' is separable. To show that E' has no subspace isomorphic to c_o, assume E' has a subspace isomorphic to c_o. If a conjugate B-space contains a subspace isomorphic to c_o, then E' contains a subspace isomorphic to the B-space of bounded sequences, ℓ_∞, by Theorem 4 of (5). Since ℓ_∞ is not separable, we have E' is not separable, which is a contradiction. Hence, T is a u.c. operator. Since E'' is separable, then E' is almost reflexive and, thus, T is a weak Cauchy operator. From the fact that E'' is the direct sum of J(E) and ℓ, we have that E is not reflexive. Thus, E' is not reflexive and T is not weakly compact. The mapping T is not completely continuous since an assumption otherwise would contradict Theorem 1.14.

Example 1.21 There exists an operator that is u.c. but not weak Cauchy nor completely continuous. Let T be the identity map on X, that is the direct sum of E' and ℓ. Since E' and ℓ are separable, then X is separable. Hence, X' is a separable conjugate space and contains no subspace isomorphic to c_o. From Theorem 1.12 T is a u.c. operator. For any bounded sequence (a_n) and (b_n) in E' and ℓ respectively, the sequence $(a_n \oplus b_n)$ is bounded in X. If X is almost reflexive, then E' and ℓ are almost reflexive. Since ℓ is not almost reflexive, it follows that X is not almost reflexive. Hence, T is not weak Cauchy. Also, T is not completely continuous since weak Cauchy sequences do not correspond to Cauchy sequences in E'.

Example 1.22 There exists an operator T that is weak Cauchy but not u.c. Let T be the identity on the B-space c_o. Since c_o is almost reflexive, T is weak Cauchy. From Theorem 1.12 it follows that T is not a u.c. operator.

Before defining the Dunford-Pettis property and proving some characterizations of it, we shall need the following preliminaries.

Grothendieck (16) introduced four types of limited sets. Only two of these types will be used in characterizing the Dunford-Pettis property; the others are given here for completeness and future reference.

Definition 1.23 Let X be a B-space and X' its conjugate space.

(a) Let A' be a subset of X'. The set A' is w-limited in X' if

$$\lim_{n} \sup_{A'} | x' (x_n) | = 0$$

for every sequence (x_n) in X which is weakly convergent to 0.

(b) Let A be a subset of X. The set A is $\widetilde{w}$-limited in X if

$$\lim_{n} \sup_{A} | x'_n (x) | = 0$$

for every sequence (x'_n) in X' which is weakly convergent to zero.

(c) Let A' be a subset of X'. The set A' is w.u.c.-limited if

$$\lim_{n} \sup_{A'} x' (x_n) = 0$$

for every w.u.c. series Σx_n in X.

(d) Let A be a subset of X. The set A is $\widetilde{\text{w.u.c.}}$-limited if

$$\lim_{n} \sup_{A} x'_n (x) = 0$$

for every w.u.c. series $\Sigma x'_n$ in X'.

The idea of a limited set can be used to characterize a completely continuous operator. This characterization will be useful in studying the Dunford-Pettis property.

Theorem 1.24 Let T be a map from X to Y and T' be its adjoint map.

(a) T is completely continuous if and only if T' maps bounded sets into w-limited sets in X'.

(b) T' is completely continuous if and only if T maps bounded sets into $\tilde{w}$-limited sets in Y.

Proof. Since the proof of both parts of this theorem are similar, only a proof of part (a) will be given. Assume T is completely continuous. Let A' be a bounded set in Y' and (x_n) a sequence in X that is weakly convergent to zero. Since T is completely continuous, the sequence (Tx_n) is norm convergent to zero. Thus, for y' in A'

$$\lim_{n} \sup_{A'} | y' (T x_n) | = 0,$$

but since $y'(Tx_n) = T'y'(x_n)$, it follows that T'(A') is a w-limited set in X'.

Assume T' maps bounded sets into w-limited sets in X'. Let (x_n) be a weak Cauchy sequence in X. From the characterization given by Grothendieck (16) for weak Cauchy sequences, we have for any subsequences (k) and (m) of the sequence (n) of indices, the sequence $(x_k - x_m)$ converges weakly to zero. Let y' be in Y'; thus, by hypothesis

$$\lim_{n} \sup | y'(T(x_k - x_m)) | = \lim_{n} \sup | T'y'(x_k - x_m) | = 0.$$

Therefore, $(T(x_k - x_m))$ is a Cauchy sequence in Y and Y is a B-space. Thus, (Tx_n) is norm convergent and T is completely continuous.

Dunford-Pettis Property

In 1953, Grothendieck (16) defined the concept of the Dunford-Pettis property. The Dunford-Pettis property affords a sort of "axiomatization" of relatively deep characteristics of weakly compact operators acting on spaces of continuous or integrable functions (see Theorems 2.2 and 2.23). The basic notions for the following characterizations are to be found in (38), (22) and (23).

Definition 1.25 A B-space X has the Dunford-Pettis (D. P.) property if for every B-space Y and every weakly compact operator T from X to Y, T maps weak Cauchy sequences of X into Cauchy sequences in the norm topology of Y.

Theorem 1.26 Let X be a B-space. The following conditions are equivalent.

(a) The space X has the D. P. property.

(b) For every B-space Y, if T is a weakly compact operator from X to Y, then $\lim |Tx_n| = 0$ for every sequence (x_n) in X that converges weakly to zero.

(c) The condition (b) is satisfied for $Y = c_o$.

(d) For every sequence (x_n) in X that converges weakly to zero and for every sequence (x'_n) in X' that converges weakly to zero, then $\lim x'_n(x_n) = 0$.

(e) If (x_n) and (x'_n) are weak Cauchy sequences in X and X' respectively, then the $\lim x'_n x_n$ exists.

(f) Given any B-space Y, every weakly compact operator T from X to Y is also a completely continuous operator.

(g) Given any B-space Y, then every weak Cauchy operator T' from

Y' to X' is such that T is completely continuous.

(h) Every weakly sequentially compact set in X' is w-limited in X'.

(i) Every weakly sequentially compact set in X is $\tilde{w}$-limited in X.

(j) For every B-space Y, every weakly compact operator T from Y to X is such that T' is completely continuous.

<u>Proof</u>. Since the proof is lengthy, we shall sketch the plan of attack. We shall establish the following implications:

$$h \rightarrow f \rightarrow a \rightarrow j \rightarrow i \rightarrow e \rightarrow b \rightarrow c \rightarrow d \rightarrow g \rightarrow h$$

Now to execute our plan.

(h) implies (f). Let Y be a B-space and T a weakly compact operator from X to Y. Thus by a theorem of Gantmacher (11, VI.4.8) the conjugate map T' from Y' to X' is weakly compact. Let H be a bounded set in Y'. Since T' is weakly compact, T'(H) is weakly sequentially compact in X'. Hence, T'(H) is w-limited in X' and Theorem 1.24 implies that T is completely continuous.

(f) implies (a). This implication follows directly from the definition of completely continuous.

(a) implies (j). Let Y be a B-space and T a weak Cauchy operator from Y to X. The conjugate map T' from X' to Y' will now be shown to be completely continuous. It will suffice to show that $\lim |T'x_n'| = 0$, for every sequence (x_n') in X' which converges weakly to zero (16, page 138).

Let (x_n') be a sequence in X' that converges weakly to zero and $\limsup |T'(x_n')| = t$ where t is a non-negative real number. The Hahn-Banach Theorem implies that there exists a sequence (y_n) in Y such that $|y_n| = 1$ and $T'(x_n') = |T'x_n'|$. Define $x_n = T(y_n)$ for each n. Since T is weak Cauchy, we may assume without loss of generality (for otherwise we could replace this sequence by a subsequence) that (x_n) is a weak Cauchy sequence. Thus,

$$\limsup x'_n x_n = \limsup x'_n(Ty_n) = \limsup T'x'_n y_n = \limsup |T'x'_n| = t.$$

Now to show that t must be zero. Let (m) be a subsequence of the indices of (x'_n) such that $|x'_m x_n| < t/2$. Such a subsequence (m) will exist since x_n is weak Cauchy. Thus,

$$x'_m x_m = x'_m (x_m - x_n) + x'_m x_n .$$

Since (x_n) is a weak Cauchy sequence, then $(x_m - x_n)$ converges weakly to zero (16, page 138). Hence,

$$t = \limsup |x'_n x| \leq \limsup |x'_m(x_m - x_n)| + \limsup |x'_m x_n| \leq t/2.$$

Therefore, $t = 0$ and T' is completely continuous.

(j) implies (i). Let A be a weakly sequentially compact set in X. Consider the B-space $\ell(A)$, which is the set of all scalar-valued functions whose norm, given by $|x| = \Sigma \{ |x(a)| : a \in A \text{ and } x(a) \neq 0 \}$, is finite. Let $\{ e_a : a \in A \}$ be the collection of characteristic functions of the singleton set $\{a\}$. Thus for each a in A, e_a is in $\ell(A)$. Define T from $\ell(A)$ to X by $T(\Sigma x(a)e_a) = \Sigma x(a)Te_a$. The linear operator T is weakly compact since A is weakly sequentially compact (7, page 54). Therefore, T' is a completely continuous operator. Let (x'_n) be a sequence in X' that converges weakly to zero. Hence,

$$\lim_n \sup_A |T'x'_n(a)| = \lim_n \sup_A |x'_n(a)| = 0.$$

Therefore, A is $\widetilde{W}$-limited in X.

(i) implies (e). Let x_n and x'_n be weak Cauchy sequences in X and X', respectively. Let e_n denote the n-th unit base vector in c_o. Define T from c_o to X as follows:

$$T e_n = x_n \quad .$$

Thus, the conjugate map, T', from X' to ℓ is completely continuous (31). Hence, for each x' in X'

$$T'x'(e_n) = x'(Te_n) = x'x_n \quad .$$

Thus, for each n, $T'x'_n = x'_n x_n$. Since T' is completely continuous it follows that $(T'x_n) = (x'_n x_n)$ is a norm-convergent sequence in ℓ. Therefore, $\lim x'_n x_n$ exists.

(e) implies (b). Let Y be a B-space and T a weakly compact operator from X to Y. Let (x_n) be a sequence that converges weakly to zero and $\lim\sup |Tx_n| = t$ where t is a non-negative real number. The Hahn-Banach Theorem implies that there exists a sequence (y'_n) in Y' such that $|y'_n| = 1$ and $y'_n(Tx_n) = |Tx_n|$. Define $x'_n = T'y'_n$. By a theorem of Gantmacher, T' is weakly compact. Theorem 1.10 implies that T' is a weak Cauchy operator. Hence without loss of generality we may assume that (x'_n) is a weak Cauchy sequence (for otherwise we could replace this sequence by a subsequence). Thus,

$$\lim\sup x'_n x_n = \lim\sup Ty'_n x_n = \lim\sup y'_n Tx_n = \lim\sup |Tx_n| = t \ .$$

It will now be shown that t must be zero. There exist subsequences (y_k) and (y'_k) of (x_n) and (x'_n), respectively, such that

$$\lim |y'_k(y_k)| = t \quad .$$

Since the sequence (x_n) converges weakly to zero, there exists a subsequence (z_m) of the sequence (y_k) such that $|z'_m(z_m)| < t/2$. Thus, we can write

$$z'_m z_m = (z'_m - y'_k) z_m + y'_k z_m .$$

Since (x'_n) is a weak Cauchy sequence, $(z'_m - y'_k)$ is a sequence that converges weakly to zero. Hence,

$$t = \lim |z'_m z_m| \leq \limsup |(z'_m - y'_k) z_m| + \limsup |y'_k z_m| \leq t/2 .$$

Therefore, $t = 0$ and $\lim |Tx_n| = 0$.

(b) implies (c). Since c_o is a B-space, condition (c) is a special case of condition (b).

(c) implies (d). Let (x'_n) be an arbitrary sequence in the conjugate space X' that converges weakly to zero. Let J be the natural embedding map of X into X''. For any x'' in $J(X)$, we have $x''x'_n = x'_n x$ and the sequence $(x'_n x)$ converges to zero. Define a map T from X to c_o as $T(x) = (x'_n x)$. This mapping is linear. Since (x'_n) is pointwise bounded and for each n, x_n is a continuous linear functional on X, it follows that (x'_n) is equicontinuous from (47, XI.2.3). This means that the $\{x'_n\}$ is uniformly bounded. Hence, T is a continuous linear map. The following will show that T' is weakly compact. Let e'_n denote the n-th unit base vector in $\ell = c'_o$. For T' the adjoint operator of T,

$$T' e'_n = e_n T = x'_n \qquad \text{for all } x \text{ in } X .$$

Let $\quad \text{bal}(e'_n) \quad$ represent the balanced hull of the sequence (e'_n),

$\quad \overline{\text{bal}}(e'_n) \quad$ the closed balanced hull of the sequence (e'_n), and $\text{conv}(\overline{\text{bal}}(e_n'))$ the convex hull of the set $\overline{\text{bal}}(e_n')$.

Let S be the closed unit sphere in ℓ. For x in S, $x = \Sigma\, t_i e_i'$ where the ℓ-norm of x is less than or equal to one, thus $|t_i| \leq 1$ for all i. Let

$$a_n = \sum_1^n t_i .$$

Thus,

$$\frac{1}{a_n} \sum_1^n t_i e_i \quad \text{is a sequence of elements in} \quad \text{conv}\,(\overline{\text{bal}}\,(e_n')) .$$

Hence,

$$\frac{1}{a}\, x \quad \text{is in the} \quad \overline{\text{conv}}\,(\overline{\text{bal}}\,(e_n')) = H \quad \text{where} \quad \lim a_n = a.$$

Therefore, $a(\frac{1}{a}\, x) = x$ is in H. Since S is a closed balanced convex set, we have S = H. Hence,

$$T'(S) \subset \overline{T'\,(\text{conv}\,(\text{bal}\,(e_n')))} = \overline{\text{conv}}\,(\overline{\text{bal}}\,(x_n'))$$

and $\overline{\text{bal}}(x_n')$ is weakly compact by (47, page 177). The Krien-Smulian Theorem (11, V.6.4) implies $\overline{T'(S)}$ is weakly compact in X'. Thus, T' is a weakly compact operator. By a theorem of Gantmacher T is a weakly compact operator. Let (x_m) be an arbitrary sequence in X that is weakly convergent to zero. Thus by hypothesis,

$$\lim |\, T\, x_m \,| = \lim \sup |\, x_n'\, x_m \,| = 0 .$$

Therefore, $\lim x_n'\, x_n = 0$.

(d) implies (g). Let Y be a B-space and T' be a weak Cauchy operator from Y' to X'. Now to show that T is completely continuous, it will suffice to show that for every sequence (x_n) in X that converges weakly to zero, then $\lim |Tx_n| = 0$. Let $\lim\sup |Tx_n| = t$ where t is a non-negative real number. The Hahn-Banach Theorem implies there exists a sequence (y'_n) in Y' such that $|y'_n| = 1$ and $y'_n(Tx_n) = |Tx_n|$. Define $x'_n = T'(y'_n)$. The remainder of the argument is similar to that used in (e) implies (b) to show that t must be zero.

(g) implies (h). First it will be noticed that if Y is a B-space and T a weakly compact operator from X to Y, then T' is a weakly compact operator (11, VI.4.8). From Theorem 1.10 it follows that T' is also a weak Cauchy operator.

Let K' be a weakly sequentially compact set in X'. Let B(K') be the B-space of all bounded scalar-valued functions on K' with the sup-norm. Define T from X to B(K') by $Tx(k') = k'(x)$. Since K' is weakly sequentially compact, T is weakly compact by (46). From the above observation it follows that T' is a weak Cauchy operator. Therefore, T is completely continuous. Let (x_n) be a sequence in X that converges weakly to zero, then (Tx_n) is norm-convergent to zero. Hence,

$$\lim_n \sup_{K'} |Tx_n(k')| = \lim_n \sup_{K'} |k'(x)| = 0 .$$

Therefore, K' is w-limited in X'.

CHAPTER II

PROPERTIES OF THE DUNFORD-PETTIS PROPERTY

The objective of this chapter is to investigate the basic properties of the Dunford-Pettis property. It is shown that isomorphisms, finite topological direct sums, and complemented subspaces are among the permanence properties of the Dunford-Pettis property. Among the non-permanence properties are subspaces, quotient spaces, inductive and projective limit spaces. Some generalizations of the $C(S)$ and $L_1(\mu)$ spaces are considered. At the end of this chapter there are two tables showing whether some of the common Banach spaces possess the Dunford-Pettis property.

The first theorem in this chapter will deal with a class of B-spaces that do not possess the D. P. property.

Theorem 2.1 No infinite dimensional reflexive B-space possesses the D. P. property.

Proof. Let X be a infinite dimensional reflexive B-space. Let T be the identity map on X. Since X is reflexive, it follows that T is weakly compact. Suppose T is completely continuous. From Theorem 1.14 we have T is compact. Therefore, X is finite dimensional which is a contradiction. Hence, T is not completely continuous and X does not possess the D. P. property.

It would now seem proper to ask if there exist any B-spaces, other than finite dimensional ones, that possess the D. P. property. The next theorem that relies heavily on measure theory will show that B-spaces that possess the D. P. property are quite numerous.

Theorem 2.2 Let S be a compact Hausdorff space, then the B-space of all continuous scalar-valued functions on S, C(S), possesses the D. P. property.

Proof. Let Y be a B-space and T a weakly compact operator from C(S) to Y. Let (f_n) be a weak Cauchy sequence in C(S). Let g belong to the conjugate space of C(S); thus, $(g(f_n))$ is a Cauchy sequence of scalars and $\sup_n |g(f_n)|$ is finite. The $\sup_n |f_n|$ is finite from (11, II.3.20). Define for each s in S $f(s) = \lim f_n(s)$. The limit function f is bounded and measurable on S. Using the representation theorem of Dunford and Schwartz (11, VI.7.3), it follows that there exists a vector measure μ defined on the Borel sets of S such that

$$T\, f_n = \int_S f_n \, d\mu \, .$$

It follows that

$$\lim T\, f_n = T\, f$$

and (Tf_n) is a convergent sequence in Y from the dominated convergence theorem (11, IV.10.10). Therefore, C(S) has the D. P. property.

In the study of B-spaces the conjugate space plays an important role. Two natural questions at this point would be that if X' has the D. P. property, then does X possess the D. P. property and vice versa. Our next theorem will answer the first question positively, but the second part is an open question.

Theorem 2.3 If the space X', the conjugate space of a B-space X, has the D. P. property, then the space X has the D. P. property.

Proof. Let (x_n) and (x_n') be sequences in X and X', respectively, that converge weakly to zero. Let J be the natural embedding of X into X''. Thus, $J(x_n) = \hat{x}_n$ is a sequence in X'' that converges weakly to zero. Since $\hat{x}_n(x_n') = x_n'(x_n)$ and X' has the D. P. property, this implies that $\lim x_n'(x_n) = 0$. Then from Theorem 1.26 part (d), X possesses the D. P. property.

Some Permanence Properties

If the D. P. property happened to be a hereditary property, then there would be some very nice results. As one might gather from Theorem 2.1, this is not the case (see remarks after Theorem 2.7). What conditions are needed on a linear subspace in order to ensure that it will have the D. P. property? In order to answer this question we need the following definition.

Definition 2.4 Let X be a B-space, L a linear subspace of X and M an algebraic complement of L relative to X. The map $(a,b) \rightarrow a + b$ is a continuous algebraic isomorphism of $L \times M$ onto X. If the map has a continuous inverse, then L and M are said to be topological complements of each other relative to X.

For a B-space X that is the algebraic direct sum of closed subspaces L and M, the subspaces are topological complements of each other relative to X (12, page 66). Even for a B-space X there will in general exist in X closed linear subspaces L that admit, relative to X, no topological complement. Kothe (29, page 424) has proven that c_o

is a closed subspace of ℓ_∞ that has no topological complement.

Theorem 2.5 If X possesses the D. P. property and H is a linear subspace of X admitting a topological complement relative to X, then H possesses the D. P. property.

Proof. The hypothesis on H implies there exists a continuous projection P of X onto H. Let Y be a B-space and T a weakly compact operator from H to Y. Thus TP is a weakly compact operator from X to Y by (11, VI.4.5). Hence TP is completely continuous. Since T agrees with the restriction of TP to H, it follows that T is completely continuous and H has the D. P. property.

The following definition is included in this paper to avoid any possible confusion that might arise when the statement is made that two B-spaces are isomorphic.

Definition 2.6 An isomorphism between two B-spaces is a linear homeomorphism. An isometric isomorphism between two B-spaces X and Y is an isomorphism U between X and Y such that $|Ux| = |x|$.

With the above definitions we are now able to prove that the D. P. property is preserved under isomorphisms.

Theorem 2.7 Let X and Y be isomorphic B-spaces. If X has the D. P. property, then so does Y.

Proof. Let U be an isomorphism from X to Y. Let W be a B-space and T a weakly compact operator from Y to W. Since U is continuous, TU is a weakly compact operator from X to W. Let (y_n) be a weak Cauchy sequence in Y. Thus, there exists a sequence (x_n) in X such that

$U(x_n) = y_n$. For each f in X', fU^{-1} is in Y' and it follows that (x_n) is a weak Cauchy sequence in X. Hence, TU is completely continuous and $TU(x_n) = T(y_n)$ is norm convergent. Therefore, Y has the D. P. property.

Let X be an infinite dimensional reflexive B-space. From the universality property of the B-space C(S), there exists a compact Hausdorff space S such that X is isometrically isomorphic to a linear subspace W of C(S). From Theorems 2.1 and 2.7 it is easily seen that the D. P. property is not hereditary.

Definition 2.8 Let X and Y be B-spaces over the same field of scalars. Let $W = X \oplus Y$ be the algebraic direct sum of X and Y. Let W have either of the norms

$$|(x,y)| = \max(|x|, |y|) \qquad \text{or}$$

$$|(x,y)| = (|x|^p + |y|^p)^{1/p} \qquad 1 \leq p < \infty,$$

and W becomes a B-space. The space W obtained in this manner is called the direct sum of the two B-spaces X and Y.

The extension to any finite number of summands is immediate. The direct sum of a denumerable number of B-spaces can be made into a Frechet space, but in general not into a B-space. Also, one can define the direct sum of an arbitrary family of linear topological spaces, but it is ordinarily not a metric space even if the summands are. Either of the norms is equivalent to the product topology on W, as pointed out in (11, page 89). Also, we have the property

$$(X \oplus Y)' = X' \oplus Y'.$$

Theorem 2.9 Let X_1 and X_2 be B-spaces and X be the direct sum of X_1 and X_2. Let T be a continuous linear map from X to a B-space W. The map T is weakly compact if and only if T_1 (which is the restriction of T to X_1) and T_2 (the restriction of T to X_2) are both weakly compact.

Proof. If T is a weakly compact operator, then so is its restriction to X_1 and similarly to X_2.

Assume T_1 and T_2 are weakly compact operators. Let A be the closed unit sphere in X and P_i be the projection of X onto X_i $(i = 1, 2)$. Thus, $P_i(A)$ $(i = 1, 2)$ is bounded and convex. Hence, $T_iP_i(A)$ $(i = 1, 2)$ is convex and conditionally weakly compact since T_i $(i = 1, 2)$ is weakly compact. With the closure (cl) being taken in the weak topology we have

$$\text{cl }(T(A)) = \text{cl }(T_1P_1(A) + T_2P_2(A)) = \text{cl }(T_1P_1(A)) + \text{cl }(T_2P_2(A))$$

is weakly compact by (11, page 415). Therefore, T is a weakly compact operator.

Theorem 2.10 Let X_1 and X_2 be B-spaces and X be the direct sum of X_1 and X_2. Let T be a continuous linear map from X to a B-space Y. Let T_i be the restriction of T to X_i $(i = 1, 2)$. Then, T is completely continuous if and only if both T_1 and T_2 are completely continuous.

Proof. Assume T is completely continuous. Clearly, T_1 and T_2 are also completely continuous.

Assume T_1 and T_2 are completely continuous. Let (x_n) be a weak Cauchy sequence in X. Let P_i $(i = 1, 2)$ be the projection of X onto X_i. Each P_i will preserve weak Cauchy sequences, thus $P_i(x_n)$ is a weak Cauchy sequence in X_i. Since T_i $(i = 1, 2)$ is a completely continuous operator, $T_1P_1(x_n)$ is a Cauchy sequence in X_1. But

$$T x_n = T_1 P_1 (x_n) + T_2 P_2 (x_n)$$

is a Cauchy sequence in X. Therefore, T is completely continuous.

With the above characterizations of weakly compact and completely continuous operators we are able to prove the following very useful theorem.

<u>Theorem 2.11</u> Let X_1 and X_2 be B-spaces and X be the direct sum of X_1 and X_2. The space X will have the D. P. property if and only if X_1 and X_2 have the D. P. property.

<u>Proof.</u> Assume X_1 and X_2 have the D. P. property. Let W be a B-space and T a weakly compact operator from X to W. Let P_i be the projection of X onto X_i $(i = 1, 2)$. Since X and each X_i are B-spaces and each P_i is a linear continuous map, P_i is also weakly continuous. Let (z_n) be a sequence in X that converges weakly to zero, thus

$$P_1(z_n) = P_1(x_n + y_n) = x_n$$

and

$$P_2(z_n) = P_2(x_n + y_n) = y_n$$

converge weakly to zero. Hence,

$$T (z_n) = T_1 P_1 (z_n) + T_2 P_2 (z_n) .$$

Since T_1 and T_2 are weakly compact and X_1 and X_2 have the D. P. property, it follows that $T_1(x_n)$ and $T_2(y_n)$ are norm convergent to zero. Thus, $T(z_n)$ is also norm convergent to zero and T is completely continuous. Therefore, X has the D. P. property.

Assume X_1 and X_2 have the D. P. property. Let T be a weakly compact operator on X. By Theorem 2.9 it follows that T_1 and T_2 are weakly compact operators. Since X_1 and X_2 have property V, then T_1 and T_2 are completely continuous. From Theorem 2.10, T is completely continuous. Therefore, X has the D. P. property.

In the study of the permanence properties of the D. P. property, it has been seen that the D. P. property is preserved under complemented subspaces, isomorphisms and finite direct sums. In the field of functional analysis the quotient spaces, inductive limit spaces and projective limit spaces are familar tools. The question arises whether the D. P. property is preserved under the formation of these spaces.

<u>Example 2.12</u> If a B-space X has the D. P. property, then a quotient space of X will not necessarily have the D. P. property. Since ℓ_2 is an infinite dimensional reflexive space that is also separable, there exists a closed subspace N of ℓ such that ℓ_2 is isomorphic to the quotient space of ℓ by N (29, page 280). The space ℓ has the D. P. property. The space ℓ/N does not have the D. P. property since ℓ_2 is reflexive and the D. P. property is preserved under isomorphisms.

The inductive limits space will now be defined. Then the question of whether it is a permanence property of the D. P. property will be examined. A more complete discussion of these spaces can be found in (29, page 219) and (40, page 76).

<u>Definition 2.13</u> Let X_γ be a collection of locally convex spaces and X be a vector space. Let T_γ be a collection of linear maps from

X_γ into X such that $\cup T_\gamma(X_\gamma)$ spans X. Let B be the collection of all convex and balanced subsets U of X such that for each γ, $T_\gamma^{-1}(U)$ is a neighborhood in X_γ; thus, B is a base for the linear topology on X which makes all the T_γ continuous. The convex space X with this topology is called the inductive limit of the locally convex spaces X_γ by the mappings T_γ.

An extreme case of an inductive limit space is the quotient space. For if $X = Y/M$ and T is the canonical mapping of Y onto X, the quotient topology is the finest linear topology making T continuous. From Example 2.12 it is seen that the inductive limit space is not a permanence property of the D. P. property.

<u>Definition 2.14</u> Let X be a vector space and X_γ a collection of locally convex spaces. Let T_γ be a collection of linear mappings of X into X_γ such that if x is in X and x is non-zero, then there exists some γ such that $T_\gamma(x) \neq 0$. Let V_γ be a base of convex and balanced neighborhoods in X_γ, the finite intersections of the sets $T_\gamma^{-1}(v_\gamma)$ where v_γ is in V_γ forms a base B of convex and balanced neighborhoods for X. This topology is the coarsest topology on X compatible with the algebraic structure under which all the T_γ are continuous. The locally convex space X with this topology is the projective limit of the convex spaces X_γ by the mappings T_γ.

One example of a projective limit is the weak topology on any weakly convex space X, obtained by taking for the collection of T_γ the set of all continuous linear functionals on X.

<u>Example 2.15</u> An example of a projective limit is the induced topology on a vector subspace M of a convex space X; it is the

coarsest topology making the identity mapping of M into X continuous. Let M be a reflexive space. Thus, M is isometric to a subspace of C(S) where S is a compact Hausdorff space, (47, page 241). The space C(S) has the D. P. property. But M is the projective limit space of C(S) under the identity map on M. The space M does not have the D. P. property. Therefore, the projective limit is not a permanence property of the D. P. property.

Abstract L- and M-Spaces

We shall now consider two classes of B-spaces, that will include most of the familar B-spaces found in an introductory course in functional analysis. Such B-spaces have some very nice properties. The first of these will be the abstract M-space. Such spaces were introduced and studied axiomatically by G. Birkhoff (6). Kakutani has given some representation theorems for both the abstract L- and M-spaces. The following definition of an abstract M-space is given by Kakutani (26).

Definition 2.16 A B-space X is called an abstract M-space if there is defined a relation $x \geq y$ (or equivalently, $y \leq x$) for some pairs of elements x, y in X and if it satisfies the following conditions for x, y, z, w in X and t a real scalar.

(a) $x \geq y$ and $y \geq x$ implies $x = y$.

(b) $x \geq y$ and $y \geq z$ implies $x \geq z$.

(c) $x \geq y$ and $t \geq 0$ implies $tx \geq ty$.

(d) $x \geq y$ implies $x + z \geq y + z$.

(e) To any pair of elements x, y in X, there exists a maximum $z = x \vee y$ such that $z \geq x$, $z \geq y$ and $w \geq z$ for any w with $w \geq x$ and $w \geq y$.

(f) To any pair of elements x, y in X, there exists a minimum $w = x \wedge y$ such that $w \leq x$, $w \leq y$ and $z \leq w$ for any z with $z \leq x$ and $z \leq y$.

(g) $x_n \geq y_n$, x_n converges to x and y_n converges to y implies $x \geq y$.

(h) $x \wedge y = 0$ implies $|x + y| = |x - y|$.

(i) $x \geq 0$ and $y \geq 0$ implies $|x \vee y| = \max(|x|, |y|)$.

The aforementioned conditions (a)-(i) mean that X is a linear lattice.

Definition 2.17 If there exists a non-negative element e such that $|e| = 1$ and for any x such that $|x| \leq 1$ implies $x \leq e$, then e is called a unit element.

Let us now consider some examples of abstract M-spaces. One must notice that the existence of a unit element is not assumed in an abstract M-space.

The foremost example of abstract M-spaces will be given by the space C(H) of all bounded continuous real-valued functions x(h) defined on a Hausdorff space H where the norm is given by

$$|x| = \sup \{ x(h) : h \text{ is in } H \}.$$

The order is defined as

$$x \geq y \quad \text{if and only if}$$

$$x(h) \geq y(h) \quad \text{for any } h \text{ in } H.$$

Also, if we take an arbitrary set A and consider A as a discrete topological space, then the space C(A) is nothing more than M(A) of all bounded real-valued functions x(h) defined on A (with the same norm and partial ordering as in the case of C(H)).

Consider the subspace $M(A, \mu)$ of $M(A)$ consisting of all bounded measurable real-valued functions x(a) defined on A (where measurability is with respect to a measure μ defined on A); this is also an example of an abstract M-space. If sets in A of measure zero are neglected, then the space $N(A, \mu)$ is again an example of an abstract M-space. The norm for $N(A, \mu)$ is

$$|x| = \text{ess. sup} \left\{ x(a) : a \text{ is in } A \right\}.$$

The order is given by

$$x \geq y \qquad \text{if and only if}$$

$$x(a) \geq y(a) \qquad \text{almost everywhere on } A.$$

We must remember that two functions which differ from each other only on a set of measure zero are considered to be the same element of $N(A, \mu)$.

Among the examples given above we find they all have a unit, namely the constant function one. The following example will afford us with an abstract M-space that does not have a unit element. Consider a special subspace of C(H), where H is a Hausdorff space. If we consider only those functions x(t) of C(H) that vanish at a given point h_o in H, then the space $C(H, h_o)$ of all such functions with the norm and partial ordering the same as C(H) will be an abstract M-space. Consider the space C(H, 0) where H is the closed interval between 0 and 1. This space does not have a unit element.

Since C(S), where S is a compact Hausdorff space, has the D. P.

property, we see (viewing Theorem 2.7) that it is important to be able to recognize spaces that are isomorphic to C(S). Kakutani (26) proved the following theorem.

<u>Theorem 2.18</u> For any abstract M-space X with a unit, there exists a compact Hausdorff space S such that X is isometrically isomorphic to the real B-space C(S).

From this theorem we notice that any abstract M-space will possess the D. P. property. For example, the abstract M-space, ℓ_∞ , has a unit and the appropriate S of Theorem 2.18 is the Stone-Cech compactification of the positive integers.

Much of the motivation for the study of abstract L-spaces was derived from the applications of the theory of Hermitian operators on a Hilbert space. Kakutani (27) gives the following definition for an abstract L-space.

<u>Definition 2.19</u> A B-space X is an abstract L-space if there is defined a relation $x \geq y$ (or equivalently $y \leq x$) for some pairs of elements x, y in X and if it satisfies the following conditions for x, y, z, w in X and t a real scalar.

(a) $x \geq y$ and $y \geq x$ implies $x = y$.

(b) $x \geq y$ and $y \geq z$ implies $x \geq z$.

(c) $x \geq y$ and $t \geq 0$ implies $tx \geq ty$.

(d) $x \geq y$ implies $x + z \geq y + z$.

(e) To any pair of elements x, y in X, there exists a maximum $z = x \vee y$ such that $z \geq x$, $z \geq y$ and $w \geq z$ for any w with $w \geq x$ and $w \geq y$.

(f) To any pair of elements x and y, there exists a minimum $w = x \wedge y$ such that $w \leq x$, $w \leq y$ and $z \leq w$ for any z with $z \leq x$ and $z \leq y$.

(g) $x_n \geq y_n$, x_n converges to x and y_n converges to y implies $x \geq y$.

(h) $x \geq 0$ and $y \geq 0$ implies $|x + y| = |x| + |y|$.

(i) $x \wedge y = 0$ implies $|x + y| = |x - y|$.

Such B-spaces were introduced axiomatically by G. Birkhoff (6) as abstractions from the concrete B-spaces of Lebesque integrable functions on a measure space.

Definition 2.20 An abstract L-space has a unit if there exists an element e for which $x > 0$ implies $x \wedge e > 0$.

Since many of the familiar B-spaces that are studied in a course in functional analysis are abstract L-spaces, it might be profitable to consider some examples of these spaces.

For an example of an abstract L-space, consider a set H where a completely additive measure is defined. The totality of all real-valued measurable functions x(h) that are absolutely integrable on H constitutes a B-space L(H) with

$$|x| = \int_H |x(h)| \, dh$$

as its norm. The order is given by

$$x \geq y \quad \text{if and only if}$$

$$x(h) \geq y(h) \quad \text{almost everywhere on H.}$$

Two elements x, y in $L(H)$ are equal if and only if

$$x(h) = y(h) \qquad \text{almost everywhere on } H.$$

Also, the constant function $x(h) = 1$ serves as a unit. Hence, $L(H)$ is an abstract L-space with unit.

The following B-spaces given by Dunford and Schwartz (11, IV) are also examples of abstract L-spaces: $rca(S, \Sigma)$, $ba(S, \Sigma)$, $NBV(I)$, and $BV(I)$. We shall give the order for the first space, $rca(S, \Sigma)$. Let

$$|\mu| = \text{total variation of } \mu(E) =$$

$$\sup\{\mu(E) : E \text{ is a subset of } S\} - \inf\{\mu(E) : E \text{ is a subset of } S\}$$

for all Borel sets E of S. The partial ordering is given by

$$\mu \geq \nu \qquad \text{if and only if}$$

$$\mu(E) \geq \nu(E) \qquad \text{for any Borel set } E \text{ a subset of } S.$$

Thus, $rca(S, \Sigma)$ becomes an abstract L-space.

As seen in our first example, $L(H)$ is an abstract L-space. Kakutani (27) addressed his paper to the converse problem, i.e. is it possible to represent any abstract L-space by a concrete abstract L-space of the form $L(H)$. He was able to give a positive answer to this question. We shall only state his results here.

<u>Theorem 2.21</u> Given an abstract L-space with a unit, there exists a totally disconnected compact topological space S and a countably additive measure μ defined on the Borel field Σ of S such that the abstract L-space is isometrically isomorphic to the real B-space

$L_1(S, \Sigma, \mu)$.

The following theorem, also due to Kakutani, shows a relationship between the abstract L-spaces and the abstract M-spaces.

Theorem 2.22 The conjugate space of an abstract M-space is an abstract L-space. The conjugate of an abstract L-space is an abstract M-space with unit.

We are now in a position to give a partial result to the open question that is the converse of Theorem 2.3. This result will be very useful in compiling the tables at the end of this chapter.

Theorem 2.23 If X is an abstract L-space, then X' has the D. P. property.

Proof. Let X be an abstract L-space; thus, X' is an abstract M-space with unit by Theorem 2.22. Using Theorem 2.18 we have X is isometrically isomorphic to C(S) for some compact Hausdorff space, S. It then follows from Theorems 2.2 and 2.7 that X' has the D. P. property.

Some Generalizations of C(S) and $L_1(\mu)$

There are several ways of generalizing the space C(S) where S is a compact Hausdorff space. One such method is seen by letting S be a locally compact Hausdorff space, and let $C_o(S)$ be the B-space of continuous scalar-valued functions f on S that tend to zero at infinity (in the sense that the set

$$\{ s \in S : |f(s)| > \epsilon \}$$

is relatively compact in S for each $\epsilon > 0$). The norm on $C_o(S)$ is given by

$$|f| = \text{Sup}\left\{ |f(s)| : s \in S \right\}.$$

One will notice that for a compact Hausdorff space S that $C_o(S)$ and $C(S)$ agree. The probability density functions of statistics belong to $C_o(R)$ where R is the real numbers. The question now arises does $C_o(S)$, for S a locally compact Hausdorff space, have the D. P. property. Edwards (12, page 637) has shown that this space has the D. P. property.

A generalization of $C(S)$ where S is a compact Hausdorff space will now be given. Let X be a B-space and S a compact Hausdorff space. The space $C(S, X)$ is the set of all X valued continuous functions on S. Thus, $C(S, X)$ with the sup-norm is a B-space. This space has been studied by Swartz (45), Batt and Berg (4), and Pelczynski (36).

Theorem 2.24 Let S be a compact Hausdorff space and X a B-space. If $C(S, X)$ has the D. P. property, then X has the D. P. property.

Proof. Let Y be a B-space and T a weakly compact operator from X to Y. Fix a s in S and pick a f in $C(S)$ such that $f(s) = 1$ and $|f| = 1$. Define the map U from $C(S, X)$ to X by

$$Ug = g(s) \qquad \text{for all } g \text{ in } C(S, X).$$

It follows from the Uniform Boundedness Principle that U is a continuous operator. Thus, TU is a weakly compact operator (11, VI.4.5). The space $C(S, X)$ having the D. P. property implies that the operator TU is a completely continuous operator. Let V_x be a map from the reals R into X defined by $V_x(a) = ax$. Thus, $V_x f$ is in $C(S, X)$. If (x_n)

is a weak Cauchy sequence, then $(V_{x_n} f)$ is a weak Cauchy sequence. This follows since

$$\left| V_{x_n} f - V_{x_m} f \right| = |f| \left| V_{x_n} - V_{x_m} \right| =$$

$$\sup \{ |ax_n - ax_m| : a \text{ is in the unit disk of } R \} \leq |x_n - x_m| .$$

Hence, if $x \in X$, then $Tx = TU(V_x f)$ and T is completely continuous. Therefore, X has the D. P. property.

The converse of the above theorem is an important open question. Grothendieck (17) established that the space $C(S, X)$ is isomorphic to the weak tensor product of $C(S)$ and X. Thus, if one could solve this open problem, it would help in solving the open problem involving tensor products of spaces that have the D. P. property.

Dobrakov (10) and Alexander and Swartz (1) have considered this problem. Their attention has been directed at the following space. Let S be a locally compact Hausdorff space and X a B-space. Let $C_o(S, X)$ denote the B-space of all X valued continuous functions on S tending to zero at infinity with the usual sup-norm.

Before considering Dobrakov's partial result on this open problem, there is a need to develop some notation. A non-void class H of subsets of S is called a semi-tribe on δ-ring if H is closed under set difference, finite unions, and countable intersections. A non-void class H of subsets of S is called a tribe or σ-ring if H is closed under set difference and countable unions. Let X and Y be B-spaces and $L(X, Y)$ denote the B-space of all bounded linear operators from X to Y. Let B_o be the semi-tribe generated by the compact subsets of S that are G_δ sets and define m from B_o to $L(X, Y)$, which is an operator valued measure countably additive in the norm topology in $L(X, Y)$, i.e. for every x in X $m(\cdot)x$ is a countably additive vector measure. Denote by $G(B_o)$ the σ-ring or tribe generated by B_o . A B_o-simple function on S with values in X is a function of the form

$$f = \sum_{i=1}^{k} x_i C_{E_i} \quad \text{for } x_i \text{ in } X,\ E_i \text{ in } B_o \text{ and } E_i \cap E_j = \phi \text{ for } i \neq j.$$

Here, C_E denotes the characteristic function of the set E in S. The integral of a B_o-simple function on an E in $G(B_o)$ is defined as

$$\int_E f\, dm = \sum_{i=1}^{k} x_i\, m\,(\,E \cap E_i\,)\,.$$

Denote by F the set of all B_o-simple functions on S with values in X. For a function f from S to X and a set A a subset of S, define

$$|f|_A = \sup \left\{ |f(x)| \, : \, x \text{ in } A \right\}.$$

Define on $G(B_o)$ the non-negative set function $\hat{m}$, called the semi-variation of the measure m, by the equality

$$\hat{m}(E) = \sup \left\{ \left| \int_E f\, dm \right| \, : \, f \text{ in } F,\ |f|_E \leq 1 \right\} \quad \text{for } E \text{ in } G(B_o).$$

The function $\bar{m}$ is a monotone and countably subadditive set function on $G(B_o)$ (8, page 53). For every function f in F and every set E in $G(B_o)$ we have

$$\left| \int_E f\, dm \right| \leq |f|_E\, \hat{m}(E) \qquad (8, \text{ page } 109).$$

Denote by $\bar{F}$ the closure of F in the norm $|\cdot|_S$ in the B-space of all bounded X valued functions on S and $\hat{m}(S) = \sup \{\hat{m}(E) : E \text{ in } B_o\}$ is finite. Thus, the integral defined on F can be extended to $\bar{F}$ as follows. For f in $\bar{F}$ and for a set E in $G(B_o)$ define

$$\int_E f\,dm = \lim \int_E f_n\,dm \qquad \text{where } f_n \text{ is in } F \text{ and } |f_n - f|_S \to 0.$$

Also, $C_o(S, X)$ is a subset of $\bar{F}$. A wide class of bounded linear operators T from $C_o(S, X)$ into Y can be represented in the form

$$T\,f = \int_S f\,dm$$

for a measure m (9).

Dobrakov (10) has proved the basic theorems on representation of bounded linear operators on $C_o(S, X)$ in the form of an integral with respect to a Baire operator valued measure. It is also pointed out that the dual space of $C_o(S, X)$ is isometrically isomorphic to the space $\text{cabv}(G(B_o), X')$ of countably additive X' valued vector measures with bounded variations.

With these preliminaries we now give Dobrakov's partial result to the difficult problem, which is as follows: If X has the D. P. property, does the space $C_o(S, X)$ also have the D. P. property?

<u>Theorem 2.25</u> a) If S is a discrete topological space and X has the D. P. property, then $C_o(S, X)$ has the D. P. property.

b) If weak and norm convergence of sequences coincide in X, then for any locally compact Hausdorff topological space S, $C_o(S, X)$ has the D. P. property.

<u>Proof.</u> Let (f_n) be a weak Cauchy sequence; thus, there exists a M such that

$$|f_n|_S \leq M \qquad \text{for all } n.$$

Let T be a weakly compact operator from $C_o(S, X)$ into an arbitrary

B-space Y. The operator T can be represented in the form

$$Tf = \int_S f \ dm$$

where m is a Baire operator valued measure on $G(B_o)$ with $\hat{m}(S) = |T|$ whose values are weakly compact operators from $L(X, Y)$ and its semi-variation $\hat{m}$ is continuous on $G(B_o)$ (10). Using this representation we extend the operator T from $C_o(S, X)$ to $\bar{F}$ without increasing its norm. Let $\epsilon > 0$, define for each n,

$$A_n = \{ s \epsilon S : |f_n(s)| \geq \frac{\epsilon}{6(1 + |T|)} \} .$$

Thus, each A_n is a compact subset of S and each A_n is also a G_δ set (20, page 221). Since $G(B_o)$ is a σ-ring it follows that

$$A = \bigcup_1^\infty A_n \qquad \text{belongs to } G(B_o) .$$

Let C_B denote the characteristic function on B. Thus,

$$\left| \int_S f_n C_{S-A} \, dm \right| \leq \left| f_n C_{S-A} \right|_{S-A} \hat{m}(S) \leq \left| f_n \right|_{S-A} \left| T \right|$$

$$|f_n|_{S-A} |T| \leq \frac{\epsilon |T|}{6(1 + |T|)} \leq \frac{\epsilon}{6} \qquad \text{for every n.}$$

a) Assume that S is a discrete topological space and X has the D. P. property. Hence, A must be a countable set, $\{a_1, a_2, \ldots\}$ in S. The semivariation being continuous on $G(B_o)$ implies that for any decreasing sequence of sets E_n such that each E_n belongs to $G(B_o)$ and $\cap E_n = \phi$, $\lim \hat{m}(E_n) = 0$. Consider $B_n = \{a_{n+1}, a_{n+2}, \ldots\}$, B_n belongs to $G(B_o)$ and $\cap B_n = \phi$; hence, there exists a K such that for

$$B_K = \{ a_{K+1}, a_{K+2}, \ldots \}$$

and for every n, then

$$\left| \int_{B_K} f_n \, dm \right| \leq \frac{\epsilon}{6} .$$

For every $i = 1, 2, \ldots, K$ the sequence $(f_n(a_i))$ is weak Cauchy (10). Since $m(\{a_i\})$ is a weakly compact operator on $L(X, Y)$ and X has the D. P. property, there exists a N such that for any n, p greater than N we have

$$\left| \int_{A-B_K} (f_n - f_p) \, dm \right| < \frac{\epsilon}{6} .$$

Therefore for $\epsilon > 0$ choose $Q = \max (N, K)$ and we have

$$\left| Uf_n - Uf_p \right| = \left| \int_S (f_n - f_p) \, dm \right|$$

$$\left| \int_S (f_n - f_p) dm \right| \leq \left| \int_{A-B_K} (f_n - f_p) dm \right| + \left| \int_{B_k} (f_n - f_p) dm \right| + \left| \int_{S-A} (f_n - f_p) dm \right|$$

$$\left| \int_{A-B_K} (f_n - f_p) dm \right| + \left| \int_{B_K} (f_n - f_p) dm \right| + \left| \int_{S-A} (f_n - f_p) dm \right| \leq \frac{\epsilon}{2} .$$

Hence, U transforms a weak Cauchy sequence into a norm Cauchy sequence and $C_o(S, X)$ has the D. P. property.

b) Assume weak and norm convergence of sequences coincide in X. Thus, $|m(E)| \leq \hat{m}(E)$ for every set E in $G(B_o)$ (8, page 52). There exists a finite non-negative countably additive measure λ on $G(B_o)$ with the properties:

$$\lambda(E) \leq |m(E)| \qquad \text{and}$$

$$\lim_{\lambda(E) \to 0} |m(E)| = 0 \qquad \text{for } E \text{ in } G(B_o)$$

by (11, IV.10.5). If $\lambda(N) = 0$ for N in $G(B_o)$, then $|m(N)| = 0$ and, therefore, $\hat{m}(N) = 0$. Suppose

$$\lim_{\lambda(E)\to 0} \hat{m}(E) \neq 0 \qquad \text{for } E \text{ in } G(B_o).$$

Then, there exists an $\epsilon > 0$ and a sequence of sets A_k in $G(B_o)$, $k = 1, 2, \ldots$, with $\lambda(A_k) < \frac{1}{2^k}$ and $\hat{m}(A_k) > \epsilon$. Put

$$B_k = \bigcup_{i=k}^{\infty} A_i \qquad \text{and} \qquad B = \bigcap_{k=1}^{\infty} B_k .$$

Since λ is a finite non-negative countably additive measure on $G(B_o)$, $\lambda(B) = 0$. While $\hat{m}(B) \geq \hat{m}(B_k) - \hat{m}(B_k - B) > \epsilon$ for sufficiently large k is implied by the monotonicity and continuity of $\hat{m}$ on $G(B_o)$, which is a contradiction. Therefore, λ has the desired properties. Choose a $\delta > 0$ such that $\lambda(E) < \delta$ for E in $G(B_o)$, which implies $\hat{m}(E) < \epsilon/6M$. Since the sequence (f_n) is weak Cauchy in $C_o(S, X)$, for each x' in X' and each point s in S there is a finite limit

$$\lim_{n\to\infty} x' f_n(s)$$

(10). Since the weak and the strong convergences of sequences coincide in X, for each point s in S there exists a limit $\lim f_n(s) = f(s)$ in X. Be Egoroff's Theorem for the measure λ there is a set F in $G(B_o)$ with $\lambda(F) < \delta$ such that on $A-F$ the sequence (f_n) converges uniformly to the function f. Choose q such that for $n, p \geq q$ it is

$$\left| f_n - f_p \right|_{A-F} \leq \frac{\epsilon}{6(1 + \hat{m}(S))} ,$$

Then from the inequality

$$\left|Tf_n - Tf_p\right| \leq \left|\int_{A-F}(f_n - f_p)dm\right| + \left|\int_F (f_n - f_p)dm\right| + \frac{\epsilon}{3} \quad \text{for } n, p=1,2,\ldots$$

it follows that for $n, p \geq q$ then $|Tf_n - Tf_p| \leq \epsilon$. Therefore, $C_o(S, X)$ has the D. P. property for any locally compact Hausdorff space.

The affirmative answer to the open problem can be given for spaces other than those used in the previous theorem. If X is isometrically isomorphic to some $C_o(T)$ where T is locally compact, then $C_o(S, C_o(T))$ is isomorphic to $C_o(S \times T)$ (17). Since $S \times T$ is locally compact, as pointed out earlier $C_o(S \times T)$ will possess the D. P. property and so will $C_o(S, X)$.

Alexander and Swartz (1) have also obtained partial results on this open problem. Their method does not involve the use of integration theory. Let S be the one-point compactification of the positive integers. The space $C(S, X)$ will denote the B-space of all X valued convergent sequences with the sup norm,

$$\| (x_i) \| = \sup \left\{ |x_i| : i = 1, 2, \ldots \right\} .$$

Before giving their partial result it will be necessary to develop some notation.

The first thing to be done is to give a characterization of completely continuous operators on $C(S, X)$. Foias and Singer (13) have proved that a bounded linear operator T on $C(S, X)$ into a B-space Y has a unique representation. Using duality notation, this representation is as follows:

$$\langle y', Tx \rangle = \langle y', T_o(\lim x_n) \rangle + \Sigma \langle y', T_n x_n \rangle$$

where $x = (x_n)$ is in $C(S, X)$, y' in Y', T_o maps X into Y'', and T_n

maps X into Y are bounded linear operators and the series $\Sigma\langle y', T_n x_n\rangle$ having the property $\Sigma\ |y'T_n|$ is finite.

Theorem 2.26 Let X and Y be B-spaces. The linear subspace of all completely continuous operators in $L(X, Y)$ is closed in the norm topology of $L(X, Y)$.

Proof. Let T_n be a sequence of completely continuous operators from X to Y such that T_n converges to T where T belongs to $L(X, Y)$. Let (x_n) be a weak Cauchy sequence in X; thus, there exists a K such that $|x_n| \leq K$ for all n. Let $\epsilon > 0$ be given. Since T_n converges to T, this implies there exists a N such that

$$\left| T_N - T \right| < \frac{\epsilon}{3K} .$$

Each T_n being completely continuous implies that the sequence $(T_N x_n)$ is norm convergent and there exists an M such that for all $n, m \geq M$ implies

$$\left| T_N(x_n - x_m) \right| < \frac{\epsilon}{3} .$$

Thus for $n, m \geq M$

$$\left|Tx_n - Tx_m\right| \leq \left|(T - T_N)x_n\right| + \left|(T_N - T)x_m\right| + \left|T_N(x_n - x_m)\right|$$

$$\left|(T-T_N)x_n\right| + \left|(T_N-T)x_m\right| + \left|T_N(x_n-x_m)\right| \leq \left|T-T_N\right|\left|x_n\right| + \left|T_N-T\right|\ \left|x_m\right| + \frac{\epsilon}{3}$$

$$\left|T - T_N\right|\ \left|x_n\right| + \left|T_N - T\right|\ \left|x_m\right| + \frac{\epsilon}{3} < \epsilon .$$

Therefore, (Tx_n) is a norm convergent sequence and T is completely continuous.

Now a characterization of completely continuous operators can be given using Foias and Singer's representation theorem.

Theorem 2.27 Let S be the one-point compactification of the positive integers and Y a B-space. A bounded linear operator T from $C(S, X)$ into Y is completely continuous if and only if using the Foias and Singer representation for T

a) each T_j is completely continuous for $j = 0, 1, 2, \ldots$

b) the series ΣT_j is such that $\Sigma T_j x_j$ converges for each sequence (x_j) in X, $|x_j| \leq 1$.

Proof. Assume T is a completely continuous operator. For $j \geq 1$ denote P_j the bounded linear operator from X into $C(S, X)$ defined by

$$P_j(x) = (\delta_{nj}\, x)_{n=1}^{\infty}$$

where δ_{nj} is the Kronecker delta function. Let (x_j) be a sequence in X such that $|x_j| \leq 1$ and define

$$z_n = \sum_{j=1}^{n} P_j\, x_j \quad ,$$

which belongs to $C(S, X)$. The dual of $C(S, X)$ can be identified with the space $\ell(X')$ of all absolutely summable X' valued sequences so that if $x' = (x'_n)$ is in $C(S, X)'$, then

$$\langle x', z_n \rangle = \sum_{j=1}^{n} \langle x'_j, x_j \rangle$$

converges to $\Sigma \langle x'_j, x_j \rangle$ (13) and (10). Hence, (z_n) is a weak Cauchy sequence in $C(S, X)$ and, thus, by hypothesis

$$(T\, z_n) = (\sum_{j=1}^{n} T_j\, x_j)$$

is norm convergent where x_j is identified with the sequence of all zeros except in the j-th position. Therefore (b) is satisfied.

For $j \geq 1$ and $x \in X$ we have $TP_j x = T_j x$, which is completely continuous since T is completely continuous. In order to show that T_o is completely continuous, consider the continuous linear operator Q from X into $C(S, X)$ defined by $Qx = (x, x, x, \ldots)$. From the Foias and Singer representation we have

$$\widehat{Tx}(y') = T_o(\lim x_n)(y') + \Sigma \widehat{T_n x_n}(y') .$$

From condition (b) we have $T_o(\lim x_n) = \widehat{Tx} - \hat{y}$ where $\hat{y} = \Sigma \widehat{T_n x_n}$. Therefore, $T_o(\lim x_n)$ is in Y and T_o belongs to $L(X, Y)$. In fact for $x \in X$,

$$T_o(x) = TQx - \Sigma T_n x .$$

The series $\Sigma T_j x$ converges uniformly for all $|x| \leq 1$ (3). That is to say that T_o is the limit in the norm topology on $L(X, Y)$ of the sequence of completely continuous operators

$$(TQ - \sum_{j=1}^{n} T_j) .$$

Thus, T_o is completely continuous since the completely continuous operators are closed in the norm topology of $L(X, Y)$.

Assume conditions (a) and (b) are true. For each n, let X_n be a map from $C(S, X)$ into Y defined by

$$S_n x = T_o(\lim x_n) + \sum_{j=1}^{n} T_j x_j \qquad \text{where } x = (x_n) .$$

From condition (b) and the Foias and Singer representation it follows that $T_o(\lim x_n)$ belongs to Y. Condition (a) implies that each S_n is a completely continuous operator. The series $\Sigma T_j x_j$ converges uniformly in Y for all $|x_j| \leq 1$ and S_n converges to T in the norm topology of $L(C(S, X), Y)$ by condition (b) and (3). Now since the completely continuous operators are closed in the norm topology of $L(C(S, X), Y)$, it follows that T is a completely continuous operator.

The following theorem is another partial result to the open question posed earlier. This proof does not involve the use of integration theory.

<u>Theorem 2.28</u> Let S be the one-point compactification of the positive integers. If X has the D. P. property, then $C(S, X)$ has the D. P. property.

<u>Proof.</u> Let T be a weakly compact operator from $C(S, X)$ into Y. Let T be written in the Foias and Singer representation. Each T_j is weakly compact for $j = 0, 1, 2, \ldots$ and the series $\Sigma\, T_j$ is such that $\Sigma\, T_j x_j$ converges for each sequence (x_j) in X, $|x_j| \leq 1$ (3). From the hypothesis it follows that each T_j is completely continuous for $j = 0, 1, 2, \ldots$. Theorem 2.27 implies that T is a completely continuous operator. Therefore, $C(S, X)$ has the D. P. property.

We shall now consider generalizations of the $L_p(\mu)$, $1 \leq p \leq \infty$, spaces. Lindenstrass and Pelczynski (33) introduced a B-space, $\mathfrak{L}(p)$ whose finite-dimensional subspaces are close to the finite-dimensional subspaces of $L_p(\mu)$. In order to make this more precise we shall need the following definitions.

<u>Definition 2.29</u> Let X and Y be B-spaces. Let $L(X, Y)$ be the B-space of all operators from X into Y with the usual operator norm. The

distance $d(X, Y)$ between the B-spaces X and Y is defined as

$$d(X, Y) = \inf \left\{ |T|\ |T^{-1}| : T \text{ in } L(X, Y) \right\} .$$

If no such T exists, i.e. X and Y are not isomorphic, $d(X, Y)$ is taken to be ∞ .

It should be mentioned that d is not a metric. Also, two B-spaces X and Y are "near" if $d(X, Y)$ is close to 1.

Definition 2.30 A B-space X is a $\mathcal{P}(p, \lambda)$-space $1 \leq p \leq \infty$, $1 \leq \lambda < \infty$, provided that for every finite-dimensional subspace B of X there is a finite-dimensional subspace E of X containing B such that $d(E, \ell_p^n) \leq \lambda$ where $n = \dim E$. A B-space X is a $\mathcal{P}(p)$-space, $1 \leq p \leq \infty$, if there exists a $\lambda \geq 1$ such that X is a $\mathcal{P}(p, \lambda)$-space.

The $\mathcal{P}(p)$-space is a generalization of a $L_p(\mu)$-space. Let X be a $L_p(\mu)$-space for $1 \leq p < \infty$. Let $\{X_i\}$ be a decomposition of the measure space X into n disjoint measurable sets of finite measure. Let B_n be the linear span of the characteristic functions on this decomposition. Since the simple functions are dense in X,

$$X = \overline{\bigcup_{n=1}^{\infty} B_n} .$$

Since X is separable, it follows that X is a $\mathcal{P}(p)$-space for any λ greater than 1 (34).

Let X be an abstract M-space. From Theorem 2.21 the dual X' of X is isometric to an abstract L-space; thus, X' is isometric to $L_1(\mu)$ for some measure μ. The space X is a $\mathcal{P}(\infty, \lambda)$-space for any λ greater than 1 (34). Therefore, X is a $\mathcal{P}(\infty)$-space. In particular, C(S) where S is a compact Hausdorff space, is a $\mathcal{P}(\infty)$-space.

Among the properties that the $\mathcal{L}(p)$-spaces possess, one is that they are reflexive for $1 < p < \infty$. Also, the conjugate of a $\mathcal{L}(1)$-space is a $\mathcal{L}(\infty)$-space and vice-versa. It will now be shown that any $\mathcal{L}(1)$-space has the D. P. property.

<u>Theorem 2.31</u> If X is a $\mathcal{L}(1)$-space, then X has the D. P. property.

<u>Proof.</u> There exists a $L_1(\mu)$-space Z and operators T from X to Z and P from Z to X'' such that PT is the canonical embedding of X' in X'' (33). Let H be the canonical embedding of X' into X'''. Consider the operators

$$\begin{array}{ccccccc} X' & \xrightarrow{H} & X''' & \xrightarrow{P'} & Z' & \xrightarrow{T'} & X' \\ & & \downarrow{\scriptstyle J'} & & & & \\ & & X' & & & & \end{array}$$

Thus, J'H is the identity operator on X' and $(PT)'H = T'P'H = J'H$. Thus, $Z' = \text{image}\ (P') \oplus \ker\ (T')$ and $T'P' = I$ on X', which implies that T' is onto and P' is one-to-one. Hence, $P'H(X')$ is a complemented subspace of Z' and X' is isomorphic to $P'H(X')$. From Theorems 2.23 and 2.5 it follows that X' has the D. P. property. Therefore, X has the D. P. property by Theorem 2.3.

From the fact that the conjugate of any $\mathcal{L}(\infty)$-space is a $\mathcal{L}(1)$-space and above theorem, we see that any $\mathcal{L}(\infty)$-space will also possess the D. P. property.

The following theorem will give a sufficient condition in terms of the $\mathcal{L}(1)$- and $\mathcal{L}(\infty)$-spaces to ensure a space does not have the D. P. property.

<u>Theorem 2.32</u> If X is a B-space that is isomorphic to a subspace

of a $\mathcal{P}(1)$-space and also isomorphic to a quotient space of a $\mathcal{P}(\infty)$-space, then X does not possess the D. P. property.

Proof. Let Y be a $\mathcal{P}(1)$-space such that X is isomorphic to a subspace of Y. Since X is isomorphic to a quotient space of a $\mathcal{P}(\infty)$-space, there exists an operator from a $\mathcal{P}(\infty)$-space Z onto X. Consider the operator T as an operator from Z into Y. Hence, there exists an operator U from a Hilbert space H onto X (33). Let W be the orthogonal complement of the kernel of U. Thus, X is isomorphic to H/W. Therefore, X is a reflexive space and by Theorem 2.1 X does not possess the D. P. property.

It will be necessary at this point to define the tensor product of two B-spaces. A more complete study of this subject can be found in Schaefer (42), Day 7, or Robertson and Robertson (40). The first definition will be that of the tensor product of two linear spaces.

Definition 2.33 Let X and Y be linear spaces over the same field. Let $B(X, Y)$ be the linear space of all bilinear maps on $X \times Y$. For each pair (x, y) in $X \times Y$, the mapping $U_{xy}(f) = f(x, y)$ is a linear map on $B(X, Y)$ and hence an element of the algebraic dual $B(X, Y)'$. The mapping $\varphi(x, y) = U_{xy}$ is a bilinear map from $X \times Y$ into $B(X, Y)'$. The linear hull of $\varphi(X \times Y)$ in $B(X, Y)'$ is the tensor product of X and Y which is denoted by $X \otimes Y$.

It is a common practice to denote the element $U_{x,y}$ in $X \otimes Y$ by $x \otimes y$, thus each element $x \otimes y$ is a finite sum $\Sigma \lambda_i(x_i \otimes y_i)$.

If X and Y are locally convex spaces, then there are numerous ways to induce a topology in $X \otimes Y$ relative to the given topologies in original spaces and to the tensor product structure. A good discussion on this can be found in (7), (43) or (42).

Definition 2.34 Let X and Y be B-spaces. Let $X \hat{\otimes} Y$ be the completion of the algebraic tensor product of X and Y with the norm

$$\left| \sum_{i=1}^{n} x_i \otimes y_i \right| = \inf \left\{ \sum_{k=1}^{m} |x_k| \, |y_k| \right\}$$

where the inf is taken over the set of all expressions

$$\sum_{k=1}^{m} x_k \otimes y_k \quad \text{equivalent to} \quad \sum_{i=1}^{n} x_i \otimes y_i \, .$$

The B-space $X \hat{\otimes} Y$ is the projective tensor product of X and Y.

Let $X \hat{\hat{\otimes}} Y$ be the completion of the algebraic tensor product of X and Y with the norm

$$\left| \sum_{i=1}^{n} x_i \otimes y_i \right| = \sup \left\{ \sum_{i=1}^{n} f(x_i) g(y_i) : f \in X', \, g \in Y', \, |f| \leq 1, \, |g| \leq 1 \right\} .$$

The B-space $X \hat{\hat{\otimes}} Y$ is the weak tensor product of X and Y.

There is a relationship between the space C(S, X) and the weak tensor product of C(S) and X. Grothendieck (17) showed that these spaces were isomorphic.

Grothendieck's (18) paper, which considered a closs of (L)-spaces and (C)-spaces, takes on more interest when it is noted that Stegall and Retherford (44) have proved the (L)-spaces are equivalent to the $\mathfrak{L}(1)$-spaces. This characterization will now be given for completeness, but the proof will be omitted.

Theorem 2.35 A B-space is a $\mathfrak{L}(1)$-space if and only if for every space F and closed subspace G of F the natural injection of $G \hat{\otimes} X$ into $F \hat{\otimes} X$ is an isomorphism.

With the above characterization we can now give a class of B-spaces that will have the D. P. property. The projective tensor product of any two of them will also possess the D. P. property.

<u>Theorem 2.36</u> If X and Y are $\mathscr{P}(1)$-spaces, then the projective tensor product of X and Y has the D. P. property.

<u>Proof.</u> Let E be an arbitrary B-space and F a closed subspace of E. Since X is a $\mathscr{P}(1)$-space, the natural injection of $F \hat{\otimes} X$ into $E \hat{\otimes} X$ is an isomorphism. Thus $F \hat{\otimes} X$ is a closed subspace of $E \hat{\otimes} X$ (44). Since Y is a $\mathscr{P}(1)$-space we have that the injection $(F \hat{\otimes} X) \hat{\otimes} Y$ into $(E \hat{\otimes} X) \hat{\otimes} Y$ is an isomorphism. Since the injection of $(F \hat{\otimes} X) \hat{\otimes} Y$ into $(E \hat{\otimes} X) \hat{\otimes} Y$ is the same as the injection of $F \hat{\otimes} (X \hat{\otimes} Y)$ into $E \hat{\otimes} (X \hat{\otimes} Y)$, $X \hat{\otimes} Y$ is a $\mathscr{P}(1)$-space. Therefore, $X \hat{\otimes} Y$ has the D. P. property.

Any $\mathscr{P}(\infty)$-space will possess the D. P. property since its conjugate is a $\mathscr{P}(1)$-space. The weak tensor product of two $\mathscr{P}(\infty)$-spaces is again a $\mathscr{P}(\infty)$-space (21). Thus, we have the following theorem.

<u>Theorem 2.37</u> If X and Y are $\mathscr{P}(\infty)$-spaces, then the weak tensor product of X and Y has the D. P. property.

Using the fact that C(S, X) is isomorphic to the weak tensor product of C(S) and X we have the following.

<u>Theorem 2.38</u> If X is a $\mathscr{P}(\infty)$-space, then, for any compact Hausdorff space S, C(S, X) has the D. P. property.

To conclude this chapter, a table of familar B-spaces will be given and whether these spaces possess the D. P. property. In the construction of the following two tables the references will justify the conclusions on whether the space has the D. P. property. All spaces, except the last two in Table I and the last three in Table II are discussed in (11, IV). It should be mentioned that any finite dimensional B-space will possess the D. P. property, but these will not be included in the tables.

TABLE I

SOME BANACH SPACES WITH THE DUNFORD-PETTIS PROPERTY

Space	Reference
ℓ_1	Theorem 2.23, 2.3
ℓ_∞	Theorem 2.23, 2.3
c	Theorem 2.3, $c' = \ell_1$
c_o	Theorem 2.3, $c_o' = \ell_1$
bv	IV.13.11, Theorem 2.10, 2.7
bv_o	IV.13.11, Theorem 2.7
bs	IV.13.13, Theorem 2.7
cs	IV.13.12, Theorem 2.3
$B(S, \Sigma)$	IV.6.18, Theorem 2.2
B(S)	IV.6.18, Theorem 2.2
C(S)	Theorem 2.2
$ba(S, \Sigma)$	Theorem 2.23
$rca(s, \Sigma)$	Theorem 2.23
$L_1(S, \Sigma, \mu)$	Theorem 2.23, 2.3
$L_\infty(S, \Sigma, \mu)$	V.8.11, Theorem 2.7
BV(I)	Theorem 2.23, 2.3
NBV(I)	Theorem 2.23, 2.3
AC(I)	IV.12.3
AP	IV.7.6, IV.6.18, Theorem 2.2
$C^m(\Omega)$	(12, page 640)
$C_o(S)$	(12, page 637)

TABLE II

SOME BANACH SPACES THAT DO NOT POSSESS THE DUNFORD-PETTIS PROPERTY

Space	Reference
$\ell_p \quad (1 < p < \infty)$	Theorem 2.1
$L_p(S, \Sigma, \mu)\ (1 < p < \infty)$	Theorem 2.1
Hilbert space	Theorem 2.1
E (defined on page 10)	Theorem 3.29
E'	Theorem 3.29
$E\ '$	Theorem 2.11

CHAPTER III

APPLICATIONS OF THE DUNFORD-PETTIS PROPERTY AND SOME SIMILAR PROPERTIES

The purpose of Chapter III will be to give a brief introduction to the Dieudonné, V property and the weak Cauchy V property for B-spaces. The relationship between the properties V, Dieudonné and Dunford-Pettis, will be given. These different properties will be used to examine the conditions under which certain classes of operators on a B space will agree. There are two tables at the end of this chapter that give some sufficient conditions on a B space to determine whether it will possess the D. P. property.

Dieudonné Property

For spaces of continuous functions, Grothendieck (16) isolated a property similar to the D. P. property and subjected it to a similar process of axiomatization. He named this the Dieudonné property. We shall frame our definition of this property on the basis of Theorem 3.3, but there will be need for the following theorem.

Theorem 3.1 Let X and Y be B-spaces and G a collection of bounded subsets of X. Let H be the linear subspace of X'' generated by the weak star $(\sigma(X'', X'))$ closure in X'' of sets A in G and T a continuous linear map from X into Y. Assume $X \subset H$. The following conditions are equivalent:

(a) For each A in G, T(A) is a set whose closure in the weak topology of Y is compact.

(b) $T''(H) \subset J(Y)$, where J is the natural embedding of Y into Y''.

Proof. Assume condition (a) is true. Let A belong to G; thus, the weak closure of T(A) is compact in the weak topology of Y. Let $\bar{A}$ be the $\sigma(H, X')$ closure of A. Also T'' is continuous for $\sigma(X'', X')$ and $\sigma(Y'', Y')$. Let J_X and J_Y be the natural imbedding maps of X and Y respectively. Thus,

$$T''(J_X(\bar{A})) = J_Y(T(\bar{A})) \subset J_Y(\overline{T(A)})$$

or

$$T''(\bar{A}) \subset \overline{T(A)} .$$

Hence, $T''(\bar{A})$ is contained in the $\sigma(Y'', Y')$ closure of T(A). Since T(A) is conditionally compact in the weak topology on Y, it follows that $T''(\bar{A})$ is contained in the $\sigma(Y, Y')$ closure of T(A). The $\sigma(X, X')$ closure of $T''(\bar{A})$ is a subset of a conditionally compact set and, therefore, $T''(\bar{A})$ is conditionally compact in the $\sigma(Y, Y')$ topology of Y. Since

$$H = \bigcup_{A \in G} \bar{A} ,$$

we have

$$T''(H) \subset \bigcup_{A \in G} T''(A) \subset Y .$$

Assume (b) is true. As we have noticed, T'' is continuous for $\sigma(X'', X')$ and $\sigma(Y'', Y')$. Let A be a bounded set in G. Let A^o

be the polar of A. Hence, $A \subset A^{\circ\circ} \cap H$, but $A^{\circ\circ}$ is the convex balanced $\sigma(X'', X')$ closure of A, which is bounded, and, therefore, $A^{\circ\circ}$ is $\sigma(X'', X')$ compact (47, page 240). Hence, $A^{\circ\circ}$ is $\sigma(X, X')$ compact and the weak closure of A is also $\sigma(X, X')$ compact. Thus by hypothesis, $T(A) = T''(A)$ and we have that the weak closure of T(A) is compact in the weak topology on Y (11, I.5.7).

By choosing G in Theorem 3.1 to be the set of all bounded subsets of X, then we have the following characterization of weakly compact operators.

Corollary 3.2 Let X and Y be B-spaces and T a continuous linear map from X to Y. The following are equivalent:

(a) T is weakly compact.

(b) $T''(X'')$ is a subset of J(Y), where J is the natural embedding of Y into Y''.

Theorem 3.3 Let X be a B-space, G a collection of sequences of X, each of which is $\sigma(X'', X')$ convergent in X'', and H the linear subspace of X'' generated by X and the limits of members of G. The following conditions are equivalent:

(a) Any continuous linear map T of X into a B-space Y that transforms members of G into weakly convergent sequences in Y is a weakly compact operator.

(b) Any continuous linear map T of X into a B space Y such that $T''(H) \subset Y$ then $T''(X'') \subset Y$.

Proof. Assume condition (a) is true and the hypothesis of (b) is fulfilled. Let (x_i) be a member of G. By hypothesis, (x_i) is

$\sigma(X'', X')$ convergent in X'' to some x'' in H. Then, $T(x_i) = T''(x_i)$ is weakly convergent to $T''(x'')$ in Y since T'' is continuous for $\sigma(X'', X')$ and $\sigma(Y'', Y')$. In all cases we have $T''(H) \subset Y$. Hence, by Corollary 3.2 we have $T(X'') \subset Y$. The argument is reversible to show that (b) implies (a).

Now consider G to be the set of all weak Cauchy sequences in a B space X Now we shall show that each weak Cauchy sequence is $\sigma(X'', X')$ convergent in X''. Without loss of generality let S be the closed unit sphere that contains the weak Cauchy sequence (x_i). Since the closed unit sphere D'' in X'' is the $\sigma(X'', X')$ closure of D and also D'' is bounded, then D'' is $\sigma(X'', X')$ compact. Therefore, the weak Cauchy sequence in X is $\sigma(X'', X')$ convergent in X''. Also, if (x_n) is $\sigma(X'', X')$ convergent in X'', then (x_n) is $\sigma(X, X')$ convergent in X.

With the above theorems and discussion we are in a position to define the Dieudonné property.

Definition 3.4 A B-space X has the Dieudonné property (D. property) if for every B-space Y and every continuous linear map T from X to Y that transforms weak Cauchy sequences to weak convergent sequences in Y, then T is a weakly compact operator.

From Theorem 3.3 we see that condition (b) is a characterization of the D. property. One might introduce a Dieudonné type property relative to any set G of directed families satisfying the conditions imposed on Theorem 3.3. The smaller the set G is the stronger the associated Dieudonné property. The D. property enjoys the same properties as that of the D. P. property given in Theorems 2.5 and

2.11. The proof of these properties is similar to the ones given for the D. P. property and will therefore be omitted.

The following theorem gives sufficient conditions on a space to ensure that it will possess the D. property.

Theorem 3.5 If X be an almost reflexive space, then X possesses the D. property.

Proof. Let Y be a B-space and T a continuous linear map from X to Y. Let (x_n) be a bounded sequence in X. Now show that $(T(x_n))$ is weakly sequentially compact. Let (y_m) be a sequence in the set (Tx_n). Thus, there exists a bounded set of x_m such that

$$T(x_m) = y_m \qquad \text{for each } m .$$

Since X is almost reflexive, there exists a weak Cauchy subsequence (x_k) of (x_m). Thus, if $(T(x_k))$ is weakly convergent, then it follows that T is weakly compact. Hence, X possesses the D. property.

The above theorem is a generalization of some work by Edwards (12, page 646) We have the following interesting corrollaries.

Corollary 3.6 Every reflexive space has the D. property.

Corollary 3.7 If a B-space X is such that the bounded sets in X are weakly metrizable, then X possesses the D. property.

Proof. Let X be such a B-space. Thus, X' is separable (11, V.5.2). Therefore, it follows that X is almost reflexive by (31).

Theorem 3.8 Any B-space X which is weakly complete and possesses the D. property is a reflexive space.

Proof. Let T be the identity map on X. Thus, T sends weak Cauchy sequences into weakly convergent sequences. Hence, T is weakly compact. Since X has the D. property, T is a weakly compact operator. Thus, X is a reflexive space.

A consequence of this theorem is that the only abstract L spaces that possess the D. property are those that are reflexive. Thus, L does not have the D. property unless it is finite dimensional.

One final remark on the D. property. The following is an example of a B-space that possesses the D. property but is not almost reflexive. One of the main results of Grothendieck (16) is that for any compact Hausdorff space S then C(S) has the D. property. Using the definition of almost reflexive and part nine of the main theorem in Pelczynski and Semadeni (39), it is readily seen that C([0, 1]) is not almost reflexive.

Property V

A consequence of Theorem 1.15 is that every weakly compact linear operator between B spaces sends weakly unconditionally convergent (w.u.c.) series into an unconditionally convergent (u.c.) series. Pelczynski (36) studied the converse of the above problem. He defined a property V and made a systematic study of this property.

Definition 3.9 A B space X has property V if it satisfies one of the following conditions:

(a) For every B space Y, every u.c. operator T from X to Y is also a weakly compact operator.

(b) For every subset K' of X' which satisfies the condition

$$\lim_{n} \sup_{x' \in K'} x'(x_n) = 0$$

for every w.u.c. series Σx_n in X, K' is weakly sequentially compact in the weak star topology of X'.

The two conditions in Definition 3.9 are proved to be equivalent by Pelczynski (36). Some of the basic properties of the property V are proved in his paper. One of the main results by Pelczynski in (36) is that for any compact Hausdorff space S, C(S) has property V. It is easily seen that every reflexive space has property V from (11, VI.4.3).

Weak Cauchy V Property

Howard (22) has proven that given B-spaces X and Y with T' a weak Cauchy operator from Y' to X', then T is a u.c. operator from X to Y. This led him to define a property that he calls the weak Cauchy V property.

<u>Definition 3.10</u> A B-space X has the weak Cauchy V property if it satisfies one of the equivalent conditions:

(a) Given any B space Y, every u.c. operator T from X to Y is such that T' is weak Cauchy from Y' to X'.

(b) For every subset K' of X' satisfying the condition

$$\lim_{n} \sup_{x' \in K'} x'(x_n) = 0$$

for every w.u.c. series Σx_n in X, K' has a weak Cauchy sequence.

The proof that these two conditions are equivalent can be found in

(22). Since every weakly compact operator is also a weak Cauchy operator, there is a possibility of a relationship between the weak Cauchy V property and property V. The following theorem gives a condition that is needed on a B-space in order to ensure the equivalence of the weak Cauchy V property and property V.

Theorem 3.11 A B-space X has property V if and only if X has the weak Cauchy V property and X' is weakly complete.

Proof. Assume X has property V. Thus, X' is weakly complete (36). Let T be a u.c. operator on X; then, T is weakly compact. Since T' is also weakly compact, Theorem 1.10 implies that T' is weak Cauchy. Therefore, X has the weak Cauchy V property.

Assume X' is weakly complete and X has the weak Cauchy V property. Let T be a u.c. operator on X. The weak Cauchy V property implies that T' is a weak Cauchy operator. For a bounded set A in Y', T'(A) is bounded in X' since T' is continuous. Since T' is a weak Cauchy operator, T'(A) has a weak Cauchy subsequence. It follows that T'(A) is weakly sequentially compact since X' is weakly complete. Hence, T' is weakly compact and so is T. Therefore, X has the property V.

For weakly complete B-spaces we can give a characterization of the weak Cauchy V property in terms of its conjugate space.

Theorem 3.12 Let X be a weakly complete B space. Then X has the weak Cauchy V property if and only if X' is almost reflexive.

Proof. Assume X has the weak Cauchy V property. By Orlicz's Theorem every w.u.c. series is also a u.c. series (35). Thus, every bounded set in X' will satisfy the condition given in Definition 3.10

(36). Since X has the weak Cauchy V property, it follows that each bounded set in X' has a weak Cauchy sequence. Therefore, X' is almost reflexive.

Assume X' is almost reflexive. Since X is weakly complete, the w.u.c. and u.c. series are equivalent in X Thus, the closed unit sphere S' in X' is w.u.c.-limited by (5). Since X' is almost reflexive, it follows that X has the weak Cauchy V property.

The weak Cauchy V along with the property V can be used to give sufficient conditions for a B space not to possess the D. P. property.

Theorem 3.13 If X is almost reflexive and X' has property V, then X does not possess the D. P. property.

Proof. Since X' has property V, it follows that X is weakly complete from propositions 4 and 6 of (36). From the hypothesis, X is almost reflexive. From the Eberlein-Smulian Theorem any weakly complete and almost reflexive space is reflexive. Therefore, X does not have the D. P. property by Theorem 2.1.

Comparison of the Dieudonné, Dunford-Pettis and V Properties

Before considering some of the applications of these different properties, we shall examine the relationship between them.

Theorem 3.14 If X is a B-space which has property V, then X has property D.

Proof. It will suffice to show that any operator T that sends weak Cauchy sequences into weakly convergent sequences is a u.c.

operator.

Let T be such an operator from X to Y. Let Σx_n be a w.u.c. series in X. Let ΣTy_k be a subseries of ΣTx_n. Thus, there exists a w.u.c. subseries Σx_k of Σx_n such that $Tx_k = Ty_k$. Hence, the sequence

$$\left(\sum_{k=1}^{n} x_k \right)$$

is weak Cauchy. From the property of T, it follows that

$$\left(T \sum_{k=1}^{n} x_k \right)$$

is weakly convergent in Y. Therefore, every subseries of ΣTx_n is weakly convergent in Y Hence, T is a u.c. operator.

This is the only general relationship that exists between these properties. The following examples will verify this.

Example 3.15 If X has the D. property, then X does not necessarily have the property V. Consider the space E defined in Example 1.20. Since E is almost reflexive, it follows from Theorem 3.5 that E has the D. property. Assume E has the property V. From Theorem 3.11, E' is weakly complete. Also, E' is almost reflexive. The Eberlien-Smulian Theorem implies that E' is reflexive. Therefore, E is reflexive, which is a contradiction. Hence, E does not have the property V.

Example 3.16 If X has the D. property, then X does not necessarily have the D. P. property. Consider the space ℓ_2. Since ℓ_2 is almost reflexive, Theorem 3.5 implies ℓ_2 has the D. property. Theorem 2.1 yields that ℓ_2 does not have the D. P. property.

Example 3.17 If X has the D. P. property, then X does not necessarily have the D. property. Consider the space ℓ. This space has the D. P. property from Theorem 2.23. From Theorem 3.8 it follows that ℓ does not have the D. property.

Example 3.18 If X has the property V, then X does not necessarily have the D. P. property. Let X be an infinite dimensional reflexive B-space. The space X has property V (11, VI.4.3). From Theorem 2.1, X does not have the D. P. property.

Example 3.19 If X has the D. P. property, then X does not necessarily have property V. The space ℓ has the D. P property by Theorem 2.23. Assume ℓ has the V property. The conjugate space of ℓ is ℓ_∞ which is not weakly complete by (11, IV 13.5). This contradicts Theorem 3.11, which implies that ℓ_∞ is weakly complete. Therefore, ℓ does not have the property V.

Applications of These Properties

Five different operators were defined in chapter one and their relation to each other was given in general for a B-space. Now we plan on investigating what happens to these classes of operators when some of the properties discussed in this chapter are added to the domain or range space. Also, with the addition that a B-space has one or some combination of these properties, some classic and interesting results are easily obtained.

The next theorem gives conditions on the domain, X, and range, Y, spaces in order that all continuous operators from X to Y are exactly the weakly compact operators.

Theorem 3.20 Let X be a B-space that possesses the D. property and Y a B-space that is weakly complete. If T is a continuous linear map from X to Y, then T is weakly compact.

Proof. The continuity of T implies that T transforms weak Cauchy sequences into the same type sequences. From the property of Y, these weak Cauchy sequences are weakly convergent. Thus, it follows that T is weakly compact since X has the D. property.

If it is known that X has both the D. and D. P. property (for example, C(S) where S is a compact Hausdorff space), then an even more remarkable statement can be made about any continuous linear map. The following theorem gives this result.

Theorem 3.21 Let X, Y be B-spaces such that Y is a B-space which is weakly complete and X possesses both the D. and D. P. properties. If T is a continuous linear map from X to Y, then T is completely continuous.

Proof. From Theorem 3.20, T is weakly compact. Since X has the D. P. property, T is completely continuous.

Now consider a space which possesses the D. P. and V properties. The following theorem gives some equivalences between operators.

Theorem 3.22 Let X be a B-space which possesses the V and D. P. properties. Let Y be any B-space and T be any continuous linear map from X to Y, then the following are equivalent:

(a) T is a u.c. operator.

(b) T is a weakly compact operator.

(c) T is a completely continuous operator.

Proof. (a) implies (b). This implication follows since X has property V.

(b) implies (c). Since X has the D. P. property we see that this implication is true.

(c) implies (a). Using Theorem 1.13 this result follows immediately.

Some examples of spaces that possess both the V and D. P. properties are B(S), C(S), c, c_o, ℓ_∞, and $L_\infty(S, \Sigma, \mu)$. An interesting question now arises. Is it possible for the class of u.c. operators to coincide with the compact operators? The following theorem answers this question positively.

Theorem 3.23 Let X be a B-space that is almost reflexive and possesses the V and D. P. properties. If T is a u.c. operator from X to any B-space Y, then T is also a compact operator.

Proof. Let T be a u.c. operator. Using Theorem 3.22 we have T is completely continuous. Now show that T is compact. Since X is almost reflexive, any bounded sequence in X will contain a weak Cauchy subsequence (x_n). Now (x_n) is weak Cauchy if and only if $(x_k - x_m)$ converges weakly to zero for each subsequence (x_m) and (x_k) of (x_n). Hence, $(Tx_k - Tx_m)$ converges to zero and this is true if and only if (Tx_n) is a Cauchy sequence. By the completeness of Y, it follows that (Tx_n) is convergent. Therefore, T is a compact operator.

Some examples of spaces that are almost reflexive and possess the V and D P. properties are c_o, c, and C(S) where S is a compact

Hausdorff dispersed space (that is, S is dispersed if it contains no non-void closed subset E which is dense in itself).

One can obtain results similar to those of Theorem 3.22 by using the weak Cauchy V property along with the D. P. property.

Theorem 3.24 Let X be a B-space that possesses both the weak Cauchy V and D. P. properties. Let Y be any B-space and T a continuous linear map from X to Y; then, the following are equivalent:

(a) T is a u.c. operator.

(b) T' is a weak Cauchy operator.

(c) T is a completely continuous operator.

Proof. (a) implies (b). From the fact that X has the weak Cauchy V property, this implication follows readily.

(b) implies (c). This implication follows from Theorem 1.26 part (g) and X possessing the D. P. property.

(c) implies (a). This is easily seen by Theorem 1.13.

Now we shall investigate the conditions under which norm and weak convergence correspond in a B-space. In the next two theorems we shall find a class of B-spaces where these two types of convergence will agree.

Theorem 3.25 Let X be a B-space, then the following conditions are equivalent:

(a) Weak and norm convergence correspond in X.

(b) Every operator T from X to ℓ_∞ (the space of bounded sequences) is completely continuous.

(c) For every sequence (x_n) that converges weakly to zero in X and for every bounded sequence (x_n') in X'

$$\lim x_n' (x_n) = 0 .$$

<u>Proof.</u> (a) implies (b). Let (x_n) be a sequence in X that converges weakly to zero. By hypothesis, (x_n) is norm convergent to zero. Since T is continuous, (Tx_n) is norm convergent. Therefore, T is completely continuous.

(b) implies (c). Let (x_n') be a bounded sequence in X'. Define $T(x) = x_n'(x)$ for x in X. By the Uniform Boundedness Principle we have $x_n'(x)$ is an element in ℓ_∞. By assumption, T is completely continuous. Let (x_m) be a sequence in X that converges weakly to zero. Thus,

$$\lim_m \left| T x_m \right| = \lim_m \sup_n \left| x_n' (x_m) \right| = 0 .$$

Hence, $\lim x_n' (x_n) = 0$.

(c) implies (a). It will suffice to show that every map T from X to an arbitrary B-space Y is completely continuous. Let (x_n) be a sequence in X that converges weakly to zero. Let (y_n') be a sequence in Y' such that

$$\left|y_n'\right| = 1 \quad \text{and} \quad y_n'(Tx_n) = \left|Tx_n\right| \qquad \text{for each } n.$$

Define

$$x_n' = T'y_n' .$$

Since T' is continuous, the sequence (x_n') is bounded and

$$\lim \left| T x_n \right| = \lim \left| y_n' (T x_n) \right| = \lim \left| T' y_n' (x_n) \right|$$

$$\lim \left| T' y_n' (x_n) \right| = \lim \left| x_n' (x_n) \right| = 0 .$$

Hence, T is completely continuous. Thus, the identity on X is completely continuous and it follows that weak and norm convergence correspond in X.

In view of this theorem we can give a condition on the conjugate space such that the D. P. property will be equivalent to the corresponding of the norm and weak convergence in the space.

Theorem 3.26 Let X' be almost reflexive. The following are equivalent:

(a) Weak and norm convergence correspond in X.

(b) X has the D. P property.

Proof. Assume X has the D. P. property. Let T be the identity from X to X. Thus, T' is the identity from X' to X'. The map T' is weak Cauchy since X' is almost reflexive. From Theorem 1.26 part (g) it follows that T is completely continuous. Therefore, weak and norm convergence correspond in X.

The converse follows from Theorem 3.25 and Theorem 1.26 part (d).

The following are some consequences of these two theorems. Given any B-space such that weak and norm convergence agree, this space will have the D. P. property. From Theorems 2.1 and 3.26 we can see that for any reflexive space the weak and norm convergence do not agree.

Theorem 3.27 Let X be almost reflexive. The following conditions are equivalent:

(a) Weak and norm convergence correspond on X'.

(b) X has the D. P. property.

(c) X' has the D. P. property.

Proof. (a) implies (c). This implication follows from Theorem 3.25.

(c) implies (b). This implication follows from Theorem 2.3.

(b) implies (a). Let T be the identity map from X to X. Since X is almost reflexive, T is weak Cauchy. Since X has the D. P. property, T' (which is the identity from X' to X') is completely continuous. Therefore, weak and norm convergence correspond on X'.

The above theorem has some interesting corollaries. Among these is a result of Pelczynski given in the following corollary. Corollary 3.29 then gives sufficient conditions to ensure that a space does not possess the D. P. property.

Corollary 3.28 Let X be a B-space that possesses the D. P. property. Let A be a complemented subspace of X and suppose that the annihilator

$$A^{\perp} = \left\{ x' : x' \in X',\ x'(A) = 0 \right\}$$

is separable; weak and norm convergence correspond on $A^{\perp}$

Proof. Let P be a projection from X onto A. Thus, the quotient space X/A is linearly homeomorphic with the kernel of P, ker P. Since the subspace ker P is complementary to A, it is complemented in X. Therefore, ker P has the D. P. property from Theorem 2.5. The conjugate space to X/A can be naturally identified with $A^{\perp}$. Since $A^{\perp}$ is separable, we have that (ker P)' = $A^{\perp}$ and (ker P)' is separable. The ker P is almost reflexive by (31). Using Theorem 3.27, it follows that weak and norm convergence agree on (ker P)' = $A^{\perp}$.

Corollary 3.29 If X is an infinite dimensional B-space such that X and X' are both almost reflexive, then X does not have the D. P. property.

Proof. Assume X has the D. P. property. Theorem 3.27 implies that X' is weakly complete. By the Eberlien-Smulian Theorem, any weakly complete and almost reflexive space is reflexive. Since no infinite dimensional reflexive Banach space has the D. P property by Theorem 2.1, this implies X does not have the D. P. property. This is a contradiction.

The Dunford-Pettis Property and Weakly Compactly Generated Spaces

Some of the recent work that has been done with the D. P. property is by Rosenthal (41). He also used the concept of weakly compactly generated B-spaces that was developed and studied by Lindenstrauss (32).

Definition 3.30 A B-space X is weakly compactly generated, denoted by W.C.G. if there exists a weakly compact subset of X whose linear span is dense in X.

Among the basic properties that Lindenstrauss proved were these: complemented subspaces of a W.C.G. B-space are also W.C.G., and if X is a W.C.G. B-space and Y is isomorphic to X, then Y is a W.C.G. B-space. Further properties of W.C.G. B-spaces can be found in (32).

Since the closed unit sphere in a reflexive space is weakly compact, it follows that a reflexive space is a W.C.G. space. Let X be a separable Banach space and (x_n) be a countable dense subset of X.

Define for each n,

$$y_n = \frac{x_n}{n \mid x_n \mid} .$$

Thus, $(y_n) \cup \{0\}$ is a compact set and, thus, weakly compact. Therefore, X is a W.C.G. space. Hence, the W.C.G. spaces are generalizations of the reflexive and separable Banach spaces.

Rosenthal (41) has taken the concept of W.C.G. B-spaces and generalized a result given by Grothendieck.

Theorem 3.31 Let X be a B-space that possesses the D. P. property. If X is isomorphic to a subspace of a weakly compactly generated conjugate B-space, then weak and norm convergence correspond in X.

Proof. First, we shall observe that for any sequences (x_n) and (x'_n) in X and X', respectively, such that (x_n) converges weakly to zero and (x'_n) is weak Cauchy, then $x'_n(x)$ converges to zero. Suppose not, thus without loss of generality (otherwise a subsequence could be used) assume $x'_n(x_n)$ converges to L where L is non-zero. Since (x_n) converges weakly to zero, we may choose a subsequence (x_{nk}) of (x_n) such that

$$\lim x'_k (x_{nk}) = 0 .$$

Hence, $(x_n - x_{nk})$ converges weakly to zero (16). Since X has the D P. property, it follows that

$$(x'_n - x'_{nk}) (x_{nk}) \qquad \text{converges to zero.}$$

Thus,

$$x'_{nk} (x_{nk}) \qquad \text{converges to zero,}$$

which is a contradiction.

Since the D. P. property is preserved by isomorphism (Theorem 2.7), we may assume there exists a B-space Y such that Y' is W.C.G. and X is a subset of Y'. Let (x_n) be a sequence in X that converges weakly to zero and suppose (x_n) does not converge to zero. Thus without loss of generality (passing to a subsequence if necessary), we may assume there exists a t greater than zero such that

$$|x_n| > t \qquad \text{for all n.}$$

For each n, choose a y_n in Y such that

$$|y_n| = 1 \quad \text{and} \quad |x_n(y_n)| > t .$$

This is possible since S is a subset of Y'. The unit ball of Y'' is weak star sequentially compact (2). Hence, there exists a subsequence (y_m) of (y_n) and a y'' in Y'' such that (y_m) converges weak star to y''. Thus, (y_m) is a weak Cauchy sequence. Define a map T from Y to X' as follows:

$$Ty(x) = x(y) \qquad \text{for all y in Y and x in X.}$$

Since X is a subset of Y', T is a continuous map and (Ty_m) is a weak Cauchy sequence in X'. From our above observations we find that

$$\lim (Ty_m)(x_m) = \lim x_m(y_m) = 0,$$

which is a contradiction. The theorem follows from the fact that a sequence (x_n) in X is weak (norm) Cauchy if and only if for every pair of its subsequences (x_k) and (x_m),

$$(x_m - x_k) \qquad \text{converges weakly (in norm) to zero.}$$

Rosenthal also used the D. P. property to help him generalize the result of Gelfand that $L_1([0, 1])$ is not isomorphic to a subspace of a separable conjugate space.

<u>Theorem 3.32</u> Let μ be a measure and X be a complemented subspace of $L_1(\mu)$. If μ is finite and X is isomorphic to a conjugate B-space, or more generally, if μ is arbitrary and X is isomorphic to a subspace of a W.C.G. conjugate B-space, then weak Cauchy sequences in X are norm convergent and X is isomorphic to a complemented subspace of $L_1([0, 1])$.

<u>Proof.</u> First consider μ finite. From Theorem 2.23 $L_1(\mu)$ has the D. P. property and is also W.C.G. since $L_2(\mu)$ injects densely into $L_1(\mu)$. Its complemented subspace is also W.C.G. and possesses the D. P. property. Thus, if X is isomorphic to a subspace of a W.C.G. conjugate space, Theorem 3.31 implies that weak Cauchy sequences converge in the norm topology of X. Thus by Eberlien's Theorem, X is separable. Now choose a subspace of $L_1(\mu)$ containing X and isomorphic to $L_1(\nu)$ for some separable measure ν. Hence, for such a ν, $L_1(\nu)$ is isomorphic to a complemented subspace of $L_1([0, 1])$ by (20, page 123).

For a general measure μ, the above argument and Rosenthal's (41) Lemma 1.3 yields that if X is isomorphic to a subspace of a W.C.G. B-space, then there exists a finite measure ν and a subspace Z of $L_1(\mu)$ with Z isomorphic to $L_1(\nu)$ and X a subset of Z.

The next theorem provides an elementary proof that every weakly compact subset of $L_\infty(\mu)$, for a finite measure μ, is separable. Also every W.C.G. subspace of $L_\infty(\mu)$ will be separable.

Theorem 3.33 If X is a B-space that is W.C.G. and possesses the D. P. property, then every weakly compact subset of X' is separable.

Proof. First we shall observe that if K is a weakly compact subset of the B-space X and T is a map from X' to C(K) defined by

$$Tx'(k) = x'(k) \qquad \text{for all } x' \text{ in } X' \text{ and } k' \text{ in } K,$$

then it follows that T is weakly compact from (11, page 490).

Now let K be a weakly compact subset of X'. From the above observation and letting X = X' and T map X'' to C(K) as above, we have TJ is a weakly compact operator from X to C(K) where J is the natural imbedding of X into X''. Since X is W.C.G., let G be a weakly compact subset of X that generates X. Since X possesses the D. P. property, TJ(G) is a compact subset of C(K) and, thus, a separable subset. The subspace TJ(X) of C(K) is separable since G generates X. Let A be the smallest closed subalgebra of C(K) which contains TJ(G) and the constants, then A is also separable. Let s and k be distinct points of K. Since a compact Hausdorff space is normal, it follows from Urysohn's Lemma that TJ(X) separates the points of K; hence, so does A. The Stone-Weierstrass Theorem implies that A is equal to C(K). Thus, K is metrizable in its weak topology by (11, V.5.1). Therefore, K is separable.

The above theorem has the following corollary. This corollary will provide a sufficient condition for a B-space not to posses the D. P. property.

Corollary 3.34 If X is a B-space such that X' is weakly compactly generated and non-separable, then X does not possess the D. P. property.

Proof. Assume X has the D. P. property. Since X' is weakly compactly generated, the closed unit sphere in X'' is weak-star sequentially compact (2). Let (x_n) be a bounded sequence in X; thus, $(\hat{x}_n)$ is a bounded sequence in X''. Hence, there exists a subsequence of $(\hat{x}_n)$ which is weak-star convergent, say $(\hat{x}_m)$. Therefore, (x_m) will be a subsequence of (x_n) that is weak Cauchy in X. Thus, it is seen that X is almost reflexive. From Theorem 3.27 it follows that weak and norm convergence correspond in X'. The space X', that is W.C.G., is also compactly generated and is, therefore, separable. This is a contradiction and it follows that X does not have the D. P. property.

It might be mentioned that the Dieudonné and D. P. property have been very helpful in the study of vector-valued Radon measures. The interested reader can find a discussion of this along with more references on the subject in (12).

The following two tables represent a collection of sufficient conditions to determine whether a B-space possesses the D. P. property. These conditions are a result of the research done on this thesis and are given in tabular form for easy reference.

TABLE III

SUFFICIENT CONDITIONS FOR A BANACH SPACE X TO HAVE THE DUNFORD-PETTIS PROPERTY

(a) The conjugate space of X, X', has the Dunford-Pettis property. (Theorem 2.3)

(b) The weak and norm convergence correspond in X. (Theorem 3.25)

(c) The weak and norm convergence correspond in X'. (Theorems 3.25, 2.3)

(d) The space X isomorphic to a C(S) space where S is a compact Hausdorff space. (Theorems 2.2, 2.7)

(e) The space X is isomorphic to an abstract L-space. (Theorems 2.23, 2.7)

(f) The space X is an abstract M-space with unit. (Theorem 2.18)

(g) Every weakly sequentially compact set in X' is w-limited in X'. (Theorem 1.26)

(h) Every weakly sequentially compact set in X is $\widetilde{w}$-limited in X. (Theorem 1.26)

(i) The space X is a $\mathfrak{L}(1)$-space. (Theorem 2.31)

(j) The space X is a $\mathfrak{L}(\infty)$-space. (Theorem 2.3)

(k) The space C(S, X), S is a compact Hausdorff space, has the D. P. property. (Theorem 2.24)

TABLE IV

SUFFICIENT CONDITIONS FOR A BANACH SPACE X NOT TO POSSESS THE DUNFORD-PETTIS PROPERTY

(a) The space X be isomorphic to any infinite dimensional reflexive B-space. (Theorems 2.1, 2.7)

(b) The space X isomorphic to a subspace of a weakly compactly generated conjugate B-space such that weak and norm convergence do not correspond in X. (Theorem 3.31)

(c) The space X' is almost reflexive and weak and norm convergence do not correspond in X. (Theorem 3.26)

(d) The space X and X' are almost reflexive. (Corollary 3.29)

(e) The space X' is weakly compactly generated and non-separable. (Corollary 3.34)

(f) The space X' has property V and X' is separable. (Theorem 3.13)

(g) The space X is weakly complete and has property D. (Theorems 3.8, 2.1)

(h) The space X' has property V and X is almost reflexive. (Theorem 3.13)

(i) The space X is isomorphic to a subspace of a $\mathfrak{L}(1)$-space and also isomorphic to a quotient space of a $\mathfrak{L}(\infty)$-space. (Theorem 2.32)

CHAPTER IV

SUMMARY AND SOME OPEN QUESTIONS

The main purpose of this thesis is to give some characterizations of the Dunford-Pettis property and examine some of its applications in the field of functional analysis. Some of these characterizations are given in terms of operators. There are two main results given in Chapter I. One is the Venn diagram, which demonstrates the inter-relations among the compact, weakly compact, weak Cauchy, completely continuous, and unconditionally converging operators. The other is given by Theorem 1.26, which consists of the characterizations of the Dunford-Pettis property.

Some of the very basic properties of the Dunford-Pettis property are given in Chapter II. It is pointed out that some of the permanence properties of the Dunford-Pettis property consist of isomorphism, complemented subspaces, and finite topological direct sums. Among the non-permanence properties are subspaces, quotient spaces, inductive and projective limit spaces. Tables I and II are given at the end of Chapter II and show whether some of the common B-spaces encountered in an introductory course in functional analysis have the Dunford-Pettis property.

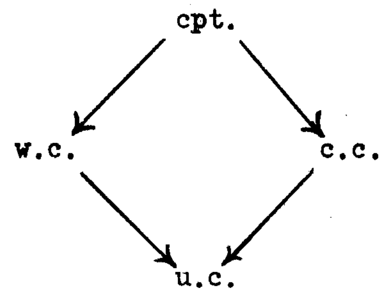
In Chapter III properties similar to the Dunford-Pettis property are defined. The relationship between the property V, Dieudonné and Dunford-Pettis properties is exhibited in this chapter. These different properties are used to help investigate when certain classes of operators on B-spaces will agree. It is shown that for a certain class of B-spaces the Dunford-Pettis property can be used to characterize the property that weak and norm convergence correspond. Table III at the end of Chapter III is a collection of sufficient conditions to determine that a B-space will possess the Dunford-Pettis property. Table IV is a collection of sufficient conditions to determine that a B-space will not possess the Dunford-Pettis property.

Four well-known operators have been mentioned in this paper, namely, compact (cpt.), weakly compact (w.c.), completely continuous (c.c.), and unconditionally converging (u.c.) operators. This paper contains a study of the interrelationships among these operators and their use in giving characterizations of the Dunford-Pettis property. These operators have received much consideration in the past few years since many properties of a B-space can be described in terms of them. Pelczynski (36) and Grothendieck (16) have shown that the property V and the Dunford-Pettis property have been very helpful in the field of functional analysis. The following is an extension of the concept of property V and Dunford-Pettis property.

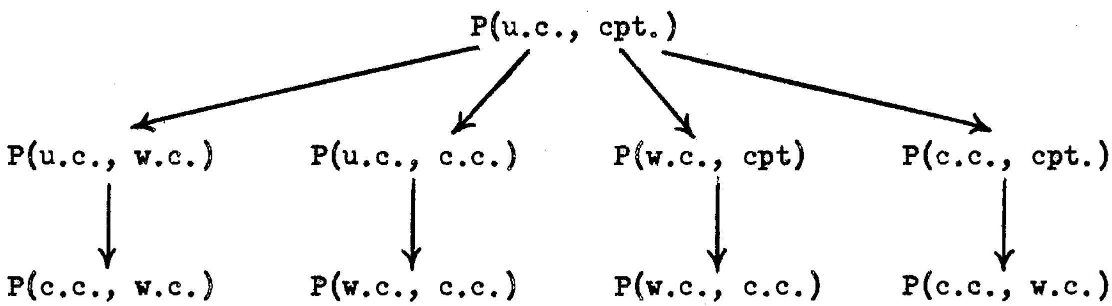
<u>Definition 4.1</u> A B-space X has the P(a,b) property if for every B-space Y every a-type operator from X to Y is also a b-type operator.

The Dunford-Pettis property and property V can be represented in terms of this new terminology by P(w.c., c.c.) and P(u.c., w.c.), respectively. It will be noticed that the reflexive spaces have the P(c.c., w.c.), which is in a sense the converse of the Dunford-Pettis property.

From Chapter I it is seen that the following relationships always hold for operators.



With these implications, one can easily construct the following:



Grothendieck (16) studied many properties of limited sets and pointed out that many of these properties can be stated in terms of the Mackey topology on the conjugate space. If it is possible to get a connection between the P(a, b) property and limited sets, then it seems possible that some of the problems that are encountered in using limited sets could possible be overcome by using operators. Similarly, some problems involving the Mackey topology on the conjugate space could be viewed in terms of operators. Howard (23) and Grothendeick (16) have studied limited sets and as a result of their studies we are able to relate operators to limited sets. Given next will be results from their studies.

1. P(u.c., cpt.) is equivalent to every w.u.c.-limited set being sequentially compact in X'.
2. P(u.c., w.c.) is equivalent to every w.u.c.-limited set being weakly sequentially compact in X'.
3. P(u.c., c.c.) is equivalent to every w.u.c.-limited set being w-limited in X'.
4. P(w.c., cpt.) is equivalent to every weakly sequentially compact set being sequentially compact in X'.
5. P(c.c., cpt.) is equivalent to every w-limited set being sequentially compact in X'.
6. P(w.c., c.c.) is equivalent to every weakly sequentially compact set being w-limited in X'.
7. P(c.c., w.c.) is equivalent to every w-limited set being weakly sequentially compact in X'.

The P(a, b) property could be subjected to a systematic study by using different classes of operators as Grothendeick (16) did the Dunford-Pettis property and Pelczynski (36) the property V. It will now be shown how some of the P(a, b) properties are related by using compact, weakly compact, unconditionally converging, and completely continuous operators.

<u>Theorem 4.2</u> If X is a B-space and possesses the P(u.c., cpt.) property, then X possesses the P(w.c., cpt.) property.

<u>Proof.</u> Let T be a weakly compact operator on X. From Theorem 1.15 it is seen that T is a u.c. operator and, thus, T is a compact operator. Therefore, X has the P(w.c., cpt.) property.

Theorem 4.3 If X is an abstract M-space with unit, then X possesses both the P(w.c., c.c.) and P(c.c., w.c.) properties.

Proof. From Theorem 2.18 X is isometrically isomorphic to a C(S) space where S is a compact Hausdorff space. The class of weakly compact and completely continuous operators coincide for this space as seen by Theorem 3.22.

Theorem 4.4 If X is a B-space and possesses the P(c.c., cpt.) property, then X possesses P(c.c., w.c.) property.

Proof. Let T be a completely continuous operator on X. Since X has the P(c.c., cpt.) property, T is a compact operator. From Theorem 1.8, T is also weakly compact. Therefore, X has the P(c.c., w.c.) property.

Theorem 4.5 If X is a B-space that is almost reflexive, then X has the P(c.c., cpt.) property.

Proof. Let T be a completely continuous operator from X to an arbitrary B-space Y. Since X is almost reflexive, any bounded sequence in X contains a weak Cauchy subsequence (x_n). Now (x_n) is weak Cauchy if and only if $(x_k - x_{k+1})$ converges weakly to zero for each subsequence (x_k) of (x_n). Since T is completely continuous, $(Tx_k - Tx_{k+1})$ converges to zero for each subsequence (x_k), which is equivalent to (Tx_n) being a Cauchy sequence. Since Y is complete, it follows that (Tx_n) is convergent. Hence, every bounded sequence is mapped into a sequence that has a convergent subsequence. Therefore, T is a compact operator.

Theorem 4.6 If X is a B-space which has the P(w.c., cpt.) property, then X possesses the P(w.c., c.c.) property.

Proof. Let T be a weakly compact operator on X. Since X has the P(w.c., cpt.) property., T is a compact operator. From Theorem 1.11, T is completely continuous. Therefore, X has the P(w.c., c.c.) property.

Theorem 4.7 If X is a B-space that possesses both the P(u.c., w.c.) and P(w.c., c.c.) properties, then X possesses the P(u.c., c.c.) property.

Proof. Let T be an unconditionally converging operator on X. Since X has the P(u.c., w.c.) property, T is weakly compact. It follows that T is a completely continuous operator from the fact that X has the P(w.c., c.c.) property.

Theorem 4.8 If X is an almost reflexive B-space that possesses the P(w.c., c.c.) property, then X' has the P(w.c., c.c.) property.

Proof. This is a restatement of Theorem 3.27 in the new terminology.

Theorem 4.9 If X is an almost reflexive B-space that possesses both the P(u.c., w.c.) and P(w.c., c.c.) properties, then X possesses the P(u.c., cpt.) property.

Proof. Let T be a u.c. operator on X. From Theorem 4.7 it follows that T is completely continuous. Since X is almost reflexive, Theorem 4.5 implies that T is a compact operator. Therefore, X possesses the P(u.c., cpt.) property.

<u>Theorem 4.10</u> If X is an almost reflexive B-space that possesses the P(w.c., c.c.) property, then X has the P(w.c., cpt.) property.

<u>Proof.</u> Let T be a weakly compact operator on X. Since X has the P(w.c., c.c.) property, T is a completely continuous operator. From Theorem 4.5 it follows that T is a compact operator.

Several examples will now be given in order to make Table V more meaningful. These examples will provide some of the needed counter-examples for the table.

<u>Example 4.11</u> The space ℓ_∞ is a C(S) space where S is a compact Hausdorff space. Let T be the identity on ℓ_∞'. Thus, T is a u.c. operator if and only if ℓ_∞' has no subspace isomorphic to c_0. Since ℓ_∞' is an abstract L-space, ℓ_∞' is weakly complete. By Orclicz's Theorem it follows that every w.u.c. series is also a u.c. series. Therefore, ℓ_∞' does not contain a subspace isomorphic to c_0 (5). Hence, T is a u.c. operator. Since weak and norm convergence do not correspond on ℓ_∞', T is not a completely continuous operator.

<u>Example 4.12</u> The Banach and Mazur Theorem states that given any separable B-space X, there exists an operator T that maps ℓ onto X. Let X = ℓ_2 and T be a operator from ℓ onto ℓ_2. Since ℓ_2 is reflexive, T is weakly compact. Hence, T' is weakly compact and has a bounded inverse on ℓ_2. Thus, T' is not compact by (15).

<u>Example 4.13</u> The space ℓ is the conjugate of a C(S) space where S is a compact Hausdorff space. Let T be the identity on ℓ. The map T is completely continuous since weak and norm convergence correspond on ℓ. Since ℓ is not reflexive, T is not weakly compact. From Theorem

1.8 it follows that T is not compact.

Example 4.14 The space c_o is almost reflexive (31). The space c_o has an unconditional basis (7, page 73). It follows then that c_o has the property V from (36).

Example 4.15 Any operator with domain ℓ will be completely continuous (31).

Example 4.16 Using the operator T given in example 4.12, it is seen that T'' maps ℓ'_∞ into ℓ_2 . Since T' is weakly compact, then T'' is weakly compact. The operator T' is not compact; thus, T'' is not compact. The space ℓ'_∞ has the P(w.c., c.c.) property; hence, T'' is also completely continuous.

Example 4.17 Assume ℓ'_∞ has the P(u.c., w.c.) property. From Theorem 3.14 ℓ'_∞ has the D. property. Since ℓ'_∞ is weakly complete, it must be reflexive by Theorem 3.8. This is a contradiction. Therefore, ℓ'_∞ does not have the P(u.c., w.c.) property.

Example 4.18 Since ℓ is a conjugate B space, then ℓ is complemented in its second conjugate, namely ℓ'_∞ . Hence, there exists a continuous projection T from ℓ'_∞ onto ℓ. The range of T being ℓ implies that T is completely continuous (31). Assume T is weakly compact. The conjugate operator T' is also weakly compact. The operator T' has a bounded inverse (14, page 61). Let S be the closed unit sphere in ℓ_∞ ; thus T'(S) is weakly sequentially compact. Since $(T')^{-1}$ is continuous, $T'^{-1}(T'(S)) = S$ is weakly sequentially compact. By the Eberlian-Smulian Theorem S is compact in the weak topology. Thus, ℓ_∞ must be reflexive, which implies ℓ_1 is also reflexive, which is a contradiction.

Therefore, T is not compact.

Example 4.19 The space B_3 is not reflexive. Assume B_3 has the P(u.c., w.c.) property. From Theorem 3.11 it follows that B_3' is weakly complete. Since B_3'' is separable, B_3' is almost reflexive. From the Eberlian-Smulian Theorem it follows that B_3' is reflexive. This implies that B_3 is reflexive, which is a contradiction. Therefore, B_3 does not possess the P(u.c., w.c.) property. A similar argument could be used to show that E does not possess the P(u.c., w.c.) property.

Example 4.20 The space B_3'' is separable which implies that B_3' is separable. Assume B_3' contains a subspace isomorphic to c_o. Hence, B_3' also contains a subspace isomorphic to ℓ_∞ (5). Thus, B_3' is not separable, which is a contradiction. Therefore, the identity on B_3' is a u.c. operator. Since B_3' is not reflexive, the identity is not weakly compact. Therefore, B_3' does not possess the P(u.c., w.c.) property. A similar argument could be used on E' to show that it does not have the P(u.c., w.c.) property.

Example 4.21 The spaces B_3 and B_3' are both almost reflexive. From Corollary 3.29 it follows that B_3 does not have the P(w.c., c.c.) property. If B_3' had the P(w.c., c.c.) property this would contradict Theorem 2.3. Similarly for the spaces E and E'.

Example 4.22 Assume B_3 has the P(u.c., c.c.) property. Let T be an unconditionally converging operator on B_3. Thus T is a completely continuous operator. Since B_3 is almost reflexive, Theorem 4.5 implies that T is a compact operator. Hence, B_3 has the P(u.c., w.c.) property which is a contradiction. Therefore, B_3 does not have the P(u.c., c.c.)

property. A similar argument could be used for the spaces B_3', E, and E'.

Example 4.23 If either X or Y is reflexive, then every operator from X to Y is weakly compact (11, VI.4.3).

Example 4.24 Let T be the identity map on any infinite dimensional reflexive space. The map T is weakly compact, but not compact.

Example 4.25 For S a dispersed space, the space C(S) is almost reflexive (39).

Example 4.26 The B-space ℓ_∞ can be identified with the space C(S) where S is the Stone-Cech compactification of the positive integers. The conjugate space of ℓ_∞ is an abstract L-space and thus S is not dispersed (39). Therefore, S contains a non-void perfect set. The space ℓ_2 is separable; hence, there exists a continuous linear map T such that T maps ℓ_∞ onto ℓ_2 (30). This map is weakly compact since ℓ_2 is reflexive. Assume T is a compact operator. The conjugate map T' is also compact. The operator T' has a bounded inverse (14, page 61). Let S be the closed unit sphere in ℓ_2, thus, T'(S) is sequentially compact. Since $(T')^{-1}$ is continuous, $(T')^{-1} T'(S) = S$ is a closed and sequentially compact set in ℓ_2 and is, therefore, compact (11, I.6.13). Therefore, ℓ_2 must be finite dimensional which is a contradiction.

TABLE V

BANACH SPACES AND THE P(a, b) PROPERTIES

Space	P(u.c., cpt.)	P(u.c., w.c.)	P(u.c., c.c.)
C(S) S compact Hausdorff	No (Ex. 4.26)	Yes (Th. 3.22)	Yes (Th. 4.7)
(C(S))' S compact Hausdorff	No (Ex. 1.17)	No (Ex. 1.17)	No (Ex. 4.11)
c_0	Yes (Th. 4.9)	Yes (Ex. 4.14)	Yes (Th. 4.7)
ℓ_1	No (Ex. 1.17	No (Ex. 1.17)	Yes (Ex. 4.15)
ℓ_∞	No (Ex. 4.26)	Yes (Th. 3.22)	Yes (Th. 4.7)
ℓ'_∞	No (Ex. 4.16)	No (Ex. 4.17)	No (Ex. 4.11)
E	No (Th. 4.2)	No (Ex. 4.19)	No (Ex. 4.22)
E'	No (Ex. 1.20)	No (Ex. 1.20)	No (Ex. 1.20)
B_3 (ref. (13))	No (Th. 4.2)	No (Ex. 4.19)	No (Ex. 4.22)
B'_3	No (Th. 4.2)	No (Ex. 4.20)	No (Ex. 4.22)
Reflexive	No (Ex. 4.24)	Yes (Ex. 4.23)	No (Ex. 4.24)
Almost Reflexive	--	--	--
C(S), S dispersed	Yes (Th. 4.9)	Yes (Th. 3.22)	Yes (Th. 4.7)

P(w.c., cpt.)	P(c.c., cpt.)	P(w.c., c.c.)	P(c.c., w.c.)
No (Ex. 4.26)	No (Ex. 4.26)	Yes (Th. 2.2)	Yes (Th. 4.3)
No (Ex. 4.12)	No (Ex. 4.13)	Yes (Th. 2.22)	No (Ex. 1.17)
Yes (Th. 4.10	Yes (Th. 4.5)	Yes (Th. 2.3)	Yes (Th. 4.5)
No (Ex. 4.12)	No (Ex. 1.18)	Yes (Th. 2.22)	No (Ex. 4.13)
No (Ex. 4.26)	No (Ex. 4.26)	Yes (Th. 2.2)	Yes (Th. 4.3)
No (Ex. 4.16)	No (Ex. 4.16)	Yes (Th. 2.22)	No (Ex. 4.19)
No (Th. 4.6)	Yes (Th. 4.5)	No (Ex. 4.21)	Yes (Th. 4.5)
No (Th. 4.6)	Yes (Th. 4.5)	No (Ex. 4.21)	Yes (Th. 4.5)
No (Th. 4.6)	Yes (Th. 4.5)	No (Ex. 4.21)	Yes (Th. 4.5)
No (Th. 4.6)	Yes (Th. 4.5)	No (Ex. 4.21)	Yes (Th. 4.5)
No (Ex. 4.24)	Yes (Th. 4.5)	No (Th. 2.1)	Yes (Th. 4.5)
--	Yes (Th. 4.5)	--	Yes (Th. 4.5)
Yes (Th. 4.10)	Yes (Th. 4.5)	Yes (Th. 2.2)	Yes (Th. 4.3)

Open Questions and Partial Results

The remaining part of this chapter will deal with some of the open questions that involve the D. P. property.

(a) One of the outstanding open questions dealing with the D. P. property is the conjecture that if X has the D. P. property, then X' has the D. P. property. Partial results on this open question are given by Theorems 2.18, 2.23, 2.31 and 3.27. From these results it can be seen that for a possible counterexample we shall need a B-space that is not one of the familiar ones studied in an introductory course in functional analysis.

(b) Pelczynski (38) introduces the notions of weakly compact polynomial and multilinear operators. He investigated conditions on a B-space, X, under which every weakly compact polynomial operator on X can map weak Cauchy sequences into strong Cauchy sequences. Any B-space that has this property is said to have the polynomial Dunford-Pettis (P. D. P.) property. The conjecture was that necessary and sufficient conditions for X to have the P. D. P. property is that X possess the D. P. property. It was proved that every polynomial (multilinear) operator with real or complex values defined on a space that possessed the D. P. property has the P. D. P. property.
The following are open questions:

(i) Does every B-space satisfying the D. P. property also have the P. D. P. property?

(ii) Let X and Y be B-spaces with the D. P. property. Does the projective tensor product of X and Y have the D. P. property?

It follows from Corollary 5 in Pelczynski's (38) work that the positive answer to (ii) implies a positive answer to (i). A partial result to question (ii) is given by Theorem 2.36.

(c) Dobrakov (10) has posed the following open question. Let S be a locally compact Hausdorff space and X a B-space. Let $C_o(S, X)$ be the B-space of all X valued continuous functions on T tending to zero at infinity with the sup-norm. The important open question and at the same time very difficult one is as follows: If X has the Dunford-Pettis property, does $C_o(S, X)$ also have this property? Partial results to this question are given by Theorems 2.25, 2.28, and 2.38.

(d) The last open question mentioned here deals with Theorem 3.31. From this theorem and the Eberlien-Smulian Theorem it follows that if X satisfies all the hypothesis of Theorem 3.31, then every weakly compact subset of X is norm-compact and, therefore, separable. Thus, if X is assumed to be W.C.G., then X must be separable. Rosenthal (41) has conjectured that separability of X should follow without this additional assumption.

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