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Michel GRANGER, David MOND,
Alicia NIETO-REYES & Mathias SCHULZE

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LINEAR FREE DIVISORS AND THE GLOBAL LOGARITHMIC COMPARISON THEOREM

by Michel GRANGER, David MOND,
Alicia NIETO-REYES & Mathias SCHULZE (*)

ABSTRACT. — A complex hypersurface D in $\mathbb{C}^n$ is a *linear free divisor* (LFD) if its module of logarithmic vector fields has a global basis of linear vector fields. We classify all LFDs for n at most 4.

By analogy with Grothendieck's comparison theorem, we say that the *global logarithmic comparison theorem* (GLCT) holds for D if the complex of global logarithmic differential forms computes the complex cohomology of $\mathbb{C}^n \setminus D$. We develop a general criterion for the GLCT for LFDs and prove that it is fulfilled whenever the Lie algebra of linear logarithmic vector fields is reductive. For n at most 4, we show that the GLCT holds for all LFDs.

We show that LFDs arising naturally as discriminants in quiver representation spaces (of real Schur roots) fulfill the GLCT. As a by-product we obtain a topological proof of a theorem of V. Kac on the number of irreducible components of such discriminants.

Keywords: Free divisor, prehomogeneous vector space, De Rham cohomology, logarithmic comparison theorem, Lie algebra cohomology, quiver representation.
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Résumé. — Une hypersurface complexe de $\mathbb{C}^n$ est appelée un *diviseur linéairement libre* (ou DLL) si son module de champs de vecteur logarithmiques a une base globale formée de champs de vecteurs linéaires. Nous classifions tous les DLL pour n au plus égal à 4.

Par analogie avec le théorème de comparaison de Grothendieck, on dit que le *théorème de comparaison logarithmique global* (ou TCLG) est vrai pour D si le complexe des formes différentielles logarithmiques globales permet de calculer la cohomologie de $\mathbb{C}^n \setminus D$ à coefficients complexes. Nous mettons en évidence un critère général pour qu'un DLL ait la propriété TCLG, et nous démontrons que ce critère s'applique lorsque l'algèbre de Lie des champs de vecteurs logarithmiques linéaires est réductive. Pour n inférieur ou égal à 4, nous montrons que le TCLG est vrai pour tous les DLL.

Nous montrons que les DLL qui apparaissent naturellement comme discriminants dans les espaces de représentations de carquois pour des racines de Schur réelles satisfont au TCLG. Comme corollaire nous obtenons une démonstration topologique d'un résultat de V. Kac sur le nombre de composantes irréductibles de tels discriminants.

1. Introduction

We denote by $\mathscr{O} = \mathscr{O}_{\mathbb{C}^n}$ the sheaf of holomorphic functions on $\mathbb{C}^n$, by $\mathfrak{m}_p \subseteq \mathscr{O}_p$ the maximal ideal at $p \in \mathbb{C}^n$, by $\mathrm{Der} = \mathrm{Der}_{\mathbb{C}}(\mathscr{O})$ the sheaf of $\mathbb{C}$-linear derivations of $\mathscr{O}$ (or *holomorphic vector fields*) on $\mathbb{C}^n$, and by $\Omega^\bullet = \Omega^\bullet_{\mathbb{C}^n}$ the complex of sheaves of holomorphic differential forms. We shall frequently use a local or global coordinate system $x = x_1, \dots, x_n$ on $\mathbb{C}^n$ and then denote by $\partial = \partial_1, \dots, \partial_n$ the corresponding operators of partial derivatives $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$. Note that $\mathrm{Der} = \bigoplus \mathscr{O} \cdot \partial_i$ is a free $\mathscr{O}$-module of rank n.

Let $D \subseteq \mathbb{C}^n$ be a reduced divisor. K. Saito [22] associated to D the (coherent) sheaf of *logarithmic vector fields* $\mathrm{Der}(-\log D) \subseteq \mathrm{Der}$ and the complex of (coherent) sheaves $\Omega^\bullet(\log D) \subseteq \Omega^\bullet(*D)$ of *logarithmic differential forms* along D. For a (local or global) defining equation $\Delta \in \mathscr{O}$ of the germ D, $\delta \in \mathrm{Der}$ is in $\mathrm{Der}(-\log D)$ if $\delta(\Delta) \in \mathscr{O} \cdot \Delta$, and $\omega \in \Omega^\bullet[\Delta^{-1}]$ is in $\Omega^\bullet(\log D)$ if $\Delta \cdot \omega, \Delta \cdot d\omega \in \Omega^\bullet$. Note that $\mathrm{Der}(-\log D)$ contains the *annihilator* $\mathrm{Der}(-\log \Delta)$ of Δ defined by the condition $\delta(\Delta) = 0$. Saito showed that $\mathrm{Der}(-\log D)$ and $\Omega^1(\log D)$ are reflexive and mutually dual and introduced the following important class of divisors.

Definition 1.1. — *A divisor D is called* free *if* $\mathrm{Der}(-\log D)$, *or equivalently* $\Omega^1(\log D)$, *is a locally free $\mathscr{O}$-module, necessarily of rank n.*

We will be concerned in this article with the following subclass of divisors.

Definition 1.2. — *A free divisor D is called* linear *if* $\Gamma(\mathbb{C}^n, \mathrm{Der}(-\log D))$ *admits a basis $\delta_1, \dots, \delta_n$ such that each δ_i has linear coefficients with respect to the $\mathscr{O}$-basis $\partial_1, \dots, \partial_n$ of* Der *or equivalently each δ_i is homogeneous of degree zero with respect to the standard degree defined by* $\deg x_i = 1 = -\deg \partial_i$ *on the variables and generators of* Der.

Saito's criterion [22, Thm. 1.8.(ii)] implies the following fundamental observation.

Lemma 1.3. — *If $\delta_1, \dots, \delta_n$ is a basis of $\Gamma(\mathbb{C}^n, \mathrm{Der}(-\log D))$ for a linear free divisor D, then the homogeneous polynomial $\Delta = \det((\delta_i(x_j))_{i,j}) \in \mathbb{C}[x]$ of degree n is a global defining equation for D.*

Note that because $\mathrm{Der}(-\log D)$ can have no members of negative degree, D cannot be isomorphic to the product of $\mathbb{C}$ with a lower dimensional divisor. It turns out that linear free divisors are relatively abundant; the authors believe that in the current paper and in [3], recipes are given which allow the straightforward construction of more free divisors than have been described in the sum of all previous papers.

Examples 1.4. —
The normal crossing divisor $D = \{x_1 \cdots x_n = 0\} \subseteq \mathbb{C}^n$ is a linear free divisor where

$$x_1\partial_1, \dots, x_n\partial_n$$

is a basis of $\mathrm{Der}(-\log D)$. Up to isomorphism it is the only example among hyperplane arrangements, cf. [20, Ch. 4].
In the space $B_{2,3}$ of binary cubics, the discriminant D, which consists of binary cubics having a repeated root, is a linear free divisor. For $f(u,v) = xu^3 + yu^2v + zuv^2 + wv^3$ has a repeated root if and only if its Jacobian ideal does not contain any power of the maximal ideal (u,v), and this in turn holds if and only if the four cubics

$$u\partial_u f, v\partial_u f, u\partial_v f, v\partial_v f$$

are linearly dependent. Writing the coefficients of these four cubics as the columns of the 4×4 matrix

$$A := \begin{pmatrix} 3x & 0 & y & 0 \\ 2y & 3x & 2z & y \\ z & 2y & 3w & 2z \\ 0 & z & 0 & 3w \end{pmatrix}$$

we conclude that D has equation $\det A = 0$. After division by 3 this determinant is

$$-y^2z^2 + 4wy^3 + 4xz^3 - 18wxyz + 27w^2x^2.$$

In fact each of the columns of this matrix determines a vector field in $\mathrm{Der}(-\log D)$; for the group $\mathrm{Gl}_2(\mathbb{C})$ acts linearly on $B_{2,3}$ by composition on the right, and, up to a sign, the four columns here are the infinitesimal generators of this action corresponding to a basis of $\mathfrak{gl}_2(\mathbb{C})$. Each is tangent to D, since the action preserves D.

Further examples of irreducible linear free divisors can be found (though not under this name) in the paper [23] of Sato and Kimura. Besides our example, two, of ambient dimension 12 and 40, are described in [23, §5, Prop. 11, 15], and by repeated application of castling transformations, cf. [23, §2], it is possible to generate infinitely many more, of higher dimensions.

In Section 5 of this paper we describe a number of further examples of linear free divisors, and in Section 6 we prove some results about linear bases for the module $\Gamma(\mathbb{C}^n, \mathrm{Der}(-\log D))$, and go on to classify all linear free divisors in dimension $n \leqslant 4$.

Linear free divisors provide a new insight into a conjecture of H. Terao [26, Conj. 3.1] relating the cohomology of the complement of certain divisors D to the cohomology of the complex $\Omega^\bullet(\log D)$ of forms with logarithmic poles along D. For linear free divisors, the link between the complex $\Gamma(\mathbb{C}^n, \Omega^\bullet(\log D))$ and $H^*(\mathbb{C}^n \smallsetminus D)$ can be understood as follows.

Definition 1.5. — *For a linear free divisor D defined by $\Delta \in \mathbb{C}[x]$, we consider the subgroup*

$$G_D := \{A \in \mathrm{Gl}_n(\mathbb{C}) \mid A(D) = D\} = \{A \in \mathrm{Gl}_n(\mathbb{C}) \mid \Delta \circ A \in \mathbb{C} \cdot \Delta\}$$

with identity component G_D° and Lie algebra $\mathfrak{g}_D$. We call D reductive *if G_D°, or equivalently $\mathfrak{g}_D$, is reductive.*

It turns out that $\mathbb{C}^n \smallsetminus D$ is a single orbit of G_D° with finite isotropy group, so $H^*(\mathbb{C}^n \smallsetminus D; \mathbb{C})$ is isomorphic to the cohomology of G_D°; this is explained in Section 2. Moreover, $H^*(\Gamma(\mathbb{C}^n, \Omega^\bullet(\log D)))$ coincides with the Lie algebra cohomology of $\mathfrak{g}_D$ with complex coefficients. For compact connected Lie groups G, a well-known argument shows that the Lie algebra cohomology coincides with the topological cohomology of the group. For linear free divisors the group G_D° is never compact, but the isomorphism also holds good for the larger class of reductive groups, and for a significant class of linear free divisors, G_D° is indeed reductive. In Section 3 we prove our main result:

Theorem 1.6. — *If D is a reductive linear free divisor then*

$$H^*(\Gamma(\mathbb{C}^n, \Omega^\bullet(\log D))) \simeq H^*(\mathbb{C}^n \smallsetminus D; \mathbb{C}). \tag{1.1}$$

Among linear free divisors to which it applies are those arising as discriminants in representation spaces of quivers, as discussed in detail in [3] and briefly in Section 4 below.

Terao's conjecture remains open, though it has been answered in the affirmative for a very large class of arrangements in [29], using a technique developed in [7]. For general free divisors, a local result from which the global isomorphism of (1.1) follows holds when imposing the following additional hypothesis.

Definition 1.7. — *A divisor D is called* quasihomogeneous *at $p \in D$ if the germ (D, p) admits a local defining equation $\Delta \in \mathscr{O}_p$ that is weighted homogeneous with respect to weights $w_1, \dots, w_n \in \mathbb{Q}_+$ in some local coordinate system $x_1, \dots, x_n$ centred at p. Dividing $w_1, \dots, w_n$ by the weighted degree of Δ, note that the preceding condition means that $\chi(\Delta) = \Delta$ where $\chi = \sum_{i=1}^n w_i x_i \partial_i \in \mathrm{Der}(-\log D)_p$. D is called* locally quasihomogeneous *if it is quasihomogeneous at p for all $p \in D$. We say* homogeneous *instead of* quasihomogeneous *if $w = 1, \dots, 1$.*

Theorem 1.8 ([7]). — *Let $D \subseteq \mathbb{C}^n$ be a locally quasihomogeneous free divisor, let $U = \mathbb{C}^n \smallsetminus D$, and let $j : U \to \mathbb{C}^n$ be inclusion. Then the de Rham morphism*

$$\Omega^\bullet_X(\log D) \to \mathrm{R}j_* \mathbb{C}_U \tag{1.2}$$

is a quasi-isomorphism.

Grothendieck's *Comparison Theorem* [12] asserts that a similar quasi-isomorphism holds for any divisor D, if instead of logarithmic poles we allow meromorphic poles of arbitrary order along D. Because of this similarity, we refer to the quasi-isomorphism of (1.2) as the Logarithmic Comparison Theorem (LCT) and to the global isomorphism (1.1) as the Global Logarithmic Comparison Theorem (GLCT). Several authors have further investigated the range of validity of LCT, and established interesting links with the theory of $\mathscr{D}$-modules, in particular in [4], [6], [11], [27], and [28].

Local quasihomogeneity was introduced in [7] as a technical device to make possible an inductive proof of the isomorphism in 1.8. Subsequently it turned out to have a deeper connection with the theorem. In particular by [5], for plane curves the logarithmic comparison theorem holds if and only if all singularities are quasihomogeneous. The situation in higher dimensions remains unclear. There is as yet no counterexample to the conjecture that LCT is equivalent to the following weaker condition.

Definition 1.9. — *A divisor D is called* Euler homogeneous *at $p \in D$ if there is a germ of vector field $\chi \in \mathfrak{m}_p \cdot \mathrm{Der}_p$ such that $\chi(\Delta) = \Delta$ for some local defining equation $\Delta \in \mathscr{O}_p$ of the germ (D,p). In this case, χ is called an* Euler vector field *for D at p. D is called* strongly Euler homogeneous *if it is Euler homogeneous at p for all $p \in D$.*

Remark 1.10. — The Euler homogeneity of D is independent of the choice of an equation. If χ is an Euler vector field at p for D defined by $\Delta \in \mathscr{O}_p$, and $u \in \mathscr{O}_p^*$ is a unit, then the defining equation $u\Delta$ of D at p satisfies an equation

$$(\chi(u)+u)^{-1}u\chi(u\Delta) = (\chi(u)+u)^{-1}(\chi(u)+u)u\Delta = u\Delta$$

with Euler vector field $(\chi(u)+u)^{-1}u\chi$.

In Section 7 we examine the examples described in Sections 5 and 6 with respect to local quasihomogeneity and strong Euler homogeneity. It turns out that all linear free divisors in dimension $n \leqslant 4$ are locally quasihomogeneous and there is no linear free divisor which we know not to be strongly Euler homogeneous. The optimistic reader could therefore conjecture that all linear free divisors are strongly Euler homogeneous, and also fulfil LCT and so also GLCT. We do not know any counter-example to these statements.

In Subsection 7.1 we give examples of quivers Q and dimension vectors $\mathbf{d}$ for which the discriminant in $\mathrm{Rep}(Q,\mathbf{d})$ is a linear free divisor but is not locally quasihomogeneous. In such cases Theorem 1.8 therefore does not apply, but Theorem 1.6 does.

In Subsection 7.2, we show that a linear free divisor does not need to be reductive for LCT to hold. However we do not know whether reductiveness of the group implies LCT for linear free divisors. The property of being a linear free divisor is not local, and our proof of GLCT here is quite different from the proof of LCT in [7].

The fact that linear free divisors in $\mathbb{C}^n$ arise as the complement of the open orbit of an n-dimensional connected algebraic subgroup of $\mathrm{Gl}_n(\mathbb{C})$, means that there is some overlap between the topic of this paper and of the paper [23], where Sato and Kimura classify irreducible *prehomogeneous vector spaces*, that is, triples (G,ρ,V), where ρ is an irreducible representation of the algebraic group G on V, in which there is an open orbit. However, the hypothesis of irreducibility means that the overlap is slight. Any linear free divisor arising as the complement of the open orbit in an irreducible prehomogeneous vector space is necessarily irreducible by [23, §4, Prop. 12], whereas among our examples and in our low-dimensional classification (in Section 6) all the linear free divisors except one (Example 1.4(1.4)) are reducible. Even where G is reductive, the passage from irreducible to reducible representations in this context is by no means trivial, including as it does substantial parts of the theory of representations of quivers.

2. Linear free divisors and subgroups of $\mathrm{Gl}_n(\mathbb{C})$

A degree zero vector field $\delta \in \mathrm{Der}$ can be identified with an $n \times n$ matrix $A = (a_{i,j})_{i,j} \in \mathbb{C}^{n\times n}$ by $\delta = \sum_{i,j} x_i a_{i,j} \partial_j = xA\partial^t$. Under this identification, the commutator of square matrices corresponds to the Lie bracket of vector fields.

Let $D \subseteq \mathbb{C}^n$ be a reduced divisor defined by a homogeneous polynomial $\Delta \in \mathbb{C}[x]$ of degree d.

Definition 2.1. — *We denote by*

$$\mathrm{L}_D := \{xA\partial^t \mid xA\partial^t(\Delta) \in \mathbb{C} \cdot \Delta\} \subseteq \Gamma(\mathbb{C}^n, \mathrm{Der}(-\log D))$$

the Lie algebra of degree zero global logarithmic vector fields.

Recall from Definition 1.2, that D is linear free if L_D contains a basis of $\mathrm{Der}(-\log D)$, and recall G_D° from Definition 1.5.

Lemma 2.2. — *G_D° is an algebraic subgroup of $\mathrm{Gl}_n(\mathbb{C})$ and*

$$\mathfrak{g}_D = \{A \mid xA^t\partial^t \in \mathrm{L}_D\}.$$

Proof. — Clearly G_D is a subgroup of $\mathrm{Gl}_n(\mathbb{C})$ and defined by a system of polynomial (determinantal) equations. Thus G_D and hence also G_D° is an algebraic subgroup of $\mathrm{Gl}_n(\mathbb{C})$. The Lie algebra of G_D° consists of all $n \times n$-matrices A such that

$$\Delta \circ (I + A\varepsilon) = a(\varepsilon) \cdot \Delta \in \mathbb{C}[\varepsilon] \cdot \Delta$$

where $\mathbb{C}[\varepsilon] = \mathbb{C}[t]/\langle t^2\rangle \ni [t] =: \varepsilon$. Taylor expansion of this equation with respect to ε yields

$$\Delta + \partial(\Delta) \cdot A \cdot x^t \cdot \varepsilon = (a(0) + a'(0) \cdot \varepsilon) \cdot \Delta$$

and hence $a(0) = 1$ and, by transposing the ε-coefficient, $xA^t\partial^t \in \mathrm{L}_D$. The argument can be reversed to prove the converse by setting

$$a(\varepsilon) := 1 + (xA^t\partial^t(\Delta)/\Delta) \cdot \varepsilon.$$

□

Lemma 2.3. — *The complement* $\mathbb{C}^n \smallsetminus D$ *of a linear free divisor is an orbit of* G_D° *with finite isotropy groups.*

Proof. — For $p \in \mathbb{C}^n$, the orbit $G_D^\circ \cdot p$ is a smooth locally closed subset of $\mathbb{C}^n$ whose boundary is a union of strictly lower dimensional orbits, cf. [14, Prop. 8.3]. The orbit map $G_D^\circ \to G_D^\circ \cdot p$ sends $I_n + A\varepsilon$ to $p + pA^t\varepsilon$ and induces a tangent map

$$\mathfrak{g}_D \twoheadrightarrow T_p(G_D^\circ \cdot p), \quad A \mapsto pA^t. \tag{2.1}$$

For $p \notin D$, $\mathrm{Der}(-\log D)(p)$ and hence also $\mathrm{L}_D(p)$ is n-dimensional. Then by Lemma 2.2 and (2.1) $T_p G_D^\circ \cdot p$ and hence $G_D^\circ \cdot p$ are n-dimensional which implies the finiteness of the isotropy group of p in G_D°. As this holds for all $p \notin D$, the boundary of $G_D^\circ \cdot p$ must be D and then $G_D^\circ \cdot p = \mathbb{C}^n \smallsetminus D$. □

Reversing our point of view we might try to find algebraic subgroups $G \subseteq \mathrm{Gl}_n(\mathbb{C})$ that define linear free divisors. This requires by definition that G is n-dimensional and connected and by Lemma 2.3 that there is an open orbit. The complement D is then a candidate for a free divisor. Indeed D is a divisor: comparing with (2.1), D is defined by the *discriminant determinant*

$$\Delta = \det \begin{pmatrix} A_1 x^t & \cdots & A_n x^t \end{pmatrix}$$

where $A_1, \ldots, A_n$ is a basis of the Lie algebra $\mathfrak{g}$ of G and we denote by $f = \Delta_{\mathrm{red}}$ the reduced equation of D. As the entries of the defining polynomial are linear, Δ is a homogeneous polynomial of degree n. Thus, if Δ is not reduced, D can not be linear free. We shall see examples where this happens in the next section. On the other hand, Saito's criterion [22, Lem. 1.9] shows the following.

Lemma 2.4. — *Let the* n*-dimensional algebraic group* G *act linearly on* $\mathbb{C}^n$ *with an open orbit. If* Δ *is reduced then* D *is a linear free divisor.* □

As a first step towards our main result, we now describe the cohomology of $\mathbb{C}^n \smallsetminus D$ in terms of G_D°.

Proposition 2.5. — *Suppose that* $D \subseteq \mathbb{C}^n$ *is a linear free divisor and let* $G_{D,p}^\circ$ *be the (finite) isotropy group of* $p \in \mathbb{C}^n \smallsetminus D$ *in* G_D°*. Then*

$$H^*(\mathbb{C}^n \smallsetminus D; \mathbb{C}) = H^*(G_D^\circ; \mathbb{C})^{G_{D,p}^\circ} = H^*(G_D^\circ; \mathbb{C}).$$

Proof. — By Lemma 2.3, $\mathbb{C}^n \smallsetminus D \cong G_D^\circ / G_{D,p}^\circ$ with finite $G_{D,p}^\circ$ and the first equality follows. The second equality holds because G_D° is path connected, which means that left translation by $g \in G_{D,p}^\circ$ is homotopic to the identity and thus induces the identity map on cohomology. □

Remark 2.6. — The argument for the second equality also shows that if G_D° is a finite quotient of the connected Lie group G then $H^*(\mathbb{C}^n \smallsetminus D;\mathbb{C}) \simeq H^*(G;\mathbb{C})$. We will use this below in calculating the cohomology of $\mathbb{C}^n \smallsetminus D$.

3. Cohomology of the complement and Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra. The complex of Lie algebra cochains with coefficients in the complex representation V of $\mathfrak{g}$ has kth term $\bigwedge_{\mathbb{C}}^k \operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}, V) \cong \operatorname{Hom}_{\mathbb{C}}(\bigwedge_{\mathbb{C}}^k \mathfrak{g}, V)$, and differential $d_L : \bigwedge_{\mathbb{C}}^k \operatorname{Hom}(\mathfrak{g}, V) \to \bigwedge_{\mathbb{C}}^{k+1} \operatorname{Hom}(\mathfrak{g}, V)$ defined by

$$\begin{aligned}(d_L\omega)&(v_1 \wedge \cdots \wedge v_{k+1})\\ &= \sum_{i<j}(-1)^{i+j}\omega([v_i, v_j] \wedge v_1 \cdots \wedge \widehat{v_i} \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_{k+1})\\ &\quad+ \sum_i (-1)^{i+1} v_i \cdot \omega(v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_{k+1}).\end{aligned} \tag{3.1}$$

The cohomology of this complex is the *Lie algebra cohomology* of $\mathfrak{g}$ with coefficients in V and will be denoted $H_A^*(\mathfrak{g}; V)$.

The exterior derivative of a differential k-form satisfies an identical formula:

$$\begin{aligned}d\omega&(\chi_1 \wedge \cdots \wedge \chi_{k+1})\\ &= \sum_{i<j}(-1)^{i+j}\omega([\chi_i, \chi_j] \wedge \chi_1 \wedge \cdots \wedge \widehat{\chi_i} \wedge \cdots \wedge \widehat{\chi_j} \wedge \cdots \wedge \chi_{k+1})\\ &\quad+ \sum_i (-1)^{i+1} \chi_i \cdot \omega(\chi_1 \wedge \cdots \wedge \widehat{\chi_i} \wedge \cdots \wedge \chi_{k+1}).\end{aligned} \tag{3.2}$$

Here the χ_i are vector fields.

When D is a free divisor and $V = \mathscr{O}_p$ for some $p \in D$, it is tempting to conclude from the comparison of (3.1) and (3.2) that the complex $\Omega^\bullet(\log D)$ coincides with the complex of Lie algebra cohomology, with coefficients in $\mathscr{O}_p$, of the Lie algebra $\operatorname{Der}(-\log D)_p$. For $\Omega^1(\log D)_p$ is the dual of $\operatorname{Der}(-\log D)_p$, and $\Omega^k(\log D) = \bigwedge^k \Omega^1(\log D)$. However, this identification is incorrect, since, in the complex $\Omega^\bullet(\log D)$, both exterior powers and Hom are taken over the ring of coefficients $\mathscr{O}$, rather than over $\mathbb{C}$, as in the complex of Lie algebra cochains. The cohomology of $\Omega^\bullet(\log D)_p$ is instead the *Lie algebroid* cohomology of $\operatorname{Der}(-\log D)_p$ with coefficients in $\mathscr{O}_p$. Nevertheless, when D is a linear free divisor, there is the following important link between these two complexes.

Recall L_D from Definition 2.1.

Lemma 3.1. — *Let D be a linear free divisor. The complex*

$$\Gamma(\mathbb{C}^n, \Omega^\bullet(\log D))_0$$

of global homogeneous differential forms of degree zero coincides with the complex $\bigwedge_{\mathbb{C}}^\bullet \mathrm{Hom}(\mathrm{L}_D, \mathbb{C})$ of Lie algebra cochains with coefficients in $\mathbb{C}$.

Proof. — First we establish a natural isomorphism between the corresponding terms of the two complexes. We have

$$\begin{aligned}\Omega^1(\log D) = \mathrm{Hom}_{\mathscr{O}}(\mathrm{Der}(-\log D), \mathscr{O}) &= \mathrm{Hom}_{\mathscr{O}}(\mathrm{L}_D \otimes_{\mathbb{C}} \mathscr{O}, \mathscr{O}) \\ &= \mathrm{Hom}_{\mathbb{C}}(\mathrm{L}_D, \mathbb{C}) \otimes_{\mathbb{C}} \mathscr{O}.\end{aligned}$$

Since $\mathrm{Hom}_{\mathbb{C}}(\mathrm{L}_D, \mathbb{C})$ is purely of degree zero, and the degree zero part of $\mathscr{O}$ consists just of $\mathbb{C}$, the degree zero part of $\Gamma(\mathbb{C}^n, \Omega^1(\log D))$ is

$$\Gamma(\mathbb{C}^n, \Omega^1(\log D))_0 = \mathrm{Hom}_{\mathbb{C}}(\mathrm{L}_D, \mathbb{C}).$$

Since moreover $\Gamma(\mathbb{C}^n, \Omega^1(\log D))$ has no part of negative degree, it follows that

$$\Gamma(\mathbb{C}^n, \Omega^k(\log D))_0 = \Gamma\big(\mathbb{C}^n, \bigwedge_{\mathscr{O}}^k \Omega^1(\log D)\big)_0 = \bigwedge_{\mathbb{C}}^k \mathrm{Hom}_{\mathbb{C}}(\mathrm{L}_D, \mathbb{C}).$$

Next, we show that the coboundary operators are the same. Because we are working with constant coefficients, the second sum on the right in (3.1) vanishes. Let $\chi_1, \ldots, \chi_{k+1} \in \mathrm{L}_D$. Then for $\omega \in \Gamma(\mathbb{C}^n, \Omega^k(\log D))_0$ and $i \in \{1, \ldots, k+1\}$, $\omega(\chi_1 \wedge \cdots \wedge \widehat{\chi_i} \wedge \cdots \wedge \chi_{k+1})$ is a constant. It follows that the second sum on the right in (3.2) vanishes. Thus, the coboundary operator d_L and the exterior derivative d coincide. □

More generally let us consider weights $w = w_1, \ldots, w_n \in \mathbb{Q}_+$ and assign the weight w_i (resp. $-w_i$) to x_i and dx_i (resp. to ∂_i). Then the set of homogeneous vector fields or differential forms of a given degree is well defined.

Lemma 3.2. — *Suppose that the divisor $D \subseteq \mathbb{C}^n$ is quasihomogeneous with respect to weights $w = w_1, \ldots, w_n \in \mathbb{Q}_+$. Then the following holds for any open set $U \subseteq \mathbb{C}^n$:*

1. *If $\omega \in \Gamma(U, \Omega^k(\log D))$ is w-homogeneous, then $L_\chi(\omega) = \deg_w(\omega)\omega$, where L_χ is the Lie derivative with respect the Euler vector field $\chi = \sum_{i=1}^n w_i x_i \partial_i$.*
2. *For any closed $\omega \in \Gamma(U, \Omega^k(\log D))$ with decomposition $\omega = \sum_{j \geqslant j_0} \omega_j$ into w-homogeneous parts, $\omega - \omega_0$ is exact.*

(3) *If* $\Gamma(U,\Omega^k(\log D))_r \subseteq \Gamma(U,\Omega^k(\log D))$ *denotes the subspace of w-homogeneous forms of w-degree r, then*

$$\Gamma(U,\Omega^\bullet(\log D))_0 \hookrightarrow \Gamma(U,\Omega^\bullet(\log D))$$

is a quasi-isomorphism.

Proof. —

(1) is a straightforward calculation, using Cartan's formula $L_\chi(\omega) = d\iota_\chi\omega + \iota_\chi d\omega$, where ι_χ is contraction by χ.

(2) follows, for, if ω is closed, so is ω_j for every j, and thus

$$\omega - \omega_0 = \sum_{0\neq j\geqslant j_0} \omega_j = L_\chi\Big(\sum_{0\neq j\geqslant j_0} \frac{\omega_j}{j}\Big) = d\Big(\iota_\chi\Big(\sum_{0\neq j\geqslant j_0} \frac{\omega_j}{j}\Big)\Big).$$

(3) is now an immediate consequence.

□

From Lemma 3.1 and Lemma 3.2(3) applied to $U = \mathbb{C}^n$ we deduce the following

Proposition 3.3. — *Let* $D \subseteq \mathbb{C}^n$ *be a linear free divisor. Then*

$$H^*(\Gamma(\mathbb{C}^n,\Omega^\bullet(\log D))) \cong H^*_A(\mathrm{L}_D;\mathbb{C}). \qquad \square$$

Recall G_D° and $\mathfrak{g}_D$ from Definition 1.5. From Propositions 2.5 and 3.3 we deduce

Corollary 3.4. — *The global logarithmic comparison theorem holds for a linear free divisor D if and only if*

$$H^*(G_D^\circ;\mathbb{C}) \cong H^*_A(\mathfrak{g}_D;\mathbb{C}). \tag{3.3}$$ □

There is such an isomorphism if G is a connected compact real Lie group with Lie algebra $\mathfrak{g}$ (which is not our situation here). Left translation around the group gives rise to an isomorphism of complexes

$$T : \bigwedge^\bullet \mathfrak{g}^* \to \big(\Omega^\bullet(G)^G, d\big)$$

where $\mathfrak{g}^* = \mathrm{Hom}_\mathbb{R}(\mathfrak{g},\mathbb{R})$ and $\Omega^\bullet(G)^G$ is the complex of left-invariant real-valued differential forms on G. Composing this with the inclusion

$$\big(\Omega^\bullet(G)^G, d\big) \to \big(\Omega^\bullet(G), d\big) \tag{3.4}$$

and taking cohomology gives a morphism

$$\tau_G\colon H^*_A(\mathfrak{g}_D;\mathbb{R}) \to H^*(G;\mathbb{R}). \tag{3.5}$$

If G is compact, (3.5) is an isomorphism. For from each closed k-form ω we obtain a left-invariant closed k-form ω_A by averaging:

$$\omega_A := \frac{1}{|G|}\int_G \ell_g^*(\omega)d\mu_L,$$

where μ_L is a left-invariant measure and $|G|$ is the volume of G with respect to this measure. As G is path-connected, for each $g \in G$ ℓ_g is homotopic to the identity, so ω and $\ell_g^*(\omega)$ are equal in cohomology. It follows from this that ω and ω_A are also equal in cohomology.

Of course, this does not apply directly in any of the cases discussed here, since G_D is not compact. Nevertheless if G_D is a reductive group, the complexified morphism (3.5) is an isomorphism. We now briefly outline the necessary definitions. Let G_0 be a compact Lie group. Then ([19, §5.4, Thm. 10]) G_0 has a faithful real representation. It follows ([19, §3.4, Thm. 5]) that G_0 has an affine real algebraic group structure. This allows its complexification.

Definition 3.5. —

(1) *The complex Lie algebra representation is* reductive *if it is the direct sum of a semi-simple ideal and a diagonalizable ideal.*
(2) *The complex linear algebraic group G is* reductive *if its Lie algebra (representation) is reductive.*

The term "reductive" is due to the fact that these groups are characterised, among complex algebraic groups, by the complete reducibility of every finite-dimensional complex representation. Chapter 5 of [19] establishes a bijection between compact Lie groups and reductive complex linear algebraic groups:

Theorem 3.6 ([19, §5.2, Thm. 5]). — *On any compact Lie group K there exists a unique real algebraic group structure, whose complexification $K(\mathbb{C})$ is reductive. Any reductive complex algebraic group possesses an algebraic compact real form (of which it is therefore the complexification).* □

The significance of this notion for us derives from the following fact:

Theorem 3.7 ([19, §5.2 Thm. 2]). — *Let G be a complex reductive algebraic group with an n-dimensional compact real form K. Then G is diffeomorphic to $K \times \mathbb{R}^n$.* □

Corollary 3.8. — *If G is a connected reductive complex algebraic group with complex Lie algebra $\mathfrak{g}$ then*

$$H_A^*(\mathfrak{g};\mathbb{C}) \simeq H^*(G;\mathbb{C}).$$

Proof. — Let K be a compact real form of G. By 3.7, inclusion of K into $G = K(\mathbb{C})$ induces an isomorphism on cohomology. The Lie algebra $\mathfrak{g}$ of G is the complexification of the Lie algebra $\mathfrak{k}$ of K, so we have

$$H_A^*(\mathfrak{g};\mathbb{C}) \simeq H^*(\mathfrak{k};\mathbb{R}) \otimes \mathbb{C} \simeq H^*(K;\mathbb{R}) \otimes \mathbb{C} \simeq H^*(K;\mathbb{C}) \simeq H^*(G;\mathbb{C}),$$

where the second isomorphism comes from the isomorphism (3.5). □

From Corollary 3.4, Definition 3.5, and Corollary 3.8 we now conclude Theorem 1.6 as announced in the introduction: the Global Logarithmic Comparison Theorem holds for all reductive linear free divisors.

Using the reductiveness of the group $\mathrm{Gl}_n(\mathbb{C})$, we will show in the next section that the group G_D is reductive for divisors obtained as discriminants in the representation spaces of quivers. The subgroup $B_n \subseteq \mathrm{Gl}_n(\mathbb{C})$ of upper triangular matrices is not reductive, and appears as the group G_D in Example 5.1 which shows that reductivity is not necessary for the GLCT to hold.

4. Linear free divisors in quiver representation spaces

The following discussion summarises part of [3]. A quiver Q is a finite connected oriented graph; it consists of a set Q_0 of nodes and a set Q_1 of arrows joining some of them. For each arrow $\varphi \in Q_1$ we denote by $t\varphi$ (for "tail") and $h\varphi$ (for "head") the nodes where it starts and finishes. A (complex) representation V of Q is a choice of complex vector space V_α for each node $\alpha \in Q_0$ and linear map $V(\varphi) : V_{t\varphi} \to V_{h\varphi}$ for each arrow $\varphi \in Q_1$. For a fixed *dimension vector*

$$\mathbf{d} = (d_\alpha)_{\alpha\in Q_0} := (\dim V_\alpha)_{\alpha\in Q_0}.$$

and a choice of bases for the V_α, $\alpha \in Q_0$, the *representation space* of the quiver Q of dimension $\mathbf{d}$ is

$$\mathrm{Rep}(Q,\mathbf{d}) := \prod_{\varphi\in Q_1} \mathrm{Hom}(\mathbb{C}^{d_{t\varphi}}, \mathbb{C}^{d_{h\varphi}}) \cong \prod_{\varphi\in Q_1} \mathrm{Hom}(V_{t\varphi}, V_{h\varphi}).$$

On this space the *quiver group*

$$\mathrm{Gl}(Q,\mathbf{d}) := \prod_{\alpha\in Q_0} \mathrm{Gl}_{d_\alpha}(\mathbb{C}) \cong \prod_{\alpha\in Q_0} \mathrm{Gl}(V_\alpha)$$

acts, by

$$\big((g_\alpha)_{\alpha\in Q_0} \cdot V\big)_\varphi := g_{h\varphi} V(\varphi) g_{t\varphi}^{-1}. \tag{4.1}$$

This action factors through the group

$$Z := \mathbb{C}^* \cdot (I_{d_\alpha})_{\alpha \in Q_0} \subseteq \mathrm{Z}(\mathrm{Gl}(Q, \mathbf{d})) \tag{4.2}$$

in the center of $\mathrm{Gl}(Q, \mathbf{d})$ where $I_{d_\alpha} \in \mathrm{Gl}_{d_\alpha}(\mathbb{C})$ is the unit matrix. The group $\mathrm{Gl}(Q, \mathbf{d})/Z$ is reductive as, choosing a vertex $x_0 \in Q_0$, we can consider it as a central quotient

$$\mathrm{Gl}(Q, \mathbf{d})/Z \cong \Big(\mathrm{Sl}_{d_{x_0}}(\mathbb{C}) \times \prod_{x \in Q_0 \smallsetminus \{x_0\}} \mathrm{Gl}_{d_x}(\mathbb{C})\Big)\Big/\big(\mu_{d_{x_0}} \cdot \prod I_{d_x}\big) \tag{4.3}$$

where $\mu_k \subseteq \mathbb{C}^*$ denotes the cyclic subgroup of order k. It acts faithfully on $\mathrm{Rep}(Q, \mathbf{d})$. For $\mathrm{Rep}(Q, \mathbf{d})$ and $\mathrm{Gl}(Q, \mathbf{d})/Z$ to play the role of $\mathbb{C}^n$ and G_D as in Section 2, we must require

$$\begin{aligned} \sum_{n \in N} d_n^2 - \sum_{\varphi \in A} d_{t\varphi} d_{h\varphi} \\ = \dim_{\mathbb{C}} \mathrm{Gl}(Q, \mathbf{d}) - \dim_{\mathbb{C}} \mathrm{Rep}(Q, \mathbf{d}) = \dim Z = 1. \end{aligned} \tag{4.4}$$

But this equality is not yet sufficient: it is also necessary that $\mathrm{Gl}(Q, \mathbf{d})/Z$ has an open orbit. This occurs if the general representation in $\mathrm{Rep}(Q, \mathbf{d})$ is indecomposable. If both this last condition and (4.4) hold, $\mathbf{d}$ is called a *real Schur root* of Q. In this case, there is a single open orbit, and the *discriminant determinant* Δ defines its complement D, a divisor called the *discriminant*. This is the consequence of a result due to Kraft and Riedtmann [17, §2.6], which asserts that if the general representation is indecomposable it has only scalar endomorphisms. Then

$$\mathrm{Gl}(Q, \mathbf{d})/Z \cong G_D = G_D^\circ. \tag{4.5}$$

The above discussion combined with Theorem 1.6 proves the following

THEOREM 4.1. — *If* $\mathbf{d}$ *is a real Schur root of a quiver* Q *and the discriminant* D *in* $\mathrm{Rep}(Q, \mathbf{d})$ *is reduced then* D *is a linear free divisor that satisfies GLCT.*

In [3] it is shown that if, moreover, Q is a *Dynkin quiver*, i.e. its underlying unoriented graph is a Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8, then Δ is always reduced, and thus defines a linear free divisor. The significance of the Dynkin quivers is, that by a theorem of Gabriel [9], they are the quivers of *finite type*, i.e. the number of $\mathrm{Gl}(Q, \mathbf{d})$ orbits in $\mathrm{Rep}(Q, \mathbf{d})$ is finite. It is this that guarantees that Δ is always reduced, cf. [3, Prop. 5.4]. It also implies that every root of a Dynkin quiver is a real Schur root.

Corollary 4.2. — *If* $\mathbf{d}$ *is a (real Schur) root of a Dynkin quiver* Q *then the discriminant* D *in* $\operatorname{Rep}(Q,\mathbf{d})$ *is a linear free divisor that satisfies GLCT.*

Remark 4.3. — The argument showing that GLCT holds for the free divisors arising as discriminants in quiver representation spaces yields a simple topological proof of a theorem of V. Kac [16, p. 153] (see also [24]): When $\mathbf{d}$ is a sincere (i.e. $\mathbf{d}_x > 0$ for all $x \in Q_0$) real Schur root of a quiver Q with no oriented cycles, the discriminant in $\operatorname{Rep}(Q,\mathbf{d})$ has $|Q_0| - 1$ irreducible components. The proof is this: the number of irreducible components of a divisor in a complex vector space is equal to the rank of H^1 of the complement. From Theorem 1.6 we know that $H^1(\operatorname{Rep}(Q,\mathbf{d}) \smallsetminus D; \mathbb{C}) \simeq H^1_A(\mathfrak{g}_D; \mathbb{C})$; as by (4.3)

$$\mathfrak{g}_D \simeq \mathfrak{sl}_{d_{x_0}}(\mathbb{C}) \oplus \bigoplus_{x \in Q_0 \smallsetminus \{x_0\}} \mathfrak{gl}_{d_x}(\mathbb{C}),$$

it follows that

$$H^1(\operatorname{Rep}(Q,\mathbf{d}) \smallsetminus D; \mathbb{C}) \simeq 0 \oplus \bigoplus_{x \in Q_0 \smallsetminus \{x_0\}} H^1(\mathfrak{gl}_{d_x}(\mathbb{C}); \mathbb{C})$$

and so has rank $|Q_0| - 1$.

Another simple algebraic proof of Kac's theorem was pointed out to us by the referee. It consists in determining the dimension of the vector space of rational function on $\mathbb{C}^n$ with zeroes and poles along D only and lifting them to the group G_D.

5. Examples of linear free divisors

The conclusion of Section 2 guides our search for linear free divisors. Our first example shows that the implication in Theorem 1.6 is not an equivalence.

We denote by

$$E_{ij} = (\delta_{i,k} \cdot \delta_{j,l})_{k,l} \in \mathfrak{gl}_n(\mathbb{C}) \tag{5.1}$$

the elementary matrix with 1 in the ith row and jth column and 0 elsewhere.

5.1. A non-reductive example satisfying GLCT

Example 5.1. — For $n \geqslant 2$, the group B_n of $n \times n$ invertible upper triangular matrices is not reductive. It acts on the space $\operatorname{Sym}_n(\mathbb{C})$ of symmetric matrices by transpose conjugation:

$$B \cdot S = B^t S B.$$

Under the corresponding infinitesimal action, the matrix b in the Lie algebra $\mathfrak{b}_n$ gives rise to the vector field χ_b defined by

$$\chi_b(S) = b^t S + Sb.$$

The dimensions of B_n and $\mathrm{Sym}_n(\mathbb{C})$ are equal. The discriminant determinant Δ is reduced and defines a linear free divisor $D = V(\Delta)$.

To see this, consider an elementary matrix $E_{ij} \in \mathfrak{b}_n$ and let χ_{ij} be the corresponding vector field on $\mathrm{Sym}_n(\mathbb{C})$. If I is the $n \times n$ identity matrix, then $\chi_{ij}(I) = E_{ji} + E_{ij}$. The vectors $\chi_{ij}(I)$ for $1 \leqslant i \leqslant j \leqslant n$ are therefore linearly independent, and $\Delta(I) \neq 0$.

For an $n \times n$ matrix A, let A_j be the $j \times j$ matrix obtained by deleting the last $n-j$ rows and columns of A, and let $\det_j(A) = \det(A_j)$. If $B \in B_n$ and $S \in \mathrm{Sym}_n(\mathbb{C})$, then because B is upper triangular, $(B^t SB)_j = B_j^t S_j B_j$, and so $\det_j(B^t SB) = \det_j(B_j)^2 \det_j(S)$. It follows that the hypersurface $D_j := \{\det_j = 0\}$ is invariant under the action, and the infinitesimal action of B_n on $\mathrm{Sym}_n(\mathbb{C})$ is tangent to each. Thus Δ vanishes on each of them. The sum of the degrees of the D_j as j ranges from 1 to n is equal to $\dim \mathrm{Sym}_n(\mathbb{C})$, and so coincides with the degree of Δ. Hence Δ is reduced, and we conclude, by Lemma 2.4, that $D = D_1 \cup \cdots \cup D_n$ is a linear free divisor. In particular, when $n = 2$, $D \subsetneq \mathrm{Sym}_2(\mathbb{C}) = \mathbb{C}^3$ is the union of a quadric cone and one of its tangent planes.

We now give a proof that GLCT holds for D, in the spirit of the proofs of the preceding section, even though D is not reductive. In fact LCT already follows, by Theorem 1.8, from local quasihomogeneity, which we prove in Subsection 7.2 below.

Proposition 5.2. — *GLCT holds for the discriminant D of the action of B_n on* $\mathrm{Sym}_n(\mathbb{C})$ *in Example 5.1.*

Proof. — The group G_D° is a finite quotient of the group B_n of upper-triangular matrices in $\mathrm{Gl}_n(\mathbb{C})$. There is a deformation retraction of B_n to the maximal torus T consisting of its diagonal matrices, and, with respect to the standard coordinates a_{ij} on matrix space, it follows that $H^*(B_n)$ is isomorphic to the free exterior algebra on the forms da_{ii}/a_{ii}. Each of these is left-invariant, and it follows that the map $\tau_{B_n} : H_A^*(\mathfrak{b}_n;\mathbb{C}) \to H^*(B_n;\mathbb{C})$ from (3.5) is an epimorphism.

Similarly, the Lie algebra complex $\bigwedge^\bullet \mathfrak{b}_n^*$ has a contracting homotopy to its semisimple part. We may consider it as the complex of left-invariant forms on the group B_n. Assign weights $w_1, \dots, w_n$ to the columns and weights $-w_1, \dots, -w_n$ to the rows. This gives the elementary matrix $E_{ij} \in \mathfrak{b}_n$ the weight $w_i - w_j$. If $\varepsilon_{i,j} \in \mathfrak{b}_n^*$ denotes the dual basis and we assign the weight 0 to $\mathbb{C}$ then $\mathrm{wt}(\varepsilon_{i,j}) = -\mathrm{wt}(E_{i,j})$. With respect to the resulting gradings of $\mathfrak{b}_n$ and $\mathfrak{b}_n^*$, both the Lie bracket and the differential d_L of the complex $\bigwedge^\bullet \mathfrak{b}_n^*$ are homogeneous of degree 0, cf. (3.1).

Let $E = \sum_i w_i E_{ii}$, and let $\iota_E : \bigwedge^\bullet \mathfrak{b}_n^* \to \bigwedge^\bullet \mathfrak{b}_n^*$ be the operation of *contraction by* E defined by

$$(\iota_E \omega)(v_1 \wedge \cdots \wedge v_k) := \omega(E \wedge v_1 \wedge \cdots \wedge v_k).$$

Observe that for each generator $E_{ij} \in \mathfrak{b}_n$ we have

$$[E, E_{ij}] = (w_i - w_j) \cdot E_{ij} = \mathrm{wt}(E_{ij}) \cdot E_{ij}. \tag{5.2}$$

We claim that the operation

$$L_E := \iota_E d_L + d_L \iota_E,$$

of taking the Lie derivative along E has the effect of multiplying each homogeneous element of $\bigwedge^\bullet \mathfrak{b}_n^*$ by its w-degree. Indeed the operation L_E is a derivation of degree zero on $\bigwedge^\bullet \mathfrak{b}_n^*$, and the result on 1 forms,

$$L_E(\varepsilon_{i,j}) = (w_j - w_i)\varepsilon_{i,j},$$

is therefore sufficient and can be easily checked by direct calculation.

Thus L_E defines a contracting homotopy from $\bigwedge^\bullet \mathfrak{b}_n^*$ to its w-degree 0 part $\left(\bigwedge^\bullet \mathfrak{b}_n^*\right)_0$, by exactly the same calculation as in Lemma 3.2, but with $\Gamma(U, \Omega^k(\log D))$ and L_χ replaced respectively by $\bigwedge^\bullet \mathfrak{b}_n^*$ and L_E. If we choose $w_1 < \cdots < w_n$ then all off-diagonal members of the basis $\{\varepsilon_{i,j}\}_{1 \leqslant i \leqslant j \leqslant n}$ of $\mathfrak{b}_n^*$ have strictly positive w-degree. It follows that

$$\bigwedge^\bullet \mathfrak{b}_n^* \simeq \Big(\bigwedge^\bullet \mathfrak{b}_n^*\Big)_0 = \bigwedge^\bullet \langle \varepsilon_{1,1}, \dots, \varepsilon_{n,n} \rangle = \bigwedge^\bullet \mathfrak{t}^*$$

where $\mathfrak{t}$ is the Lie algebra of the torus T above. The differential d_L is zero on this subcomplex, showing that τ_{B_n} is an isomorphism. □

5.2. Discriminants of quiver representations

The following example, due to Ragnar-Olaf Buchweitz, is of the type discussed in Section 4.

Example 5.3. — In the space $M_{n,n+1}(\mathbb{C})$ of $n \times (n+1)$ matrices, let D be the divisor defined by the vanishing of the product of the maximal minors. That is, for each matrix $A \in M_{n,n+1}(\mathbb{C})$, let A_j be A minus its j'th column, and let $\Delta_j(A) = \det(A_j)$. Then

$$D = \{A \in M_{n,n+1}(\mathbb{C}) : \delta = \prod_{i=1}^{n+1} \Delta_j(A) = 0\}.$$

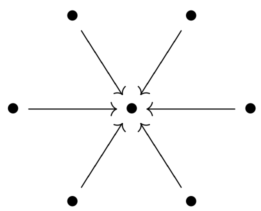
It is a linear free divisor. Here, as the group G in Remark 2.6 we may take the product $Gl_n(\mathbb{C}) \times \{\mathrm{diag}(1, \lambda_1, \ldots, \lambda_n) : \lambda_1, \ldots, \lambda_n \in \mathbb{C}^*\}$, acting by

$$\big(A, \mathrm{diag}(1, \lambda_1, \ldots, \lambda_n)\big) \cdot M = A \cdot M \cdot \mathrm{diag}(1, \lambda_1, \ldots, \lambda_n)^{-1}$$

The placing of the 1 in the first entry of the diagonal matrices is rather arbitrary; it could be placed instead in any other fixed position on the diagonal. That D is a linear free divisor follows from the fact that

(1) the complement of D is a single orbit, so the discriminant determinant is not identically zero, and
(2) the degree of D is equal to the dimension of G_D, so the discriminant determinant is reduced.

In our Example 5.3, $M_{n,n+1}(\mathbb{C})$ is the representation space of the star quiver



consisting of one sink and $n+1$ sources, with dimension vector assigning dimension n to the sink and 1 to each of the sources: Once we have chosen a basis for each V_α, each $V(\varphi)$ is represented by an $n \times 1$ matrix; together they make up an $n \times (n+1)$ matrix. So the basis identifies $\mathrm{Rep}(Q, \mathbf{d}) = M_{n,n+1}(\mathbb{C})$ and then

$$\mathrm{Gl}(Q, \mathbf{d}) = \mathrm{Gl}_n(\mathbb{C}) \times \mathrm{Gl}_1(\mathbb{C})^{n+1} \tag{5.3}$$

and the action in (4.1) becomes

$$(A, \lambda_1, \ldots, \lambda_{n+1}) \cdot M = AM\,\mathrm{diag}(\lambda_1^{-1}, \ldots, \lambda_{n+1}^{-1}). \tag{5.4}$$

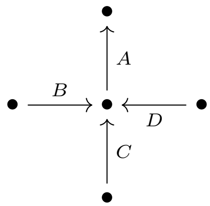
From (4.2), (4.3), and (4.5), result isomorphisms

$$\mathrm{Gl}(Q, \mathbf{d}) \cong G_D \times Z, \quad Z = \mathbb{C}^* \cdot (I_n, (1, \ldots, 1)), \tag{5.5}$$

defined by normalizing an arbitrary element in the second factor.

Many more examples of linear free divisors can be found by similar means in representation spaces of quivers. The next example, also from [3], is a non Dynkin quiver where finiteness of orbits fails and Δ is not reduced.

Example 5.4. — Consider the star quiver of Example 5.3 with $n = 3$ with $\mathbf{d} = (3, 1, 1, 1, 1)$, as before, and now reverse the direction of one of the arrows.



The four hypersurfaces

$$\det(AB) = 0, \det(AC) = 0, \det(AD) = 0, \det(BCD) = 0,$$

are invariant under the action of the subgroup $G_D \subseteq \mathrm{Gl}(Q, \mathbf{d})$ of Example 5.3. However, the last of these is made up of infinitely many orbits: if the images of B, C and D lie in a plane P, then together with $\ker(A \cap P)$ they determine a cross-ratio. The discriminant determinant is equal, up to a scalar factor, to

$$\Delta = \det(AB) \cdot \det(AC) \cdot \det(AD) \cdot (\det(BCD))^2.$$

5.3. Incomplete collections of maximal minors

In the space $M_{m,n}(\mathbb{C})$ of $m \times n$ matrices with $n > m + 1$, the product of all of the maximal minors no longer defines a linear free divisor, by reason of its degree. However, certain collections of n maximal minors do define free divisors. There is a simple procedure for generating infinitely many such collections, first described in [18]:

The space $M_{m,n}(\mathbb{C})$ can still be viewed as $\mathrm{Rep}(Q, \mathbf{d})$ where Q is the star quiver of Example 5.3 with 1 sink and n sources, and $\mathbf{d} = (m, 1, \ldots, 1)$. As before, the quiver group $\mathrm{Gl}(Q, \mathbf{d})$ acts with 1-dimensional kernel Z, but now

$$\dim \mathrm{Gl}(Q, \mathbf{d}) - 1 = m^2 + n - 1 < mn = \dim M_{m,n}(\mathbb{C}),$$

making an open orbit impossible. Therefore we replace $\mathrm{Gl}(Q, \mathbf{d})$ by a group

$$G := \mathrm{Gl}_m(\mathbb{C}) \times G_R \tag{5.6}$$

with $\dim G/Z = \dim M_{m,n}$ by augmenting the second factor in (5.3) to a group $G_R \subseteq \mathrm{Gl}_n(\mathbb{C})$ with $\dim G_R = mn - m^2 + 1$. To construct G_R, we consider an auxiliary quiver $\tilde{Q} = (Q_0, Q_1)$ with $Q_0 = \{1, \dots, n\}$ and $Q_1 \subseteq Q_0^2$ satisfying the following conditions:

- $|Q_1| = mn - m^2 + 1$;
- $(i, j) \in Q_1$ and $(j, k) \in Q_1$ implies that $(i, k) \in Q_1$.
- $(i, i) \in Q_1$ for all $i \in Q_0$;

These conditions are exactly those we need for the following formula:

$$G_R := \mathbb{C}^{Q_1} \smallsetminus \{\det = 0\} \subseteq \mathbb{C}^{n\times n} \smallsetminus \{\det = 0\} = \mathrm{Gl}_n(\mathbb{C}) \tag{5.7}$$

to define a group. We write $(Q_0, Q_1) =: Q(G_R)$. This group G_R is generated by $\mathrm{Gl}_1(\mathbb{C})^n = \mathrm{diag}(\mathbb{C}^*, \dots, \mathbb{C}^*)$ and $mn - m^2 - n + 1$ supplementary elementary matrices $I_n + \mathbb{C} \cdot E_{i,j}$ with $(i, j) \in Q_1$ and $i \neq j$, cf. (5.1).

The action of G on $M_{m,n}(\mathbb{C})$ extends that in (5.4) by right multiplication of G_R and factors through G/Z with $Z = \mathbb{C} \cdot (I_m, I_n)$ which is, as in (4.3), a central quotient

$$G/Z \cong (\mathrm{Sl}_m(\mathbb{C}) \times G_R)/\mu_m \cdot (I_m, I_n) \tag{5.8}$$

where $\mu_k \subseteq \mathbb{C}^*$ denotes the cyclic subgroup of order k.

Proposition 5.5. —
If the discriminant determinant Δ of the action of G is not identically zero and the action of G preserves the divisors of zeros of precisely n distinct $m \times m$ minors, then the union of these divisors is a linear free divisor $D = V(\Delta)$.
If the action of G preserves the divisor of zeros of more than n distinct $m \times m$ minors then Δ is identically zero.

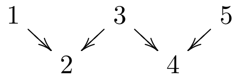
Proof. — Any algebraic set preserved by the action of G is contained in $V(\Delta)$. By construction, if Δ is not identically zero then its degree is mn. If moreover the action of G preserves the zero set of n distinct $m \times m$ minors then Δ is reduced and Lemma 2.4 shows that $V(\Delta)$ is a linear free divisor. □

Lemma 5.6. — *Right multiplication by $I_n + \mathbb{C} \cdot E_{i,j}$ preserves the divisor defined by an $m \times m$ minor if and only if the minor either contains column i of the generic matrix or does not contain column j.*

Proof. — Suppose that the $m \times m$ submatrix M' of the generic $m \times n$ matrix M contains column j but not column i. Let p be a point of $\{\det M' = 0\}$ at which M' has rank $m - 1$ and the matrix M'' obtained from M' by replacing column j by column i has rank m. Then $\det(M' \cdot (I_n + E_{i,j}))(p) \neq 0$.

That is, $\cdot(I_n + \mathbb{C}\cdot E_{i,j})$ does not preserve $\{\det M' = 0\}$. Similarly, $\det M'$ is clearly invariant under $\cdot(I_n + \mathbb{C}\cdot E_{i,j})$ if M' contains both columns i and j. $\square$

Example 5.7. — Let $m = 3$ and $n = 6$. The quiver



determines admissible minors M_{123}, M_{345}, M_{135}, M_{136}, M_{156}, M_{356}, and the associated divisor (the zero locus of their product) is a linear free divisor. Other linear free divisors that can be constructed by these methods are listed, for small values of m, n, in the preprint version [10] of this article. They are not in general reductive.

PROPOSITION 5.8. — *Let $D = V(\Delta)$ be a linear free divisor as constructed above. If $Q(G_R)$ has no oriented loops then GLCT holds for D.*

Proof. — As in the proof of Proposition 5.2, one can show that

$$\tau_{G_R} : H^*_A(\mathfrak{g}_R;\mathbb{C}) \to H^*(G_R;\mathbb{C})$$

from (3.5) is an isomorphism. Here the absence of oriented loops serves as a replacement for the upper triangularity in the preceding proof. Indeed, if there are no oriented loops in $Q(G_R)$, it is possible to order the vertices of $Q(G_R)$, and thus the rows of the matrices in $M_{m,n}$, so that $i < j$ whenever there is an arrow from i to j. This puts all of the matrices of G_R into upper triangular form. It follows both that G_R has a deformation retraction to its maximal torus T consisting of diagonal matrices, and that the same contracting homotopy as in the proof of 5.2 shows that the inclusion $\bigwedge^\bullet \mathfrak{t}^* \to \bigwedge^\bullet \mathfrak{g}_R^*$ is a homotopy equivalence. Thus $H^*(T) : H^*_A(\mathfrak{g}_R;\mathbb{C}) \to H^*(G_R;\mathbb{C})$ is an isomorphism.

Also for $G = \mathrm{Sl}_m(\mathbb{C})$ the map τ_G from (3.5) is an isomorphism. So by applying the the Künneth formulas for both Lie algebra and complex cohomology, the same holds for $G = \mathrm{Sl}_m(\mathbb{C}) \times G_R$. By (5.8), G/Z is connected as a finite quotient of the connected group $\mathrm{Sl}_m(\mathbb{C})\times G_R$. By Proposition 5.5 G_D° is then also a connected finite quotient of G/Z hence of $\mathrm{Sl}_m(\mathbb{C}) \times G_R$, and GLCT holds for D by Corollary 3.4. $\square$

6. Classification in small dimensions

6.1. Structure of logarithmic vector fields

Let $\delta, \xi \in \mathrm{Der}$ and let $g \in \mathscr{O}$. To emphasise the action of δ on $\mathscr{O}$ and on Der, in place of $dg(\delta)$ we write $\delta(g)$, and in place of $[\delta, \xi]$ we write $\delta(\xi) = \mathrm{ad}_\delta(\xi)$. The degree k parts of $\Gamma(\mathbb{C}^n, \mathrm{Der})$ and $\Gamma(\mathbb{C}^n, \mathrm{Der}(-\log D))$ with respect to $\deg(x_i) = 1 = -\deg(\partial_i)$ will be denoted by $\Gamma(\mathbb{C}^n, \mathrm{Der})_k$ and $\Gamma(\mathbb{C}^n, \mathrm{Der}(-\log D))_k$ respectively. For $\delta \in \Gamma(\mathbb{C}^n, \mathrm{Der})_0$, we write δ_S for its semisimple part and δ_N for its nilpotent part.

Let $D \subseteq \mathbb{C}^n$ be a linear free divisor defined by the homogeneous polynomial $\Delta = \det((\delta_i(x_j))_{i,j}) \in \mathbb{C}[x]$ of degree n as in Lemma 1.3 where $\delta = \delta_1, \dots, \delta_n$ is a global degree 0 basis of $\mathrm{Der}(-\log D)$. Then $\delta_i(\Delta) \in \mathbb{C} \cdot \Delta$ and there is the standard Euler vector field $\chi = \sum_i x_i \partial_i \in \langle \delta_1, \dots, \delta_n \rangle_{\mathbb{C}}$. Since $\chi(\Delta)/\Delta = n \neq 0$, we can assume that $\delta_1 = \chi$ and $\delta_i(\Delta) = 0$ for $i = 2, \dots, n$. So $\delta_2, \dots, \delta_n$ is a global degree 0 basis of the annihilator $\mathrm{Der}(-\log \Delta)$ of Δ which is a direct factor of $\mathrm{Der}(-\log D)$.

Since χ vanishes only at the origin, the origin of the affine coordinate system $x = x_1, \dots, x_n$ is uniquely determined. A coordinate change between two degree 0 bases of $\mathrm{Der}(-\log D)$ can always be chosen linear. Among all possible linear coordinate changes, let $s + 1$ be the maximal number of linearly independent diagonal logarithmic vector fields.

Theorem 6.1. — *There exists a global degree* 0 *basis* χ, $\sigma_1, \dots, \sigma_s$, $\nu_1, \dots, \nu_{n-s-1}$ *of* $\mathrm{Der}(-\log D)$ *such that*

(1) $\chi(\sigma_i) = 0$ *and* $\chi(\nu_j) = 0$,
(2) *the* σ_i *are simultaneously diagonalizable with eigenvalues in* $\mathbb{Q}$ *and* $\sigma_i(\Delta) = 0$,
(3) *the* ν_j *are nilpotent and* $\nu_j(\Delta) = 0$,
(4) $\sigma_i(\nu_j) \in \mathbb{Q} \cdot \nu_j$ *and* $\sum_j \sigma_i(\nu_j)/\nu_j + \mathrm{trace}(\sigma_i) = 0$.
(5) *If* $\delta \in \Gamma(\mathbb{C}^n, \mathrm{Der}(-\log D))_0$ *with* $\sigma_i(\delta) = 0$ *for* $i = 1, \dots, s$ *then* $\delta_S \in \langle \sigma_1, \dots, \sigma_s \rangle_{\mathbb{C}}$.

Moreover, $s \geqslant 1$ *and if* $s = n - 1$ *then* $\Delta = x_1 \cdots x_n$ *defines a normal crossing divisor.*

Proof. — It is easy to check that the formal coordinate changes used in [11] reduce to *linear* coordinate changes in the case of *linear* free divisors. Thus (1)-(3), (5), and the first part of (4) follow from [11, Thm. 5.4].

For the second part of (4), we set $\delta_1, \dots, \delta_n = \chi, \sigma_1, \dots, \sigma_s, \nu_1 \dots, \nu_{n-s-1}$ and rewrite Δ as

$$\Delta = \sum_{\alpha \in S_n} \operatorname{sign}(\alpha) \cdot \delta_1(x_{\alpha_1}) \cdots \delta_n(x_{\alpha_n}). \tag{6.1}$$

Let us choose coordinates in which all σ_i are diagonal: $\sigma_i = \sum_j w_{i,j} x_j \partial_j$. The equation $\sigma_i(\Delta) = 0$ means that Δ is weighted homogeneous of degree zero when we assign to the variable x_j the weight $w_{i,j} = \sigma_i(x_j)/x_j$. The weighted degree of $\delta_j(x_k)$ is then $\sigma_i(\delta_j)/\delta_j + w_{i,k}$. This implies the second part of (4), since each term in the sum (6.1) has the same weighted degree $\sum_j \left(\sigma_i(\delta_j)/\delta_j + w_{i\alpha_j}\right) = \sum_j \sigma_i(\delta_j)/\delta_j + \operatorname{trace}(\sigma_i)$.

Now assume that $s = 0$. Then the vector space generated by the ν_i is entirely made of nilpotent elements and we can apply Engel's Theorem [25, I.4], and $\nu_1, \dots, \nu_{n-1}$ can be chosen upper triangular. But then Δ is clearly divisible by the square of the first variable x_1 and hence not reduced. So $s = 0$ is impossible.

If $s = n-1$, then Δ must be a monomial and hence $\Delta = x_1 \cdots x_n$ defines a normal crossing divisor. □

Remark 6.2. — In Theorem 6.1, one can perform the Gauss algorithm on the diagonals of $\sigma_1, \dots, \sigma_s$. Then $\sigma_i \equiv x_i \partial_i \mod \sum_{j=s+1}^n \mathbb{C} \cdot x_j \partial_j$.

We shall frequently use the following simple fact.

LEMMA 6.3. — *Let $\sigma = \sum_i w_i x_i \partial_i$. Then $x_i \partial_j$ is an eigenvector of* ad_σ *for the eigenvalue $w_i - w_j$.*

6.2. The case $s = n - 2$

LEMMA 6.4. — *Let $s = n - 2$ in the situation of Theorem 6.1. Then $-\sigma_k(\nu_1)/\nu_1 = \operatorname{trace}(\sigma_k) \neq 0$ for some k and ν_1 has, after normalization, two entries equal to* 1 *and all other entries equal to* 0.

Proof. — If $s = n - 2$ then for any $\sigma \in \{\sigma_1, \dots, \sigma_{n-2}\}, \sigma(\nu_1)/\nu_1 + \operatorname{trace}(\sigma) = 0$. Hence, a monomial $x_i \partial_j$ in ν_1 gives a relation $w_i - w_j + \sum_k w_k = 0$ on the diagonal entries $w_1, \dots, w_n$ of σ. Since 3 of these relations with $i \neq j$ and also $\sigma_1, \dots, \sigma_{n-2}$ are linearly independent, ν_1 can have at most 2 nonzero nondiagonal entries. If $\sigma_k(\nu_1)/\nu_1 \neq 0$ for some k then ν_1 is strictly triangular with at most 2 nonzero entries after ordering the diagonal of σ_k. If ν_1 has only one nonzero entry then Δ is divisible by the square of a variable, a contradiction. Both nonzero entries of ν_1 can be normalized to 1. If $\sigma_k(\nu_1)/\nu_1 = 0$ for all k then the nonzero entries of ν_1 are in a 2-dimensional simultaneous eigenspace of $\chi, \sigma_1, \dots, \sigma_{n-2}$. Otherwise, there are 3 linearly independent relations $w_{i_1} = w_{j_1}$, $w_{i_2} = w_{j_2}$, $\sum_k w_k = 0$ on the diagonal entries of $\sigma_1, \dots, \sigma_{n-2}$, a contradiction to the linear independence of these vector fields. But then ν_1 has only one nonzero entry after a linear coordinate change, a contradiction as before. $\square$

To simplify the notation, we shall write $\equiv$ for equivalence modulo $\mathbb{C}^*$. By Lemma 6.4, we may assume that $\nu_1 = x_k\partial_1 + x_l\partial_2$ where $1 \neq k \neq l \neq 2$. Then

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ a_{2,1}x_1 & a_{2,2}x_2 & a_{2,3}x_3 & \cdots & a_{2,n}x_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1,1}x_1 & a_{n-1,2}x_2 & a_{n-1,3}x_3 & \cdots & a_{1,n}x_n \\ x_k & x_l & 0 & \cdots & 0 \end{vmatrix} \equiv (x_2x_k - x_1x_l)x_3\cdots x_n.$$

As Δ is reduced, there are, up to coordinate changes, only two non-normal-crossing cases:

6.2.1. $k = 2$, $l = 3$

Then Δ comes from the linear free divisor of Example 5.1 in dimension 3:

$$\Delta = (x_2^2 - x_1x_3)x_3\cdots x_n \equiv \begin{vmatrix} x_1 & x_2 & x_3 \\ 4x_1 & x_2 & -2x_3 \\ 2x_2 & x_3 & 0 \end{vmatrix} \cdot x_4\cdots x_n.$$

6.2.2. $k = 3$, $l = 4$

Then Δ comes from a linear free divisor in dimension 4:

$$\Delta = (x_2x_3 - x_1x_4)x_3\cdots x_n \equiv \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & 2x_2 & -x_3 & 0 \\ 2x_1 & x_2 & 0 & -x_4 \\ x_3 & x_4 & 0 & 0 \end{vmatrix} \cdot x_5\cdots x_n.$$

6.3. Classification up to dimension 4

We consider the situation of Theorem 6.1 and abbreviate $x, y, z, w = x_1, x_2, x_3, x_4$. By the results of Section 6.2, we may assume that $s = 1$ and $n = 4$. Let us first assume that $\Gamma(\mathbb{C}^n, \mathrm{Der}(-\log D))_0$ is a nonsolvable Lie algebra and hence $\langle \sigma_1, \nu_1, \nu_2 \rangle = \mathfrak{sl}_2$.

Recall that by [25, IV.4], $\mathbb{C}^4$ is a direct sum of irreducible $\mathfrak{sl}_2$-modules W_m of dimension $m+1$ and that W_m is represented in a basis $e_0, \ldots, e_m$ by

$$\sigma_1(e_i) = (-m + 2i)e_i, \quad \nu_1(e_i) = (i+1)e_{i+1}, \quad \nu_2(e_i) = (m - i + 1)e_{i-1}.$$

So there are 3 types of $\mathfrak{sl}_2$-representations. The first two cases are $\mathbb{C}^4 = W_2 \oplus W_0$ and $\mathbb{C}^4 = W_1 \oplus W_1$, which lead to a zero and a nonreduced determinant of the form $\Delta \equiv (xw - yz)^2$ respectively. But W_3 gives the nontrivial linear free divisor

$$\Delta = \begin{vmatrix} x & y & z & w \\ -3x & -y & z & 3w \\ y & 2z & 3w & 0 \\ 0 & 3x & 2y & z \end{vmatrix} \equiv y^2z^2 - 4xz^3 - 4y^3w + 18xyzw - 27x^2w^2$$

isomorphic to the discriminant in the space of binary cubics described in Example 1.4.(1.4).

Now, assume that $\Gamma(\mathbb{C}^n, \mathrm{Der}(-\log D))_0$ is a solvable Lie algebra. Then, by Lie's Theorem [25, I.7], ν_1 and ν_2 can be chosen triangular and also $[\nu_1, \nu_2]$ is triangular. Hence, $[\nu_1, \nu_2] \in \langle \nu_1, \nu_2 \rangle$ and even $[\nu_1, \nu_2] = 0$ by nilpotency of ad_{ν_1} and ad_{ν_2}.

We set

$$\sigma_1 = ax\partial_x + by\partial_y + cz\partial_z + dw\partial_w$$

and start a case by case discussion with respect to the cardinality of $\{a, b, c, d\}$. In the following we shall omit the details that can be found in the preprint version [10] of this article.

6.3.1. $2 \leqslant |\{a, b, c, d\}| \leqslant 3$

In each case we refine to subcases depending on whether $\sigma_1(\nu_i)/\nu_i$, $i = 1, 2$, is zero or not. All these subcases lead to $\Delta = 0$ or Δ being divisible by a square of a variable, in contradiction to our assumptions.

6.3.2. $|\{a,b,c,d\}| = 4$

Since ν_1 and ν_2 are σ_1-homogeneous and might be chosen triangular, $\sigma(\nu_1)/\nu_1 \neq 0 \neq \sigma(\nu_2)/\nu_2$ and hence ν_1 and ν_2 have at most 3 nonzero entries by Lemma 6.3. Using, if necessary, permutations of the basis vectors, we have only to consider the following two cases:

6.3.2.1. ν_1 has one nonzero term or ν_1 and ν_2 have at most two nonzero terms. In both cases, a detailed discussion leads to the contradiction that Δ is divisible by a square of a variable.

6.3.2.2. ν_1 has three nonzero terms. This turns out to be the only case that leads to a linear free divisor with solvable Lie algebra and $s = 1$. We may assume that

$$\nu_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which implies also that $(a,b,c,d) = a \cdot (1,1,1,1) + (0,\lambda,2\lambda,3\lambda)$ for some $0 \neq \lambda \in \mathbb{Q}$. The relation $[\nu_1,\nu_2] = 0$ then implies that

$$\nu_2 = p \cdot \nu_1 + \begin{pmatrix} 0 & 0 & q & r \\ 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So using the σ_1-homogeneity of ν_2 the only case which was not yet considered is

$$\nu_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Delta \equiv \begin{vmatrix} x & y & z & w \\ 0 & \lambda y & 2\lambda z & 3\lambda w \\ 0 & x & y & z \\ 0 & 0 & x & y \end{vmatrix} \equiv x(y^3 - 3xyz + 3x^2 w).$$

The trace equation in Theorem 6.1.4 for σ_1 reads $-\lambda - 2\lambda + 4a + 6\lambda = 0$ or $a = -\frac{3}{4}\lambda$. Setting $\lambda = 4$, we obtain $\sigma_1 = -3x\partial_x + y\partial_y + 5z\partial_z + 9w\partial_w$.

6.4. Summary of the classification up to dimension 4

The following table summarizes our classification of linear free divisors up to dimension 4. The matrices are interpreted row-wise as bases of $\mathrm{Der}(-\log D)$.

n	Δ	$\mathrm{Der}(-\log D)$	$\mathfrak{g}_D$	reductive?
1	x	$\begin{pmatrix} x \end{pmatrix}$	$\mathbb{C}$	Yes
2	xy	$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	$\mathbb{C}^2$	Yes
3	xyz	$\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$	$\mathbb{C}^3$	Yes
3	$(y^2+xz)z$	$\begin{pmatrix} x & y & z \\ 4x & y & -2z \\ -2y & z & 0 \end{pmatrix}$	$\mathfrak{b}_2$	No
4	$xyzw$	$\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{pmatrix}$	$\mathbb{C}^4$	Yes
4	$(y^2+xz)zw$	$\begin{pmatrix} x & y & z & 0 \\ 4x & y & -2z & 0 \\ -2y & z & 0 & 0 \\ 0 & 0 & 0 & w \end{pmatrix}$	$\mathbb{C} \oplus \mathfrak{b}_2$	No
4	$(yz+xw)zw$	$\begin{pmatrix} x & 0 & 0 & -w \\ 0 & y & 0 & w \\ 0 & 0 & z & w \\ z & -w & 0 & 0 \end{pmatrix}$	$\mathbb{C}^2 \oplus \mathfrak{g}_0$	No
4	$x(y^3-3xyz+3x^2w)$	$\begin{pmatrix} x & y & z & w \\ 0 & y & 2z & 3w \\ 0 & x & y & z \\ 0 & 0 & x & y \end{pmatrix}$	$\mathbb{C} \oplus \mathfrak{g}$	No
4	$y^2z^2-4xz^3-4y^3w+$ $18xyzw-27w^2x^2$	$\begin{pmatrix} 3x & 2y & z & 0 \\ 0 & 3x & 2y & z \\ y & 2z & 3w & 0 \\ 0 & y & 2z & 3w \end{pmatrix}$	$\mathfrak{gl}_2(\mathbb{C})$	Yes

The annihilators of Δ in the Lie algebras for $\Delta = (yz+xw)zw$ and $\Delta = x(y^3-3xyz+3x^2w)$ are described in [15, Ch. I, §4]. The former is the direct sum of $\mathbb{C}$ and the non-Abelian Lie algebra $\mathfrak{g}_0$ of dimension 2, and the latter is the 3-dimensional Lie algebra $\mathfrak{g}$ characterized as having 2-dimensional Abelian derived algebra $\mathfrak{g}'$, on which the adjoint action of a basis vector outside $\mathfrak{g}'$ is semi-simple with eigenvalues 1 and 2. Straightforward computations show that the two *groups* G_D° are, respectively, the set of 4×4 matrices of the form

$$\begin{pmatrix} x^{-1}y^{-2} & 0 & z & 0 \\ 0 & x^{-2}y^{-1} & 0 & -x^{-1}yz \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \end{pmatrix}$$

and

$$\begin{pmatrix} x^{-3} & 0 & 0 & 0 \\ y & x & 0 & 0 \\ z & x^4y & x^5 & 0 \\ x^3yz-\frac{1}{3}x^6y^3 & x^4z & x^8y & x^9 \end{pmatrix}$$

with $x,y\in\mathbb{C}^*$ and $z\in\mathbb{C}$ in the first and $x\in\mathbb{C}^*, y,z\in\mathbb{C}$ in the second.

7. Strong Euler homogeneity and local quasihomogeneity

In this section, we investigate linear free divisors with respect to the properties of local quasihomogeneity and strong Euler homogeneity from Definitions 1.7 and 1.9.

The following reformulation of the definition of local quasihomogeneity is a direct consequence of the Poincaré–Dulac Theorem [1, Ch. 3, §3.2] and Artin's Approximation Theorem [2].

Theorem 7.1. — *A divisor D is locally quasihomogeneous if and only if, at any $p\in D$, there is an Euler vector field χ for D at p whose degree zero part χ_0 has strictly positive eigenvalues.*

We denote by $\mathscr{D}=\mathscr{D}_{\mathbb{C}^n}$ the sheaf of germs of linear differential operators with holomorphic coefficients on $\mathbb{C}^n$. It is naturally equipped with an increasing filtration F of coherent $\mathscr{O}$-modules by the order of differential operators and we denote by $\sigma(P)$ the symbol of $P\in\mathscr{D}$ in $\operatorname{gr}_F\mathscr{D}$. Note that $\operatorname{gr}_F\mathscr{D}_p\cong\mathscr{O}_p[\partial]=\mathbb{C}\{x\}[\partial]$ in a local coordinate system x at p, where we identify $\sigma(\partial_i)=\partial_i$.The following property is closely related to local quasihomogeneity.

Definition 7.2. — *A free divisor D is called* Koszul free *if, at any $p\in D$, there exists a basis $\delta_1,\dots,\delta_n$ of* $\operatorname{Der}(-\log D)_p$ *such that $\sigma(\delta_1),\dots,\sigma(\delta_n)$ is a regular sequence in $\operatorname{gr}_F\mathscr{D}_p$.*

Koszul freeness can be interpreted geometrically in terms of the *logarithmic stratification*, introduced by K. Saito [22], which is the partition of D into the integral varieties of the distribution $\operatorname{Der}(-\log D)$. As it is not always locally finite, the term "stratification" is a misnomer, but is generally used. If D_α is a stratum of the logarithmic stratification and $p \in D_\alpha$ then $T_p D_\alpha = \operatorname{Der}(-\log D)(p)$. The graded ring $\operatorname{gr}_F \mathscr{D}_p \cong \mathscr{O}_p[\partial]$ contains $\operatorname{Der}_p = \oplus_{i=1}^n \mathscr{O}_p \partial_i$ and can be identified with the ring of functions on the cotangent space $T^*(\mathbb{C}^n, p)$ of the germ $(\mathbb{C}^n, p)$, polynomial on the fibers and analytic on the base.

Definition 7.3. — *The* logarithmic characteristic variety $L_{\mathbb{C}^n}(D)$ *of* D *is the variety in* $T^*\mathbb{C}^n$ *defined by the image of* $\operatorname{Der}(-\log D)$ *in* $\operatorname{gr}_F \mathscr{D}$.

Thus D is Koszul free at p if and only if $L_{\mathbb{C}^n}(D)$ is purely n-dimensional [4, 1.8]. Moreover, by [22, 3.16], $L_{\mathbb{C}^n}(D)$ is the union, over all strata $D_\alpha \subseteq D$ in the logarithmic stratification of D, of the conormal bundle $T^*_{D_\alpha}\mathbb{C}^n$ of D_α, each of which is n-dimensional. This proves the following result.

Theorem 7.4. — *A free divisor D is Koszul free if and only if the logarithmic stratification is locally finite.*

A locally quasihomogeneous free divisor is Koszul free [4, 4.3], and thus the failure of Koszul-freeness serves as a computable criterion for the failure of local quasihomogeneity.

As in Section 6.1, let $D \subseteq \mathbb{C}^n$ be a linear free divisor and $\chi, \delta_2, \ldots, \delta_n$ a global degree 0 basis of $\operatorname{Der}(-\log D)$ with $\delta_i(\Delta) = 0$ for $\Delta = \det(S)$ where

$$S := \begin{pmatrix} \chi(x_j) \\ \delta_i(x_j) \end{pmatrix}_{i,j}$$

The arguments which follow are valid as well for an arbitrary germ of a free divisor $D \subseteq (\mathbb{C}^n, 0)$ with a germ of an Euler vector field $\chi \in \operatorname{Der}_0$ at 0.

The following criterion gives a method to test strong Euler homogeneity algorithmically. The reduced variety S_k defined by the $(k+1) \times (k+1)$-minors of the $n \times n$-matrix S is the union of logarithmic strata of dimension at most k. In more invariant terms, S_k is the variety of zeros of the $(n-k-1)$'st Fitting ideal of the $\mathscr{O}_{\mathbb{C}^n}$-module $\operatorname{Der} / \operatorname{Der}(-\log D)$. Thus a free divisor D is Koszul free if and only if $\dim S_k \leqslant k$ for all k. Note that $S_n = \mathbb{C}^n$, $S_{n-1} = D$, and $S_{n-2} = \operatorname{Sing}(D)$. For a linear free divisor, $S_0 = \{0\}$ because of the presence of the Euler vector field. Since $\dim \operatorname{Sing}(D) < \dim D$, it follows that linear free divisors are Koszul free in dimension $n \leqslant 3$.

In order to characterize strong Euler homogeneity, we also consider the reduced variety $T_k \supseteq S_k$ defined by the $(k+1) \times (k+1)$-minors of the $(n-1) \times n$-matrix $T := (\delta_i(x_j))_{i,j}$. Again, this is the variety defined by a Fitting ideal, this time the $(n-k-2)$nd Fitting ideal of the module

$\mathrm{Der} / \mathrm{Der}(-\log \Delta)$. Note that, by definition,

$$S = \begin{pmatrix} x \\ T \end{pmatrix}.$$

LEMMA 7.5. — *D is strongly Euler homogeneous if and only if $S_k = T_k$ for $0 \leqslant k \leqslant n-2$.*

Proof. — A vector field $\delta \in \left(\mathfrak{m}_p \cdot \chi + \sum_i \mathscr{O}_p \cdot \delta_i\right) \cap \mathfrak{m}_p \cdot \mathrm{Der}_p$ is not an Euler vector field at $p \in D$ since $\delta(\Delta) \in \mathfrak{m}_p \cdot \Delta$, and indeed $\delta(u \cdot \Delta) \in \mathfrak{m}_p \cdot u\Delta$ for any unit u. Hence, an Euler vector field η for D at p must be of the form $\eta = a_0 \cdot \chi + \sum_i a_i \cdot \delta_i \in \mathfrak{m}_p \cdot \mathrm{Der}_p$ with $a_0(0) \neq 0$. This means that $\chi(p) \in \sum_i \mathbb{C} \cdot \delta_i(p)$, and the matrices S and T have equal rank at p. Conversely if $\chi(p) = \sum \lambda_i \delta_i(p)$ then up to multiplication by a scalar, $\chi - \sum_i \lambda_i \delta_i$ is an Euler field for D at p. □

Remark 7.6. — The proof of Lemma 7.5 shows that the question of local quasihomogeneity is much more complicated: The degree zero parts of Euler vector fields at a point $p \in D$ are the degree zero parts of vector fields $a_1 \cdot \chi + \sum_i a_i \cdot \delta_i$ where $a_1, \dots, a_n$ are linear forms such that $a_1(p) \cdot \chi(p) + \sum_i a_i(p) \cdot \delta_i(p) = 0$.

For $k = 1, \dots, n$, let $M_k = (-1)^{k+1} \det(\delta_i(x_j))_{j \neq k}$.

LEMMA 7.7. — *For $k = 1, \dots, n$, $\partial_k(\Delta) = n \cdot M_k$. In particular, $S_k = T_k$ for $k = n-2$.*

Proof. — Since

$$S \begin{pmatrix} \partial_1(\Delta) \\ \vdots \\ \partial_n(\Delta) \end{pmatrix} = \begin{pmatrix} \chi \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix} (\Delta) = \begin{pmatrix} n \cdot \Delta \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

we obtain, by canceling Δ,

$$\begin{pmatrix} \partial_1(\Delta) \\ \vdots \\ \partial_n(\Delta) \end{pmatrix} = \check{S} \begin{pmatrix} n \\ 0 \\ \vdots \\ 0 \end{pmatrix} = n \cdot \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix}$$

where $\check{S}$ denotes the cofactor matrix of S. □

LEMMA 7.8. — *$S_0 = T_0$.*

Proof. — Assume that $T_0 \neq S_0 = \{0\}$. By homogeneity, T_0 contains the x_n-axis after an appropriate linear coordinate change. Then T is independent of x_n. Writing $x' = x_1, \dots, x_{n-1}$, we have that $\Delta = g + x_n \cdot \Delta'$ where g and $\Delta' := M_n$ depend only on x'. Since Δ does not depend on fewer variables, we must have $\Delta' \neq 0$. For $i = 2, \dots, n$, let $\delta'_i := \sum_{j=1}^{n-1} \delta_i(x_j)\partial_j$ be the projection of δ_i to the $\mathbb{C}[x]$-module with basis $\partial' := \partial_1, \dots, \partial_{n-1}$. Then, for $i = 2, \dots, n$, $\delta'_i(\Delta') = 0$, as it is the coefficient of x_n in $\delta_i(\Delta) = 0$. Since the rank of the $\mathbb{C}[x']$-annihilator of $\partial_1(\Delta'), \dots, \partial_{n-1}(\Delta')$ is strictly smaller than $n-1$, there must be a relation $\sum_{i=2}^n a_i\delta'_i = 0$ for some homogeneous polynomials $a_i \in \mathbb{C}[x']$. But since $\delta_2, \dots, \delta_n$ are independent over $\mathbb{C}[x]$, $\sum_{i=2}^n a_i\delta_i(x_n) \neq 0$ and hence $0 = \sum_{i=2}^n a_i\delta_i(\Delta) = \left(\sum_{i=2}^n a_i\delta_i(x_n)\right) \cdot \Delta'$, contradicting the fact that $\Delta' \neq 0$. □

LEMMA 7.9. — *Let D be strongly Euler homogeneous. Then D is locally quasihomogeneous on the complement of S_{n-3}. In particular, D is locally quasihomogeneous if $S_{n-3} = \{0\}$.*

Proof. — By [22, 3.5], $(D, p) = (D', p') \times (\mathbb{C}^{n-2}, 0)$ for $p \in S_{n-2} \smallsetminus S_{n-3}$ where $(D', p') \subseteq (\mathbb{C}^2, 0)$ is strongly Euler homogeneous by [11, 3.2]. As the germ of a curve, (D', p') has an isolated singularity. Then (D', p'), and hence (D, p), are quasihomogeneous, by Saito's theorem [21]. □

THEOREM 7.10. — *Every linear free divisor in dimension $n \leqslant 4$ is locally quasihomogeneous and hence LCT and GLCT hold.*

Proof. — By Lemmas 7.7 and 7.8, $S_1 = T_1$ if $n = 3$ and $S_0 = T_0$. If $n \leqslant 3$ then D is strongly Euler homogeneous by Lemma 7.5 and so locally quasihomogeneous by Lemma 7.9.

For $n = 4$ analogous arguments yield $S_0 = T_0 = \{0\}$ and $S_2 = T_2$. Now we use the classification in Subsection 6.4 and a case by case study: In each case, one can verify that $S_1 = T_1$ and construct an Euler vector field with positive eigenvalues in degree zero at each point of $S_1 \smallsetminus S_0$. Again this is sufficient for local quasihomogeneity by Lemma 7.9. For $\Delta = (yz + xw)zw$, $S_1 = \{xy = z = w = 0\}$ and $2\chi - \sigma + \frac{x-\xi}{\xi}\sigma$, where $\sigma = 2x\partial_x + y\partial_y - w\partial_w$, is an Euler vector field at $(\xi, 0, 0, 0) \notin S_0$ with eigenvalues $2, 1, 2, 3$ in degree zero. For $\Delta = x(y^3 - 3xyz + 3x^2w)$, $S_1 = \{x = y = z = 0\}$ and $9\chi - \sigma + \frac{w-\omega}{\omega}\sigma$, where $\sigma = -3x\partial_x + y\partial_y + 5z\partial_z + 9w\partial_w$, is an Euler vector field at $(0, 0, 0, \omega) \notin S_0$ with eigenvalues $12, 8, 4, 9$ in degree zero. The remaining cases are trivial.

By [7], local quasihomogeneity implies that LCT and hence, by taking global sections, GLCT holds. □

7.1. Example 5.3 again

In this subsection, we study the linear free divisor in Example 5.3 in detail and show that if $n > 2$ it is not Koszul free and hence not locally quasihomogeneous. However we will see that it is strongly Euler homogeneous, like all other linear free divisors whose strong Euler homogeneity has been investigated.

Denote by $x_{i,j}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n+1$, the coordinates on the space of $n \times (n+1)$-matrices $M_{n,n+1}$. The Lie group $G = \mathrm{Gl}_n(\mathbb{C}) \times \mathrm{Gl}_1(\mathbb{C})^{n+1}$ acts on $M_{n,n+1}$ by left matrix multiplication of $\mathrm{Gl}_n(\mathbb{C})$ and multiplication of the jth factor $\mathrm{Gl}_1(\mathbb{C}) = \mathbb{C}^*$ on the jth column of members of $M_{n,n+1}$. By Lemma 2.2, $\mathrm{Der}(-\log D)$ is generated by the infinitesimal action of the Lie algebra of G and hence a basis of $\mathrm{Der}(-\log D)$ is extracted from the set of n^2+n+1 vector fields

$$\xi_{i,j} = \sum_{k=1}^{n+1} x_{i,k}\partial_{j,k}, \text{ for } 1 \leqslant i,j \leqslant n, \tag{7.1}$$
$$\xi_i = \sum_{l=1}^{n} x_{l,i}\partial_{l,i}, \text{ for } 1 \leqslant i \leqslant n+1,$$

by omitting one because of the relation

$$\chi = \sum_{i=1}^{n} \xi_{i,i} = \sum_{j=1}^{n+1} \xi_j$$

corresponding to the Lie algebra of the kernel of the action. Note that the vector field $\xi_{i,i}$ resp. ξ_j is the Euler vector field related to the ith row resp. to the jth column of the general $n \times (n+1)$-matrix and that χ is the global Euler vector field on $M_{n,n+1}$.

Since the determinant Δ_j has degree one with respect to each line and to each column except the jth column for which the degree is zero, the degree of Δ equals $n+1$ with respect to a row and n with respect to a column. These considerations yield

$$\xi_{i,i}(\Delta) = (n+1)\Delta, \quad \xi_j(\Delta) = n\Delta, \quad \xi_{i,j}(\Delta) = 0 \text{ for } i \neq j,$$

and one can easily derive a basis of the vector fields annihilating Δ.

The following lemma is self evident by definition of the action of G on $M_{n,n+1}$ and we shall use it implicitly. In particular, the rank of a G-orbit is well-defined as the rank of any of its elements.

Lemma 7.11. —

Two matrices in $M_{n,n+1}$ having the same row space are in the same G-orbit. Similarly two matrices given by lists of column vectors $A = C_1, \dots,$ C_{n+1} and $A' = C'_1, \dots, C'_{n+1}$ are in the same G-orbit if there is a $\lambda_j \in \mathbb{C}^*$ such that $C'_j = \lambda_j C_j$ for all $j = 1, \dots, n+1$.
If A and A' are in the same G-orbit in $M_{n,n+1}$ then any submatrix of A consisting of columns $C_{i_1}, \dots, C_{i_p}$ has the same rank as the submatrix of A' consisting of the corresponding columns $C'_{i_1}, \dots, C'_{i_p}$ of A'. In particular A and A' have the same rank.

By the left action of $\mathrm{Gl}_n(\mathbb{C}) \subseteq G$, any G-orbit in rank r contains, up to permutation of columns, an element of the form

$$\begin{pmatrix} 1 & \dots & 0 & x_{1,r+1} & \dots & x_{1,n+1} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & 1 & x_{r,r+1} & \dots & x_{r,n+1} \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \tag{7.2}$$

By using also the action of G, we may assume that $x_{i,r+1} \in \{0,1\}$: If $x_{i,r+1} \neq 0$ then one can divide the ith row by $x_{i,r+1}$ and multiply the ith column by $x_{i,r+1}$. Thus there is only a finite number of maximal rank G-orbits including the generic orbit for which all $x_{i,n+1}$ equal 1.

Proposition 7.12. —
There are only finitely many G-orbits in $M_{2,3}$, and the linear free divisor $D \subseteq M_{2,3}$ is locally quasihomogeneous(1).
The number of G-orbits in the linear free divisor $D \subseteq M_{3,4}$ is infinite. In particular, the set of G-orbits in D is not locally finite, and D is not Koszul free and hence not locally quasihomogeneous.

Proof. —
The first statement follows, by Gabriel's theorem [9], from the fact that we are considering the representation space of a Dynkin quiver, here of type D_4. In fact, in the case of $M_{2,3}$, the only orbits which remain to be considered are $\{0\}$ and the rank one orbits which contain, up to permutation of columns, one of the typical elements:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(1) See Remark 7.19 below.

At each point $x \neq 0$, D is isomorphic to the product of the germ at x of the orbit I_x of x, and the germ at x of $D' := D \cap T$, where T is a smooth transversal to I_x of complementary dimension. Since T is logarithmically transverse to D in the neighborhood of x, D' is a free divisor. By the Cancellation Property for products of analytic spaces [13], (D', x) is determined up to isomorphism by the fact that $(D, x) \simeq I_x \times (D', x)$, so it does not matter which transversal to I_x we choose. In the table below, we take T to be affine. The local equations of D shown in the last column are simply the restriction of the original equation of D to the transversal T. By inspection of these equations, D is locally quasihomogeneous.

Representative	Transversal	Local Equation
$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & x_{12} & 0\\ 0 & x_{22} & 1\end{pmatrix}$	$x_{21}x_{22} = 0$
$\begin{pmatrix}1 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 1\\ x_{21} & x_{22} & 0\end{pmatrix}$	$x_{21}x_{22}(x_{22} - x_{21}) = 0$
$\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 1 & x_{13}\\ 0 & x_{22} & x_{23}\end{pmatrix}$	$x_{22}x_{23}(x_{23} - x_{22}x_{13}) = 0$
$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}1 & x_{12} & x_{13}\\ 0 & x_{22} & x_{23}\end{pmatrix}$	$x_{22}x_{23}(x_{12}x_{23} - x_{22}x_{13}) = 0$

In the case of $M_{3,4}$, consider the stratum in D consisting of matrices of rank 2. The four columns span a 2-dimensional plane, and assuming they are pairwise independent, determine four lines in this plane. The cross ratio of these four lines is a G_D invariant: quadruples spanning the same plane, but with different cross-ration, cannot be equivalent. Thus there are infinitely many orbits. Now by [4, 4.3] D is not locally quasihomogeneous. □

Proposition 7.13. — *The linear free divisor $D \subseteq M_{n,n+1}$ from Example 5.3 is strongly Euler homogeneous for any n.*

Proof. — Let us consider a rank r orbit of G in $M_{n,n+1}$. If $r < n$, we can find a point A in this orbit with a zero row, say row number i. Then the Euler vector field $\xi_{i,i}$ of this row is an Euler vector field at A.

If $r = n$ we can assume that A is of the form (7.2) with $x_{i,n+1} = 1$ for $1 \leqslant i \leqslant s$ and $x_{i,n+1} = 0$ for $s+1 \leqslant i \leqslant n$ for some $s \leqslant n$. Then by (7.1) the space parametrized by the variables $x_{i,n+1}$ with $s+1 \leqslant i \leqslant n$ is a smooth transversal to the orbit at A and the restricted equation of D is just $x_{s+1,n} \cdots x_{n,n+1} = 0$. Thus D is normal crossing and hence strongly Euler homogeneous. □

7.2. Example 5.1 again

In this subsection, we show that the linear free divisors in Example 5.1 are locally quasihomogeneous and hence Koszul free by [4, 4.3]. By [7], this implies that LCT holds although the defining group is not reductive.

We denote by $x_{i,j}$, $1 \leqslant i \leqslant j \leqslant n$, the coordinates on the space of symmetric $n \times n$-matrices $\mathrm{Sym}_n(\mathbb{C}) \subseteq M_{n,n}$. Let $D \subseteq \mathrm{Sym}_n(\mathbb{C})$ be the divisor defined by the product

$$\Delta = \det_1 \cdots \det_n$$

of minors

$$\det_k = \begin{vmatrix} x_{1,1} & \cdots & x_{1,k} \\ \vdots & & \vdots \\ x_{k,1} & \cdots & x_{k,k} \end{vmatrix}.$$

By Example 5.1, the group $B_n \subseteq \mathrm{Gl}_n(\mathbb{C})$ of upper triangular matrices acts on $\mathrm{Sym}_n(\mathbb{C})$ by transpose conjugation

$$B \cdot S = B^t S B, \text{ for } B \in B_n, S \in \mathrm{Sym}_n(\mathbb{C})$$

and the discriminant D is a linear free divisor. Thus, $\mathrm{Der}(-\log D)$ can be identified with the Lie algebra of B_n and has a basis consisting of the $\frac{1}{2}n(n+1)$ vector fields

$$\xi_{i,j} = x_{1,i}\frac{\partial}{\partial x_{1,j}} + \cdots + x_{i,i}\frac{\partial}{\partial x_{i,j}} + \cdots + 2x_{i,j}\frac{\partial}{\partial x_{j,j}} + \cdots + x_{i,n}\frac{\partial}{\partial x_{j,n}} \quad \text{for } 1 \leqslant i \leqslant j \leqslant n.$$

It may be helpful to view this as the symmetric matrix

$$\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 & x_{1,i} & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & x_{i,i} & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & x_{i,j-1} & 0 & \cdots & 0 \\ x_{1,i} & \cdots & x_{i,i} & \cdots & x_{i,j-1} & 2x_{i,j} & x_{i,j+1} & \cdots & x_{i,n} \\ 0 & \cdots & 0 & \cdots & 0 & x_{i,j+1} & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & x_{i,n} & 0 & \cdots & 0 \end{pmatrix}$$

in which all the nonzero elements lie in the j'th row and the j'th column. Note that the Euler vector field is

$$\chi = \frac{1}{2}\sum_{i=1}^{n} \xi_{i,i}.$$

For $i < j$, $\xi_{i,j}$ is nilpotent, so that $\xi_{i,j}(\Delta) = 0$. The vector field $\xi_{i,i}$ is the infinitesimal generator of the $\mathbb{C}^*$ action in which the i'th row and column are simultaneously multiplied by $\lambda \in \mathbb{C}^*$. It follows that each determinant $\det_k$ with $k \geqslant i$ is homogeneous of degree 2 with respect to $\xi_{i,i}$, and we conclude that

$$\xi_{i,i}(\Delta) = 2(n-i+1)\Delta, \quad \xi_{i,j}(\Delta) = 0 \text{ for } i < j.$$

Lemma 7.14. — *There are finitely many B_n-orbits in* $\mathrm{Sym}_n(\mathbb{C})$.

Proof. — If $i \leqslant j$, the pair of elementary row and column operations ("add c times column i to column j", "add c times row j to row i") can be effected by the action of B_n. By such operations any symmetric matrix may be brought to a normal form with at most a single nonzero element in each row and column. Another operation in B_n changes each of these nonzero elements to a 1. Thus there are only finitely many B_n-orbits in $\mathrm{Sym}_n(\mathbb{C})$. □

By the discussion at the start of Section 7, it follows that D is Koszul free. In fact this will also follow from

Proposition 7.15. — *The linear free divisor D of Example 5.1 associated with the action of B_n on* $\mathrm{Sym}_n(\mathbb{C})$ *is locally quasi-homogeneous.*

To prove this, it is enough to show that at each point S of D there is an element of $\mathrm{Der}(-\log D)_S$ which vanishes at S and whose linear part is diagonal with positive eigenvalues. This is the result of the proposition below. In what follows we fix a symmetric matrix S such that $s_{i,j} \in \{0,1\}$, with at most one nonzero coefficient in each row and column. By Lemma 7.14, each B_n orbit contains such a matrix, and local quasihomogeneity is preserved by the B_n action, so it is enough to construct a vector field of the required form in the neighborhood of each such matrix S.

Lemma 7.16. — *Assume that $s_{i,j} = 1$ with $i \leqslant j$, then for each pair (k,ℓ) in the set*

$$\{(i,j),(i,j+1),\dots,(i,n)\} \cup \{(i+1,j),\dots,(j,j),(j,j+1),\dots,(j,n)\}$$

there is a vector field $v_{k,\ell}$, vanishing at S, such that

(i) $v_{k,\ell}(\Delta) \in \mathscr{O} \cdot \Delta$, *and*

(ii) *the linear part of* $v_{k,\ell}$ *at* S *is equal to* $(x_{k,\ell} - s_{k,\ell})\frac{\partial}{\partial x_{k,\ell}}$, *and in particular is diagonal.*

Proof. —

(1) If $(k,\ell) = (i,\ell)$ with $j < \ell$, then

$$v_{i,\ell} = x_{i,\ell}\xi_{j,\ell} = x_{i,\ell}\big[x_{i,j}\partial_{i,\ell} \mod \mathfrak{m}_S \operatorname{Der}\big] = x_{i,\ell}\partial_{i,\ell} \mod \mathfrak{m}_S^2 \operatorname{Der}.$$

(2) If $(k,\ell) = (i,j)$, then if $i < j$

$$\begin{aligned} v_{i,j} = (x_{i,j}-1)\xi_{j,j} &= (x_{i,j}-1)\big[x_{i,j}\partial_{i,j} \mod \mathfrak{m}_S \operatorname{Der}\big] \\ &= (x_{i,j}-1)\partial_{i,j} \mod \mathfrak{m}_S^2 \operatorname{Der}. \end{aligned}$$

and if $i = j$

$$v_{i,i} = \frac{1}{2}(x_{i,i}-1)\xi_{i,i} = (x_{i,i}-1)\partial_{i,i} \mod \mathfrak{m}_S^2 \operatorname{Der}.$$

(3) If $(k,\ell) = (k,j)$, with $i < k < j$ then

$$v_{k,j} = x_{k,j}\xi_{i,k} = x_{k,j}\big[x_{i,j}\partial_{k,j} + \mathfrak{m}_S \operatorname{Der}\big] = x_{k,j}\partial_{k,j} \mod \mathfrak{m}_S^2 \operatorname{Der}.$$

(4) If $(k,\ell) = (j,j)$ with $i < j$, then

$$v_{j,j} = \frac{1}{2}x_{j,j}\xi_{i,j} = \frac{1}{2}x_{j,j}\big[2x_{i,j}\partial_{j,j} \mod \mathfrak{m}_S \operatorname{Der}\big] = x_{j,j}\partial_{j,j} \mod \mathfrak{m}_S^2 \operatorname{Der}.$$

(5) If $(k,\ell) = (j,\ell)$ with $j < \ell$, then

$$v_{j,\ell} = x_{j,\ell}\xi_{i,\ell} = x_{j,\ell}\big[x_{i,j}\partial_{j,\ell} \mod \mathfrak{m}_S \operatorname{Der}\big] = x_{j,\ell}\partial_{j,\ell} \mod \mathfrak{m}_S^2 \operatorname{Der}.$$

□

LEMMA 7.17. — *For each* $i \in \{1,\dots,n\}$ *there is a vector field* v_i *vanishing at* S, *such that*

(i) $v_i(\Delta) \in \mathscr{O} \cdot \Delta$

(ii) *the linear part of* v_i *at* S *is* $\sum_{k,\ell} \lambda_{k,\ell}(x_{k,\ell} - s_{k,\ell})\frac{\partial}{\partial x_{k,\ell}}$ *where* $\lambda_{k,\ell} = 0$ *if* $k > i$ *and* $\lambda_{i,\ell} > 0$ *if* $\ell \geqslant i$; *in particular it is diagonal.*

In other words we have a triangular-type system of diagonal linear parts with positive terms on the i'th row and zeros on rows after the i'th.

Proof. — If $s_{i,j} = 0$ for any $j \geqslant i$, and $s_{k,i} = 0$ for any $k \leqslant i$, we can take $v_i = \xi_{i,i}$.

If $s_{k,i} = 1$, with $k \leqslant i$, then we may apply Lemma 7.16 and a linear combination of the vector fields $v_{i,i}, v_{i,i+1}, \dots, v_{i,n}$ does the trick.

Finally if $s_{i,j} = 1$ for some $j > i$, we observe that $\xi_{i,i} - \xi_{j,j}$, is diagonal and has non zero positive eigenvalues in the positions

$$\{(i,i),\dots,(i,j-1)\} \cup \{(i,j+1),\dots(i,n)\}.$$

Then we see that the vector field

$$v_i = v_{i,j} + \xi_{i,i} - \xi_{j,j} + v_{i+1,j} + \cdots + v_{j,j} + v_{j,j+1} + \cdots + v_{j,n}$$

does the trick since by adding $v_{i,j}$ we complete the row i by a positive eigenvalue at (i,j), and we cancel with the help of the appropriate $v_{k,\ell}$ all the negative eigenvalues with row indices $k > i$. □

PROPOSITION 7.18. — *There is an Euler vector field v, $v(\Delta) \in \mathscr{O} \cdot \Delta$ vanishing at S, with linear part diagonal and having only strictly positive eigenvalues.*

Proof. — We construct v, by decreasing induction on i, as a linear combination $\alpha_n v_n + \cdots + \alpha_1 v_1$ with positive coefficients, with $\alpha_i > 0$ large enough following the choice of $\alpha_n, \ldots, \alpha_{i+1}$. By construction we have $v(\Delta) = \lambda\Delta$ with $\lambda \in \mathscr{O}$. □

This completes the proof of Proposition 7.15.

Remark 7.19. — To conclude, we mention a recent theorem of Fehér and Patakfalvi. In [8] they prove that the discriminant D in the representation space of a root of a Dynkin quiver is locally quasihomogeneous. Their theorem ([8, Thm. 5.2]) is stated in terms of the Incidence Property that is the subject of their paper, but their proof consists essentially of the contruction of the requisite $\mathbb{C}^*$-action. As a consequence, the LCT holds for these discriminants, by Theorem 1.8.

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Michel GRANGER
Université d'Angers
Département de Mathématiques
2 bd. Lavoisier
49045 Angers (France)
granger@univ-angers.fr

David MOND
University of Warwick
Mathematics Institute
Coventry CV47AL (England)
D.M.Q.Mond@warwick.ac.uk

Alicia NIETO-REYES
Universidad de Cantabria
Departamento de Matematicas,
Estadistica y Computacion (Spain)
alicia.nieto@unican.es

Mathias SCHULZE
Oklahoma State University
Department of Mathematics
Stillwater, OK 74078 (United States)
mschulze@math.okstate.edu