Pre-calc-ulus

openstax™

Precalculus

SENIOR CONTRIBUTING AUTHOR

Jay Abramson, Arizona State University

openstax™

5

Trigonometric Functions

Figure 1 The tide rises and falls at regular, predictable intervals. (credit: Andrea Schaffer, Flickr)

CHAPTER OUTLINE

Introduction

Life is dense with phenomena that repeat in regular intervals. Each day, for example, the tides rise and fall in response to the gravitational pull of the moon. Similarly, the progression from day to night occurs as a result of Earth's rotation, and the pattern of the seasons repeats in response to Earth's revolution around the sun. Outside of nature, many stocks that mirror a company's profits are influenced by changes in the economic business cycle.

In mathematics, a function that repeats its values in regular intervals is known as a periodic function. The graphs of such functions show a general shape reflective of a pattern that keeps repeating. This means the graph of the function has the same output at exactly the same place in every cycle. And this translates to all the cycles of the function having exactly the same length. So, if we know all the details of one full cycle of a true periodic function, then we know the state of the function's outputs at all times, future and past. In this chapter, we will investigate various examples of periodic functions.

LEARNING OBJECTIVES

In this section, you will:

- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.
- Find the length of a circular arc.
- Use linear and angular speed to describe motion on a circular path.

5.1 ANGLES

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.

Drawing Angles in Standard Position

Properly defining an angle first requires that we define a ray. A **ray** consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its endpoint and any other point on it. The ray in **Figure 1** can be named as ray EF, or in symbol form $\overrightarrow{EF}$.

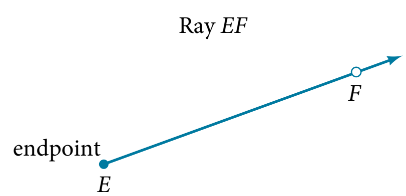


Figure 1

An **angle** is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. The angle in **Figure 2** is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle *DEF*, or in symbol form $\angle DEF$.

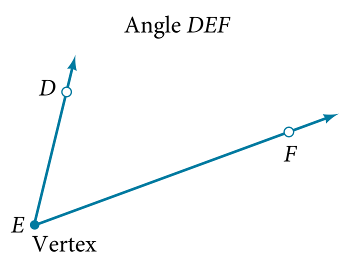


Figure 2

Greek letters are often used as variables for the measure of an angle. **Table 1** is a list of Greek letters commonly used to represent angles, and a sample angle is shown in **Figure 3**.

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

Table 1

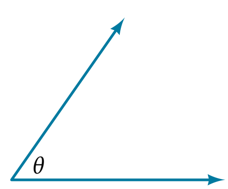


Figure 3 Angle theta, shown as $\angle\theta$

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial side,** and the rotated ray is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in **Figure 4**.

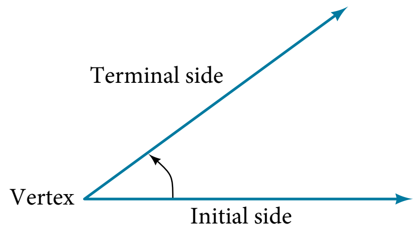


Figure 4

As we discussed at the beginning of the section, there are many applications for angles, but in order to use them correctly, we must be able to measure them. The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One **degree** is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit "degrees" after the number, or include the degree symbol °. For example, 90 degrees $= 90°$.

To formalize our work, we will begin by drawing angles on an x-y coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible. An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive x-axis. See **Figure 5**.

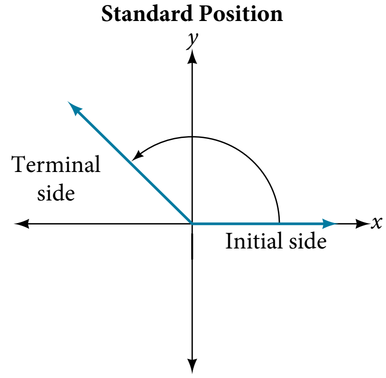


Figure 5

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

Drawing an angle in standard position always starts the same way—draw the initial side along the positive x-axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by 360°. For example, to draw a 90° angle, we calculate that $\frac{90°}{360°} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive x-axis. To draw a 360° angle, we calculate that $\frac{360°}{360°} = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive x-axis. In this case, the initial side and the terminal side overlap. See **Figure 6**.

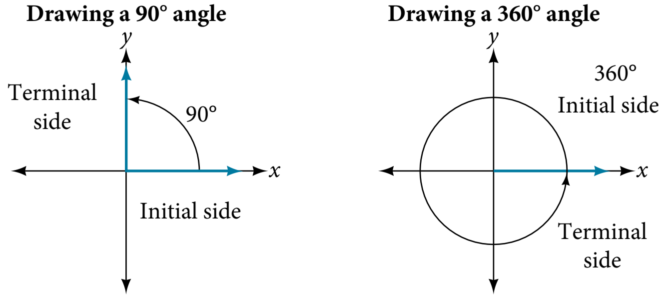


Figure 6

Since we define an angle in standard position by its initial side, we have a special type of angle whose terminal side lies on an axis, a **quadrantal angle**. This type of angle can have a measure of 0°, 90°, 180°, 270° or 360°. See **Figure 7**.

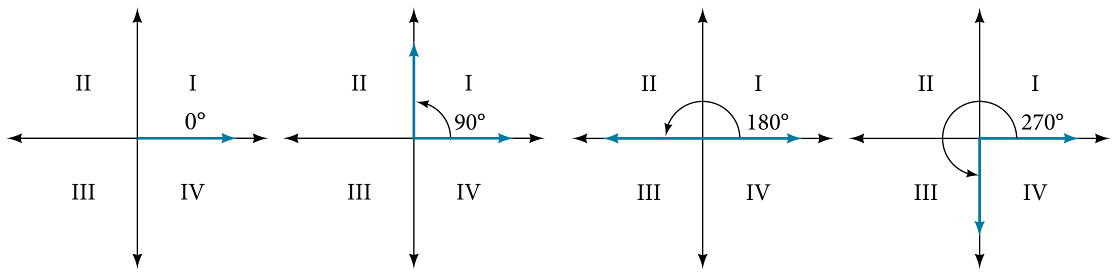


Figure 7 Quadrantal angles have a terminal side that lies along an axis. Examples are shown.

> ***quadrantal angles***
>
> Quadrantal angles are angles in standard position whose terminal side lies on an axis, including 0°, 90°, 180°, 270°, or 360°.

How To...

Given an angle measure in degrees, draw the angle in standard position.

1. Express the angle measure as a fraction of 360°.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive x-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Example 1 Drawing an Angle in Standard Position Measured in Degrees

a. Sketch an angle of 30° in standard position.
b. Sketch an angle of −135° in standard position.

Solution

a. Divide the angle measure by 360°.

$$\frac{30°}{360°} = \frac{1}{12}$$

To rewrite the fraction in a more familiar fraction, we can recognize that

$$\frac{1}{12} = \frac{1}{3}\left(\frac{1}{4}\right)$$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at 30° as in **Figure 8**.

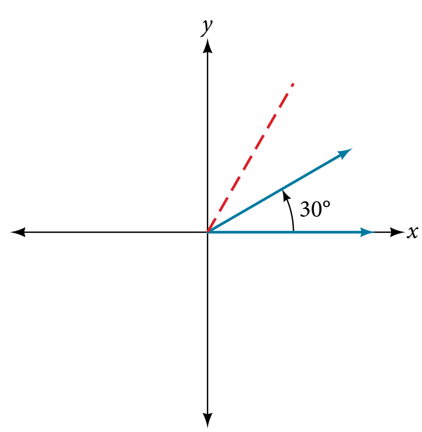


Figure 8

b. Divide the angle measure by 360°.

$$\frac{-135°}{360°} = -\frac{3}{8}$$

In this case, we can recognize that

$$-\frac{3}{8} = -\frac{3}{2}\left(\frac{1}{4}\right)$$

Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in **Figure 9**.

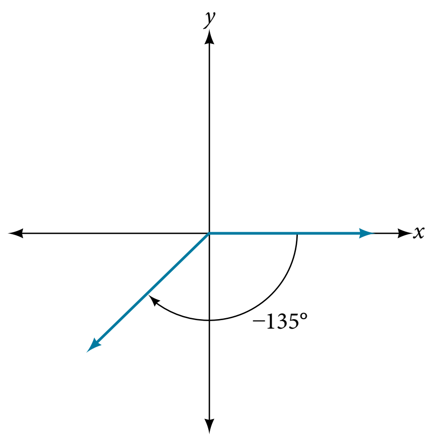


Figure 9

Try It #1

Show an angle of 240° on a circle in standard position.

Converting Between Degrees and Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An arc may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the circumference of that circle.

The circumference of a circle is $C = 2\pi r$. If we divide both sides of this equation by r, we create the ratio of the circumference to the radius, which is always 2π regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in **Figure 10**.

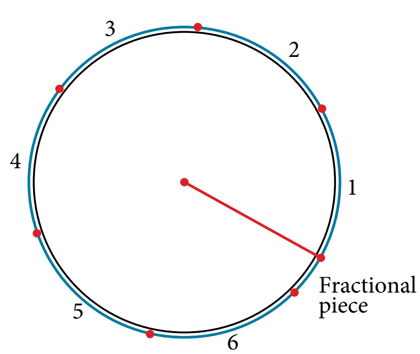


Figure 10

This brings us to our new angle measure. One **radian** is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$2\pi \text{ radians} = 360°$$

$$\pi \text{ radians} = \frac{360°}{2} = 180°$$

$$1 \text{ radian} = \frac{180°}{\pi} \approx 57.3°$$

See **Figure 11**. Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3 radians. In fact, radian measure is dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel out.

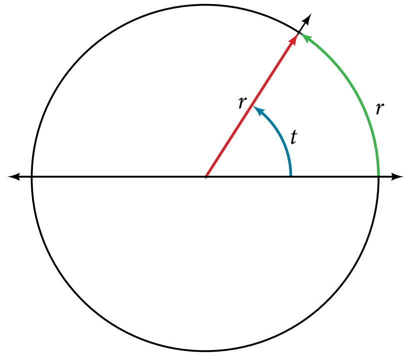


Figure 11 The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.

Relating Arc Lengths to Radius

An **arc length** s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius.

This ratio, called the radian measure, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s to the radius r. See **Figure 12**.

$$s = r\theta$$

$$\theta = \frac{s}{r}$$

If $s = r$, then $\theta = \frac{r}{r} = 1$ radian.

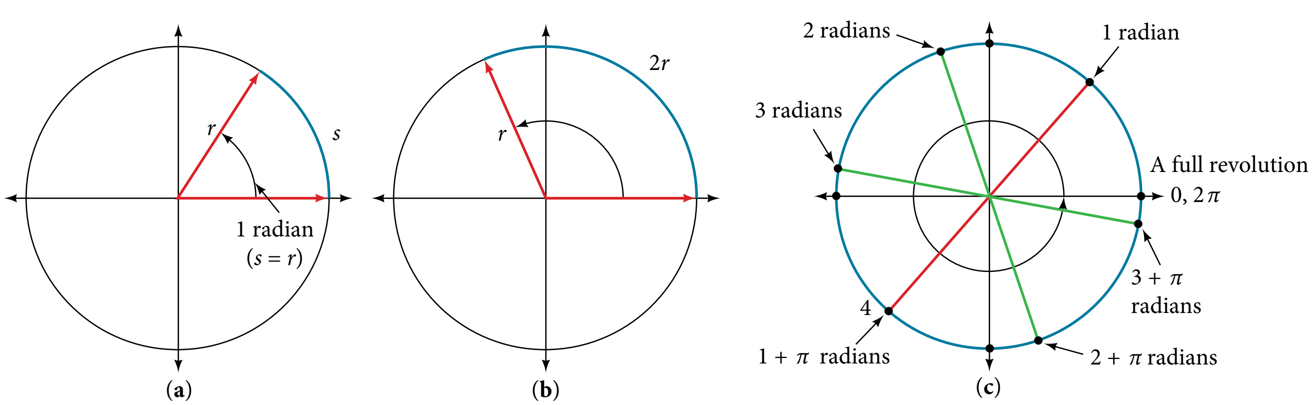


Figure 12 (a) In an angle of 1 radian, the arc length s equals the radius r.
(b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π or about 6.28 radians.

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$, where r is the radius. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$. Now we draw a 45°angle on the two circles, as in **Figure 13**.

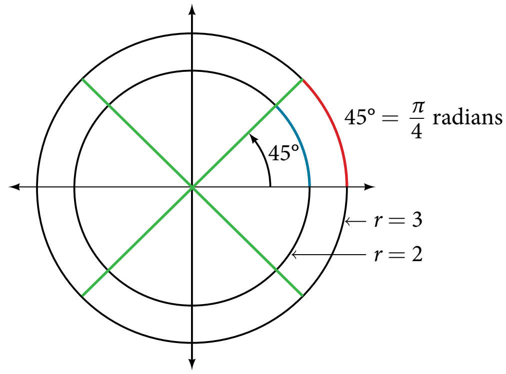


Figure 13 A 45° angle contains one-eighth of the circumference of a circle, regardless of the radius.

Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\text{Smaller circle:} \quad \frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi$$

$$\text{Larger circle:} \quad \frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc length and radius differ.

> ***radians***
>
> One **radian** is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals 2π radians. A half revolution (180°) is equivalent to π radians.
>
> The **radian measure** of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

Q & A...

A measure of 1 radian looks to be about 60°. Is that correct?

Yes. It is approximately 57.3°. Because 2π radians equals 360°, 1 radian equals $\frac{360°}{2\pi} \approx 57.3°$.

Using Radians

Because radian measure is the ratio of two lengths, it is a unitless measure. For example, in **Figure 12**, suppose the radius was 2 inches and the distance along the arc was also 2 inches. When we calculate the radian measure of the angle, the "inches" cancel, and we have a result without units. Therefore, it is not necessary to write the label "radians" after a radian measure, and if we see an angle that is not labeled with "degrees" or the degree symbol, we can assume that it is a radian measure.

Considering the most basic case, the unit circle (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360°. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$1 \text{ rotation} = 360° = 2\pi \text{ radians}$$

$$\frac{1}{2} \text{rotation} = 180° = \pi \text{ radians}$$

$$\frac{1}{4} \text{rotation} = 90° = \frac{\pi}{2} \text{ radians}$$

Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in **Figure 14**. Memorizing these angles will be very useful as we study the properties associated with angles.

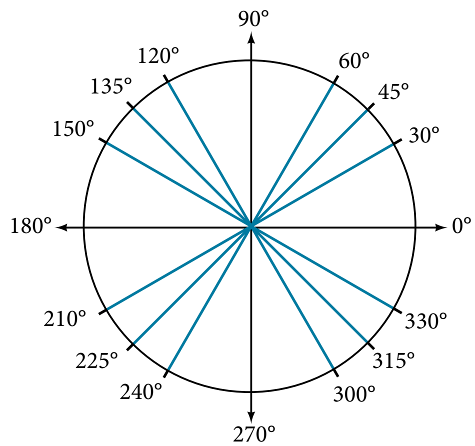


Figure 14 Commonly encountered angles measured in degrees

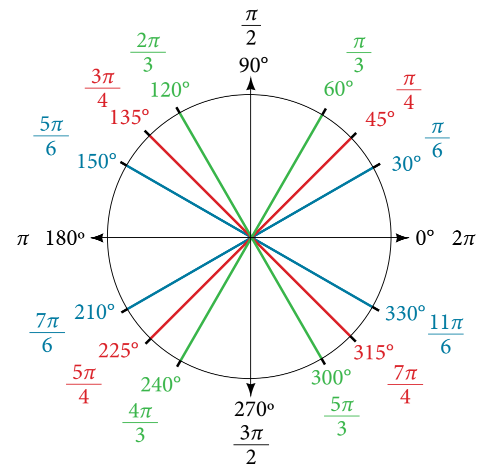


Figure 15 Commonly encountered angles measured in radians

Now, we can list the corresponding radian values for the common measures of a circle corresponding to those listed in **Figure 14**, which are shown in **Figure 15**. Be sure you can verify each of these measures.

Example 2 Finding a Radian Measure

Find the radian measure of one-third of a full rotation.

Solution For any circle, the arc length along such a rotation would be one-third of the circumference. We know that

$$1 \text{ rotation} = 2\pi r$$

So,

$$s = \frac{1}{3}(2\pi r)$$
$$= \frac{2\pi r}{3}$$

The radian measure would be the arc length divided by the radius.

$$\text{radian measure} = \frac{\frac{2\pi r}{3}}{r}$$
$$= \frac{2\pi r}{3r}$$
$$= \frac{2\pi}{3}$$

Try It #2

Find the radian measure of three-fourths of a full rotation.

Converting Between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π.

$$\frac{\text{Degrees}}{180} = \frac{\text{Radians}}{\pi}$$

converting between radians and degrees

To convert between degrees and radians, use the proportion

$$\frac{\theta}{180} = \frac{\theta^R}{\pi}$$

Example 3 Converting Radians to Degrees

Convert each radian measure to degrees.

a. $\frac{\pi}{6}$ **b.** 3

Solution Because we are given radians and we want degrees, we should set up a proportion and solve it.

a. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi}$$

$$\frac{\theta}{180} = \frac{\frac{\pi}{6}}{\pi}$$

$$\theta = \frac{180}{6}$$

$$\theta = 30°$$

b. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi}$$

$$\frac{\theta}{180} = \frac{3}{\pi}$$

$$\theta = \frac{3(180)}{\pi}$$

$$\theta \approx 172°$$

Try It #3

Convert $-\frac{3\pi}{4}$ radians to degrees.

Example 4 Converting Degrees to Radians

Convert 15 degrees to radians.

Solution In this example, we start with degrees and want radians, so we again set up a proportion and solve it, but we substitute the given information into a different part of the proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi}$$

$$\frac{15}{180} = \frac{\theta^R}{\pi}$$

$$\frac{15\pi}{180} = \theta^R$$

$$\frac{\pi}{12} = \theta^R$$

Analysis *Another way to think about this problem is by remembering that* $30° = \frac{\pi}{6}$*. Because* $15° = \frac{1}{2}(30°)$*, we can find that* $\frac{1}{2}\left(\frac{\pi}{6}\right)$ *is* $\frac{\pi}{12}$.

Try It #4

Convert 126° to radians.

Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0° to 360°, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

It is possible for more than one angle to have the same terminal side. Look at **Figure 16**. The angle of 140° is a positive angle, measured counterclockwise. The angle of −220° is a negative angle, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are coterminal angles. Every angle greater than 360° or less than 0° is coterminal with an angle between 0° and 360°, and it is often more convenient to find the coterminal angle within the range of 0° to 360° than to work with an angle that is outside that range.

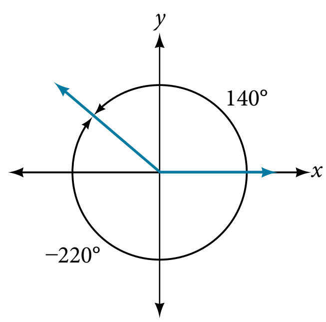


Figure 16 An angle of 140° and an angle of −220° are coterminal angles.

Any angle has infinitely many coterminal angles because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location. For example, 100° and 460° are coterminal for this reason, as is −260°. Recognizing that any angle has infinitely many coterminal angles explains the repetitive shape in the graphs of trigonometric functions.

An angle's reference angle is the measure of the smallest, positive, acute angle t formed by the terminal side of the angle t and the horizontal axis. Thus positive reference angles have terminal sides that lie in the first quadrant and can be used as models for angles in other quadrants. See **Figure 17** for examples of reference angles for angles in different quadrants.

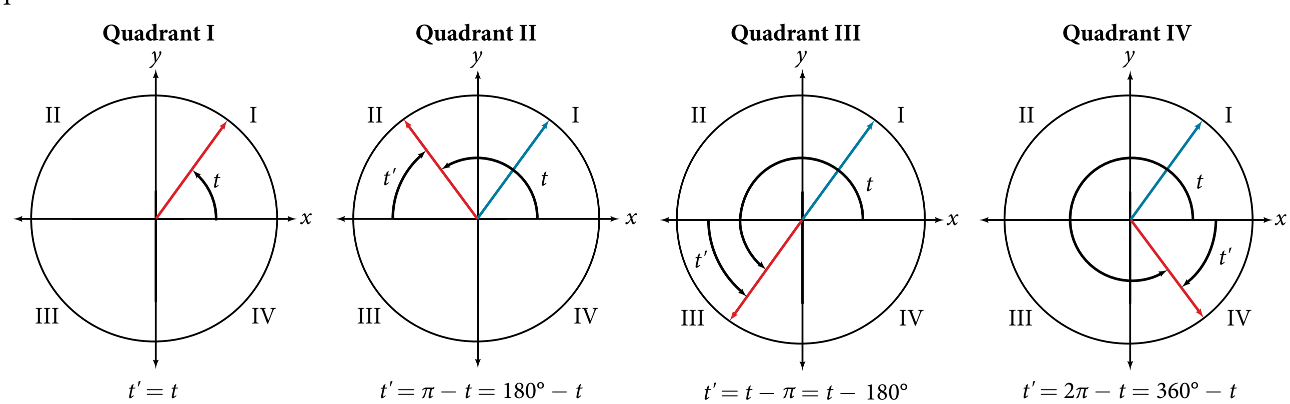


Figure 17

> ***coterminal and reference angles***
>
> Coterminal angles are two angles in standard position that have the same terminal side.
> An angle's **reference angle** is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

How To...

Given an angle greater than 360°, find a coterminal angle between 0° and 360°.

1. Subtract 360° from the given angle.
2. If the result is still greater than 360°, subtract 360° again till the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 5 Finding an Angle Coterminal with an Angle of Measure Greater Than 360°

Find the least positive angle θ that is coterminal with an angle measuring 800°, where $0° \leq \theta < 360°$.

Solution An angle with measure 800° is coterminal with an angle with measure $800 - 360 = 440°$, but 440° is still greater than 360°, so we subtract 360° again to find another coterminal angle: $440 - 360 = 80°$.

The angle $\theta = 80°$ is coterminal with 800°. To put it another way, 800° equals 80° plus two full rotations, as shown in **Figure 18**.

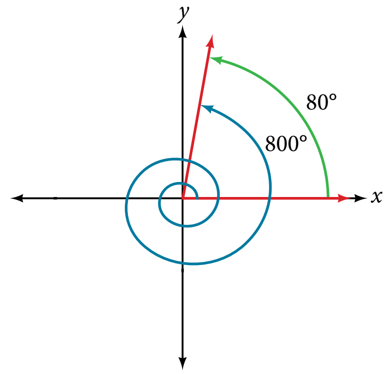


Figure 18

Try It #5

Find an angle α that is coterminal with an angle measuring 870°, where $0° \leq \alpha < 360°$.

How To...

Given an angle with measure less than 0°, find a coterminal angle having a measure between 0° and 360°.

1. Add 360° to the given angle.
2. If the result is still less than 0°, add 360° again until the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 6 Finding an Angle Coterminal with an Angle Measuring Less Than 0°

Show the angle with measure −45° on a circle and find a positive coterminal angle α such that $0° \leq \alpha < 360°$.

Solution Since 45° is half of 90°, we can start at the positive horizontal axis and measure clockwise half of a 90° angle.

Because we can find coterminal angles by adding or subtracting a full rotation of 360°, we can find a positive coterminal angle here by adding 360°:

$$-45° + 360° = 315°$$

We can then show the angle on a circle, as in **Figure 19**.

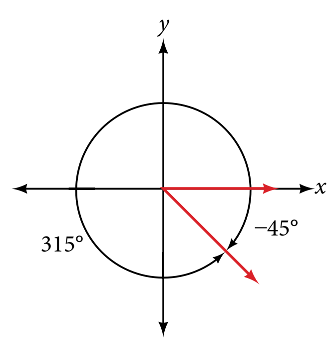


Figure 19

Try It #6

Find an angle β that is coterminal with an angle measuring −300° such that $0° \leq \beta < 360°$.

Finding Coterminal Angles Measured in Radians

We can find coterminal angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

How To…

Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Example 7 Finding Coterminal Angles Using Radians

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \le \beta < 2\pi$.

Solution When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians:

$$\begin{aligned}\frac{19\pi}{4} - 2\pi &= \frac{19\pi}{4} - \frac{8\pi}{4} \\ &= \frac{11\pi}{4}\end{aligned}$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation:

$$\begin{aligned}\frac{11\pi}{4} - 2\pi &= \frac{11\pi}{4} - \frac{8\pi}{4} \\ &= \frac{3\pi}{4}\end{aligned}$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$, as shown in **Figure 20**.

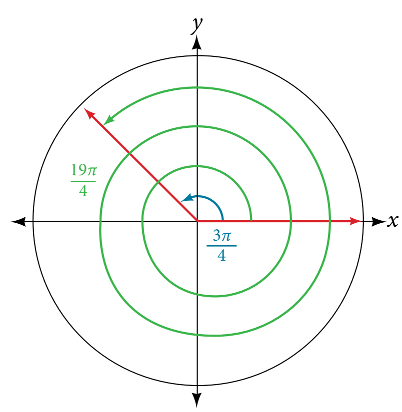


Figure 20

Try It #7

Find an angle of measure θ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \le \theta < 2\pi$.

Determining the Length of an Arc

Recall that the radian measure θ of an angle was defined as the ratio of the arc length s of a circular arc to the radius r of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arc length along a circle, given an angle.

arc length on a circle

In a circle of radius r, the length of an arc s subtended by an angle with measure θ in radians, shown in **Figure 21**, is

$$s = r\theta$$

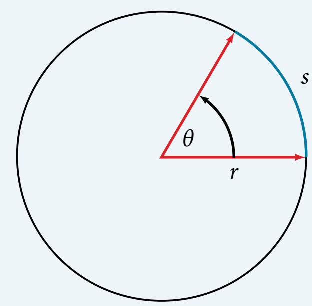


Figure 21

How To…

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r by the radian measure of $\theta : s = r\theta$.

Example 8 Finding the Length of an Arc

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

a. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?

b. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

Solution

a. Let's begin by finding the circumference of Mercury's orbit.

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi(36 \text{ million miles}) \\ &\approx 226 \text{ million miles} \end{aligned}$$

Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the distance traveled:

$$(0.0114)226 \text{ million miles} = 2.58 \text{ million miles}$$

b. Now, we convert to radians:

$$\begin{aligned} \text{radian} &= \frac{\text{arclength}}{\text{radius}} \\ &= \frac{2.58 \text{ million miles}}{36 \text{ million miles}} \\ &= 0.0717 \end{aligned}$$

Try It #8

Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

Finding the Area of a Sector of a Circle

In addition to arc length, we can also use angles to find the area of a sector of a circle. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. Recall that the area of a circle with radius r can be found using the formula $A = \pi r^2$. If the two radii form an angle of θ, measured in radians, then $\frac{\theta}{2\pi}$ is the ratio of the angle measure to the measure of a full rotation and is also, therefore, the ratio of the area of the sector to the area of the circle. Thus, the **area of a sector** is the fraction $\frac{\theta}{2\pi}$ multiplied by the entire area. (Always remember that this formula only applies if θ is in radians.)

$$\begin{aligned} \text{Area of sector} &= \left(\frac{\theta}{2\pi}\right)\pi r^2 \\ &= \frac{\theta\pi r^2}{2\pi} \\ &= \frac{1}{2}\theta r^2 \end{aligned}$$

area of a sector

The **area of a sector** of a circle with radius r subtended by an angle θ, measured in radians, is

$$A = \frac{1}{2}\theta r^2$$

See **Figure 22**.

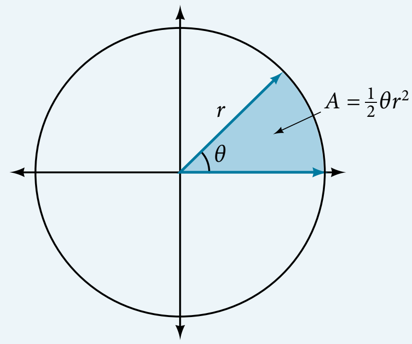


Figure 22 The area of the sector equals half the square of the radius times the central angle measured in radians.

How To...

Given a circle of radius r, find the area of a sector defined by a given angle θ.

1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r: $A = \frac{1}{2}\theta r^2$.

Example 9 Finding the Area of a Sector

An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in **Figure 23**. What is the area of the sector of grass the sprinkler waters?

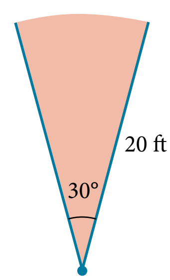


Figure 23 The sprinkler sprays 20 ft within an arc of 30°.

Solution First, we need to convert the angle measure into radians. Because 30 degrees is one of our special angles, we already know the equivalent radian measure, but we can also convert:

$$\begin{aligned} 30 \text{ degrees} &= 30 \cdot \frac{\pi}{180} \\ &= \frac{\pi}{6} \text{ radians} \end{aligned}$$

The area of the sector is then

$$\begin{aligned} \text{Area} &= \frac{1}{2}\left(\frac{\pi}{6}\right)(20)^2 \\ &\approx 104.72 \end{aligned}$$

So the area is about 104.72 ft^2.

Try It #9

In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover? Write the answer in radian measure to two decimal places.

Use Linear and Angular Speed to Describe Motion on a Circular Path

In addition to finding the area of a sector, we can use angles to describe the speed of a moving object. An object traveling in a circular path has two types of speed. **Linear speed** is speed along a straight path and can be determined by the distance it moves along (its **displacement**) in a given time interval. For instance, if a wheel with radius 5 inches rotates once a second, a point on the edge of the wheel moves a distance equal to the circumference, or 10π inches, every second. So the linear speed of the point is 10π in./s. The equation for linear speed is as follows where v is linear speed, s is displacement, and t is time.

$$v = \frac{s}{t}$$

Angular speed results from circular motion and can be determined by the angle through which a point rotates in a given time interval. In other words, angular speed is angular rotation per unit time. So, for instance, if a gear makes a full rotation every 4 seconds, we can calculate its angular speed as $\frac{360 \text{ degrees}}{4 \text{ seconds}} = 90$ degrees per second. Angular speed can be given in radians per second, rotations per minute, or degrees per hour for example. The equation for angular speed is as follows, where ω (read as omega) is angular speed, θ is the angle traversed, and t is time.

$$\omega = \frac{\theta}{t}$$

Combining the definition of angular speed with the arc length equation, $s = r\theta$, we can find a relationship between angular and linear speeds. The angular speed equation can be solved for θ, giving $\theta = \omega t$. Substituting this into the arc length equation gives:

$$\begin{aligned} s &= r\theta \\ &= r\omega t \end{aligned}$$

Substituting this into the linear speed equation gives:

$$\begin{aligned} v &= \frac{s}{t} \\ &= \frac{r\omega t}{t} \\ &= r\omega \end{aligned}$$

angular and linear speed

As a point moves along a circle of radius r, its **angular speed**, ω, is the angular rotation θ per unit time, t.

$$\omega = \frac{\theta}{t}$$

The **linear speed**, v, of the point can be found as the distance traveled, arc length s, per unit time, t.

$$v = \frac{s}{t}$$

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation

$$v = r\omega$$

This equation states that the angular speed in radians, ω, representing the amount of rotation occurring in a unit of time, can be multiplied by the radius r to calculate the total arc length traveled in a unit of time, which is the definition of linear speed.

How To...

Given the amount of angle rotation and the time elapsed, calculate the angular speed.

1. If necessary, convert the angle measure to radians.
2. Divide the angle in radians by the number of time units elapsed: $\omega = \frac{\theta}{t}$.
3. The resulting speed will be in radians per time unit.

Example 10 Finding Angular Speed

A water wheel, shown in **Figure 24**, completes 1 rotation every 5 seconds. Find the angular speed in radians per second.

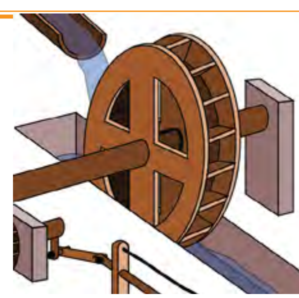


Figure 24

Solution The wheel completes 1 rotation, or passes through an angle of 2π radians in 5 seconds, so the angular speed would be $\omega = \frac{2\pi}{5} \approx 1.257$ radians per second.

Try It #10

An old vinyl record is played on a turntable rotating clockwise at a rate of 45 rotations per minute. Find the angular speed in radians per second.

How To…

Given the radius of a circle, an angle of rotation, and a length of elapsed time, determine the linear speed.

1. Convert the total rotation to radians if necessary.
2. Divide the total rotation in radians by the elapsed time to find the angular speed: apply $\omega = \frac{\theta}{t}$.
3. Multiply the angular speed by the length of the radius to find the linear speed, expressed in terms of the length unit used for the radius and the time unit used for the elapsed time: apply $v = r\omega$.

Example 11 Finding a Linear Speed

A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

Solution Here, we have an angular speed and need to find the corresponding linear speed, since the linear speed of the outside of the tires is the speed at which the bicycle travels down the road.

We begin by converting from rotations per minute to radians per minute. It can be helpful to utilize the units to make this conversion:

$$180\,\frac{\cancel{\text{rotations}}}{\text{minute}} \cdot \frac{2\pi\text{ radians}}{\cancel{\text{rotation}}} = 360\pi\,\frac{\text{radians}}{\text{minute}}$$

Using the formula from above along with the radius of the wheels, we can find the linear speed:

$$\begin{aligned} v &= (14\text{ inches})\left(360\pi\,\frac{\text{radians}}{\text{minute}}\right) \\ &= 5040\pi\,\frac{\text{inches}}{\text{minute}} \end{aligned}$$

Remember that radians are a unitless measure, so it is not necessary to include them.

Finally, we may wish to convert this linear speed into a more familiar measurement, like miles per hour.

$$5040\pi\,\frac{\cancel{\text{inches}}}{\cancel{\text{minute}}} \cdot \frac{1\ \cancel{\text{feet}}}{12\ \cancel{\text{inches}}} \cdot \frac{1\text{ mile}}{5280\ \cancel{\text{feet}}} \cdot \frac{60\ \cancel{\text{minutes}}}{1\text{ hour}} \approx 14.99\text{ miles per hour (mph)}$$

Try It #11

A satellite is rotating around Earth at 0.25 radians per hour at an altitude of 242 km above Earth. If the radius of Earth is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

Access these online resources for additional instruction and practice with angles, arc length, and areas of sectors.

- Angles in Standard Position (http://openstaxcollege.org/l/standardpos)
- Angle of Rotation (http://openstaxcollege.org/l/angleofrotation)
- Coterminal Angles (http://openstaxcollege.org/l/coterminal)
- Determining Coterminal Angles (http://openstaxcollege.org/l/detcoterm)
- Positive and Negative Coterminal Angles (http://openstaxcollege.org/l/posnegcoterm)
- Radian Measure (http://openstaxcollege.org/l/radianmeas)
- Coterminal Angles in Radians (http://openstaxcollege.org/l/cotermrad)
- Arc Length and Area of a Sector (http://openstaxcollege.org/l/arclength)

5.1 SECTION EXERCISES

VERBAL

1. Draw an angle in standard position. Label the vertex, initial side, and terminal side.

2. Explain why there are an infinite number of angles that are coterminal to a certain angle.

3. State what a positive or negative angle signifies, and explain how to draw each.

4. How does radian measure of an angle compare to the degree measure? Include an explanation of 1 radian in your paragraph.

5. Explain the differences between linear speed and angular speed when describing motion along a circular path.

For the following exercises, draw an angle in standard position with the given measure.

6. $30°$ **7.** $300°$ **8.** $-80°$ **9.** $135°$ **10.** $-150°$ **11.** $\frac{2\pi}{3}$

12. $\frac{7\pi}{4}$ **13.** $\frac{5\pi}{6}$ **14.** $\frac{\pi}{2}$ **15.** $-\frac{\pi}{10}$ **16.** $415°$ **17.** $-120°$

18. $-315°$ **19.** $\frac{22\pi}{3}$ **20.** $-\frac{\pi}{6}$ **21.** $-\frac{4\pi}{3}$

For the following exercises, refer to **Figure 25**. Round to two decimal places.

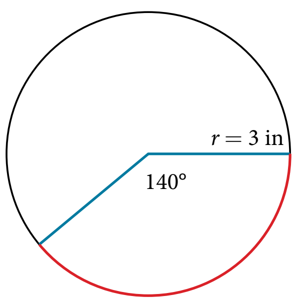


Figure 25

22. Find the arc length.

23. Find the area of the sector.

For the following exercises, refer to **Figure 26**. Round to two decimal places.

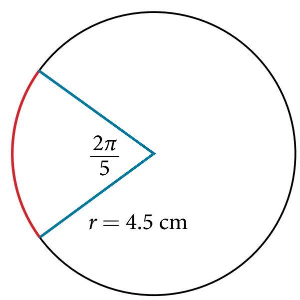


Figure 26

24. Find the arc length.

25. Find the area of the sector.

ALGEBRAIC

For the following exercises, convert angles in radians to degrees.

26. $\frac{3\pi}{4}$ radians **27.** $\frac{\pi}{9}$ radians **28.** $-\frac{5\pi}{4}$ radians **29.** $\frac{\pi}{3}$ radians

30. $-\frac{7\pi}{3}$ radians **31.** $-\frac{5\pi}{12}$ radians **32.** $\frac{11\pi}{6}$ radians

For the following exercises, convert angles in degrees to radians.

33. $90°$ **34.** $100°$ **35.** $-540°$ **36.** $-120°$

37. $180°$ **38.** $-315°$ **39.** $150°$

For the following exercises, use to given information to find the length of a circular arc. Round to two decimal places.

40. Find the length of the arc of a circle of radius 12 inches subtended by a central angle of $\frac{\pi}{4}$ radians.

41. Find the length of the arc of a circle of radius 5.02 miles subtended by the central angle of $\frac{\pi}{3}$.

42. Find the length of the arc of a circle of diameter 14 meters subtended by the central angle of $\frac{5\pi}{6}$.

43. Find the length of the arc of a circle of radius 10 centimeters subtended by the central angle of $50°$.

44. Find the length of the arc of a circle of radius 5 inches subtended by the central angle of $220°$.

45. Find the length of the arc of a circle of diameter 12 meters subtended by the central angle is $63°$.

For the following exercises, use the given information to find the area of the sector. Round to four decimal places.

46. A sector of a circle has a central angle of 45° and a radius 6 cm.

47. A sector of a circle has a central angle of 30° and a radius of 20 cm.

48. A sector of a circle with diameter 10 feet and an angle of $\frac{\pi}{2}$ radians.

49. A sector of a circle with radius of 0.7 inches and an angle of π radians.

For the following exercises, find the angle between 0° and 360° that is coterminal to the given angle.

50. $-40°$

51. $-110°$

52. 700°

53. 1400°

For the following exercises, find the angle between 0 and 2π in radians that is coterminal to the given angle.

54. $-\frac{\pi}{9}$

55. $\frac{10\pi}{3}$

56. $\frac{13\pi}{6}$

57. $\frac{44\pi}{9}$

REAL-WORLD APPLICATIONS

58. A truck with 32-inch diameter wheels is traveling at 60 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

59. A bicycle with 24-inch diameter wheels is traveling at 15 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

60. A wheel of radius 8 inches is rotating 15°/s. What is the linear speed v, the angular speed in RPM, and the angular speed in rad/s?

61. A wheel of radius 14 inches is rotating 0.5 rad/s. What is the linear speed v, the angular speed in RPM, and the angular speed in deg/s?

62. A CD has diameter of 120 millimeters. When playing audio, the angular speed varies to keep the linear speed constant where the disc is being read. When reading along the outer edge of the disc, the angular speed is about 200 RPM (revolutions per minute). Find the linear speed.

63. When being burned in a writable CD-R drive, the angular speed of a CD varies to keep the linear speed constant where the disc is being written. When writing along the outer edge of the disc, the angular speed of one drive is about 4,800 RPM (revolutions per minute). Find the linear speed if the CD has diameter of 120 millimeters.

64. A person is standing on the equator of Earth (radius 3960 miles). What are his linear and angular speeds?

65. Find the distance along an arc on the surface of Earth that subtends a central angle of 5 minutes $\left(1 \text{ minute} = \frac{1}{60} \text{ degree}\right)$. The radius of Earth is 3,960 mi.

66. Find the distance along an arc on the surface of Earth that subtends a central angle of 7 minutes $\left(1 \text{ minute} = \frac{1}{60} \text{ degree}\right)$. The radius of Earth is 3,960 miles.

67. Consider a clock with an hour hand and minute hand. What is the measure of the angle the minute hand traces in 20 minutes?

EXTENSIONS

68. Two cities have the same longitude. The latitude of city A is 9.00 degrees north and the latitude of city B is 30.00 degree north. Assume the radius of the earth is 3960 miles. Find the distance between the two cities.

69. A city is located at 40 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

70. A city is located at 75 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

71. Find the linear speed of the moon if the average distance between the earth and moon is 239,000 miles, assuming the orbit of the moon is circular and requires about 28 days. Express answer in miles per hour.

72. A bicycle has wheels 28 inches in diameter. A tachometer determines that the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is travelling down the road.

73. A car travels 3 miles. Its tires make 2640 revolutions. What is the radius of a tire in inches?

74. A wheel on a tractor has a 24-inch diameter. How many revolutions does the wheel make if the tractor travels 4 miles?

LEARNING OBJECTIVES

In this section, you will:

- Find function values for the sine and cosine of 30° or $\left(\frac{\pi}{6}\right)$, 45° or $\left(\frac{\pi}{4}\right)$ and 60° or $\left(\frac{\pi}{3}\right)$.
- Identify the domain and range of sine and cosine functions.
- Use reference angles to evaluate trigonometric functions.

5.2 UNIT CIRCLE: SINE AND COSINE FUNCTIONS

Figure 1 The Singapore Flyer is the world's tallest Ferris wheel. (credit: "Vibin JK"/Flickr)

Looking for a thrill? Then consider a ride on the Singapore Flyer, the world's tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.

Finding Function Values for the Sine and Cosine

To define our trigonometric functions, we begin by drawing a unit circle, a circle centered at the origin with radius 1, as shown in **Figure 2**. The angle (in radians) that t intercepts forms an arc of length s. Using the formula $s = rt$, and knowing that $r = 1$, we see that for a unit circle, $s = t$.

Recall that the x- and y-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y). The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$.

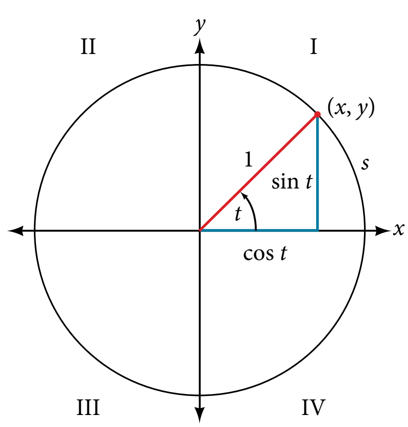


Figure 2 Unit circle where the central angle is t radians

unit circle

A **unit circle** has a center at (0, 0) and radius 1. In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle t.

Let (x, y) be the endpoint on the unit circle of an arc of arc length s. The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions

Now that we have our unit circle labeled, we can learn how the (x, y) coordinates relate to the arc length and angle. The **sine function** relates a real number t to the y-coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t. In **Figure 2**, the sine is equal to y. Like all functions, the sine function has an input and an output. Its input is the measure of the angle; its output is the y-coordinate of the corresponding point on the unit circle.

The **cosine function** of an angle t equals the x-value of the endpoint on the unit circle of an arc of length t. In **Figure 3**, the cosine is equal to x.

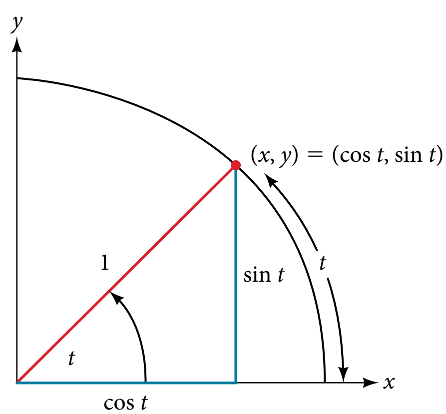


Figure 3

Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos(t))^2$. Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

sine and cosine functions

If t is a real number and a point (x, y) on the unit circle corresponds to an angle of t, then

$$\cos t = x$$
$$\sin t = y$$

How To...

Given a point $P(x, y)$ on the unit circle corresponding to an angle of t, find the sine and cosine.

1. The sine of t is equal to the y-coordinate of point P: $\sin t = y$.
2. The cosine of t is equal to the x-coordinate of point P: $\cos t = x$.

Example 1 Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t, as shown in **Figure 4.** Find $\cos(t)$ and $\sin(t)$.

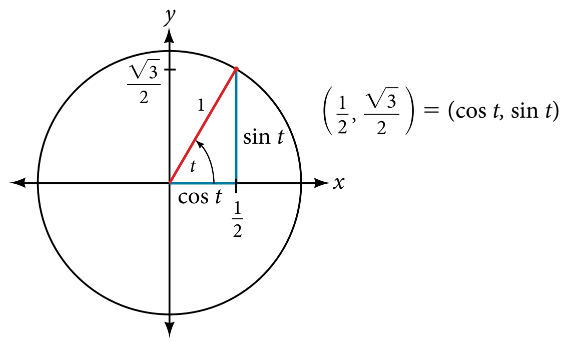


Figure 4

Solution We know that $\cos t$ is the x-coordinate of the corresponding point on the unit circle and $\sin t$ is the y-coordinate of the corresponding point on the unit circle. So:

$$x = \cos t = \frac{1}{2}$$
$$y = \sin t = \frac{\sqrt{3}}{2}$$

Try It #1

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in **Figure 5**. Find $\cos t$ and $\sin t$.

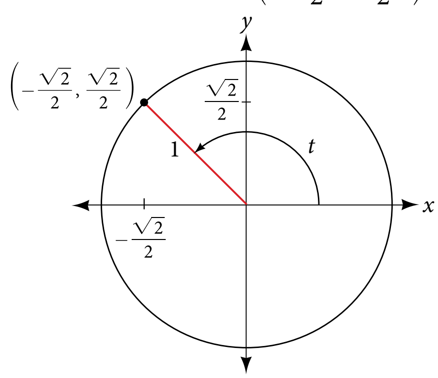


Figure 5

Finding Sines and Cosines of Angles on an Axis

For quadrantral angles, the corresponding point on the unit circle falls on the x- or y-axis. In that case, we can easily calculate cosine and sine from the values of x and y.

Example 2 Calculating Sines and Cosines along an Axis

Find cos(90°) and sin(90°).

Solution Moving 90° counterclockwise around the unit circle from the positive x-axis brings us to the top of the circle, where the (x, y) coordinates are (0, 1), as shown in **Figure 6**.

Using our definitions of cosine and sine,

$$x = \cos t = \cos(90°) = 0$$
$$y = \sin t = \sin(90°) = 1$$

The cosine of 90° is 0; the sine of 90° is 1.

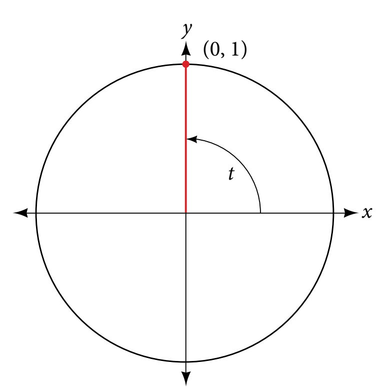


Figure 6

Try It #2

Find cosine and sine of the angle π.

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the **Pythagorean Identity**. See **Figure 7**.

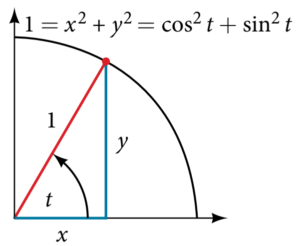


Figure 7

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

Pythagorean Identity

The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1$$

How To…

Given the sine of some angle t and its quadrant location, find the cosine of t.

1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
2. Solve for $\cos(t)$.
3. Choose the solution with the appropriate sign for the x-values in the quadrant where t is located.

Example 3 Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

Solution If we drop a vertical line from the point on the unit circle corresponding to t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See **Figure 8**.

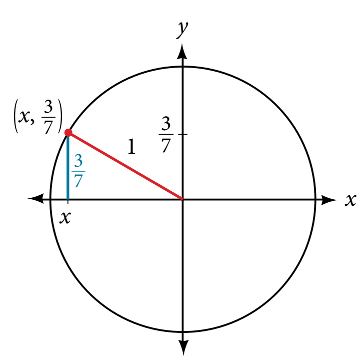


Figure 8

Substituting the known value for sine into the Pythagorean Identity,

$$\cos^2(t) + \sin^2(t) = 1$$

$$\cos^2(t) + \frac{9}{49} = 1$$

$$\cos^2(t) = \frac{40}{49}$$

$$\cos(t) = \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7}$$

Because the angle is in the second quadrant, we know the x-value is a negative real number, so the cosine is also negative. So $\cos(t) = -\frac{2\sqrt{10}}{7}$

Try It #3

If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees. We can also calculate sines and cosines of the special angles using the Pythagorean Identity and our knowledge of triangles.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of 45° or $\frac{\pi}{4}$, as shown in **Figure 9**. A 45° – 45° – 90° triangle is an isosceles triangle, so the x- and y-coordinates of the corresponding point on the circle are the same. Because the x- and y-values are the same, the sine and cosine values will also be equal.

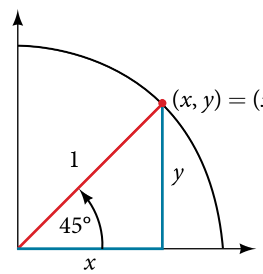


Figure 9

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first quadrantal angle. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1. So, the right triangle formed below the line $y = x$ has sides x and y ($y = x$), and a radius = 1. See **Figure 10**.

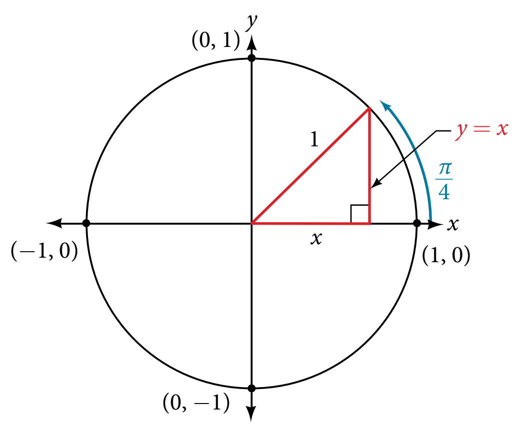


Figure 10

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

Substituting $y = x$, we get

$$x^2 + x^2 = 1$$

Combining like terms we get

$$2x^2 = 1$$

And solving for x, we get

$$x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

In quadrant I, $x = \frac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or 45 degrees,

$$(x, y) = (x, x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}}$$

$$\cos t = \frac{1}{\sqrt{2}},\ \sin t = \frac{1}{\sqrt{2}}$$

If we then rationalize the denominators, we get

$$\begin{aligned}\cos t &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

$$\begin{aligned}\sin t &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of 45° are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of 30°, or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of 30°, and another at an angle of −30°, as shown in **Figure 11**. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60°, as shown in **Figure 12**.

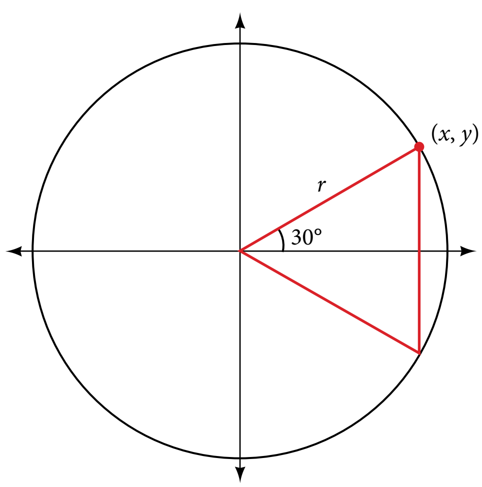


Figure 11

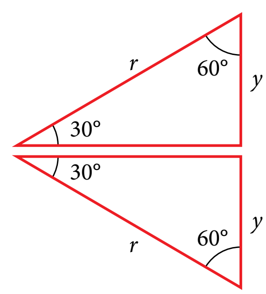


Figure 12

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}r$$

And since $r = 1$ in our unit circle,

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1) \\ &= \frac{1}{2}\end{aligned}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\cos^2 \frac{\pi}{6} + \sin^2\left(\frac{\pi}{6}\right) = 1$$

$$\cos^2\left(\frac{\pi}{6}\right) + \left(\frac{1}{2}\right)^2 = 1$$

$$\cos^2\left(\frac{\pi}{6}\right) = \frac{3}{4} \qquad \text{Use the square root property.}$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\pm\sqrt{3}}{\pm\sqrt{4}} = \frac{\sqrt{3}}{2} \qquad \text{Since } y \text{ is positive, choose the positive root.}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of 30° are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. At $t = \frac{\pi}{3}$ (60°), the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD, as shown in **Figure 13**. Angle A has measure 60°. At point B, we draw an angle ABC with measure of 60°. We know the angles in a triangle sum to 180°, so the measure of angle C is also 60°. Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.

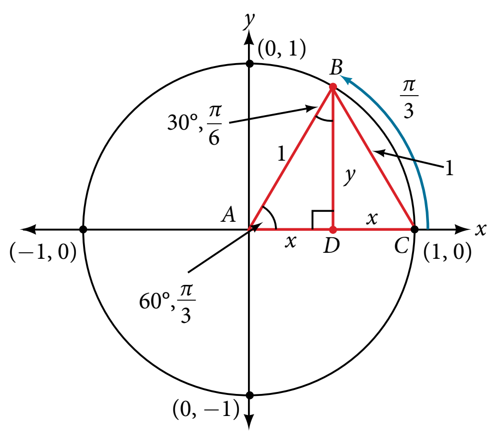


Figure 13

The measure of angle ABD is 30°. So, if double, angle ABC is 60°. BD is the perpendicular bisector of AC, so it cuts AC in half. This means that AD is $\frac{1}{2}$ the radius, or $\frac{1}{2}$. Notice that AD is the x-coordinate of point B, which is at the intersection of the 60° angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2 + y^2 = 1$$

Substituting $x = \frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$

Solving for y, we get

$$\frac{1}{4} + y^2 = 1$$

$$y^2 = 1 - \frac{1}{4}$$

$$y^2 = \frac{3}{4}$$

$$y = \pm\frac{\sqrt{3}}{2}$$

Since $t = \frac{\pi}{3}$ has the terminal side in quadrant I where the y-coordinate is positive, we choose $y = \frac{\sqrt{3}}{2}$, the positive value.

At $t = \frac{\pi}{3}$ (60°), the (x, y) coordinates for the point on a circle of radius 1 at an angle of 60° are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$$

$$\cos t = \frac{1}{2}, \sin t = \frac{\sqrt{3}}{2}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. **Table 1** summarizes these values.

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Table 1

Figure 14 shows the common angles in the first quadrant of the unit circle.

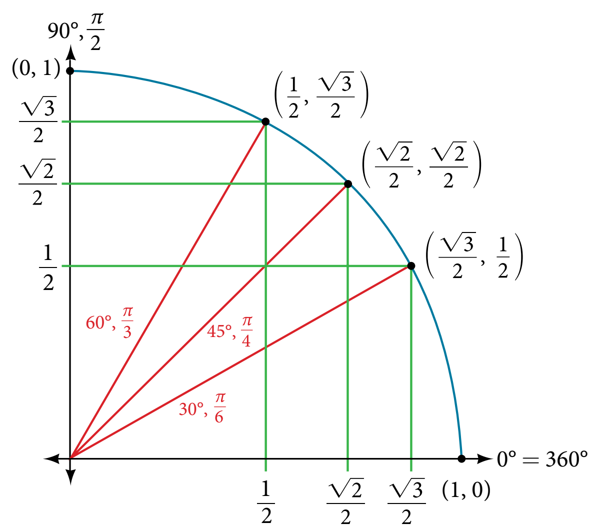


Figure 14

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the special angles, we turn to a computer or calculator. **Be aware**: Most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When we evaluate cos(30) on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To...

Given an angle in radians, use a graphing calculator to find the cosine.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the **COS** key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press **ENTER**.

Example 4 Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos\left(\frac{5\pi}{3}\right)$ using a graphing calculator or computer.

Solution Enter the following keystrokes:

$$\textbf{COS}(5 \times \pi \div 3)\ \textbf{ENTER}$$

$$\cos\left(\frac{5\pi}{3}\right) = 0.5$$

Analysis *We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sign of 20°, for example, by including the conversion factor to radians as part of the input:*

$$\textbf{SIN}(20 \times \pi \div 180)\ \textbf{ENTER}$$

Try It #4

Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x, y, and r, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x-axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the unit circle, as shown in **Figure 15**. The bounds of the x-coordinate are $[-1, 1]$.

The bounds of the y-coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

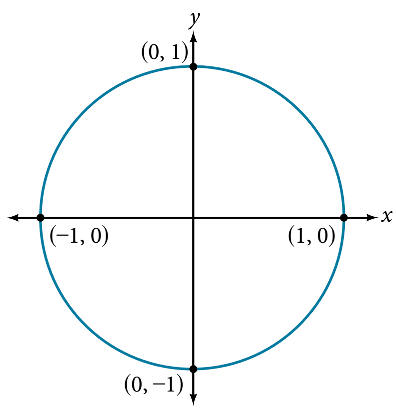


Figure 15

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the y-coordinate on the unit circle, the other angle with the same sine will share the same y-value, but have the opposite x-value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same x-value but will have the opposite y-value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in **Figure 16**, angle α has the same sine value as angle t; the cosine values are opposites. Angle β has the same cosine value as angle t; the sine values are opposites.

$$\sin(t) = \sin(\alpha) \quad \text{and } \cos(t) = -\cos(\alpha)$$

$$\sin(t) = -\sin(\beta) \text{ and } \cos(t) = \cos(\beta)$$

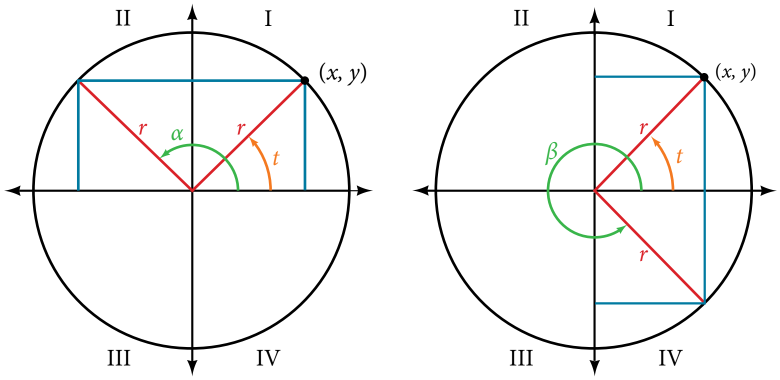


Figure 16

Recall that an angle's reference angle is the acute angle, t, formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and 90°, or 0 and $\frac{\pi}{2}$ radians. As we can see from **Figure 17**, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

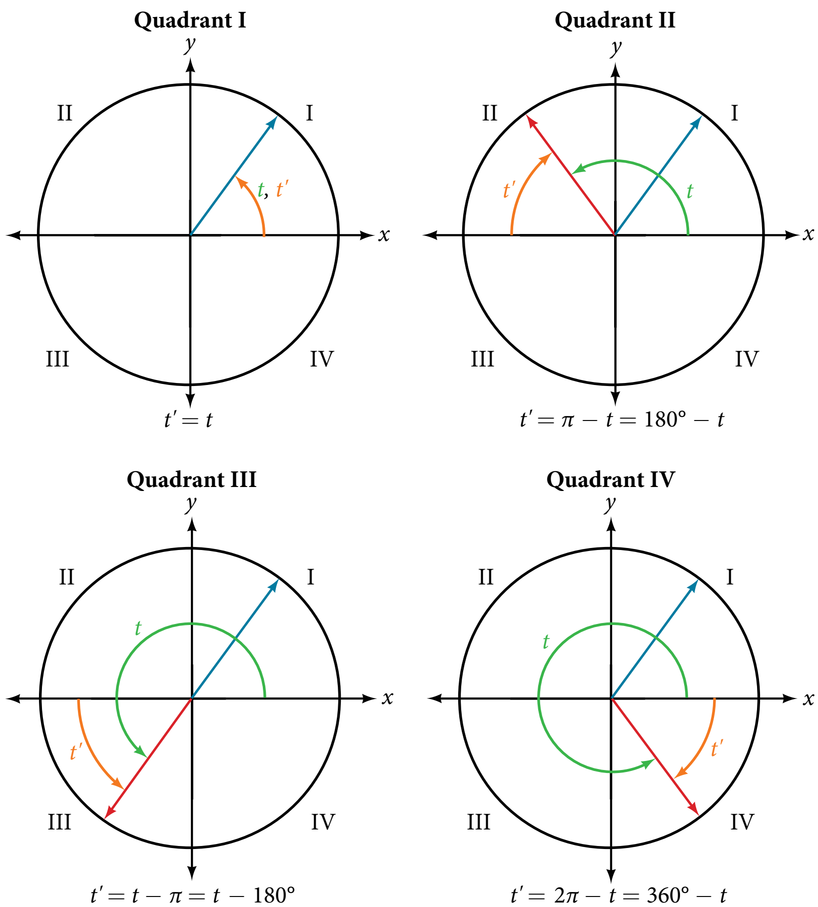


Figure 17

How To...

Given an angle between 0 and 2π, find its reference angle.

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180° - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360° - t$.
4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Example 5 **Finding a Reference Angle**

Find the reference angle of 225° as shown in **Figure 18**.

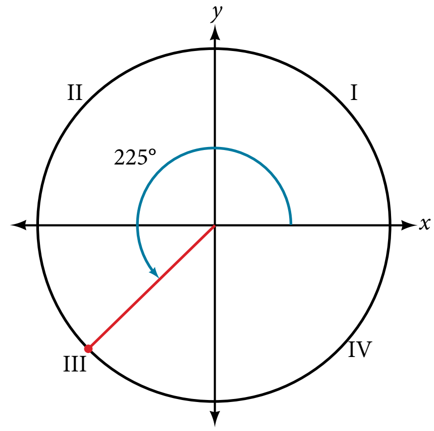


Figure 18

Solution Because 225° is in the third quadrant, the reference angle is

$$|(180° - 225°)| = |-45°| = 45°$$

Try It #5

Find the reference angle of $\frac{5\pi}{3}$.

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a negative angle. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the reference angle of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x-values in that quadrant. The sine will be positive or negative depending on the sign of the y-values in that quadrant.

using reference angles to find cosine and sine

Angles have cosines and sines with the same absolute value as cosines and sines of their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To…

Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle.

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

Example 6 **Using Reference Angles to Find Sine and Cosine**

a. Using a reference angle, find the exact value of $\cos(150°)$ and $\sin(150°)$.

b. Using the reference angle, find $\cos\frac{5\pi}{4}$ and $\sin\frac{5\pi}{4}$.

Solution

a. 150° is located in the second quadrant. The angle it makes with the x-axis is $180° - 150° = 30°$, so the reference angle is 30°.

This tells us that 150° has the same sine and cosine values as 30°, except for the sign. We know that

$$\cos(30°) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(30°) = \frac{1}{2}.$$

Since 150° is in the second quadrant, the x-coordinate of the point on the circle is negative, so the cosine value is negative. The y-coordinate is positive, so the sine value is positive.

$$\cos(150°) = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(150°) = \frac{1}{2}$$

b. $\frac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$. The cosine and sine of $\frac{\pi}{4}$ are both $\frac{\sqrt{2}}{2}$. In the third quadrant, both x and y are negative, so:

$$\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} \text{ and } \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$

Try It #6

a. Use the reference angle of 315° to find $\cos(315°)$ and $\sin(315°)$.

b. Use the reference angle of $-\frac{\pi}{6}$ to find $\cos\left(-\frac{\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$.

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in **Figure 19**. Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant.

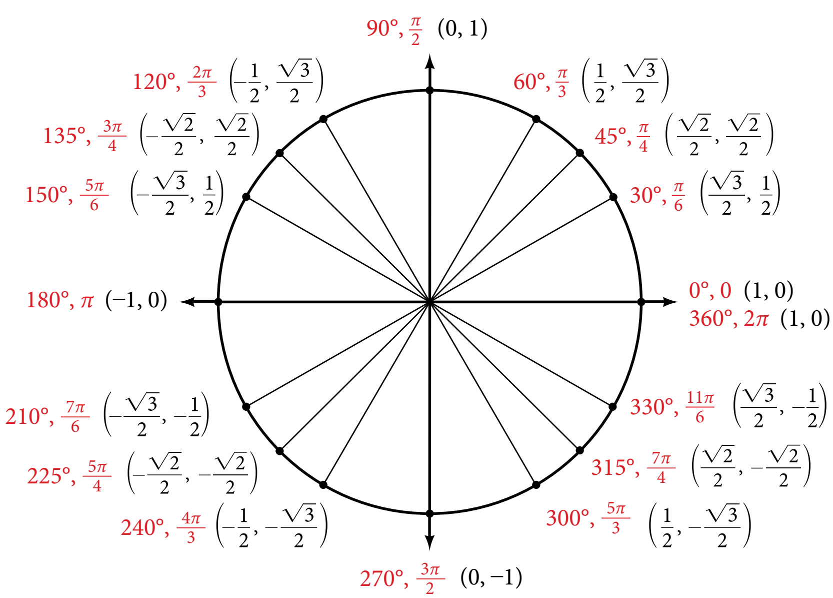


Figure 19 Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the identities

$$x = \cos t \qquad\qquad y = \sin t$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the reference angle, and give them the signs corresponding to the y- and x-values of the quadrant.

How To…

Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point.

1. Find the reference angle by measuring the smallest angle to the x-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Example 7 **Using the Unit Circle to Find Coordinates**

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Solution We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the x-axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π.

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$$

Next, we will find the cosine and sine of the reference angle:

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \text{ and } \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos\,\theta$ and $y = \sin\,\theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

Try It #7

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Access these online resources for additional instruction and practice with sine and cosine functions.

- Trigonometric Functions Using the Unit Circle (http://openstaxcollege.org/l/trigunitcir)
- Sine and Cosine from the Unit Circle (http://openstaxcollege.org/l/sincosuc)
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six (http://openstaxcollege.org/l/sincosmult)
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four (http://openstaxcollege.org/l/sincosmult4)
- Trigonometric Functions Using Reference Angles (http://openstaxcollege.org/l/trigrefang)

5.2 SECTION EXERCISES

VERBAL

1. Describe the unit circle.

2. What do the x- and y-coordinates of the points on the unit circle represent?

3. Discuss the difference between a coterminal angle and a reference angle.

4. Explain how the cosine of an angle in the second quadrant differs from the cosine of its reference angle in the unit circle.

5. Explain how the sine of an angle in the second quadrant differs from the sine of its reference angle in the unit circle.

ALGEBRAIC

For the following exercises, use the given sign of the sine and cosine functions to find the quadrant in which the terminal point determined by t lies.

6. $\sin(t) < 0$ and $\cos(t) < 0$

7. $\sin(t) > 0$ and $\cos(t) > 0$

8. $\sin(t) > 0$ and $\cos(t) < 0$

9. $\sin(t) < 0$ and $\cos(t) > 0$

For the following exercises, find the exact value of each trigonometric function.

10. $\sin \frac{\pi}{2}$

11. $\sin \frac{\pi}{3}$

12. $\cos \frac{\pi}{2}$

13. $\cos \frac{\pi}{3}$

14. $\sin \frac{\pi}{4}$

15. $\cos \frac{\pi}{4}$

16. $\sin \frac{\pi}{6}$

17. $\sin \pi$

18. $\sin \frac{3\pi}{2}$

19. $\cos \pi$

20. $\cos 0$

21. $\cos \frac{\pi}{6}$

22. $\sin 0$

NUMERIC

For the following exercises, state the reference angle for the given angle.

23. $240°$

24. $-170°$

25. $100°$

26. $-315°$

27. $135°$

28. $\frac{5\pi}{4}$

29. $\frac{2\pi}{3}$

30. $\frac{5\pi}{6}$

31. $\frac{-11\pi}{3}$

32. $\frac{-7\pi}{4}$

33. $\frac{-\pi}{8}$

For the following exercises, find the reference angle, the quadrant of the terminal side, and the sine and cosine of each angle. If the angle is not one of the angles on the unit circle, use a calculator and round to three decimal places.

34. $225°$

35. $300°$

36. $320°$

37. $135°$

38. $210°$

39. $120°$

40. $250°$

41. $150°$

42. $\frac{5\pi}{4}$

43. $\frac{7\pi}{6}$

44. $\frac{5\pi}{3}$

45. $\frac{3\pi}{4}$

46. $\frac{4\pi}{3}$

47. $\frac{2\pi}{3}$

48. $\frac{5\pi}{6}$

49. $\frac{7\pi}{4}$

For the following exercises, find the requested value.

50. If $\cos(t) = \frac{1}{7}$ and t is in the 4^{th} quadrant, find $\sin(t)$.

51. If $\cos(t) = \frac{2}{9}$ and t is in the 1^{st} quadrant, find $\sin(t)$.

52. If $\sin(t) = \frac{3}{8}$ and t is in the 2^{nd} quadrant, find $\cos(t)$.

53. If $\sin(t) = -\frac{1}{4}$ and t is in the 3^{rd} quadrant, find $\cos(t)$.

54. Find the coordinates of the point on a circle with radius 15 corresponding to an angle of $220°$.

55. Find the coordinates of the point on a circle with radius 20 corresponding to an angle of $120°$.

56. Find the coordinates of the point on a circle with radius 8 corresponding to an angle of $\frac{7\pi}{4}$.

57. Find the coordinates of the point on a circle with radius 16 corresponding to an angle of $\frac{5\pi}{9.}$

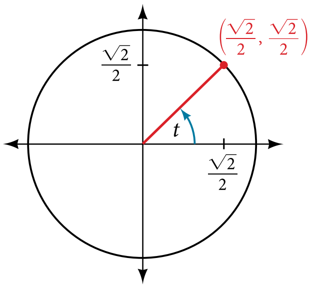
58. State the domain of the sine and cosine functions.

59. State the range of the sine and cosine functions.

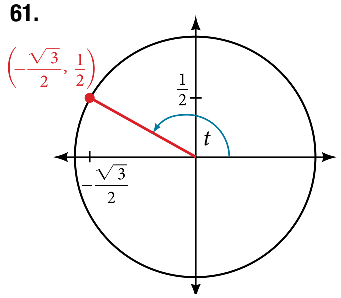
GRAPHICAL

For the following exercises, use the given point on the unit circle to find the value of the sine and cosine of *t*.

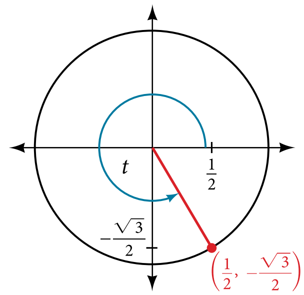
60.



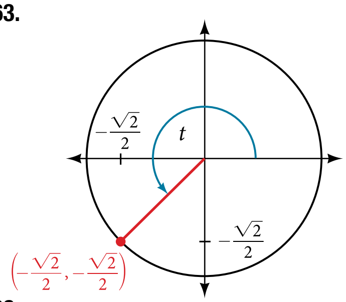
61.



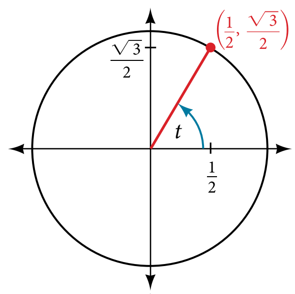
62.



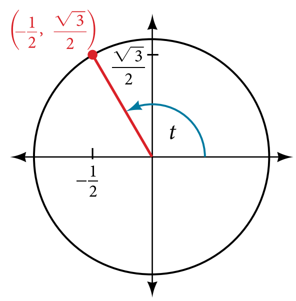
63.



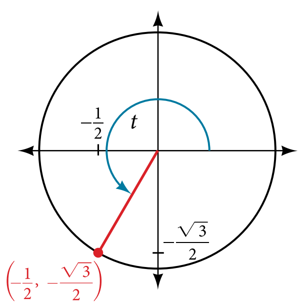
64.



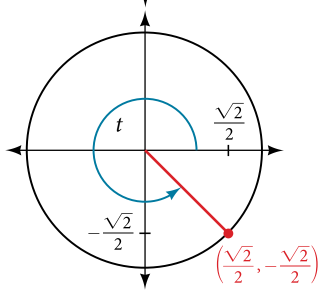
65.



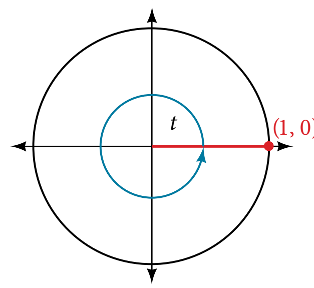
66.



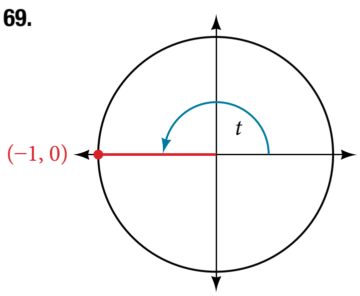
67.



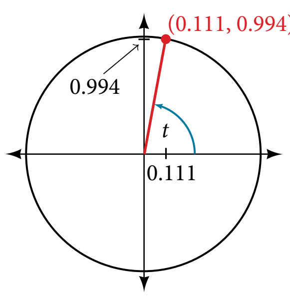
68.



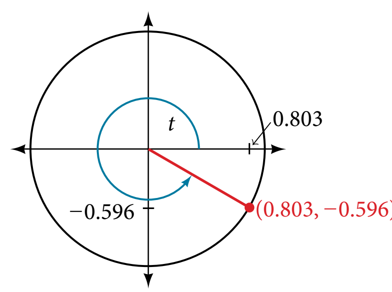
69.



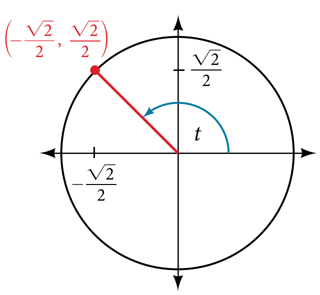
70.



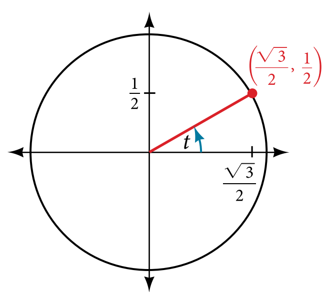
71.



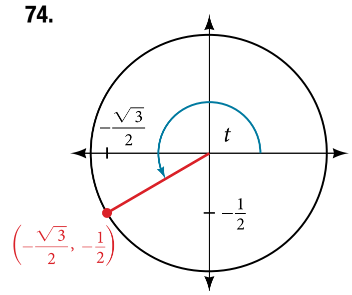
72.



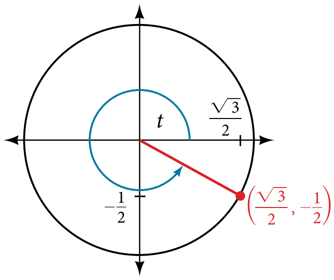
73.



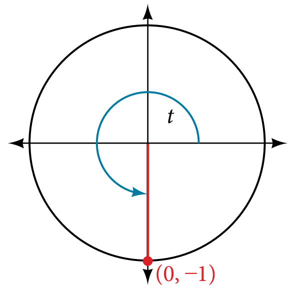
74.



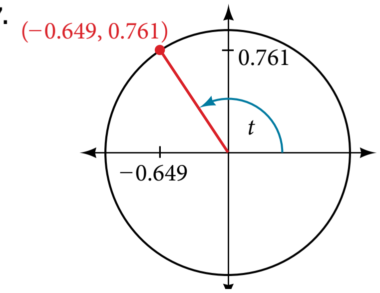
75.



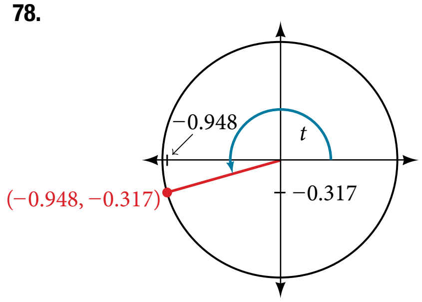
76.



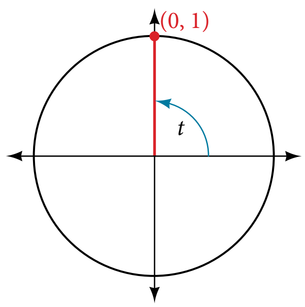
77.



78.



79.



TECHNOLOGY

For the following exercises, use a graphing calculator to evaluate.

80. $\sin \frac{5\pi}{9}$

81. $\cos \frac{5\pi}{9}$

82. $\sin \frac{\pi}{10}$

83. $\cos \frac{\pi}{10}$

84. $\sin \frac{3\pi}{4}$

85. $\cos \frac{3\pi}{4}$

86. $\sin 98°$

87. $\cos 98°$

88. $\cos 310°$

89. $\sin 310°$

EXTENSIONS

90. $\sin\left(\frac{11\pi}{3}\right)\cos\left(\frac{-5\pi}{6}\right)$

91. $\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{5\pi}{3}\right)$

92. $\sin\left(-\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{2}\right)$

93. $\sin\left(\frac{-9\pi}{4}\right)\cos\left(\frac{-\pi}{6}\right)$

94. $\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{-\pi}{3}\right)$

95. $\sin\left(\frac{7\pi}{4}\right)\cos\left(\frac{-2\pi}{3}\right)$

96. $\cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{2\pi}{3}\right)$

97. $\cos\left(\frac{-\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)$

98. $\sin\left(\frac{-5\pi}{4}\right)\sin\left(\frac{11\pi}{6}\right)$

99. $\sin(\pi)\sin\left(\frac{\pi}{6}\right)$

REAL-WORLD APPLICATIONS

For the following exercises, use this scenario: A child enters a carousel that takes one minute to revolve once around. The child enters at the point (0, 1), that is, on the due north position. Assume the carousel revolves counter clockwise.

100. What are the coordinates of the child after 45 seconds?

101. What are the coordinates of the child after 90 seconds?

102. What is the coordinates of the child after 125 seconds?

103. When will the child have coordinates (0.707, –0.707) if the ride lasts 6 minutes? (There are multiple answers.)

104. When will the child have coordinates (−0.866, −0.5) if the ride last 6 minutes?

LEARNING OBJECTIVES

In this section, you will:

- Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.
- Use reference angles to evaluate the trigonometric functions secant, cosecant, tangent, and cotangent.
- Use properties of even and odd trigonometric functions.
- Recognize and use fundamental identities.
- Evaluate trigonometric functions with a calculator.

5.3 THE OTHER TRIGONOMETRIC FUNCTIONS

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is $\frac{1}{12}$ or less, regardless of its length. A tangent represents a ratio, so this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions. We have already defined the sine and cosine functions of an angle. Though sine and cosine are the trigonometric functions most often used, there are four others. Together they make up the set of six trigonometric functions. In this section, we will investigate the remaining functions.

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

To define the remaining functions, we will once again draw a unit circle with a point (x, y) corresponding to an angle of t, as shown in **Figure 1**. As with the sine and cosine, we can use the (x, y) coordinates to find the other functions.

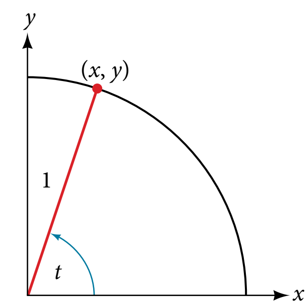


Figure 1

The first function we will define is the tangent. The **tangent** of an angle is the ratio of the y-value to the x-value of the corresponding point on the unit circle. In **Figure 1**, the tangent of angle t is equal to $\frac{y}{x}, x \neq 0$. Because the y-value is equal to the sine of t, and the x-value is equal to the cosine of t, the tangent of angle t can also be defined as $\frac{\sin t}{\cos t}$, $\cos t \neq 0$. The tangent function is abbreviated as tan. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The **secant** function is the reciprocal of the cosine function. In **Figure 1**, the secant of angle t is equal to $\frac{1}{\cos t} = \frac{1}{x}, x \neq 0$. The secant function is abbreviated as sec.
- The **cotangent** function is the reciprocal of the tangent function. In **Figure 1**, the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}, y \neq 0$. The cotangent function is abbreviated as cot.
- The **cosecant** function is the reciprocal of the sine function. In **Figure 1**, the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}, y \neq 0$. The cosecant function is abbreviated as csc.

tangent, secant, cosecant, and cotangent functions

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\tan t = \frac{y}{x}, x \neq 0 \qquad \sec t = \frac{1}{x}, x \neq 0$$
$$\csc t = \frac{1}{y}, y \neq 0 \qquad \cot t = \frac{x}{y}, y \neq 0$$

Example 1 Finding Trigonometric Functions from a Point on the Unit Circle

The point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is on the unit circle, as shown in **Figure 2**. Find sin t, cos t, tan t, sec t, csc t, and cot t.

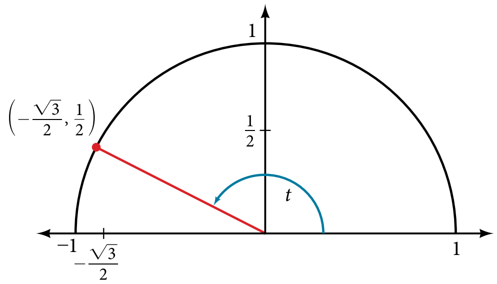


Figure 2

Solution Because we know the (x, y) coordinates of the point on the unit circle indicated by angle t, we can use those coordinates to find the six functions:

$$\sin t = y = \frac{1}{2}$$

$$\cos t = x = -\frac{\sqrt{3}}{2}$$

$$\tan t = \frac{y}{x} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\sec t = \frac{1}{x} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\csc t = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot t = \frac{x}{y} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) = -\sqrt{3}$$

Try It #1

The point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is on the unit circle, as shown in **Figure 3**. Find sin t, cos t, tan t, sec t, csc t, and cot t.

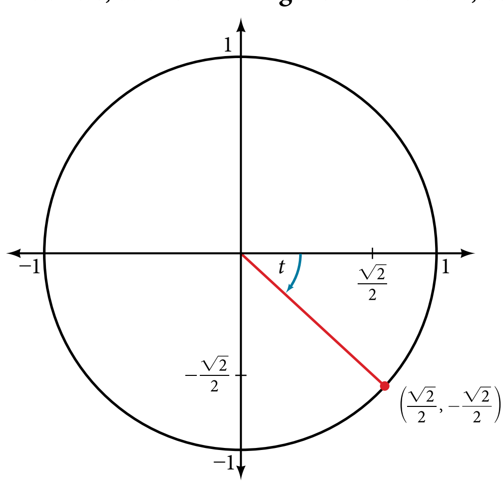


Figure 3

Example 2 Finding the Trigonometric Functions of an Angle

Find sin t, cos t, tan t, sec t, csc t, and cot t when $t = \frac{\pi}{6}$.

Solution We have previously used the properties of equilateral triangles to demonstrate that $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\tan \frac{\pi}{6} = \frac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{6}}$$

$$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sec \frac{\pi}{6} = \frac{1}{\cos \frac{\pi}{6}}$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\csc \frac{\pi}{6} = \frac{1}{\sin \frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot \frac{\pi}{6} = \frac{\cos \frac{\pi}{6}}{\sin \frac{\pi}{6}}$$

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

Try It #2

Find sin t, cos t, tan t, sec t, csc t, and cot t when $t = \frac{\pi}{3}$.

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x equal to the cosine and y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in **Table 1**.

Angle	**0**	$\frac{\pi}{6}$**, or 30°**	$\frac{\pi}{4}$**, or 45°**	$\frac{\pi}{3}$**, or 60°**	$\frac{\pi}{2}$**, or 90°**
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Table 1

Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the reference angle formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by x- and y-values in the original quadrant. **Figure 4** shows which functions are positive in which quadrant.

To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "A Smart Trig Class." Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is "**A**," **a**ll of the six trigonometric functions are positive. In quadrant II, "**S**mart," only **s**ine and its reciprocal function, cosecant, are positive. In quadrant III, "**T**rig," only **t**angent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, "**C**lass," only **c**osine and its reciprocal function, secant, are positive.

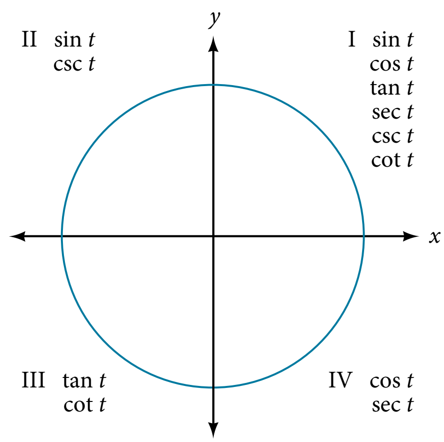


Figure 4

How To...

Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions.

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Example 3 Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

Solution The angle between this angle's terminal side and the x-axis is $\frac{\pi}{6}$, so that is the reference angle. Since $-\frac{5\pi}{6}$ is in the third quadrant, where both x and y are negative, cosine, sine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$$\cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}, \quad \sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2}, \quad \tan\left(-\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{3}$$

$$\sec\left(-\frac{5\pi}{6}\right) = -\frac{2\sqrt{3}}{3}, \quad \csc\left(-\frac{5\pi}{6}\right) = -2, \quad \cot\left(-\frac{5\pi}{6}\right) = \sqrt{3}$$

Try It #3

Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

Using Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard. Consider the function $f(x) = x^2$, shown in **Figure 5**. The graph of the function is symmetrical about the y-axis. All along the curve, any two points with opposite x-values have the same function value. This matches the result of calculation: $(4)^2 = (-4)^2$, $(-5)^2 = (5)^2$, and so on. So $f(x) = x^2$ is an even function, a function such that two inputs that are opposites have the same output. That means $f(-x) = f(x)$.

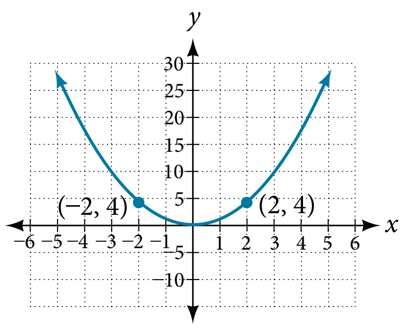


Figure 5 The function $f(x) = x^2$ is an even function.

Now consider the function $f(x) = x^3$, shown in **Figure 6**. The graph is not symmetrical about the y-axis. All along the graph, any two points with opposite x-values also have opposite y-values. So $f(x) = x^3$ is an odd function, one such that two inputs that are opposites have outputs that are also opposites. That means $f(-x) = -f(x)$.

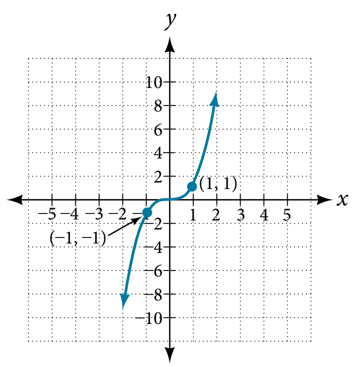


Figure 6 The function $f(x) = x^3$ is an odd function.

We can test whether a trigonometric function is even or odd by drawing a unit circle with a positive and a negative angle, as in **Figure 7**. The sine of the positive angle is y. The sine of the negative angle is $-y$. The sine function, then, is an odd function. We can test each of the six trigonometric functions in this fashion. The results are shown in **Table 2**.

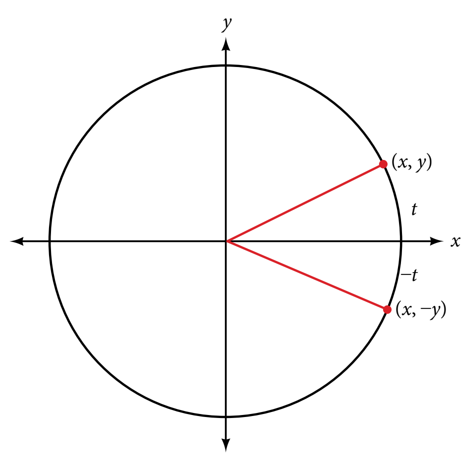


Figure 7

$\sin t = y$ $\sin(-t) = -y$ $\sin t \neq \sin(-t)$	$\cos t = x$ $\cos(-t) = x$ $\cos t = \cos(-t)$	$\tan(t) = \frac{y}{x}$ $\tan(-t) = -\frac{y}{x}$ $\tan t \neq \tan(-t)$
$\sec t = \frac{1}{x}$ $\sec(-t) = \frac{1}{x}$ $\sec t = \sec(-t)$	$\csc t = \frac{1}{y}$ $\csc(-t) = \frac{1}{-y}$ $\csc t \neq \csc(-t)$	$\cot t = \frac{x}{y}$ $\cot(-t) = \frac{x}{-y}$ $\cot t \neq \cot(-t)$

Table 2

even and odd trigonometric functions

An even function is one in which $f(-x) = f(x)$. An odd function is one in which $f(-x) = -f(x)$.
Cosine and secant are even:

$$\cos(-t) = \cos t$$
$$\sec(-t) = \sec t$$

Sine, tangent, cosecant, and cotangent are odd:

$$\sin(-t) = -\sin t$$
$$\tan(-t) = -\tan t$$
$$\csc(-t) = -\csc t$$
$$\cot(-t) = -\cot t$$

Example 4 Using Even and Odd Properties of Trigonometric Functions

If the secant of angle t is 2, what is the secant of $-t$?

Solution Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

Try It #4

If the cotangent of angle t is $\sqrt{3}$, what is the cotangent of $-t$?

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

fundamental identities

We can derive some useful identities from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t} \qquad \sec t = \frac{1}{\cos t}$$

$$\csc t = \frac{1}{\sin t} \qquad \cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$$

Example 5 Using Identities to Evaluate Trigonometric Functions

a. Given $\sin(45°) = \frac{\sqrt{2}}{2,}$, $\cos(45°) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45°)$.

b. Given $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$, $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, evaluate $\sec\left(\frac{5\pi}{6}\right)$.

Solution Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

a.

$$\begin{aligned}\tan(45°) &= \frac{\sin(45°)}{\cos(45°)}\\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\\ &= 1\end{aligned}$$

b.

$$\begin{aligned}\sec\left(\frac{5\pi}{6}\right) &= \frac{1}{\left(\cos\frac{5\pi}{6}\right)}\\ &= \frac{1}{-\frac{\sqrt{3}}{2}}\\ &= \frac{-2}{\sqrt{3}}\\ &= -\frac{2\sqrt{3}}{3}\end{aligned}$$

Try It #5

Evaluate $\csc\left(\frac{7\pi}{6}\right)$.

Example 6 Using Identities to Simplify Trigonometric Expressions

Simplify $\frac{\sec t}{\tan t}$.

Solution We can simplify this by rewriting both functions in terms of sine and cosine.

$$\frac{\sec t}{\tan t} = \frac{\frac{1}{\cos t}}{\frac{\sin t}{\cos t}}$$ To divide the functions, we multiply by the reciprocal.

$$= \frac{1 \ \cos t}{\cos t \ \sin t}$$ Divide out the cosines.

$$= \frac{1}{\sin t}$$ Simplify and use the identity.

$$= \csc t$$

By showing that $\frac{\sec t}{\tan t}$ can be simplified to $\csc t$, we have, in fact, established a new identity.

$$\frac{\sec t}{\tan t} = \csc t$$

Try It #6

Simplify $(\tan t)(\cos t)$.

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternative forms of the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$:

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

$$1 + \tan^2 t = \sec^2 t$$

The other form is obtained by dividing both sides by $\sin^2 t$:

$$\frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} = \frac{1}{\sin^2 t}$$

$$\cot^2 t + 1 = \csc^2 t$$

alternate forms of the pythagorean identity

$$1 + \tan^2 t = \sec^2 t$$

$$\cot^2 t + 1 = \csc^2 t$$

Example 7 Using Identities to Relate Trigonometric Functions

If $\cos(t) = \frac{12}{13}$ and t is in quadrant IV, as shown in **Figure 8**, find the values of the other five trigonometric functions.

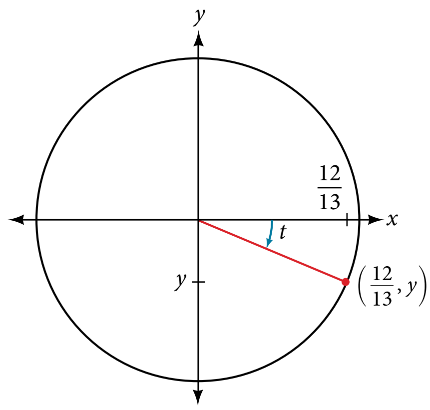


Figure 8

Solution We can find the sine using the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$, and the remaining functions by relating them to sine and cosine.

$$\left(\frac{12}{13}\right)^2 + \sin^2 t = 1$$

$$\sin^2 t = 1 - \left(\frac{12}{13}\right)^2$$

$$\sin^2 t = 1 - \frac{144}{169}$$

$$\sin^2 t = \frac{25}{169}$$

$$\sin t = \pm\sqrt{\frac{25}{169}}$$

$$\sin t = \pm\frac{\sqrt{25}}{\sqrt{169}}$$

$$\sin t = \pm\frac{5}{13}$$

The sign of the sine depends on the y-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the y-values are negative, its sine is negative, $-\frac{5}{13}$.

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{12}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{12}{13}} = \frac{13}{12}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5}$$

$$\cot t = \frac{1}{\tan \mathrm{t}} = \frac{1}{-\frac{5}{12}} = -\frac{12}{5}$$

Try It #7

If $\sec(t) = -\frac{17}{8}$ and $0 < t < \pi$, find the values of the other five functions.

As we discussed in the chapter opening, a function that repeats its values in regular intervals is known as a periodic function. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or 2π, will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x + 4) = f(x)$. This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

period of a function

The **period** P of a repeating function f is the number representing the interval such that $f(x + P) = f(x)$ for any value of x.
The period of the cosine, sine, secant, and cosecant functions is 2π.
The period of the tangent and cotangent functions is π.

Example 8 **Finding the Values of Trigonometric Functions**

Find the values of the six trigonometric functions of angle t based on **Figure 9**.

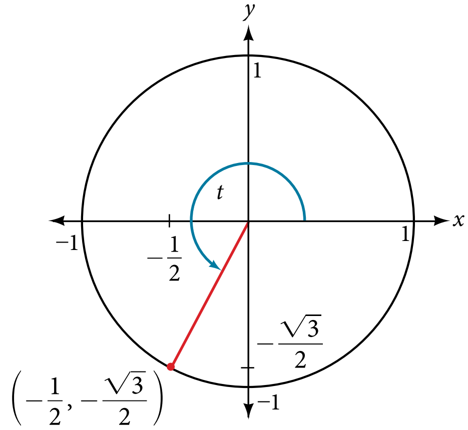


Figure 9

Solution

$$\sin t = y = -\frac{\sqrt{3}}{2}$$

$$\cos t = x = -\frac{1}{2}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{-\frac{1}{2}} = -2$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Try It #8

Find the values of the six trigonometric functions of angle t based on **Figure 10**.

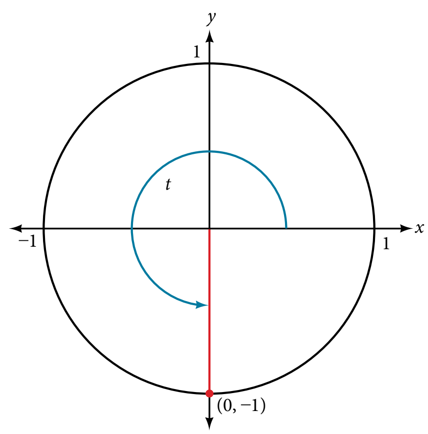


Figure 10

Example 9 Finding the Value of Trigonometric Functions

If $\sin(t) = -\frac{\sqrt{3}}{2}$ and $\cos(t) = \frac{1}{2}$, find $\sec(t)$, $\csc(t)$, $\tan(t)$, $\cot(t)$.

Solution

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{2}} = 2$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Try It #9

If $\sin(t) = \frac{\sqrt{2}}{2}$ and $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sec(t)$, $\csc(t)$, $\tan(t)$, and $\cot(t)$.

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the **TAN** key. For the reciprocal functions, there may not be any dedicated keys that say **CSC**, **SEC**, or **COT**. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor $\frac{\pi}{180}$ to convert the degrees to radians. To find the secant of 30°, we could press

$$\text{(for a scientific calculator): } \frac{1}{30 \times \frac{\pi}{180}} \text{ COS} \qquad \text{or} \qquad \text{(for a graphing calculator): } \frac{1}{\cos\left(\frac{30\pi}{180}\right)}$$

How To…

Given an angle measure in radians, use a scientific calculator to find the cosecant.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: **1 /**
3. Enter the value of the angle inside parentheses.
4. Press the **SIN** key.
5. Press the = key.

How To…

Given an angle measure in radians, use a graphing utility/calculator to find the cosecant.

1. If the graphing utility has degree mode and radian mode, set it to radian mode.
2. Enter: **1 /**
3. Press the **SIN** key.
4. Enter the value of the angle inside parentheses.
5. Press the **ENTER** key.

Example 10 Evaluating the Secant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}$.

Solution

For a scientific calculator, enter information as follows:

$$1 \,/\, (\, 5 \times \pi \,/\, 7 \,)\ \text{SIN} =$$

$$\csc\left(\frac{5\pi}{7}\right) \approx 1.279$$

Try It #10

Evaluate the cotangent of $-\frac{\pi}{8}$.

Access these online resources for additional instruction and practice with other trigonometric functions.

- Determining Trig Function Values (http://openstaxcollege.org/l/trigfuncval)
- More Examples of Determining Trig Functions (http://openstaxcollege.org/l/moretrigfun)
- Pythagorean Identities (http://openstaxcollege.org/l/pythagiden)
- Trig Functions on a Calculator (http://openstaxcollege.org/l/trigcalc)

5.3 SECTION EXERCISES

VERBAL

1. On an interval of $[0, 2\pi)$, can the sine and cosine values of a radian measure ever be equal? If so, where?

2. What would you estimate the cosine of π degrees to be? Explain your reasoning.

3. For any angle in quadrant II, if you knew the sine of the angle, how could you determine the cosine of the angle?

4. Describe the secant function.

5. Tangent and cotangent have a period of π. What does this tell us about the output of these functions?

ALGEBRAIC

For the following exercises, find the exact value of each expression.

6. $\tan \frac{\pi}{6}$

7. $\sec \frac{\pi}{6}$

8. $\csc \frac{\pi}{6}$

9. $\cot \frac{\pi}{6}$

10. $\tan \frac{\pi}{4}$

11. $\sec \frac{\pi}{4}$

12. $\csc \frac{\pi}{4}$

13. $\cot \frac{\pi}{4}$

14. $\tan \frac{\pi}{3}$

15. $\sec \frac{\pi}{3}$

16. $\csc \frac{\pi}{3}$

17. $\cot \frac{\pi}{3}$

For the following exercises, use reference angles to evaluate the expression.

18. $\tan \frac{5\pi}{6}$

19. $\sec \frac{7\pi}{6}$

20. $\csc \frac{11\pi}{6}$

21. $\cot \frac{13\pi}{6}$

22. $\tan \frac{7\pi}{4}$

23. $\sec \frac{3\pi}{4}$

24. $\csc \frac{5\pi}{4}$

25. $\cot \frac{11\pi}{4}$

26. $\tan \frac{8\pi}{3}$

27. $\sec \frac{4\pi}{3}$

28. $\csc \frac{2\pi}{3}$

29. $\cot \frac{5\pi}{3}$

30. $\tan 225°$

31. $\sec 300°$

32. $\csc 150°$

33. $\cot 240°$

34. $\tan 330°$

35. $\sec 120°$

36. $\csc 210°$

37. $\cot 315°$

38. If $\sin t = \frac{3}{4}$, and t is in quadrant II, find $\cos t$, $\sec t$, $\csc t$, $\tan t$, $\cot t$.

39. If $\cos t = -\frac{1}{3}$, and t is in quadrant III, find $\sin t$, $\sec t$, $\csc t$, $\tan t$, $\cot t$.

40. If $\tan t = \frac{12}{5}$, and $0 \le t < \frac{\pi}{2}$, find $\sin t$, $\cos t$, $\sec t$, $\csc t$, and $\cot t$.

41. If $\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2}$, find $\sec t$, $\csc t$, $\tan t$, and $\cot t$.

42. If $\sin 40° \approx 0.643$ and $\cos 40° \approx 0.766$, find $\sec 40°$, $\csc 40°$, $\tan 40°$, and $\cot 40°$.

43. If $\sin t = \frac{\sqrt{2}}{2}$, what is the $\sin(-t)$?

44. If $\cos t = \frac{1}{2}$, what is the $\cos(-t)$?

45. If $\sec t = 3.1$, what is the $\sec(-t)$?

46. If $\csc t = 0.34$, what is the $\csc(-t)$?

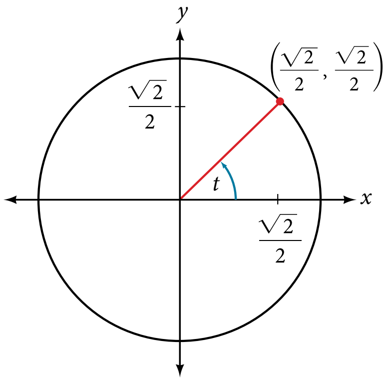
47. If $\tan t = -1.4$, what is the $\tan(-t)$?

48. If $\cot t = 9.23$, what is the $\cot(-t)$?

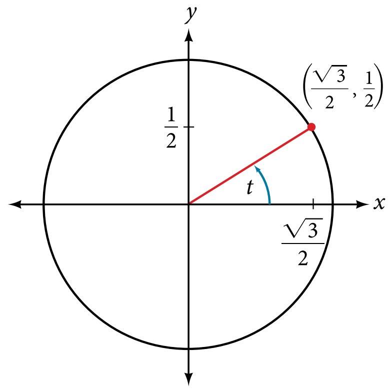
GRAPHICAL

For the following exercises, use the angle in the unit circle to find the value of the each of the six trigonometric functions.

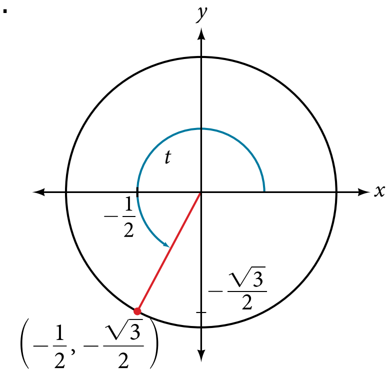
49.



50.



51.



TECHNOLOGY

For the following exercises, use a graphing calculator to evaluate.

52. $\csc \frac{5\pi}{9}$

53. $\cot \frac{4\pi}{7}$

54. $\sec \frac{\pi}{10}$

55. $\tan \frac{5\pi}{8}$

56. $\sec \frac{3\pi}{4}$

57. $\csc \frac{\pi}{4}$

58. $\tan 98°$

59. $\cot 33°$

60. $\cot 140°$

61. $\sec 310°$

EXTENSIONS

For the following exercises, use identities to evaluate the expression.

62. If $\tan(t) \approx 2.7$, and $\sin(t) \approx 0.94$, find $\cos(t)$.

63. If $\tan(t) \approx 1.3$, and $\cos(t) \approx 0.61$, find $\sin(t)$.

64. If $\csc(t) \approx 3.2$, and $\cos(t) \approx 0.95$, find $\tan(t)$.

65. If $\cot(t) \approx 0.58$, and $\cos(t) \approx 0.5$, find $\csc(t)$.

66. Determine whether the function $f(x) = 2\sin x \cos x$ is even, odd, or neither.

67. Determine whether the function $f(x) = 3\sin^2 x \cos x + \sec x$ is even, odd, or neither.

68. Determine whether the function $f(x) = \sin x - 2\cos^2 x$ is even, odd, or neither.

69. Determine whether the function $f(x) = \csc^2 x + \sec x$ is even, odd, or neither.

For the following exercises, use identities to simplify the expression.

70. $\csc t \tan t$

71. $\frac{\sec t}{\csc t}$

REAL-WORLD APPLICATIONS

72. The amount of sunlight in a certain city can be modeled by the function $h = 15 \cos\left(\frac{1}{600}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on February 10, the 42nd day of the year. State the period of the function.

73. The amount of sunlight in a certain city can be modeled by the function $h = 16\cos\left(\frac{1}{500}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on September 24, the 267th day of the year. State the period of the function.

74. The equation $P = 20\sin(2\pi t) + 100$ models the blood pressure, P, where t represents time in seconds. **a.** Find the blood pressure after 15 seconds. **b.** What are the maximum and minimum blood pressures?

75. The height of a piston, h, in inches, can be modeled by the equation $y = 2\cos x + 6$, where x represents the crank angle. Find the height of the piston when the crank angle is 55°.

76. The height of a piston, h, in inches, can be modeled by the equation $y = 2\cos x + 5$, where x represents the crank angle. Find the height of the piston when the crank angle is 55°.

LEARNING OBJECTIVES

In this section, you will:

- Use right triangles to evaluate trigonometric functions.
- Find function values for $30°\left(\frac{\pi}{6}\right)$, $45°\left(\frac{\pi}{4}\right)$, and $60°\left(\frac{\pi}{3}\right)$.
- Use cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right triangle trigonometry to solve applied problems.

5.4 RIGHT TRIANGLE TRIGONOMETRY

We have previously defined the sine and cosine of an angle in terms of the coordinates of a point on the unit circle intersected by the terminal side of the angle:

$$\cos t = x$$

$$\sin t = y$$

In this section, we will see another way to define trigonometric functions using properties of right triangles.

Using Right Triangles to Evaluate Trigonometric Functions

In earlier sections, we used a unit circle to define the trigonometric functions. In this section, we will extend those definitions so that we can apply them to right triangles. The value of the sine or cosine function of t is its value at t radians. First, we need to create our right triangle. **Figure 1** shows a point on a unit circle of radius 1. If we drop a vertical line segment from the point (x, y) to the x-axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x. We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

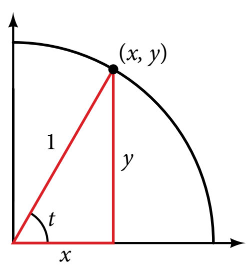


Figure 1

We know

$$\cos t = \frac{x}{1} = x$$

Likewise, we know

$$\sin t = \frac{y}{1} = y$$

These ratios still apply to the sides of a right triangle when no unit circle is involved and when the triangle is not in standard position and is not being graphed using (x, y) coordinates. To be able to use these ratios freely, we will give the sides more general names: Instead of x, we will call the side between the given angle and the right angle the **adjacent side** to angle t. (Adjacent means "next to.") Instead of y, we will call the side most distant from the given angle the **opposite side** from angle t. And instead of 1, we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in **Figure 2**.

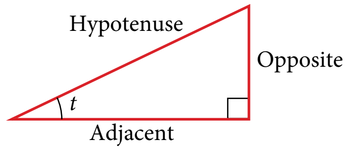


Figure 2 The sides of a right triangle in relation to angle *t*.

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of t,

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

How To…

Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle.

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent as the ratio of the opposite side to the adjacent side.

Example 1 Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in **Figure 3**, find the value of $\cos \alpha$.

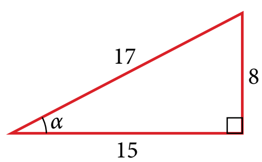


Figure 3

Solution The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so:

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{15}{17}$$

Try It #1

Given the triangle shown in **Figure 4**, find the value of $\sin t$.

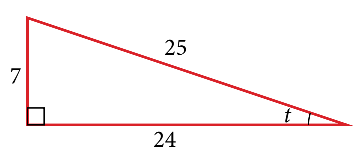


Figure 4

Relating Angles and Their Functions

When working with right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in **Figure 5**. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

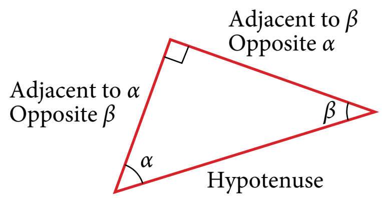


Figure 5 The side adjacent to one angle is opposite the other

We will be asked to find all six trigonometric functions for a given angle in a triangle. Our strategy is to find the sine, cosine, and tangent of the angles first. Then, we can find the other trigonometric functions easily because we know that the reciprocal of sine is cosecant, the reciprocal of cosine is secant, and the reciprocal of tangent is cotangent.

How To…

Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles.

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Example 2 Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in **Figure 6**, evaluate sin α, cos α, tan α, sec α, csc α, and cot α.

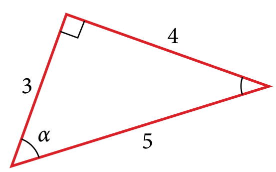


Figure 6

Solution

$$\sin \alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$

$$\cos \alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$

$$\tan \alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3}$$

$$\sec \alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3}$$

$$\csc \alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4}$$

$$\cot \alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

Try It #2

Using the triangle shown in **Figure 7**, evaluate sin t, cos t, tan t, sec t, csc t, and cot t.

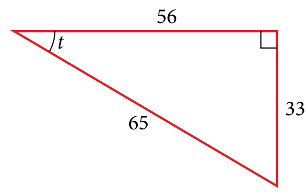


Figure 7

Finding Trigonometric Functions of Special Angles Using Side Lengths

We have already discussed the trigonometric functions as they relate to the special angles on the unit circle. Now, we can use those relationships to evaluate triangles that contain those special angles. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. We will use multiples of 30°, 60°, and 45°, however, remember that when dealing with right triangles, we are limited to angles between 0° and 90°.

Suppose we have a 30°, 60°, 90° triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a 45°, 45°, 90° triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in **Figure 8**.

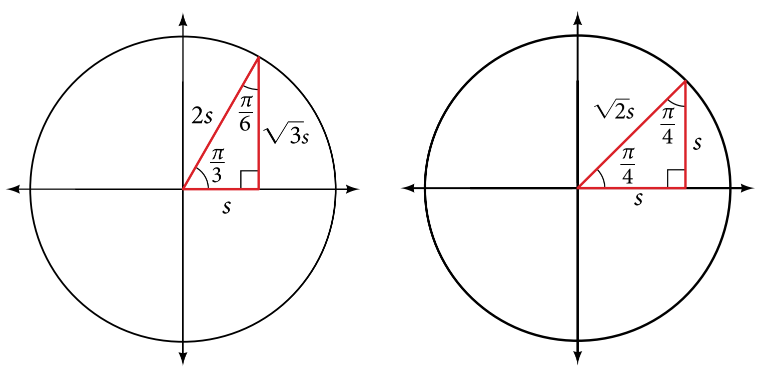


Figure 8 Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

How To...

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in **Figure 8** for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 3 Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\sin\left(\frac{\pi}{3}\right) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3}$$

$$\sec\left(\frac{\pi}{3}\right) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$
$$\csc\left(\frac{\pi}{3}\right) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\cot\left(\frac{\pi}{3}\right) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Try It #3

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$, using side lengths.

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$ namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$.

$$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

See **Figure 9.**

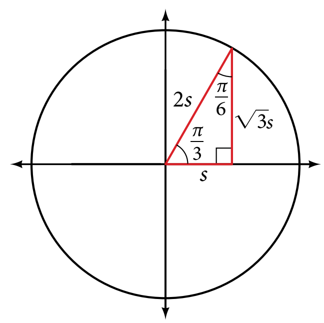


Figure 9 The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa

This result should not be surprising because, as we see from **Figure 9**, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{6}\right)$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{6}\right)$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in **Figure 10**.

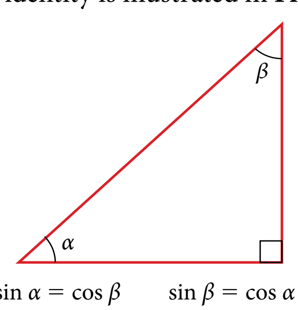


Figure 10 Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t, $\cos t = \frac{5}{13}$, then $\sin\left(\frac{\pi}{2} - t\right) = \frac{5}{13}$ as well.

cofunction identities

The cofunction identities in radians are listed in **Table 1.**

$\cos t = \sin\left(\frac{\pi}{2} - t\right)$	$\sin t = \cos\left(\frac{\pi}{2} - t\right)$
$\tan t = \cot\left(\frac{\pi}{2} - t\right)$	$\cot t = \tan\left(\frac{\pi}{2} - t\right)$
$\sec t = \csc\left(\frac{\pi}{2} - t\right)$	$\csc t = \sec\left(\frac{\pi}{2} - t\right)$

Table 1

How To...

Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 4 Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $\cos\left(\frac{\pi}{2} - t\right)$.

Solution According to the cofunction identities for sine and cosine,

$$\sin t = \cos\left(\frac{\pi}{2} - t\right).$$

So

$$\cos\left(\frac{\pi}{2} - t\right) = \frac{5}{12}.$$

Try It #4

If $\csc\left(\frac{\pi}{6}\right) = 2$, find $\sec\left(\frac{\pi}{3}\right)$.

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

How To...

Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides.

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 5 Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in **Figure 11**.

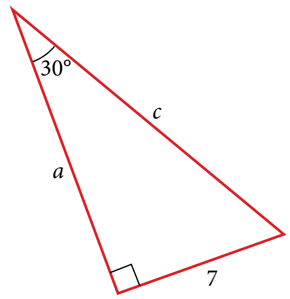


Figure 11

Solution We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30°) = \frac{7}{a}$$

We rearrange to solve for a.

$$\begin{aligned} a &= \frac{7}{\tan(30°)} \\ &\approx 12.1 \end{aligned}$$

We can use the sine to find the hypotenuse.

$$\sin(30°) = \frac{7}{c}$$

Again, we rearrange to solve for c.

$$\begin{aligned} c &= \frac{7}{\sin(30°)} \\ &\approx 14 \end{aligned}$$

Try It #5

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See **Figure 12**.

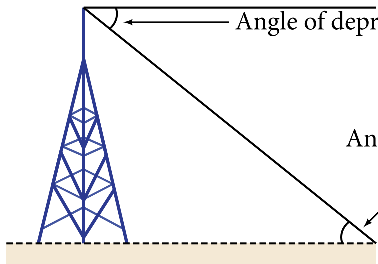


Figure 12

How To…

Given a tall object, measure its height indirectly.

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 6 Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° between a line of sight to the top of the tree and the ground, as shown in **Figure 13**. Find the height of the tree.

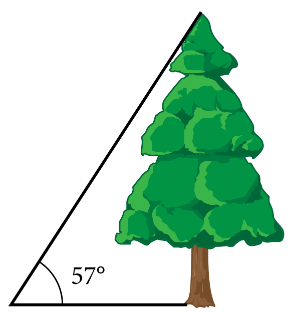


Figure 13

Solution We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of 57°, letting h be the unknown height.

$$\begin{aligned} \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} & \\ \tan(57°) &= \frac{h}{30} & \text{Solve for } h. \\ h &= 30\tan(57°) & \text{Multiply.} \\ h &\approx 46.2 & \text{Use a calculator.} \end{aligned}$$

The tree is approximately 46 feet tall.

Try It #6

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

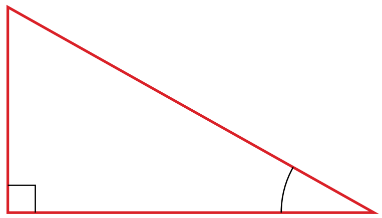
Access these online resources for additional instruction and practice with right triangle trigonometry.

- Finding Trig Functions on Calculator (http://openstaxcollege.org/l/findtrigcal)
- Finding Trig Functions Using a Right Triangle (http://openstaxcollege.org/l/trigrttri)
- Relate Trig Functions to Sides of a Right Triangle (http://openstaxcollege.org/l/reltrigtri)
- Determine Six Trig Functions from a Triangle (http://openstaxcollege.org/l/sixtrigfunc)
- Determine Length of Right Triangle Side (http://openstaxcollege.org/l/rttriside)

5.4 SECTION EXERCISES

VERBAL

1. For the given right triangle, label the adjacent side, opposite side, and hypotenuse for the indicated angle.



2. When a right triangle with a hypotenuse of 1 is placed in the unit circle, which sides of the triangle correspond to the x- and y-coordinates?

3. The tangent of an angle compares which sides of the right triangle?

4. What is the relationship between the two acute angles in a right triangle?

5. Explain the cofunction identity.

ALGEBRAIC

For the following exercises, use cofunctions of complementary angles.

6. $\cos(34°) = \sin(\_\_\_\_\_\_°)$

7. $\cos\left(\frac{\pi}{3}\right) = \sin(\_\_\_\_\_\_)$

8. $\csc(21°) = \sec(\_\_\_\_\_\_°)$

9. $\tan\left(\frac{\pi}{4}\right) = \cot(\_\_\_\_\_\_)$

For the following exercises, find the lengths of the missing sides if side a is opposite angle A, side b is opposite angle B, and side c is the hypotenuse.

10. $\cos B = \frac{4}{5}, a = 10$

11. $\sin B = \frac{1}{2}, a = 20$

12. $\tan A = \frac{5}{12}, b = 6$

13. $\tan A = 100, b = 100$

14. $\sin B = \frac{1}{\sqrt{3}}, a = 2$

15. $a = 5, \measuredangle A = 60°$

16. $c = 12, \measuredangle A = 45°$

GRAPHICAL

For the following exercises, use **Figure 14** to evaluate each trigonometric function of angle A.

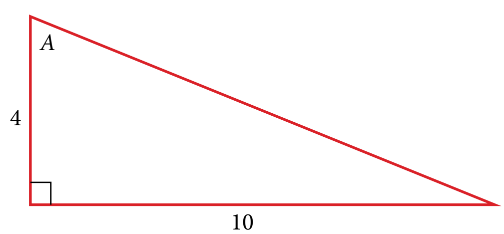


Figure 14

17. $\sin A$

18. $\cos A$

19. $\tan A$

20. $\csc A$

21. $\sec A$

22. $\cot A$

For the following exercises, use **Figure 15** to evaluate each trigonometric function of angle A.

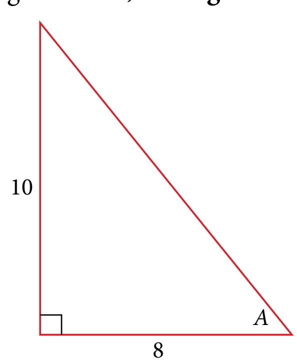


Figure 15

23. $\sin A$

24. $\cos A$

25. $\tan A$

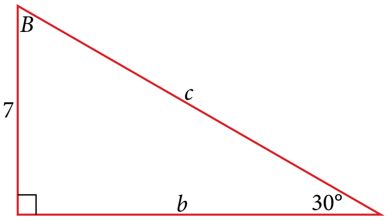
26. $\csc A$

27. $\sec A$

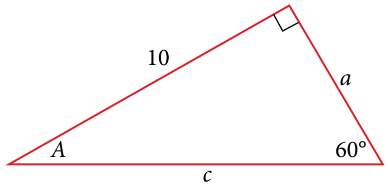
28. $\cot A$

For the following exercises, solve for the unknown sides of the given triangle.

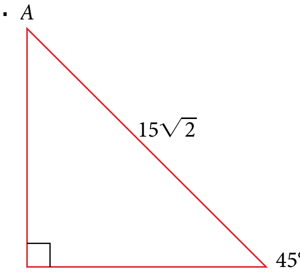
29.



30.



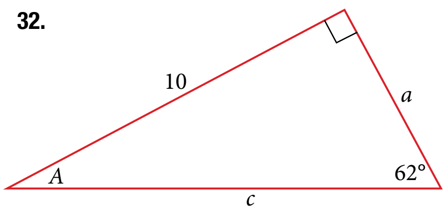
31.



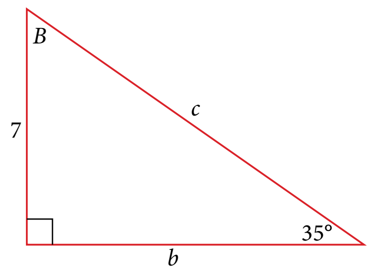
TECHNOLOGY

For the following exercises, use a calculator to find the length of each side to four decimal places.

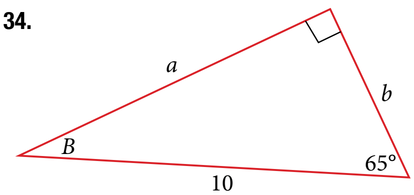
32.



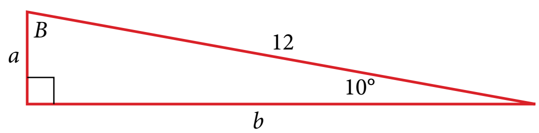
33.



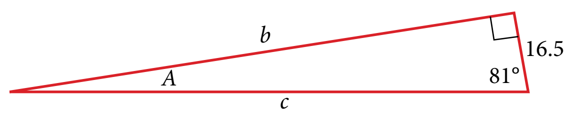
34.



35.



36.



37. $b = 15, \measuredangle B = 15°$

38. $c = 200, \measuredangle B = 5°$

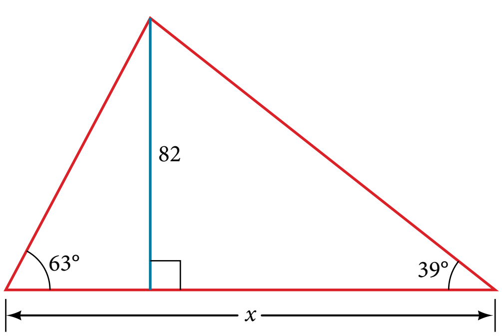
39. $c = 50, \measuredangle B = 21°$

40. $a = 30, \measuredangle A = 27°$

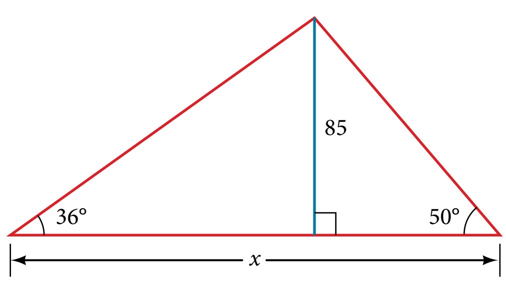
41. $b = 3.5, \measuredangle A = 78°$

EXTENSIONS

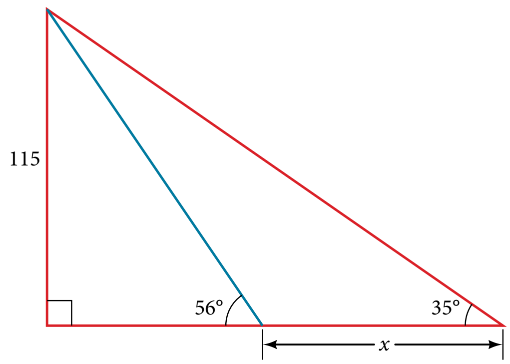
42. Find x.



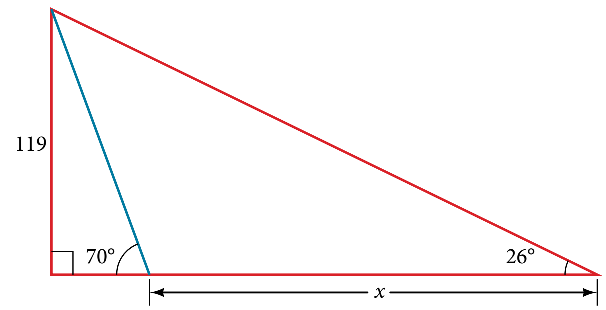
43. Find x.



44. Find x.



45. Find x.



46. A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 36°, and that the angle of depression to the bottom of the tower is 23°. How tall is the tower?

47. A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 43°, and that the angle of depression to the bottom of the tower is 31°. How tall is the tower?

48. A 200-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 15°, and that the angle of depression to the bottom of the tower is 2°. How far is the person from the monument?

49. A 400-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 18°, and that the angle of depression to the bottom of the monument is 3°. How far is the person from the monument?

50. There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be 40°. From the same location, the angle of elevation to the top of the antenna is measured to be 43°. Find the height of the antenna.

51. There is lightning rod on the top of a building. From a location 500 feet from the base of the building, the angle of elevation to the top of the building is measured to be 36°. From the same location, the angle of elevation to the top of the lightning rod is measured to be 38°. Find the height of the lightning rod.

REAL-WORLD APPLICATIONS

52. A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

53. A 23-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

54. The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

55. The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.

56. Assuming that a 370-foot tall giant redwood grows vertically, if I walk a certain distance from the tree and measure the angle of elevation to the top of the tree to be 60°, how far from the base of the tree am I?

CHAPTER 5 REVIEW

Key Terms

adjacent side in a right triangle, the side between a given angle and the right angle

angle the union of two rays having a common endpoint

angle of depression the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

angular speed the angle through which a rotating object travels in a unit of time

arc length the length of the curve formed by an arc

area of a sector area of a portion of a circle bordered by two radii and the intercepted arc; the fraction $\frac{\theta}{2\pi}$ multiplied by the area of the entire circle

cosecant the reciprocal of the sine function: on the unit circle, $\csc t = \frac{1}{y}, y \neq 0$

cosine function the x-value of the point on a unit circle corresponding to a given angle

cotangent the reciprocal of the tangent function: on the unit circle, $\cot t = \frac{x}{y}, y \neq 0$

coterminal angles description of positive and negative angles in standard position sharing the same terminal side

degree a unit of measure describing the size of an angle as one-360th of a full revolution of a circle

hypotenuse the side of a right triangle opposite the right angle

identities statements that are true for all values of the input on which they are defined

initial side the side of an angle from which rotation begins

linear speed the distance along a straight path a rotating object travels in a unit of time; determined by the arc length

measure of an angle the amount of rotation from the initial side to the terminal side

negative angle description of an angle measured clockwise from the positive x-axis

opposite side in a right triangle, the side most distant from a given angle

period the smallest interval P of a repeating function f such that $f(x + P) = f(x)$

positive angle description of an angle measured counterclockwise from the positive x-axis

Pythagorean Identity a corollary of the Pythagorean Theorem stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

quadrantal angle an angle whose terminal side lies on an axis

radian the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle

radian measure the ratio of the arc length formed by an angle divided by the radius of the circle

ray one point on a line and all points extending in one direction from that point; one side of an angle

reference angle the measure of the acute angle formed by the terminal side of the angle and the horizontal axis

secant the reciprocal of the cosine function: on the unit circle, $\sec t = \frac{1}{x}, x \neq 0$

sine function the y-value of the point on a unit circle corresponding to a given angle

standard position the position of an angle having the vertex at the origin and the initial side along the positive x-axis

tangent the quotient of the sine and cosine: on the unit circle, $\tan t = \frac{y}{x}, x \neq 0$

terminal side the side of an angle at which rotation ends

unit circle a circle with a center at $(0, 0)$ and radius 1.

vertex the common endpoint of two rays that form an angle

Key Equations

arc length	$s = r\theta$
area of a sector	$A = \frac{1}{2}\theta r^2$
angular speed	$\omega = \frac{\theta}{t}$
linear speed	$v = \frac{s}{t}$
linear speed related to angular speed	$v = r\omega$
cosine	$\cos t = x$
sine	$\sin t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$
tangent function	$\tan t = \frac{\sin t}{\cos t}$
secant function	$\sec t = \frac{1}{\cos t}$
cosecant function	$\csc t = \frac{1}{\sin t}$
cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$
cofunction identities	$\cos t = \sin\left(\frac{\pi}{2} - t\right)$
	$\sin t = \cos\left(\frac{\pi}{2} - t\right)$
	$\tan t = \cot\left(\frac{\pi}{2} - t\right)$
	$\cot t = \tan\left(\frac{\pi}{2} - t\right)$
	$\sec t = \csc\left(\frac{\pi}{2} - t\right)$
	$\csc t = \sec\left(\frac{\pi}{2} - t\right)$

Key Concepts

5.1 Angles

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive x-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive x-axis and then place the terminal side according to the fraction of a full rotation the angle represents. See **Example 1**.
- In addition to degrees, the measure of an angle can be described in radians. See **Example 2**.
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{\theta^R}{\pi}$. See **Example 3** and **Example 4**.
- Two angles that have the same terminal side are called coterminal angles.
- We can find coterminal angles by adding or subtracting 360° or 2π. See **Example 5** and **Example 6**.
- Coterminal angles can be found using radians just as they are for degrees. See **Example 7**.
- The length of a circular arc is a fraction of the circumference of the entire circle. See **Example 8**.

- The area of sector is a fraction of the area of the entire circle. See **Example 9**.
- An object moving in a circular path has both linear and angular speed.
- The angular speed of an object traveling in a circular path is the measure of the angle through which it turns in a unit of time. See **Example 10**.
- The linear speed of an object traveling along a circular path is the distance it travels in a unit of time. See **Example 11**.

5.2 Unit Circle: Sine and Cosine Functions

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t whereas the cosine of an angle t equals the x-value of the endpoint. See **Example 1**.
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis. See **Example 2**.
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles. See **Example 3**.
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known. See **Example 4**.
- The domain of the sine and cosine functions is all real numbers.
- The range of both the sine and cosine functions is $[-1, 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the x- and y-values in the quadrant of the original angle.
- An angle's reference angle is the size angle, t, formed by the terminal side of the angle t and the horizontal axis. See **Example 5**.
- Reference angles can be used to find the sine and cosine of the original angle. See **Example 6**.
- Reference angles can also be used to find the coordinates of a point on a circle. See **Example 7**.

5.3 The Other Trigonometric Functions

- The tangent of an angle is the ratio of the y-value to the x-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.
- The six trigonometric functions can be found from a point on the unit circle. See **Example 1**.
- Trigonometric functions can also be found from an angle. See **Example 2**.
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles. See **Example 3**.
- A function is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions. See **Example 4**.
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.
- Identities can be used to evaluate trigonometric functions. See **Example 5** and **Example 6**.
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities. See **Example 7**.
- The trigonometric functions repeat at regular intervals.
- The period P of a repeating function f is the smallest interval such that $f(x + P) = f(x)$ for any value of x.
- The values of trigonometric functions of special angles can be found by mathematical analysis.
- To evaluate trigonometric functions of other angles, we can use a calculator or computer software. See **Example 8**.

5.4 Right Triangle Trigonometry

- We can define trigonometric functions as ratios of the side lengths of a right triangle. See **Example 1**.
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See **Example 2**.
- We can evaluate the trigonometric functions of special angles, knowing the side lengths of the triangles in which they occur. See **Example 3**.
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See **Example 4**.
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See **Example 5**.
- Right-triangle trigonometry permits the measurement of inaccessible heights and distances.
- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See **Example 6**.

CHAPTER 5 REVIEW EXERCISES

ANGLES

For the following exercises, convert the angle measures to degrees.

1. $\frac{\pi}{4}$

2. $-\frac{5\pi}{3}$

For the following exercises, convert the angle measures to radians.

3. $-210°$

4. $180°$

5. Find the length of an arc in a circle of radius 7 meters subtended by the central angle of 85°.

6. Find the area of the sector of a circle with diameter 32 feet and an angle of $\frac{3\pi}{5}$ radians.

For the following exercises, find the angle between 0° and 360° that is coterminal with the given angle.

7. $420°$

8. $-80°$

For the following exercises, find the angle between 0 and 2π in radians that is coterminal with the given angle.

9. $-\frac{20\pi}{11}$

10. $\frac{14\pi}{5}$

For the following exercises, draw the angle provided in standard position on the Cartesian plane.

11. $-210°$

12. $75°$

13. $\frac{5\pi}{4}$

14. $-\frac{\pi}{3}$

15. Find the linear speed of a point on the equator of the earth if the earth has a radius of 3,960 miles and the earth rotates on its axis every 24 hours. Express answer in miles per hour.

16. A car wheel with a diameter of 18 inches spins at the rate of 10 revolutions per second. What is the car's speed in miles per hour?

UNIT CIRCLE: SINE AND COSINE FUNCTIONS

17. Find the exact value of $\sin \frac{\pi}{3}$.

18. Find the exact value of $\cos \frac{\pi}{4}$.

19. Find the exact value of $\cos \pi$.

20. State the reference angle for 300°.

21. State the reference angle for $\frac{3\pi}{4}$.

22. Compute cosine of 330°.

23. Compute sine of $\frac{5\pi}{4}$.

24. State the domain of the sine and cosine functions.

25. State the range of the sine and cosine functions.

THE OTHER TRIGONOMETRIC FUNCTIONS

For the following exercises, find the exact value of the given expression.

26. $\cos \frac{\pi}{6}$

27. $\tan \frac{\pi}{4}$

28. $\csc \frac{\pi}{3}$

29. $\sec \frac{\pi}{4}$

For the following exercises, use reference angles to evaluate the given expression.

30. $\sec \frac{11\pi}{3}$

31. $\sec 315°$

32. If $\sec(t) = -2.5$, what is the $\sec(-t)$?

33. If $\tan(t) = -0.6$, what is the $\tan(-t)$?

34. If $\tan(t) = \frac{1}{3}$, find $\tan(t - \pi)$.

35. If $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sin(t + 2\pi)$.

36. Which trigonometric functions are even?

37. Which trigonometric functions are odd?

RIGHT TRIANGLE TRIGONOMETRY

For the following exercises, use side lengths to evaluate.

38. $\cos \frac{\pi}{4}$

39. $\cot \frac{\pi}{3}$

40. $\tan \frac{\pi}{6}$

41. $\cos\left(\frac{\pi}{2}\right) = \sin(\_\_\_\_\_°)$

42. $\csc(18°) = \sec(\_\_\_\_\_°)$

For the following exercises, use the given information to find the lengths of the other two sides of the right triangle.

43. $\cos B = \frac{3}{5}, a = 6$

44. $\tan A = \frac{5}{9}, b = 6$

For the following exercises, use **Figure 1** to evaluate each trigonometric function.

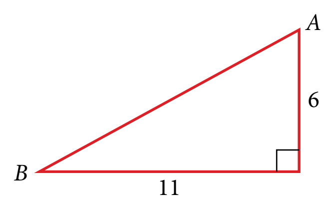


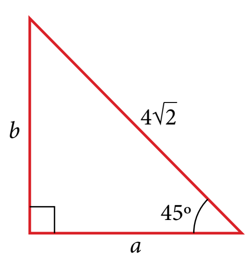
Figure 1

45. $\sin A$

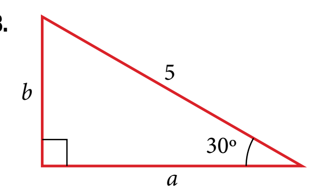
46. $\tan B$

For the following exercises, solve for the unknown sides of the given triangle.

47.



48.



49. A 15-ft ladder leans against a building so that the angle between the ground and the ladder is 70°. How high does the ladder reach up the side of the building?

50. The angle of elevation to the top of a building in Baltimore is found to be 4 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

CHAPTER 5 PRACTICE TEST

1. Convert $\frac{5\pi}{6}$ radians to degrees.

2. Convert $-620°$ to radians.

3. Find the length of a circular arc with a radius 12 centimeters subtended by the central angle of 30°.

4. Find the area of the sector with radius of 8 feet and an angle of $\frac{5\pi}{4}$ radians.

5. Find the angle between 0° and 360° that is coterminal with 375°.

6. Find the angle between 0 and 2π in radians that is coterminal with $-\frac{4\pi}{7}$.

7. Draw the angle 315° in standard position on the Cartesian plane.

8. Draw the angle $-\frac{\pi}{6}$ in standard position on the Cartesian plane.

9. A carnival has a Ferris wheel with a diameter of 80 feet. The time for the Ferris wheel to make one revolution is 75 seconds. What is the linear speed in feet per second of a point on the Ferris wheel? What is the angular speed in radians per second?

10. Find the exact value of $\sin \frac{\pi}{6}$.

11. Compute sine of 240°.

12. State the domain of the sine and cosine functions.

13. State the range of the sine and cosine functions.

14. Find the exact value of $\cot \frac{\pi}{4}$.

15. Find the exact value of $\tan \frac{\pi}{3}$.

16. Use reference angles to evaluate $\csc \frac{7\pi}{4}$.

17. Use reference angles to evaluate tan 210°.

18. If $\csc t = 0.68$, what is the $\csc(-t)$?

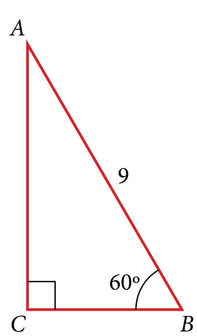
19. If $\cos t = \frac{\sqrt{3}}{2}$, find $\cos(t - 2\pi)$.

20. Which trigonometric functions are even?

21. Find the missing angle: $\cos\left(\frac{\pi}{6}\right) = \sin(\_\_\_\_\_\_)$

22. Find the missing sides of the triangle ABC: $\sin B = \frac{3}{4}, c = 12$

23. Find the missing sides of the triangle.



24. The angle of elevation to the top of a building in Chicago is found to be 9 degrees from the ground at a distance of 2,000 feet from the base of the building. Using this information, find the height of the building.

6

Periodic Functions

Figure 1 (credit: "Maxxer_", Flickr)

CHAPTER OUTLINE

Introduction

Each day, the sun rises in an easterly direction, approaches some maximum height relative to the celestial equator, and sets in a westerly direction. The celestial equator is an imaginary line that divides the visible universe into two halves in much the same way Earth's equator is an imaginary line that divides the planet into two halves. The exact path the sun appears to follow depends on the exact location on Earth, but each location observes a predictable pattern over time.

The pattern of the sun's motion throughout the course of a year is a periodic function. Creating a visual representation of a periodic function in the form of a graph can help us analyze the properties of the function. In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

LEARNING OBJECTIVES

In this section, you will:

- Graph variations of $y = \sin(x)$ and $y = \cos(x)$.
- Use phase shifts of sine and cosine curves.

6.1 GRAPHS OF THE SINE AND COSINE FUNCTIONS

Figure 1 Light can be separated into colors because of its wavelike properties. (credit: "wonderferret"/ Flickr)

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow.

Light waves can be represented graphically by the sine function. In the chapter on **Trigonometric Functions**, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the x- and y-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. We can create a table of values and use them to sketch a graph. **Table 1** lists some of the values for the sine function on a unit circle.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Table 1

Plotting the points from the table and continuing along the x-axis gives the shape of the sine function. See **Figure 2**.

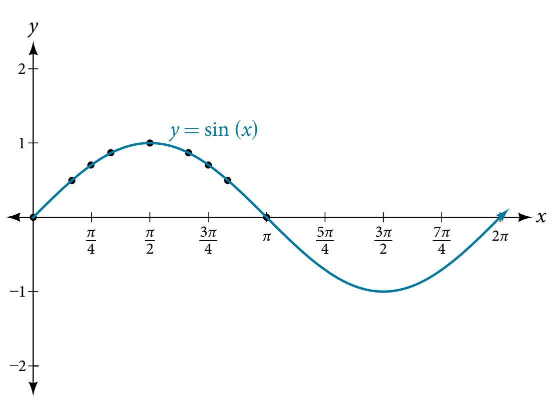


Figure 2 The sine function

Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in quadrants III and IV on the unit circle. See **Figure 3**.

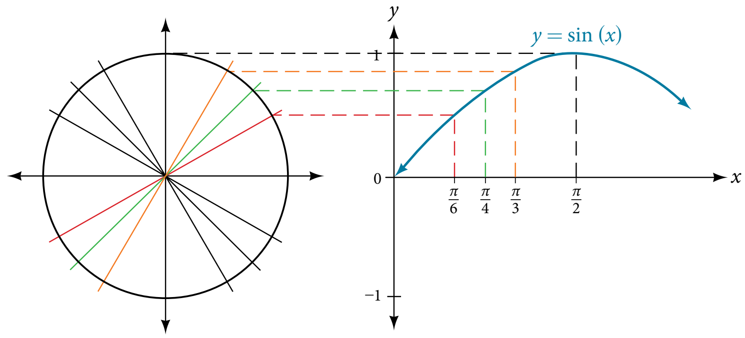


Figure 3 Plotting values of the sine function

Now let's take a similar look at the cosine function. Again, we can create a table of values and use them to sketch a graph. **Table 2** lists some of the values for the cosine function on a unit circle.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Table 2

As with the sine function, we can plots points to create a graph of the cosine function as in **Figure 4**.

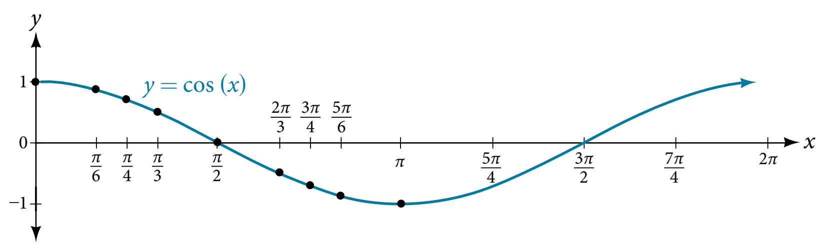


Figure 4 The cosine function

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

In both graphs, the shape of the graph repeats after 2π, which means the functions are periodic with a period of 2π. A **periodic function** is a function for which a specific horizontal shift, P, results in a function equal to the original function: $f(x + P) = f(x)$ for all values of x in the domain of f. When this occurs, we call the smallest such horizontal shift with $P > 0$ the period of the function. **Figure 5** shows several periods of the sine and cosine functions.

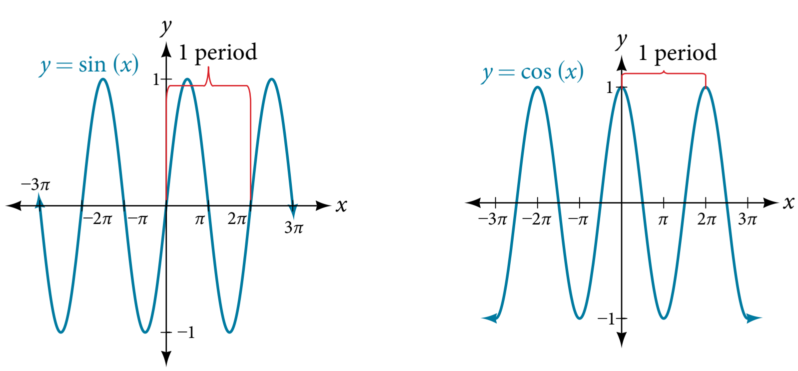


Figure 5

Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries. As we can see in **Figure 6**, the sine function is symmetric about the origin. Recall from **The Other Trigonometric Functions** that we determined from the unit circle that the sine function is an odd function because $\sin(-x) = -\sin x$. Now we can clearly see this property from the graph.

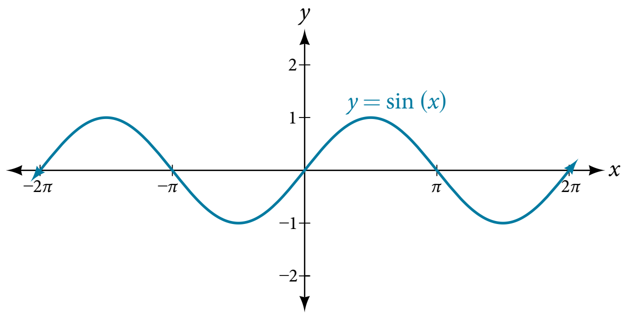


Figure 6 Odd symmetry of the sine function

Figure 7 shows that the cosine function is symmetric about the y-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos(-x) = \cos x$.

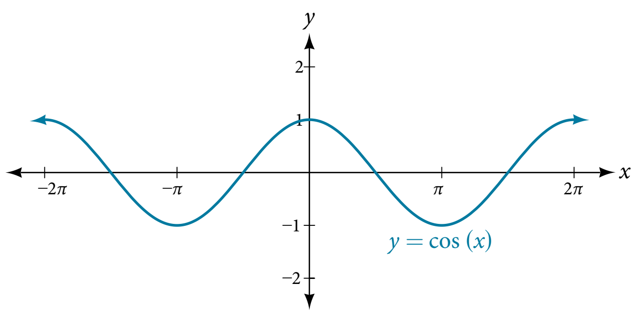


Figure 7 Even symmetry of the cosine function

characteristics of sine and cosine functions

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- The graph of $y = \sin x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos x$ is symmetric about the y- axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or cosine function is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$y = A\sin(Bx - C) + D$$

and

$$y = A\cos(Bx - C) + D$$

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \frac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is 2π, which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is π and the graph is compressed. If $f(x) = \sin\left(\frac{x}{2}\right)$, then $B = \frac{1}{2}$, so the period is 4π and the graph is stretched. Notice in **Figure 8** how the period is indirectly related to $|B|$.

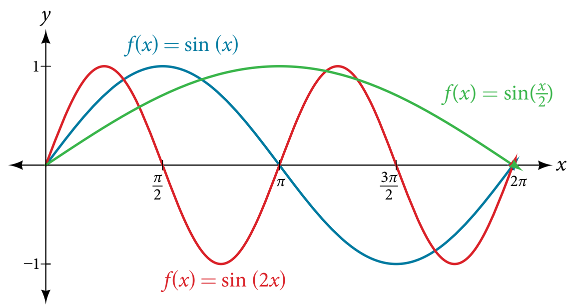


Figure 8

> ***period of sinusoidal functions***
>
> If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms
>
> $$y = A\sin(Bx) \qquad y = A\cos(Bx)$$
>
> The period is $\frac{2\pi}{|B|}$.

Example 1 **Identifying the Period of a Sine or Cosine Function**

Determine the period of the function $f(x) = \sin\left(\frac{\pi}{6}x\right)$.

Solution Let's begin by comparing the equation to the general form $y = A\sin(Bx)$.

In the given equation, $B = \frac{\pi}{6}$, so the period will be

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{6}} \\ &= 2\pi \cdot \frac{6}{\pi} \\ &= 12 \end{aligned}$$

Try It #1

Determine the period of the function $g(x) = \left(\cos\frac{x}{3}\right)$.

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the vertical **midline** of the graph, which is the line $x = D$; because $D = 0$ in this case, the midline is the x-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4\sin x$ is twice the amplitude of $f(x) = 2\sin x$. If $|A| < 1$, the function is compressed. **Figure 9** compares several sine functions with different amplitudes.

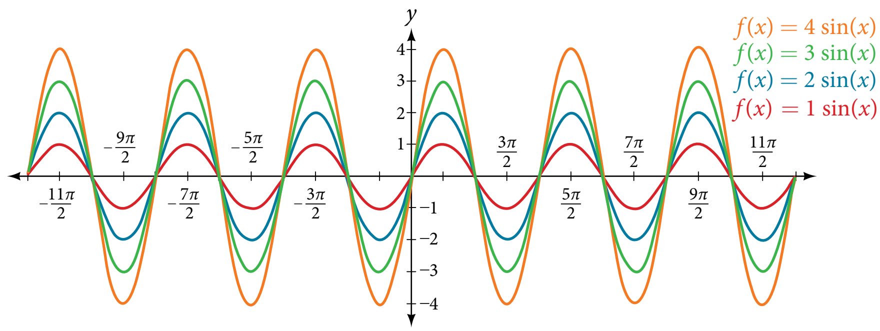


Figure 9

> ***amplitude of sinusoidal functions***
>
> If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms
>
> $$y = A\sin(Bx) \text{ and } y = A\cos(Bx)$$
>
> The **amplitude** is A, and the vertical height from the **midline** is $|A|$. In addition, notice in the example that
>
> $$|A| = \text{amplitude} = \frac{1}{2}|\text{maximum} - \text{minimum}|$$

Example 2 Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

Solution Let's begin by comparing the function to the simplified form $y = A\sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched.

Analysis *The negative value of A results in a reflection across the x-axis of the sine function, as shown in* ***Figure 10****.*

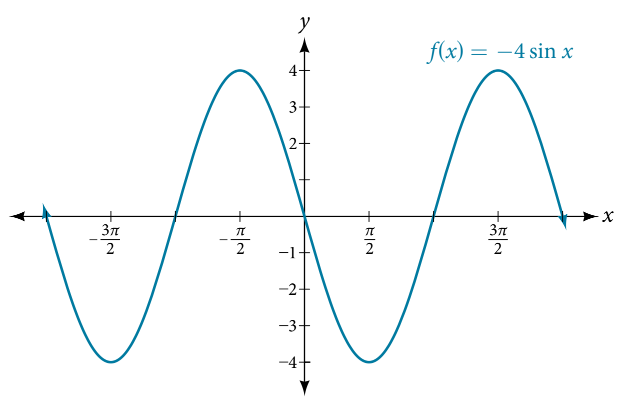


Figure 10

Try It #2

What is the amplitude of the sinusoidal function $f(x) = \frac{1}{2}\sin(x)$? Is the function stretched or compressed vertically?

Analyzing Graphs of Variations of $y = \sin x$ and $y = \cos x$

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D. Recall the general form:

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D$$

or

$$y = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \text{ and } y = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D$$

The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left. The greater the value of $|C|$, the more the graph is shifted. **Figure 11** shows that the graph of $f(x) = \sin(x - \pi)$ shifts to the right by π units, which is more than we see in the graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.

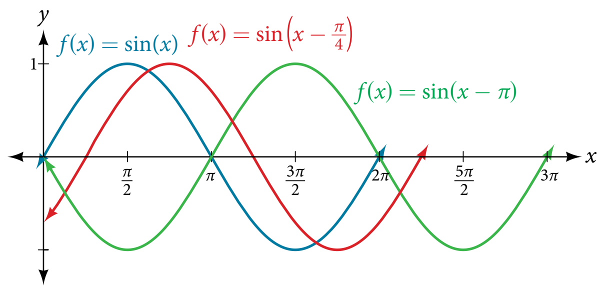


Figure 11

While C relates to the horizontal shift, D indicates the vertical shift from the midline in the general formula for a sinusoidal function. See **Figure 12**. The function $y = \cos(x) + D$ has its midline at $y = D$.

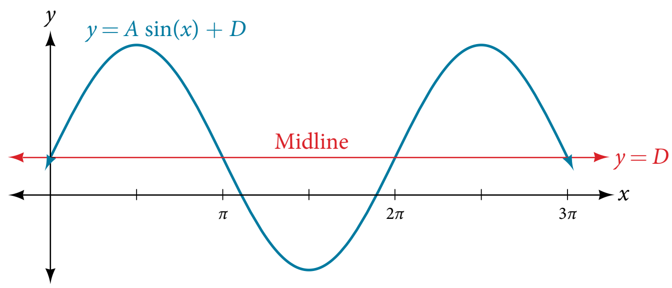


Figure 12

Any value of D other than zero shifts the graph up or down. **Figure 13** compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

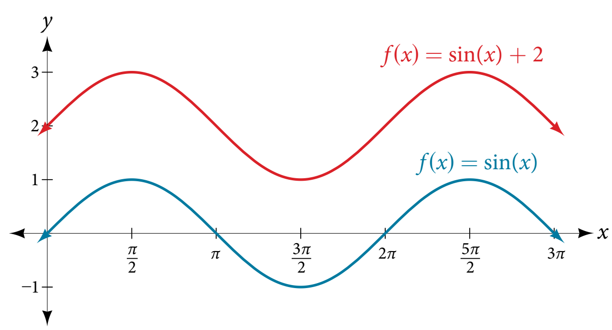


Figure 13

variations of sine and cosine functions

Given an equation in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, $\frac{C}{B}$ is the **phase shift** and D is the vertical shift.

Example 3 **Identifying the Phase Shift of a Function**

Determine the direction and magnitude of the phase shift for $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$.

Solution Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

In the given equation, notice that $B = 1$ and $C = -\frac{\pi}{6}$. So the phase shift is

$$\begin{aligned}\frac{C}{B} &= -\frac{\frac{\pi}{6}}{1}\\ &= -\frac{\pi}{6}\end{aligned}$$

or $\frac{\pi}{6}$ units to the left.

Analysis *We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore* $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$ *can be rewritten as* $f(x) = \sin\left(x - \left(-\frac{\pi}{6}\right)\right) - 2.$ *If the value of C is negative, the shift is to the left.*

Try It #3

Determine the direction and magnitude of the phase shift for $f(x) = 3\cos\left(x - \frac{\pi}{2}\right)$.

Example 4 Identifying the Vertical Shift of a Function

Determine the direction and magnitude of the vertical shift for $f(x) = \cos(x) - 3$.

Solution Let's begin by comparing the equation to the general form $y = A\cos(Bx - C) + D$.

In the given equation, $D = -3$ so the shift is 3 units downward.

Try It #4

Determine the direction and magnitude of the vertical shift for $f(x) = 3\sin(x) + 2$.

How To...

Given a sinusoidal function in the form $f(x) = A\sin(Bx - C) + D$, identify the midline, amplitude, period, and phase shift.

1. Determine the amplitude as $|A|$.
2. Determine the period as $P = \frac{2\pi}{|B|}$.
3. Determine the phase shift as $\frac{C}{B}$.
4. Determine the midline as $y = D$.

Example 5 Identifying the Variations of a Sinusoidal Function from an Equation

Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x) + 1$.

Solution Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

$A = 3$, so the amplitude is $|A| = 3$.

Next, $B = 2$, so the period is $P = \frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi$.

There is no added constant inside the parentheses, so $C = 0$ and the phase shift is $\frac{C}{B} = \frac{0}{2} = 0$.

Finally, $D = 1$, so the midline is $y = 1$.

Analysis *Inspecting the graph, we can determine that the period is* π*, the midline is* $y = 1$*, and the amplitude is 3. See* ***Figure 14***.

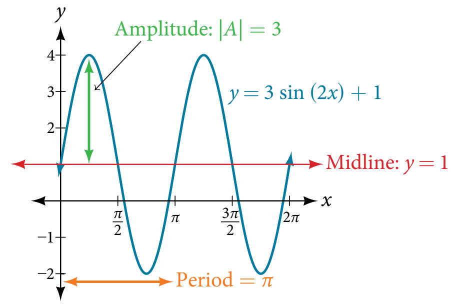


Figure 14

Try It #5

Determine the midline, amplitude, period, and phase shift of the function $y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$.

Example 6 Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the cosine function in **Figure 15**.

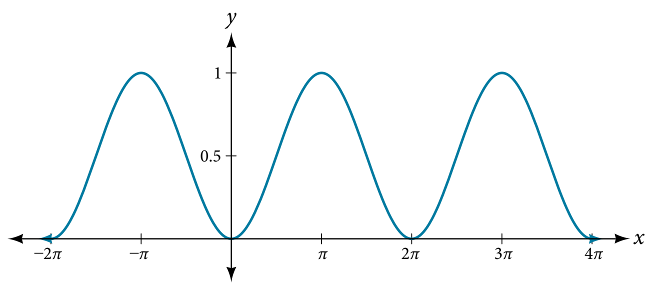


Figure 15

Solution To determine the equation, we need to identify each value in the general form of a sinusoidal function.

$$y = A\sin(Bx - C) + D \qquad y = A\cos(Bx - C) + D$$

The graph could represent either a sine or a cosine function that is shifted and/or reflected. When $x = 0$, the graph has an extreme point, $(0, 0)$. Since the cosine function has an extreme point for $x = 0$, let us write our equation in terms of a cosine function.

Let's start with the midline. We can see that the graph rises and falls an equal distance above and below $y = 0.5$. This value, which is the midline, is D in the equation, so $D = 0.5$.

The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So $|A| = 0.5$. Another way we could have determined the amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so $|A| = \frac{1}{2} = 0.5$. Also, the graph is reflected about the x-axis so that $A = -0.5$.

The graph is not horizontally stretched or compressed, so $B = 1$; and the graph is not shifted horizontally, so $C = 0$.

Putting this all together,

$$g(x) = -0.5\cos(x) + 0.5$$

Try It #6

Determine the formula for the sine function in **Figure 16**.

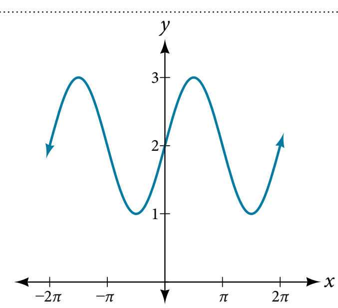


Figure 16

Example 7 Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in **Figure 17**.

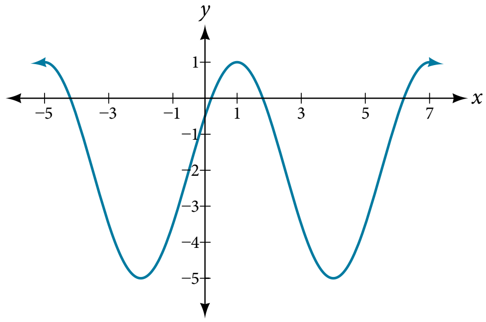


Figure 17

Solution With the highest value at 1 and the lowest value at −5, the midline will be halfway between at −2. So $D = -2$.

The distance from the midline to the highest or lowest value gives an amplitude of $|A| = 3$.

The period of the graph is 6, which can be measured from the peak at $x = 1$ to the next peak at $x = 7$, or from the distance between the lowest points. Therefore, $P = \frac{2\pi}{|B|} = 6$. Using the positive value for B, we find that

$$B = \frac{2\pi}{P} = \frac{2\pi}{6} = \frac{\pi}{3}$$

So far, our equation is either $y = 3\sin\left(\frac{\pi}{3}x - C\right) - 2$ or $y = 3\cos\left(\frac{\pi}{3}x - C\right) - 2$. For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes

$$y = 3\cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) - 2 \text{ or } y = -3\cos\left(\frac{\pi}{3}x + \frac{2\pi}{3}\right) - 2$$

Again, these functions are equivalent, so both yield the same graph.

Try It #7

Write a formula for the function graphed in **Figure 18**.

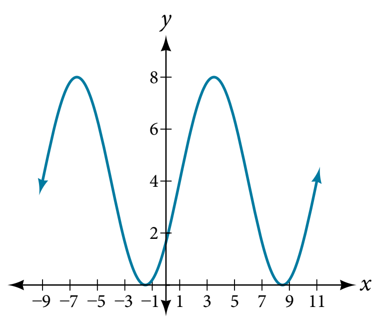


Figure 18

Graphing Variations of $y = \sin x$ and $y = \cos x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D,$$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.

How To...

Given the function $y = A\sin(Bx)$, sketch its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \dfrac{2\pi}{|B|}$.
3. Start at the origin, with the function increasing to the right if A is positive or decreasing if A is negative.
4. At $x = \dfrac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the x-axis at $x = \dfrac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \dfrac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the x-axis at $x = \dfrac{\pi}{2|B|}$.

Example 8 Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2\sin\left(\dfrac{\pi x}{2}\right)$.

Solution Let's begin by comparing the equation to the form $y = A\sin(Bx)$.

Step 1. We can see from the equation that $A = -2$, so the amplitude is 2.

$$|A| = 2$$

Step 2. The equation shows that $B = \dfrac{\pi}{2}$, so the period is

$$\begin{aligned} P &= \frac{2\pi}{\frac{\pi}{2}} \\ &= 2\pi \cdot \frac{2}{\pi} \\ &= 4 \end{aligned}$$

Step 3. Because A is negative, the graph descends as we move to the right of the origin.

Step 4–7. The x-intercepts are at the beginning of one period, $x = 0$, the horizontal midpoints are at $x = 2$ and at the end of one period at $x = 4$.

The quarter points include the minimum at $x = 1$ and the maximum at $x = 3$. A local minimum will occur 2 units below the midline, at $x = 1$, and a local maximum will occur at 2 units above the midline, at $x = 3$. **Figure 19** shows the graph of the function.

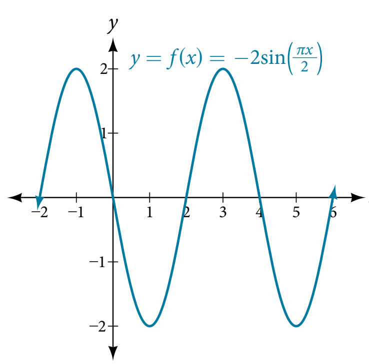


Figure 19

Try It #8

Sketch a graph of $g(x) = -0.8\cos(2x)$. Determine the midline, amplitude, period, and phase shift.

How To...

Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph.

1. Express the function in the general form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.
2. Identify the amplitude, $|A|$.
3. Identify the period, $P = \dfrac{2\pi}{|B|}$.
4. Identify the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $f(x) = A\sin(Bx)$ shifted to the right or left by $\dfrac{C}{B}$ and up or down by D.

Example 9 Graphing a Transformed Sinusoid

Sketch a graph of $f(x) = 3\sin\left(\dfrac{\pi}{4}x - \dfrac{\pi}{4}\right)$.

Solution

Step 1. The function is already written in general form: $f(x) = 3\sin\left(\dfrac{\pi}{4}x - \dfrac{\pi}{4}\right)$. This graph will have the shape of a sine function, starting at the midline and increasing to the right.

Step 2. $|A| = |3| = 3$. The amplitude is 3.

Step 3. Since $|B| = \left|\dfrac{\pi}{4}\right| = \dfrac{\pi}{4}$, we determine the period as follows.

$$P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{4}} = 2\pi \cdot \frac{4}{\pi} = 8$$

The period is 8.

Step 4. Since $C = \dfrac{\pi}{4}$, the phase shift is

$$\frac{C}{B} = \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1.$$

The phase shift is 1 unit.

Step 5. **Figure 20** shows the graph of the function.

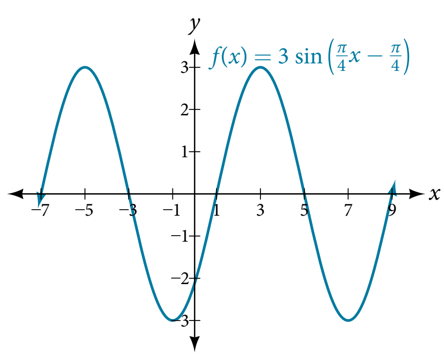


Figure 20 A horizontally compressed, vertically stretched, and horizontally shifted sinusoid

Try It #9

Draw a graph of $g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

Example 10 Identifying the Properties of a Sinusoidal Function

Given $y = -2\cos\left(\frac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

Solution Begin by comparing the equation to the general form and use the steps outlined in **Example 9**.

$$y = A\cos(Bx - C) + D$$

Step 1. The function is already written in general form.

Step 2. Since $A = -2$, the amplitude is $|A| = 2$.

Step 3. $|B| = \frac{\pi}{2}$, so the period is $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$. The period is 4.

Step 4. $C = -\pi$, so we calculate the phase shift as $\frac{C}{B} = \frac{-\pi}{\frac{\pi}{2}} = -\pi \cdot \frac{2}{\pi} = -2$. The phase shift is -2.

Step 5. $D = 3$, so the midline is $y = 3$, and the vertical shift is up 3.

Since A is negative, the graph of the cosine function has been reflected about the x-axis. **Figure 21** shows one cycle of the graph of the function.

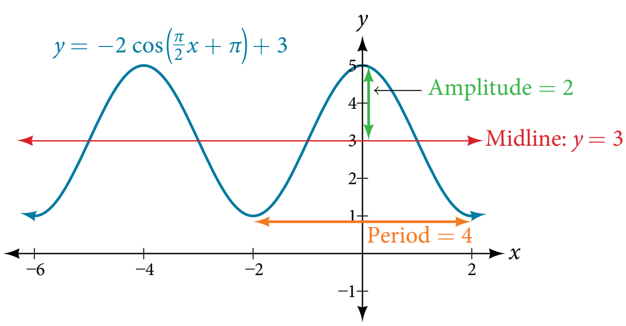


Figure 21

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter, circular motion can be modeled using either the sine or cosine function.

Example 11 Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

Solution Recall that, for a point on a circle of radius r, the y-coordinate of the point is $y = r\sin(x)$, so in this case, we get the equation $y(x) = 3\sin(x)$. The constant 3 causes a vertical stretch of the y-values of the function by a factor of 3, which we can see in the graph in **Figure 22**.

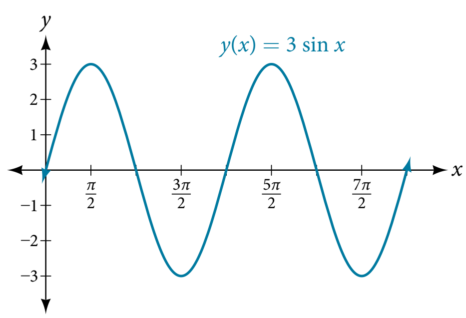


Figure 22

Analysis *Notice that the period of the function is still* 2π*; as we travel around the circle, we return to the point* (3, 0) *for* $x = 2\pi, 4\pi, 6\pi, \ldots$ *Because the outputs of the graph will now oscillate between* –3 *and* 3*, the amplitude of the sine wave is* 3*.*

Try It #10

What is the amplitude of the function $f(x) = 7\cos(x)$? Sketch a graph of this function.

Example 12 Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled P, as shown in **Figure 23**. Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

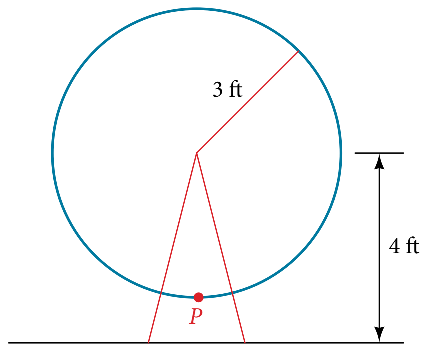


Figure 23

Solution Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in **Figure 24**.

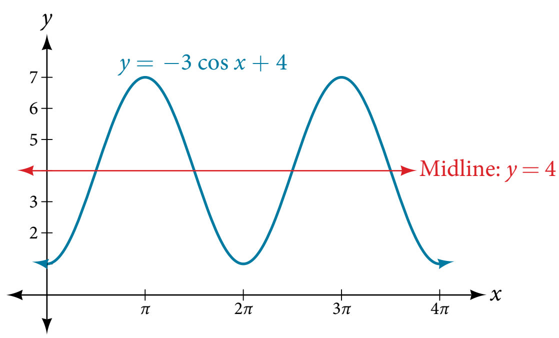


Figure 24

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a sine function starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$

Try It #11

A weight is attached to a spring that is then hung from a board, as shown in **Figure 25**. As the spring oscillates up and down, the position y of the weight relative to the board ranges from -1 in. (at time $x = 0$) to -7 in. (at time $x = \pi$) below the board. Assume the position of y is given as a sinusoidal function of x. Sketch a graph of the function, and then find a cosine function that gives the position y in terms of x.

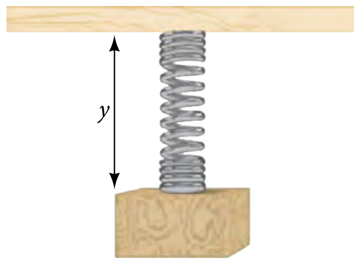


Figure 25

Example 13 **Determining a Rider's Height on a Ferris Wheel**

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A = 67.5$
- Midline: 69.5, so $D = 69.5$
- Period: 30, so $B = \frac{2\pi}{30} = \frac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider's height would be

$$y = -\,67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$

where t is in minutes and y is measured in meters.

Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

- Amplitude and Period of Sine and Cosine (http://openstaxcollege.org/l/ampperiod)
- Translations of Sine and Cosine (http://openstaxcollege.org/l/translasincos)
- Graphing Sine and Cosine Transformations (http://openstaxcollege.org/l/transformsincos)
- Graphing the Sine Function (http://openstaxcollege.org/l/graphsinefunc)

6.1 SECTION EXERCISES

VERBAL

1. Why are the sine and cosine functions called periodic functions?

2. How does the graph of $y = \sin x$ compare with the graph of $y = \cos x$? Explain how you could horizontally translate the graph of $y = \sin x$ to obtain $y = \cos x$.

3. For the equation $A\cos(Bx + C) + D$, what constants affect the range of the function and how do they affect the range?

4. How does the range of a translated sine function relate to the equation $y = A\sin(Bx + C) + D$?

5. How can the unit circle be used to construct the graph of $f(t) = \sin t$?

GRAPHICAL

For the following exercises, graph two full periods of each function and state the amplitude, period, and midline. State the maximum and minimum y-values and their corresponding x-values on one period for $x > 0$. Round answers to two decimal places if necessary.

6. $f(x) = 2\sin x$

7. $f(x) = \frac{2}{3}\cos x$

8. $f(x) = -3\sin x$

9. $f(x) = 4\sin x$

10. $f(x) = 2\cos x$

11. $f(x) = \cos(2x)$

12. $f(x) = 2\sin\left(\frac{1}{2}x\right)$

13. $f(x) = 4\cos(\pi x)$

14. $f(x) = 3\cos\left(\frac{6}{5}x\right)$

15. $y = 3\sin(8(x + 4)) + 5$

16. $y = 2\sin(3x - 21) + 4$

17. $y = 5\sin(5x + 20) - 2$

For the following exercises, graph one full period of each function, starting at $x = 0$. For each function, state the amplitude, period, and midline. State the maximum and minimum y-values and their corresponding x-values on one period for $x > 0$. State the phase shift and vertical translation, if applicable. Round answers to two decimal places if necessary.

18. $f(t) = 2\sin\left(t - \frac{5\pi}{6}\right)$

19. $f(t) = -\cos\left(t + \frac{\pi}{3}\right) + 1$

20. $f(t) = 4\cos\left(2\left(t + \frac{\pi}{4}\right)\right) - 3$

21. $f(t) = -\sin\left(\frac{1}{2}t + \frac{5\pi}{3}\right)$

22. $f(x) = 4\sin\left(\frac{\pi}{2}(x - 3)\right) + 7$

23. Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in **Figure 26**.

24. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in **Figure 27.**

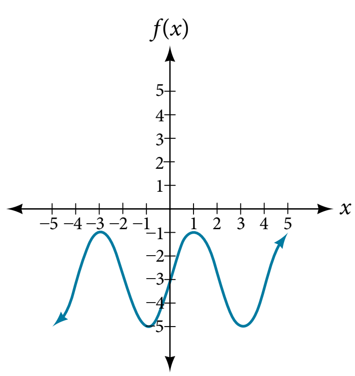


Figure 26

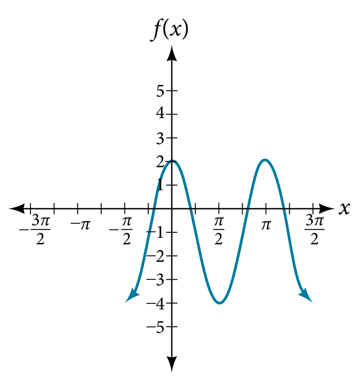


Figure 27

25. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in **Figure 28**.

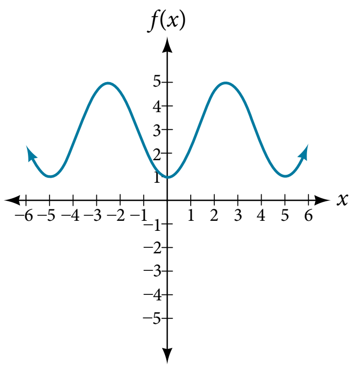


Figure 28

26. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in **Figure 29**.

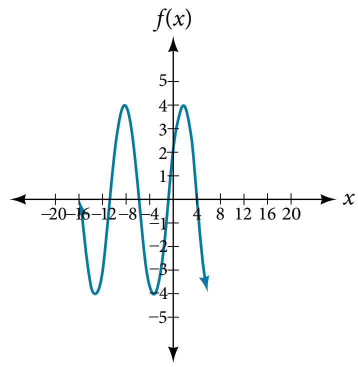


Figure 29

27. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in **Figure 30**.

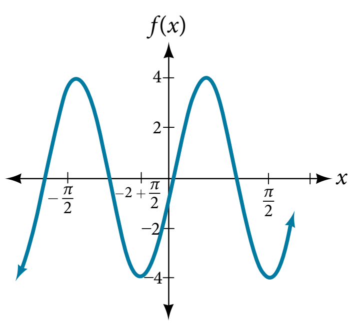


Figure 30

28. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in **Figure 31**.

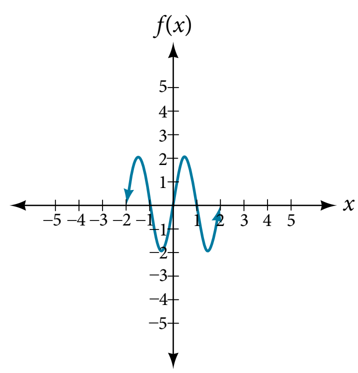


Figure 31

29. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in **Figure 32**.

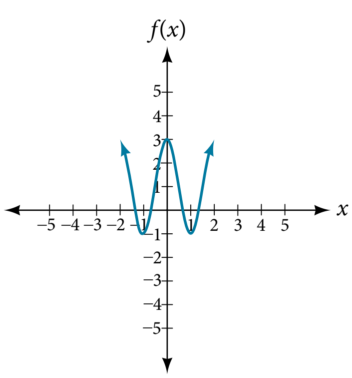


Figure 32

30. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in **Figure 33**.

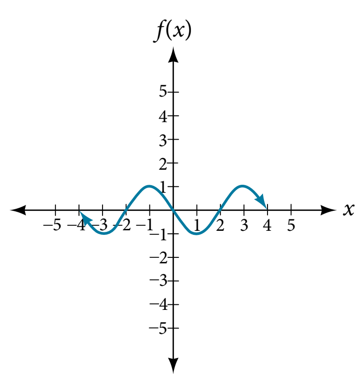


Figure 33

ALGEBRAIC

For the following exercises, let $f(x) = \sin x$.

31. On $[0, 2\pi)$, solve $f(x) = 0$.

32. On $[0, 2\pi)$, solve $f(x) = \frac{1}{2}$.

33. Evaluate $f\left(\frac{\pi}{2}\right)$.

34. On $[0, 2\pi)$, $f(x) = \frac{\sqrt{2}}{2}$. Find all values of x.

35. On $[0, 2\pi)$, the maximum value(s) of the function occur(s) at what x-value(s)?

36. On $[0, 2\pi)$, the minimum value(s) of the function occur(s) at what x-value(s)?

37. Show that $f(-x) = -f(x)$. This means that $f(x) = \sin x$ is an odd function and possesses symmetry with respect to ________________.

For the following exercises, let $f(x) = \cos x$.

38. On $[0, 2\pi)$, solve the equation $f(x) = \cos x = 0$.

39. On $[0, 2\pi)$, solve $f(x) = \frac{1}{2}$.

40. On $[0, 2\pi)$, find the x-intercepts of $f(x) = \cos x$.

41. On $[0, 2\pi)$, find the x-values at which the function has a maximum or minimum value.

42. On $[0, 2\pi)$, solve the equation $f(x) = \frac{\sqrt{3}}{2}$.

TECHNOLOGY

43. Graph $h(x) = x + \sin x$ on $[0, 2\pi]$. Explain why the graph appears as it does.

44. Graph $h(x) = x + \sin x$ on $[-100, 100]$. Did the graph appear as predicted in the previous exercise?

45. Graph $f(x) = x \sin x$ on $[0, 2\pi]$ and verbalize how the graph varies from the graph of $f(x) = \sin x$.

46. Graph $f(x) = x \sin x$ on the window $[-10, 10]$ and explain what the graph shows.

47. Graph $f(x) = \frac{\sin x}{x}$ on the window $[-5\pi, 5\pi]$ and explain what the graph shows.

REAL-WORLD APPLICATIONS

48. A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function $h(t)$ gives a person's height in meters above the ground t minutes after the wheel begins to turn.

- **a.** Find the amplitude, midline, and period of $h(t)$.
- **b.** Find a formula for the height function $h(t)$.
- **c.** How high off the ground is a person after 5 minutes?

LEARNING OBJECTIVES

In this section, you will:

- Analyze the graph of $y = \tan x$.
- Graph variations of $y = \tan x$.
- Analyze the graphs of $y = \sec x$ and $y = \csc x$.
- Graph variations of $y = \sec x$ and $y = \csc x$.
- Analyze the graph of $y = \cot x$.
- Graph variations of $y = \cot x$.

6.2 GRAPHS OF THE OTHER TRIGONOMETRIC FUNCTIONS

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and other trigonometric functions.

Analyzing the Graph of $y = \tan x$

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that

$$\tan x = \frac{\sin x}{\cos x}$$

The period of the tangent function is π because the graph repeats itself on intervals of $k\pi$ where k is a constant. If we graph the tangent function on $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

$$\begin{aligned}
\tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{Definition of tangent.}\\
&= \frac{-\sin x}{\cos x} && \text{Sine is an odd function, cosine is even.}\\
&= -\frac{\sin x}{\cos x} && \text{The quotient of an odd and an even function is odd.}\\
&= -\tan x && \text{Definition of tangent.}
\end{aligned}$$

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in **Table 1**.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
tan(x)	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Table 1

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when $\frac{\pi}{3} < x < \frac{\pi}{2}$, we can use a table to look for a trend. Because $\frac{\pi}{3} \approx 1.05$ and $\frac{\pi}{2} \approx 1.57$, we will evaluate x at radian measures $1.05 < x < 1.57$ as shown in **Table 2**.

x	1.3	1.5	1.55	1.56
$\tan x$	3.6	14.1	48.1	92.6

Table 2

As x approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan x$ is an odd function, we see the corresponding table of negative values in **Table 3**.

x	−1.3	−1.5	−1.55	−1.56
$\tan x$	−3.6	−14.1	−48.1	−92.6

Table 3

We can see that, as x approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of x for which $\cos x = 0$. For example, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{3\pi}{2}\right) = 0$. At these values, the tangent function is undefined, so the graph of $y = \tan x$ has discontinuities at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes. **Figure 1** represents the graph of $y = \tan x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from π to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

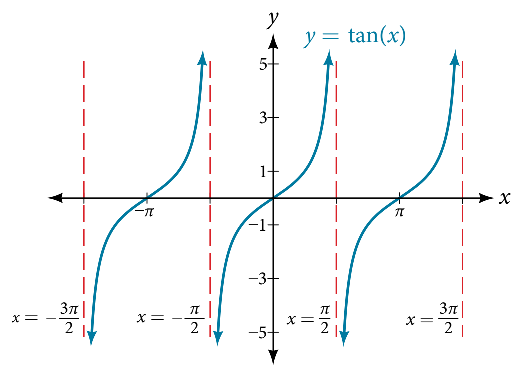


Figure 1 Graph of the tangent function

Graphing Variations of $y = \tan x$

As with the sine and cosine functions, the tangent function can be described by a general equation.

$$y = A\tan(Bx)$$

We can identify horizontal and vertical stretches and compressions using values of A and B. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

features of the graph of $y = A\tan(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$.
- The domain is all real numbers x, where $x \neq \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$ such that k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$, where k is an integer.
- $y = A\tan(Bx)$ is an odd function.

Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the tangent function to quickly sketch a graph of any stretched and/or compressed tangent function of the form $f(x) = A\tan(Bx)$. We focus on a single period of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function's domain if we wish. Our limited domain is then the interval $\left(-\frac{P}{2}, \frac{P}{2}\right)$ and the graph has vertical asymptotes at $\pm\frac{P}{2}$ where $P = \frac{\pi}{B}$. On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the graph will come up from the left asymptote at $x = -\frac{\pi}{2}$, cross through the origin, and continue to increase as it approaches the right asymptote at $x = \frac{\pi}{2}$. To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through. For example, we can use

$$f\left(\frac{P}{4}\right) = A\tan\left(B\frac{P}{4}\right) = A\tan\left(B\frac{\pi}{4B}\right) = A$$

because $\tan\left(\frac{\pi}{4}\right) = 1$.

How To...

Given the function $f(x) = A\tan(Bx)$, graph one period.

1. Identify the stretching factor, $|A|$.
2. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
3. Draw vertical asymptotes at $x = -\frac{P}{2}$ and $x = \frac{P}{2}$.
4. For $A > 0$, the graph approaches the left asymptote at negative output values and the right asymptote at positive output values (reverse for $A < 0$).
5. Plot reference points at $\left(\frac{P}{4}, A\right)$, $(0, 0)$, and $\left(-\frac{P}{4}, -A\right)$, and draw the graph through these points.

Example 1 **Sketching a Compressed Tangent**

Sketch a graph of one period of the function $y = 0.5\tan\left(\frac{\pi}{2}x\right)$.

Solution First, we identify A and B.

$$\begin{gathered} y = 0.5\tan\left(\frac{\pi}{2}x\right) \\ \uparrow \qquad \nearrow \\ y = A\tan(Bx) \end{gathered}$$

Because $A = 0.5$ and $B = \frac{\pi}{2}$, we can find the stretching/compressing factor and period. The period is $\frac{\pi}{\frac{\pi}{2}} = 2$, so the asymptotes are at $x = \pm 1$. At a quarter period from the origin, we have

$$\begin{aligned} f(0.5) &= 0.5\tan\left(\frac{0.5\pi}{2}\right) \\ &= 0.5\tan\left(\frac{\pi}{4}\right) \\ &= 0.5 \end{aligned}$$

This means the curve must pass through the points $(0.5, 0.5)$, $(0, 0)$, and $(-0.5, -0.5)$. The only inflection point is at the origin. **Figure 2** shows the graph of one period of the function.

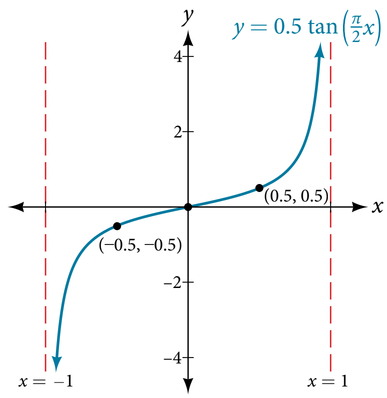


Figure 2

Try It #1

Sketch a graph of $f(x) = 3\tan\left(\dfrac{\pi}{6}x\right)$.

Graphing One Period of a Shifted Tangent Function

Now that we can graph a tangent function that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add C and D to the general form of the tangent function.

$$f(x) = A\tan(Bx - C) + D$$

The graph of a transformed tangent function is different from the basic tangent function tan x in several ways:

features of the graph of $y = A\tan(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{\pi}{|B|}$.
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\tan(Bx)$ is an odd function because it is the quotient of odd and even functions (sine and cosine respectively).

How To…

Given the function $y = A\tan(Bx - C) + D$, sketch the graph of one period.

1. Express the function given in the form $y = A\tan(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{\pi}{|B|}$.
4. Identify C and determine the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\dfrac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$, where k is an odd integer.
7. Plot any three reference points and draw the graph through these points.

Example 2 Graphing One Period of a Shifted Tangent Function

Graph one period of the function $y = -2\tan(\pi x + \pi) - 1$.

Solution

Step 1. The function is already written in the form $y = A\tan(Bx - C) + D$.

Step 2. $A = -2$, so the stretching factor is $|A| = 2$.

Step 3. $B = \pi$, so the period is $P = \dfrac{\pi}{|B|} = \dfrac{\pi}{\pi} = 1$.

Step 4. $C = -\pi$, so the phase shift is $\dfrac{C}{B} = \dfrac{-\pi}{\pi} = -1$.

Step 5-7. The asymptotes are at $x = -\dfrac{3}{2}$ and $x = -\dfrac{1}{2}$ and the three recommended reference points are $(-1.25, 1)$, $(-1, -1)$, and $(-0.75, -3)$. The graph is shown in **Figure 3**.

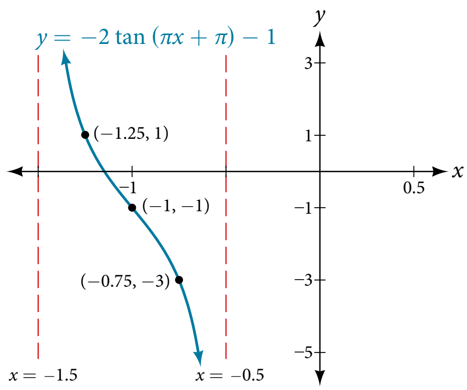


Figure 3

Analysis *Note that this is a decreasing function because $A < 0$.*

Try It #2

How would the graph in **Example 2** look different if we made $A = 2$ instead of -2?

How To…

Given the graph of a tangent function, identify horizontal and vertical stretches.

1. Find the period P from the spacing between successive vertical asymptotes or x-intercepts.
2. Write $f(x) = A\tan\left(\dfrac{\pi}{P}x\right)$.
3. Determine a convenient point $(x, f(x))$ on the given graph and use it to determine A.

Example 3 Identifying the Graph of a Stretched Tangent

Find a formula for the function graphed in **Figure 4**.

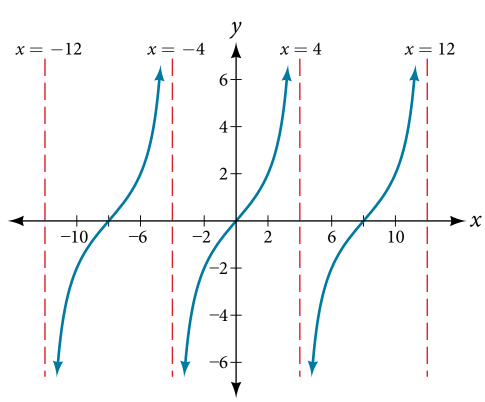


Figure 4 A stretched tangent function

Solution The graph has the shape of a tangent function.

Step 1. One cycle extends from -4 to 4, so the period is $P = 8$. Since $P = \frac{\pi}{|B|}$, we have $B = \frac{\pi}{P} = \frac{\pi}{8}$.

Step 2. The equation must have the form $f(x) = A\tan\left(\frac{\pi}{8}x\right)$.

Step 3. To find the vertical stretch A, we can use the point (2, 2).

$$2 = A\tan\left(\frac{\pi}{8} \cdot 2\right) = A\tan\left(\frac{\pi}{4}\right)$$

Because $\tan\left(\frac{\pi}{4}\right) = 1$, $A = 2$.

This function would have a formula $f(x) = 2\tan\left(\frac{\pi}{8}x\right)$.

Try It #3

Find a formula for the function in **Figure 5**.

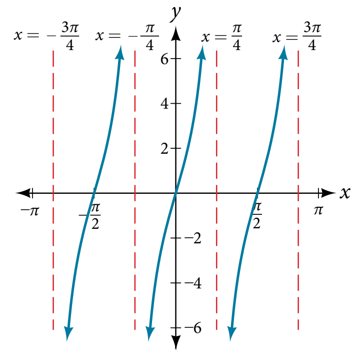


Figure 5

Analyzing the Graphs of $y = \sec x$ and $y = \csc x$

The secant was defined by the reciprocal identity $\sec x = \frac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\frac{\pi}{2}, \frac{3\pi}{2}$, etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See **Figure 6**. The graph of the cosine is shown as a blue wave so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of x where the cosine graph crosses the x-axis; we show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x) = \sec x$.

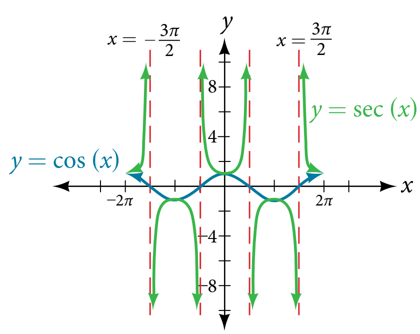


Figure 6 Graph of the secant function, $f(x) = \sec x = \frac{1}{\cos x}$

As we did for the tangent function, we will again refer to the constant $|A|$ as the stretching factor, not the amplitude.

features of the graph of $y = A\sec(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

Similar to the secant, the cosecant is defined by the reciprocal identity $\csc x = \dfrac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \csc x$ by observing the graph of the sine function because these two functions are reciprocals of one another. See **Figure 7**. The graph of sine is shown as a blue wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the cosecant function decreases.

The cosecant graph has vertical asymptotes at each value of x where the sine graph crosses the x-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, $\csc(-x) = -\csc x$.

The graph of cosecant, which is shown in **Figure 7**, is similar to the graph of secant.

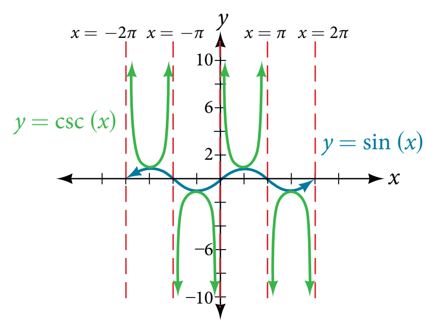


Figure 7 The graph of the cosecant function, $f(x) = \csc x = \dfrac{1}{\sin x}$

features of the graph of $y = A\csc(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

Graphing Variations of $y = \sec x$ and $y = \csc x$

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions. The equations become the following.

$$y = A\sec(Bx - C) + D \qquad y = A\csc(Bx - C) + D$$

features of the graph of $y = A\sec(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

features of the graph of $y = A\csc(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$, where k is an integer.
- There is no amplitude.
- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

How To...

Given a function of the form $y = A\sec(Bx)$, graph one period.

1. Express the function given in the form $y = A\sec(Bx)$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{2\pi}{|B|}$.
4. Sketch the graph of $y = A\cos(Bx)$.
5. Use the reciprocal relationship between $y = \cos x$ and $y = \sec x$ to draw the graph of $y = A\sec(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 4 **Graphing a Variation of the Secant Function**

Graph one period of $f(x) = 2.5\sec(0.4x)$.

Solution

Step 1. The given function is already written in the general form, $y = A\sec(Bx)$.

Step 2. $A = 2.5$ so the stretching factor is 2.5.

Step 3. $B = 0.4$ so $P = \dfrac{2\pi}{0.4} = 5\pi$. The period is 5π units.

Step 4. Sketch the graph of the function $g(x) = 2.5\cos(0.4x)$.

Step 5. Use the reciprocal relationship of the cosine and secant functions to draw the cosecant function.

Steps 6–7. Sketch two asymptotes at $x = 1.25\pi$ and $x = 3.75\pi$. We can use two reference points, the local minimum at $(0, 2.5)$ and the local maximum at $(2.5\pi, -2.5)$. **Figure 8** shows the graph.

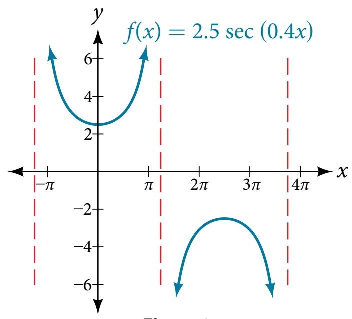


Figure 8

Try It #4

Graph one period of $f(x) = -2.5\sec(0.4x)$.

Q & A…

Do the vertical shift and stretch/compression affect the secant's range?

Yes. The range of $f(x) = A\sec(Bx - C) + D$ is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.

How To…

Given a function of the form $f(x) = A\sec(Bx - C) + D$, graph one period.

1. Express the function given in the form $y = A\sec(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\frac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\sec(Bx)$ but shift it to the right by $\frac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$, where k is an odd integer.

Example 5 Graphing a Variation of the Secant Function

Graph one period of $y = 4\sec\left(\frac{\pi}{3}x - \frac{\pi}{2}\right) + 1$.

Solution

Step 1. Express the function given in the form $y = 4\sec\left(\frac{\pi}{3}x - \frac{\pi}{2}\right) + 1$.

Step 2. The stretching/compressing factor is $|A| = 4$.

Step 3. The period is

$$\begin{aligned}\frac{2\pi}{|B|} &= \frac{2\pi}{\frac{\pi}{3}}\\ &= \frac{2\pi}{1}\cdot\frac{3}{\pi}\\ &= 6\end{aligned}$$

Step 4. The phase shift is

$$\begin{aligned}\frac{C}{B} &= \frac{\frac{\pi}{2}}{\frac{\pi}{3}}\\ &= \frac{\pi}{2}\cdot\frac{3}{\pi}\\ &= 1.5\end{aligned}$$

Step 5. Draw the graph of $y = A\sec(Bx)$, but shift it to the right by $\frac{C}{B} = 1.5$ and up by $D = 6$.

Step 6. Sketch the vertical asymptotes, which occur at $x = 0$, $x = 3$, and $x = 6$. There is a local minimum at $(1.5, 5)$ and a local maximum at $(4.5, -3)$. **Figure 9** shows the graph.

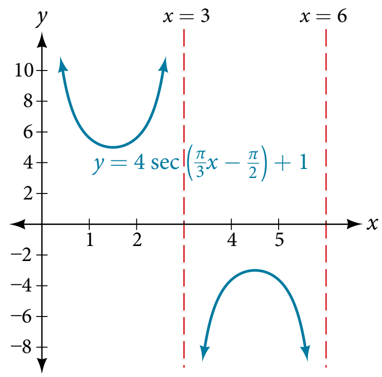


Figure 9

Try It #5

Graph one period of $f(x) = -6\sec(4x + 2) - 8$.

Q & A...

The domain of csc x was given to be all x such that $x \neq k\pi$ for any integer k. Would the domain of $y = A\csc(Bx - C) + D$ be $x \neq \frac{C + k\pi}{B}$?

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function's input.

How To...

Given a function of the form $y = A\csc(Bx)$, graph one period.

1. Express the function given in the form $y = A\csc(Bx)$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \frac{2\pi}{|B|}$.
4. Draw the graph of $y = A\sin(Bx)$.
5. Use the reciprocal relationship between $y = \sin x$ and $y = \csc x$ to draw the graph of $y = A\csc(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 6 Graphing a Variation of the Cosecant Function

Graph one period of $f(x) = -3\csc(4x)$.

Solution

Step 1. The given function is already written in the general form, $y = A\csc(Bx)$.

Step 2. $|A| = |-3| = 3$, so the stretching factor is 3.

Step 3. $B = 4$, so $P = \frac{2\pi}{4} = \frac{\pi}{2}$. The period is $\frac{\pi}{2}$ units.

Step 4. Sketch the graph of the function $g(x) = -3\sin(4x)$.

Step 5. Use the reciprocal relationship of the sine and cosecant functions to draw the cosecant function.

Steps 6–7. Sketch three asymptotes at $x = 0$, $x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$. We can use two reference points, the local maximum at $\left(\frac{\pi}{8}, -3\right)$ and the local minimum at $\left(\frac{3\pi}{8}, 3\right)$. **Figure 10** shows the graph.

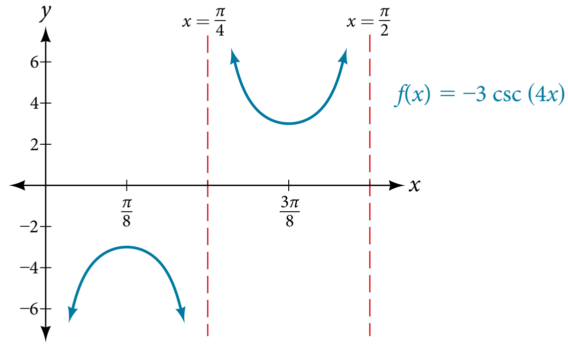


Figure 10

Try It #6

Graph one period of $f(x) = 0.5\csc(2x)$.

How To…

Given a function of the form $f(x) = A\csc(Bx - C) + D$, graph one period.

1. Express the function given in the form $y = A\csc(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\frac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\csc(Bx)$ but shift it to the right by $\frac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.

Example 7 Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant

Sketch a graph of $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$. What are the domain and range of this function?

Solution

Step 1. Express the function given in the form $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$.

Step 2. Identify the stretching/compressing factor, $|A| = 2$.

Step 3. The period is $\frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = \frac{2\pi}{1} \cdot \frac{2}{\pi} = 4$.

Step 4. The phase shift is $\frac{0}{\frac{\pi}{2}} = 0$.

Step 5. Draw the graph of $y = A\csc(Bx)$ but shift it up $D = 1$.

Step 6. Sketch the vertical asymptotes, which occur at $x = 0$, $x = 2$, $x = 4$.

The graph for this function is shown in **Figure 11**.

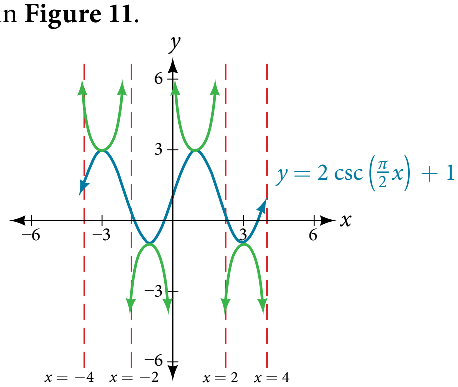


Figure 11 A transformed cosecant function

Analysis *The vertical asymptotes shown on the graph mark off one period of the function, and the local extrema in this interval are shown by dots. Notice how the graph of the transformed cosecant relates to the graph of* $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1$, *shown as the blue wave.*

Try It #7

Given the graph of $f(x) = 2\cos\left(\frac{\pi}{2}x\right) + 1$ shown in **Figure 12**, sketch the graph of $g(x) = 2\sec\left(\frac{\pi}{2}x\right) + 1$ on the same axes.

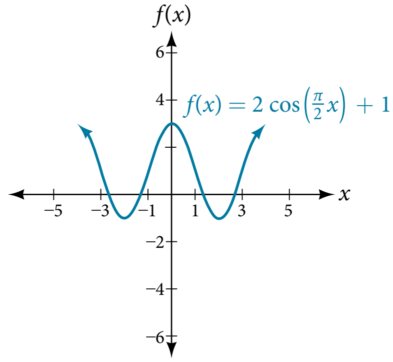


Figure 12

Analyzing the Graph of $y = \cot x$

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $\cot x = \frac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph $y = \cot x$ by observing the graph of the tangent function because these two functions are reciprocals of one another. See **Figure 13**. Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of x where $\tan x = 0$; we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of x where $\tan x = 0$, and $\cot x = 0$ at all values of x where $\tan x$ has its vertical asymptotes.

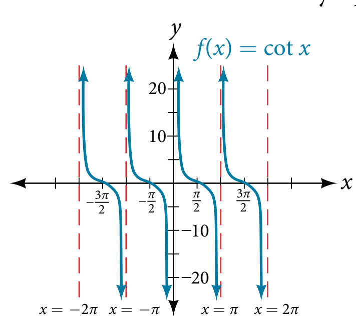


Figure 13 The cotangent function

features of the graph of $y = A\cot(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$.
- The domain is $x \neq \frac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{|B|}k$, where k is an integer.
- $y = A\cot(Bx)$ is an odd function.

Graphing Variations of $y = \cot x$

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes the following.

$$y = A\cot(Bx - C) + D$$

features of the graph of $y = A\cot(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{\pi}{|B|}$.
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$, where k is an integer.
- There is no amplitude.
- $y = A\cot(Bx)$ is an odd function because it is the quotient of even and odd functions (cosine and sine, respectively)

How To...

Given a modified cotangent function of the form $f(x) = A\cot(Bx)$, graph one period.

1. Express the function in the form $f(x) = A\cot(Bx)$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \dfrac{\pi}{|B|}$.
4. Draw the graph of $y = A\tan(Bx)$.
5. Plot any two reference points.
6. Use the reciprocal relationship between tangent and cotangent to draw the graph of $y = A\cot(Bx)$.
7. Sketch the asymptotes.

Example 8 **Graphing Variations of the Cotangent Function**

Determine the stretching factor, period, and phase shift of $y = 3\cot(4x)$, and then sketch a graph.

Solution

Step 1. Expressing the function in the form $f(x) = A\cot(Bx)$ gives $f(x) = 3\cot(4x)$.

Step 2. The stretching factor is $|A| = 3$.

Step 3. The period is $P = \dfrac{\pi}{4}$.

Step 4. Sketch the graph of $y = 3\tan(4x)$.

Step 5. Plot two reference points. Two such points are $\left(\dfrac{\pi}{16}, 3\right)$ and $\left(\dfrac{3\pi}{16}, -3\right)$.

Step 6. Use the reciprocal relationship to draw $y = 3\cot(4x)$.

Step 7. Sketch the asymptotes, $x = 0$, $x = \dfrac{\pi}{4}$.

The blue graph in **Figure 14** shows $y = 3\tan(4x)$ and the red graph shows $y = 3\cot(4x)$.

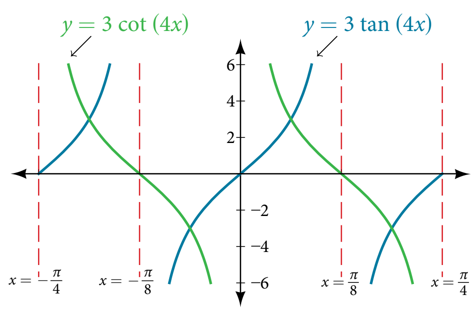


Figure 14

How To…

Given a modified cotangent function of the form $f(x) = A\cot(Bx - C) + D$, graph one period.

1. Express the function in the form $f(x) = A\cot(Bx - C) + D$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the asymptotes $x = \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.
7. Plot any three reference points and draw the graph through these points.

Example 9 Graphing a Modified Cotangent

Sketch a graph of one period of the function $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.

Solution

Step 1. The function is already written in the general form $f(x) = A\cot(Bx - C) + D$.

Step 2. $A = 4$, so the stretching factor is 4.

Step 3. $B = \frac{\pi}{8}$, so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{\frac{\pi}{8}} = 8$.

Step 4. $C = \frac{\pi}{2}$, so the phase shift is $\frac{C}{B} = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4$.

Step 5. We draw $f(x) = 4\tan\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.

Step 6-7. Three points we can use to guide the graph are (6, 2), (8, −2), and (10, −6). We use the reciprocal relationship of tangent and cotangent to draw $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.

Step 8. The vertical asymptotes are $x = 4$ and $x = 12$.

The graph is shown in **Figure 15**.

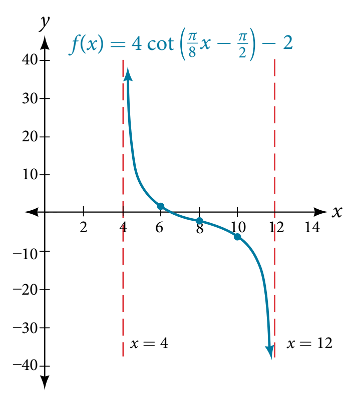


Figure 15 One period of a modified cotangent function

Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let's return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the tangent function.

Example 10 Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function $y = 5\tan\left(\frac{\pi}{4}t\right)$ marks the distance in the movement of a light beam from the top of a police car across a wall where t is the time in seconds and y is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the stretching factor and period.

b. Graph on the interval [0, 5].

c. Evaluate $f(1)$ and discuss the function's value at that input.

Solution

a. We know from the general form of $y = A\tan(Bt)$ that $|A|$ is the stretching factor and $\frac{\pi}{B}$ is the period.

$$y = \underset{\substack{\uparrow \\ A}}{5}\tan\left(\underset{\substack{\uparrow \\ B}}{\frac{\pi}{4}}t\right)$$

Figure 16

We see that the stretching factor is 5. This means that the beam of light will have moved 5 ft after half the period.

The period is $\frac{\pi}{\frac{\pi}{4}} = \frac{\pi}{1} \cdot \frac{4}{\pi} = 4$. This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

b. To graph the function, we draw an asymptote at $t = 2$ and use the stretching factor and period. See **Figure 17**

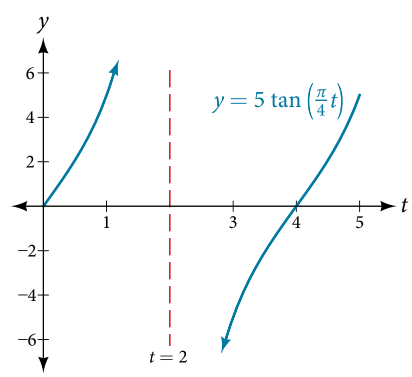


Figure 17

c. Period: $f(1) = 5\tan\left(\frac{\pi}{4}(1)\right) = 5(1) = 5$; after 1 second, the beam of light has moved 5 ft from the spot across from the police car.

Access these online resources for additional instruction and practice with graphs of other trigonometric functions.

- Graphing the Tangent Function (http://openstaxcollege.org/l/graphtangent)
- Graphing Cosecant and Secant Functions (http://openstaxcollege.org/l/graphcscsec)
- Graphing the Cotangent Function (http://openstaxcollege.org/l/graphcot)

6.2 SECTION EXERCISES

VERBAL

1. Explain how the graph of the sine function can be used to graph $y = \csc x$.

2. How can the graph of $y = \cos x$ be used to construct the graph of $y = \sec x$?

3. Explain why the period of $\tan x$ is equal to π.

4. Why are there no intercepts on the graph of $y = \csc x$?

5. How does the period of $y = \csc x$ compare with the period of $y = \sin x$?

ALGEBRAIC

For the following exercises, match each trigonometric function with one of the graphs in **Figure 18**.

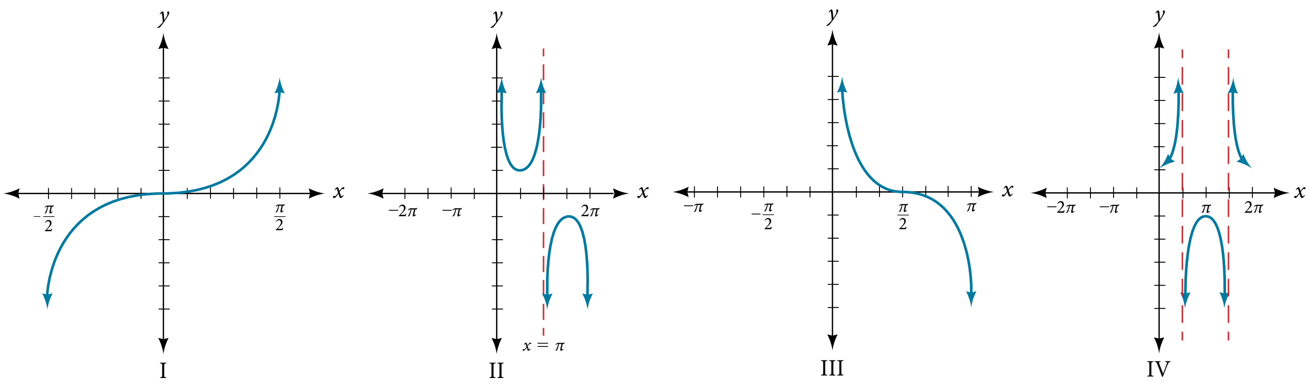


Figure 18

6. $f(x) = \tan x$

7. $f(x) = \sec x$

8. $f(x) = \csc x$

9. $f(x) = \cot x$

For the following exercises, find the period and horizontal shift of each of the functions.

10. $f(x) = 2\tan(4x - 32)$

11. $h(x) = 2\sec\left(\frac{\pi}{4}(x + 1)\right)$

12. $m(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right)$

13. If $\tan x = -1.5$, find $\tan(-x)$.

14. If $\sec x = 2$, find $\sec(-x)$.

15. If $\csc x = -5$, find $\csc(-x)$.

16. If $x\sin x = 2$, find $(-x)\sin(-x)$.

For the following exercises, rewrite each expression such that the argument x is positive.

17. $\cot(-x)\cos(-x) + \sin(-x)$

18. $\cos(-x) + \tan(-x)\sin(-x)$

GRAPHICAL

For the following exercises, sketch two periods of the graph for each of the following functions. Identify the stretching factor, period, and asymptotes.

19. $f(x) = 2\tan(4x - 32)$

20. $h(x) = 2\sec\left(\frac{\pi}{4}(x + 1)\right)$

21. $m(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right)$

22. $j(x) = \tan\left(\frac{\pi}{2}x\right)$

23. $p(x) = \tan\left(x - \frac{\pi}{2}\right)$

24. $f(x) = 4\tan(x)$

25. $f(x) = \tan\left(x + \frac{\pi}{4}\right)$

26. $f(x) = \pi\tan(\pi x - \pi) - \pi$

27. $f(x) = 2\csc(x)$

28. $f(x) = -\frac{1}{4}\csc(x)$

29. $f(x) = 4\sec(3x)$

30. $f(x) = -3\cot(2x)$

31. $f(x) = 7\sec(5x)$

32. $f(x) = \frac{9}{10}\csc(\pi x)$

33. $f(x) = 2\csc\left(x + \frac{\pi}{4}\right) - 1$

34. $f(x) = -\sec\left(x - \frac{\pi}{3}\right) - 2$

35. $f(x) = \frac{7}{5}\csc\left(x - \frac{\pi}{4}\right)$

36. $f(x) = 5\left(\cot\left(x + \frac{\pi}{2}\right) - 3\right)$

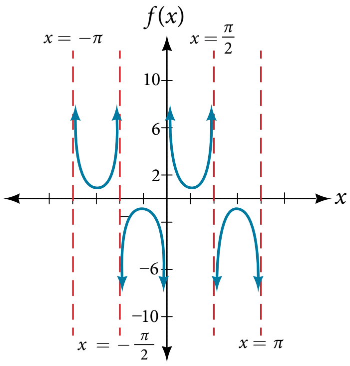
For the following exercises, find and graph two periods of the periodic function with the given stretching factor, $|A|$, period, and phase shift.

37. A tangent curve, $A = 1$, period of $\frac{\pi}{3}$; and phase shift $(h, k) = \left(\frac{\pi}{4}, 2\right)$

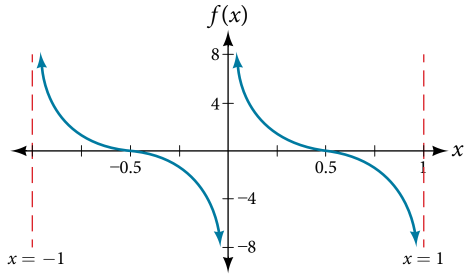
38. A tangent curve, $A = -2$, period of $\frac{\pi}{4}$, and phase shift $(h, k) = \left(-\frac{\pi}{4}, -2\right)$

For the following exercises, find an equation for the graph of each function.

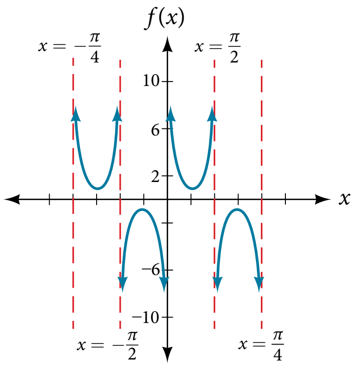
39.



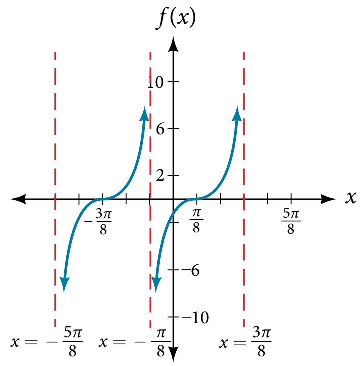
40.



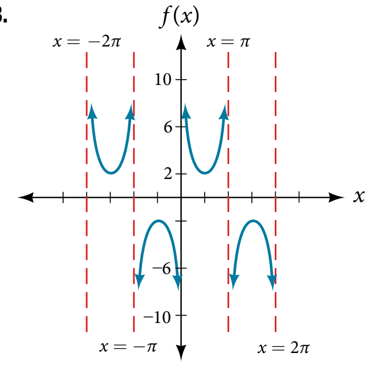
41.



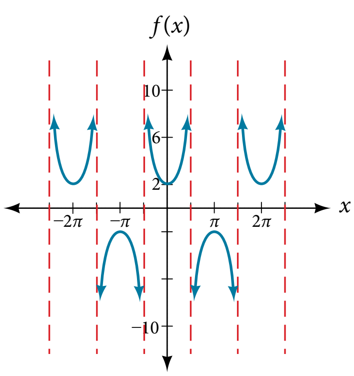
42.



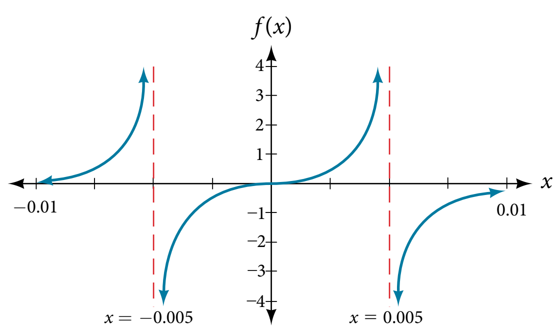
43.



44.



45.



TECHNOLOGY

For the following exercises, use a graphing calculator to graph two periods of the given function. Note: most graphing calculators do not have a cosecant button; therefore, you will need to input $\csc x$ as $\frac{1}{\sin x}$.

46. $f(x) = |\csc(x)|$

47. $f(x) = |\cot(x)|$

48. $f(x) = 2^{\csc(x)}$

49. $f(x) = \dfrac{\csc(x)}{\sec(x)}$

50. Graph $f(x) = 1 + \sec^2(x) - \tan^2(x)$. What is the function shown in the graph?

51. $f(x) = \sec(0.001x)$

52. $f(x) = \cot(100\pi x)$

53. $f(x) = \sin^2 x + \cos^2 x$

REAL-WORLD APPLICATIONS

54. The function $f(x) = 20\tan\left(\frac{\pi}{10}x\right)$ marks the distance in the movement of a light beam from a police car across a wall for time x, in seconds, and distance $f(x)$, in feet.

- **a.** Graph on the interval $[0, 5]$.
- **b.** Find and interpret the stretching factor, period, and asymptote.
- **c.** Evaluate $f(1)$ and $f(2.5)$ and discuss the function's values at those inputs.

55. Standing on the shore of a lake, a fisherman sights a boat far in the distance to his left. Let x, measured in radians, be the angle formed by the line of sight to the ship and a line due north from his position. Assume due north is 0 and x is measured negative to the left and positive to the right. (See **Figure 19**.) The boat travels from due west to due east and, ignoring the curvature of the Earth, the distance $d(x)$, in kilometers, from the fisherman to the boat is given by the function $d(x) = 1.5\sec(x)$.

- **a.** What is a reasonable domain for $d(x)$?
- **b.** Graph $d(x)$ on this domain.
- **c.** Find and discuss the meaning of any vertical asymptotes on the graph of $d(x)$.
- **d.** Calculate and interpret $d\left(-\frac{\pi}{3}\right)$. Round to the second decimal place.
- **e.** Calculate and interpret $d\left(\frac{\pi}{6}\right)$. Round to the second decimal place.
- **f.** What is the minimum distance between the fisherman and the boat? When does this occur?

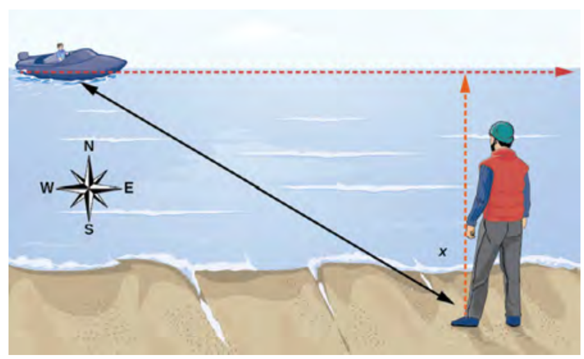


Figure 19

56. A laser rangefinder is locked on a comet approaching Earth. The distance $g(x)$, in kilometers, of the comet after x days, for x in the interval 0 to 30 days, is given by $g(x) = 250{,}000\csc\left(\frac{\pi}{30}x\right)$.

- **a.** Graph $g(x)$ on the interval $[0, 35]$.
- **b.** Evaluate $g(5)$ and interpret the information.
- **c.** What is the minimum distance between the comet and Earth? When does this occur? To which constant in the equation does this correspond?
- **d.** Find and discuss the meaning of any vertical asymptotes.

57. A video camera is focused on a rocket on a launching pad 2 miles from the camera. The angle of elevation from the ground to the rocket after x seconds is $\frac{\pi}{120}x$.

- **a.** Write a function expressing the altitude $h(x)$, in miles, of the rocket above the ground after x seconds. Ignore the curvature of the Earth.
- **b.** Graph $h(x)$ on the interval $(0, 60)$.
- **c.** Evaluate and interpret the values $h(0)$ and $h(30)$.
- **d.** What happens to the values of $h(x)$ as x approaches 60 seconds? Interpret the meaning of this in terms of the problem.

LEARNING OBJECTIVES

In this section, you will:

- Understand and use the inverse sine, cosine, and tangent functions.
- Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- Use a calculator to evaluate inverse trigonometric functions.
- Find exact values of composite functions with inverse trigonometric functions.

6.3 INVERSE TRIGONOMETRIC FUNCTIONS

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions.

Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa, as summarized in **Figure 1**.

Trig Functions	Inverse Trig Functions
Domain: Measure of an angle	Domain: Ratio
Range: Ratio	Range: Measure of an angle

Figure 1

For example, if $f(x) = \sin x$, then we would write $f^{-1}(x) = \sin^{-1} x$. Be aware that $\sin^{-1} x$ does not mean $\frac{1}{\sin x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, then $\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right)$.
- Since $\cos(\pi) = -1$, then $\pi = \cos^{-1}(-1)$.
- Since $\tan\left(\frac{\pi}{4}\right) = 1$, then $\frac{\pi}{4} = \tan^{-1}(1)$.

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the domain of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0. **Figure 2** shows the graph of the sine function limited to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the graph of the cosine function limited to $[0, \pi]$.

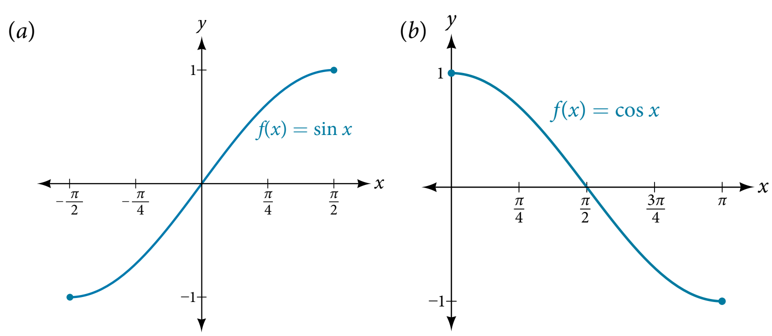


Figure 2 (*a*) Sine function on a restricted domain of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; (*b*) Cosine function on a restricted domain of $[0, \pi]$

Figure 3 shows the graph of the tangent function limited to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

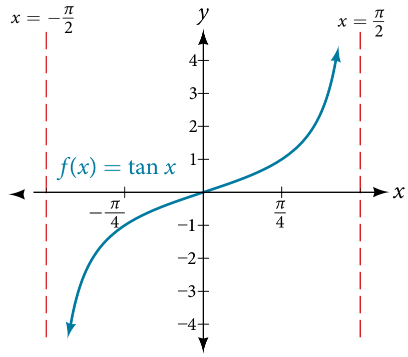


Figure 3 Tangent function on a restricted domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values, and most importantly, each results in a one-to-one function that is invertible. The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one vertical asymptote to the next instead of being divided into two parts by an asymptote.

On these restricted domains, we can define the inverse trigonometric functions.

- The **inverse sine function** $y = \sin^{-1} x$ means $x = \sin y$. The inverse sine function is sometimes called the **arcsine function**, and notated arcsin*x*.

$$y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

- The **inverse cosine function** $y = \cos^{-1} x$ means $x = \cos y$. The inverse cosine function is sometimes called the **arccosine** function, and notated arccos x.

$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$$

- The **inverse tangent function** $y = \tan^{-1} x$ means $x = \tan y$. The inverse tangent function is sometimes called the **arctangent** function, and notated arctan x.

$$y = \tan^{-1} x \text{ has domain } (-\infty, \infty) \text{ and range } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

The graphs of the inverse functions are shown in **Figure 4**, **Figure 5**, and **Figure 6**. Notice that the output of each of these inverse functions is a number, an angle in radian measure. We see that $\sin^{-1} x$ has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$, and $\tan^{-1} x$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

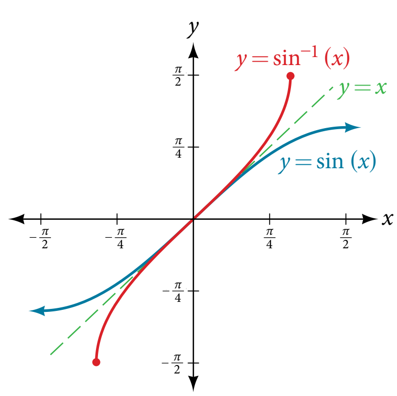


Figure 4 The sine function and inverse sine (or arcsine) function

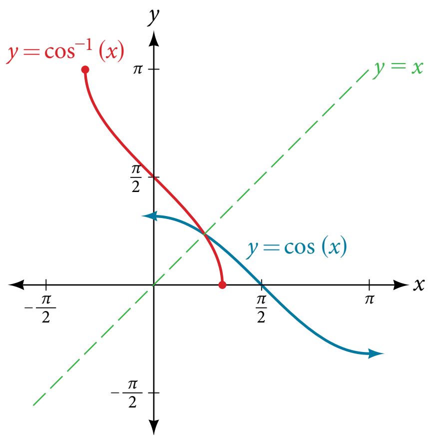


Figure 5 The cosine function and inverse cosine (or arccosine) function

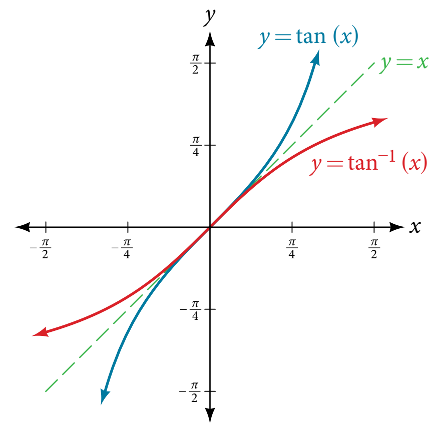


Figure 6 The tangent function and inverse tangent (or arctangent) function

relations for inverse sine, cosine, and tangent functions

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan y = x$, then $\tan^{-1} x = y$.

Example 1 Writing a Relation for an Inverse Function

Given $\sin\left(\frac{5\pi}{12}\right) \approx 0.96593$, write a relation involving the inverse sine.

Solution Use the relation for the inverse sine. If $\sin y = x$, then $\sin^{-1} x = y$.

In this problem, $x = 0.96593$, and $y = \frac{5\pi}{12}$.

$$\sin^{-1}(0.96593) \approx \frac{5\pi}{12}$$

Try It #1

Given $\cos(0.5) \approx 0.8776$, write a relation involving the inverse cosine.

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°), and $\frac{\pi}{3}$ (60°), and their reflections into other quadrants.

How To...

Given a "special" input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x, depending on which corresponds to the given inverse function.

Example 2 Evaluating Inverse Trigonometric Functions for Special Input Values

Evaluate each of the following.

a. $\sin^{-1}\left(\frac{1}{2}\right)$ **b.** $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ **c.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **d.** $\tan^{-1}(1)$

Solution

a. Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as determining the angle that would have a sine value of $\frac{1}{2}$. In other words, what angle x would satisfy $\sin(x) = \frac{1}{2}$? There are multiple values that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the answer will be $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Remember that the inverse is a function, so for each input, we will get exactly one output.

b. To evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

c. To evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, we are looking for an angle in the interval $[0, \pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

d. Evaluating $\tan^{-1}(1)$, we are looking for an angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is $\tan^{-1}(1) = \frac{\pi}{4}$.

Try It #2

Evaluate each of the following.

a. $\sin^{-1}(-1)$ **b.** $\tan^{-1}(-1)$ **c.** $\cos^{-1}(-1)$ **d.** $\cos^{-1}\left(\frac{1}{2}\right)$

Using a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate inverse trigonometric functions that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions. These may be labeled, for example, **SIN-1**, **ARCSIN**, or **ASIN**.

In the previous chapter, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trigonometric functions, we can solve for the angles of a right triangle given two sides, and we can use a calculator to find the values to several decimal places.

In these examples and exercises, the answers will be interpreted as angles and we will use θ as the independent variable. The value displayed on the calculator may be in degrees or radians, so be sure to set the mode appropriate to the application.

Example 3 Evaluating the Inverse Sine on a Calculator

Evaluate $\sin^{-1}(0.97)$ using a calculator.

Solution Because the output of the inverse function is an angle, the calculator will give us a degree value if in degree mode and a radian value if in radian mode. Calculators also use the same domain restrictions on the angles as we are using.

In radian mode, $\sin^{-1}(0.97) \approx 1.3252$. In degree mode, $\sin^{-1}(0.97) \approx 75.93°$. Note that in calculus and beyond we will use radians in almost all cases.

Try It #3

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

How To…

Given two sides of a right triangle like the one shown in **Figure 7**, find an angle.

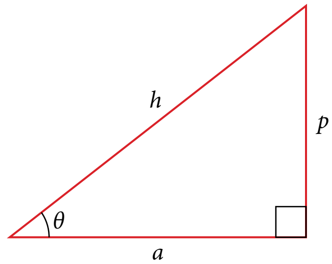


Figure 7

1. If one given side is the hypotenuse of length h and the side of length a adjacent to the desired angle is given, use the equation $\theta = \cos^{-1}\left(\frac{a}{h}\right)$.
2. If one given side is the hypotenuse of length h and the side of length p opposite to the desired angle is given, use the equation $\theta = \sin^{-1}\left(\frac{p}{h}\right)$
3. If the two legs (the sides adjacent to the right angle) are given, then use the equation $\theta = \tan^{-1}\left(\frac{p}{a}\right)$.

Example 4 Applying the Inverse Cosine to a Right Triangle

Solve the triangle in **Figure 8** for the angle θ.

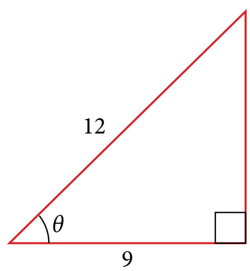


Figure 8

Solution Because we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\cos\theta = \frac{9}{12}$$

$$\theta = \cos^{-1}\left(\frac{9}{12}\right) \qquad \text{Apply definition of the inverse.}$$

$$\theta \approx 0.7227 \text{ or about } 41.4096^\circ \qquad \text{Evaluate.}$$

Try It #4

Solve the triangle in **Figure 9** for the angle θ.

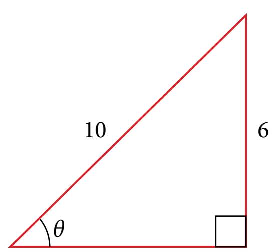


Figure 9

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let $f(x)$ and $g(x)$ be two different trigonometric functions belonging to the set $\{\sin(x), \cos(x), \tan(x)\}$ and let $f^{-1}(y)$ and $g^{-1}(y)$ be their inverses.

Evaluating Compositions of the Form $f(f^{-1}(y))$ and $f^{-1}(f(x))$

For any trigonometric function, $f(f^{-1}(y)) = y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}. However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

compositions of a trigonometric function and its inverse

$$\sin(\sin^{-1} x) = x \text{ for } -1 \leq x \leq 1$$
$$\cos(\cos^{-1} x) = x \text{ for } -1 \leq x \leq 1$$
$$\tan(\tan^{-1} x) = x \text{ for } -\infty < x < \infty$$

$$\sin^{-1}(\sin x) = x \text{ only for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$\cos^{-1}(\cos x) = x \text{ only for } 0 \leq x \leq \pi$$
$$\tan^{-1}(\tan x) = x \text{ only for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Q & A...

Is it correct that $\sin^{-1}(\sin x) = x$?

No. This equation is correct if x belongs to the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but sine is defined for all real input values, and for x outside the restricted interval, the equation is not correct because its inverse always returns a value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The situation is similar for cosine and tangent and their inverses. For example, $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$.

How To...

Given an expression of the form $f^{-1}(f(\theta))$ where $f(\theta) = \sin\theta, \cos\theta$, or $\tan\theta$, evaluate.

1. If θ is in the restricted domain of f, then $f^{-1}(f(\theta)) = \theta$.
2. If not, then find an angle ϕ within the restricted domain of f such that $f(\phi) = f(\theta)$. Then $f^{-1}(f(\theta)) = \phi$.

Example 5 Using Inverse Trigonometric Functions

Evaluate the following:

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$ **b.** $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ **c.** $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$ **d.** $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Solution

a. $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

b. $\frac{2\pi}{3}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$, so $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$.

c. $\frac{2\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$.

d. $-\frac{\pi}{3}$ is not in $[0, \pi]$, but $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ because cosine is an even function. $\frac{\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

Try It #5

Evaluate $\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)$ and $\tan^{-1}\left(\tan\left(\frac{11\pi}{9}\right)\right)$.

Evaluating Compositions of the Form $f^{-1}(g(x))$

Now that we can compose a trigonometric function with its inverse, we can explore how to evaluate a composition of a trigonometric function and the inverse of another trigonometric function. We will begin with compositions of the form $f^{-1}(g(x))$. For special values of x, we can exactly evaluate the inner function and then the outer, inverse function. However, we can find a more general approach by considering the relation between the two acute angles of a right triangle where one is θ, making the other $\frac{\pi}{2} - \theta$. Consider the sine and cosine of each angle of the right triangle in **Figure 10**.

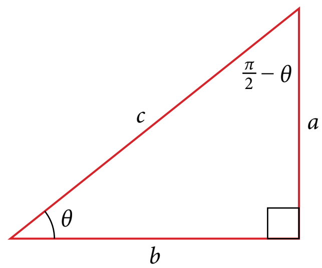


Figure 10 Right triangle illustrating the cofunction relationships

Because $\cos\theta = \frac{b}{c} = \sin\left(\frac{\pi}{2} - \theta\right)$, we have $\sin^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$ if $0 \leq \theta \leq \pi$. If θ is not in this domain, then we need to find another angle that has the same cosine as θ and does belong to the restricted domain; we then subtract this angle from $\frac{\pi}{2}$. Similarly, $\sin\theta = \frac{a}{c} = \cos\left(\frac{\pi}{2} - \theta\right)$, so $\cos^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$ if $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. These are just the function-cofunction relationships presented in another way.

How To…

Given functions of the form $\sin^{-1}(\cos x)$ and $\cos^{-1}(\sin x)$, evaluate them.

1. If x is in $[0, \pi]$, then $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$.
2. If x is not in $[0, \pi]$, then find another angle y in $[0, \pi]$ such that $\cos y = \cos x$.
$$\sin^{-1}(\cos x) = \frac{\pi}{2} - y$$
3. If x is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$.
4. If x is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then find another angle y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = \sin x$.
$$\cos^{-1}(\sin x) = \frac{\pi}{2} - y$$

Example 6 Evaluating the Composition of an Inverse Sine with a Cosine

Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$

a. by direct evaluation. **b.** by the method described previously.

Solution

a. Here, we can directly evaluate the inside of the composition.

$$\begin{aligned}\cos\left(\frac{13\pi}{6}\right) &= \cos\left(\frac{\pi}{6} + 2\pi\right)\\ &= \cos\left(\frac{\pi}{6}\right)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Now, we can evaluate the inverse function as we did earlier.

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

b. We have $x = \frac{13\pi}{6}$, $y = \frac{\pi}{6}$, and

$$\begin{aligned}\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right) &= \frac{\pi}{2} - \frac{\pi}{6}\\ &= \frac{\pi}{3}\end{aligned}$$

Try It #6

Evaluate $\cos^{-1}\left(\sin\left(-\frac{11\pi}{4}\right)\right)$.

Evaluating Compositions of the Form $f(g^{-1}(x))$

To evaluate compositions of the form $f(g^{-1}(x))$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, we have exact formulas, such as $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$. When we need to use them, we can derive these formulas by using the trigonometric relations between the angles and sides of a right triangle, together with the use of Pythagorean's relation between the lengths of the sides. We can use the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$, to solve for one when given the other. We can also use the inverse trigonometric functions to find compositions involving algebraic expressions.

Example 7 Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

Solution Beginning with the inside, we can say there is some angle such that $\theta = \cos^{-1}\left(\frac{4}{5}\right)$, which means $\cos\theta = \frac{4}{5}$, and we are looking for $\sin\theta$. We can use the Pythagorean identity to do this.

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{Use our known value for cosine.}$$

$$\sin^2\theta + \left(\frac{4}{5}\right)^2 = 1 \quad \text{Solve for sine.}$$

$$\sin^2\theta = 1 - \frac{16}{25}$$

$$\sin\theta = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

Since $\theta = \cos^{-1}\left(\frac{4}{5}\right)$ is in quadrant I, $\sin\theta$ must be positive, so the solution is $\frac{3}{5}$. See **Figure 11**.

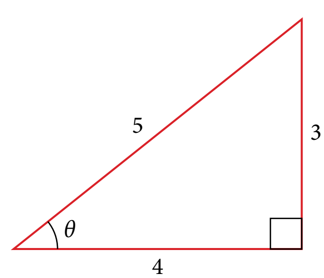


Figure 11 Right triangle illustrating that if $\cos\theta = \frac{4}{5}$, then $\sin\theta = \frac{3}{5}$

We know that the inverse cosine always gives an angle on the interval $[0, \pi]$, so we know that the sine of that angle must be positive; therefore $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) = \sin\theta = \frac{3}{5}$.

Try It #7

Evaluate $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

Example 8 Evaluating the Composition of a Sine with an Inverse Tangent

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Solution While we could use a similar technique as in **Example 6**, we will demonstrate a different technique here. From the inside, we know there is an angle such that $\tan\theta = \frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle, as shown in **Figure 12**.

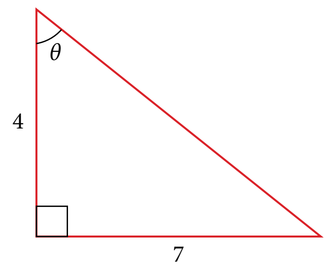


Figure 12 A right triangle with two sides known

Using the Pythagorean Theorem, we can find the hypotenuse of this triangle.

$$4^2 + 7^2 = \text{hypotenuse}^2$$
$$\text{hypotenuse} = \sqrt{65}$$

Now, we can evaluate the sine of the angle as the opposite side divided by the hypotenuse.

$$\sin\theta = \frac{7}{\sqrt{65}}$$

This gives us our desired composition.

$$\begin{aligned}\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right) &= \sin\theta \\ &= \frac{7}{\sqrt{65}} \\ &= \frac{7\sqrt{65}}{65}\end{aligned}$$

Try It #8

Evaluate $\cos\left(\sin^{-1}\left(\frac{7}{9}\right)\right)$.

Example 9 Finding the Cosine of the Inverse Sine of an Algebraic Expression

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ for $-3 \le x \le 3$.

Solution We know there is an angle θ such that $\sin\theta = \frac{x}{3}$.

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= 1 && \text{Use the Pythagorean Theorem.} \\ \left(\frac{x}{3}\right)^2 + \cos^2\theta &= 1 && \text{Solve for cosine.} \\ \cos^2\theta &= 1 - \frac{x^2}{9} \\ \cos\theta &= \pm\sqrt{\frac{9 - x^2}{9}} = \pm\frac{\sqrt{9 - x^2}}{3}\end{aligned}$$

Because we know that the inverse sine must give an angle on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can deduce that the cosine of that angle must be positive.

$$\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \frac{\sqrt{9 - x^2}}{3}$$

Try It #9

Find a simplified expression for $\sin(\tan^{-1}(4x))$ for $-\frac{1}{4} \le x \le \frac{1}{4}$.

Access this online resource for additional instruction and practice with inverse trigonometric functions.

- Evaluate Expressions Involving Inverse Trigonometric Functions (http://openstaxcollege.org/l/evalinverstrig)

6.3 SECTION EXERCISES

VERBAL

1. Why do the functions $f(x) = \sin^{-1} x$ and $g(x) = \cos^{-1} x$ have different ranges?

2. Since the functions $y = \cos x$ and $y = \cos^{-1} x$ are inverse functions, why is $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$ not equal to $-\frac{\pi}{6}$?

3. Explain the meaning of $\frac{\pi}{6} = \arcsin(0.5)$.

4. Most calculators do not have a key to evaluate $\sec^{-1}(2)$. Explain how this can be done using the cosine function or the inverse cosine function.

5. Why must the domain of the sine function, $\sin x$, be restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for the inverse sine function to exist?

6. Discuss why this statement is incorrect: $\arccos(\cos x) = x$ for all x.

7. Determine whether the following statement is true or false and explain your answer: $\arccos(-x) = \pi - \arccos x$.

ALGEBRAIC

For the following exercises, evaluate the expressions.

8. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

9. $\sin^{-1}\left(-\frac{1}{2}\right)$

10. $\cos^{-1}\left(\frac{1}{2}\right)$

11. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

12. $\tan^{-1}(1)$

13. $\tan^{-1}(-\sqrt{3})$

14. $\tan^{-1}(-1)$

15. $\tan^{-1}(\sqrt{3})$

16. $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

For the following exercises, use a calculator to evaluate each expression. Express answers to the nearest hundredth.

17. $\cos^{-1}(-0.4)$

18. $\arcsin(0.23)$

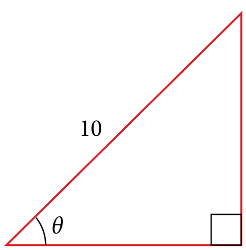
19. $\arccos\left(\frac{3}{5}\right)$

20. $\cos^{-1}(0.8)$

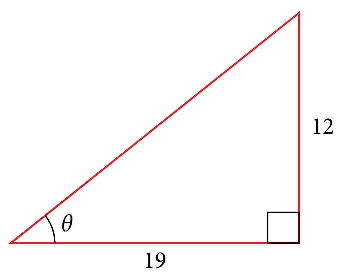
21. $\tan^{-1}(6)$

For the following exercises, find the angle θ in the given right triangle. Round answers to the nearest hundredth.

22.



23.



For the following exercises, find the exact value, if possible, without a calculator. If it is not possible, explain why.

24. $\sin^{-1}(\cos(\pi))$

25. $\tan^{-1}(\sin(\pi))$

26. $\cos^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

27. $\tan^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

28. $\sin^{-1}\left(\cos\left(\frac{-\pi}{2}\right)\right)$

29. $\tan^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$

30. $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

31. $\tan^{-1}\left(\sin\left(\frac{-5\pi}{2}\right)\right)$

32. $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

33. $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

34. $\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

35. $\cos\left(\tan^{-1}\left(\frac{12}{5}\right)\right)$

36. $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

For the following exercises, find the exact value of the expression in terms of x with the help of a reference triangle.

37. $\tan(\sin^{-1}(x-1))$

38. $\sin(\cos^{-1}(1-x))$

39. $\cos\left(\sin^{-1}\left(\frac{1}{x}\right)\right)$

40. $\cos(\tan^{-1}(3x-1))$

41. $\tan\left(\sin^{-1}\left(x+\frac{1}{2}\right)\right)$

EXTENSIONS

For the following exercise, evaluate the expression without using a calculator. Give the exact value.

42. $$\frac{\sin^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\cos^{-1}(1)}{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)+\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(0)}$$

For the following exercises, find the function if $\sin t=\frac{x}{x+1}$.

43. $\cos t$

44. $\sec t$

45. $\cot t$

46. $\cos\left(\sin^{-1}\left(\frac{x}{x+1}\right)\right)$

47. $\tan^{-1}\left(\frac{x}{\sqrt{2x+1}}\right)$

GRAPHICAL

48. Graph $y=\sin^{-1}x$ and state the domain and range of the function.

49. Graph $y=\arccos x$ and state the domain and range of the function.

50. Graph one cycle of $y=\tan^{-1}x$ and state the domain and range of the function.

51. For what value of x does $\sin x=\sin^{-1}x$? Use a graphing calculator to approximate the answer.

52. For what value of x does $\cos x=\cos^{-1}x$? Use a graphing calculator to approximate the answer.

REAL-WORLD APPLICATIONS

53. Suppose a 13-foot ladder is leaning against a building, reaching to the bottom of a second-floor window 12 feet above the ground. What angle, in radians, does the ladder make with the building?

54. Suppose you drive 0.6 miles on a road so that the vertical distance changes from 0 to 150 feet. What is the angle of elevation of the road?

55. An isosceles triangle has two congruent sides of length 9 inches. The remaining side has a length of 8 inches. Find the angle that a side of 9 inches makes with the 8-inch side.

56. Without using a calculator, approximate the value of arctan(10,000). Explain why your answer is reasonable.

57. A truss for the roof of a house is constructed from two identical right triangles. Each has a base of 12 feet and height of 4 feet. Find the measure of the acute angle adjacent to the 4-foot side.

58. The line $y=\frac{3}{5}x$ passes through the origin in the x,y-plane. What is the measure of the angle that the line makes with the positive x-axis?

59. The line $y=-\frac{3}{7}x$ passes through the origin in the x,y-plane. What is the measure of the angle that the line makes with the negative x-axis?

60. What percentage grade should a road have if the angle of elevation of the road is 4 degrees? (The percentage grade is defined as the change in the altitude of the road over a 100-foot horizontal distance. For example a 5% grade means that the road rises 5 feet for every 100 feet of horizontal distance.)

61. A 20-foot ladder leans up against the side of a building so that the foot of the ladder is 10 feet from the base of the building. If specifications call for the ladder's angle of elevation to be between 35 and 45 degrees, does the placement of this ladder satisfy safety specifications?

62. Suppose a 15-foot ladder leans against the side of a house so that the angle of elevation of the ladder is 42 degrees. How far is the foot of the ladder from the side of the house?

CHAPTER 6 REVIEW

Key Terms

amplitude the vertical height of a function; the constant A appearing in the definition of a sinusoidal function

arccosine another name for the inverse cosine; $\arccos x = \cos^{-1} x$

arcsine another name for the inverse sine; $\arcsin x = \sin^{-1} x$

arctangent another name for the inverse tangent; $\arctan x = \tan^{-1} x$

inverse cosine function the function $\cos^{-1} x$, which is the inverse of the cosine function and the angle that has a cosine equal to a given number

inverse sine function the function $\sin^{-1} x$, which is the inverse of the sine function and the angle that has a sine equal to a given number

inverse tangent function the function $\tan^{-1} x$, which is the inverse of the tangent function and the angle that has a tangent equal to a given number

midline the horizontal line $y = D$, where D appears in the general form of a sinusoidal function

periodic function a function $f(x)$ that satisfies $f(x + P) = f(x)$ for a specific constant P and any value of x

phase shift the horizontal displacement of the basic sine or cosine function; the constant $\frac{C}{B}$

sinusoidal function any function that can be expressed in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$

Key Equations

Sinusoidal functions	$f(x) = A\sin(Bx - C) + D$
	$f(x) = A\cos(Bx - C) + D$
Shifted, compressed, and/or stretched tangent function	$y = A\tan(Bx - C) + D$
Shifted, compressed, and/or stretched secant function	$y = A\sec(Bx - C) + D$
Shifted, compressed, and/or stretched cosecant function	$y = A\csc(Bx - C) + D$
Shifted, compressed, and/or stretched cotangent function	$y = A\cot(Bx - C) + D$

Key Concepts

6.1 Graphs of the Sine and Cosine Functions

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of 2π.
- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the y-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \frac{2\pi}{|B|}$ See **Example 1**.
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed. See **Example 2**.
- The value $\frac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift. See **Example 3**.
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline. See **Example 4**.
- Combinations of variations of sinusoidal functions can be detected from an equation. See **Example 5**.
- The equation for a sinusoidal function can be determined from a graph. See **Example 6** and **Example 7**.
- A function can be graphed by identifying its amplitude and period. See **Example 8** and **Example 9**.
- A function can also be graphed by identifying its amplitude, period, phase shift, and horizontal shift. See **Example 10**.
- Sinusoidal functions can be used to solve real-world problems. See **Example 11**, **Example 12**, and **Example 13**.

6.2 Graphs of the Other Trigonometric Functions

- The tangent function has period π.
- $f(x) = A\tan(Bx - C) + D$ is a tangent with vertical and/or horizontal stretch/compression and shift. See **Example 1**, **Example 2**, and **Example 3**.
- The secant and cosecant are both periodic functions with a period of 2π. $f(x) = A\sec(Bx - C) + D$ gives a shifted, compressed, and/or stretched secant function graph. See **Example 4** and **Example 5**.
- $f(x) = A\csc(Bx - C) + D$ gives a shifted, compressed, and/or stretched cosecant function graph. See **Example 6** and **Example 7**.
- The cotangent function has period π and vertical asymptotes at $0, \pm\pi, \pm 2\pi, \ldots$
- The range of cotangent is $(-\infty, \infty)$, and the function is decreasing at each point in its range.
- The cotangent is zero at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
- $f(x) = A\cot(Bx - C) + D$ is a cotangent with vertical and/or horizontal stretch/compression and shift. See **Example 8** and **Example 9**.
- Real-world scenarios can be solved using graphs of trigonometric functions. See **Example 10**.

6.3 Inverse Trigonometric Functions

- An inverse function is one that "undoes" another function. The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- Because the trigonometric functions are not one-to-one on their natural domains, inverse trigonometric functions are defined for restricted domains.
- For any trigonometric function $f(x)$, if $x = f^{-1}(y)$, then $f(x) = y$. However, $f(x) = y$ only implies $x = f^{-1}(y)$ if x is in the restricted domain of f. See **Example 1.**
- Special angles are the outputs of inverse trigonometric functions for special input values; for example, $\frac{\pi}{4} = \tan^{-1}(1)$ and $\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right)$. See **Example 2**.
- A calculator will return an angle within the restricted domain of the original trigonometric function. See **Example 3**.
- Inverse functions allow us to find an angle when given two sides of a right triangle. See **Example 4**.
- In function composition, if the inside function is an inverse trigonometric function, then there are exact expressions; for example, $\sin(\cos^{-1}(x)) = \sqrt{1 - x^2}$. See **Example 5**.
- If the inside function is a trigonometric function, then the only possible combinations are $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ if $0 \le x \le \pi$ and $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. See **Example 6** and **Example 7**.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, draw a reference triangle to assist in determining the ratio of sides that represents the output of the trigonometric function. See **Example 8**.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, you may use trig identities to assist in determining the ratio of sides. See **Example 9**.

CHAPTER 6 REVIEW EXERCISES

GRAPHS OF THE SINE AND COSINE FUNCTIONS

For the following exercises, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

1. $f(x) = -3\cos x + 3$

2. $f(x) = \frac{1}{4}\sin x$

3. $f(x) = 3\cos\left(x + \frac{\pi}{6}\right)$

4. $f(x) = -2\sin\left(x - \frac{2\pi}{3}\right)$

5. $f(x) = 3\sin\left(x - \frac{\pi}{4}\right) - 4$

6. $f(x) = 2\left(\cos\left(x - \frac{4\pi}{3}\right) + 1\right)$

7. $f(x) = 6\sin\left(3x - \frac{\pi}{6}\right) - 1$

8. $f(x) = -100\sin(50x - 20)$

GRAPHS OF THE OTHER TRIGONOMETRIC FUNCTIONS

For the following exercises, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

9. $f(x) = \tan x - 4$

10. $f(x) = 2\tan\left(x - \frac{\pi}{6}\right)$

11. $f(x) = -3\tan(4x) - 2$

12. $f(x) = 0.2\cos(0.1x) + 0.3$

For the following exercises, graph two full periods. Identify the period, the phase shift, the amplitude, and asymptotes.

13. $f(x) = \frac{1}{3}\sec x$

14. $f(x) = 3\cot x$

15. $f(x) = 4\csc(5x)$

16. $f(x) = 8\sec\left(\frac{1}{4}x\right)$

17. $f(x) = \frac{2}{3}\csc\left(\frac{1}{2}x\right)$

18. $f(x) = -\csc(2x + \pi)$

For the following exercises, use this scenario: The population of a city has risen and fallen over a 20-year interval. Its population may be modeled by the following function: $y = 12{,}000 + 8{,}000\sin(0.628x)$, where the domain is the years since 1980 and the range is the population of the city.

19. What is the largest and smallest population the city may have?

20. Graph the function on the domain of [0, 40].

21. What are the amplitude, period, and phase shift for the function?

22. Over this domain, when does the population reach 18,000? 13,000?

23. What is the predicted population in 2007? 2010?

For the following exercises, suppose a weight is attached to a spring and bobs up and down, exhibiting symmetry.

24. Suppose the graph of the displacement function is shown in **Figure 1**, where the values on the x-axis represent the time in seconds and the y-axis represents the displacement in inches. Give the equation that models the vertical displacement of the weight on the spring.

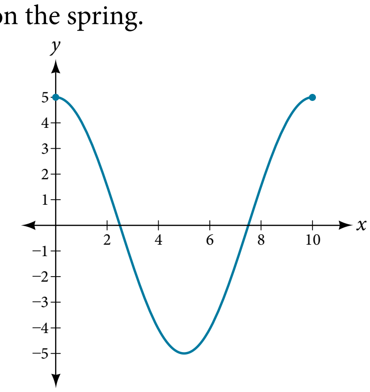


Figure 1

25. At time $= 0$, what is the displacement of the weight?

26. At what time does the displacement from the equilibrium point equal zero?

27. What is the time required for the weight to return to its initial height of 5 inches? In other words, what is the period for the displacement function?

INVERSE TRIGONOMETRIC FUNCTIONS

For the following exercises, find the exact value without the aid of a calculator.

28. $\sin^{-1}(1)$

29. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

30. $\tan^{-1}(-1)$

31. $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

32. $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

33. $\sin^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)$

34. $\cos^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

35. $\sin\left(\sec^{-1}\left(\frac{3}{5}\right)\right)$

36. $\cot\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

37. $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$

38. $\sin\left(\cos^{-1}\left(\frac{x}{x+1}\right)\right)$

39. Graph $f(x) = \cos x$ and $f(x) = \sec x$ on the interval $[0, 2\pi)$ and explain any observations.

40. Graph $f(x) = \sin x$ and $f(x) = \csc x$ and explain any observations.

41. Graph the function $f(x) = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ on the interval $[-1, 1]$ and compare the graph to the graph of $f(x) = \sin x$ on the same interval. Describe any observations.

CHAPTER 6 PRACTICE TEST

For the following exercises, sketch the graph of each function for two full periods. Determine the amplitude, the period, and the equation for the midline.

1. $f(x) = 0.5\sin x$

2. $f(x) = 5\cos x$

3. $f(x) = 5\sin x$

4. $f(x) = \sin(3x)$

5. $f(x) = -\cos\left(x + \frac{\pi}{3}\right) + 1$

6. $f(x) = 5\sin\left(3\left(x - \frac{\pi}{6}\right)\right) + 4$

7. $f(x) = 3\cos\left(\frac{1}{3}x - \frac{5\pi}{6}\right)$

8. $f(x) = \tan(4x)$

9. $f(x) = -2\tan\left(x - \frac{7\pi}{6}\right) + 2$

10. $f(x) = \pi\cos(3x + \pi)$

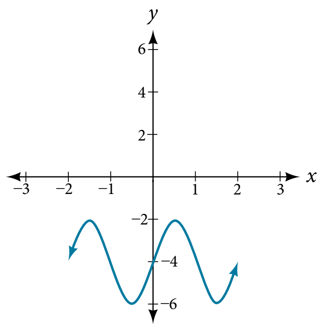
11. $f(x) = 5\csc(3x)$

12. $f(x) = \pi\sec\left(\frac{\pi}{2}x\right)$

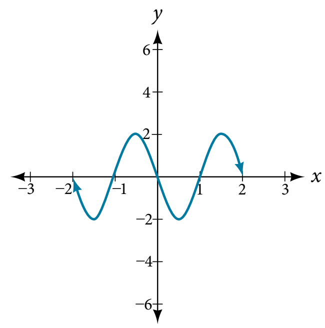
13. $f(x) = 2\csc\left(x + \frac{\pi}{4}\right) - 3$

For the following exercises, determine the amplitude, period, and midline of the graph, and then find a formula for the function.

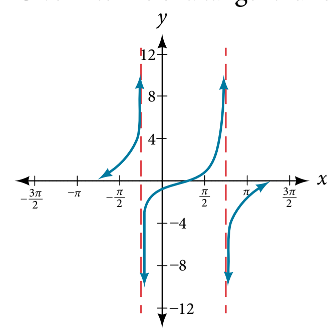
14. Give in terms of a sine function.



15. Give in terms of a sine function.



16. Give in terms of a tangent function.



For the following exercises, find the amplitude, period, phase shift, and midline.

17. $y = \sin\left(\frac{\pi}{6}x + \pi\right) - 3$

18. $y = 8\sin\left(\frac{7\pi}{6}x + \frac{7\pi}{2}\right) + 6$

19. The outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 68°F at midnight and the high and low temperatures during the day are 80°F and 56°F, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

20. Water is pumped into a storage bin and empties according to a periodic rate. The depth of the water is 3 feet at its lowest at 2:00 a.m. and 71 feet at its highest, which occurs every 5 hours. Write a cosine function that models the depth of the water as a function of time, and then graph the function for one period.

For the following exercises, find the period and horizontal shift of each function.

21. $g(x) = 3\tan(6x + 42)$

22. $n(x) = 4\csc\left(\frac{5\pi}{3}x - \frac{20\pi}{3}\right)$

23. Write the equation for the graph in **Figure 1** in terms of the secant function and give the period and phase shift.

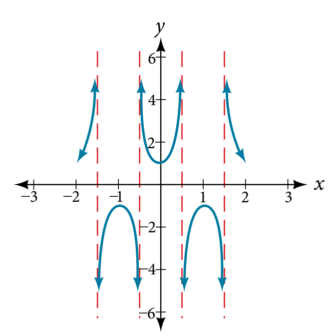


Figure 1

24. If $\tan x = 3$, find $\tan(-x)$.

25. If $\sec x = 4$, find $\sec(-x)$.

For the following exercises, graph the functions on the specified window and answer the questions.

26. Graph $m(x) = \sin(2x) + \cos(3x)$ on the viewing window $[-10, 10]$ by $[-3, 3]$. Approximate the graph's period.

27. Graph $n(x) = 0.02\sin(50\pi x)$ on the following domains in x: $[0, 1]$ and $[0, 3]$. Suppose this function models sound waves. Why would these views look so different?

28. Graph $f(x) = \dfrac{\sin x}{x}$ on $[-0.5, 0.5]$ and explain any observations.

For the following exercises, let $f(x) = \dfrac{3}{5}\cos(6x)$.

29. What is the largest possible value for $f(x)$?

30. What is the smallest possible value for $f(x)$?

31. Where is the function increasing on the interval $[0, 2\pi]$?

For the following exercises, find and graph one period of the periodic function with the given amplitude, period, and phase shift.

32. Sine curve with amplitude 3, period $\dfrac{\pi}{3}$, and phase shift $(h, k) = \left(\dfrac{\pi}{4}, 2\right)$

33. Cosine curve with amplitude 2, period $\dfrac{\pi}{6}$, and phase shift $(h, k) = \left(-\dfrac{\pi}{4}, 3\right)$

For the following exercises, graph the function. Describe the graph and, wherever applicable, any periodic behavior, amplitude, asymptotes, or undefined points.

34. $f(x) = 5\cos(3x) + 4\sin(2x)$

35. $f(x) = e^{\sin t}$

For the following exercises, find the exact value.

36. $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

37. $\tan^{-1}\left(\sqrt{3}\right)$

38. $\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

39. $\cos^{-1}(\sin(\pi))$

40. $\cos^{-1}\left(\tan\left(\dfrac{7\pi}{4}\right)\right)$

41. $\cos(\sin^{-1}(1 - 2x))$

42. $\cos^{-1}(-0.4)$

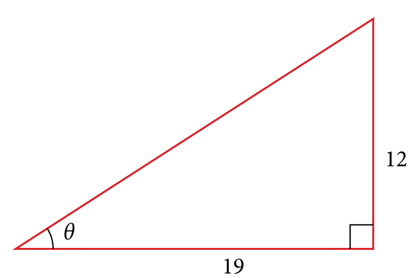
43. $\cos(\tan^{-1}(x^2))$

For the following exercises, suppose $\sin t = \dfrac{x}{x+1}$.

44. $\tan t$

45. $\csc t$

46. Given **Figure 2**, find the measure of angle θ to three decimal places. Answer in radians.



For the following exercises, determine whether the equation is true or false.

47. $\arcsin\left(\sin\left(\dfrac{5\pi}{6}\right)\right) = \dfrac{5\pi}{6}$

48. $\arccos\left(\cos\left(\dfrac{5\pi}{6}\right)\right) = \dfrac{5\pi}{6}$

49. The grade of a road is 7%. This means that for every horizontal distance of 100 feet on the road, the vertical rise is 7 feet. Find the angle the road makes with the horizontal in radians.

7

Trigonometric Identities and Equations

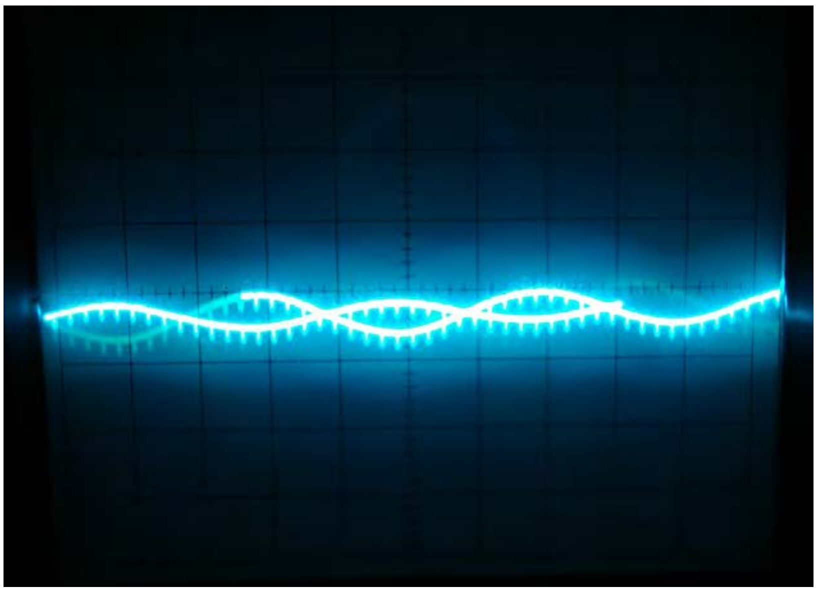


Figure 1 A sine wave models disturbance. (credit: modification of work by Mikael Altemark, Flickr).

CHAPTER OUTLINE

Introduction

Math is everywhere, even in places we might not immediately recognize. For example, mathematical relationships describe the transmission of images, light, and sound. The sinusoidal graph in **Figure 1** models music playing on a phone, radio, or computer. Such graphs are described using trigonometric equations and functions. In this chapter, we discuss how to manipulate trigonometric equations algebraically by applying various formulas and trigonometric identities. We will also investigate some of the ways that trigonometric equations are used to model real-life phenomena.

LEARNING OBJECTIVES

In this section, you will:

- Verify the fundamental trigonometric identities.
- Simplify trigonometric expressions using algebra and the identities.

7.1 SOLVING TRIGONOMETRIC EQUATIONS WITH IDENTITIES

Figure 1 International passports and travel documents

In espionage movies, we see international spies with multiple passports, each claiming a different identity. However, we know that each of those passports represents the same person. The trigonometric identities act in a similar manner to multiple passports—there are many ways to represent the same trigonometric expression. Just as a spy will choose an Italian passport when traveling to Italy, we choose the identity that applies to the given scenario when solving a trigonometric equation.

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

Verifying the Fundamental Trigonometric Identities

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations, just as factoring, finding common denominators, and using special formulas are the basic tools of solving algebraic equations. In fact, we use algebraic techniques constantly to simplify trigonometric expressions. Basic properties and formulas of algebra, such as the difference of squares formula and the perfect squares formula, will simplify the work involved with trigonometric expressions and equations. We already know that all of the trigonometric functions are related because they all are defined in terms of the unit circle. Consequently, any trigonometric identity can be written in many ways.

To verify the trigonometric identities, we usually start with the more complicated side of the equation and essentially rewrite the expression until it has been transformed into the same expression as the other side of the equation. Sometimes we have to factor expressions, expand expressions, find common denominators, or use other algebraic strategies to obtain the desired result. In this first section, we will work with the fundamental identities: the Pythagorean identities, the even-odd identities, the reciprocal identities, and the quotient identities.

We will begin with the **Pythagorean identities** (see **Table 1**), which are equations involving trigonometric functions based on the properties of a right triangle. We have already seen and used the first of these identifies, but now we will also use additional identities.

Pythagorean Identities		
$\sin^2\theta + \cos^2\theta = 1$	$1 + \cot^2\theta = \csc^2\theta$	$1 + \tan^2\theta = \sec^2\theta$

Table 1

The second and third identities can be obtained by manipulating the first. The identity $1 + \cot^2 \theta = \csc^2 \theta$ is found by rewriting the left side of the equation in terms of sine and cosine.

Prove: $1 + \cot^2 \theta = \csc^2 \theta$

$$\begin{aligned} 1 + \cot^2 \theta &= \left(1 + \frac{\cos^2 \theta}{\sin^2 \theta}\right) && \text{Rewrite the left side.} \\ &= \left(\frac{\sin^2 \theta}{\sin^2 \theta}\right) + \left(\frac{\cos^2 \theta}{\sin^2 \theta}\right) && \text{Write both terms with the common denominator.} \\ &= \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta} \\ &= \frac{1}{\sin^2 \theta} \\ &= \csc^2 \theta \end{aligned}$$

Similarly, $1 + \tan^2 \theta = \sec^2 \theta$ can be obtained by rewriting the left side of this identity in terms of sine and cosine. This gives

$$\begin{aligned} 1 + \tan^2 \theta &= 1 + \left(\frac{\sin \theta}{\cos \theta}\right)^2 && \text{Rewrite left side.} \\ &= \left(\frac{\cos \theta}{\cos \theta}\right)^2 + \left(\frac{\sin \theta}{\cos \theta}\right)^2 && \text{Write both terms with the common denominator.} \\ &= \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} \\ &= \frac{1}{\cos^2 \theta} \\ &= \sec^2 \theta \end{aligned}$$

The next set of fundamental identities is the set of **even-odd identities**. The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle and determine whether the identity is odd or even. (See **Table 2**).

Even-Odd Identities		
$\tan(-\theta) = -\tan \theta$	$\sin(-\theta) = -\sin \theta$	$\cos(-\theta) = \cos \theta$
$\cot(-\theta) = -\cot \theta$	$\csc(-\theta) = -\csc \theta$	$\sec(-\theta) = \sec \theta$

Table 2

Recall that an odd function is one in which $f(-x) = -f(x)$ for all x in the domain of f. The sine function is an odd function because $\sin(-\theta) = -\sin \theta$. The graph of an odd function is symmetric about the origin. For example, consider corresponding inputs of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. The output of $\sin\left(\frac{\pi}{2}\right)$ is opposite the output of $\sin\left(-\frac{\pi}{2}\right)$. Thus,

$$\sin\left(\frac{\pi}{2}\right) = 1$$

and

$$\begin{aligned} \sin\left(-\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right) \\ &= -1 \end{aligned}$$

This is shown in **Figure 2**.

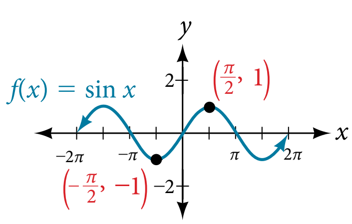


Figure 2 Graph of $y = \sin \theta$

Recall that an even function is one in which

$$f(-x) = f(x) \text{ for all } x \text{ in the domain of } f$$

The graph of an even function is symmetric about the y-axis. The cosine function is an even function because $\cos(-\theta) = \cos\theta$. For example, consider corresponding inputs $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. The output of $\cos\left(\frac{\pi}{4}\right)$ is the same as the output of $\cos\left(-\frac{\pi}{4}\right)$. Thus,

$$\cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$
$$\approx 0.707$$

See **Figure 3**.

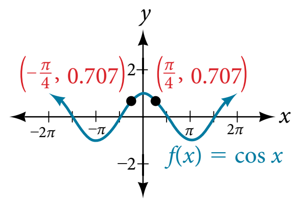


Figure 3 Graph of $y = \cos\theta$

For all θ in the domain of the sine and cosine functions, respectively, we can state the following:

- Since $\sin(-\theta) = -\sin\theta$, sine is an odd function.
- Since, $\cos(-\theta) = \cos\theta$, cosine is an even function.

The other even-odd identities follow from the even and odd nature of the sine and cosine functions. For example, consider the tangent identity, $\tan(-\theta) = -\tan\theta$. We can interpret the tangent of a negative angle as $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$. Tangent is therefore an odd function, which means that $\tan(-\theta) = -\tan(\theta)$ for all θ in the domain of the tangent function.

The cotangent identity, $\cot(-\theta) = -\cot\theta$, also follows from the sine and cosine identities. We can interpret the cotangent of a negative angle as $\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$. Cotangent is therefore an odd function, which means that $\cot(-\theta) = -\cot(\theta)$ for all θ in the domain of the cotangent function.

The cosecant function is the reciprocal of the sine function, which means that the cosecant of a negative angle will be interpreted as $\csc(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta$. The cosecant function is therefore odd.

Finally, the secant function is the reciprocal of the cosine function, and the secant of a negative angle is interpreted as $\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta$. The secant function is therefore even.
To sum up, only two of the trigonometric functions, cosine and secant, are even. The other four functions are odd, verifying the even-odd identities.

The next set of fundamental identities is the set of **reciprocal identities**, which, as their name implies, relate trigonometric functions that are reciprocals of each other. See **Table 3**.

Reciprocal Identities	
$\sin\theta = \frac{1}{\csc\theta}$	$\csc\theta = \frac{1}{\sin\theta}$
$\cos\theta = \frac{1}{\sec\theta}$	$\sec\theta = \frac{1}{\cos\theta}$
$\tan\theta = \frac{1}{\cot\theta}$	$\cot\theta = \frac{1}{\tan\theta}$

Table 3

The final set of identities is the set of quotient identities, which define relationships among certain trigonometric functions and can be very helpful in verifying other identities. See **Table 4**.

Quotient Identities	
$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$

Table 4

The reciprocal and quotient identities are derived from the definitions of the basic trigonometric functions.

summarizing trigonometric identities

The **Pythagorean identities** are based on the properties of a right triangle.

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \cot^2\theta = \csc^2\theta$$
$$1 + \tan^2\theta = \sec^2\theta$$

The **even-odd identities** relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle.

$$\tan(-\theta) = -\tan\theta$$
$$\cot(-\theta) = -\cot\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\csc(-\theta) = -\csc\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\sec(-\theta) = \sec\theta$$

The **reciprocal identities** define reciprocals of the trigonometric functions.

$$\sin\theta = \frac{1}{\csc\theta}$$
$$\cos\theta = \frac{1}{\sec\theta}$$
$$\tan\theta = \frac{1}{\cot\theta}$$
$$\csc\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$

The **quotient identities** define the relationship among the trigonometric functions.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Example 1 Graphing the Equations of an Identity

Graph both sides of the identity $\cot\theta = \frac{1}{\tan\theta}$. In other words, on the graphing calculator, graph $y = \cot\theta$ and $y = \frac{1}{\tan\theta}$.

Solution See **Figure 4**.

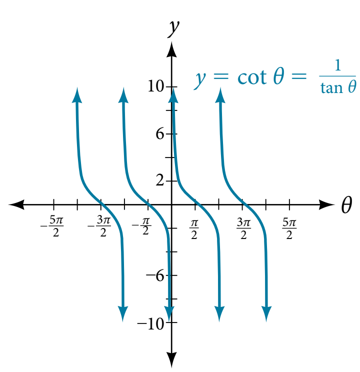


Figure 4

Analysis *We see only one graph because both expressions generate the same image. One is on top of the other. This is a good way to prove any identity. If both expressions give the same graph, then they must be identities.*

How To…

Given a trigonometric identity, verify that it is true.

1. Work on one side of the equation. It is usually better to start with the more complex side, as it is easier to simplify than to build.
2. Look for opportunities to factor expressions, square a binomial, or add fractions.
3. Noting which functions are in the final expression, look for opportunities to use the identities and make the proper substitutions.
4. If these steps do not yield the desired result, try converting all terms to sines and cosines.

Example 2 Verifying a Trigonometric Identity

Verify $\tan\theta\cos\theta = \sin\theta$.

Solution We will start on the left side, as it is the more complicated side:

$$\begin{aligned}\tan\theta\cos\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta\\ &= \left(\frac{\sin\theta}{\cancel{\cos\theta}}\right)\cancel{\cos\theta}\\ &= \sin\theta\end{aligned}$$

Analysis *This identity was fairly simple to verify, as it only required writing* $\tan\theta$ *in terms of* $\sin\theta$ *and* $\cos\theta$*.*

Try It #1

Verify the identity $\csc\theta\cos\theta\tan\theta = 1$.

Example 3 Verifying the Equivalency Using the Even-Odd Identities

Verify the following equivalency using the even-odd identities:

$$(1+\sin x)[1+\sin(-x)] = \cos^2 x$$

Solution Working on the left side of the equation, we have

$$\begin{aligned}(1+\sin x)[1+\sin(-x)] &= (1+\sin x)(1-\sin x) && \text{Since } \sin(-x) = -\sin x\\ &= 1-\sin^2 x && \text{Difference of squares}\\ &= \cos^2 x && \cos^2 x = 1-\sin^2 x\end{aligned}$$

Example 4 Verifying a Trigonometric Identity Involving $\sec^2\theta$

Verify the identity $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$

Solution As the left side is more complicated, let's begin there.

$$\begin{aligned}\frac{\sec^2\theta-1}{\sec^2\theta} &= \frac{(\tan^2\theta+1)-1}{\sec^2\theta} && \sec^2\theta = \tan^2\theta+1\\ &= \frac{\tan^2\theta}{\sec^2\theta}\\ &= \tan^2\theta\left(\frac{1}{\sec^2\theta}\right)\\ &= \tan^2\theta(\cos^2\theta) && \cos^2\theta = \frac{1}{\sec^2\theta}\\ &= \left(\frac{\sin^2\theta}{\cos^2\theta}\right)(\cos^2\theta) && \tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}\\ &= \left(\frac{\sin^2\theta}{\cancel{\cos^2\theta}}\right)(\cancel{\cos^2\theta})\\ &= \sin^2\theta\end{aligned}$$

There is more than one way to verify an identity. Here is another possibility. Again, we can start with the left side.

$$\begin{aligned}\frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{\sec^2\theta}{\sec^2\theta} - \frac{1}{\sec^2\theta}\\ &= 1 - \cos^2\theta\\ &= \sin^2\theta\end{aligned}$$

Analysis *In the first method, we used the identity* $\sec^2\theta = \tan^2\theta + 1$ *and continued to simplify. In the second method, we split the fraction, putting both terms in the numerator over the common denominator. This problem illustrates that there are multiple ways we can verify an identity. Employing some creativity can sometimes simplify a procedure. As long as the substitutions are correct, the answer will be the same.*

Try It #2

Show that $\dfrac{\cot\theta}{\csc\theta} = \cos\theta$.

Example 5 Creating and Verifying an Identity

Create an identity for the expression $2\tan\theta\sec\theta$ by rewriting strictly in terms of sine.

Solution There are a number of ways to begin, but here we will use the quotient and reciprocal identities to rewrite the expression:

$$\begin{aligned}2\tan\theta\sec\theta &= 2\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right)\\ &= \frac{2\sin\theta}{\cos^2\theta}\\ &= \frac{2\sin\theta}{1-\sin^2\theta} && \text{Substitute } 1-\sin^2\theta \text{ for } \cos^2\theta\end{aligned}$$

Thus,

$$2\tan\theta\sec\theta = \frac{2\sin\theta}{1-\sin^2\theta}$$

Example 6 Verifying an Identity Using Algebra and Even/Odd Identities

Verify the identity:

$$\frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} = \cos\theta - \sin\theta$$

Solution Let's start with the left side and simplify:

$$\begin{aligned}\frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} &= \frac{[\sin(-\theta)]^2 - [\cos(-\theta)]^2}{\sin(-\theta) - \cos(-\theta)}\\ &= \frac{(-\sin\theta)^2 - (\cos\theta)^2}{-\sin\theta - \cos\theta} && \sin(-x) = -\sin x \text{ and } \cos(-x) = \cos x\\ &= \frac{(\sin\theta)^2 - (\cos\theta)^2}{-\sin\theta - \cos\theta} && \text{Difference of squares}\\ &= \frac{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)}{-(\sin\theta + \cos\theta)}\\ &= \frac{(\sin\theta - \cos\theta)\cancel{(\sin\theta + \cos\theta)}}{-\cancel{(\sin\theta + \cos\theta)}}\\ &= \cos\theta - \sin\theta\end{aligned}$$

Try It #3

Verify the identity $\dfrac{\sin^2\theta - 1}{\tan\theta\sin\theta - \tan\theta} = \dfrac{\sin\theta + 1}{\tan\theta}$.

Example 7 Verifying an Identity Involving Cosines and Cotangents

Verify the identity: $(1 - \cos^2 x)(1 + \cot^2 x) = 1$.

Solution We will work on the left side of the equation

$$\begin{aligned}(1 - \cos^2 x)(1 + \cot^2 x) &= (1 - \cos^2 x)\left(1 + \frac{\cos^2 x}{\sin^2 x}\right) \\ &= (1 - \cos^2 x)\left(\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\right) && \text{Find the common denominator.} \\ &= (1 - \cos^2 x)\left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\right) \\ &= (\sin^2 x)\left(\frac{1}{\sin^2 x}\right) \\ &= 1\end{aligned}$$

Using Algebra to Simplify Trigonometric Expressions

We have seen that algebra is very important in verifying trigonometric identities, but it is just as critical in simplifying trigonometric expressions before solving. Being familiar with the basic properties and formulas of algebra, such as the difference of squares formula, the perfect square formula, or substitution, will simplify the work involved with trigonometric expressions and equations.

For example, the equation $(\sin x + 1)(\sin x - 1) = 0$ resembles the equation $(x + 1)(x - 1) = 0$, which uses the factored form of the difference of squares. Using algebra makes finding a solution straightforward and familiar. We can set each factor equal to zero and solve. This is one example of recognizing algebraic patterns in trigonometric expressions or equations.

Another example is the difference of squares formula, $a^2 - b^2 = (a - b)(a + b)$, which is widely used in many areas other than mathematics, such as engineering, architecture, and physics. We can also create our own identities by continually expanding an expression and making the appropriate substitutions. Using algebraic properties and formulas makes many trigonometric equations easier to understand and solve.

Example 8 Writing the Trigonometric Expression as an Algebraic Expression

Write the following trigonometric expression as an algebraic expression: $2\cos^2 \theta + \cos \theta - 1$.

Solution Notice that the pattern displayed has the same form as a standard quadratic expression, $ax^2 + bx + c$. Letting $\cos \theta = x$, we can rewrite the expression as follows:

$$2x^2 + x - 1$$

This expression can be factored as $(2x - 1)(x + 1)$. If it were set equal to zero and we wanted to solve the equation, we would use the zero factor property and solve each factor for x. At this point, we would replace x with $\cos \theta$ and solve for θ.

Example 9 Rewriting a Trigonometric Expression Using the Difference of Squares

Rewrite the trigonometric expression: $4 \cos^2 \theta - 1$.

Solution Notice that both the coefficient and the trigonometric expression in the first term are squared, and the square of the number 1 is 1. This is the difference of squares. Thus,

$$\begin{aligned}4 \cos^2 \theta - 1 &= (2 \cos \theta)^2 - 1 \\ &= (2 \cos \theta - 1)(2 \cos \theta + 1)\end{aligned}$$

Analysis *If this expression were written in the form of an equation set equal to zero, we could solve each factor using the zero factor property. We could also use substitution like we did in the previous problem and let $\cos \theta = x$, rewrite the expression as $4x^2 - 1$, and factor $(2x - 1)(2x + 1)$. Then replace x with $\cos \theta$ and solve for the angle.*

Try It #4

Rewrite the trigonometric expression: $25 - 9 \sin^2 \theta$.

Example 10 **Simplify by Rewriting and Using Substitution**

Simplify the expression by rewriting and using identities:

$$\csc^2 \theta - \cot^2 \theta$$

Solution We can start with the Pythagorean Identity.

$$1 + \cot^2 \theta = \csc^2 \theta$$

Now we can simplify by substituting $1 + \cot^2 \theta$ for $\csc^2 \theta$. We have

$$\begin{aligned} \csc^2 \theta - \cot^2 \theta &= 1 + \cot^2 \theta - \cot^2 \theta \\ &= 1 \end{aligned}$$

Try It #5

Use algebraic techniques to verify the identity: $\dfrac{\cos \theta}{1 + \sin \theta} = \dfrac{1 - \sin \theta}{\cos \theta}$.

(Hint: Multiply the numerator and denominator on the left side by $1 - \sin \theta$.)

Access these online resources for additional instruction and practice with the fundamental trigonometric identities.

- Fundamental Trigonometric Identities (http://openstaxcollege.org/l/funtrigiden)
- Verifying Trigonometric Identities (http://openstaxcollege.org/l/verifytrigiden)

7.1 SECTION EXERCISES

VERBAL

1. We know $g(x) = \cos x$ is an even function, and $f(x) = \sin x$ and $h(x) = \tan x$ are odd functions. What about $G(x) = \cos^2 x$, $F(x) = \sin^2 x$, and $H(x) = \tan^2 x$? Are they even, odd, or neither? Why?

2. Examine the graph of $f(x) = \sec x$ on the interval $[-\pi, \pi]$. How can we tell whether the function is even or odd by only observing the graph of $f(x) = \sec x$?

3. After examining the reciprocal identity for $\sec t$, explain why the function is undefined at certain points.

4. All of the Pythagorean identities are related. Describe how to manipulate the equations to get from $\sin^2 t + \cos^2 t = 1$ to the other forms.

ALGEBRAIC

For the following exercises, use the fundamental identities to fully simplify the expression.

5. $\sin x \cos x \sec x$

6. $\sin(-x)\cos(-x)\csc(-x)$

7. $\tan x \sin x + \sec x \cos^2 x$

8. $\csc x + \cos x \cot(-x)$

9. $\dfrac{\cot t + \tan t}{\sec(-t)}$

10. $3 \sin^3 t \csc t + \cos^2 t + 2 \cos(-t)\cos t$

11. $-\tan(-x)\cot(-x)$

12. $\dfrac{-\sin(-x)\cos x \sec x \csc x \tan x}{\cot x}$

13. $\dfrac{1 + \tan^2 \theta}{\csc^2 \theta} + \sin^2 \theta + \dfrac{1}{\sec^2 \theta}$

14. $\left(\dfrac{\tan x}{\csc^2 x} + \dfrac{\tan x}{\sec^2 x}\right)\left(\dfrac{1 + \tan x}{1 + \cot x}\right) - \dfrac{1}{\cos^2 x}$

15. $\dfrac{1 - \cos^2 x}{\tan^2 x} + 2 \sin^2 x$

For the following exercises, simplify the first trigonometric expression by writing the simplified form in terms of the second expression.

16. $\dfrac{\tan x + \cot x}{\csc x}$; $\cos x$

17. $\dfrac{\sec x + \csc x}{1 + \tan x}$; $\sin x$

18. $\dfrac{\cos x}{1 + \sin x} + \tan x$; $\cos x$

19. $\dfrac{1}{\sin x \cos x} - \cot x$; $\cot x$

20. $\dfrac{1}{1 - \cos x} - \dfrac{\cos x}{1 + \cos x}$; $\csc x$

21. $(\sec x + \csc x)(\sin x + \cos x) - 2 - \cot x$; $\tan x$

22. $\dfrac{1}{\csc x - \sin x}$; $\sec x$ and $\tan x$

23. $\dfrac{1 - \sin x}{1 + \sin x} - \dfrac{1 + \sin x}{1 - \sin x}$; $\sec x$ and $\tan x$

24. $\tan x$; $\sec x$

25. $\sec x$; $\cot x$

26. $\sec x$; $\sin x$

27. $\cot x$; $\sin x$

28. $\cot x$; $\csc x$

For the following exercises, verify the identity.

29. $\cos x - \cos^3 x = \cos x \sin^2 x$

30. $\cos x(\tan x - \sec(-x)) = \sin x - 1$

31. $\dfrac{1 + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} + \dfrac{\sin^2 x}{\cos^2 x} = 1 + 2 \tan^2 x$

32. $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x$

33. $\cos^2 x - \tan^2 x = 2 - \sin^2 x - \sec^2 x$

EXTENSIONS

For the following exercises, prove or disprove the identity.

34. $\dfrac{1}{1+\cos x}-\dfrac{1}{1-\cos(-x)}=-2\cot x\csc x$

35. $\csc^2 x(1+\sin^2 x)=\cot^2 x$

36. $\left(\dfrac{\sec^2(-x)-\tan^2 x}{\tan x}\right)\left(\dfrac{2+2\tan x}{2+2\cot x}\right)-2\sin^2 x=\cos 2x$

37. $\dfrac{\tan x}{\sec x}\sin(-x)=\cos^2 x$

38. $\dfrac{\sec(-x)}{\tan x+\cot x}=-\sin(-x)$

39. $\dfrac{1+\sin x}{\cos x}=\dfrac{\cos x}{1+\sin(-x)}$

For the following exercises, determine whether the identity is true or false. If false, find an appropriate equivalent expression.

40. $\dfrac{\cos^2\theta-\sin^2\theta}{1-\tan^2\theta}=\sin^2\theta$

41. $3\sin^2\theta+4\cos^2\theta=3+\cos^2\theta$

42. $\dfrac{\sec\theta+\tan\theta}{\cot\theta+\cos\theta}=\sec^2\theta$

LEARNING OBJECTIVES

In this section, you will:

- Use sum and difference formulas for cosine.
- Use sum and difference formulas for sine.
- Use sum and difference formulas for tangent.
- Use sum and difference formulas for cofunctions.
- Use sum and difference formulas to verify identities.

7.2 SUM AND DIFFERENCE IDENTITIES

Figure 1 Mount McKinley, in Denali National Park, Alaska, rises 20,237 feet (6,168 m) above sea level. It is the highest peak in North America. (credit: Daniel A. Leifheit, Flickr)

How can the height of a mountain be measured? What about the distance from Earth to the sun? Like many seemingly impossible problems, we rely on mathematical formulas to find the answers. The trigonometric identities, commonly used in mathematical proofs, have had real-world applications for centuries, including their use in calculating long distances.

The trigonometric identities we will examine in this section can be traced to a Persian astronomer who lived around 950 AD, but the ancient Greeks discovered these same formulas much earlier and stated them in terms of chords. These are special equations or postulates, true for all values input to the equations, and with innumerable applications.

In this section, we will learn techniques that will enable us to solve problems such as the ones presented above. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word *identity*.

Using the Sum and Difference Formulas for Cosine

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. We can use the special angles, which we can review in the unit circle shown in **Figure 2**.

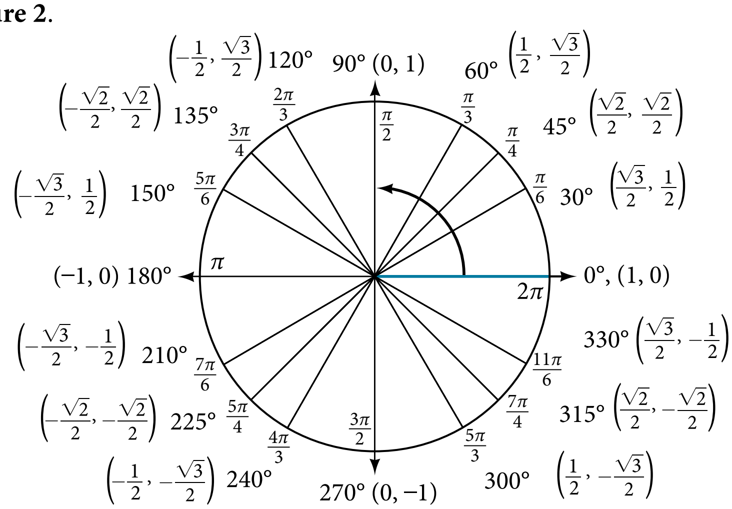


Figure 2 The Unit Circle

We will begin with the sum and difference formulas for cosine, so that we can find the cosine of a given angle if we can break it up into the sum or difference of two of the special angles. See **Table 1**.

Sum formula for cosine	$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
Difference formula for cosine	$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Table 1

First, we will prove the difference formula for cosines. Let's consider two points on the unit circle. See **Figure 3**. Point P is at an angle α from the positive x-axis with coordinates $(\cos\alpha, \sin\alpha)$ and point Q is at an angle of β from the positive x-axis with coordinates $(\cos\beta, \sin\beta)$. Note the measure of angle POQ is $\alpha - \beta$.

Label two more points: A at an angle of $(\alpha - \beta)$ from the positive x-axis with coordinates $(\cos(\alpha - \beta), \sin(\alpha - \beta))$; and point B with coordinates $(1, 0)$. Triangle POQ is a rotation of triangle AOB and thus the distance from P to Q is the same as the distance from A to B.

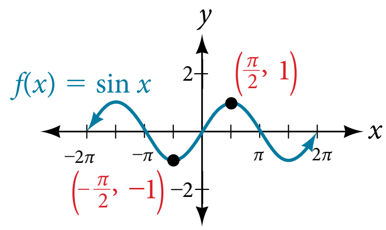


Figure 3

We can find the distance from P to Q using the distance formula.

$$\begin{aligned} d_{PQ} &= \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} \\ &= \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta} \end{aligned}$$

Then we apply the Pythagorean Identity and simplify.

$$\begin{aligned} &= \sqrt{(\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\ &= \sqrt{1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\ &= \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \end{aligned}$$

Similarly, using the distance formula we can find the distance from A to B.

$$\begin{aligned} d_{AB} &= \sqrt{(\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta) - 0)^2} \\ &= \sqrt{\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)} \end{aligned}$$

Applying the Pythagorean Identity and simplifying we get:

$$\begin{aligned} &= \sqrt{(\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)) - 2\cos(\alpha - \beta) + 1} \\ &= \sqrt{1 - 2\cos(\alpha - \beta) + 1} \\ &= \sqrt{2 - 2\cos(\alpha - \beta)} \end{aligned}$$

Because the two distances are the same, we set them equal to each other and simplify.

$$\sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} = \sqrt{2 - 2\cos(\alpha - \beta)}$$
$$2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 2 - 2\cos(\alpha - \beta)$$

Finally we subtract 2 from both sides and divide both sides by -2.

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$$

Thus, we have the difference formula for cosine. We can use similar methods to derive the cosine of the sum of two angles.

sum and difference formulas for cosine

These formulas can be used to calculate the cosine of sums and differences of angles.

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

How To…

Given two angles, find the cosine of the difference between the angles.

1. Write the difference formula for cosine.
2. Substitute the values of the given angles into the formula.
3. Simplify.

Example 1 Finding the Exact Value Using the Formula for the Cosine of the Difference of Two Angles

Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right)$.

Solution Use the formula for the cosine of the difference of two angles. We have

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\begin{aligned}
\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right) &= \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\
&= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\
&= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\
&= \frac{-\sqrt{6} - \sqrt{2}}{4}
\end{aligned}$$

Try It #1

Find the exact value of $\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$.

Example 2 Finding the Exact Value Using the Formula for the Sum of Two Angles for Cosine

Find the exact value of $\cos(75°)$.

Solution As $75° = 45° + 30°$, we can evaluate $\cos(75°)$ as $\cos(45° + 30°)$. Thus,

$$\begin{aligned}
\cos(45° + 30°) &= \cos(45°)\cos(30°) - \sin(45°)\sin(30°)\\
&= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\
&= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\
&= \frac{\sqrt{6} - \sqrt{2}}{4}
\end{aligned}$$

Try It #2

Find the exact value of $\cos(105°)$.

Using the Sum and Difference Formulas for Sine

The sum and difference formulas for sine can be derived in the same manner as those for cosine, and they resemble the cosine formulas.

sum and difference formulas for sine

These formulas can be used to calculate the sines of sums and differences of angles.

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

How To...

Given two angles, find the sine of the difference between the angles.

1. Write the difference formula for sine.
2. Substitute the given angles into the formula.
3. Simplify.

Example 3 Using Sum and Difference Identities to Evaluate the Difference of Angles

Use the sum and difference identities to evaluate the difference of the angles and show that part *a* equals part *b*.

a. $\sin(45° - 30°)$ **b.** $\sin(135° - 120°)$

Solution

a. Let's begin by writing the formula and substitute the given angles.

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(45° - 30°) = \sin(45°)\cos(30°) - \cos(45°)\sin(30°)$$

Next, we need to find the values of the trigonometric expressions.

$$\sin(45°) = \frac{\sqrt{2}}{2},\ \cos(30°) = \frac{\sqrt{3}}{2},\ \cos(45°) = \frac{\sqrt{2}}{2},\ \sin(30°) = \frac{1}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(45° - 30°) &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

b. Again, we write the formula and substitute the given angles.

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(135° - 120°) = \sin(135°)\cos(120°) - \cos(135°)\sin(120°)$$

Next, we find the values of the trigonometric expressions.

$$\sin(135°) = \frac{\sqrt{2}}{2},\ \cos(120°) = -\frac{1}{2},\ \cos(135°) = -\frac{\sqrt{2}}{2},\ \sin(120°) = \frac{\sqrt{3}}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(135° - 120°) &= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{-\sqrt{2} + \sqrt{6}}{4}\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\sin(135° - 120°) &= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{-\sqrt{2} + \sqrt{6}}{4}\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

Example 4 Finding the Exact Value of an Expression Involving an Inverse Trigonometric Function

Find the exact value of $\sin\left(\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{3}{5}\right)$.

Solution The pattern displayed in this problem is $\sin(\alpha + \beta)$. Let $\alpha = \cos^{-1}\frac{1}{2}$ and $\beta = \sin^{-1}\frac{3}{5}$. Then we can write

$$\cos\alpha = \frac{1}{2},\ 0 \le \alpha \le \pi$$

$$\sin\beta = \frac{3}{5},\ -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$$

We will use the Pythagorean identities to find $\sin\alpha$ and $\cos\beta$.

$$\begin{aligned} \sin\alpha &= \sqrt{1 - \cos^2\alpha} \\ &= \sqrt{1 - \frac{1}{4}} \\ &= \sqrt{\frac{3}{4}} \\ &= \frac{\sqrt{3}}{2} \end{aligned} \qquad \begin{aligned} \cos\beta &= \sqrt{1 - \sin^2\beta} \\ &= \sqrt{1 - \frac{9}{25}} \\ &= \sqrt{\frac{16}{25}} \\ &= \frac{4}{5} \end{aligned}$$

Using the sum formula for sine,

$$\begin{aligned} \sin\left(\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{3}{5}\right) &= \sin(\alpha + \beta) \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{\sqrt{3}}{2} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{3}{5} \\ &= \frac{4\sqrt{3} + 3}{10} \end{aligned}$$

Using the Sum and Difference Formulas for Tangent

Finding exact values for the tangent of the sum or difference of two angles is a little more complicated, but again, it is a matter of recognizing the pattern.

Finding the sum of two angles formula for tangent involves taking quotient of the sum formulas for sine and cosine and simplifying. Recall, $\tan x = \frac{\sin x}{\cos x}$, $\cos x \ne 0$.

Let's derive the sum formula for tangent.

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\ &= \frac{\dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta}} && \text{Divide the numerator and denominator by } \cos\alpha\cos\beta \\ &= \frac{\dfrac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \dfrac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\dfrac{\cancel{\cos\alpha}\cancel{\cos\beta}}{\cancel{\cos\alpha}\cancel{\cos\beta}} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \end{aligned}$$

We can derive the difference formula for tangent in a similar way.

sum and difference formulas for tangent

The sum and difference formulas for tangent are:

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

How To…

Given two angles, find the tangent of the sum of the angles.

1. Write the sum formula for tangent.
2. Substitute the given angles into the formula.
3. Simplify.

Example 5 Finding the Exact Value of an Expression Involving Tangent

Find the exact value of $\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$.

Solution Let's first write the sum formula for tangent and substitute the given angles into the formula.

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \frac{\tan\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \left(\tan\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{4}\right)\right)}$$

Next, we determine the individual tangents within the formulas:

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \qquad \tan\left(\frac{\pi}{4}\right) = 1$$

So we have

$$\begin{aligned}\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) &= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \left(\frac{1}{\sqrt{3}}\right)(1)} \\ &= \frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} \\ &= \frac{1+\sqrt{3}}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}-1}\right) \\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\end{aligned}$$

Try It #3

Find the exact value of $\tan\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$.

Example 6 Finding Multiple Sums and Differences of Angles

Given $\sin\alpha = \frac{3}{5}, 0 < \alpha < \frac{\pi}{2}, \cos\beta = -\frac{5}{13}, \pi < \beta < \frac{3\pi}{2}$, find

a. $\sin(\alpha + \beta)$ **b.** $\cos(\alpha + \beta)$ **c.** $\tan(\alpha + \beta)$ **d.** $\tan(\alpha - \beta)$

Solution We can use the sum and difference formulas to identify the sum or difference of angles when the ratio of sine, cosine, or tangent is provided for each of the individual angles. To do so, we construct what is called a reference triangle to help find each component of the sum and difference formulas.

a. To find $\sin(\alpha + \beta)$, we begin with $\sin \alpha = \frac{3}{5}$ and $0 < \alpha < \frac{\pi}{2}$. The side opposite α has length 3, the hypotenuse has length 5, and α is in the first quadrant. See **Figure 4**. Using the Pythagorean Theorem,we can find the length of side a:

$$\begin{aligned} a^2 + 3^2 &= 5^2 \\ a^2 &= 16 \\ a &= 4 \end{aligned}$$

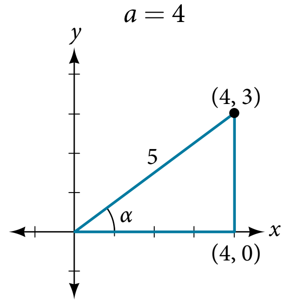


Figure 4

Since $\cos \beta = -\frac{5}{13}$ and $\pi < \beta < \frac{3\pi}{2}$, the side adjacent to β is -5, the hypotenuse is 13, and β is in the third quadrant. See **Figure 5**. Again, using the Pythagorean Theorem, we have

$$\begin{aligned} (-5)^2 + a^2 &= 13^2 \\ 25 + a^2 &= 169 \\ a^2 &= 144 \\ a &= \pm 12 \end{aligned}$$

Since β is in the third quadrant, $a = -12$.

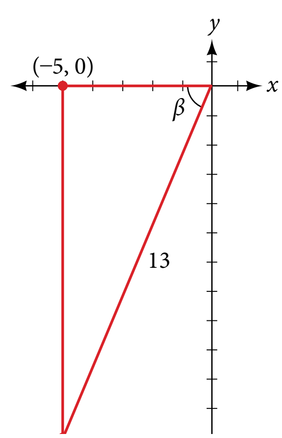


Figure 5

The next step is finding the cosine of α and the sine of β. The cosine of α is the adjacent side over the hypotenuse. We can find it from the triangle in **Figure 5**: $\cos \alpha = \frac{4}{5}$. We can also find the sine of β from the triangle in **Figure 5**, as opposite side over the hypotenuse: $\sin \beta = -\frac{12}{13}$. Now we are ready to evaluate $\sin(\alpha + \beta)$.

$$\begin{aligned} \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ &= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) \\ &= -\frac{15}{65} - \frac{48}{65} \\ &= -\frac{63}{65} \end{aligned}$$

b. We can find $\cos(\alpha+\beta)$ in a similar manner. We substitute the values according to the formula.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)\\ &= -\frac{20}{65} + \frac{36}{65}\\ &= \frac{16}{65}\end{aligned}$$

c. For $\tan(\alpha+\beta)$, if $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$, then

$$\tan\alpha = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

If $\sin\beta = -\frac{12}{13}$ and $\cos\beta = -\frac{5}{13}$, then

$$\tan\beta = \frac{\frac{-12}{13}}{\frac{-5}{13}} = \frac{12}{5}$$

Then,

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\\ &= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4}\left(\frac{12}{5}\right)}\\ &= \frac{\frac{63}{20}}{-\frac{16}{20}}\\ &= -\frac{63}{16}\end{aligned}$$

d. To find $\tan(\alpha-\beta)$, we have the values we need. We can substitute them in and evaluate.

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\\ &= \frac{\frac{3}{4} - \frac{12}{5}}{1 + \frac{3}{4}\left(\frac{12}{5}\right)}\\ &= \frac{-\frac{33}{20}}{\frac{56}{20}}\\ &= -\frac{33}{56}\end{aligned}$$

Analysis *A common mistake when addressing problems such as this one is that we may be tempted to think that α and β are angles in the same triangle, which of course, they are not. Also note that*

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

Using Sum and Difference Formulas for Cofunctions

Now that we can find the sine, cosine, and tangent functions for the sums and differences of angles, we can use them to do the same for their cofunctions. You may recall from **Right Triangle Trigonometry** that, if the sum of two positive angles is $\frac{\pi}{2}$, those two angles are complements, and the sum of the two acute angles in a right triangle is $\frac{\pi}{2}$, so they are also complements. In **Figure 6**, notice that if one of the acute angles is labeled as θ, then the other acute angle must be labeled $\left(\frac{\pi}{2} - \theta\right)$.

Notice also that $\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$: opposite over hypotenuse. Thus, when two angles are complimentary, we can say that the sine of θ equals the cofunction of the complement of θ. Similarly, tangent and cotangent are cofunctions, and secant and cosecant are cofunctions.

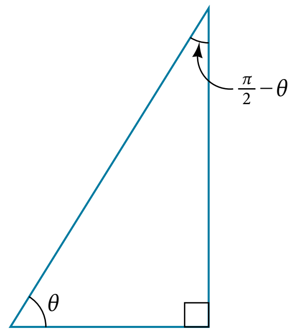


Figure 6

From these relationships, the cofunction identities are formed.

cofunction identities
The cofunction identities are summarized in **Table 2**.

$\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)$	$\tan \theta = \cot\left(\frac{\pi}{2} - \theta\right)$
$\sec \theta = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc \theta = \sec\left(\frac{\pi}{2} - \theta\right)$	$\cot \theta = \tan\left(\frac{\pi}{2} - \theta\right)$

Table 2

Notice that the formulas in the table may also be justified algebraically using the sum and difference formulas. For example, using

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta,$$

we can write

$$\begin{aligned} \cos\left(\frac{\pi}{2} - \theta\right) &= \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta \\ &= (0)\cos\theta + (1)\sin\theta \\ &= \sin\theta \end{aligned}$$

Example 7 Finding a Cofunction with the Same Value as the Given Expression

Write $\tan \frac{\pi}{9}$ in terms of its cofunction.

Solution The cofunction of $\tan \theta = \cot\left(\frac{\pi}{2} - \theta\right)$. Thus,

$$\begin{aligned} \tan\left(\frac{\pi}{9}\right) &= \cot\left(\frac{\pi}{2} - \frac{\pi}{9}\right) \\ &= \cot\left(\frac{9\pi}{18} - \frac{2\pi}{18}\right) \\ &= \cot\left(\frac{7\pi}{18}\right) \end{aligned}$$

Try It #4

Write $\sin \frac{\pi}{7}$ in terms of its cofunction.

Using the Sum and Difference Formulas to Verify Identities

Verifying an identity means demonstrating that the equation holds for all values of the variable. It helps to be very familiar with the identities or to have a list of them accessible while working the problems. Reviewing the general rules from **Solving Trigonometric Equations with Identities** may help simplify the process of verifying an identity.

How To...

Given an identity, verify using sum and difference formulas.

1. Begin with the expression on the side of the equal sign that appears most complex. Rewrite that expression until it matches the other side of the equal sign. Occasionally, we might have to alter both sides, but working on only one side is the most efficient.
2. Look for opportunities to use the sum and difference formulas.
3. Rewrite sums or differences of quotients as single quotients.
4. If the process becomes cumbersome, rewrite the expression in terms of sines and cosines.

Example 8 Verifying an Identity Involving Sine

Verify the identity $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$.

Solution We see that the left side of the equation includes the sines of the sum and the difference of angles.

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

We can rewrite each using the sum and difference formulas.

$$\begin{aligned} \sin(\alpha + \beta) + \sin(\alpha - \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ &= 2 \sin \alpha \cos \beta \end{aligned}$$

We see that the identity is verified.

Example 9 Verifying an Identity Involving Tangent

Verify the following identity.

$$\frac{\sin(\alpha - \beta)}{\cos \alpha \cos \beta} = \tan \alpha - \tan \beta$$

Solution We can begin by rewriting the numerator on the left side of the equation.

$$\begin{aligned} \frac{\sin(\alpha - \beta)}{\cos \alpha \cos \beta} &= \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\cos \alpha \cos \beta} \\ &= \frac{\sin \alpha \cancel{\cos \beta}}{\cos \alpha \cancel{\cos \beta}} - \frac{\cancel{\cos \alpha} \sin \beta}{\cancel{\cos \alpha} \cos \beta} && \text{Rewrite using a common denominator.} \\ &= \frac{\sin \alpha}{\cos \alpha} - \frac{\sin \beta}{\cos \beta} && \text{Cancel.} \\ &= \tan \alpha - \tan \beta && \text{Rewrite in terms of tangent.} \end{aligned}$$

We see that the identity is verified. In many cases, verifying tangent identities can successfully be accomplished by writing the tangent in terms of sine and cosine.

Try It #5

Verify the identity: $\tan(\pi - \theta) = -\tan\theta$.

Example 10 Using Sum and Difference Formulas to Solve an Application Problem

Let L_1 and L_2 denote two non-vertical intersecting lines, and let θ denote the acute angle between L_1 and L_2. See **Figure 7.** Show that

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 respectively. (**Hint:** Use the fact that $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$.)

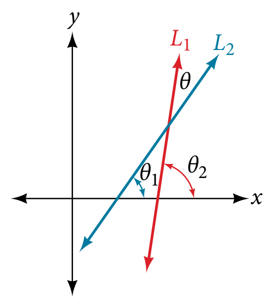


Figure 7

Solution Using the difference formula for tangent, this problem does not seem as daunting as it might.

$$\begin{aligned}\tan\theta &= \tan(\theta_2 - \theta_1)\\ &= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1\tan\theta_2}\\ &= \frac{m_2 - m_1}{1 + m_1 m_2}\end{aligned}$$

Example 11 Investigating a Guy-wire Problem

For a climbing wall, a guy-wire R is attached 47 feet high on a vertical pole. Added support is provided by another guy-wire S attached 40 feet above ground on the same pole. If the wires are attached to the ground 50 feet from the pole, find the angle α between the wires. See **Figure 8**.

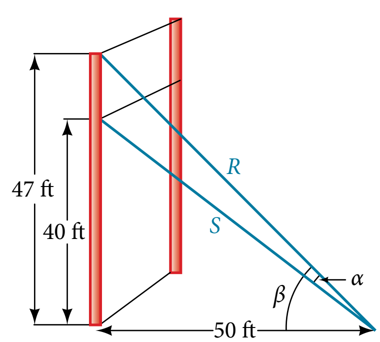


Figure 8

Solution Let's first summarize the information we can gather from the diagram. As only the sides adjacent to the right angle are known, we can use the tangent function. Notice that $\tan\beta = \frac{47}{50}$, and $\tan(\beta - \alpha) = \frac{40}{50} = \frac{4}{5}$. We can then use difference formula for tangent.

$$\tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$

Now, substituting the values we know into the formula, we have

$$\frac{4}{5} = \frac{\frac{47}{50} - \tan\alpha}{1 + \frac{47}{50}\tan\alpha}$$

$$4\left(1 + \frac{47}{50}\tan\alpha\right) = 5\left(\frac{47}{50} - \tan\alpha\right)$$

Use the distributive property, and then simplify the functions.

$$\begin{aligned} 4(1) + 4\left(\frac{47}{50}\right)\tan\alpha &= 5\left(\frac{47}{50}\right) - 5\tan\alpha \\ 4 + 3.76\tan\alpha &= 4.7 - 5\tan\alpha \\ 5\tan\alpha + 3.76\tan\alpha &= 0.7 \\ 8.76\tan\alpha &= 0.7 \\ \tan\alpha &\approx 0.07991 \\ \tan^{-1}(0.07991) &\approx .079741 \end{aligned}$$

Now we can calculate the angle in degrees.

$$\alpha \approx 0.079741\left(\frac{180}{\pi}\right) \approx 4.57°$$

Analysis *Occasionally, when an application appears that includes a right triangle, we may think that solving is a matter of applying the Pythagorean Theorem. That may be partially true, but it depends on what the problem is asking and what information is given.*

Access these online resources for additional instruction and practice with sum and difference identities.

- Sum and Difference Identities for Cosine (http://openstaxcollege.org/l/sumdifcos)
- Sum and Difference Identities for Sine (http://openstaxcollege.org/l/sumdifsin)
- Sum and Difference Identities for Tangent (http://openstaxcollege.org/l/sumdiftan)

7.2 SECTION EXERCISES

VERBAL

1. Explain the basis for the cofunction identities and when they apply.

2. Is there only one way to evaluate $\cos\left(\frac{5\pi}{4}\right)$? Explain how to set up the solution in two different ways, and then compute to make sure they give the same answer.

3. Explain to someone who has forgotten the even-odd properties of sinusoidal functions how the addition and subtraction formulas can determine this characteristic for $f(x) = \sin(x)$ and $g(x) = \cos(x)$. (Hint: $0 - x = -x$)

ALGEBRAIC

For the following exercises, find the exact value.

4. $\cos\left(\frac{7\pi}{12}\right)$

5. $\cos\left(\frac{\pi}{12}\right)$

6. $\sin\left(\frac{5\pi}{12}\right)$

7. $\sin\left(\frac{11\pi}{12}\right)$

8. $\tan\left(-\frac{\pi}{12}\right)$

9. $\tan\left(\frac{19\pi}{12}\right)$

For the following exercises, rewrite in terms of $\sin x$ and $\cos x$.

10. $\sin\left(x + \frac{11\pi}{6}\right)$

11. $\sin\left(x - \frac{3\pi}{4}\right)$

12. $\cos\left(x - \frac{5\pi}{6}\right)$

13. $\cos\left(x + \frac{2\pi}{3}\right)$

For the following exercises, simplify the given expression.

14. $\csc\left(\frac{\pi}{2} - t\right)$

15. $\sec\left(\frac{\pi}{2} - \theta\right)$

16. $\cot\left(\frac{\pi}{2} - x\right)$

17. $\tan\left(\frac{\pi}{2} - x\right)$

18. $\sin(2x)\cos(5x) - \sin(5x)\cos(2x)$

19. $\dfrac{\tan\left(\frac{3}{2}x\right) - \tan\left(\frac{7}{5}x\right)}{1 + \tan\left(\frac{3}{2}x\right)\tan\left(\frac{7}{5}x\right)}$

For the following exercises, find the requested information.

20. Given that $\sin a = \frac{2}{3}$ and $\cos b = -\frac{1}{4}$, with a and b both in the interval $\left[\frac{\pi}{2}, \pi\right)$, find $\sin(a + b)$ and $\cos(a - b)$.

21. Given that $\sin a = \frac{4}{5}$, and $\cos b = \frac{1}{3}$, with a and b both in the interval $\left[0, \frac{\pi}{2}\right)$, find $\sin(a - b)$ and $\cos(a + b)$.

For the following exercises, find the exact value of each expression.

22. $\sin\left(\cos^{-1}(0) - \cos^{-1}\left(\frac{1}{2}\right)\right)$

23. $\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

24. $\tan\left(\sin^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{1}{2}\right)\right)$

GRAPHICAL

For the following exercises, simplify the expression, and then graph both expressions as functions to verify the graphs are identical.

25. $\cos\left(\frac{\pi}{2} - x\right)$

26. $\sin(\pi - x)$

27. $\tan\left(\frac{\pi}{3} + x\right)$

28. $\sin\left(\frac{\pi}{3} + x\right)$

29. $\tan\left(\frac{\pi}{4} - x\right)$

30. $\cos\left(\frac{7\pi}{6} + x\right)$

31. $\sin\left(\frac{\pi}{4} + x\right)$

32. $\cos\left(\frac{5\pi}{4} + x\right)$

For the following exercises, use a graph to determine whether the functions are the same or different. If they are the same, show why. If they are different, replace the second function with one that is identical to the first. (Hint: think $2x = x + x$.)

33. $f(x) = \sin(4x) - \sin(3x)\cos x,\ g(x) = \sin x\cos(3x)$

34. $f(x) = \cos(4x) + \sin x\sin(3x),\ g(x) = -\cos x\cos(3x)$

35. $f(x) = \sin(3x)\cos(6x),\ g(x) = -\sin(3x)\cos(6x)$

36. $f(x) = \sin(4x),\ g(x) = \sin(5x)\cos x - \cos(5x)\sin x$

37. $f(x) = \sin(2x),\ g(x) = 2\sin x\cos x$

38. $f(\theta) = \cos(2\theta),\ g(\theta) = \cos^2\theta - \sin^2\theta$

39. $f(\theta) = \tan(2\theta),\ g(\theta) = \dfrac{\tan\theta}{1 + \tan^2\theta}$

40. $f(x) = \sin(3x)\sin x,$
$g(x) = \sin^2(2x)\cos^2 x - \cos^2(2x)\sin^2 x$

41. $f(x) = \tan(-x),\ g(x) = \dfrac{\tan x - \tan(2x)}{1 - \tan x\tan(2x)}$

TECHNOLOGY

For the following exercises, find the exact value algebraically, and then confirm the answer with a calculator to the fourth decimal point.

42. $\sin(75°)$

43. $\sin(195°)$

44. $\cos(165°)$

45. $\cos(345°)$

46. $\tan(-15°)$

EXTENSIONS

For the following exercises, prove the identities provided.

47. $\tan\left(x + \frac{\pi}{4}\right) = \dfrac{\tan x + 1}{1 - \tan x}$

48. $\dfrac{\tan(a + b)}{\tan(a - b)} = \dfrac{\sin a\cos a + \sin b\cos b}{\sin a\cos a - \sin b\cos b}$

49. $\dfrac{\cos(a + b)}{\cos a\cos b} = 1 - \tan a\tan b$

50. $\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$

51. $\dfrac{\cos(x + h) - \cos x}{h} = \cos x\,\dfrac{\cos h - 1}{h} - \sin x\dfrac{\sin h}{h}$

For the following exercises, prove or disprove the statements.

52. $\tan(u + v) = \dfrac{\tan u + \tan v}{1 - \tan u\tan v}$

53. $\tan(u - v) = \dfrac{\tan u - \tan v}{1 + \tan u\tan v}$

54. $\dfrac{\tan(x + y)}{1 + \tan x\tan x} = \dfrac{\tan x + \tan y}{1 - \tan^2 x\tan^2 y}$

55. If α, β, and γ are angles in the same triangle, then prove or disprove $\sin(\alpha + \beta) = \sin\gamma$.

56. If α, β, and γ are angles in the same triangle, then prove or disprove:
$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$.

LEARNING OBJECTIVES

In this section, you will:

- Use double-angle formulas to find exact values.
- Use double-angle formulas to verify identities.
- Use reduction formulas to simplify an expression.
- Use half-angle formulas to find exact values.

7.3 DOUBLE-ANGLE, HALF-ANGLE, AND REDUCTION FORMULAS

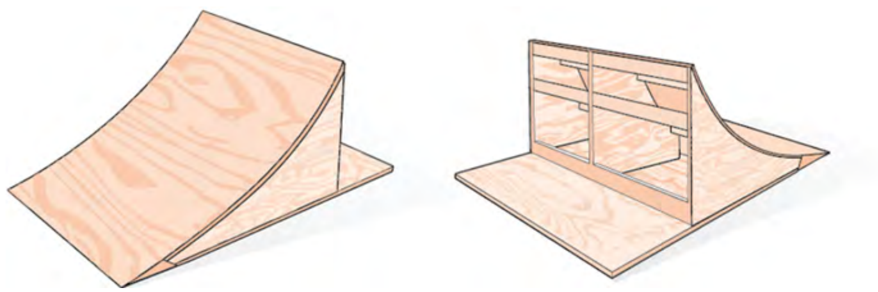


Figure 1 Bicycle ramps for advanced riders have a steeper incline than those designed for novices.

Bicycle ramps made for competition (see **Figure 1**) must vary in height depending on the skill level of the competitors. For advanced competitors, the angle formed by the ramp and the ground should be θ such that $\tan\theta = \frac{5}{3}$. The angle is divided in half for novices. What is the steepness of the ramp for novices? In this section, we will investigate three additional categories of identities that we can use to answer questions such as this one.

Using Double-Angle Formulas to Find Exact Values

In the previous section, we used addition and subtraction formulas for trigonometric functions. Now, we take another look at those same formulas. The double-angle formulas are a special case of the sum formulas, where $\alpha = \beta$. Deriving the double-angle formula for sine begins with the sum formula,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

If we let $\alpha = \beta = \theta$, then we have

$$\begin{aligned} \sin(\theta + \theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ \sin(2\theta) &= 2\sin\theta\cos\theta \end{aligned}$$

Deriving the double-angle for cosine gives us three options. First, starting from the sum formula, $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, and letting $\alpha = \beta = \theta$, we have

$$\begin{aligned} \cos(\theta + \theta) &= \cos\theta\cos\theta - \sin\theta\sin\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta \end{aligned}$$

Using the Pythagorean properties, we can expand this double-angle formula for cosine and get two more interpretations. The first one is:

$$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= (1 - \sin^2\theta) - \sin^2\theta \\ &= 1 - 2\sin^2\theta \end{aligned}$$

The second interpretation is:

$$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ &= 2\cos^2\theta - 1 \end{aligned}$$

Similarly, to derive the double-angle formula for tangent, replacing $\alpha = \beta = \theta$ in the sum formula gives

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta}$$

$$\tan(2\theta) = \frac{2\tan \theta}{1 - \tan^2 \theta}$$

double-angle formulas

The **double-angle formulas** are summarized as follows:

$$\begin{aligned} \sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ &= 1 - 2 \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ \tan(2\theta) &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \end{aligned}$$

How To...

Given the tangent of an angle and the quadrant in which it is located, use the double-angle formulas to find the exact value.

1. Draw a triangle to reflect the given information.
2. Determine the correct double-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 1 Using a Double-Angle Formula to Find the Exact Value Involving Tangent

Given that $\tan \theta = -\frac{3}{4}$ and θ is in quadrant II, find the following:

a. $\sin(2\theta)$ **b.** $\cos(2\theta)$ **c.** $\tan(2\theta)$

Solution If we draw a triangle to reflect the information given, we can find the values needed to solve the problems on the image. We are given $\tan \theta = -\frac{3}{4}$, such that θ is in quadrant II. The tangent of an angle is equal to the opposite side over the adjacent side, and because θ is in the second quadrant, the adjacent side is on the x-axis and is negative. Use the Pythagorean Theorem to find the length of the hypotenuse:

$$\begin{aligned} (-4)^2 + (3)^2 &= c^2 \\ 16 + 9 &= c^2 \\ 25 &= c^2 \\ c &= 5 \end{aligned}$$

Now we can draw a triangle similar to the one shown in **Figure 2**.

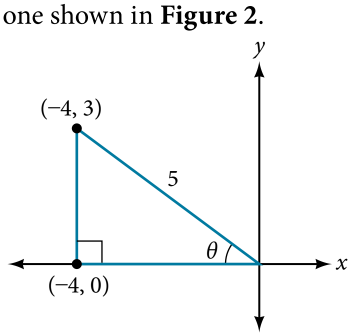


Figure 2

a. Let's begin by writing the double-angle formula for sine.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

We see that we to need to find $\sin\theta$ and $\cos\theta$. Based on **Figure 2**, we see that the hypotenuse equals 5, so $\sin\theta = \frac{3}{5}$, and $\cos\theta = -\frac{4}{5}$. Substitute these values into the equation, and simplify.

Thus,

$$\begin{aligned}\sin(2\theta) &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

b. Write the double-angle formula for cosine.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Again, substitute the values of the sine and cosine into the equation, and simplify.

$$\begin{aligned}\cos(2\theta) &= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2\\ &= \frac{16}{25} - \frac{9}{25}\\ &= \frac{7}{25}\end{aligned}$$

c. Write the double-angle formula for tangent.

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

In this formula, we need the tangent, which we were given as $\tan\theta = -\frac{3}{4}$. Substitute this value into the equation, and simplify.

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2}\\ &= \frac{-\frac{3}{2}}{1-\frac{9}{16}}\\ &= -\frac{3}{2}\left(\frac{16}{7}\right)\\ &= -\frac{24}{7}\end{aligned}$$

Try It #1

Given $\sin\alpha = \frac{5}{8}$, with θ in quadrant I, find $\cos(2\alpha)$.

Example 2 Using the Double-Angle Formula for Cosine without Exact Values

Use the double-angle formula for cosine to write $\cos(6x)$ in terms of $\cos(3x)$.

Solution

$$\begin{aligned}\cos(6x) &= \cos(2(3x))\\ &= \cos^2 3x - \sin^2 3x\\ &= 2\cos^2 3x - 1\end{aligned}$$

Analysis *This example illustrates that we can use the double-angle formula without having exact values. It emphasizes that the pattern is what we need to remember and that identities are true for all values in the domain of the trigonometric function.*

Using Double-Angle Formulas to Verify Identities

Establishing identities using the double-angle formulas is performed using the same steps we used to derive the sum and difference formulas. Choose the more complicated side of the equation and rewrite it until it matches the other side.

Example 3 Using the Double-Angle Formulas to Establish an Identity

Establish the following identity using double-angle formulas: $1 + \sin(2\theta) = (\sin\theta + \cos\theta)^2$

Solution We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned}(\sin\theta + \cos\theta)^2 &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\\ &= (\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta\\ &= 1 + 2\sin\theta\cos\theta\\ &= 1 + \sin(2\theta)\end{aligned}$$

Analysis *This process is not complicated, as long as we recall the perfect square formula from algebra:*

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

where $a = \sin\theta$ and $b = \cos\theta$. Part of being successful in mathematics is the ability to recognize patterns. While the terms or symbols may change, the algebra remains consistent.

Try It #2

Establish the identity: $\cos^4\theta - \sin^4\theta = \cos(2\theta)$.

Example 4 Verifying a Double-Angle Identity for Tangent

Verify the identity:

$$\tan(2\theta) = \frac{2}{\cot\theta - \tan\theta}$$

Solution In this case, we will work with the left side of the equation and simplify or rewrite until it equals the right side of the equation.

$$\begin{aligned}\tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta} && \text{Double-angle formula}\\ &= \frac{2\tan\theta\left(\frac{1}{\tan\theta}\right)}{(1 - \tan^2\theta)\left(\frac{1}{\tan\theta}\right)} && \text{Multiply by a term that results in desired numerator.}\\ &= \frac{2}{\frac{1}{\tan\theta} - \frac{\tan^2\theta}{\tan\theta}}\\ &= \frac{2}{\cot\theta - \tan\theta} && \text{Use reciprocal identity for } \frac{1}{\tan\theta}.\end{aligned}$$

Analysis *Here is a case where the more complicated side of the initial equation appeared on the right, but we chose to work the left side. However, if we had chosen the left side to rewrite, we would have been working backwards to arrive at the equivalency. For example, suppose that we wanted to show*

$$\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2}{\cot\theta - \tan\theta}$$

Let's work on the right side.

$$\begin{aligned}\frac{2}{\cot\theta - \tan\theta} &= \frac{2}{\frac{1}{\tan\theta} - \tan\theta}\left(\frac{\tan\theta}{\tan\theta}\right)\\ &= \frac{2\tan\theta}{\frac{1}{\cancel{\tan\theta}}(\cancel{\tan}\,\theta) - \tan\theta(\tan\theta)}\\ &= \frac{2\tan\theta}{1 - \tan^2\theta}\end{aligned}$$

When using the identities to simplify a trigonometric expression or solve a trigonometric equation, there are usually several paths to a desired result. There is no set rule as to what side should be manipulated. However, we should begin with the guidelines set forth earlier.

Try It #3

Verify the identity: $\cos(2\theta)\cos\theta = \cos^3\theta - \cos\theta\sin^2\theta$.

Use Reduction Formulas to Simplify an Expression

The double-angle formulas can be used to derive the reduction formulas, which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are especially important in higher-level math courses, calculus in particular. Also called the power-reducing formulas, three identities are included and are easily derived from the double-angle formulas.

We can use two of the three double-angle formulas for cosine to derive the reduction formulas for sine and cosine. Let's begin with $\cos(2\theta) = 1 - 2\sin^2\theta$. Solve for $\sin^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 1 - 2\sin^2\theta\\ 2\sin^2\theta &= 1 - \cos(2\theta)\\ \sin^2\theta &= \frac{1-\cos(2\theta)}{2}\end{aligned}$$

Next, we use the formula $\cos(2\theta) = 2\cos^2\theta - 1$. Solve for $\cos^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 2\cos^2\theta - 1\\ 1+\cos(2\theta) &= 2\cos^2\theta\\ \frac{1+\cos(2\theta)}{2} &= \cos^2\theta\end{aligned}$$

The last reduction formula is derived by writing tangent in terms of sine and cosine:

$$\begin{aligned}\tan^2\theta &= \frac{\sin^2\theta}{\cos^2\theta}\\ &= \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}} && \text{Substitute the reduction formulas.}\\ &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{2}{1+\cos(2\theta)}\right)\\ &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)}\end{aligned}$$

reduction formulas

The **reduction formulas** are summarized as follows:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2} \qquad \tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

Example 5 Writing an Equivalent Expression Not Containing Powers Greater Than 1

Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

Solution We will apply the reduction formula for cosine twice.

$$\begin{aligned}\cos^4 x &= (\cos^2 x)^2\\ &= \left(\frac{1+\cos(2x)}{2}\right)^2 && \text{Substitute reduction formula for } \cos^2 x.\\ &= \frac{1}{4}(1 + 2\cos(2x) + \cos^2(2x))\\ &= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1+\cos2(2x)}{2}\right) && \text{Substitute reduction formula for } \cos^2 x.\\ &= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{8} + \frac{1}{8}\cos(4x)\\ &= \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\end{aligned}$$

Analysis *The solution is found by using the reduction formula twice, as noted, and the perfect square formula from algebra.*

Example 6 **Using the Power-Reducing Formulas to Prove an Identity**

Use the power-reducing formulas to prove

$$\sin^3(2x) = \left[\frac{1}{2}\sin(2x)\right][1 - \cos(4x)]$$

Solution We will work on simplifying the left side of the equation:

$$\begin{aligned}\sin^3(2x) &= [\sin(2x)][\sin^2(2x)]\\ &= \sin(2x)\left[\frac{1-\cos(4x)}{2}\right] && \text{Substitute the power-reduction formula.}\\ &= \sin(2x)\left(\frac{1}{2}\right)[1-\cos(4x)]\\ &= \frac{1}{2}[\sin(2x)][1-\cos(4x)]\end{aligned}$$

Analysis *Note that in this example, we substituted*

$$\frac{1-\cos(4x)}{2}$$

for $\sin^2(2x)$*. The formula states*

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

We let $\theta = 2x$*, so* $2\theta = 4x$*.*

Try It #4

Use the power-reducing formulas to prove that $10\cos^4 x = \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x)$.

Using Half-Angle Formulas to Find Exact Values

The next set of identities is the set of **half-angle formulas**, which can be derived from the reduction formulas and we can use when we have an angle that is half the size of a special angle. If we replace θ with $\frac{\alpha}{2}$, the half-angle formula for sine is found by simplifying the equation and solving for $\sin\left(\frac{\alpha}{2}\right)$. Note that the half-angle formulas are preceded by a $\pm$ sign.

This does not mean that both the positive and negative expressions are valid. Rather, it depends on the quadrant in which $\frac{\alpha}{2}$ terminates.

The half-angle formula for sine is derived as follows:

$$\begin{aligned}\sin^2\theta &= \frac{1-\cos(2\theta)}{2}\\ \sin^2\left(\frac{\alpha}{2}\right) &= \frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{2}\\ &= \frac{1-\cos\alpha}{2}\\ \sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos\alpha}{2}}\end{aligned}$$

To derive the half-angle formula for cosine, we have

$$\begin{aligned}\cos^2\theta &= \frac{1+\cos(2\theta)}{2}\\ \cos^2\left(\frac{\alpha}{2}\right) &= \frac{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}{2}\\ &= \frac{1+\cos\alpha}{2}\\ \cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1+\cos\alpha}{2}}\end{aligned}$$

For the tangent identity, we have

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

$$\tan^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}$$

$$= \frac{1-\cos\alpha}{1+\cos\alpha}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

half-angle formulas

The **half-angle formulas** are as follows:

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$= \frac{\sin\alpha}{1+\cos\alpha}$$

$$= \frac{1-\cos\alpha}{\sin\alpha}$$

Example 7 Using a Half-Angle Formula to Find the Exact Value of a Sine Function

Find sin(15°) using a half-angle formula.

Solution Since $15° = \frac{30°}{2}$, we use the half-angle formula for sine:

$$\sin\frac{30°}{2} = \sqrt{\frac{1-\cos 30°}{2}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2-\sqrt{3}}{4}}$$

$$= \frac{\sqrt{2-\sqrt{3}}}{2}$$

Analysis *Notice that we used only the positive root because sin(15°) is positive.*

How To…

Given the tangent of an angle and the quadrant in which the angle lies, find the exact values of trigonometric functions of half of the angle.

1. Draw a triangle to represent the given information.
2. Determine the correct half-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 8 Finding Exact Values Using Half-Angle Identities

Given that $\tan \alpha = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

a. $\sin\left(\frac{\alpha}{2}\right)$ **b.** $\cos\left(\frac{\alpha}{2}\right)$ **c.** $\tan\left(\frac{\alpha}{2}\right)$

Solution Using the given information, we can draw the triangle shown in **Figure 3**. Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate $\sin \alpha = -\frac{8}{17}$ and $\cos \alpha = -\frac{15}{17}$.

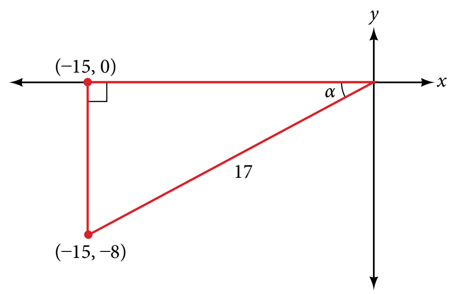


Figure 3

a. Before we start, we must remember that, if α is in quadrant III, then $180° < \alpha < 270°$, so $\frac{180°}{2} < \frac{\alpha}{2} < \frac{270°}{2}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II, since $90° < \frac{\alpha}{2} < 135°$. To find $\sin \frac{\alpha}{2}$, we begin by writing the half-angle formula for sine. Then we substitute the value of the cosine we found from the triangle in **Figure 3** and simplify.

$$\begin{aligned}
\sin \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{2}} \\
&= \pm \sqrt{\frac{1 - \left(-\frac{15}{17}\right)}{2}} \\
&= \pm \sqrt{\frac{\frac{32}{17}}{2}} \\
&= \pm \sqrt{\frac{32}{17} \cdot \frac{1}{2}} \\
&= \pm \sqrt{\frac{16}{17}} \\
&= \pm \frac{4}{\sqrt{17}} \\
&= \frac{4\sqrt{17}}{17}
\end{aligned}$$

We choose the positive value of $\sin \frac{\alpha}{2}$ because the angle terminates in quadrant II and sine is positive in quadrant II.

b. To find $\cos \frac{\alpha}{2}$, we will write the half-angle formula for cosine, substitute the value of the cosine we found from the triangle in **Figure 3**, and simplify.

$$
\begin{aligned}
\cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1+\cos\alpha}{2}}\\
&= \pm\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}}\\
&= \pm\sqrt{\frac{\frac{2}{17}}{2}}\\
&= \pm\sqrt{\frac{2}{17}\cdot\frac{1}{2}}\\
&= \pm\sqrt{\frac{1}{17}}\\
&= -\frac{\sqrt{17}}{17}
\end{aligned}
$$

We choose the negative value of $\cos\frac{\alpha}{2}$ because the angle is in quadrant II because cosine is negative in quadrant II.

c. To find $\tan\frac{\alpha}{2}$, we write the half-angle formula for tangent. Again, we substitute the value of the cosine we found from the triangle in **Figure 3** and simplify.

$$
\begin{aligned}
\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}\\
&= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{1+\left(-\frac{15}{17}\right)}}\\
&= \pm\sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}}\\
&= \pm\sqrt{\frac{32}{2}}\\
&= -\sqrt{16}\\
&= -4
\end{aligned}
$$

We choose the negative value of $\tan\frac{\alpha}{2}$ because $\frac{\alpha}{2}$ lies in quadrant II, and tangent is negative in quadrant II.

Try It #5

Given that $\sin\alpha = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos\left(\frac{\alpha}{2}\right)$.

Example 9 Finding the Measurement of a Half Angle

Now, we will return to the problem posed at the beginning of the section. A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground. Another ramp is to be constructed half as steep for novice competition. If $\tan\theta = \frac{5}{3}$ for higher-level competition, what is the measurement of the angle for novice competition?

Solution Since the angle for novice competition measures half the steepness of the angle for the high-level competition, and $\tan\theta = \frac{5}{3}$ for high-level competition, we can find $\cos\theta$ from the right triangle and the Pythagorean theorem so that we can use the half-angle identities. See **Figure 4**.

$$3^2 + 5^2 = 34$$
$$c = \sqrt{34}$$

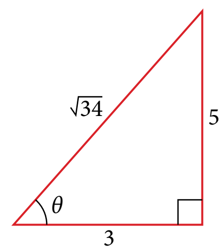


Figure 4

We see that $\cos\theta = \frac{3}{\sqrt{34}} = \frac{3\sqrt{34}}{34}$. We can use the half-angle formula for tangent: $\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$. Since $\tan\theta$ is in the first quadrant, so is $\tan\frac{\theta}{2}$. Thus,

$$\begin{aligned}
\tan\frac{\theta}{2} &= \sqrt{\frac{1-\frac{3\sqrt{34}}{34}}{1+\frac{3\sqrt{34}}{34}}}\\
&= \sqrt{\frac{\frac{34-3\sqrt{34}}{34}}{\frac{34+3\sqrt{34}}{34}}}\\
&= \sqrt{\frac{34-3\sqrt{34}}{34+3\sqrt{34}}}\\
&\approx 0.57
\end{aligned}$$

We can take the inverse tangent to find the angle: $\tan^{-1}(0.57) \approx 29.7°$. So the angle of the ramp for novice competition is $\approx 29.7°$.

Access these online resources for additional instruction and practice with double-angle, half-angle, and reduction formulas.

- Double-Angle Identities (http://openstaxcollege.org/l/doubleangiden)
- Half-Angle Identities (http://openstaxcollege.org/l/halfangleident)

7.3 SECTION EXERCISES

VERBAL

1. Explain how to determine the reduction identities from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$.

2. Explain how to determine the double-angle formula for $\tan(2x)$ using the double-angle formulas for $\cos(2x)$ and $\sin(2x)$.

3. We can determine the half-angle formula for $\tan\left(\frac{x}{2}\right) = \pm\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}$ by dividing the formula for $\sin\left(\frac{x}{2}\right)$ by $\cos\left(\frac{x}{2}\right)$. Explain how to determine two formulas for $\tan\left(\frac{x}{2}\right)$ that do not involve any square roots.

4. For the half-angle formula given in the previous exercise for $\tan\left(\frac{x}{2}\right)$, explain why dividing by 0 is not a concern. (Hint: examine the values of $\cos x$ necessary for the denominator to be 0.)

ALGEBRAIC

For the following exercises, find the exact values of a) $\sin(2x)$, b) $\cos(2x)$, and c) $\tan(2x)$ without solving for x.

5. If $\sin x = \frac{1}{8}$, and x is in quadrant I.

6. If $\cos x = \frac{2}{3}$, and x is in quadrant I.

7. If $\cos x = -\frac{1}{2}$, and x is in quadrant III.

8. If $\tan x = -8$, and x is in quadrant IV.

For the following exercises, find the values of the six trigonometric functions if the conditions provided hold.

9. $\cos(2\theta) = \frac{3}{5}$ and $90° \le \theta \le 180°$

10. $\cos(2\theta) = \frac{1}{\sqrt{2}}$ and $180° \le \theta \le 270°$

For the following exercises, simplify to one trigonometric expression.

11. $2\sin\left(\frac{\pi}{4}\right)2\cos\left(\frac{\pi}{4}\right)$

12. $4\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$

For the following exercises, find the exact value using half-angle formulas.

13. $\sin\left(\frac{\pi}{8}\right)$

14. $\cos\left(-\frac{11\pi}{12}\right)$

15. $\sin\left(\frac{11\pi}{12}\right)$

16. $\cos\left(\frac{7\pi}{8}\right)$

17. $\tan\left(\frac{5\pi}{12}\right)$

18. $\tan\left(-\frac{3\pi}{12}\right)$

19. $\tan\left(-\frac{3\pi}{8}\right)$

For the following exercises, find the exact values of a) $\sin\left(\frac{x}{2}\right)$, b) $\cos\left(\frac{x}{2}\right)$, and c) $\tan\left(\frac{x}{2}\right)$ without solving for x, when $0 \le x \le 360°$.

20. If $\tan x = -\frac{4}{3}$, and x is in quadrant IV.

21. If $\sin x = -\frac{12}{13}$, and x is in quadrant III.

22. If $\csc x = 7$, and x is in quadrant II.

23. If $\sec x = -4$, and x is in quadrant II.

For the following exercises, use **Figure 5** to find the requested half and double angles.

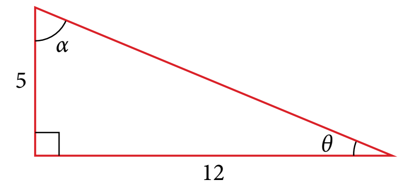


Figure 5

24. Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

25. Find $\sin(2\alpha)$, $\cos(2\alpha)$, and $\tan(2\alpha)$.

26. Find $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$.

27. Find $\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, and $\tan\left(\frac{\alpha}{2}\right)$.

For the following exercises, simplify each expression. Do not evaluate.

28. $\cos^2(28°) - \sin^2(28°)$

29. $2\cos^2(37°) - 1$

30. $1 - 2\sin^2(17°)$

31. $\cos^2(9x) - \sin^2(9x)$

32. $4\sin(8x)\cos(8x)$

33. $6\sin(5x)\cos(5x)$

For the following exercises, prove the identity given.

34. $(\sin t - \cos t)^2 = 1 - \sin(2t)$

35. $\sin(2x) = -2\sin(-x)\cos(-x)$

36. $\cot x - \tan x = 2\cot(2x)$

37. $\dfrac{\sin(2\theta)}{1 + \cos(2\theta)}\tan^2\theta = \tan^3\theta$

For the following exercises, rewrite the expression with an exponent no higher than 1.

38. $\cos^2(5x)$

39. $\cos^2(6x)$

40. $\sin^4(8x)$

41. $\sin^4(3x)$

42. $\cos^2 x\sin^4 x$

43. $\cos^4 x\sin^2 x$

44. $\tan^2 x\sin^2 x$

TECHNOLOGY

For the following exercises, reduce the equations to powers of one, and then check the answer graphically.

45. $\tan^4 x$

46. $\sin^2(2x)$

47. $\sin^2 x\cos^2 x$

48. $\tan^2 x\sin x$

49. $\tan^4 x\cos^2 x$

50. $\cos^2 x\sin(2x)$

51. $\cos^2(2x)\sin x$

52. $\tan^2\left(\frac{x}{2}\right)\sin x$

For the following exercises, algebraically find an equivalent function, only in terms of $\sin x$ and/or $\cos x$, and then check the answer by graphing both equations.

53. $\sin(4x)$

54. $\cos(4x)$

EXTENSIONS

For the following exercises, prove the identities.

55. $\sin(2x) = \dfrac{2\tan x}{1 + \tan^2 x}$

56. $\cos(2\alpha) = \dfrac{1 - \tan^2\alpha}{1 + \tan^2\alpha}$

57. $\tan(2x) = \dfrac{2\sin x\cos x}{2\cos^2 x - 1}$

58. $(\sin^2 x - 1)^2 = \cos(2x) + \sin^4 x$

59. $\sin(3x) = 3\sin x\cos^2 x - \sin^3 x$

60. $\cos(3x) = \cos^3 x - 3\sin^2 x\cos x$

61. $\dfrac{1 + \cos(2t)}{\sin(2t) - \cos t} = \dfrac{2\cos t}{2\sin t - 1}$

62. $\sin(16x) = 16\sin x\cos x\cos(2x)\cos(4x)\cos(8x)$

63. $\cos(16x) = (\cos^2(4x) - \sin^2(4x) - \sin(8x))(\cos^2(4x) - \sin^2(4x) + \sin(8x))$

LEARNING OBJECTIVES

In this section, you will:

- Express products as sums.
- Express sums as products.

7.4 SUM-TO-PRODUCT AND PRODUCT-TO-SUM FORMULAS

Figure 1 The UCLA marching band (credit: Eric Chan, Flickr).

A band marches down the field creating an amazing sound that bolsters the crowd. That sound travels as a wave that can be interpreted using trigonometric functions. For example, **Figure 2** represents a sound wave for the musical note A. In this section, we will investigate trigonometric identities that are the foundation of everyday phenomena such as sound waves.

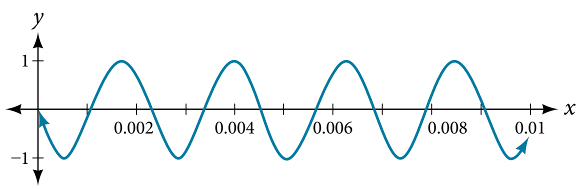


Figure 2

Expressing Products as Sums

We have already learned a number of formulas useful for expanding or simplifying trigonometric expressions, but sometimes we may need to express the product of cosine and sine as a sum. We can use the product-to-sum formulas, which express products of trigonometric functions as sums. Let's investigate the cosine identity first and then the sine identity.

Expressing Products as Sums for Cosine

We can derive the product-to-sum formula from the sum and difference identities for cosine. If we add the two equations, we get:

$$\begin{array}{r} \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta) \\ +\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta) \\ \hline 2\cos\alpha\cos\beta = \cos(\alpha-\beta)+\cos(\alpha+\beta) \end{array}$$

Then, we divide by 2 to isolate the product of cosines:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

How To...

Given a product of cosines, express as a sum.

1. Write the formula for the product of cosines.
2. Substitute the given angles into the formula.
3. Simplify.

Example 1 Writing the Product as a Sum Using the Product-to-Sum Formula for Cosine

Write the following product of cosines as a sum: $2\cos\left(\frac{7x}{2}\right)\cos\frac{3x}{2}$.

Solution We begin by writing the formula for the product of cosines:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

We can then substitute the given angles into the formula and simplify.

$$\begin{aligned} 2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right) &= (2)\left(\frac{1}{2}\right)\left[\cos\left(\frac{7x}{2}-\frac{3x}{2}\right)+\cos\left(\frac{7x}{2}+\frac{3x}{2}\right)\right] \\ &= \left[\cos\left(\frac{4x}{2}\right)+\cos\left(\frac{10x}{2}\right)\right] \\ &= \cos 2x + \cos 5x \end{aligned}$$

Try It #1

Use the product-to-sum formula to write the product as a sum or difference: $\cos(2\theta)\cos(4\theta)$.

Expressing the Product of Sine and Cosine as a Sum

Next, we will derive the product-to-sum formula for sine and cosine from the sum and difference formulas for sine. If we add the sum and difference identities, we get:

$$\begin{array}{rl} \sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta \\ + \qquad \sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta \\ \hline \sin(\alpha+\beta)+\sin(\alpha-\beta) &= 2\sin\alpha\cos\beta \end{array}$$

Then, we divide by 2 to isolate the product of cosine and sine:

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

Example 2 Writing the Product as a Sum Containing only Sine or Cosine

Express the following product as a sum containing only sine or cosine and no products: $\sin(4\theta)\cos(2\theta)$.

Solution Write the formula for the product of sine and cosine. Then substitute the given values into the formula and simplify.

$$\begin{aligned} \sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \\ \sin(4\theta)\cos(2\theta) &= \frac{1}{2}[\sin(4\theta+2\theta)+\sin(4\theta-2\theta)] \\ &= \frac{1}{2}[\sin(6\theta)+\sin(2\theta)] \end{aligned}$$

Try It #2

Use the product-to-sum formula to write the product as a sum: $\sin(x+y)\cos(x-y)$.

Expressing Products of Sines in Terms of Cosine

Expressing the product of sines in terms of cosine is also derived from the sum and difference identities for cosine. In this case, we will first subtract the two cosine formulas:

$$\begin{array}{rl} & \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ - & \cos(\alpha+\beta) = -(\cos\alpha\cos\beta - \sin\alpha\sin\beta) \\ \hline & \cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta \end{array}$$

Then, we divide by 2 to isolate the product of sines:

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

Similarly we could express the product of cosines in terms of sine or derive other product-to-sum formulas.

> ***the product-to-sum formulas***
>
> The **product-to-sum formulas** are as follows:
>
> $$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \qquad \sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$
>
> $$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \qquad \cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

Example 3 Express the Product as a Sum or Difference

Write $\cos(3\theta)\cos(5\theta)$ as a sum or difference.

Solution We have the product of cosines, so we begin by writing the related formula. Then we substitute the given angles and simplify.

$$\begin{aligned} \cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \\ \cos(3\theta)\cos(5\theta) &= \frac{1}{2}[\cos(3\theta-5\theta) + \cos(3\theta+5\theta)] \\ &= \frac{1}{2}[\cos(2\theta) + \cos(8\theta)] && \text{Use even-odd identity.} \end{aligned}$$

Try It #3

Use the product-to-sum formula to evaluate $\cos\dfrac{11\pi}{12}\cos\dfrac{\pi}{12}$.

Expressing Sums as Products

Some problems require the reverse of the process we just used. The sum-to-product formulas allow us to express sums of sine or cosine as products. These formulas can be derived from the product-to-sum identities. For example, with a few substitutions, we can derive the sum-to-product identity for sine. Let $\dfrac{u+v}{2} = \alpha$ and $\dfrac{u-v}{2} = \beta$.

Then,

$$\begin{aligned} \alpha+\beta &= \frac{u+v}{2} + \frac{u-v}{2} \\ &= \frac{2u}{2} \\ &= u \\ \alpha-\beta &= \frac{u+v}{2} - \frac{u-v}{2} \\ &= \frac{2v}{2} \\ &= v \end{aligned}$$

Thus, replacing α and β in the product-to-sum formula with the substitute expressions, we have

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \frac{1}{2}[\sin u + \sin v] \qquad \text{Substitute for } (\alpha+\beta) \text{ and } (\alpha-\beta)$$

$$2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \sin u + \sin v$$

The other sum-to-product identities are derived similarly.

sum-to-product formulas

The **sum-to-product formulas** are as follows:

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \qquad \cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

Example 4 **Writing the Difference of Sines as a Product**

Write the following difference of sines expression as a product: $\sin(4\theta) - \sin(2\theta)$.

Solution We begin by writing the formula for the difference of sines.

$$\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

Substitute the values into the formula, and simplify.

$$\begin{aligned}\sin(4\theta)-\sin(2\theta) &= 2\sin\left(\frac{4\theta-2\theta}{2}\right)\cos\left(\frac{4\theta+2\theta}{2}\right)\\ &= 2\sin\left(\frac{2\theta}{2}\right)\cos\left(\frac{6\theta}{2}\right)\\ &= 2\sin\theta\cos(3\theta)\end{aligned}$$

Try It #4

Use the sum-to-product formula to write the sum as a product: $\sin(3\theta) + \sin(\theta)$.

Example 5 **Evaluating Using the Sum-to-Product Formula**

Evaluate $\cos(15°) - \cos(75°)$.

Solution We begin by writing the formula for the difference of cosines.

$$\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Then we substitute the given angles and simplify.

$$\begin{aligned}\cos(15°)-\cos(75°) &= -2\sin\left(\frac{15°+75°}{2}\right)\sin\left(\frac{15°-75°}{2}\right)\\ &= -2\sin(45°)\sin(-30°)\\ &= -2\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Example 6 Proving an Identity

Prove the identity:

$$\frac{\cos(4t) - \cos(2t)}{\sin(4t) + \sin(2t)} = -\tan t$$

Solution We will start with the left side, the more complicated side of the equation, and rewrite the expression until it matches the right side.

$$\begin{aligned}
\frac{\cos(4t) - \cos(2t)}{\sin(4t) + \sin(2t)} &= \frac{-2\sin\left(\frac{4t + 2t}{2}\right)\sin\left(\frac{4t - 2t}{2}\right)}{2\sin\left(\frac{4t + 2t}{2}\right)\cos\left(\frac{4t - 2t}{2}\right)} \\
&= \frac{-2\sin(3t)\sin t}{2\sin(3t)\cos t} \\
&= \frac{-\cancel{2}\,\cancel{\sin(3t)}\sin t}{\cancel{2}\,\cancel{\sin(3t)}\cos t} \\
&= -\frac{\sin t}{\cos t} \\
&= -\tan t
\end{aligned}$$

Analysis *Recall that verifying trigonometric identities has its own set of rules. The procedures for solving an equation are not the same as the procedures for verifying an identity. When we prove an identity, we pick one side to work on and make substitutions until that side is transformed into the other side.*

Example 7 Verifying the Identity Using Double-Angle Formulas and Reciprocal Identities

Verify the identity $\csc^2\theta - 2 = \frac{\cos(2\theta)}{\sin^2\theta}$.

Solution For verifying this equation, we are bringing together several of the identities. We will use the double-angle formula and the reciprocal identities. We will work with the right side of the equation and rewrite it until it matches the left side.

$$\begin{aligned}
\frac{\cos(2\theta)}{\sin^2\theta} &= \frac{1 - 2\sin^2\theta}{\sin^2\theta} \\
&= \frac{1}{\sin^2\theta} - \frac{2\sin^2\theta}{\sin^2\theta} \\
&= \csc^2\theta - 2
\end{aligned}$$

Try It #5

Verify the identity $\tan\theta\cot\theta - \cos^2\theta = \sin^2\theta$.

Access these online resources for additional instruction and practice with the product-to-sum and sum-to-product identities.

- Sum to Product Identities (http://openstaxcollege.org/l/sumtoprod)
- Sum to Product and Product to Sum Identities (http://openstaxcollege.org/l/sumtpptsum)

7.4 SECTION EXERCISES

VERBAL

1. Starting with the product to sum formula $\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$, explain how to determine the formula for $\cos \alpha \sin \beta$.

2. Explain two different methods of calculating $\cos(195°)\cos(105°)$, one of which uses the product to sum. Which method is easier?

3. Explain a situation where we would convert an equation from a sum to a product and give an example.

4. Explain a situation where we would convert an equation from a product to a sum, and give an example.

ALGEBRAIC

For the following exercises, rewrite the product as a sum or difference.

5. $16\sin(16x)\sin(11x)$

6. $20\cos(36t)\cos(6t)$

7. $2\sin(5x)\cos(3x)$

8. $10\cos(5x)\sin(10x)$

9. $\sin(-x)\sin(5x)$

10. $\sin(3x)\cos(5x)$

For the following exercises, rewrite the sum or difference as a product.

11. $\cos(6t) + \cos(4t)$

12. $\sin(3x) + \sin(7x)$

13. $\cos(7x) + \cos(-7x)$

14. $\sin(3x) - \sin(-3x)$

15. $\cos(3x) + \cos(9x)$

16. $\sin h - \sin(3h)$

For the following exercises, evaluate the product using a sum or difference of two functions. Evaluate exactly.

17. $\cos(45°)\cos(15°)$

18. $\cos(45°)\sin(15°)$

19. $\sin(-345°)\sin(-15°)$

20. $\sin(195°)\cos(15°)$

21. $\sin(-45°)\sin(-15°)$

For the following exercises, evaluate the product using a sum or difference of two functions. Leave in terms of sine and cosine.

22. $\cos(23°)\sin(17°)$

23. $2\sin(100°)\sin(20°)$

24. $2\sin(-100°)\sin(-20°)$

25. $\sin(213°)\cos(8°)$

26. $2\cos(56°)\cos(47°)$

For the following exercises, rewrite the sum as a product of two functions. Leave in terms of sine and cosine.

27. $\sin(76°) + \sin(14°)$

28. $\cos(58°) - \cos(12°)$

29. $\sin(101°) - \sin(32°)$

30. $\cos(100°) + \cos(200°)$

31. $\sin(-1°) + \sin(-2°)$

For the following exercises, prove the identity.

32. $\dfrac{\cos(a+b)}{\cos(a-b)} = \dfrac{1 - \tan a \tan b}{1 + \tan a \tan b}$

33. $4\sin(3x)\cos(4x) = 2\sin(7x) - 2\sin x$

34. $\dfrac{6\cos(8x)\sin(2x)}{\sin(-6x)} = -3\sin(10x)\csc(6x) + 3$

35. $\sin x + \sin(3x) = 4\sin x \cos^2 x$

36. $2(\cos^3 x - \cos x \sin^2 x) = \cos(3x) + \cos x$

37. $2\tan x \cos(3x) = \sec x(\sin(4x) - \sin(2x))$

38. $\cos(a + b) + \cos(a - b) = 2\cos a \cos b$

NUMERIC

For the following exercises, rewrite the sum as a product of two functions or the product as a sum of two functions. Give your answer in terms of sines and cosines. Then evaluate the final answer numerically, rounded to four decimal places.

39. $\cos(58°) + \cos(12°)$

40. $\sin(2°) - \sin(3°)$

41. $\cos(44°) - \cos(22°)$

42. $\cos(176°)\sin(9°)$

43. $\sin(-14°)\sin(85°)$

TECHNOLOGY

For the following exercises, algebraically determine whether each of the given expressions is a true identity. If it is not an identity, replace the right-hand side with an expression equivalent to the left side. Verify the results by graphing both expressions on a calculator.

44. $2\sin(2x)\sin(3x) = \cos x - \cos(5x)$

45. $\dfrac{\cos(10\theta) + \cos(6\theta)}{\cos(6\theta) - \cos(10\theta)} = \cot(2\theta)\cot(8\theta)$

46. $\dfrac{\sin(3x) - \sin(5x)}{\cos(3x) + \cos(5x)} = \tan x$

47. $2\cos(2x)\cos x + \sin(2x)\sin x = 2\sin x$

48. $\dfrac{\sin(2x) + \sin(4x)}{\sin(2x) - \sin(4x)} = -\tan(3x)\cot x$

For the following exercises, simplify the expression to one term, then graph the original function and your simplified version to verify they are identical.

49. $\dfrac{\sin(9t) - \sin(3t)}{\cos(9t) + \cos(3t)}$

50. $2\sin(8x)\cos(6x) - \sin(2x)$

51. $\dfrac{\sin(3x) - \sin x}{\sin x}$

52. $\dfrac{\cos(5x) + \cos(3x)}{\sin(5x) + \sin(3x)}$

53. $\sin x\cos(15x) - \cos x\sin(15x)$

EXTENSIONS

For the following exercises, prove the following sum-to-product formulas.

54. $\sin x - \sin y = 2\sin\left(\dfrac{x-y}{2}\right)\cos\left(\dfrac{x+y}{2}\right)$

55. $\cos x + \cos y = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

For the following exercises, prove the identity.

56. $\dfrac{\sin(6x) + \sin(4x)}{\sin(6x) - \sin(4x)} = \tan(5x)\cot x$

57. $\dfrac{\cos(3x) + \cos x}{\cos(3x) - \cos x} = -\cot(2x)\cot x$

58. $\dfrac{\cos(6y) + \cos(8y)}{\sin(6y) - \sin(4y)} = \cot y\cos(7y)\sec(5y)$

59. $\dfrac{\cos(2y) - \cos(4y)}{\sin(2y) + \sin(4y)} = \tan y$

60. $\dfrac{\sin(10x) - \sin(2x)}{\cos(10x) + \cos(2x)} = \tan(4x)$

61. $\cos x - \cos(3x) = 4\sin^2 x\cos x$

62. $(\cos(2x) - \cos(4x))^2 + (\sin(4x) + \sin(2x))^2 = 4\sin^2(3x)$

63. $\tan\left(\dfrac{\pi}{4} - t\right) = \dfrac{1 - \tan t}{1 + \tan t}$

LEARNING OBJECTIVES

In this section, you will:

- Solve linear trigonometric equations in sine and cosine.
- Solve equations involving a single trigonometric function.
- Solve trigonometric equations using a calculator.
- Solve trigonometric equations that are quadratic in form.
- Solve trigonometric equations using fundamental identities.
- Solve trigonometric equations with multiple angles.
- Solve right triangle problems.

7.5 SOLVING TRIGONOMETRIC EQUATIONS

Figure 1 Egyptian pyramids standing near a modern city. (credit: Oisin Mulvihill)

Thales of Miletus (circa 625–547 BC) is known as the founder of geometry. The legend is that he calculated the height of the Great Pyramid of Giza in Egypt using the theory of *similar triangles*, which he developed by measuring the shadow of his staff. Based on proportions, this theory has applications in a number of areas, including fractal geometry, engineering, and architecture. Often, the angle of elevation and the angle of depression are found using similar triangles.

In earlier sections of this chapter, we looked at trigonometric identities. Identities are true for all values in the domain of the variable. In this section, we begin our study of trigonometric equations to study real-world scenarios such as the finding the dimensions of the pyramids.

Solving Linear Trigonometric Equations in Sine and Cosine

Trigonometric equations are, as the name implies, equations that involve trigonometric functions. Similar in many ways to solving polynomial equations or rational equations, only specific values of the variable will be solutions, if there are solutions at all. Often we will solve a trigonometric equation over a specified interval. However, just as often, we will be asked to find all possible solutions, and as trigonometric functions are periodic, solutions are repeated within each period. In other words, trigonometric equations may have an infinite number of solutions. Additionally, like rational equations, the domain of the function must be considered before we assume that any solution is valid. The period of both the sine function and the cosine function is 2π. In other words, every 2π units, the y-values repeat. If we need to find all possible solutions, then we must add $2\pi k$, where k is an integer, to the initial solution. Recall the rule that gives the format for stating all possible solutions for a function where the period is 2π:

$$\sin\theta = \sin(\theta \pm 2k\pi)$$

There are similar rules for indicating all possible solutions for the other trigonometric functions. Solving trigonometric equations requires the same techniques as solving algebraic equations. We read the equation from left to right, horizontally, like a sentence. We look for known patterns, factor, find common denominators, and substitute certain expressions with a variable to make solving a more straightforward process. However, with trigonometric equations, we also have the advantage of using the identities we developed in the previous sections.

Example 1 Solving a Linear Trigonometric Equation Involving the Cosine Function

Find all possible exact solutions for the equation $\cos \theta = \frac{1}{2}$.

Solution From the unit circle, we know that

$$\cos \theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

These are the solutions in the interval $[0, 2\pi]$. All possible solutions are given by

$$\theta = \frac{\pi}{3} \pm 2k\pi \text{ and } \theta = \frac{5\pi}{3} \pm 2k\pi$$

where k is an integer.

Example 2 Solving a Linear Equation Involving the Sine Function

Find all possible exact solutions for the equation $\sin t = \frac{1}{2}$.

Solution Solving for all possible values of t means that solutions include angles beyond the period of 2π. From **Section 7.2 Figure 2**, we can see that the solutions are $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$. But the problem is asking for all possible values that solve the equation. Therefore, the answer is

$$t = \frac{\pi}{6} \pm 2\pi k \quad \text{and} \quad t = \frac{5\pi}{6} \pm 2\pi k$$

where k is an integer.

How To...

Given a trigonometric equation, solve using algebra.

1. Look for a pattern that suggests an algebraic property, such as the difference of squares or a factoring opportunity.
2. Substitute the trigonometric expression with a single variable, such as x or u.
3. Solve the equation the same way an algebraic equation would be solved.
4. Substitute the trigonometric expression back in for the variable in the resulting expressions.
5. Solve for the angle.

Example 3 Solve the Trigonometric Equation in Linear Form

Solve the equation exactly: $2\cos \theta - 3 = -5, 0 \leq \theta < 2\pi$.

Solution Use algebraic techniques to solve the equation.

$$2\cos \theta - 3 = -5$$
$$2\cos \theta = -2$$
$$\cos \theta = -1$$
$$\theta = \pi$$

Try It #1

Solve exactly the following linear equation on the interval $[0, 2\pi)$: $2\sin x + 1 = 0$.

Solving Equations Involving a Single Trigonometric Function

When we are given equations that involve only one of the six trigonometric functions, their solutions involve using algebraic techniques and the unit circle (see **Section 7.2 Figure 2**). We need to make several considerations when the equation involves trigonometric functions other than sine and cosine. Problems involving the reciprocals of the primary trigonometric functions need to be viewed from an algebraic perspective. In other words, we will write the reciprocal function, and solve for the angles using the function. Also, an equation involving the tangent function is slightly different from one containing a sine or cosine function. First, as we know, the period of tangent is π, not 2π. Further, the domain of tangent is all real numbers with the exception of odd integer multiples of $\frac{\pi}{2}$, unless, of course, a problem places its own restrictions on the domain.

Example 4 Solving a Problem Involving a Single Trigonometric Function

Solve the problem exactly: $2\sin^2\theta - 1 = 0,\ 0 \le \theta < 2\pi$.

Solution As this problem is not easily factored, we will solve using the square root property. First, we use algebra to isolate $\sin\theta$. Then we will find the angles.

$$2\sin^2\theta - 1 = 0$$
$$2\sin^2\theta = 1$$
$$\sin^2\theta = \frac{1}{2}$$
$$\sqrt{\sin^2\theta} = \pm\sqrt{\frac{1}{2}}$$
$$\sin\theta = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Example 5 Solving a Trigonometric Equation Involving Cosecant

Solve the following equation exactly: $\csc\theta = -2,\ 0 \le \theta < 4\pi$.

Solution We want all values of θ for which $\csc\theta = -2$ over the interval $0 \le \theta < 4\pi$.

$$\csc\theta = -2$$
$$\frac{1}{\sin\theta} = -2$$
$$\sin\theta = -\frac{1}{2}$$
$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$$

Analysis *As $\sin\theta = -\frac{1}{2}$, notice that all four solutions are in the third and fourth quadrants.*

Example 6 Solving an Equation Involving Tangent

Solve the equation exactly: $\tan\left(\theta - \frac{\pi}{2}\right) = 1,\ 0 \le \theta < 2\pi$.

Solution Recall that the tangent function has a period of π. On the interval $[0, \pi)$, and at the angle of $\frac{\pi}{4}$, the tangent has a value of 1. However, the angle we want is $\left(\theta - \frac{\pi}{2}\right)$. Thus, if $\tan\left(\frac{\pi}{4}\right) = 1$, then

$$\theta - \frac{\pi}{2} = \frac{\pi}{4}$$
$$\theta = \frac{3\pi}{4} \pm k\pi$$

Over the interval $[0, 2\pi)$, we have two solutions:

$$\theta = \frac{3\pi}{4} \text{ and } \theta = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$$

Try It #2

Find all solutions for $\tan x = \sqrt{3}$.

Example 7 Identify all Solutions to the Equation Involving Tangent

Identify all exact solutions to the equation $2(\tan x + 3) = 5 + \tan x,\ 0 \le x < 2\pi$.

Solution We can solve this equation using only algebra. Isolate the expression tan x on the left side of the equals sign.

$$2(\tan x) + 2(3) = 5 + \tan x$$
$$2\tan x + 6 = 5 + \tan x$$
$$2\tan x - \tan x = 5 - 6$$
$$\tan x = -1$$

There are two angles on the unit circle that have a tangent value of -1: $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$.

Solve Trigonometric Equations Using a Calculator

Not all functions can be solved exactly using only the unit circle. When we must solve an equation involving an angle other than one of the special angles, we will need to use a calculator. Make sure it is set to the proper mode, either degrees or radians, depending on the criteria of the given problem.

Example 8 Using a Calculator to Solve a Trigonometric Equation Involving Sine

Use a calculator to solve the equation $\sin\theta = 0.8$, where θ is in radians.

Solution Make sure mode is set to radians. To find θ, use the inverse sine function. On most calculators, you will need to push the **2ND** button and then the **SIN** button to bring up the **sin⁻¹** function. What is shown on the screen is **sin⁻¹(** . The calculator is ready for the input within the parentheses. For this problem, we enter **sin⁻¹ (0.8)**, and press **ENTER**. Thus, to four decimals places,

$$\sin^{-1}(0.8) \approx 0.9273$$

The solution is

$$\theta \approx 0.9273 \pm 2\pi k$$

The angle measurement in degrees is

$$\theta \approx 53.1°$$
$$\theta \approx 180° - 53.1°$$
$$\approx 126.9°$$

Analysis *Note that a calculator will only return an angle in quadrants I or IV for the sine function, since that is the range of the inverse sine. The other angle is obtained by using $\pi - \theta$.*

Example 9 Using a Calculator to Solve a Trigonometric Equation Involving Secant

Use a calculator to solve the equation $\sec\theta = -4$, giving your answer in radians.

Solution We can begin with some algebra.

$$\sec\theta = -4$$
$$\frac{1}{\cos\theta} = -4$$
$$\cos\theta = -\frac{1}{4}$$

Check that the MODE is in radians. Now use the inverse cosine function.

$$\cos^{-1}\left(-\frac{1}{4}\right) \approx 1.8235$$
$$\theta \approx 1.8235 + 2\pi k$$

Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$, 1.8235 is between these two numbers, thus $\theta \approx 1.8235$ is in quadrant II. Cosine is also negative in quadrant III. Note that a calculator will only return an angle in quadrants I or II for the cosine function, since that is the range of the inverse cosine. See **Figure 2**.

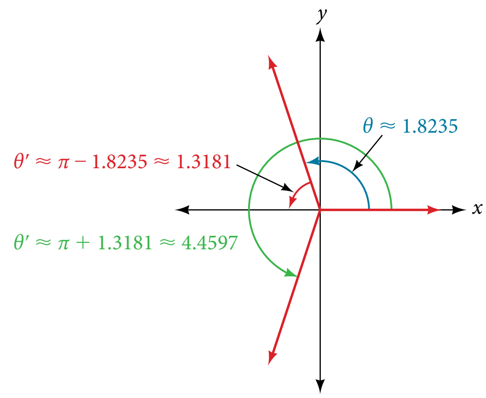


Figure 2

So, we also need to find the measure of the angle in quadrant III. In quadrant III, the reference angle is $\theta' \approx \pi - 1.8235 \approx 1.3181$. The other solution in quadrant III is $\theta' \approx \pi + 1.3181 \approx 4.4597$.

The solutions are $\theta \approx 1.8235 \pm 2\pi k$ and $\theta \approx 4.4597 \pm 2\pi k$.

Try It #3

Solve $\cos\theta = -0.2$.

Solving Trigonometric Equations in Quadratic Form

Solving a quadratic equation may be more complicated, but once again, we can use algebra as we would for any quadratic equation. Look at the pattern of the equation. Is there more than one trigonometric function in the equation, or is there only one? Which trigonometric function is squared? If there is only one function represented and one of the terms is squared, think about the standard form of a quadratic. Replace the trigonometric function with a variable such as x or u. If substitution makes the equation look like a quadratic equation, then we can use the same methods for solving quadratics to solve the trigonometric equations.

Example 10 **Solving a Trigonometric Equation in Quadratic Form**

Solve the equation exactly: $\cos^2\theta + 3\cos\theta - 1 = 0$, $0 \le \theta < 2\pi$.

Solution We begin by using substitution and replacing $\cos\theta$ with x. It is not necessary to use substitution, but it may make the problem easier to solve visually. Let $\cos\theta = x$. We have

$$x^2 + 3x - 1 = 0$$

The equation cannot be factored, so we will use the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$x = \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2}$$

$$= \frac{-3 \pm \sqrt{13}}{2}$$

Replace x with $\cos\theta$, and solve. Thus,

$$\cos\theta = \frac{-3 \pm \sqrt{13}}{2}$$

$$\theta = \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right)$$

Note that only the + sign is used. This is because we get an error when we solve $\theta = \cos^{-1}\left(\dfrac{-3 - \sqrt{13}}{2}\right)$ on a calculator, since the domain of the inverse cosine function is $[-1, 1]$. However, there is a second solution:

$$\theta = \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right)$$

$$\approx 1.26$$

This terminal side of the angle lies in quadrant I. Since cosine is also positive in quadrant IV, the second solution is

$$\begin{aligned}\theta &= 2\pi - \cos^{-1}\left(\frac{-3+\sqrt{13}}{2}\right)\\ &\approx 5.02\end{aligned}$$

Try It #4

Solve $\sin^2\theta = 2\cos\theta + 2$, $0 \le \theta \le 2\pi$. [Hint: Make a substitution to express the equation only in terms of cosine.]

Example 11 Solving a Trigonometric Equation in Quadratic Form by Factoring

Solve the equation exactly: $2\sin^2\theta - 5\sin\theta + 3 = 0$, $0 \le \theta \le 2\pi$.

Solution Using grouping, this quadratic can be factored. Either make the real substitution, $\sin\theta = u$, or imagine it, as we factor:

$$\begin{aligned}2\sin^2\theta - 5\sin\theta + 3 &= 0\\ (2\sin\theta - 3)(\sin\theta - 1) &= 0\end{aligned}$$

Now set each factor equal to zero.

$$\begin{aligned}2\sin\theta - 3 &= 0\\ 2\sin\theta &= 3\\ \sin\theta &= \frac{3}{2}\\ \sin\theta - 1 &= 0\\ \sin\theta &= 1\end{aligned}$$

Next solve for θ: $\sin\theta \ne \frac{3}{2}$, as the range of the sine function is $[-1, 1]$. However, $\sin\theta = 1$, giving the solution $\theta = \frac{\pi}{2}$.

Analysis *Make sure to check all solutions on the given domain as some factors have no solution.*

Try It #5

Solve $\sin^2\theta = 2\cos\theta + 2$, $0 \le \theta \le 2\pi$. [Hint: Make a substitution to express the equation only in terms of cosine.]

Example 12 Solving a Trigonometric Equation Using Algebra

Solve exactly:

$$2\sin^2\theta + \sin\theta = 0;\ 0 \le \theta < 2\pi$$

Solution This problem should appear familiar as it is similar to a quadratic. Let $\sin\theta = x$. The equation becomes $2x^2 + x = 0$. We begin by factoring:

$$\begin{aligned}2x^2 + x &= 0\\ x(2x+1) &= 0\end{aligned}$$

Set each factor equal to zero.

$$\begin{aligned}x &= 0\\ (2x+1) &= 0\\ x &= -\frac{1}{2}\end{aligned}$$

Then, substitute back into the equation the original expression $\sin\theta$ for x. Thus,

$$\begin{aligned}\sin\theta &= 0\\ \theta &= 0, \pi\\ \sin\theta &= -\frac{1}{2}\\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}\end{aligned}$$

The solutions within the domain $0 \le \theta < 2\pi$ are $\theta = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$.

If we prefer not to substitute, we can solve the equation by following the same pattern of factoring and setting each factor equal to zero.

$$2\sin^2\theta + \sin\theta = 0$$
$$\sin\theta(2\sin\theta + 1) = 0$$
$$\sin\theta = 0$$
$$\theta = 0, \pi$$
$$2\sin\theta + 1 = 0$$
$$2\sin\theta = -1$$
$$\sin\theta = -\frac{1}{2}$$
$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Analysis *We can see the solutions on the graph in **Figure 3**. On the interval $0 \le \theta < 2\pi$, the graph crosses the x-axis four times, at the solutions noted. Notice that trigonometric equations that are in quadratic form can yield up to four solutions instead of the expected two that are found with quadratic equations. In this example, each solution (angle) corresponding to a positive sine value will yield two angles that would result in that value.*

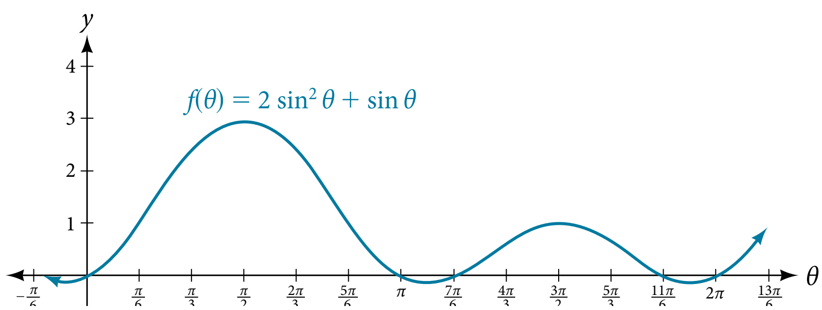


Figure 3

*We can verify the solutions on the unit circle in **Section 7.2 Figure 2** as well.*

Example 13 Solving a Trigonometric Equation Quadratic in Form

Solve the equation quadratic in form exactly: $2\sin^2\theta - 3\sin\theta + 1 = 0, 0 \le \theta < 2\pi$.

Solution We can factor using grouping. Solution values of θ can be found on the unit circle:

$$(2\sin\theta - 1)(\sin\theta - 1) = 0$$
$$2\sin\theta - 1 = 0$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$\sin\theta = 1$$
$$\theta = \frac{\pi}{2}$$

Try It #6

Solve the quadratic equation $2\cos^2\theta + \cos\theta = 0$.

Solving Trigonometric Equations Using Fundamental Identities

While algebra can be used to solve a number of trigonometric equations, we can also use the fundamental identities because they make solving equations simpler. Remember that the techniques we use for solving are not the same as those for verifying identities. The basic rules of algebra apply here, as opposed to rewriting one side of the identity to match the other side. In the next example, we use two identities to simplify the equation.

Example 14 Use Identities to Solve an Equation

Use identities to solve exactly the trigonometric equation over the interval $0 \leq x < 2\pi$.

$$\cos x \cos(2x) + \sin x \sin(2x) = \frac{\sqrt{3}}{2}$$

Solution Notice that the left side of the equation is the difference formula for cosine.

$$\begin{aligned}
\cos x \cos(2x) + \sin x \sin(2x) &= \frac{\sqrt{3}}{2} \\
\cos(x - 2x) &= \frac{\sqrt{3}}{2} && \text{Difference formula for cosine} \\
\cos(-x) &= \frac{\sqrt{3}}{2} && \text{Use the negative angle identity.} \\
\cos x &= \frac{\sqrt{3}}{2}
\end{aligned}$$

From the unit circle in **Figure 2,** we see that $\cos x = \frac{\sqrt{3}}{2}$ when $x = \frac{\pi}{6}, \frac{11\pi}{6}$.

Example 15 Solving the Equation Using a Double-Angle Formula

Solve the equation exactly using a double-angle formula: $\cos(2\theta) = \cos\theta$.

Solution We have three choices of expressions to substitute for the double-angle of cosine. As it is simpler to solve for one trigonometric function at a time, we will choose the double-angle identity involving only cosine:

$$\begin{aligned}
\cos(2\theta) &= \cos\theta \\
2\cos^2\theta - 1 &= \cos\theta \\
2\cos^2\theta - \cos\theta - 1 &= 0 \\
(2\cos\theta + 1)(\cos\theta - 1) &= 0 \\
2\cos\theta + 1 &= 0 \\
\cos\theta &= -\frac{1}{2} \\
\cos\theta - 1 &= 0 \\
\cos\theta &= 1
\end{aligned}$$

So, if $\cos\theta = -\frac{1}{2}$, then $\theta = \frac{2\pi}{3} \pm 2\pi k$ and $\theta = \frac{4\pi}{3} \pm 2\pi k$; if $\cos\theta = 1$, then $\theta = 0 \pm 2\pi k$.

Example 16 Solving an Equation Using an Identity

Solve the equation exactly using an identity: $3\cos\theta + 3 = 2\sin^2\theta$, $0 \leq \theta < 2\pi$.

Solution If we rewrite the right side, we can write the equation in terms of cosine:

$$\begin{aligned}
3\cos\theta + 3 &= 2\sin^2\theta \\
3\cos\theta + 3 &= 2(1 - \cos^2\theta) \\
3\cos\theta + 3 &= 2 - 2\cos^2\theta \\
2\cos^2\theta + 3\cos\theta + 1 &= 0 \\
(2\cos\theta + 1)(\cos\theta + 1) &= 0 \\
2\cos\theta + 1 &= 0 \\
\cos\theta &= -\frac{1}{2} \\
\theta &= \frac{2\pi}{3}, \frac{4\pi}{3} \\
\cos\theta + 1 &= 0 \\
\cos\theta &= -1 \\
\theta &= \pi
\end{aligned}$$

Our solutions are $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \pi$

Solving Trigonometric Equations with Multiple Angles

Sometimes it is not possible to solve a trigonometric equation with identities that have a multiple angle, such as $\sin(2x)$ or $\cos(3x)$. When confronted with these equations, recall that $y = \sin(2x)$ is a horizontal compression by a factor of 2 of the function $y = \sin x$. On an interval of 2π, we can graph two periods of $y = \sin(2x)$, as opposed to one cycle of $y = \sin x$. This compression of the graph leads us to believe there may be twice as many x-intercepts or solutions to $\sin(2x) = 0$ compared to $\sin x = 0$. This information will help us solve the equation.

Example 17 **Solving a Multiple Angle Trigonometric Equation**

Solve exactly: $\cos(2x) = \frac{1}{2}$ on $[0, 2\pi)$.

Solution We can see that this equation is the standard equation with a multiple of an angle. If $\cos(\alpha) = \frac{1}{2}$, we know α is in quadrants I and IV. While $\theta = \cos^{-1}\frac{1}{2}$ will only yield solutions in quadrants I and II, we recognize that the solutions to the equation $\cos\theta = \frac{1}{2}$ will be in quadrants I and IV.

Therefore, the possible angles are $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. So, $2x = \frac{\pi}{3}$ or $2x = \frac{5\pi}{3}$, which means that $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Does this make sense? Yes, because $\cos\left(2\left(\frac{\pi}{6}\right)\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

Are there any other possible answers? Let us return to our first step.
In quadrant I, $2x = \frac{\pi}{3}$, so $x = \frac{\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{\pi}{3} + 2\pi \\ &= \frac{\pi}{3} + \frac{6\pi}{3} \\ &= \frac{7\pi}{3} \end{aligned}$$

so $x = \frac{7\pi}{6}$.

One more rotation yields

$$\begin{aligned} 2x &= \frac{\pi}{3} + 4\pi \\ &= \frac{\pi}{3} + \frac{12\pi}{3} \\ &= \frac{13\pi}{3} \end{aligned}$$

$x = \frac{13\pi}{6} > 2\pi$, so this value for x is larger than 2π, so it is not a solution on $[0, 2\pi)$.

In quadrant IV, $2x = \frac{5\pi}{3}$, so $x = \frac{5\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 2\pi \\ &= \frac{5\pi}{3} + \frac{6\pi}{3} \\ &= \frac{11\pi}{3} \end{aligned}$$

so $x = \frac{11\pi}{6}$.

One more rotation yields

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 4\pi \\ &= \frac{5\pi}{3} + \frac{12\pi}{3} \\ &= \frac{17\pi}{3} \end{aligned}$$

$x = \frac{17\pi}{6} > 2\pi$, so this value for x is larger than 2π, so it is not a solution on $[0, 2\pi)$.

Our solutions are $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Note that whenever we solve a problem in the form of $\sin(nx) = c$, we must go around the unit circle n times.

Solving Right Triangle Problems

We can now use all of the methods we have learned to solve problems that involve applying the properties of right triangles and the Pythagorean Theorem. We begin with the familiar Pythagorean Theorem, $a^2 + b^2 = c^2$, and model an equation to fit a situation.

Example 18 Using the Pythagorean Theorem to Model an Equation

Use the Pythagorean Theorem, and the properties of right triangles to model an equation that fits the problem. One of the cables that anchors the center of the London Eye Ferris wheel to the ground must be replaced. The center of the Ferris wheel is 69.5 meters above the ground, and the second anchor on the ground is 23 meters from the base of the Ferris wheel. Approximately how long is the cable, and what is the angle of elevation (from ground up to the center of the Ferris wheel)? See **Figure 4.**

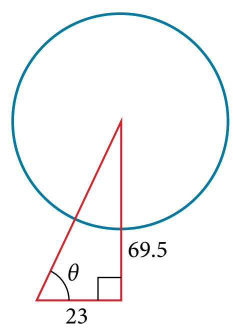


Figure 4

Solution Using the information given, we can draw a right triangle. We can find the length of the cable with the Pythagorean Theorem.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (23)^2 + (69.5)^2 &\approx 5359 \\ \sqrt{5359} &\approx 73.2 \text{ m} \end{aligned}$$

The angle of elevation is θ, formed by the second anchor on the ground and the cable reaching to the center of the wheel. We can use the tangent function to find its measure. Round to two decimal places.

$$\begin{aligned} \tan\theta &= \frac{69.5}{23} \\ \tan^{-1}\left(\frac{69.5}{23}\right) &\approx 1.2522 \\ &\approx 71.69° \end{aligned}$$

The angle of elevation is approximately 71.7°, and the length of the cable is 73.2 meters.

Example 19 Using the Pythagorean Theorem to Model an Abstract Problem

OSHA safety regulations require that the base of a ladder be placed 1 foot from the wall for every 4 feet of ladder length. Find the angle that a ladder of any length forms with the ground and the height at which the ladder touches the wall.

Solution For any length of ladder, the base needs to be a distance from the wall equal to one fourth of the ladder's length. Equivalently, if the base of the ladder is "a" feet from the wall, the length of the ladder will be $4a$ feet. See **Figure 5**.

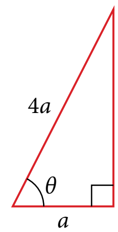


Figure 5

The side adjacent to θ is a and the hypotenuse is $4a$. Thus,

$$\cos\theta = \frac{a}{4a} = \frac{1}{4}$$

$$\cos^{-1}\left(\frac{1}{4}\right) \approx 75.5°$$

The elevation of the ladder forms an angle of 75.5° with the ground. The height at which the ladder touches the wall can be found using the Pythagorean Theorem:

$$\begin{aligned} a^2 + b^2 &= (4a)^2 \\ b^2 &= (4a)^2 - a^2 \\ b^2 &= 16a^2 - a^2 \\ b^2 &= 15a^2 \\ b &= a\sqrt{15} \end{aligned}$$

Thus, the ladder touches the wall at $a\sqrt{15}$ feet from the ground.

Access these online resources for additional instruction and practice with solving trigonometric equations.

- Solving Trigonometric Equations I (http://openstaxcollege.org/l/solvetrigeqI)
- Solving Trigonometric Equations II (http://openstaxcollege.org/l/solvetrigeqII)
- Solving Trigonometric Equations III (http://openstaxcollege.org/l/solvetrigeqIII)
- Solving Trigonometric Equations IV (http://openstaxcollege.org/l/solvetrigeqIV)
- Solving Trigonometric Equations V (http://openstaxcollege.org/l/solvetrigeqV)
- Solving Trigonometric Equations VI (http://openstaxcollege.org/l/solvetrigeqVI)

7.5 SECTION EXERCISES

VERBAL

1. Will there always be solutions to trigonometric function equations? If not, describe an equation that would not have a solution. Explain why or why not.

2. When solving a trigonometric equation involving more than one trig function, do we always want to try to rewrite the equation so it is expressed in terms of one trigonometric function? Why or why not?

3. When solving linear trig equations in terms of only sine or cosine, how do we know whether there will be solutions?

ALGEBRAIC

For the following exercises, find all solutions exactly on the interval $0 \le \theta < 2\pi$.

4. $2\sin\theta = -\sqrt{2}$

5. $2\sin\theta = \sqrt{3}$

6. $2\cos\theta = 1$

7. $2\cos\theta = -\sqrt{2}$

8. $\tan\theta = -1$

9. $\tan x = 1$

10. $\cot x + 1 = 0$

11. $4\sin^2 x - 2 = 0$

12. $\csc^2 x - 4 = 0$

For the following exercises, solve exactly on $[0, 2\pi)$.

13. $2\cos\theta = \sqrt{2}$

14. $2\cos\theta = -1$

15. $2\sin\theta = -1$

16. $2\sin\theta = -\sqrt{3}$

17. $2\sin(3\theta) = 1$

18. $2\sin(2\theta) = \sqrt{3}$

19. $2\cos(3\theta) = -\sqrt{2}$

20. $\cos(2\theta) = -\dfrac{\sqrt{3}}{2}$

21. $2\sin(\pi\theta) = 1$

22. $2\cos\left(\dfrac{\pi}{5}\theta\right) = \sqrt{3}$

For the following exercises, find all exact solutions on $[0, 2\pi)$.

23. $\sec(x)\sin(x) - 2\sin(x) = 0$

24. $\tan(x) - 2\sin(x)\tan(x) = 0$

25. $2\cos^2 t + \cos(t) = 1$

26. $2\tan^2(t) = 3\sec(t)$

27. $2\sin(x)\cos(x) - \sin(x) + 2\cos(x) - 1 = 0$

28. $\cos^2\theta = \dfrac{1}{2}$

29. $\sec^2 x = 1$

30. $\tan^2(x) = -1 + 2\tan(-x)$

31. $8\sin^2(x) + 6\sin(x) + 1 = 0$

32. $\tan^5(x) = \tan(x)$

For the following exercises, solve with the methods shown in this section exactly on the interval $[0, 2\pi)$.

33. $\sin(3x)\cos(6x) - \cos(3x)\sin(6x) = -0.9$

34. $\sin(6x)\cos(11x) - \cos(6x)\sin(11x) = -0.1$

35. $\cos(2x)\cos x + \sin(2x)\sin x = 1$

36. $6\sin(2t) + 9\sin t = 0$

37. $9\cos(2\theta) = 9\cos^2\theta - 4$

38. $\sin(2t) = \cos t$

39. $\cos(2t) = \sin t$

40. $\cos(6x) - \cos(3x) = 0$

For the following exercises, solve exactly on the interval $[0, 2\pi)$. Use the quadratic formula if the equations do not factor.

41. $\tan^2 x - \sqrt{3}\tan x = 0$

42. $\sin^2 x + \sin x - 2 = 0$

43. $\sin^2 x - 2\sin x - 4 = 0$

44. $5\cos^2 x + 3\cos x - 1 = 0$

45. $3\cos^2 x - 2\cos x - 2 = 0$

46. $5\sin^2 x + 2\sin x - 1 = 0$

47. $\tan^2 x + 5\tan x - 1 = 0$

48. $\cot^2 x = -\cot x$

49. $-\tan^2 x - \tan x - 2 = 0$

For the following exercises, find exact solutions on the interval $[0, 2\pi)$. Look for opportunities to use trigonometric identities.

50. $\sin^2 x - \cos^2 x - \sin x = 0$

51. $\sin^2 x + \cos^2 x = 0$

52. $\sin(2x) - \sin x = 0$

53. $\cos(2x) - \cos x = 0$

54. $\dfrac{2\tan x}{2 - \sec^2 x} - \sin^2 x = \cos^2 x$

55. $1 - \cos(2x) = 1 + \cos(2x)$

56. $\sec^2 x = 7$

57. $10\sin x \cos x = 6\cos x$

58. $-3\sin t = 15\cos t \sin t$

59. $4\cos^2 x - 4 = 15\cos x$

60. $8\sin^2 x + 6\sin x + 1 = 0$

61. $8\cos^2 \theta = 3 - 2\cos \theta$

62. $6\cos^2 x + 7\sin x - 8 = 0$

63. $12\sin^2 t + \cos t - 6 = 0$

64. $\tan x = 3\sin x$

65. $\cos^3 t = \cos \mathrm{t}$

GRAPHICAL

For the following exercises, algebraically determine all solutions of the trigonometric equation exactly, then verify the results by graphing the equation and finding the zeros.

66. $6\sin^2 x - 5\sin x + 1 = 0$

67. $8\cos^2 x - 2\cos x - 1 = 0$

68. $100\tan^2 x + 20\tan x - 3 = 0$

69. $2\cos^2 x - \cos x + 15 = 0$

70. $20\sin^2 x - 27\sin x + 7 = 0$

71. $2\tan^2 x + 7\tan x + 6 = 0$

72. $130\tan^2 x + 69\tan x - 130 = 0$

TECHNOLOGY

For the following exercises, use a calculator to find all solutions to four decimal places.

73. $\sin x = 0.27$

74. $\sin x = -0.55$

75. $\tan x = -0.34$

76. $\cos x = 0.71$

For the following exercises, solve the equations algebraically, and then use a calculator to find the values on the interval $[0, 2\pi)$. Round to four decimal places.

77. $\tan^2 x + 3\tan x - 3 = 0$

78. $6\tan^2 x + 13\tan x = -6$

79. $\tan^2 x - \sec x = 1$

80. $\sin^2 x - 2\cos^2 x = 0$

81. $2\tan^2 x + 9\tan x - 6 = 0$

82. $4\sin^2 x + \sin(2x)\sec x - 3 = 0$

EXTENSIONS

For the following exercises, find all solutions exactly to the equations on the interval $[0, 2\pi)$.

83. $\csc^2 x - 3\csc x - 4 = 0$

84. $\sin^2 x - \cos^2 x - 1 = 0$

85. $\sin^2 x(1 - \sin^2 x) + \cos^2 x(1 - \sin^2 x) = 0$

86. $3\sec^2 x + 2 + \sin^2 x - \tan^2 x + \cos^2 x = 0$

87. $\sin^2 x - 1 + 2\cos(2x) - \cos^2 x = 1$

88. $\tan^2 x - 1 - \sec^3 x \cos x = 0$

89. $\dfrac{\sin(2x)}{\sec^2 x} = 0$

90. $\dfrac{\sin(2x)}{2\csc^2 x} = 0$

91. $2\cos^2 x - \sin^2 x - \cos x - 5 = 0$

92. $\dfrac{1}{\sec^2 x} + 2 + \sin^2 x + 4\cos^2 x = 4$

REAL-WORLD APPLICATIONS

93. An airplane has only enough gas to fly to a city 200 miles northeast of its current location. If the pilot knows that the city is 25 miles north, how many degrees north of east should the airplane fly?

94. If a loading ramp is placed next to a truck, at a height of 4 feet, and the ramp is 15 feet long, what angle does the ramp make with the ground?

95. If a loading ramp is placed next to a truck, at a height of 2 feet, and the ramp is 20 feet long, what angle does the ramp make with the ground?

96. A woman is watching a launched rocket currently 11 miles in altitude. If she is standing 4 miles from the launch pad, at what angle is she looking up from horizontal?

97. An astronaut is in a launched rocket currently 15 miles in altitude. If a man is standing 2 miles from the launch pad, at what angle is she looking down at him from horizontal? (Hint: this is called the angle of depression.)

98. A woman is standing 8 meters away from a 10-meter tall building. At what angle is she looking to the top of the building?

99. A man is standing 10 meters away from a 6-meter tall building. Someone at the top of the building is looking down at him. At what angle is the person looking at him?

100. A 20-foot tall building has a shadow that is 55 feet long. What is the angle of elevation of the sun?

101. A 90-foot tall building has a shadow that is 2 feet long. What is the angle of elevation of the sun?

102. A spotlight on the ground 3 meters from a 2-meter tall man casts a 6 meter shadow on a wall 6 meters from the man. At what angle is the light?

103. A spotlight on the ground 3 feet from a 5-foot tall woman casts a 15-foot tall shadow on a wall 6 feet from the woman. At what angle is the light?

For the following exercises, find a solution to the word problem algebraically. Then use a calculator to verify the result. Round the answer to the nearest tenth of a degree.

104. A person does a handstand with his feet touching a wall and his hands 1.5 feet away from the wall. If the person is 6 feet tall, what angle do his feet make with the wall?

105. A person does a handstand with her feet touching a wall and her hands 3 feet away from the wall. If the person is 5 feet tall, what angle do her feet make with the wall?

106. A 23-foot ladder is positioned next to a house. If the ladder slips at 7 feet from the house when there is not enough traction, what angle should the ladder make with the ground to avoid slipping?

LEARNING OBJECTIVES

In this section, you will:

- Determine the amplitude and period of sinusoidal functions.
- Model equations and graph sinusoidal functions.
- Model periodic behavior.
- Model harmonic motion functions.

7.6 MODELING WITH TRIGONOMETRIC EQUATIONS

Figure 1 The hands on a clock are periodic: they repeat positions every twelve hours. (credit: "zoutedrop"/Flickr)

Suppose we charted the average daily temperatures in New York City over the course of one year. We would expect to find the lowest temperatures in January and February and highest in July and August. This familiar cycle repeats year after year, and if we were to extend the graph over multiple years, it would resemble a periodic function.

Many other natural phenomena are also periodic. For example, the phases of the moon have a period of approximately 28 days, and birds know to fly south at about the same time each year.

So how can we model an equation to reflect periodic behavior? First, we must collect and record data. We then find a function that resembles an observed pattern. Finally, we make the necessary alterations to the function to get a model that is dependable. In this section, we will take a deeper look at specific types of periodic behavior and model equations to fit data.

Determining the Amplitude and Period of a Sinusoidal Function

Any motion that repeats itself in a fixed time period is considered periodic motion and can be modeled by a sinusoidal function. The amplitude of a sinusoidal function is the distance from the midline to the maximum value, or from the midline to the minimum value. The midline is the average value. Sinusoidal functions oscillate above and below the midline, are periodic, and repeat values in set cycles. Recall from **Graphs of the Sine and Cosine Functions** that the period of the sine function and the cosine function is 2π. In other words, for any value of x,

$$\sin(x \pm 2\pi k) = \sin x \quad \text{and} \quad \cos(x \pm 2\pi k) = \cos x \quad \text{where } k \text{ is an integer}$$

standard form of sinusoidal equations

The general forms of a sinusoidal equation are given as

$$y = A\sin(Bt - C) + D \text{ or } y = A\cos(Bt - C) + D$$

where amplitude $= |A|$, B is related to period such that the period $= \frac{2\pi}{B}$, C is the phase shift such that $\frac{C}{B}$ denotes the horizontal shift, and D represents the vertical shift from the graph's parent graph.

Note that the models are sometimes written as $y = a\sin(\omega t \pm C) + D$ or $y = a\cos(\omega t \pm C) + D$, and period is given as $\frac{2\pi}{\omega}$.

The difference between the sine and the cosine graphs is that the sine graph begins with the average value of the function and the cosine graph begins with the maximum or minimum value of the function.

Example 1 Showing How the Properties of a Trigonometric Function Can Transform a Graph

Show the transformation of the graph of $y = \sin x$ into the graph of $y = 2\sin\left(4x - \frac{\pi}{2}\right) + 2$.

Solution Consider the series of graphs in **Figure 2** and the way each change to the equation changes the image.

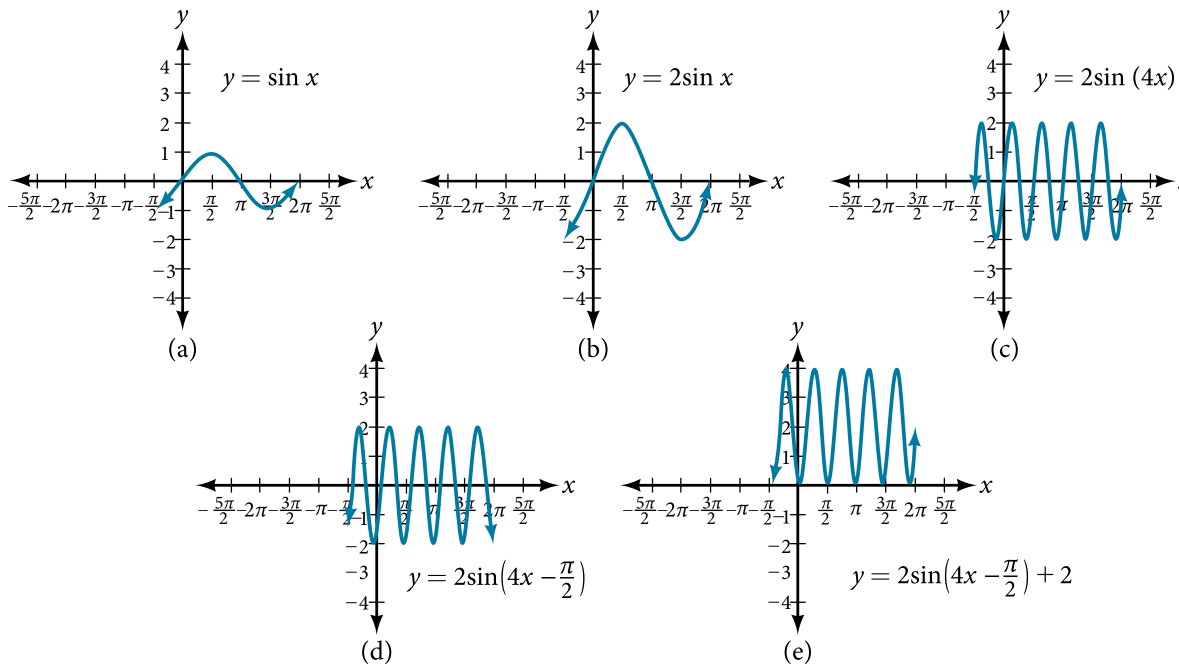


Figure 2 (*a*) The basic graph of $y = \sin x$ (*b*) Changing the amplitude from 1 to 2 generates the graph of $y = 2\sin x$. (*c*) The period of the sine function changes with the value of B, such that period $= \frac{2\pi}{B}$. Here we have $B = 4$, which translates to a period of $\frac{\pi}{2}$. The graph completes one full cycle in $\frac{\pi}{2}$ units. (*d*) The graph displays a horizontal shift equal to $\frac{C}{B}$, or $\frac{\frac{\pi}{2}}{4} = \frac{\pi}{8}$. (*e*) Finally, the graph is shifted vertically by the value of D. In this case, the graph is shifted up by 2 units.

Example 2 Finding the Amplitude and Period of a Function

Find the amplitude and period of the following functions and graph one cycle.

a. $y = 2\sin\left(\frac{1}{4}x\right)$ **b.** $y = -3\sin\left(2x + \frac{\pi}{2}\right)$ **c.** $y = \cos x + 3$

Solution We will solve these problems according to the models.

a. $y = 2\sin\left(\frac{1}{4}x\right)$ involves sine, so we use the form

$$y = A\sin(Bt - C) + D$$

We know that $|A|$ is the amplitude, so the amplitude is 2. Period is $\frac{2\pi}{B}$, so the period is

$$\begin{aligned}\frac{2\pi}{B} &= \frac{2\pi}{\frac{1}{4}}\\ &= 8\pi\end{aligned}$$

See the graph in **Figure 3.**

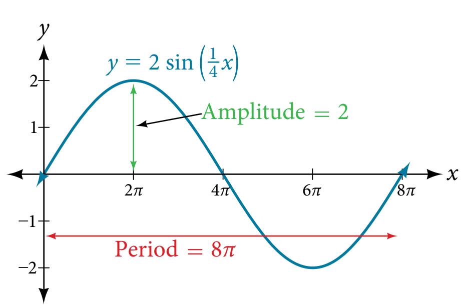


Figure 3

b. $y = -3\sin\left(2x + \frac{\pi}{2}\right)$ involves sine, so we use the form

$$y = A\sin(Bt - C) + D$$

Amplitude is $|A|$, so the amplitude is $|-3| = 3$. Since A is negative, the graph is reflected over the x-axis. Period is $\frac{2\pi}{B}$, so the period is

$$\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$$

The graph is shifted to the left by $\frac{C}{B} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$ units. See **Figure 4**.

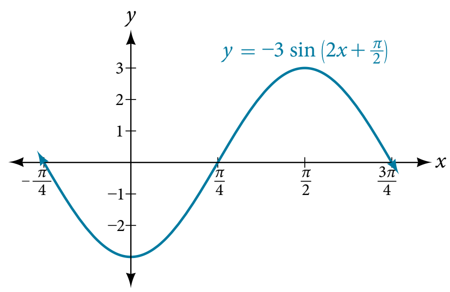


Figure 4

c. $y = \cos x + 3$ involves cosine, so we use the form

$$y = A\cos(Bt - C) + D$$

Amplitude is $|A|$, so the amplitude is 1. The period is 2π. See **Figure 5**. This is the standard cosine function shifted up three units.

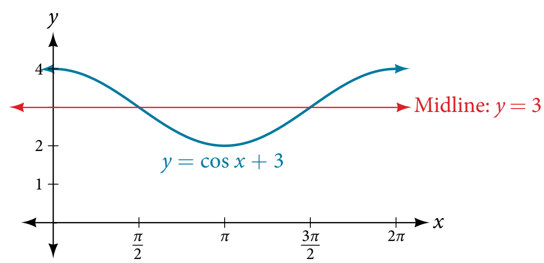


Figure 5

Try It #1

What are the amplitude and period of the function $y = 3\cos(3\pi x)$?

Finding Equations and Graphing Sinusoidal Functions

One method of graphing sinusoidal functions is to find five key points. These points will correspond to intervals of equal length representing $\frac{1}{4}$ of the period. The key points will indicate the location of maximum and minimum values. If there is no vertical shift, they will also indicate x-intercepts. For example, suppose we want to graph the function $y = \cos\theta$. We know that the period is 2π, so we find the interval between key points as follows.

$$\frac{2\pi}{4} = \frac{\pi}{2}$$

Starting with $\theta = 0$, we calculate the first y-value, add the length of the interval $\frac{\pi}{2}$ to 0, and calculate the second y-value. We then add $\frac{\pi}{2}$ repeatedly until the five key points are determined. The last value should equal the first value, as the calculations cover one full period. Making a table similar to **Table 1**, we can see these key points clearly on the graph shown in **Figure 6**.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \cos\theta$	1	0	-1	0	1

Table 1

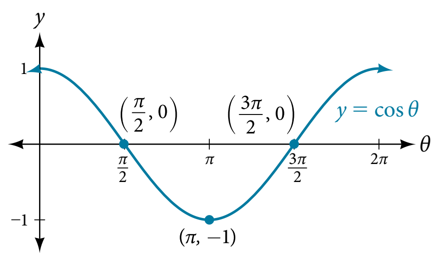


Figure 6

Example 3 Graphing Sinusoidal Functions Using Key Points

Graph the function $y = -4\cos(\pi x)$ using amplitude, period, and key points.

Solution The amplitude is $|-4| = 4$. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2$. (Recall that we sometimes refer to B as ω.) One cycle of the graph can be drawn over the interval $[0, 2]$. To find the key points, we divide the period by 4. Make a table similar to **Table 2**, starting with $x = 0$ and then adding $\frac{1}{2}$ successively to x and calculate y. See the graph in **Figure 7**.

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$y = -4\cos(\pi x)$	-4	0	4	0	-4

Table 2

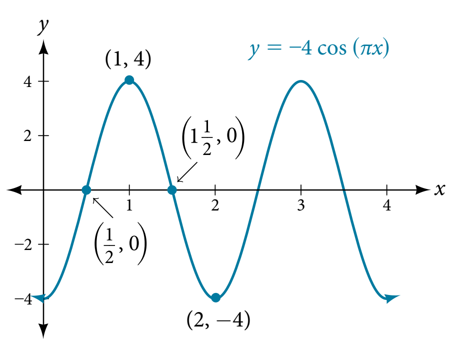


Figure 7

Try It #2

Graph the function $y = 3\sin(3x)$ using the amplitude, period, and five key points.

Modeling Periodic Behavior

We will now apply these ideas to problems involving periodic behavior.

Example 4 Modeling an Equation and Sketching a Sinusoidal Graph to Fit Criteria

The average monthly temperatures for a small town in Oregon are given in **Table 3**. Find a sinusoidal function of the form $y = A\sin(Bt - C) + D$ that fits the data (round to the nearest tenth) and sketch the graph.

Month	Temperature, ° F
January	42.5
February	44.5
March	48.5
April	52.5
May	58
June	63
July	68.5
August	69
September	64.5
October	55.5
November	46.5
December	43.5

Table 3

Solution Recall that amplitude is found using the formula

$$A = \frac{\text{largest value} - \text{smallest value}}{2}$$

Thus, the amplitude is

$$\begin{aligned} |A| &= \frac{69 - 42.5}{2} \\ &= 13.25 \end{aligned}$$

The data covers a period of 12 months, so $\frac{2\pi}{B} = 12$ which gives $B = \frac{2\pi}{12} = \frac{\pi}{6}$.

The vertical shift is found using the following equation.

$$D = \frac{\text{highest value} + \text{lowest value}}{2}$$

Thus, the vertical shift is

$$\begin{aligned} D &= \frac{69 + 42.5}{2} \\ &= 55.8 \end{aligned}$$

So far, we have the equation $y = 13.3\sin\left(\frac{\pi}{6}x - C\right) + 55.8$.

To find the horizontal shift, we input the x and y values for the first month and solve for C.

$$\begin{aligned} 42.5 &= 13.3\sin\left(\frac{\pi}{6}(1) - C\right) + 55.8 \\ -13.3 &= 13.3\sin\left(\frac{\pi}{6} - C\right) \\ -1 &= \sin\left(\frac{\pi}{6} - C\right) \quad \sin\theta = -1 \rightarrow \theta = -\frac{\pi}{2} \\ \frac{\pi}{6} - C &= -\frac{\pi}{2} \\ \frac{\pi}{6} + \frac{\pi}{2} &= C \\ &= \frac{2\pi}{3} \end{aligned}$$

We have the equation $y = 13.3\sin\left(\frac{\pi}{6}x - \frac{2\pi}{3}\right) + 55.8$. See the graph in **Figure 8**.

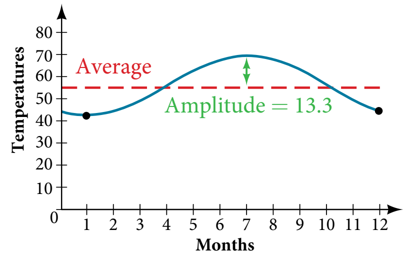


Figure 8

Example 5 Describing Periodic Motion

The hour hand of the large clock on the wall in Union Station measures 24 inches in length. At noon, the tip of the hour hand is 30 inches from the ceiling. At 3 PM, the tip is 54 inches from the ceiling, and at 6 PM, 78 inches. At 9 PM, it is again 54 inches from the ceiling, and at midnight, the tip of the hour hand returns to its original position 30 inches from the ceiling. Let y equal the distance from the tip of the hour hand to the ceiling x hours after noon. Find the equation that models the motion of the clock and sketch the graph.

Solution Begin by making a table of values as shown in **Table 4**.

x	y	Points to plot
Noon	30 in	(0, 30)
3 PM	54 in	(3, 54)
6 PM	78 in	(6, 78)
9 PM	54 in	(9, 54)
Midnight	30 in	(12, 30)

Table 4

To model an equation, we first need to find the amplitude.

$$|A| = \left|\frac{78 - 30}{2}\right| \\ = 24$$

The clock's cycle repeats every 12 hours. Thus,

$$B = \frac{2\pi}{12} \\ = \frac{\pi}{6}$$

The vertical shift is

$$D = \frac{78 + 30}{2} \\ = 54$$

There is no horizontal shift, so $C = 0$. Since the function begins with the minimum value of y when $x = 0$ (as opposed to the maximum value), we will use the cosine function with the negative value for A. In the form $y = A\cos(Bx \pm C) + D$, the equation is

$$y = -24\cos\left(\frac{\pi}{6}x\right) + 54$$

See **Figure 9**.

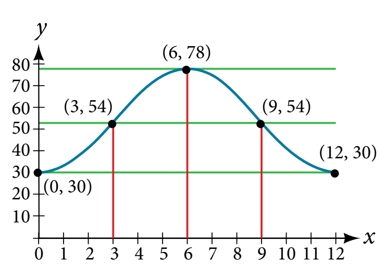


Figure 9

Example 6 Determining a Model for Tides

The height of the tide in a small beach town is measured along a seawall. Water levels oscillate between 7 feet at low tide and 15 feet at high tide. On a particular day, low tide occurred at 6 AM and high tide occurred at noon. Approximately every 12 hours, the cycle repeats. Find an equation to model the water levels.

Solution As the water level varies from 7 ft to 15 ft, we can calculate the amplitude as

$$\begin{aligned}|A| &= \left|\frac{(15-7)}{2}\right| \\ &= 4\end{aligned}$$

The cycle repeats every 12 hours; therefore, B is

$$\frac{2\pi}{12} = \frac{\pi}{6}$$

There is a vertical translation of $\frac{(15+7)}{2} = 11$. Since the value of the function is at a maximum at $t = 0$, we will use the cosine function, with the positive value for A.

$$y = 4\cos\left(\frac{\pi}{6}\right)t + 11$$

See **Figure 10**.

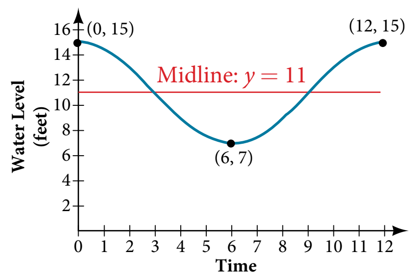


Figure 10

Try It #3

The daily temperature in the month of March in a certain city varies from a low of 24°F to a high of 40°F. Find a sinusoidal function to model daily temperature and sketch the graph. Approximate the time when the temperature reaches the freezing point 32°F. Let $t = 0$ correspond to noon.

Example 7 Interpreting the Periodic Behavior Equation

The average person's blood pressure is modeled by the function $f(t) = 20\sin(160\pi t) + 100$, where $f(t)$ represents the blood pressure at time t, measured in minutes. Interpret the function in terms of period and frequency. Sketch the graph and find the blood pressure reading.

Solution The period is given by

$$\begin{aligned}\frac{2\pi}{\omega} &= \frac{2\pi}{160\pi} \\ &= \frac{1}{80}\end{aligned}$$

In a blood pressure function, frequency represents the number of heart beats per minute. Frequency is the reciprocal of period and is given by

$$\begin{aligned}\frac{\omega}{2\pi} &= \frac{160\pi}{2\pi} \\ &= 80\end{aligned}$$

See the graph in **Figure 11**.

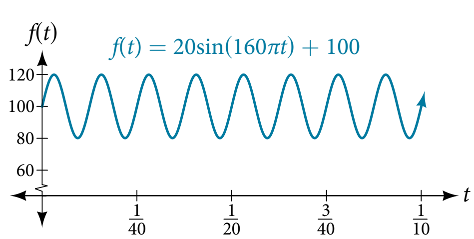


Figure 11 The blood pressure reading on the graph is $\frac{120}{80}\left(\frac{\text{maximum}}{\text{minimum}}\right)$.

Analysis *Blood pressure of* $\frac{120}{80}$ *is considered to be normal. The top number is the maximum or systolic reading, which measures the pressure in the arteries when the heart contracts. The bottom number is the minimum or diastolic reading, which measures the pressure in the arteries as the heart relaxes between beats, refilling with blood. Thus, normal blood pressure can be modeled by a periodic function with a maximum of* 120 *and a minimum of* 80.

Modeling Harmonic Motion Functions

Harmonic motion is a form of periodic motion, but there are factors to consider that differentiate the two types. While general periodic motion applications cycle through their periods with no outside interference, harmonic motion requires a restoring force. Examples of harmonic motion include springs, gravitational force, and magnetic force.

Simple Harmonic Motion

A type of motion described as simple harmonic motion involves a restoring force but assumes that the motion will continue forever. Imagine a weighted object hanging on a spring, When that object is not disturbed, we say that the object is at rest, or in equilibrium. If the object is pulled down and then released, the force of the spring pulls the object back toward equilibrium and harmonic motion begins. The restoring force is directly proportional to the displacement of the object from its equilibrium point. When $t = 0$, $d = 0$.

simple harmonic motion

We see that **simple harmonic motion** equations are given in terms of displacement:

$$d = a\cos(\omega t) \text{ or } d = a\sin(\omega t)$$

where $|a|$ is the amplitude, $\frac{2\pi}{\omega}$ is the period, and $\frac{\omega}{2\pi}$ is the frequency, or the number of cycles per unit of time.

Example 8 Finding the Displacement, Period, and Frequency, and Graphing a Function

For the given functions,
1. Find the maximum displacement of an object.
2. Find the period or the time required for one vibration.
3. Find the frequency.
4. Sketch the graph.

a. $y = 5\sin(3t)$ **b.** $y = 6\cos(\pi t)$ **c.** $y = 5\cos\left(\frac{\pi}{2}t\right)$

Solution

a. $y = 5\sin(3t)$

1. The maximum displacement is equal to the amplitude, $|a|$, which is 5.
2. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{3}$.
3. The frequency is given as $\frac{\omega}{2\pi} = \frac{3}{2\pi}$.
4. See **Figure 12**. The graph indicates the five key points.

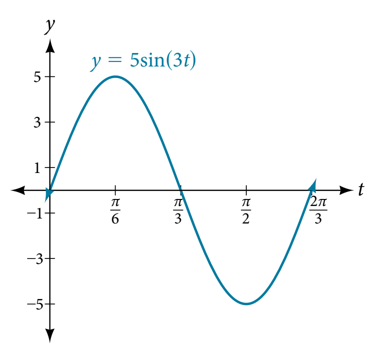


Figure 12

b. $y = 6\cos(\pi t)$

1. The maximum displacement is 6.
2. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2$.
3. The frequency is $\frac{\omega}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}$.
4. See **Figure 13**.

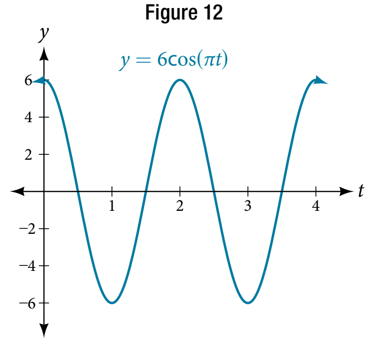


Figure 13

c. $y = 5\cos\left(\frac{\pi}{2}\right)t$

1. The maximum displacement is 5.
2. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4$.
3. The frequency is $\frac{1}{4}$.
4. See **Figure 14**.

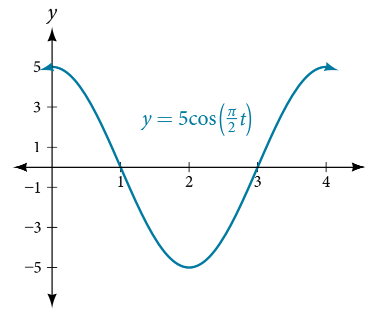


Figure 14

Damped Harmonic Motion

In reality, a pendulum does not swing back and forth forever, nor does an object on a spring bounce up and down forever. Eventually, the pendulum stops swinging and the object stops bouncing and both return to equilibrium. Periodic motion in which an energy-dissipating force, or damping factor, acts is known as damped harmonic motion. Friction is typically the damping factor.

In physics, various formulas are used to account for the damping factor on the moving object. Some of these are calculus-based formulas that involve derivatives. For our purposes, we will use formulas for basic damped harmonic motion models.

damped harmonic motion

In **damped harmonic motion**, the displacement of an oscillating object from its rest position at time t is given as

$$f(t) = ae^{-ct}\sin(\omega t) \text{ or } f(t) = ae^{-ct}\cos(\omega t)$$

where c is a damping factor, $|a|$ is the initial displacement and $\frac{2\pi}{\omega}$ is the period.

Example 9 **Modeling Damped Harmonic Motion**

Model the equations that fit the two scenarios and use a graphing utility to graph the functions: Two mass-spring systems exhibit damped harmonic motion at a frequency of 0.5 cycles per second. Both have an initial displacement of 10 cm. The first has a damping factor of 0.5 and the second has a damping factor of 0.1.

Solution At time $t = 0$, the displacement is the maximum of 10 cm, which calls for the cosine function. The cosine function will apply to both models.

We are given the frequency $f = \frac{\omega}{2\pi}$ of 0.5 cycles per second. Thus,

$$\begin{aligned} \frac{\omega}{2\pi} &= 0.5 \\ \omega &= (0.5)2\pi \\ &= \pi \end{aligned}$$

The first spring system has a damping factor of $c = 0.5$. Following the general model for damped harmonic motion, we have

$$f(t) = 10e^{-0.5t}\cos(\pi t)$$

Figure 15 models the motion of the first spring system.

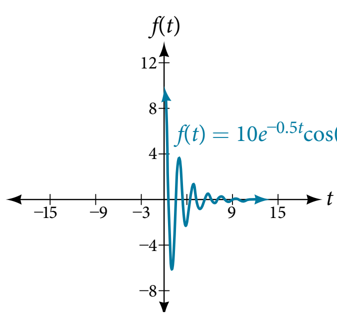


Figure 15

The second spring system has a damping factor of $c = 0.1$ and can be modeled as

$$f(t) = 10e^{-0.1t}\cos(\pi t)$$

Figure 16 models the motion of the second spring system.

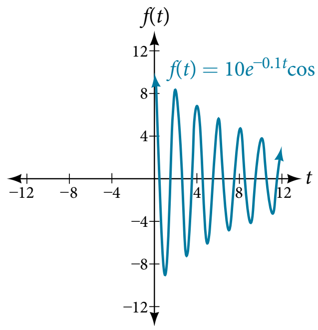


Figure 16

Analysis *Notice the differing effects of the damping constant. The local maximum and minimum values of the function with the damping factor $c = 0.5$ decreases much more rapidly than that of the function with $c = 0.1$.*

Example 10 Finding a Cosine Function that Models Damped Harmonic Motion

Find and graph a function of the form $y = ae^{-ct}\cos(\omega t)$ that models the information given.

a. $a = 20, c = 0.05, p = 4$

b. $a = 2, c = 1.5, f = 3$

Solution Substitute the given values into the model. Recall that period is $\frac{2\pi}{\omega}$ and frequency is $\frac{\omega}{2\pi}$.

a. $y = 20e^{-0.05t}\cos\left(\frac{\pi}{2}t\right)$. See **Figure 17**.

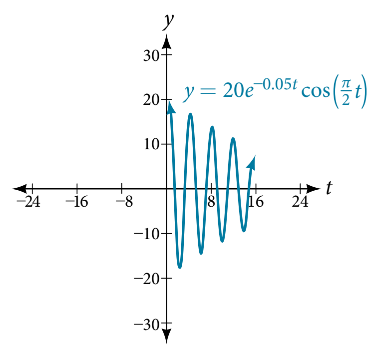


Figure 17

b. $y = 2e^{-1.5t}\cos(6\pi t)$. See **Figure 18**.

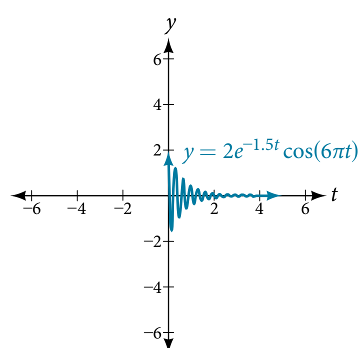


Figure 18

Try It #4

The following equation represents a damped harmonic motion model: $f(t) = 5e^{-6t}\cos(4t)$ Find the initial displacement, the damping constant, and the frequency.

Example 11 Finding a Sine Function that Models Damped Harmonic Motion

Find and graph a function of the form $y = ae^{-ct}\sin(\omega t)$ that models the information given.

a. $a = 7, c = 10, p = \frac{\pi}{6}$ **b.** $a = 0.3, c = 0.2, f = 20$

Solution Calculate the value of ω and substitute the known values into the model.

a. As period is $\frac{2\pi}{\omega}$, we have

$$\begin{aligned}\frac{\pi}{6} &= \frac{2\pi}{\omega}\\ \omega\pi &= 6(2\pi)\\ \omega &= 12\end{aligned}$$

The damping factor is given as 10 and the amplitude is 7. Thus, the model is $y = 7e^{-10t}\sin(12t)$. See **Figure 19**.

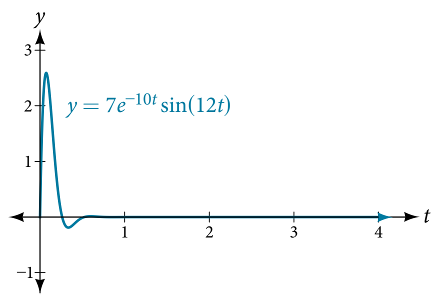


Figure 19

b. As frequency is $\frac{\omega}{2\pi}$, we have

$$\begin{aligned}20 &= \frac{\omega}{2\pi}\\ 40\pi &= \omega\end{aligned}$$

The damping factor is given as 0.2 and the amplitude is 0.3. The model is $y = 0.3e^{-0.2t}\sin(40\pi t)$. See **Figure 20**.

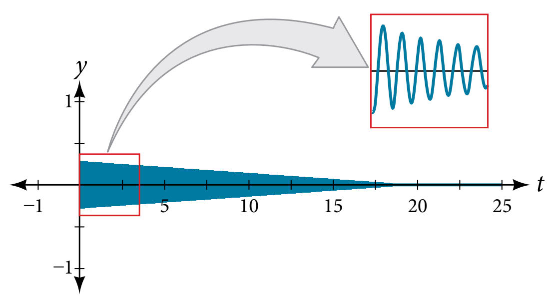


Figure 20

Analysis *A comparison of the last two examples illustrates how we choose between the sine or cosine functions to model sinusoidal criteria. We see that the cosine function is at the maximum displacement when* $t = 0$, *and the sine function is at the equilibrium point when* $t = 0$. *For example, consider the equation* $y = 20e^{-0.05t}\cos\left(\frac{\pi}{2}t\right)$ *from* ***Example 10****. We can see from the graph that when* $t = 0$, $y = 20$, *which is the initial amplitude. Check this by setting* $t = 0$ *in the cosine equation:*

$$\begin{aligned}y &= 20e^{-0.05(0)}\cos\left(\frac{\pi}{2}\right)(0)\\ &= 20(1)(1)\\ &= 20\end{aligned}$$

Using the sine function yields

$$\begin{aligned}y &= 20e^{-0.05(0)}\sin\left(\frac{\pi}{2}\right)(0)\\ &= 20(1)(0)\\ &= 0\end{aligned}$$

Thus, cosine is the correct function.

Try It #5

Write the equation for damped harmonic motion given $a = 10$, $c = 0.5$, and $p = 2$.

Example 12 Modeling the Oscillation of a Spring

A spring measuring 10 inches in natural length is compressed by 5 inches and released. It oscillates once every 3 seconds, and its amplitude decreases by 30% every second. Find an equation that models the position of the spring t seconds after being released.

Solution The amplitude begins at 5 in. and deceases 30% each second. Because the spring is initially compressed, we will write A as a negative value. We can write the amplitude portion of the function as

$$A(t) = 5(1 - 0.30)^t$$

We put $(1 - 0.30)^t$ in the form e^{ct} as follows:

$$0.7 = e^c$$
$$c = \ln 0.7$$
$$c = -0.357$$

Now let's address the period. The spring cycles through its positions every 3 seconds, this is the period, and we can use the formula to find omega.

$$3 = \frac{2\pi}{\omega}$$
$$\omega = \frac{2\pi}{3}$$

The natural length of 10 inches is the midline. We will use the cosine function, since the spring starts out at its maximum displacement. This portion of the equation is represented as

$$y = \cos\left(\frac{2\pi}{3}t\right) + 10$$

Finally, we put both functions together. Our the model for the position of the spring at t seconds is given as

$$y = -5e^{-0.357t}\cos\left(\frac{2\pi}{3}t\right) + 10$$

See the graph in **Figure 21**.

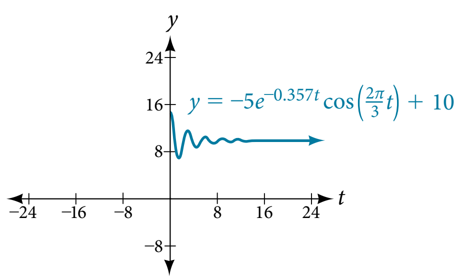


Figure 21

Try It #6

A mass suspended from a spring is raised a distance of 5 cm above its resting position. The mass is released at time $t = 0$ and allowed to oscillate. After $\frac{1}{3}$ second, it is observed that the mass returns to its highest position. Find a function to model this motion relative to its initial resting position.

Example 13 Finding the Value of the Damping Constant c According to the Given Criteria

A guitar string is plucked and vibrates in damped harmonic motion. The string is pulled and displaced 2 cm from its resting position. After 3 seconds, the displacement of the string measures 1 cm. Find the damping constant.

Solution The displacement factor represents the amplitude and is determined by the coefficient ae^{-ct} in the model for damped harmonic motion. The damping constant is included in the term e^{-ct}. It is known that after 3 seconds, the local maximum measures one-half of its original value. Therefore, we have the equation

$$ae^{-c(t+3)} = \frac{1}{2}ae^{-ct}$$

Use algebra and the laws of exponents to solve for c.

$$ae^{-c(t+3)} = \frac{1}{2}ae^{-ct}$$

$$e^{-ct} \cdot e^{-3c} = \frac{1}{2}e^{-ct} \qquad \text{Divide out } a.$$

$$e^{-3c} = \frac{1}{2} \qquad \text{Divide out } e^{-ct}.$$

$$e^{3c} = 2 \qquad \text{Take reciprocals.}$$

Then use the laws of logarithms.

$$e^{3c} = 2$$

$$3c = \ln(2)$$

$$c = \frac{\ln(2)}{3}$$

The damping constant is $\frac{\ln(2)}{3}$.

Bounding Curves in Harmonic Motion

Harmonic motion graphs may be enclosed by bounding curves. When a function has a varying amplitude, such that the amplitude rises and falls multiple times within a period, we can determine the bounding curves from part of the function.

Example 14 Graphing an Oscillating Cosine Curve

Graph the function $f(x) = \cos(2\pi x)\cos(16\pi x)$.

Solution The graph produced by this function will be shown in two parts. The first graph will be the exact function $f(x)$ (see **Figure 22**), and the second graph is the exact function $f(x)$ plus a bounding function (see **Figure 23**). The graphs look quite different.

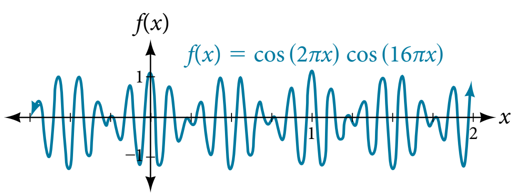


Figure 22

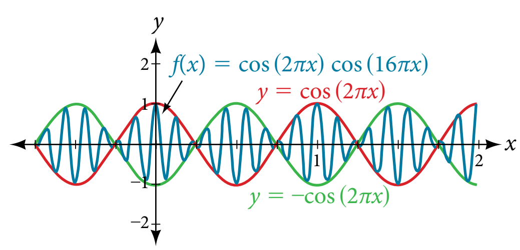


Figure 23

Analysis *The curves $y = \cos(2\pi x)$ and $y = -\cos(2\pi x)$ are bounding curves: they bound the function from above and below, tracing out the high and low points. The harmonic motion graph sits inside the bounding curves. This is an example of a function whose amplitude not only decreases with time, but actually increases and decreases multiple times within a period.*

Access these online resources for additional instruction and practice with trigonometric applications.

- Solving Problems Using Trigonometry (http://openstaxcollege.org/l/solvetrigprob)
- Ferris Wheel Trigonometry (http://openstaxcollege.org/l/ferriswheel)
- Daily Temperatures and Trigonometry (http://openstaxcollege.org/l/dailytemp)
- Simple Harmonic Motion (http://openstaxcollege.org/l/simpleharm)

7.6 SECTION EXERCISES

VERBAL

1. Explain what types of physical phenomena are best modeled by sinusoidal functions. What are the characteristics necessary?

2. What information is necessary to construct a trigonometric model of daily temperature? Give examples of two different sets of information that would enable modeling with an equation.

3. If we want to model cumulative rainfall over the course of a year, would a sinusoidal function be a good model? Why or why not?

4. Explain the effect of a damping factor on the graphs of harmonic motion functions.

ALGEBRAIC

For the following exercises, find a possible formula for the trigonometric function represented by the given table of values.

5.

x	0	3	6	9	12	15	18
y	-4	-1	2	-1	-4	-1	2

6.

x	0	2	4	6	8	10	12
y	5	1	-3	1	5	1	-3

7.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
y	2	7	2	-3	2	7	2

8.

x	0	1	2	3	4	5	6
y	1	-3	-7	-3	1	-3	-7

9.

x	0	1	2	3	4	5	6
y	-2	4	10	4	-2	4	10

10.

x	0	1	2	3	4	5	6
y	5	-3	5	13	5	-3	5

11.

x	-3	-2	-1	0	1	2	3
y	$-1-\sqrt{2}$	-1	$1-\sqrt{2}$	0	$\sqrt{2}-1$	1	$\sqrt{2}+1$

12.

x	-1	0	1	2	3	4	5
y	$\sqrt{3}-2$	0	$2-\sqrt{3}$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$2+\sqrt{3}$

GRAPHICAL

For the following exercises, graph the given function, and then find a possible physical process that the equation could model.

13. $f(x) = -30\cos\left(\frac{x\pi}{6}\right) - 20\cos^2\left(\frac{x\pi}{6}\right) + 80$ $[0, 12]$

14. $f(x) = -18\cos\left(\frac{x\pi}{12}\right) - 5\sin\left(\frac{x\pi}{12}\right) + 100$ on the interval $[0, 24]$

15. $f(x) = 10 - \sin\left(\frac{x\pi}{6}\right) + 24\tan\left(\frac{x\pi}{240}\right)$ on the interval $[0, 80]$

TECHNOLOGY

For the following exercise, construct a function modeling behavior and use a calculator to find desired results.

16. A city's average yearly rainfall is currently 20 inches and varies seasonally by 5 inches. Due to unforeseen circumstances, rainfall appears to be decreasing by 15% each year. How many years from now would we expect rainfall to initially reach 0 inches? Note, the model is invalid once it predicts negative rainfall, so choose the first point at which it goes below 0.

REAL-WORLD APPLICATIONS

For the following exercises, construct a sinusoidal function with the provided information, and then solve the equation for the requested values.

17. Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the high temperature of 105°F occurs at 5PM and the average temperature for the day is 85°F. Find the temperature, to the nearest degree, at 9AM.

18. Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the high temperature of 84°F occurs at 6PM and the average temperature for the day is 70°F. Find the temperature, to the nearest degree, at 7AM.

19. Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the temperature varies between 47°F and 63°F during the day and the average daily temperature first occurs at 10 AM. How many hours after midnight does the temperature first reach 51°F?

20. Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the temperature varies between 64°F and 86°F during the day and the average daily temperature first occurs at 12 AM. How many hours after midnight does the temperature first reach 70°F?

21. A Ferris wheel is 20 meters in diameter and boarded from a platform that is 2 meters above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 6 minutes. How much of the ride, in minutes and seconds, is spent higher than 13 meters above the ground?

22. A Ferris wheel is 45 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. How many minutes of the ride are spent higher than 27 meters above the ground? Round to the nearest second

23. The sea ice area around the North Pole fluctuates between about 6 million square kilometers on September 1 to 14 million square kilometers on March 1. Assuming a sinusoidal fluctuation, when are there less than 9 million square kilometers of sea ice? Give your answer as a range of dates, to the nearest day.

24. The sea ice area around the South Pole fluctuates between about 18 million square kilometers in September to 3 million square kilometers in March. Assuming a sinusoidal fluctuation, when are there more than 15 million square kilometers of sea ice? Give your answer as a range of dates, to the nearest day.

25. During a 90-day monsoon season, daily rainfall can be modeled by sinusoidal functions. If the rainfall fluctuates between a low of 2 inches on day 10 and 12 inches on day 55, during what period is daily rainfall more than 10 inches?

26. During a 90-day monsoon season, daily rainfall can be modeled by sinusoidal functions. A low of 4 inches of rainfall was recorded on day 30, and overall the average daily rainfall was 8 inches. During what period was daily rainfall less than 5 inches?

27. In a certain region, monthly precipitation peaks at 8 inches on June 1 and falls to a low of 1 inch on December 1. Identify the periods when the region is under flood conditions (greater than 7 inches) and drought conditions (less than 2 inches). Give your answer in terms of the nearest day.

28. In a certain region, monthly precipitation peaks at 24 inches in September and falls to a low of 4 inches in March. Identify the periods when the region is under flood conditions (greater than 22 inches) and drought conditions (less than 5 inches). Give your answer in terms of the nearest day.

For the following exercises, find the amplitude, period, and frequency of the given function.

29. The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t) = 8\sin(6\pi t)$,where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

30. The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t) = 11\sin(12\pi t)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

31. The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t) = 4\cos\left(\frac{\pi}{2}t\right)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

For the following exercises, construct an equation that models the described behavior.

32. The displacement $h(t)$, in centimeters, of a mass suspended by a spring is modeled by the function $h(t) = -5\cos(60\pi t)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

For the following exercises, construct an equation that models the described behavior.

33. A deer population oscillates 19 above and below average during the year, reaching the lowest value in January. The average population starts at 800 deer and increases by 160 each year. Find a function that models the population, P, in terms of months since January, t.

34. A rabbit population oscillates 15 above and below average during the year, reaching the lowest value in January. The average population starts at 650 rabbits and increases by 110 each year. Find a function that models the population, P, in terms of months since January, t.

35. A muskrat population oscillates 33 above and below average during the year, reaching the lowest value in January. The average population starts at 900 muskrats and increases by 7% each month. Find a function that models the population, P, in terms of months since January, t.

36. A fish population oscillates 40 above and below average during the year, reaching the lowest value in January. The average population starts at 800 fish and increases by 4% each month. Find a function that models the population, P, in terms of months since January, t.

37. A spring attached to the ceiling is pulled 10 cm down from equilibrium and released. The amplitude decreases by 15% each second. The spring oscillates 18 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

38. A spring attached to the ceiling is pulled 7 cm down from equilibrium and released. The amplitude decreases by 11% each second. The spring oscillates 20 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

39. A spring attached to the ceiling is pulled 17 cm down from equilibrium and released. After 3 seconds, the amplitude has decreased to 13 cm. The spring oscillates 14 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

40. A spring attached to the ceiling is pulled 19 cm down from equilibrium and released. After 4 seconds, the amplitude has decreased to 14 cm. The spring oscillates 13 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

For the following exercises, create a function modeling the described behavior. Then, calculate the desired result using a calculator.

41. A certain lake currently has an average trout population of 20,000. The population naturally oscillates above and below average by 2,000 every year. This year, the lake was opened to fishermen. If fishermen catch 3,000 fish every year, how long will it take for the lake to have no more trout?

42. Whitefish populations are currently at 500 in a lake. The population naturally oscillates above and below by 25 each year. If humans overfish, taking 4% of the population every year, in how many years will the lake first have fewer than 200 whitefish?

43. A spring attached to a ceiling is pulled down 11 cm from equilibrium and released. After 2 seconds, the amplitude has decreased to 6 cm. The spring oscillates 8 times each second. Find when the spring first comes between -0.1 and 0.1 cm, effectively at rest.

44. A spring attached to a ceiling is pulled down 21 cm from equilibrium and released. After 6 seconds, the amplitude has decreased to 4 cm. The spring oscillates 20 times each second. Find when the spring first comes between -0.1 and 0.1 cm, effectively at rest.

45. Two springs are pulled down from the ceiling and released at the same time. The first spring, which oscillates 8 times per second, was initially pulled down 32 cm from equilibrium, and the amplitude decreases by 50% each second. The second spring, oscillating 18 times per second, was initially pulled down 15 cm from equilibrium and after 4 seconds has an amplitude of 2 cm. Which spring comes to rest first, and at what time? Consider "rest" as an amplitude less than 0.1 cm.

46. Two springs are pulled down from the ceiling and released at the same time. The first spring, which oscillates 14 times per second, was initially pulled down 2 cm from equilibrium, and the amplitude decreases by 8% each second. The second spring, oscillating 22 times per second, was initially pulled down 10 cm from equilibrium and after 3 seconds has an amplitude of 2 cm. Which spring comes to rest first, and at what time? Consider "rest" as an amplitude less than 0.1 cm.

EXTENSIONS

47. A plane flies 1 hour at 150 mph at 22° east of north, then continues to fly for 1.5 hours at 120 mph, this time at a bearing of 112° east of north. Find the total distance from the starting point and the direct angle flown north of east.

48. A plane flies 2 hours at 200 mph at a bearing of 60°, then continues to fly for 1.5 hours at the same speed, this time at a bearing of 150°. Find the distance from the starting point and the bearing from the starting point. Hint: bearing is measured counterclockwise from north.

For the following exercises, find a function of the form $y = ab^x + c\sin\left(\frac{\pi}{2}x\right)$ that fits the given data.

49.

x	0	1	2	3
y	6	29	96	379

50.

x	0	1	2	3
y	6	34	150	746

51.

x	0	1	2	3
y	4	0	16	-40

For the following exercises, find a function of the form $y = ab^x\cos\left(\frac{\pi}{2}x\right) + c$ that fits the given data.

52.

x	0	1	2	3
y	11	3	1	3

53.

x	0	1	2	3
y	4	1	-11	1

CHAPTER 7 REVIEW

Key Terms

damped harmonic motion oscillating motion that resembles periodic motion and simple harmonic motion, except that the graph is affected by a damping factor, an energy dissipating influence on the motion, such as friction

double-angle formulas identities derived from the sum formulas for sine, cosine, and tangent in which the angles are equal

even-odd identities set of equations involving trigonometric functions such that if $f(-x) = -f(x)$, the identity is odd, and if $f(-x) = f(x)$, the identity is even

half-angle formulas identities derived from the reduction formulas and used to determine half-angle values of trigonometric functions

product-to-sum formula a trigonometric identity that allows the writing of a product of trigonometric functions as a sum or difference of trigonometric functions

Pythagorean identities set of equations involving trigonometric functions based on the right triangle properties

quotient identities pair of identities based on the fact that tangent is the ratio of sine and cosine, and cotangent is the ratio of cosine and sine

reciprocal identities set of equations involving the reciprocals of basic trigonometric definitions

reduction formulas identities derived from the double-angle formulas and used to reduce the power of a trigonometric function

simple harmonic motion a repetitive motion that can be modeled by periodic sinusoidal oscillation

sum-to-product formula a trigonometric identity that allows, by using substitution, the writing of a sum of trigonometric functions as a product of trigonometric functions

Key Equations

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \cot^2\theta = \csc^2\theta$$
$$1 + \tan^2\theta = \sec^2\theta$$

Even-odd identities

$$\tan(-\theta) = -\tan\theta$$
$$\cot(-\theta) = -\cot\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\csc(-\theta) = -\csc\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\sec(-\theta) = \sec\theta$$

Reciprocal identities

$$\sin\theta = \frac{1}{\csc\theta}$$
$$\cos\theta = \frac{1}{\sec\theta}$$
$$\tan\theta = \frac{1}{\cot\theta}$$
$$\csc\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$

Quotient identities	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
Sum Formula for Cosine	$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
Difference Formula for Cosine	$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
Sum Formula for Sine	$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
Difference Formula for Sine	$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
Sum Formula for Tangent	$\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
Difference Formula for Tangent	$\tan(\alpha-\beta) = \dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$
Cofunction identities	$\sin\theta = \cos\left(\dfrac{\pi}{2}-\theta\right)$ $\cos\theta = \sin\left(\dfrac{\pi}{2}-\theta\right)$ $\tan\theta = \cot\left(\dfrac{\pi}{2}-\theta\right)$ $\cot\theta = \tan\left(\dfrac{\pi}{2}-\theta\right)$ $\sec\theta = \csc\left(\dfrac{\pi}{2}-\theta\right)$ $\csc\theta = \sec\left(\dfrac{\pi}{2}-\theta\right)$
Double-angle formulas	$\sin(2\theta) = 2\sin\theta\cos\theta$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$ $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$
Reduction formulas	$\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$ $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$ $\tan^2\theta = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$

Half-angle formulas	$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$ $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$ $\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$
Product-to-sum Formulas	$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$ $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$ $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$ $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$
Sum-to-product Formulas	$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$ $\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ $\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
Standard form of sinusoidal equation	$y = A\sin(Bt-C)+D$ or $y = A\cos(Bt-C)+D$
Simple harmonic motion	$d = a\cos(\omega t)$ or $d = a\sin(\omega t)$
Damped harmonic motion	$f(t) = ae^{-ct}\sin(\omega t)$ or $f(t) = ae^{-ct}\cos(\omega t)$

Key Concepts

7.1 Solving Trigonometric Equations with Identities

- There are multiple ways to represent a trigonometric expression. Verifying the identities illustrates how expressions can be rewritten to simplify a problem.
- Graphing both sides of an identity will verify it. See **Example 1**.
- Simplifying one side of the equation to equal the other side is another method for verifying an identity. See **Example 2** and **Example 3**.
- The approach to verifying an identity depends on the nature of the identity. It is often useful to begin on the more complex side of the equation. See **Example 4**.
- We can create an identity by simplifying an expression and then verifying it. See **Example 5**.
- Verifying an identity may involve algebra with the fundamental identities. See **Example 6** and **Example 7**.

- Algebraic techniques can be used to simplify trigonometric expressions. We use algebraic techniques throughout this text, as they consist of the fundamental rules of mathematics. See **Example 8**, **Example 9**, and **Example 10**.

7.2 Sum and Difference Identities

- The sum formula for cosines states that the cosine of the sum of two angles equals the product of the cosines of the angles minus the product of the sines of the angles. The difference formula for cosines states that the cosine of the difference of two angles equals the product of the cosines of the angles plus the product of the sines of the angles.
- The sum and difference formulas can be used to find the exact values of the sine, cosine, or tangent of an angle. See **Example 1** and **Example 2**.
- The sum formula for sines states that the sine of the sum of two angles equals the product of the sine of the first angle and cosine of the second angle plus the product of the cosine of the first angle and the sine of the second angle. The difference formula for sines states that the sine of the difference of two angles equals the product of the sine of the first angle and cosine of the second angle minus the product of the cosine of the first angle and the sine of the second angle. See **Example 3**.
- The sum and difference formulas for sine and cosine can also be used for inverse trigonometric functions. See **Example 4**.
- The sum formula for tangent states that the tangent of the sum of two angles equals the sum of the tangents of the angles divided by 1 minus the product of the tangents of the angles. The difference formula for tangent states that the tangent of the difference of two angles equals the difference of the tangents of the angles divided by 1 plus the product of the tangents of the angles. See **Example 5**.
- The Pythagorean Theorem along with the sum and difference formulas can be used to find multiple sums and differences of angles. See **Example 6**.
- The cofunction identities apply to complementary angles and pairs of reciprocal functions. See **Example 7**.
- Sum and difference formulas are useful in verifying identities. See **Example 8** and **Example 9**.
- Application problems are often easier to solve by using sum and difference formulas. See **Example 10** and **Example 11**.

7.3 Double-Angle, Half-Angle, and Reduction Formulas

- Double-angle identities are derived from the sum formulas of the fundamental trigonometric functions: sine, cosine, and tangent. See **Example 1**, **Example 2**, **Example 3**, and **Example 4**.
- Reduction formulas are especially useful in calculus, as they allow us to reduce the power of the trigonometric term. See **Example 5** and **Example 6**.
- Half-angle formulas allow us to find the value of trigonometric functions involving half-angles, whether the original angle is known or not. See **Example 7**, **Example 8**, and **Example 9**.

7.4 Sum-to-Product and Product-to-Sum Formulas

- From the sum and difference identities, we can derive the product-to-sum formulas and the sum-to-product formulas for sine and cosine.
- We can use the product-to-sum formulas to rewrite products of sines, products of cosines, and products of sine and cosine as sums or differences of sines and cosines. See **Example 1**, **Example 2**, and **Example 3**.
- We can also derive the sum-to-product identities from the product-to-sum identities using substitution.
- We can use the sum-to-product formulas to rewrite sum or difference of sines, cosines, or products sine and cosine as products of sines and cosines. See **Example 4**.

- Trigonometric expressions are often simpler to evaluate using the formulas. See **Example 5**.
- The identities can be verified using other formulas or by converting the expressions to sines and cosines. To verify an identity, we choose the more complicated side of the equals sign and rewrite it until it is transformed into the other side. See **Example 6** and **Example 7**.

7.5 Solving Trigonometric Equations

- When solving linear trigonometric equations, we can use algebraic techniques just as we do solving algebraic equations. Look for patterns, like the difference of squares, quadratic form, or an expression that lends itself well to substitution. See **Example 1**, **Example 2**, and **Example 3**.
- Equations involving a single trigonometric function can be solved or verified using the unit circle. See **Example 4**, **Example 5**, and **Example 6**, and **Example 7**.
- We can also solve trigonometric equations using a graphing calculator. See **Example 8** and **Example 9**.
- Many equations appear quadratic in form. We can use substitution to make the equation appear simpler, and then use the same techniques we use solving an algebraic quadratic: factoring, the quadratic formula, etc. See **Example 10**, **Example 11**, **Example 12**, and **Example 13**.
- We can also use the identities to solve trigonometric equation. See **Example 14**, **Example 15**, and **Example 16**.
- We can use substitution to solve a multiple-angle trigonometric equation, which is a compression of a standard trigonometric function. We will need to take the compression into account and verify that we have found all solutions on the given interval. See **Example 17**.
- Real-world scenarios can be modeled and solved using the Pythagorean Theorem and trigonometric functions. See **Example 18**.

7.6 Modeling with Trigonometric Equations

- Sinusoidal functions are represented by the sine and cosine graphs. In standard form, we can find the amplitude, period, and horizontal and vertical shifts. See **Example 1** and **Example 2**.
- Use key points to graph a sinusoidal function. The five key points include the minimum and maximum values and the midline values. See **Example 3**.
- Periodic functions can model events that reoccur in set cycles, like the phases of the moon, the hands on a clock, and the seasons in a year. See **Example 4**, **Example 5**, **Example 6** and **Example 7**.
- Harmonic motion functions are modeled from given data. Similar to periodic motion applications, harmonic motion requires a restoring force. Examples include gravitational force and spring motion activated by weight. See **Example 8**.
- Damped harmonic motion is a form of periodic behavior affected by a damping factor. Energy dissipating factors, like friction, cause the displacement of the object to shrink. See **Example 9**, **Example 10**, **Example 11**, **Example 12**, and **Example 13**.
- Bounding curves delineate the graph of harmonic motion with variable maximum and minimum values. See **Example 14**.

CHAPTER 7 REVIEW EXERCISES

SOLVING TRIGONOMETRIC EQUATIONS WITH IDENTITIES

For the following exercises, find all solutions exactly that exist on the interval $[0, 2\pi)$.

1. $\csc^2 t = 3$

2. $\cos^2 x = \dfrac{1}{4}$

3. $2\sin\theta = -1$

4. $\tan x \sin x + \sin(-x) = 0$

5. $9\sin\omega - 2 = 4\sin^2\omega$

6. $1 - 2\tan(\omega) = \tan^2(\omega)$

For the following exercises, use basic identities to simplify the expression.

7. $\sec x \cos x + \cos x - \dfrac{1}{\sec x}$

8. $\sin^3 x + \cos^2 x \sin x$

For the following exercises, determine if the given identities are equivalent.

9. $\sin^2 x + \sec^2 x - 1 = \dfrac{(1 - \cos^2 x)(1 + \cos^2 x)}{\cos^2 x}$

10. $\tan^3 x \csc^2 x \cot^2 x \cos x \sin x = 1$

SUM AND DIFFERENCE IDENTITIES

For the following exercises, find the exact value.

11. $\tan\left(\dfrac{7\pi}{12}\right)$

12. $\cos\left(\dfrac{25\pi}{12}\right)$

13. $\sin(70°)\cos(25°) - \cos(70°)\sin(25°)$

14. $\cos(83°)\cos(23°) + \sin(83°)\sin(23°)$

For the following exercises, prove the identity.

15. $\cos(4x) - \cos(3x)\cos x = \sin^2 x - 4\cos^2 x \sin^2 x$

16. $\cos(3x) - \cos^3 x = -\cos x \sin^2 x - \sin x \sin(2x)$

For the following exercise, simplify the expression.

17. $\dfrac{\tan\left(\frac{1}{2}x\right) + \tan\left(\frac{1}{8}x\right)}{1 - \tan\left(\frac{1}{8}x\right)\tan\left(\frac{1}{2}x\right)}$

For the following exercises, find the exact value.

18. $\cos\left(\sin^{-1}(0) - \cos^{-1}\left(\dfrac{1}{2}\right)\right)$

19. $\tan\left(\sin^{-1}(0) + \sin^{-1}\left(\dfrac{1}{2}\right)\right)$

DOUBLE-ANGLE, HALF-ANGLE, AND REDUCTION FORMULAS

For the following exercises, find the exact value.

20. Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\cos\theta = -\dfrac{1}{3}$ and θ is in the interval $\left[\dfrac{\pi}{2}, \pi\right]$

21. Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\sec\theta = -\dfrac{5}{3}$ and θ is in the interval $\left[\dfrac{\pi}{2}, \pi\right]$

22. $\sin\left(\dfrac{7\pi}{8}\right)$

23. $\sec\left(\dfrac{3\pi}{8}\right)$

For the following exercises, use **Figure 1** to find the desired quantities.

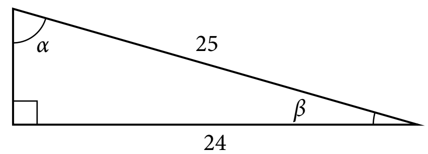


Figure 1

24. $\sin(2\beta)$, $\cos(2\beta)$, $\tan(2\beta)$, $\sin(2\alpha)$, $\cos(2\alpha)$, and $\tan(2\alpha)$

25. $\sin\left(\frac{\beta}{2}\right)$, $\cos\left(\frac{\beta}{2}\right)$, $\tan\left(\frac{\beta}{2}\right)$, $\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, and $\tan\left(\frac{\alpha}{2}\right)$

For the following exercises, prove the identity.

26. $\frac{2\cos(2x)}{\sin(2x)} = \cot x - \tan x$

27. $\cot x \cos(2x) = -\sin(2x) + \cot x$

For the following exercises, rewrite the expression with no powers.

28. $\cos^2 x \sin^4(2x)$

29. $\tan^2 x \sin^3 x$

SUM-TO-PRODUCT AND PRODUCT-TO-SUM FORMULAS

For the following exercises, evaluate the product for the given expression using a sum or difference of two functions. Write the exact answer.

30. $\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

31. $2\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{5\pi}{6}\right)$

32. $2\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{3}\right)$

For the following exercises, evaluate the sum by using a product formula. Write the exact answer.

33. $\sin\left(\frac{\pi}{12}\right) - \sin\left(\frac{7\pi}{12}\right)$

34. $\cos\left(\frac{5\pi}{12}\right) + \cos\left(\frac{7\pi}{12}\right)$

For the following exercises, change the functions from a product to a sum or a sum to a product.

35. $\sin(9x)\cos(3x)$

36. $\cos(7x)\cos(12x)$

37. $\sin(11x) + \sin(2x)$

38. $\cos(6x) + \cos(5x)$

SOLVING TRIGONOMETRIC EQUATIONS

For the following exercises, find all exact solutions on the interval $[0, 2\pi)$.

39. $\tan x + 1 = 0$

40. $2\sin(2x) + \sqrt{2} = 0$

For the following exercises, find all exact solutions on the interval $[0, 2\pi)$.

41. $2\sin^2 x - \sin x = 0$

42. $\cos^2 x - \cos x - 1 = 0$

43. $2\sin^2 x + 5\sin x + 3 = 0$

44. $\cos x - 5\sin(2x) = 0$

45. $\frac{1}{\sec^2 x} + 2 + \sin^2 x + 4\cos^2 x = 0$

For the following exercises, simplify the equation algebraically as much as possible. Then use a calculator to find the solutions on the interval $[0, 2\pi)$. Round to four decimal places.

46. $\sqrt{3}\cot^2 x + \cot x = 1$

47. $\csc^2 x - 3\csc x - 4 = 0$

For the following exercises, graph each side of the equation to find the zeroes on the interval $[0, 2\pi)$.

48. $20\cos^2 x + 21\cos x + 1 = 0$

49. $\sec^2 x - 2\sec x = 15$

MODELING WITH TRIGONOMETRIC EQUATIONS

For the following exercises, graph the points and find a possible formula for the trigonometric values in the given table.

50.

x	0	1	2	3	4	5
y	1	6	11	6	1	6

51.

x	0	1	2	3	4	5
y	-2	1	-2	-5	-2	1

52.

x	-3	-2	-1	0	1	2	3
y	$3+2\sqrt{2}$	3	$2\sqrt{2}-1$	1	$3-2\sqrt{2}$	-1	$-1-2\sqrt{2}$

53. A man with his eye level 6 feet above the ground is standing 3 feet away from the base of a 15-foot vertical ladder. If he looks to the top of the ladder, at what angle above horizontal is he looking?

54. Using the ladder from the previous exercise, if a 6-foot-tall construction worker standing at the top of the ladder looks down at the feet of the man standing at the bottom, what angle from the horizontal is he looking?

For the following exercises, construct functions that model the described behavior.

55. A population of lemmings varies with a yearly low of 500 in March. If the average yearly population of lemmings is 950, write a function that models the population with respect to t, the month.

56. Daily temperatures in the desert can be very extreme. If the temperature varies from 90°F to 30°F and the average daily temperature first occurs at 10 AM, write a function modeling this behavior.

For the following exercises, find the amplitude, frequency, and period of the given equations.

57. $y = 3\cos(x\pi)$

58. $y = -2\sin(16x\pi)$

For the following exercises, model the described behavior and find requested values.

59. An invasive species of carp is introduced to Lake Freshwater. Initially there are 100 carp in the lake and the population varies by 20 fish seasonally. If by year 5, there are 625 carp, find a function modeling the population of carp with respect to t, the number of years from now.

60. The native fish population of Lake Freshwater averages 2500 fish, varying by 100 fish seasonally. Due to competition for resources from the invasive carp, the native fish population is expected to decrease by 5% each year. Find a function modeling the population of native fish with respect to t, the number of years from now. Also determine how many years it will take for the carp to overtake the native fish population.

CHAPTER 7 PRACTICE TEST

For the following exercises, simplify the given expression.

1. $\cos(-x)\sin x \cot x + \sin^2 x$

2. $\sin(-x)\cos(-2x) - \sin(-x)\cos(-2x)$

For the following exercises, find the exact value.

3. $\cos\left(\frac{7\pi}{12}\right)$

4. $\tan\left(\frac{3\pi}{8}\right)$

5. $\tan\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + \tan^{-1}\sqrt{3}\right)$

6. $2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$

For the following exercises, find all exact solutions to the equation on $[0, 2\pi)$.

7. $\cos^2 x - \sin^2 x - 1 = 0$

8. $\cos^2 x = \cos x\ 4\sin^2 x + 2\sin x - 3 = 0$

9. $\cos(2x) + \sin^2 x = 0$

10. $2\sin^2 x - \sin x = 0$

11. Rewrite the expression as a product instead of a sum: $\cos(2x) + \cos(-8x)$.

12. Find all solutions of $\tan(x) - \sqrt{3} = 0$.

13. Find the solutions of $\sec^2 x - 2\sec x = 15$ on the interval $[0, 2\pi)$ algebraically; then graph both sides of the equation to determine the answer.

14. Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\cot\theta = -\frac{3}{4}$ and θ is on the interval $\left[\frac{\pi}{2}, \pi\right]$.

15. Find $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$ given $\cos\theta = \frac{7}{25}$ and θ is in quadrant IV.

16. Rewrite the expression $\sin^4 x$ with no powers greater than 1.

For the following exercises, prove the identity.

17. $\tan^3 x - \tan x \sec^2 x = \tan(-x)$

18. $\sin(3x) - \cos x \sin(2x) = \cos^2 x \sin x - \sin^3 x$

19. $\frac{\sin(2x)}{\sin x} - \frac{\cos(2x)}{\cos x} = \sec x$

20. Plot the points and find a function of the form $y = A\cos(Bx + C) + D$ that fits the given data.

x	0	1	2	3	4	5
y	−2	2	−2	2	−2	2

21. The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t) = \frac{1}{4}\sin(120\pi t)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

22. A woman is standing 300 feet away from a 2,000-foot building. If she looks to the top of the building, at what angle above horizontal is she looking? A bored worker looks down at her from the 15th floor (1500 feet above her). At what angle is he looking down at her? Round to the nearest tenth of a degree.

23. Two frequencies of sound are played on an instrument governed by the equation $n(t) = 8\cos(20\pi t)\cos(1{,}000\pi t)$. What are the period and frequency of the "fast" and "slow" oscillations? What is the amplitude?

24. The average monthly snowfall in a small village in the Himalayas is 6 inches, with the low of 1 inch occurring in July. Construct a function that models this behavior. During what period is there more than 10 inches of snowfall?

25. A spring attached to a ceiling is pulled down 20 cm. After 3 seconds, wherein it completes 6 full periods, the amplitude is only 15 cm. Find the function modeling the position of the spring t seconds after being released. At what time will the spring come to rest? In this case, use 1 cm amplitude as rest.

26. Water levels near a glacier currently average 9 feet, varying seasonally by 2 inches above and below the average and reaching their highest point in January. Due to global warming, the glacier has begun melting faster than normal. Every year, the water levels rise by a steady 3 inches. Find a function modeling the depth of the water t months from now. If the docks are 2 feet above current water levels, at what point will the water first rise above the docks?

8

Further Applications of Trigonometry

Figure 1 General Sherman, the world's largest living tree. (credit: Mike Baird, Flickr)

CHAPTER OUTLINE

Introduction

The world's largest tree by volume, named General Sherman, stands 274.9 feet tall and resides in Northern California.[27] Just how do scientists know its true height? A common way to measure the height involves determining the angle of elevation, which is formed by the tree and the ground at a point some distance away from the base of the tree. This method is much more practical than climbing the tree and dropping a very long tape measure.

In this chapter, we will explore applications of trigonometry that will enable us to solve many different kinds of problems, including finding the height of a tree. We extend topics we introduced in **Trigonometric Functions** and investigate applications more deeply and meaningfully.

27 Source: National Park Service. "The General Sherman Tree." http://www.nps.gov/seki/naturescience/sherman.htm. Accessed April 25, 2014.

LEARNING OBJECTIVES

In this section, you will:

- Use the Law of Sines to solve oblique triangles.
- Find the area of an oblique triangle using the sine function.
- Solve applied problems using the Law of Sines.

8.1 NON-RIGHT TRIANGLES: LAW OF SINES

Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 35 degrees, whereas the angle of elevation measured by the second station is 15 degrees. How can we determine the altitude of the aircraft? We see in **Figure 1** that the triangle formed by the aircraft and the two stations is not a right triangle, so we cannot use what we know about right triangles. In this section, we will find out how to solve problems involving non-right triangles.

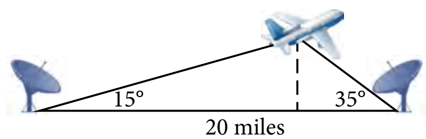


Figure 1

Using the Law of Sines to Solve Oblique Triangles

In any triangle, we can draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

Any triangle that is not a right triangle is an **oblique triangle**. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

1. **ASA (angle-side-angle)** We know the measurements of two angles and the included side. See **Figure 2**.

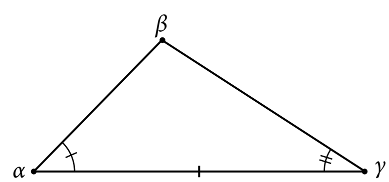


Figure 2

2. **AAS (angle-angle-side)** We know the measurements of two angles and a side that is not between the known angles. See **Figure 3**.

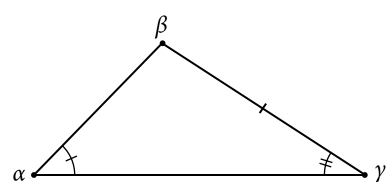


Figure 3

3. **SSA (side-side-angle)** We know the measurements of two sides and an angle that is not between the known sides. See **Figure 4**.

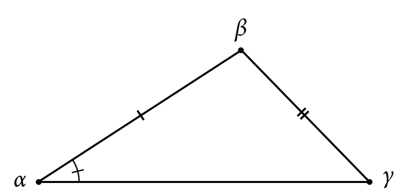


Figure 4

Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let's see how this statement is derived by considering the triangle shown in **Figure 5**.

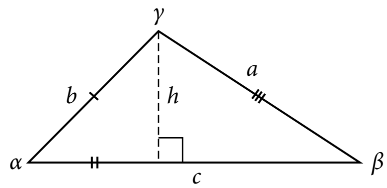


Figure 5

Using the right triangle relationships, we know that $\sin\alpha = \frac{h}{b}$ and $\sin\beta = \frac{h}{a}$. Solving both equations for h gives two different expressions for h.

$$h = b\sin\alpha \text{ and } h = a\sin\beta$$

We then set the expressions equal to each other.

$$b\sin\alpha = a\sin\beta$$

$$\left(\frac{1}{ab}\right)(b\sin\alpha) = (a\sin\beta)\left(\frac{1}{ab}\right) \qquad \text{Multiply both sides by } \frac{1}{ab}.$$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

Similarly, we can compare the other ratios.

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \text{ and } \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Collectively, these relationships are called the **Law of Sines.**

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Note the standard way of labeling triangles: angle α (alpha) is opposite side a; angle β (beta) is opposite side b; and angle γ (gamma) is opposite side c. See **Figure 6**.

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.

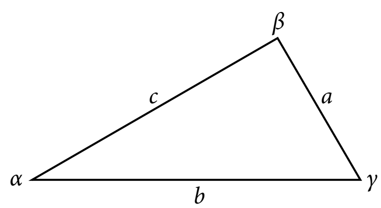


Figure 6

Law of Sines

Given a triangle with angles and opposite sides labeled as in **Figure 6**, the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal. The **Law of Sines** is based on proportions and is presented symbolically two ways.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

To solve an oblique triangle, use any pair of applicable ratios.

Example 1 **Solving for Two Unknown Sides and Angle of an AAS Triangle**

Solve the triangle shown in **Figure 7** to the nearest tenth.

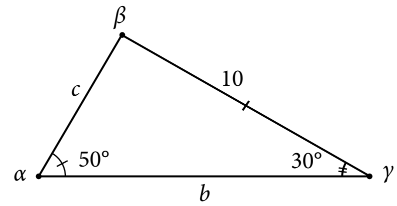


Figure 7

Solution The three angles must add up to 180 degrees. From this, we can determine that

$$\begin{aligned}\beta &= 180° - 50° - 30° \\ &= 100°\end{aligned}$$

To find an unknown side, we need to know the corresponding angle and a known ratio. We know that angle $\alpha = 50°$ and its corresponding side $a = 10$. We can use the following proportion from the Law of Sines to find the length of c.

$$\frac{\sin(50°)}{10} = \frac{\sin(30°)}{c}$$

$$c\frac{\sin(50°)}{10} = \sin(30°) \qquad \text{Multiply both sides by } c.$$

$$c = \sin(30°)\frac{10}{\sin(50°)} \qquad \text{Multiply by the reciprocal to isolate } c.$$

$$c \approx 6.5$$

Similarly, to solve for b, we set up another proportion.

$$\frac{\sin(50°)}{10} = \frac{\sin(100°)}{b}$$

$$b\sin(50°) = 10\sin(100°) \qquad \text{Multiply both sides by } b.$$

$$b = \frac{10\sin(100°)}{\sin(50°)} \qquad \text{Multiply by the reciprocal to isolate } b.$$

$$b \approx 12.9$$

Therefore, the complete set of angles and sides is

$$\begin{aligned}\alpha &= 50° & a &= 10 \\ \beta &= 100° & b &\approx 12.9 \\ \gamma &= 30° & c &\approx 6.5\end{aligned}$$

Try It #1

Solve the triangle shown in **Figure 8** to the nearest tenth.

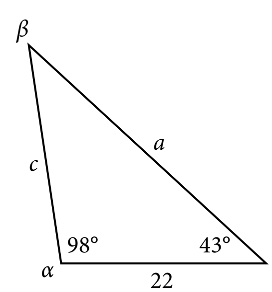


Figure 8

Using The Law of Sines to Solve SSA Triangles

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an **ambiguous case**. Triangles classified as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.

possible outcomes for SSA triangles

Oblique triangles in the category SSA may have four different outcomes. **Figure 9** illustrates the solutions with the known sides a and b and known angle α.

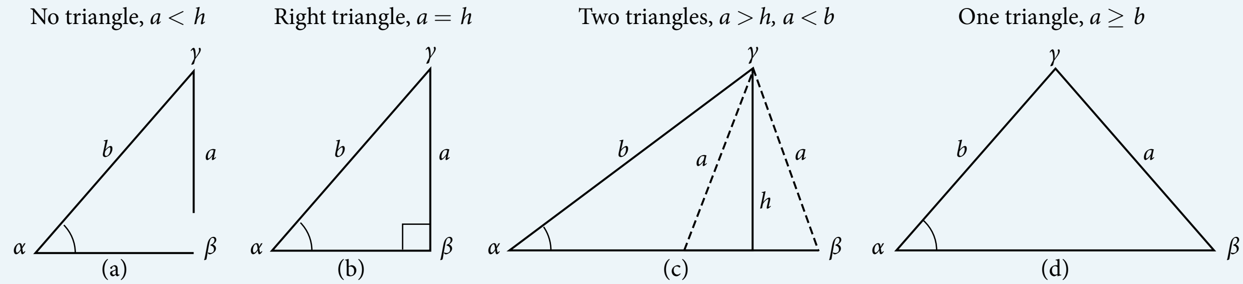


Figure 9

Example 2 Solving an Oblique SSA Triangle

Solve the triangle in **Figure 10** for the missing side and find the missing angle measures to the nearest tenth.

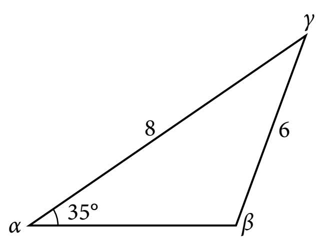


Figure 10

Solution Use the Law of Sines to find angle β and angle γ, and then side c. Solving for β, we have the proportion

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$
$$\frac{\sin(35°)}{6} = \frac{\sin\beta}{8}$$
$$\frac{8\sin(35°)}{6} = \sin\beta$$
$$0.7648 \approx \sin\beta$$
$$\sin^{-1}(0.7648) \approx 49.9°$$
$$\beta \approx 49.9°$$

However, in the diagram, angle β appears to be an obtuse angle and may be greater than 90°. How did we get an acute angle, and how do we find the measurement of β? Let's investigate further. Dropping a perpendicular from γ and viewing the triangle from a right angle perspective, we have **Figure 11**. It appears that there may be a second triangle that will fit the given criteria.

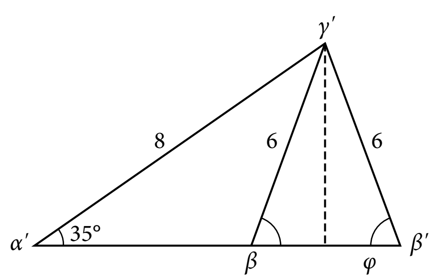


Figure 11

The angle supplementary to β is approximately equal to 49.9°, which means that $\beta = 180° - 49.9° = 130.1°$. (Remember that the sine function is positive in both the first and second quadrants.) Solving for γ, we have

$$\gamma = 180° - 35° - 130.1° \approx 14.9°$$

We can then use these measurements to solve the other triangle. Since γ' is supplementary to α and β, we have

$$\gamma' = 180° - 35° - 49.9° \approx 95.1°$$

Now we need to find c and c'.

We have

$$\frac{c}{\sin(14.9°)} = \frac{6}{\sin(35°)}$$
$$c = \frac{6\sin(14.9°)}{\sin(35°)} \approx 2.7$$

Finally,

$$\frac{c'}{\sin(95.1°)} = \frac{6}{\sin(35°)}$$
$$c' = \frac{6\sin(95.1°)}{\sin(35°)} \approx 10.4$$

To summarize, there are two triangles with an angle of 35°, an adjacent side of 8, and an opposite side of 6, as shown in **Figure 12**.

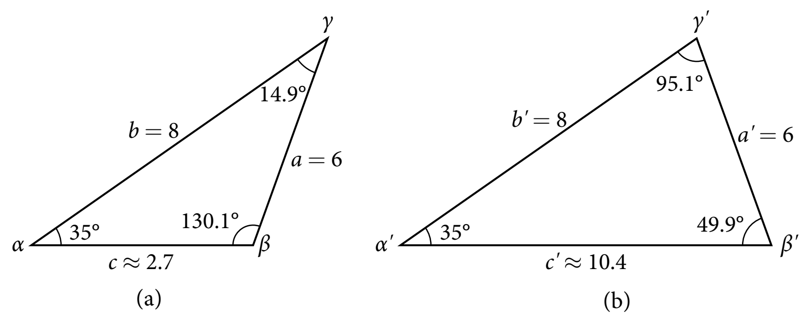


Figure 12

However, we were looking for the values for the triangle with an obtuse angle β. We can see them in the first triangle (a) in **Figure 12**.

Try It #2

Given $\alpha = 80°$, $a = 120$, and $b = 121$, find the missing side and angles. If there is more than one possible solution, show both.

Example 3 Solving for the Unknown Sides and Angles of a SSA Triangle

In the triangle shown in **Figure 13**, solve for the unknown side and angles. Round your answers to the nearest tenth.

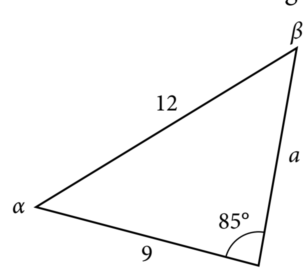


Figure 13

Solution In choosing the pair of ratios from the Law of Sines to use, look at the information given. In this case, we know the angle $\gamma = 85°$, and its corresponding side $c = 12$, and we know side $b = 9$. We will use this proportion to solve for β.

$$\frac{\sin(85°)}{12} = \frac{\sin \beta}{9} \quad \text{Isolate the unknown.}$$
$$\frac{9\sin(85°)}{12} = \sin \beta$$

To find β, apply the inverse sine function. The inverse sine will produce a single result, but keep in mind that there may be two values for β. It is important to verify the result, as there may be two viable solutions, only one solution (the usual case), or no solutions.

$$\beta = \sin^{-1}\left(\frac{9\sin(85°)}{12}\right)$$
$$\beta \approx \sin^{-1}(0.7471)$$
$$\beta \approx 48.3°$$

In this case, if we subtract β from 180°, we find that there may be a second possible solution. Thus, $\beta = 180° - 48.3° \approx 131.7°$. To check the solution, subtract both angles, 131.7° and 85°, from 180°. This gives

$$\alpha = 180° - 85° - 131.7° \approx -36.7°,$$

which is impossible, and so $\beta \approx 48.3°$.

To find the remaining missing values, we calculate $\alpha = 180° - 85° - 48.3° \approx 46.7°$. Now, only side a is needed. Use the Law of Sines to solve for a by one of the proportions.

$$\frac{\sin(85°)}{12} = \frac{\sin(46.7°)}{a}$$
$$a\frac{\sin(85°)}{12} = \sin(46.7°)$$
$$a = \frac{12\sin(46.7°)}{\sin(85°)} \approx 8.8$$

The complete set of solutions for the given triangle is

$$\alpha \approx 46.7° \qquad a \approx 8.8$$
$$\beta \approx 48.3° \qquad b = 9$$
$$\gamma = 85° \qquad c = 12$$

Try It #3

Given $\alpha = 80°$, $a = 100$, $b = 10$, find the missing side and angles. If there is more than one possible solution, show both. Round your answers to the nearest tenth.

Example 4 Finding the Triangles That Meet the Given Criteria

Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.

Solution Using the given information, we can solve for the angle opposite the side of length 10. See **Figure 14**.

$$\frac{\sin\alpha}{10} = \frac{\sin(50°)}{4}$$
$$\sin\alpha = \frac{10\sin(50°)}{4}$$
$$\sin\alpha \approx 1.915$$

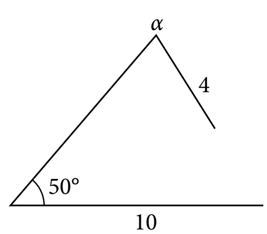


Figure 14

We can stop here without finding the value of α. Because the range of the sine function is $[-1, 1]$, it is impossible for the sine value to be 1.915. In fact, inputting $\sin^{-1}(1.915)$ in a graphing calculator generates an **ERROR DOMAIN**. Therefore, no triangles can be drawn with the provided dimensions.

Try It #4

Determine the number of triangles possible given $a = 31$, $b = 26$, $\beta = 48°$.

Finding the Area of an Oblique Triangle Using the Sine Function

Now that we can solve a triangle for missing values, we can use some of those values and the sine function to find the area of an oblique triangle. Recall that the area formula for a triangle is given as Area $= \frac{1}{2}bh$, where b is base and h is height. For oblique triangles, we must find h before we can use the area formula. Observing the two triangles in **Figure 15**, one acute and one obtuse, we can drop a perpendicular to represent the height and then apply the trigonometric property $\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}}$ to write an equation for area in oblique triangles. In the acute triangle, we have $\sin \alpha = \frac{h}{c}$ or $c\sin \alpha = h$. However, in the obtuse triangle, we drop the perpendicular outside the triangle and extend the base b to form a right triangle. The angle used in calculation is α', or $180 - \alpha$.

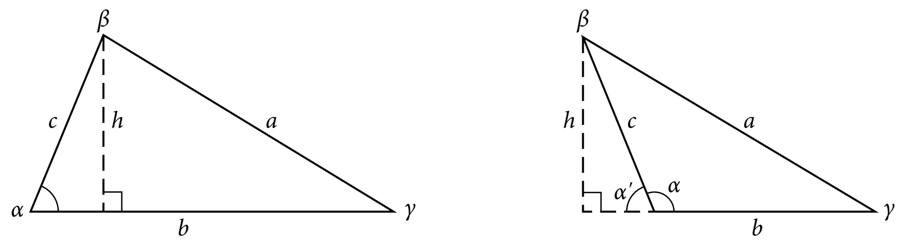


Figure 15

Thus,

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}b(c\sin \alpha)$$

Similarly,

$$\text{Area} = \frac{1}{2}a(b\sin \gamma) = \frac{1}{2}a(c\sin \beta)$$

area of an oblique triangle

The formula for the area of an oblique triangle is given by

$$\begin{aligned}\text{Area} &= \frac{1}{2}bc\sin \alpha \\ &= \frac{1}{2}ac\sin \beta \\ &= \frac{1}{2}ab\sin \gamma\end{aligned}$$

This is equivalent to one-half of the product of two sides and the sine of their included angle.

Example 5 Finding the Area of an Oblique Triangle

Find the area of a triangle with sides $a = 90$, $b = 52$, and angle $\gamma = 102°$. Round the area to the nearest integer.

Solution Using the formula, we have

$$\text{Area} = \frac{1}{2}ab\sin \gamma$$

$$\text{Area} = \frac{1}{2}(90)(52)\sin(102°)$$

$$\text{Area} \approx 2289 \text{ square units}$$

Try It #5

Find the area of the triangle given $\beta = 42°$, $a = 7.2$ ft, $c = 3.4$ ft. Round the area to the nearest tenth.

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

Example 6 Finding an Altitude

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in **Figure 16**. Round the altitude to the nearest tenth of a mile.

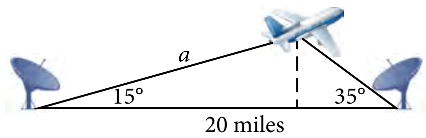


Figure 16

Solution To find the elevation of the aircraft, we first find the distance from one station to the aircraft, such as the side *a*, and then use right triangle relationships to find the height of the aircraft, *h*.

Because the angles in the triangle add up to 180 degrees, the unknown angle must be $180° - 15° - 35° = 130°$. This angle is opposite the side of length 20, allowing us to set up a Law of Sines relationship.

$$\frac{\sin(130°)}{20} = \frac{\sin(35°)}{a}$$

$$a\sin(130°) = 20\sin(35°)$$

$$a = \frac{20\sin(35°)}{\sin(130°)}$$

$$a \approx 14.98$$

The distance from one station to the aircraft is about 14.98 miles.

Now that we know *a*, we can use right triangle relationships to solve for *h*.

$$\sin(15°) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin(15°) = \frac{h}{a}$$

$$\sin(15°) = \frac{h}{14.98}$$

$$h = 14.98\sin(15°)$$

$$h \approx 3.88$$

The aircraft is at an altitude of approximately 3.9 miles.

Try It #6

The diagram shown in **Figure 17** represents the height of a blimp flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point *A*, is 70°, the angle of elevation from the northern end zone, point *B*, is 62°, and the distance between the viewing points of the two end zones is 145 yards.

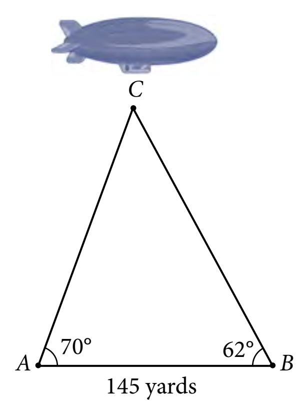


Figure 17

Access the following online resources for additional instruction and practice with trigonometric applications.

- Law of Sines: The Basics (http://openstaxcollege.org/l/sinesbasic)
- Law of Sines: The Ambiguous Case (http://openstaxcollege.org/l/sinesambiguous)

8.1 SECTION EXERCISES

VERBAL

1. Describe the altitude of a triangle.

2. Compare right triangles and oblique triangles.

3. When can you use the Law of Sines to find a missing angle?

4. In the Law of Sines, what is the relationship between the angle in the numerator and the side in the denominator?

5. What type of triangle results in an ambiguous case?

ALGEBRAIC

For the following exercises, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve each triangle, if possible. Round each answer to the nearest tenth.

6. $\alpha = 43°, \gamma = 69°, a = 20$

7. $\alpha = 35°, \gamma = 73°, c = 20$

8. $\alpha = 60°, \beta = 60°, \gamma = 60°$

9. $a = 4, \alpha = 60°, \beta = 100°$

10. $b = 10, \beta = 95°, \gamma = 30°$

For the following exercises, use the Law of Sines to solve for the missing side for each oblique triangle. Round each answer to the nearest hundredth. Assume that angle A is opposite side a, angle B is opposite side b, and angle C is opposite side c.

11. Find side b when $A = 37°, B = 49°, c = 5$.

12. Find side a when $A = 132°, C = 23°, b = 10$.

13. Find side c when $B = 37°, C = 21, b = 23$.

For the following exercises, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Determine whether there is no triangle, one triangle, or two triangles. Then solve each triangle, if possible. Round each answer to the nearest tenth.

14. $\alpha = 119°, a = 14, b = 26$

15. $\gamma = 113°, b = 10, c = 32$

16. $b = 3.5, c = 5.3, \gamma = 80°$

17. $a = 12, c = 17, \alpha = 35°$

18. $a = 20.5, b = 35.0, \beta = 25°$

19. $a = 7, c = 9, \alpha = 43°$

20. $a = 7, b = 3, \beta = 24°$

21. $b = 13, c = 5, \gamma = 10°$

22. $a = 2.3, c = 1.8, \gamma = 28°$

23. $\beta = 119°, b = 8.2, a = 11.3$

For the following exercises, use the Law of Sines to solve, if possible, the missing side or angle for each triangle or triangles in the ambiguous case. Round each answer to the nearest tenth.

24. Find angle A when $a = 24, b = 5, B = 22°$.

25. Find angle A when $a = 13, b = 6, B = 20°$.

26. Find angle B when $A = 12°, a = 2, b = 9$.

For the following exercises, find the area of the triangle with the given measurements. Round each answer to the nearest tenth.

27. $a = 5, c = 6, \beta = 35°$

28. $b = 11, c = 8, \alpha = 28°$

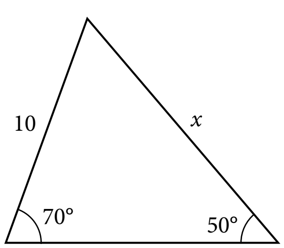
29. $a = 32, b = 24, \gamma = 75°$

30. $a = 7.2, b = 4.5, \gamma = 43°$

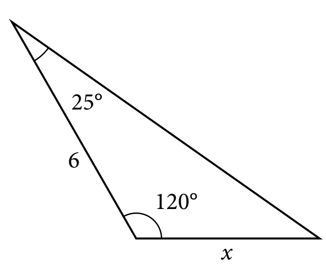
GRAPHICAL

For the following exercises, find the length of side x. Round to the nearest tenth.

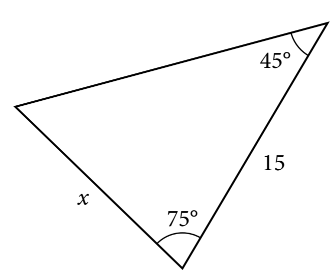
31.



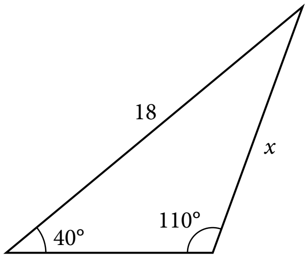
32.



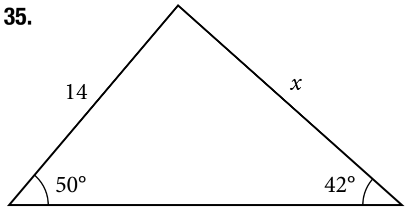
33.



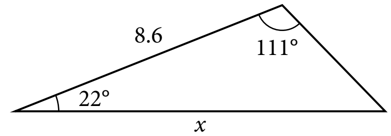
34.



35.

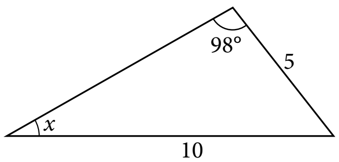


36.

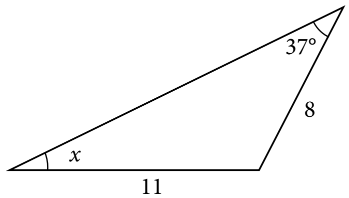


For the following exercises, find the measure of angle *x*, if possible. Round to the nearest tenth.

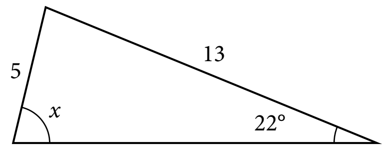
37.



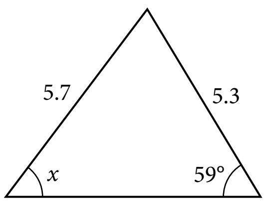
38.



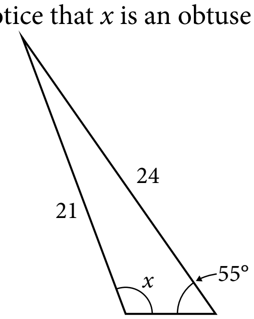
39.



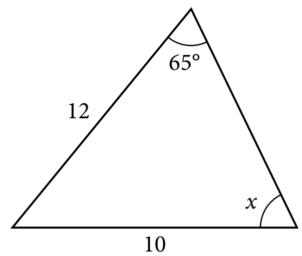
40.



41. Notice that *x* is an obtuse angle.

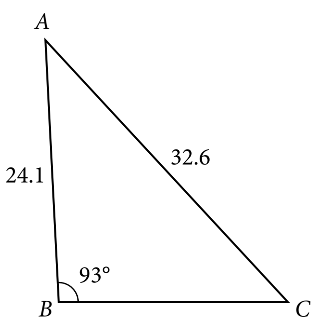


42.



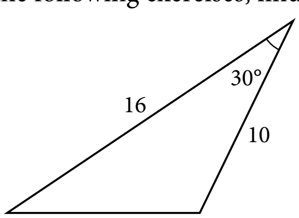
For the following exercises, solve the triangle. Round each answer to the nearest tenth.

43.

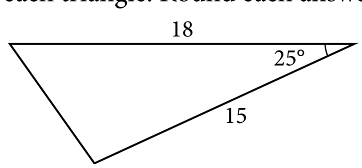


For the following exercises, find the area of each triangle. Round each answer to the nearest tenth.

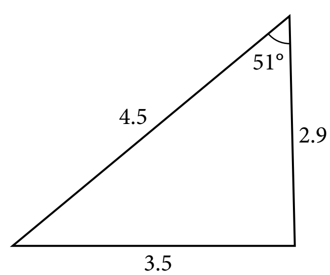
44.



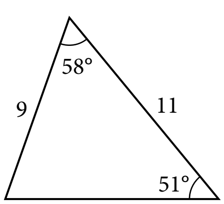
45.



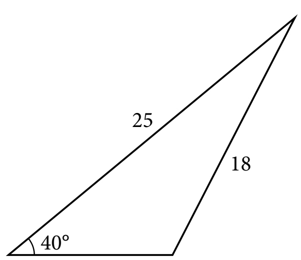
46.



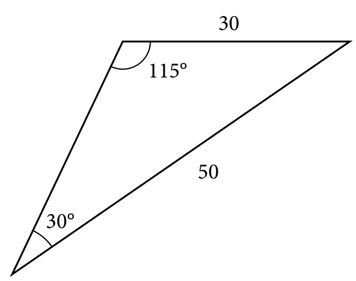
47.



48.



49.



EXTENSIONS

50. Find the radius of the circle in **Figure 18**. Round to the nearest tenth.

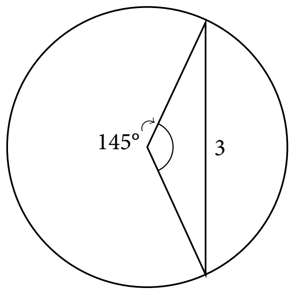


Figure 18

51. Find the diameter of the circle in **Figure 19**. Round to the nearest tenth.

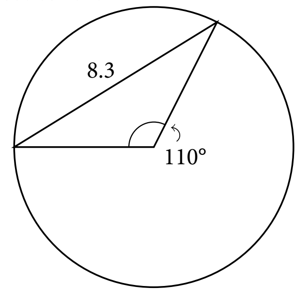


Figure 19

52. Find $m\angle ADC$ in **Figure 20**. Round to the nearest tenth.

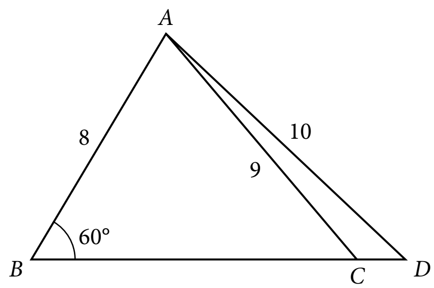


Figure 20

53. Find AD in **Figure 21**. Round to the nearest tenth.

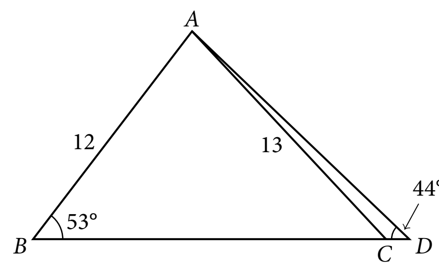


Figure 21

54. Solve both triangles in **Figure 22**. Round each answer to the nearest tenth.

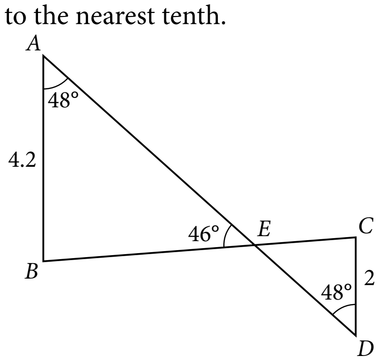


Figure 22

55. Find AB in the parallelogram shown in **Figure 23**.

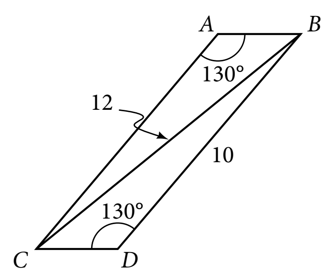


Figure 23

56. Solve the triangle in **Figure 24**. (Hint: Draw a perpendicular from H to JK). Round each answer to the nearest tenth.

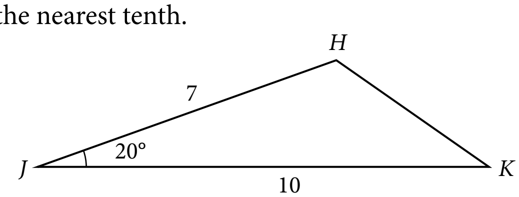


Figure 24

57. Solve the triangle in **Figure 25**. (Hint: Draw a perpendicular from N to LM). Round each answer to the nearest tenth.

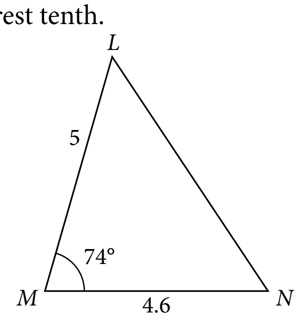


Figure 25

58. In **Figure 26**, *ABCD* is not a parallelogram. $\angle m$ is obtuse. Solve both triangles. Round each answer to the nearest tenth.

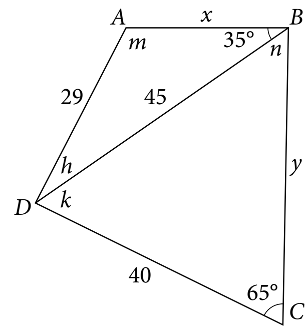


Figure 26

REAL-WORLD APPLICATIONS

59. A pole leans away from the sun at an angle of 7° to the vertical, as shown in **Figure 27**. When the elevation of the sun is 55°, the pole casts a shadow 42 feet long on the level ground. How long is the pole? Round the answer to the nearest tenth.

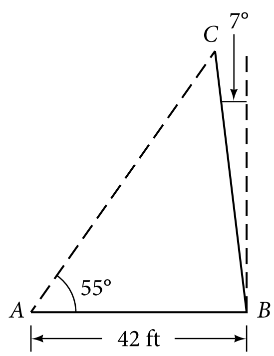


Figure 27

60. To determine how far a boat is from shore, two radar stations 500 feet apart find the angles out to the boat, as shown in **Figure 28**. Determine the distance of the boat from station *A* and the distance of the boat from shore. Round your answers to the nearest whole foot.

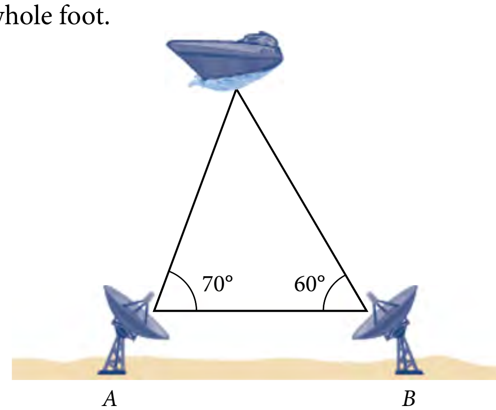


Figure 28

61. **Figure 29** shows *a* satellite orbiting Earth. The satellite passes directly over two tracking stations *A* and *B,* which are 69 miles apart. When the satellite is on one side of the two stations, the angles of elevation at *A* and *B* are measured to be 86.2° and 83.9°, respectively. How far is the satellite from station *A* and how high is the satellite above the ground? Round answers to the nearest whole mile.

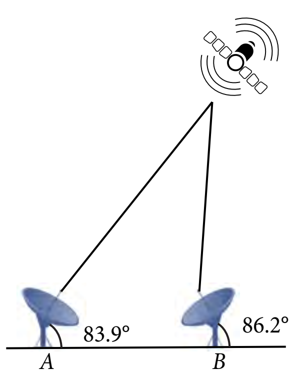


Figure 29

62. A communications tower is located at the top of *a* steep hill, as shown in **Figure 30**. The angle of inclination of the hill is 67°. *A* guy wire is to be attached to the top of the tower and to the ground, 165 meters downhill from the base of the tower. The angle formed by the guy wire and the hill is 16°. Find the length of the cable required for the guy wire to the nearest whole meter.

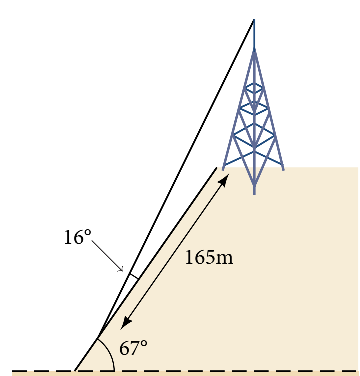


Figure 30

63. The roof of a house is at a 20° angle. An 8-foot solar panel is to be mounted on the roof and should be angled 38° relative to the horizontal for optimal results. (See **Figure 31**). How long does the vertical support holding up the back of the panel need to be? Round to the nearest tenth.

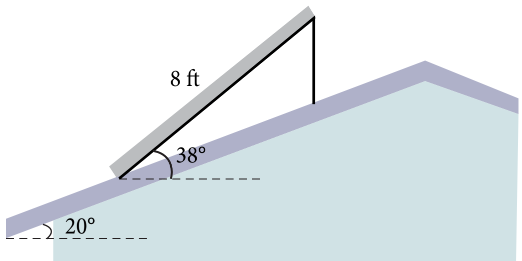


Figure 31

64. Similar to an angle of elevation, an *angle of depression* is the acute angle formed by a horizontal line and an observer's line of sight to an object below the horizontal. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 6.6 km apart, to be 37° and 44°, as shown in **Figure 32**. Find the distance of the plane from point *A* to the nearest tenth of a kilometer.

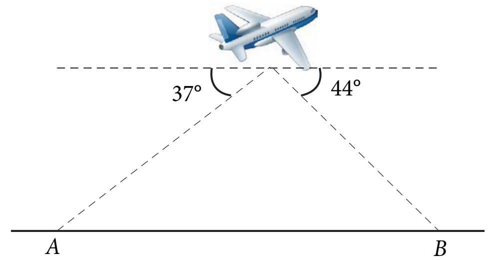


Figure 32

65. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 4.3 km apart, to be 32° and 56°, as shown in **Figure 33**. Find the distance of the plane from point *A* to the nearest tenth of a kilometer.

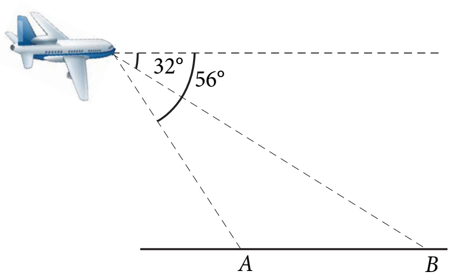


Figure 33

66. In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be 39°. They then move 300 feet closer to the building and find the angle of elevation to be 50°. Assuming that the street is level, estimate the height of the building to the nearest foot.

67. In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be 35°. They then move 250 feet closer to the building and find the angle of elevation to be 53°. Assuming that the street is level, estimate the height of the building to the nearest foot.

68. Points *A* and *B* are on opposite sides of a lake. Point *C* is 97 meters from *A*. The measure of angle *BAC* is determined to be 101°, and the measure of angle *ACB* is determined to be 53°. What is the distance from *A* to *B*, rounded to the nearest whole meter?

69. A man and a woman standing $3\frac{1}{2}$ miles apart spot a hot air balloon at the same time. If the angle of elevation from the man to the balloon is 27°, and the angle of elevation from the woman to the balloon is 41°, find the altitude of the balloon to the nearest foot.

70. Two search teams spot a stranded climber on a mountain. The first search team is 0.5 miles from the second search team, and both teams are at an altitude of 1 mile. The angle of elevation from the first search team to the stranded climber is 15°. The angle of elevation from the second search team to the climber is 22°. What is the altitude of the climber? Round to the nearest tenth of a mile.

71. A street light is mounted on a pole. *A* 6-foot-tall man is standing on the street a short distance from the pole, casting a shadow. The angle of elevation from the tip of the man's shadow to the top of his head of 28°. *A* 6-foot-tall woman is standing on the same street on the opposite side of the pole from the man. The angle of elevation from the tip of her shadow to the top of her head is 28°. If the man and woman are 20 feet apart, how far is the street light from the tip of the shadow of each person? Round the distance to the nearest tenth of a foot.

72. Three cities, *A, B,* and *C,* are located so that city *A* is due east of city *B*. If city *C* is located 35° west of north from city *B* and is 100 miles from city *A* and 70 miles from city *B,* how far is city *A* from city *B*? Round the distance to the nearest tenth of a mile.

73. Two streets meet at an 80° angle. At the corner, a park is being built in the shape of a triangle. Find the area of the park if, along one road, the park measures 180 feet, and along the other road, the park measures 215 feet.

74. Brian's house is on a corner lot. Find the area of the front yard if the edges measure 40 and 56 feet, as shown in **Figure 34**.

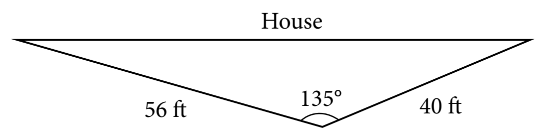


Figure 34

75. The Bermuda triangle is a region of the Atlantic Ocean that connects Bermuda, Florida, and Puerto Rico. Find the area of the Bermuda triangle if the distance from Florida to Bermuda is 1030 miles, the distance from Puerto Rico to Bermuda is 980 miles, and the angle created by the two distances is 62°.

76. A yield sign measures 30 inches on all three sides. What is the area of the sign?

77. Naomi bought a modern dining table whose top is in the shape of a triangle. Find the area of the table top if two of the sides measure 4 feet and 4.5 feet, and the smaller angles measure 32° and 42°, as shown in **Figure 35**.

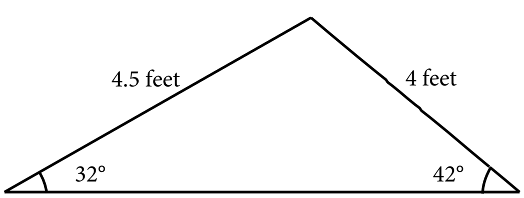


Figure 35

LEARNING OBJECTIVES

In this section, you will:

- Use the Law of Cosines to solve oblique triangles.
- Solve applied problems using the Law of Cosines.
- Use Heron's formula to find the area of a triangle.

8.2 NON-RIGHT TRIANGLES: LAW OF COSINES

Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles as shown in **Figure 1**. How far from port is the boat?

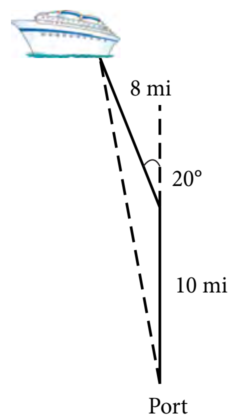


Figure 1

Unfortunately, while the Law of Sines enables us to address many non-right triangle cases, it does not help us with triangles where the known angle is between two known sides, a SAS (side-angle-side) triangle, or when all three sides are known, but no angles are known, a SSS (side-side-side) triangle. In this section, we will investigate another tool for solving oblique triangles described by these last two cases.

Using the Law of Cosines to Solve Oblique Triangles

The tool we need to solve the problem of the boat's distance from the port is the **Law of Cosines**, which defines the relationship among angle measurements and side lengths in oblique triangles. Three formulas make up the Law of Cosines. At first glance, the formulas may appear complicated because they include many variables. However, once the pattern is understood, the Law of Cosines is easier to work with than most formulas at this mathematical level.

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the **Generalized Pythagorean Theorem**, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle ABC is placed in the coordinate plane with vertex A at the origin, side c drawn along the x-axis, and vertex C located at some point (x, y) in the plane, as illustrated in **Figure 2**. Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.

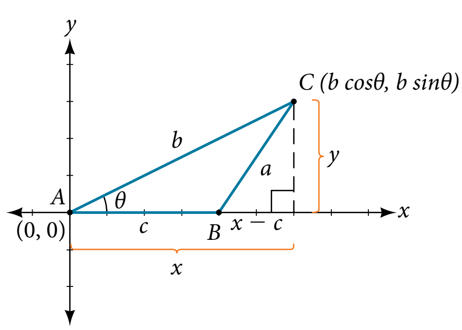


Figure 2

We can drop a perpendicular from C to the x-axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that

$$\cos\theta = \frac{x(\text{adjacent})}{b(\text{hypotenuse})} \text{ and } \sin\theta = \frac{y(\text{opposite})}{b(\text{hypotenuse})}$$

In terms of θ, $x = b\cos\theta$ and $y = b\sin\theta$. The (x, y) point located at C has coordinates $(b\cos\theta, b\sin\theta)$. Using the side $(x - c)$ as one leg of a right triangle and y as the second leg, we can find the length of hypotenuse a using the Pythagorean Theorem. Thus,

$$\begin{aligned}
a^2 &= (x - c)^2 + y^2 & \\
&= (b\cos\theta - c)^2 + (b\sin\theta)^2 & \text{Substitute } (b\cos\theta) \text{ for } x \text{ and } (b\sin\theta) \text{ for } y. \\
&= (b^2\cos^2\theta - 2bc\cos\theta + c^2) + b^2\sin^2\theta & \text{Expand the perfect square.} \\
&= b^2\cos^2\theta + b^2\sin^2\theta + c^2 - 2bc\cos\theta & \text{Group terms noting that } \cos^2\theta + \sin^2\theta = 1. \\
&= b^2(\cos^2\theta + \sin^2\theta) + c^2 - 2bc\cos\theta & \text{Factor out } b^2. \\
a^2 &= b^2 + c^2 - 2bc\cos\theta &
\end{aligned}$$

The formula derived is one of the three equations of the Law of Cosines. The other equations are found in a similar fashion.

Keep in mind that it is always helpful to sketch the triangle when solving for angles or sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

Law of Cosines

The **Law of Cosines** states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle. For triangles labeled as in **Figure 3**, with angles α, β, and γ, and opposite corresponding sides a, b, and c, respectively, the Law of Cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

To solve for a missing side measurement, the corresponding opposite angle measure is needed.

When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

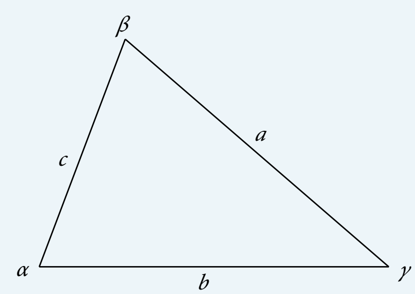


Figure 3

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

How To…

Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle.

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.
2. Apply the Law of Cosines to find the length of the unknown side or angle.
3. Apply the Law of Sines or Cosines to find the measure of a second angle.
4. Compute the measure of the remaining angle.

Example 1 Finding the Unknown Side and Angles of a SAS Triangle

Find the unknown side and angles of the triangle in **Figure 4**.

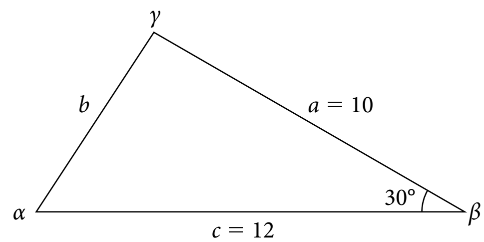


Figure 4

Solution First, make note of what is given: two sides and the angle between them. This arrangement is classified as SAS and supplies the data needed to apply the Law of Cosines.

Each one of the three laws of cosines begins with the square of an unknown side opposite a known angle. For this example, the first side to solve for is side b, as we know the measurement of the opposite angle β.

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$b^2 = 10^2 + 12^2 - 2(10)(12)\cos(30°)$$ Substitute the measurements for the known quantities.

$$b^2 = 100 + 144 - 240\left(\frac{\sqrt{3}}{2}\right)$$ Evaluate the cosine and begin to simplify.

$$b^2 = 244 - 120\sqrt{3}$$

$$b = \sqrt{244 - 120\sqrt{3}}$$ Use the square root property.

$$b \approx 6.013$$

Because we are solving for a length, we use only the positive square root. Now that we know the length b, we can use the Law of Sines to fill in the remaining angles of the triangle. Solving for angle α, we have

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

$$\frac{\sin\alpha}{10} = \frac{\sin(30°)}{6.013}$$

$$\sin\alpha = \frac{10\sin(30°)}{6.013}$$ Multiply both sides of the equation by 10.

$$\alpha = \sin^{-1}\left(\frac{10\sin(30°)}{6.013)}\right)$$ Find the inverse sine of $\frac{10\sin(30°)}{6.013}$.

$$\alpha \approx 56.3°$$

The other possibility for α would be $\alpha = 180° - 56.3° \approx 123.7°$. In the original diagram, α is adjacent to the longest side, so α is an acute angle and, therefore, 123.7° does not make sense. Notice that if we choose to apply the Law of Cosines, we arrive at a unique answer. We do not have to consider the other possibilities, as cosine is unique for angles between 0° and 180°. Proceeding with $\alpha \approx 56.3°$, we can then find the third angle of the triangle.

$$\gamma = 180° - 30° - 56.3° \approx 93.7°$$

The complete set of angles and sides is

$$\alpha \approx 56.3° \qquad a = 10$$
$$\beta = 30° \qquad b \approx 6.013$$
$$\gamma \approx 93.7° \qquad c = 12$$

Try It #1

Find the missing side and angles of the given triangle: $\alpha = 30°$, $b = 12$, $c = 24$.

Example 2 Solving for an Angle of a SSS Triangle

Find the angle α for the given triangle if side $a = 20$, side $b = 25$, and side $c = 18$.

Solution For this example, we have no angles. We can solve for any angle using the Law of Cosines. To solve for angle α, we have

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$20^2 = 25^2 + 18^2 - 2(25)(18)\cos\alpha \quad \text{Substitute the appropriate measurements.}$$

$$400 = 625 + 324 - 900\cos\alpha \quad \text{Simplify in each step.}$$

$$400 = 949 - 900\cos\alpha$$

$$-549 = -900\cos\alpha \quad \text{Isolate } \cos\alpha.$$

$$\frac{-549}{-900} = \cos\alpha$$

$$0.61 \approx \cos\alpha$$

$$\cos^{-1}(0.61) \approx \alpha \quad \text{Find the inverse cosine.}$$

$$\alpha \approx 52.4°$$

See **Figure 5**.

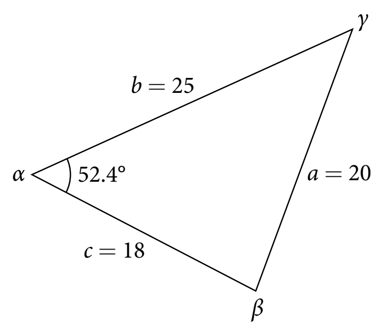


Figure 5

Analysis *Because the inverse cosine can return any angle between* 0 *and* 180 *degrees, there will not be any ambiguous cases using this method.*

Try It #2

Given $a = 5$, $b = 7$, and $c = 10$, find the missing angles.

Solving Applied Problems Using the Law of Cosines

Just as the Law of Sines provided the appropriate equations to solve a number of applications, the Law of Cosines is applicable to situations in which the given data fits the cosine models. We may see these in the fields of navigation, surveying, astronomy, and geometry, just to name a few.

Example 3 Using the Law of Cosines to Solve a Communication Problem

On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points. Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6,000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5,050 feet from the first tower and 2,420 feet from the second tower. Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.

Solution For simplicity, we start by drawing a diagram similar to **Figure 6** and labeling our given information.

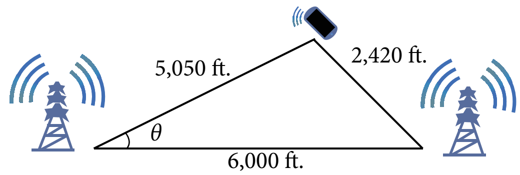


Figure 6

Using the Law of Cosines, we can solve for the angle θ. Remember that the Law of Cosines uses the square of one side to find the cosine of the opposite angle. For this example, let $a = 2420$, $b = 5050$, and $c = 6000$. Thus, θ corresponds to the opposite side $a = 2420$.

$$a^2 = b^2 + c^2 - 2bc\cos\theta$$
$$(2420)^2 = (5050)^2 + (6000)^2 - 2(5050)(6000)\cos\theta$$
$$(2420)^2 - (5050)^2 - (6000)^2 = -2(5050)(6000)\cos\theta$$
$$\frac{(2420)^2 - (5050)^2 - (6000)^2}{-2(5050)(6000)} = \cos\theta$$
$$\cos\theta \approx 0.9183$$
$$\theta \approx \cos^{-1}(0.9183)$$
$$\theta \approx 23.3°$$

To answer the questions about the phone's position north and east of the tower, and the distance to the highway, drop a perpendicular from the position of the cell phone, as in **Figure 7**. This forms two right triangles, although we only need the right triangle that includes the first tower for this problem.

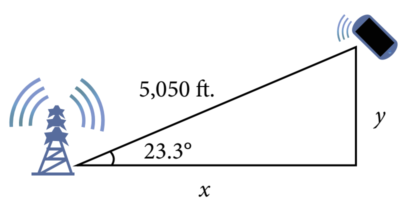


Figure 7

Using the angle $\theta = 23.3°$ and the basic trigonometric identities, we can find the solutions. Thus

$$\cos(23.3°) = \frac{x}{5050}$$
$$x = 5050\cos(23.3°)$$
$$x \approx 4638.15 \text{ feet}$$

$$\sin(23.3°) = \frac{y}{5050}$$
$$y = 5050\sin(23.3°)$$
$$y \approx 1997.5 \text{ feet}$$

The cell phone is approximately 4,638 feet east and 1,998 feet north of the first tower, and 1,998 feet from the highway.

Example 4 Calculating Distance Traveled Using a SAS Triangle

Returning to our problem at the beginning of this section, suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat? The diagram is repeated here in **Figure 8**.

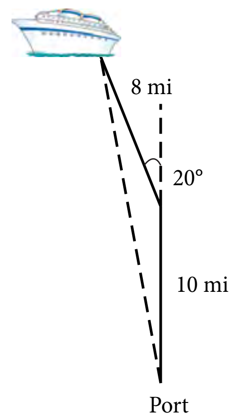


Figure 8

Solution The boat turned 20 degrees, so the obtuse angle of the non-right triangle is the supplemental angle, $180° - 20° = 160°$. With this, we can utilize the Law of Cosines to find the missing side of the obtuse triangle—the distance of the boat to the port.

$$\begin{aligned} x^2 &= 8^2 + 10^2 - 2(8)(10)\cos(160°) \\ x^2 &= 314.35 \\ x &= \sqrt{314.35} \\ x &\approx 17.7 \text{ miles} \end{aligned}$$

The boat is about 17.7 miles from port.

Using Heron's Formula to Find the Area of a Triangle

We already learned how to find the area of an oblique triangle when we know two sides and an angle. We also know the formula to find the area of a triangle using the base and the height. When we know the three sides, however, we can use Heron's formula instead of finding the height. Heron of Alexandria was a geometer who lived during the first century *A*.D. He discovered a formula for finding the area of oblique triangles when three sides are known.

> ***Heron's formula***
> Heron's formula finds the area of oblique triangles in which sides a, b, and c are known.
>
> $$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$
>
> where $s = \dfrac{(a+b+c)}{2}$ is one half of the perimeter of the triangle, sometimes called the semi-perimeter.

Example 5 Using Heron's Formula to Find the Area of a Given Triangle

Find the area of the triangle in **Figure 9** using Heron's formula.

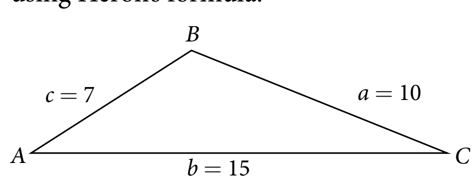


Figure 9

Solution First, we calculate s.

$$s = \frac{(a+b+c)}{2}$$

$$s = \frac{(10+15+7)}{2} = 16$$

Then we apply the formula.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$
$$\text{Area} = \sqrt{16(16-10)(16-15)(16-7)}$$
$$\text{Area} \approx 29.4$$

The area is approximately 29.4 square units.

Try It #3

Use Heron's formula to find the area of a triangle with sides of lengths $a = 29.7$ ft, $b = 42.3$ ft, and $c = 38.4$ ft.

Example 6 Applying Heron's Formula to a Real-World Problem

A Chicago city developer wants to construct a building consisting of artist's lofts on a triangular lot bordered by Rush Street, Wabash Avenue, and Pearson Street. The frontage along Rush Street is approximately 62.4 meters, along Wabash Avenue it is approximately 43.5 meters, and along Pearson Street it is approximately 34.1 meters. How many square meters are available to the developer? See **Figure 10** for a view of the city property.

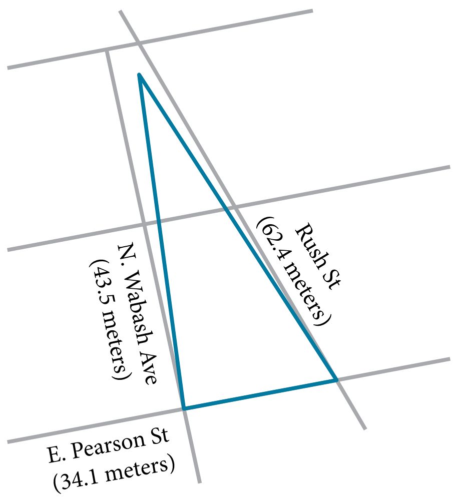


Figure 10

Solution Find the measurement for s, which is one-half of the perimeter.

$$s = \frac{62.4 + 43.5 + 34.1}{2}$$
$$s = 70 \text{ m}$$

Apply Heron's formula.

$$\text{Area} = \sqrt{70(70-62.4)(70-43.5)(70-34.1)}$$
$$\text{Area} = \sqrt{506{,}118.2}$$
$$\text{Area} \approx 711.4$$

The developer has about 711.4 square meters.

Try It #4

Find the area of a triangle given $a = 4.38$ ft , $b = 3.79$ ft, and $c = 5.22$ ft.

Access these online resources for additional instruction and practice with the Law of Cosines.

- Law of Cosines (http://openstaxcollege.org/l/lawcosines)
- Law of Cosines: Applications (http://openstaxcollege.org/l/cosineapp)
- Law of Cosines: Applications 2 (http://openstaxcollege.org/l/cosineapp2)

8.2 SECTION EXERCISES

VERBAL

1. If you are looking for a missing side of a triangle, what do you need to know when using the Law of Cosines?

2. If you are looking for a missing angle of a triangle, what do you need to know when using the Law of Cosines?

3. Explain what s represents in Heron's formula.

4. Explain the relationship between the Pythagorean Theorem and the Law of Cosines.

5. When must you use the Law of Cosines instead of the Pythagorean Theorem?

ALGEBRAIC

For the following exercises, assume α is opposite side a, β is opposite side b, and γ is opposite side c. If possible, solve each triangle for the unknown side. Round to the nearest tenth.

6. $\gamma = 41.2°$, $a = 2.49$, $b = 3.13$

7. $\alpha = 120°$, $b = 6$, $c = 7$

8. $\beta = 58.7°$, $a = 10.6$, $c = 15.7$

9. $\gamma = 115°$, $a = 18$, $b = 23$

10. $\alpha = 119°$, $a = 26$, $b = 14$

11. $\gamma = 113°$, $b = 10$, $c = 32$

12. $\beta = 67°$, $a = 49$, $b = 38$

13. $\alpha = 43.1°$, $a = 184.2$, $b = 242.8$

14. $\alpha = 36.6°$, $a = 186.2$, $b = 242.2$

15. $\beta = 50°$, $a = 105$, $b = 45$

For the following exercises, use the Law of Cosines to solve for the missing angle of the oblique triangle. Round to the nearest tenth.

16. $a = 42$, $b = 19$, $c = 30$; find angle A.

17. $a = 14$, $b = 13$, $c = 20$; find angle C.

18. $a = 16$, $b = 31$, $c = 20$; find angle B.

19. $a = 13$, $b = 22$, $c = 28$; find angle A.

20. $a = 108$, $b = 132$, $c = 160$; find angle C.

For the following exercises, solve the triangle. Round to the nearest tenth.

21. $A = 35°$, $b = 8$, $c = 11$

22. $B = 88°$, $a = 4.4$, $c = 5.2$

23. $C = 121°$, $a = 21$, $b = 37$

24. $a = 13$, $b = 11$, $c = 15$

25. $a = 3.1$, $b = 3.5$, $c = 5$

26. $a = 51$, $b = 25$, $c = 29$

For the following exercises, use Heron's formula to find the area of the triangle. Round to the nearest hundredth.

27. Find the area of a triangle with sides of length 18 in, 21 in, and 32 in. Round to the nearest tenth.

28. Find the area of a triangle with sides of length 20 cm, 26 cm, and 37 cm. Round to the nearest tenth.

29. $a = \frac{1}{2}\ m$, $b = \frac{1}{3}\ m$, $c = \frac{1}{4}\ m$

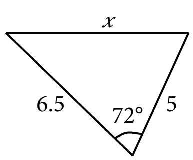
30. $a = 12.4$ ft, $b = 13.7$ ft, $c = 20.2$ ft

31. $a = 1.6$ yd, $b = 2.6$ yd, $c = 4.1$ yd

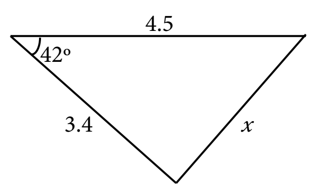
GRAPHICAL

For the following exercises, find the length of side x. Round to the nearest tenth.

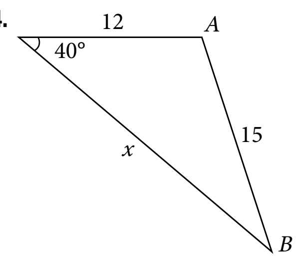
32.



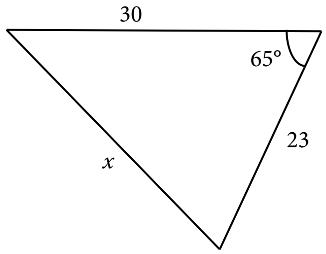
33.



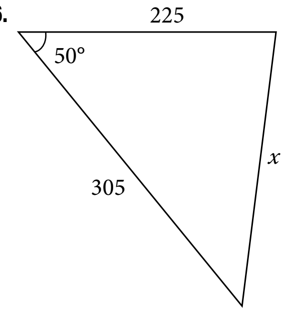
34.



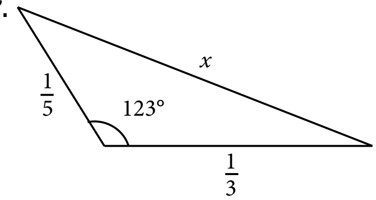
35.



36.

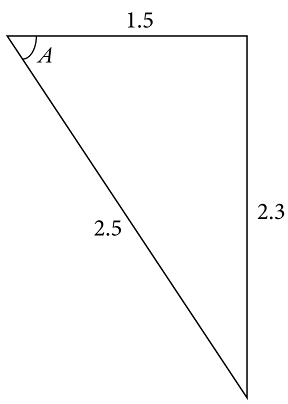


37.

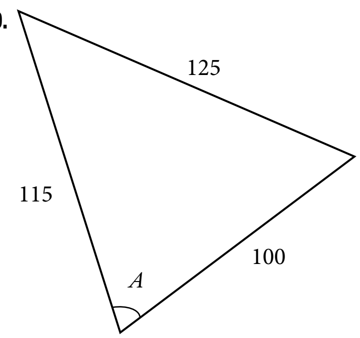


For the following exercises, find the measurement of angle A.

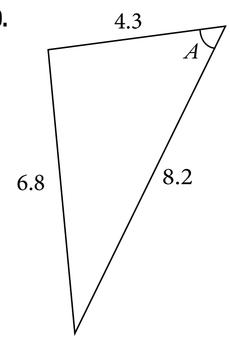
38.



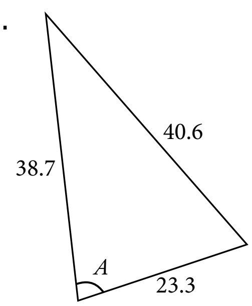
39.



40.



41.



42. Find the measure of each angle in the triangle shown in **Figure 11**. Round to the nearest tenth.

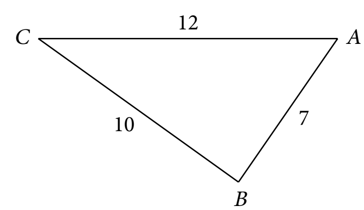
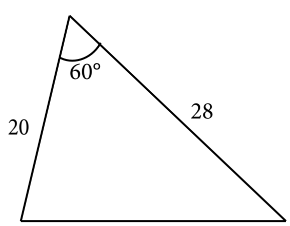


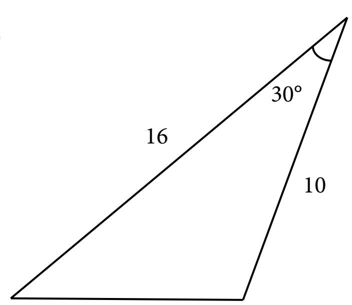
Figure 11

For the following exercises, solve for the unknown side. Round to the nearest tenth.

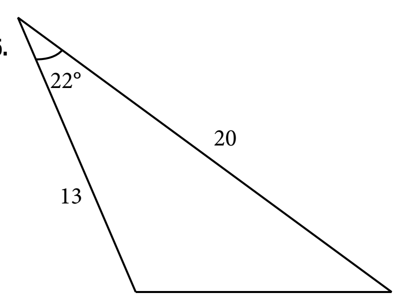
43.



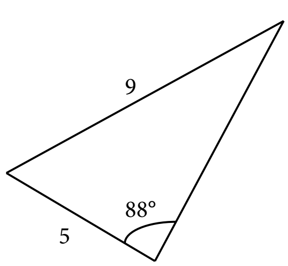
44.



45.

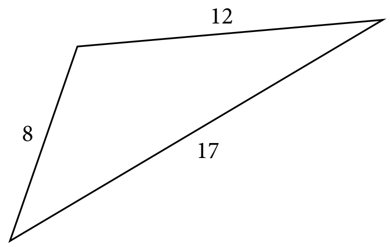


46.

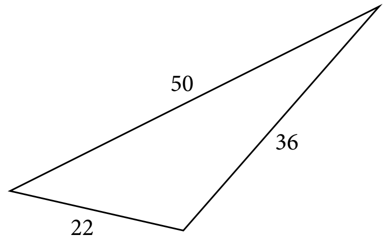


For the following exercises, find the area of the triangle. Round to the nearest hundredth.

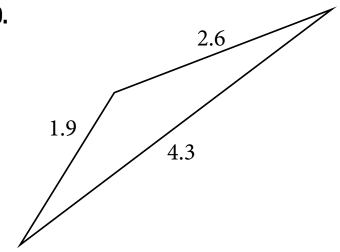
47.



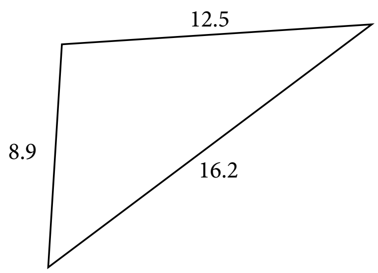
48.



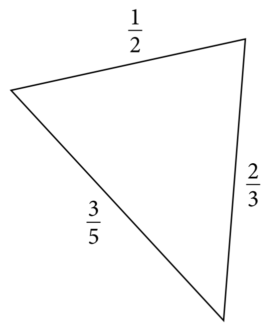
49.



50.



51.



EXTENSIONS

52. A parallelogram has sides of length 16 units and 10 units. The shorter diagonal is 12 units. Find the measure of the longer diagonal.

53. The sides of a parallelogram are 11 feet and 17 feet. The longer diagonal is 22 feet. Find the length of the shorter diagonal.

54. The sides of a parallelogram are 28 centimeters and 40 centimeters. The measure of the larger angle is 100°. Find the length of the shorter diagonal.

55. A regular octagon is inscribed in a circle with a radius of 8 inches. (See **Figure 12**.) Find the perimeter of the octagon.

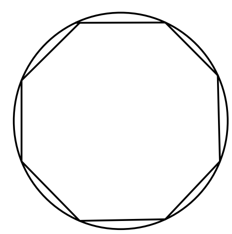


Figure 12

56. A regular pentagon is inscribed in a circle of radius 12 cm. (See **Figure 13**.) Find the perimeter of the pentagon. Round to the nearest tenth of a centimeter.

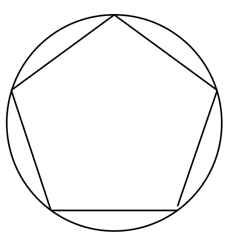


Figure 13

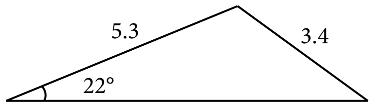
For the following exercises, suppose that $x^2 = 25 + 36 - 60\cos(52)$ represents the relationship of three sides of a triangle and the cosine of an angle.

57. Draw the triangle.

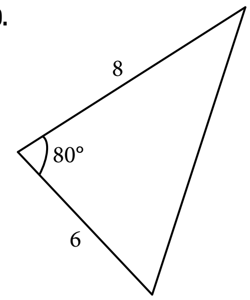
58. Find the length of the third side.

For the following exercises, find the area of the triangle.

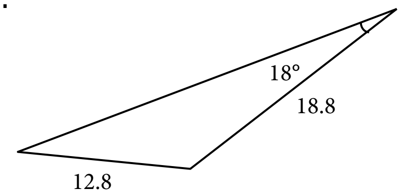
59.



60.



61.



REAL-WORLD APPLICATIONS

62. A surveyor has taken the measurements shown in **Figure 14**. Find the distance across the lake. Round answers to the nearest tenth.

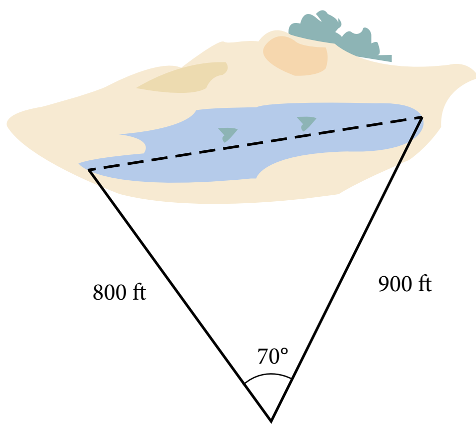


Figure 14

63. A satellite calculates the distances and angle shown in **Figure 15** (not to scale). Find the distance between the two cities. Round answers to the nearest tenth.

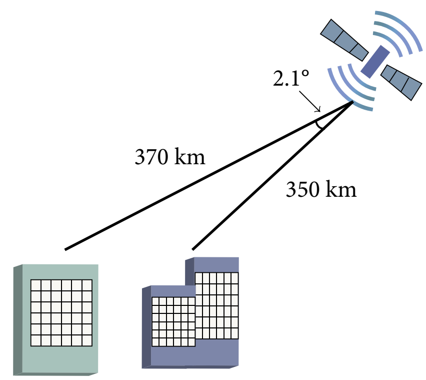


Figure 15

64. An airplane flies 220 miles with a heading of 40°, and then flies 180 miles with a heading of 170°. How far is the plane from its starting point, and at what heading? Round answers to the nearest tenth.

65. A 113-foot tower is located on a hill that is inclined 34° to the horizontal, as shown in **Figure 16**. *A* guy-wire is to be attached to the top of the tower and anchored at a point 98 feet uphill from the base of the tower. Find the length of wire needed.

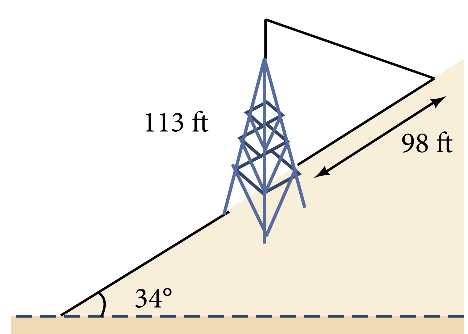


Figure 16

66. Two ships left a port at the same time. One ship traveled at a speed of 18 miles per hour at a heading of 320°. The other ship traveled at a speed of 22 miles per hour at a heading of 194°. Find the distance between the two ships after 10 hours of travel.

67. The graph in **Figure 17** represents two boats departing at the same time from the same dock. The first boat is traveling at 18 miles per hour at a heading of 327° and the second boat is traveling at 4 miles per hour at a heading of 60°. Find the distance between the two boats after 2 hours.

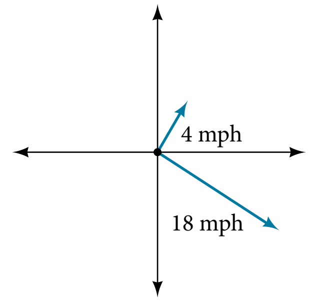


Figure 17

68. A triangular swimming pool measures 40 feet on one side and 65 feet on another side. These sides form an angle that measures 50°. How long is the third side (to the nearest tenth)?

69. A pilot flies in a straight path for 1 hour 30 min. She then makes a course correction, heading 10° to the right of her original course, and flies 2 hours in the new direction. If she maintains a constant speed of 680 miles per hour, how far is she from her starting position?

70. Los Angeles is 1,744 miles from Chicago, Chicago is 714 miles from New York, and New York is 2,451 miles from Los Angeles. Draw a triangle connecting these three cities, and find the angles in the triangle.

71. Philadelphia is 140 miles from Washington, *D*.C., Washington, *D*.C. is 442 miles from Boston, and Boston is 315 miles from Philadelphia. Draw a triangle connecting these three cities and find the angles in the triangle.

72. Two planes leave the same airport at the same time. One flies at 20° east of north at 500 miles per hour. The second flies at 30° east of south at 600 miles per hour. How far apart are the planes after 2 hours?

73. Two airplanes take off in different directions. One travels 300 mph due west and the other travels 25° north of west at 420 mph. After 90 minutes, how far apart are they, assuming they are flying at the same altitude?

74. A parallelogram has sides of length 15.4 units and 9.8 units. Its area is 72.9 square units. Find the measure of the longer diagonal.

75. The four sequential sides of a quadrilateral have lengths 4.5 cm, 7.9 cm, 9.4 cm, and 12.9 cm. The angle between the two smallest sides is 117°. What is the area of this quadrilateral?

76. The four sequential sides of a quadrilateral have lengths 5.7 cm, 7.2 cm, 9.4 cm, and 12.8 cm. The angle between the two smallest sides is 106°. What is the area of this quadrilateral?

77. Find the area of a triangular piece of land that measures 30 feet on one side and 42 feet on another; the included angle measures 132°. Round to the nearest whole square foot.

78. Find the area of a triangular piece of land that measures 110 feet on one side and 250 feet on another; the included angle measures 85°. Round to the nearest whole square foot.

LEARNING OBJECTIVES

In this section, you will:

- View vectors geometrically.
- Find magnitude and direction.
- Perform vector addition and scalar multiplication.
- Find the component form of a vector.
- Find the unit vector in the direction of *v*.
- Perform operations with vectors in terms of *i* and *j*.
- Find the dot product of two vectors.

8.8 VECTORS

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour, as shown in **Figure 1**. What are the ground speed and actual bearing of the plane?

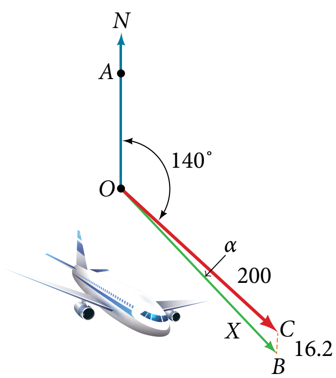


Figure 1

Ground speed refers to the speed of a plane relative to the ground. Airspeed refers to the speed a plane can travel relative to its surrounding air mass. These two quantities are not the same because of the effect of wind. In an earlier section, we used triangles to solve a similar problem involving the movement of boats. Later in this section, we will find the airplane's groundspeed and bearing, while investigating another approach to problems of this type. First, however, let's examine the basics of vectors.

A Geometric View of Vectors

A **vector** is a specific quantity drawn as a line segment with an arrowhead at one end. It has an **initial point**, where it begins, and a **terminal point**, where it ends. A vector is defined by its **magnitude**, or the length of the line, and its direction, indicated by an arrowhead at the terminal point. Thus, a vector is a directed line segment. There are various symbols that distinguish vectors from other quantities:

- Lower case, boldfaced type, with or without an arrow on top such as $\boldsymbol{v}$, $\boldsymbol{u}$, $\boldsymbol{w}$, $\vec{\boldsymbol{v}}$, $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{w}}$.
- Given initial point P and terminal point Q, a vector can be represented as $\overrightarrow{PQ}$. The arrowhead on top is what indicates that it is not just a line, but a directed line segment.
- Given an initial point of $(0, 0)$ and terminal point (a, b), a vector may be represented as $\langle a, b\rangle$.

This last symbol $\langle a, b\rangle$ has special significance. It is called the **standard position**. The position vector has an initial point $(0, 0)$ and a terminal point $\langle a, b\rangle$. To change any vector into the position vector, we think about the change in the x-coordinates and the change in the y-coordinates. Thus, if the initial point of a vector $\overrightarrow{CD}$ is $C(x_1, y_1)$ and the terminal point is $D(x_2, y_2)$, then the position vector is found by calculating

$$\begin{aligned}\overrightarrow{AB} &= \langle x_2 - x_1, y_2 - y_1\rangle \\ &= \langle a, b\rangle\end{aligned}$$

In **Figure 2**, we see the original vector $\overrightarrow{CD}$ and the position vector $\overrightarrow{AB}$.

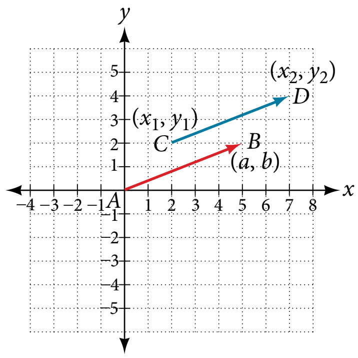


Figure 2

> ***properties of vectors***
> A vector is a directed line segment with an initial point and a terminal point. Vectors are identified by magnitude, or the length of the line, and direction, represented by the arrowhead pointing toward the terminal point. The position vector has an initial point at (0, 0) and is identified by its terminal point $\langle a, b \rangle$.

Example 1 Find the Position Vector

Consider the vector whose initial point is $P(2, 3)$ and terminal point is $Q(6, 4)$. Find the position vector.

Solution The position vector is found by subtracting one x-coordinate from the other x-coordinate, and one y-coordinate from the other y-coordinate. Thus

$$\begin{aligned} v &= \langle 6 - 2, 4 - 3 \rangle \\ &= \langle 4, 1 \rangle \end{aligned}$$

The position vector begins at (0, 0) and terminates at (4, 1). The graphs of both vectors are shown in **Figure 3**.

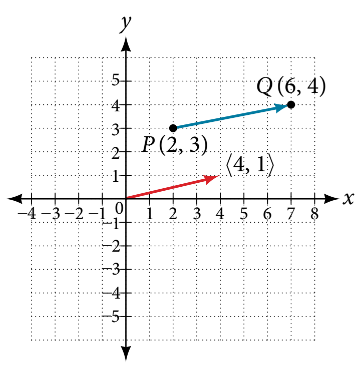


Figure 3

We see that the position vector is $\langle 4, 1 \rangle$.

Example 2 Drawing a Vector with the Given Criteria and Its Equivalent Position Vector

Find the position vector given that vector v has an initial point at $(-3, 2)$ and a terminal point at (4, 5), then graph both vectors in the same plane.

Solution The position vector is found using the following calculation:

$$\begin{aligned} v &= \langle 4 - (-3), 5 - 2 \rangle \\ &= \langle 7, 3 \rangle \end{aligned}$$

Thus, the position vector begins at (0, 0) and terminates at (7, 3). See **Figure 4**.

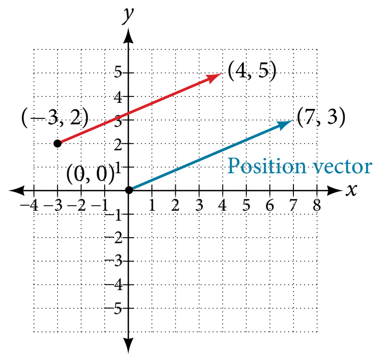


Figure 4

Try It #1

Draw a vector v that connects from the origin to the point (3, 5).

Finding Magnitude and Direction

To work with a vector, we need to be able to find its magnitude and its direction. We find its magnitude using the Pythagorean Theorem or the distance formula, and we find its direction using the inverse tangent function.

magnitude and direction of a vector

Given a position vector $v = \langle a, b \rangle$, the magnitude is found by $|v| = \sqrt{a^2 + b^2}$. The direction is equal to the angle formed with the x-axis, or with the y-axis, depending on the application. For a position vector, the direction is found by $\tan\theta = \left(\frac{b}{a}\right) \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$, as illustrated in **Figure 5.**

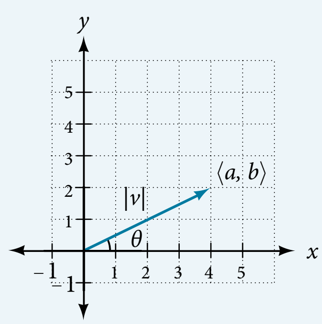


Figure 5

Two vectors v and u are considered equal if they have the same magnitude and the same direction. Additionally, if both vectors have the same position vector, they are equal.

Example 3 Finding the Magnitude and Direction of a Vector

Find the magnitude and direction of the vector with initial point $P(-8, 1)$ and terminal point $Q(-2, -5)$. Draw the vector.

Solution First, find the position vector.

$$\begin{aligned} u &= \langle -2, -(-8), -5-1 \rangle \\ &= \langle 6, -6 \rangle \end{aligned}$$

We use the Pythagorean Theorem to find the magnitude.

$$\begin{aligned} |u| &= \sqrt{(6)^2 + (-6)^2} \\ &= \sqrt{72} \\ &= 6\sqrt{2} \end{aligned}$$

The direction is given as

$$\tan\theta = \frac{-6}{6} = -1 \Rightarrow \theta = \tan^{-1}(-1)$$
$$= -45°$$

However, the angle terminates in the fourth quadrant, so we add 360° to obtain a positive angle. Thus, $-45° + 360° = 315°$. See **Figure 6**.

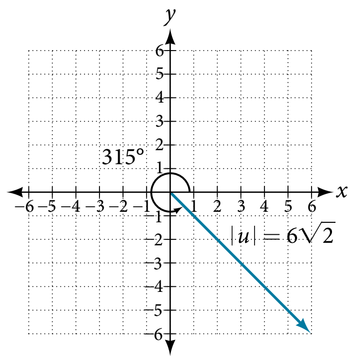


Figure 6

Example 4 Showing That Two Vectors Are Equal

Show that vector v with initial point at $(5, -3)$ and terminal point at $(-1, 2)$ is equal to vector u with initial point at $(-1, -3)$ and terminal point at $(-7, 2)$. Draw the position vector on the same grid as v and u. Next, find the magnitude and direction of each vector.

Solution As shown in **Figure 7**, draw the vector v starting at initial $(5, -3)$ and terminal point $(-1, 2)$. Draw the vector u with initial point $(-1, -3)$ and terminal point $(-7, 2)$. Find the standard position for each.

Next, find and sketch the position vector for v and u. We have

$$\begin{aligned} v &= \langle -1-5, 2-(-3)\rangle \\ &= \langle -6, 5\rangle \\ u &= \langle -7-(-1), 2-(-3)\rangle \\ &= \langle -6, 5\rangle \end{aligned}$$

Since the position vectors are the same, v and u are the same.

An alternative way to check for vector equality is to show that the magnitude and direction are the same for both vectors. To show that the magnitudes are equal, use the Pythagorean Theorem.

$$\begin{aligned} |v| &= \sqrt{(-1-5)^2 + (2-(-3))^2} \\ &= \sqrt{(-6)^2 + (5)^2} \\ &= \sqrt{36+25} \\ &= \sqrt{61} \\ |u| &= \sqrt{(-7-(-1))^2 + (2-(-3))^2} \\ &= \sqrt{(-6)^2 + (5)^2} \\ &= \sqrt{36+25} \\ &= \sqrt{61} \end{aligned}$$

As the magnitudes are equal, we now need to verify the direction. Using the tangent function with the position vector gives

$$\tan\theta = -\frac{5}{6} \Rightarrow \theta = \tan^{-1}\left(-\frac{5}{6}\right)$$
$$= -39.8°$$

However, we can see that the position vector terminates in the second quadrant, so we add 180°. Thus, the direction is $-39.8° + 180° = 140.2°$.

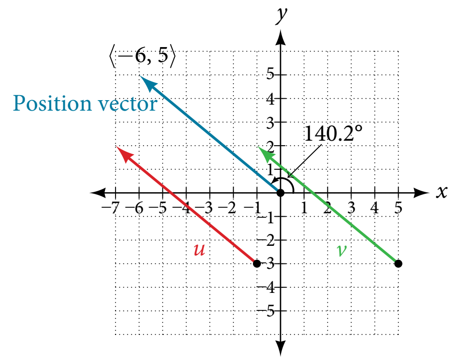


Figure 7

Performing Vector Addition and Scalar Multiplication

Now that we understand the properties of vectors, we can perform operations involving them. While it is convenient to think of the vector $u = \langle x, y\rangle$ as an arrow or directed line segment from the origin to the point (x, y), vectors can be situated anywhere in the plane. The sum of two vectors u and v, or **vector addition**, produces a third vector $u+ v$, the **resultant** vector.

To find $u + v$, we first draw the vector u, and from the terminal end of u, we drawn the vector v. In other words, we have the initial point of v meet the terminal end of u. This position corresponds to the notion that we move along the first vector and then, from its terminal point, we move along the second vector. The sum $u + v$ is the resultant vector because it results from addition or subtraction of two vectors. The resultant vector travels directly from the beginning of u to the end of v in a straight path, as shown in **Figure 8**.

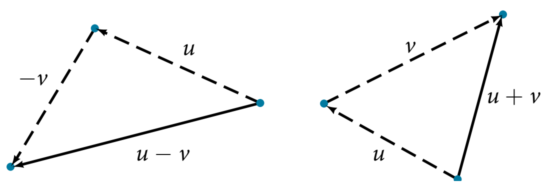


Figure 8

Vector subtraction is similar to vector addition. To find $u - v$, view it as $u + (-v)$. Adding $-v$ is reversing direction of v and adding it to the end of u. The new vector begins at the start of u and stops at the end point of $-v$. See **Figure 9** for a visual that compares vector addition and vector subtraction using parallelograms.

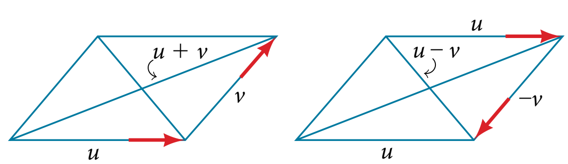


Figure 9

Example 5 Adding and Subtracting Vectors

Given $u = \langle 3, -2\rangle$ and $v = \langle -1, 4\rangle$, find two new vectors $u + v$, and $u - v$.

Solution To find the sum of two vectors, we add the components. Thus,

$$\begin{aligned} u + v &= \langle 3, -2\rangle + \langle -1, 4\rangle \\ &= \langle 3 + (-1), -2 + 4\rangle \\ &= \langle 2, 2\rangle \end{aligned}$$

See **Figure 10(a)**.

To find the difference of two vectors, add the negative components of v to u. Thus,

$$\begin{aligned} u + (-v) &= \langle 3, -2\rangle + \langle 1, -4\rangle \\ &= \langle 3 + 1, -2 + (-4)\rangle \\ &= \langle 4, -6\rangle \end{aligned}$$

See **Figure 10(b)**.

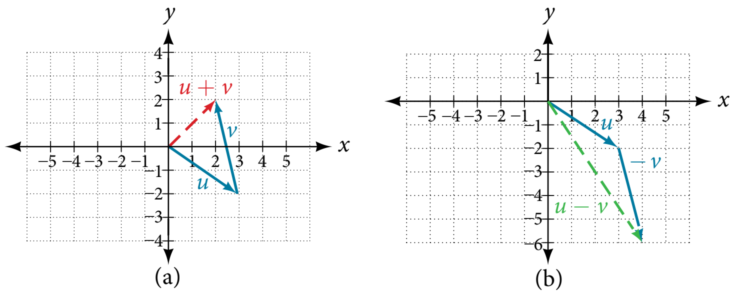


Figure 10 (a) Sum of two vectors (b) Difference of two vectors

Multiplying By a Scalar

While adding and subtracting vectors gives us a new vector with a different magnitude and direction, the process of multiplying a vector by a **scalar**, a constant, changes only the magnitude of the vector or the length of the line. Scalar multiplication has no effect on the direction unless the scalar is negative, in which case the direction of the resulting vector is opposite the direction of the original vector.

> ***scalar multiplication***
> **Scalar multiplication** involves the product of a vector and a scalar. Each component of the vector is multiplied by the scalar. Thus, to multiply $v = \langle a, b\rangle$ by k, we have
>
> $$kv = \langle ka, kb\rangle$$
>
> Only the magnitude changes, unless k is negative, and then the vector reverses direction.

Example 6 Performing Scalar Multiplication

Given vector $v = \langle 3, 1\rangle$, find $3v$, $\frac{1}{2}v$, and $-v$.

Solution See **Figure 11** for a geometric interpretation. If $v = \langle 3, 1\rangle$, then

$$\begin{aligned} 3v &= \langle 3 \cdot 3, 3 \cdot 1\rangle \\ &= \langle 9, 3\rangle \\ \frac{1}{2}v &= \left\langle \frac{1}{2} \cdot 3, \frac{1}{2} \cdot 1\right\rangle \\ &= \left\langle \frac{3}{2}, \frac{1}{2}\right\rangle \\ -v &= \langle -3, -1\rangle \end{aligned}$$

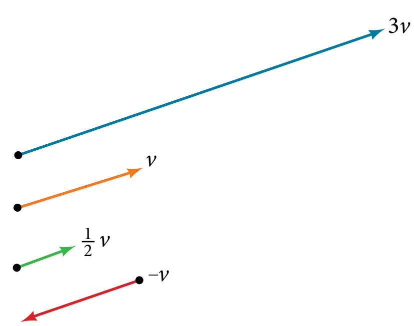


Figure 11

Analysis *Notice that the vector $3v$ is three times the length of v, $\frac{1}{2}v$ is half the length of v, and $-v$ is the same length of v, but in the opposite direction.*

Try It #2

Find the scalar multiple $3u$ given $u = \langle 5, 4 \rangle$.

Example 7 **Using Vector Addition and Scalar Multiplication to Find a New Vector**

Given $u = \langle 3, -2 \rangle$ and $v = \langle -1, 4 \rangle$, find a new vector $w = 3u + 2v$.

Solution First, we must multiply each vector by the scalar.

$$\begin{aligned} 3u &= 3\,\langle 3, -2 \rangle \\ &= \langle 9, -6 \rangle \\ 2v &= 2\,\langle -1, 4 \rangle \\ &= \langle -2, 8 \rangle \end{aligned}$$

Then, add the two together.

$$\begin{aligned} w &= 3u + 2v \\ &= \langle 9, -6 \rangle + \langle -2, 8 \rangle \\ &= \langle 9 - 2, -6 + 8 \rangle \\ &= \langle 7, 2 \rangle \end{aligned}$$

So, $w = \langle 7, 2 \rangle$.

Finding Component Form

In some applications involving vectors, it is helpful for us to be able to break a vector down into its components. Vectors are comprised of two components: the horizontal component is the x direction, and the vertical component is the y direction. For example, we can see in the graph in **Figure 12** that the position vector $\langle 2, 3 \rangle$ comes from adding the vectors v_1 and v_2. We have v_1 with initial point (0, 0) and terminal point (2, 0).

$$\begin{aligned} v_1 &= \langle 2 - 0, 0 - 0 \rangle \\ &= \langle 2, 0 \rangle \end{aligned}$$

We also have v_2 with initial point (0, 0) and terminal point (0, 3).

$$\begin{aligned} v_2 &= \langle 0 - 0, 3 - 0 \rangle \\ &= \langle 0, 3 \rangle \end{aligned}$$

Therefore, the position vector is

$$\begin{aligned} v &= \langle 2 + 0, 3 + 0 \rangle \\ &= \langle 2, 3 \rangle \end{aligned}$$

Using the Pythagorean Theorem, the magnitude of v_1 is 2, and the magnitude of v_2 is 3. To find the magnitude of v, use the formula with the position vector.

$$\begin{aligned} |v| &= \sqrt{|v_1|^2 + |v_2|^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$

The magnitude of v is $\sqrt{13}$. To find the direction, we use the tangent function $\tan\theta = \frac{y}{x}$.

$$\begin{aligned} \tan\theta &= \frac{v_2}{v_1} \\ \tan\theta &= \frac{3}{2} \\ \theta &= \tan^{-1}\left(\frac{3}{2}\right) = 56.3° \end{aligned}$$

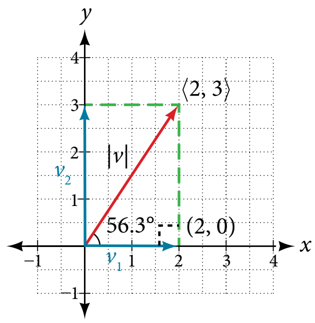


Figure 12

Thus, the magnitude of v is $\sqrt{13}$ and the direction is 56.3° off the horizontal.

Example 8 **Finding the Components of the Vector**

Find the components of the vector v with initial point (3, 2) and terminal point (7, 4).

Solution First find the standard position.

$$\begin{aligned} v &= \langle 7 - 3, 4 - 2 \rangle \\ &= \langle 4, 2 \rangle \end{aligned}$$

See the illustration in **Figure 13**.

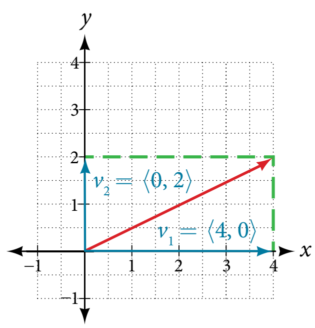


Figure 13

The horizontal component is $v_1 = \langle 4, 0 \rangle$ and the vertical component is $v_2 = \langle 0, 2 \rangle$.

Finding the Unit Vector in the Direction of *v*

In addition to finding a vector's components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a **unit vector**. We can then preserve the direction of the original vector while simplifying calculations.

Unit vectors are defined in terms of components. The horizontal unit vector is written as $i = \langle 1, 0 \rangle$ and is directed along the positive horizontal axis. The vertical unit vector is written as $j = \langle 0, 1 \rangle$ and is directed along the positive vertical axis. See **Figure 14**.

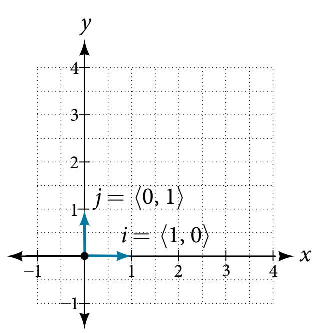


Figure 14

the unit vectors

If v is a nonzero vector, then $\frac{v}{|v|}$ is a unit vector in the direction of v. Any vector divided by its magnitude is a unit vector. Notice that magnitude is always a scalar, and dividing by a scalar is the same as multiplying by the reciprocal of the scalar.

Example 9 Finding the Unit Vector in the Direction of *v*

Find a unit vector in the same direction as $v = \langle -5, 12 \rangle$.

Solution First, we will find the magnitude.

$$\begin{aligned} |v| &= \sqrt{(-5)^2 + (12)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

Then we divide each component by $|v|$, which gives a unit vector in the same direction as v:

$$\frac{v}{|v|} = -\frac{5}{13}i + \frac{12}{13}j$$

or, in component form

$$\frac{v}{|v|} = \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle$$

See **Figure 15**.

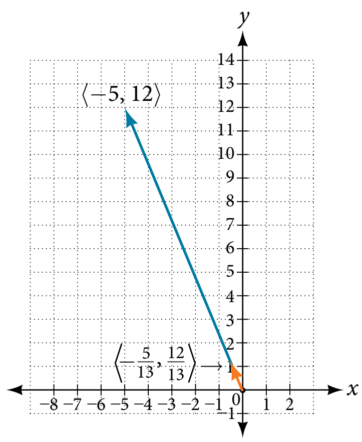


Figure 15

Verify that the magnitude of the unit vector equals 1. The magnitude of $-\frac{5}{13}i + \frac{12}{13}j$ is given as

$$\begin{aligned} \sqrt{\left(-\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2} &= \sqrt{\frac{25}{169} + \frac{144}{169}} \\ &= \sqrt{\frac{169}{169}} \\ &= 1 \end{aligned}$$

The vector $u = \frac{5}{13}i + \frac{12}{13}j$ is the unit vector in the same direction as $v = \langle -5, 12 \rangle$.

Performing Operations with Vectors in Terms of *i* and *j*

So far, we have investigated the basics of vectors: magnitude and direction, vector addition and subtraction, scalar multiplication, the components of vectors, and the representation of vectors geometrically. Now that we are familiar with the general strategies used in working with vectors, we will represent vectors in rectangular coordinates in terms of i and j.

vectors in the rectangular plane

Given a vector v with initial point $P = (x_1, y_1)$ and terminal point $Q = (x_2, y_2)$, v is written as

$$v = (x_2 - x_1)i + (y_1 - y_2)j$$

The position vector from (0, 0) to (a, b), where $(x_2 - x_1) = a$ and $(y_2 - y_1) = b$, is written as $v = ai + bj$. This vector sum is called a linear combination of the vectors i and j.

The magnitude of $v = ai + bj$ is given as $|v| = \sqrt{a^2 + b^2}$. See **Figure 16**.

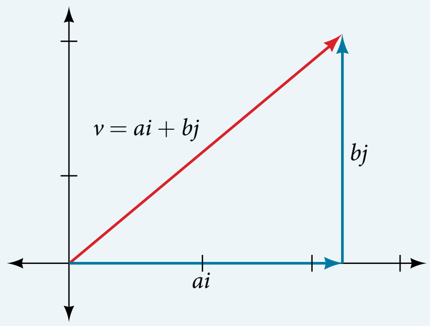


Figure 16

Example 10 Writing a Vector in Terms of *i* and *j*

Given a vector v with initial point $P = (2, -6)$ and terminal point $Q = (-6, 6)$, write the vector in terms of i and j.

Solution Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$\begin{aligned} v &= (x_2 - x_1)i + (y_2 - y_1)j \\ &= (-6 - 2)i + (6 - (-6))j \\ &= -8i + 12j \end{aligned}$$

Example 11 Writing a Vector in Terms of *i* and *j* Using Initial and Terminal Points

Given initial point $P_1 = (-1, 3)$ and terminal point $P_2 = (2, 7)$, write the vector v in terms of i and j.

Solution Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$\begin{aligned} v &= (x_2 - x_1)i + (y_2 - y_1)j \\ v &= (2 - (-1))i + (7 - 3)j \\ &= 3i + 4j \end{aligned}$$

Try It #3

Write the vector u with initial point $P = (-1, 6)$ and terminal point $Q = (7, -5)$ in terms of i and j.

Performing Operations on Vectors in Terms of *i* and *j*

When vectors are written in terms of i and j, we can carry out addition, subtraction, and scalar multiplication by performing operations on corresponding components.

adding and subtracting vectors in rectangular coordinates

Given $v = ai + bj$ and $u = ci + dj$, then

$$\begin{aligned} v + u &= (a + c)i + (b + d)j \\ v - u &= (a - c)i + (b - d)j \end{aligned}$$

Example 12 Finding the Sum of the Vectors

Find the sum of $v_1 = 2i - 3j$ and $v_2 = 4i + 5j$.

Solution According to the formula, we have

$$\begin{aligned} v_1 + v_2 &= (2 + 4)i + (-3 + 5)j \\ &= 6i + 2j \end{aligned}$$

Calculating the Component Form of a Vector: Direction

We have seen how to draw vectors according to their initial and terminal points and how to find the position vector. We have also examined notation for vectors drawn specifically in the Cartesian coordinate plane using i and j. For any of these vectors, we can calculate the magnitude. Now, we want to combine the key points, and look further at the ideas of magnitude and direction.

Calculating direction follows the same straightforward process we used for polar coordinates. We find the direction of the vector by finding the angle to the horizontal. We do this by using the basic trigonometric identities, but with $|v|$ replacing r.

vector components in terms of magnitude and direction
Given a position vector $v = \langle x, y\rangle$ and a direction angle θ,

$$\cos\theta = \frac{x}{|v|} \quad \text{and} \quad \sin\theta = \frac{y}{|v|}$$

$$x = |v|\cos\theta \qquad y = |v|\sin\theta$$

Thus, $v = xi + yj = |v|\cos\theta i + |v|\sin\theta j$, and magnitude is expressed as $|v| = \sqrt{x^2 + y^2}$.

Example 13 Writing a Vector in Terms of Magnitude and Direction

Write a vector with length 7 at an angle of 135° to the positive x-axis in terms of magnitude and direction.

Solution Using the conversion formulas $x = |v|\cos\theta i$ and $y = |v|\sin\theta j$, we find that

$$\begin{aligned} x &= 7\cos(135°)i \\ &= -\frac{7\sqrt{2}}{2} \\ y &= 7\sin(135°)j \\ &= \frac{7\sqrt{2}}{2} \end{aligned}$$

This vector can be written as $v = 7\cos(135°)i + 7\sin(135°)j$ or simplified as

$$v = -\frac{7\sqrt{2}}{2}i + \frac{7\sqrt{2}}{2}j$$

Try It #4

A vector travels from the origin to the point (3, 5). Write the vector in terms of magnitude and direction.

Finding the Dot Product of Two Vectors

As we discussed earlier in the section, scalar multiplication involves multiplying a vector by a scalar, and the result is a vector. As we have seen, multiplying a vector by a number is called scalar multiplication. If we multiply a vector by a vector, there are two possibilities: the *dot product* and the *cross product*. We will only examine the dot product here; you may encounter the cross product in more advanced mathematics courses.

The dot product of two vectors involves multiplying two vectors together, and the result is a scalar.

dot product

The **dot product** of two vectors $v = \langle a, b \rangle$ and $u = \langle c, d \rangle$ is the sum of the product of the horizontal components and the product of the vertical components.

$$v \cdot u = ac + bd$$

To find the angle between the two vectors, use the formula below.

$$\cos \theta = \frac{v}{|v|} \cdot \frac{u}{|u|}$$

Example 14 Finding the Dot Product of Two Vectors

Find the dot product of $v = \langle 5, 12 \rangle$ and $u = \langle -3, 4 \rangle$.

Solution Using the formula, we have

$$\begin{aligned} v \cdot u &= \langle 5, 12 \rangle \cdot \langle -3, 4 \rangle \\ &= 5 \cdot (-3) + 12 \cdot 4 \\ &= -15 + 48 \\ &= 33 \end{aligned}$$

Example 15 Finding the Dot Product of Two Vectors and the Angle between Them

Find the dot product of $v_1 = 5i + 2j$ and $v_2 = 3i + 7j$. Then, find the angle between the two vectors.

Solution Finding the dot product, we multiply corresponding components.

$$\begin{aligned} v_1 \cdot v_2 &= \langle 5, 2 \rangle \cdot \langle 3, 7 \rangle \\ &= 5 \cdot 3 + 2 \cdot 7 \\ &= 15 + 14 \\ &= 29 \end{aligned}$$

To find the angle between them, we use the formula $\cos \theta = \frac{v}{|v|} \cdot \frac{u}{|u|}$

$$\begin{aligned} \frac{v}{|v|} \cdot \frac{u}{|u|} &= \left\langle \frac{5}{\sqrt{29}} + \frac{2}{\sqrt{29}} \right\rangle \cdot \left\langle \frac{3}{\sqrt{58}} + \frac{7}{\sqrt{58}} \right\rangle \\ &= \frac{5}{\sqrt{29}} \cdot \frac{3}{\sqrt{58}} + \frac{2}{\sqrt{29}} \cdot \frac{7}{\sqrt{58}} \\ &= \frac{15}{\sqrt{1682}} + \frac{14}{\sqrt{1682}} = \frac{29}{\sqrt{1682}} \\ &= 0.707107 \end{aligned}$$

$$\cos^{-1}(0.707107) = 45°$$

See **Figure 17**.

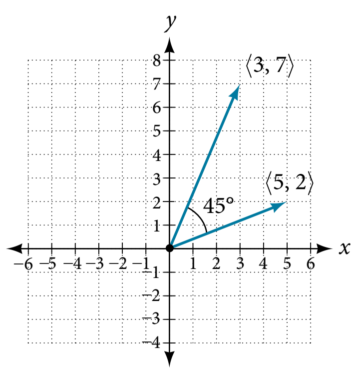


Figure 17

Example 16 Finding the Angle between Two Vectors

Find the angle between $u = \langle -3, 4 \rangle$ and $v = \langle 5, 12 \rangle$.

Solution Using the formula, we have

$$\theta = \cos^{-1}\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right)$$

$$\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right) = \frac{-3i + 4j}{5} \cdot \frac{5i + 12j}{13}$$

$$= \left(-\frac{3}{5} \cdot \frac{5}{13}\right) + \left(\frac{4}{5} \cdot \frac{12}{13}\right)$$

$$= -\frac{15}{65} + \frac{48}{65}$$

$$= \frac{33}{65}$$

$$\theta = \cos^{-1}\left(\frac{33}{65}\right)$$

$$= 59.5°$$

See **Figure 18**.

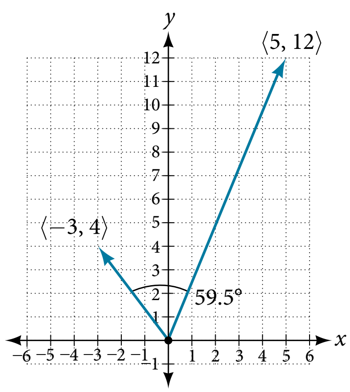


Figure 18

Example 17 Finding Ground Speed and Bearing Using Vectors

We now have the tools to solve the problem we introduced in the opening of the section.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour. What are the ground speed and actual bearing of the plane? See **Figure 19**.

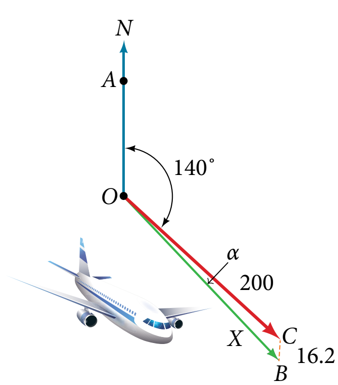


Figure 19

Solution The ground speed is represented by x in the diagram, and we need to find the angle α in order to calculate the adjusted bearing, which will be $140° + \alpha$.

Notice in **Figure 19**, that angle *BCO* must be equal to angle *AOC* by the rule of alternating interior angles, so angle *BCO* is 140°. We can find x by the Law of Cosines:

$$\begin{aligned} x^2 &= (16.2)^2 + (200)^2 - 2(16.2)(200)\cos(140°) \\ x^2 &= 45{,}226.41 \\ x &= \sqrt{45,226.41} \\ x &= 212.7 \end{aligned}$$

The ground speed is approximately 213 miles per hour. Now we can calculate the bearing using the Law of Sines.

$$\begin{aligned} \frac{\sin\alpha}{16.2} &= \frac{\sin(140°)}{212.7} \\ \sin\alpha &= \frac{16.2\sin(140°)}{212.7} \\ &= 0.04896 \\ \sin^{-1}(0.04896) &= 2.8° \end{aligned}$$

Therefore, the plane has a SE bearing of $140° + 2.8° = 142.8°$. The ground speed is 212.7 miles per hour.

Access these online resources for additional instruction and practice with vectors.

- Introduction to Vectors (http://openstaxcollege.org/l/introvectors)
- Vector Operations (http://openstaxcollege.org/l/vectoroperation)
- The Unit Vector (http://openstaxcollege.org/l/unitvector)

8.8 SECTION EXERCISES

VERBAL

1. What are the characteristics of the letters that are commonly used to represent vectors?

2. How is a vector more specific than a line segment?

3. What are i and j, and what do they represent?

4. What is component form?

5. When a unit vector is expressed as $\langle a, b\rangle$, which letter is the coefficient of the i and which the j?

ALGEBRAIC

6. Given a vector with initial point $(5, 2)$ and terminal point $(-1, -3)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form $\langle a, b\rangle$.

7. Given a vector with initial point $(-4, 2)$ and terminal point $(3, -3)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form $\langle a, b\rangle$.

8. Given a vector with initial point $(7, -1)$ and terminal point $(-1, -7)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form $\langle a, b\rangle$.

For the following exercises, determine whether the two vectors u and v are equal, where u has an initial point P_1 and a terminal point P_2 and v has an initial point P_3 and a terminal point P_4.

9. $P_1 = (5, 1)$, $P_2 = (3, -2)$, $P_3 = (-1, 3)$, and $P_4 = (9, -4)$

10. $P_1 = (2, -3)$, $P_2 = (5, 1)$, $P_3 = (6, -1)$, and $P_4 = (9, 3)$

11. $P_1 = (-1, -1)$, $P_2 = (-4, 5)$, $P_3 = (-10, 6)$, and $P_4 = (-13, 12)$

12. $P_1 = (3, 7)$, $P_2 = (2, 1)$, $P_3 = (1, 2)$, and $P_4 = (-1, -4)$

13. $P_1 = (8, 3)$, $P_2 = (6, 5)$, $P_3 = (11, 8)$, and $P_4 = (9, 10)$

14. Given initial point $P_1 = (-3, 1)$ and terminal point $P_2 = (5, 2)$, write the vector v in terms of i and j.

15. Given initial point $P_1 = (6, 0)$ and terminal point $P_2 = (-1, -3)$, write the vector v in terms of i and j.

For the following exercises, use the vectors $u = i + 5j$, $v = -2i - 3j$, and $w = 4i - j$.

16. Find $u + (v - w)$

17. Find $4v + 2u$

For the following exercises, use the given vectors to compute $u + v$, $u - v$, and $2u - 3v$.

18. $u = \langle 2, -3\rangle$, $v = \langle 1, 5\rangle$

19. $u = \langle -3, 4\rangle$, $v = \langle -2, 1\rangle$

20. Let $v = -4i + 3j$. Find a vector that is half the length and points in the same direction as v.

21. Let $v = 5i + 2j$. Find a vector that is twice the length and points in the opposite direction as v.

For the following exercises, find a unit vector in the same direction as the given vector.

22. $a = 3i + 4j$

23. $b = -2i + 5j$

24. $c = 10i - j$

25. $d = -\frac{1}{3}i + \frac{5}{2}j$

26. $u = 100i + 200j$

27. $u = -14i + 2j$

For the following exercises, find the magnitude and direction of the vector, $0 \le \theta < 2\pi$.

28. $\langle 0, 4\rangle$

29. $\langle 6, 5\rangle$

30. $\langle 2, -5\rangle$

31. $\langle -4, -6\rangle$

32. Given $u = 3i - 4j$ and $v = -2i + 3j$, calculate $u \cdot v$.

33. Given $u = -i - j$ and $v = i + 5j$, calculate $u \cdot v$.

34. Given $u = \langle -2, 4\rangle$ and $v = \langle -3, 1\rangle$, calculate $u \cdot v$.

35. Given $u = \langle -1, 6\rangle$ and $v = \langle 6, -1\rangle$, calculate $u \cdot v$.

GRAPHICAL

For the following exercises, given v, draw v, $3v$ and $\frac{1}{2}v$.

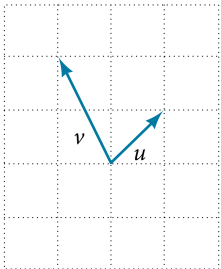
36. $\langle 2, -1 \rangle$

37. $\langle -1, 4 \rangle$

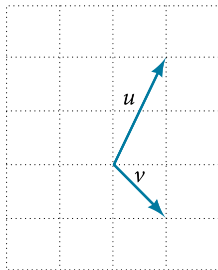
38. $\langle -3, -2 \rangle$

For the following exercises, use the vectors shown to sketch $u + v$, $u - v$, and $2u$.

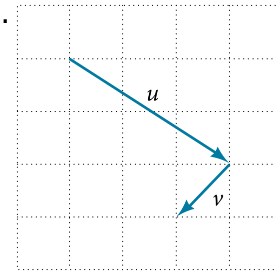
39.



40.

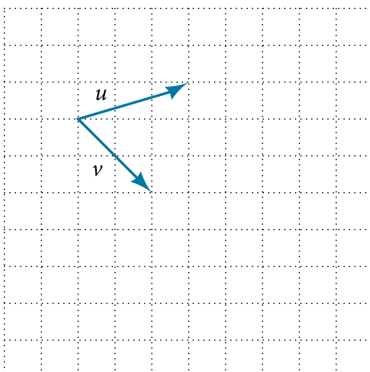


41.

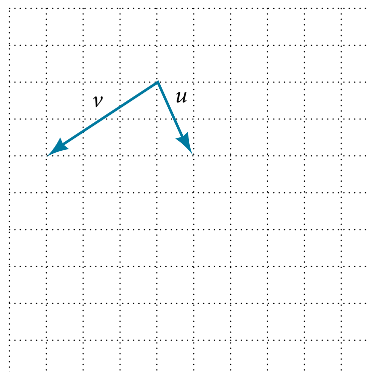


For the following exercises, use the vectors shown to sketch $2u + v$.

42.

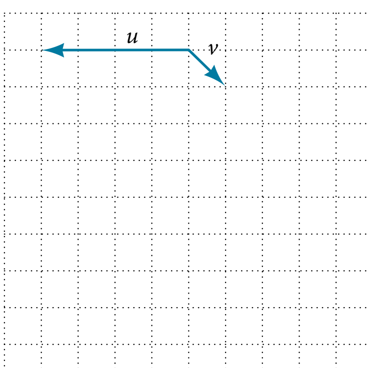


43.

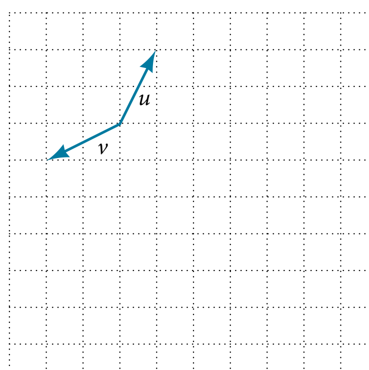


For the following exercises, use the vectors shown to sketch $u - 3v$.

44.

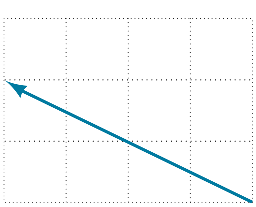


45.

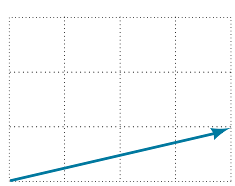


For the following exercises, write the vector shown in component form.

46.



47.



48. Given initial point $P_1 = (2, 1)$ and terminal point $P_2 = (-1, 2)$, write the vector v in terms of i and j, then draw the vector on the graph.

49. Given initial point $P_1 = (4, -1)$ and terminal point $P_2 = (-3, 2)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

50. Given initial point $P_1 = (3, 3)$ and terminal point $P_2 = (-3, 3)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

EXTENSIONS

For the following exercises, use the given magnitude and direction in standard position, write the vector in component form.

51. $|v| = 6, \theta = 45\,°$

52. $|v| = 8, \theta = 220°$

53. $|v| = 2, \theta = 300°$

54. $|v| = 5, \theta = 135°$

55. A 60-pound box is resting on a ramp that is inclined 12°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.

b. Find the magnitude of the component of the force that is parallel to the ramp.

56. A 25-pound box is resting on a ramp that is inclined 8°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.

b. Find the magnitude of the component of the force that is parallel to the ramp.

57. Find the magnitude of the horizontal and vertical components of a vector with magnitude 8 pounds pointed in a direction of 27° above the horizontal. Round to the nearest hundredth.

58. Find the magnitude of the horizontal and vertical components of the vector with magnitude 4 pounds pointed in a direction of 127° above the horizontal. Round to the nearest hundredth.

59. Find the magnitude of the horizontal and vertical components of a vector with magnitude 5 pounds pointed in a direction of 55° above the horizontal. Round to the nearest hundredth.

60. Find the magnitude of the horizontal and vertical components of the vector with magnitude 1 pound pointed in a direction of 8° above the horizontal. Round to the nearest hundredth.

REAL-WORLD APPLICATIONS

61. A woman leaves home and walks 3 miles west, then 2 miles southwest. How far from home is she, and in what direction must she walk to head directly home?

62. A boat leaves the marina and sails 6 miles north, then 2 miles northeast. How far from the marina is the boat, and in what direction must it sail to head directly back to the marina?

63. A man starts walking from home and walks 4 miles east, 2 miles southeast, 5 miles south, 4 miles southwest, and 2 miles east. How far has he walked? If he walked straight home, how far would he have to walk?

64. A woman starts walking from home and walks 4 miles east, 7 miles southeast, 6 miles south, 5 miles southwest, and 3 miles east. How far has she walked? If she walked straight home, how far would she have to walk?

65. A man starts walking from home and walks 3 miles at 20° north of west, then 5 miles at 10° west of south, then 4 miles at 15° north of east. If he walked straight home, how far would he have to the walk, and in what direction?

66. A woman starts walking from home and walks 6 miles at 40° north of east, then 2 miles at 15° east of south, then 5 miles at 30° south of west. If she walked straight home, how far would she have to walk, and in what direction?

67. An airplane is heading north at an airspeed of 600 km/hr, but there is a wind blowing from the southwest at 80 km/hr. How many degrees off course will the plane end up flying, and what is the plane's speed relative to the ground?

68. An airplane is heading north at an airspeed of 500 km/hr, but there is a wind blowing from the northwest at 50 km/hr. How many degrees off course will the plane end up flying, and what is the plane's speed relative to the ground?

69. An airplane needs to head due north, but there is a wind blowing from the southwest at 60 km/hr. The plane flies with an airspeed of 550 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

70. An airplane needs to head due north, but there is a wind blowing from the northwest at 80 km/hr. The plane flies with an airspeed of 500 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

71. As part of a video game, the point (5, 7) is rotated counterclockwise about the origin through an angle of 35°. Find the new coordinates of this point.

72. As part of a video game, the point (7, 3) is rotated counterclockwise about the origin through an angle of 40°. Find the new coordinates of this point.

73. Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 10 mph relative to the car, and the car is traveling 25 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

74. Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 8 mph relative to the car, and the car is traveling 45 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

75. A 50-pound object rests on a ramp that is inclined 19°. Find the magnitude of the components of the force parallel to and perpendicular to (normal) the ramp to the nearest tenth of a pound.

76. Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it upward, and 5 pounds acting on it 45° from the horizontal. What single force is the resultant force acting on the body?

77. Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it $-135°$ from the horizontal, and 5 pounds acting on it directed 150° from the horizontal. What single force is the resultant force acting on the body?

78. The condition of equilibrium is when the sum of the forces acting on a body is the zero vector. Suppose a body has a force of 2 pounds acting on it to the right, 5 pounds acting on it upward, and 3 pounds acting on it 45° from the horizontal. What single force is needed to produce a state of equilibrium on the body?

79. Suppose a body has a force of 3 pounds acting on it to the left, 4 pounds acting on it upward, and 2 pounds acting on it 30° from the horizontal. What single force is needed to produce a state of equilibrium on the body? Draw the vector.

CHAPTER 8 REVIEW

Key Terms

altitude a perpendicular line from one vertex of a triangle to the opposite side, or in the case of an obtuse triangle, to the line containing the opposite side, forming two right triangles

ambiguous case a scenario in which more than one triangle is a valid solution for a given oblique SSA triangle

Archimedes' spiral a polar curve given by $r = \theta$. When multiplied by a constant, the equation appears as $r = a\theta$. As $r = \theta$, the curve continues to widen in a spiral path over the domain.

argument the angle associated with a complex number; the angle between the line from the origin to the point and the positive real axis

cardioid a member of the limaçon family of curves, named for its resemblance to a heart; its equation is given as $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$, where $\frac{a}{b} = 1$

convex limaçon a type of one-loop limaçon represented by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ such that $\frac{a}{b} \geq 2$

De Moivre's Theorem formula used to find the nth power or nth roots of a complex number; states that, for a positive integer n, z^n is found by raising the modulus to the nth power and multiplying the angles by n

dimpled limaçon a type of one-loop limaçon represented by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ such that $1 < \frac{a}{b} < 2$

dot product given two vectors, the sum of the product of the horizontal components and the product of the vertical components

Generalized Pythagorean Theorem an extension of the Law of Cosines; relates the sides of an oblique triangle and is used for SAS and SSS triangles

initial point the origin of a vector

inner-loop limaçon a polar curve similar to the cardioid, but with an inner loop; passes through the pole twice; represented by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a < b$

Law of Cosines states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle

Law of Sines states that the ratio of the measurement of one angle of a triangle to the length of its opposite side is equal to the remaining two ratios of angle measure to opposite side; any pair of proportions may be used to solve for a missing angle or side

lemniscate a polar curve resembling a figure 8 and given by the equation $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta$, $a \neq 0$

magnitude the length of a vector; may represent a quantity such as speed, and is calculated using the Pythagorean Theorem

modulus the absolute value of a complex number, or the distance from the origin to the point (x, y); also called the amplitude

oblique triangle any triangle that is not a right triangle

one-loop limaçon a polar curve represented by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ such that $a > 0$, $b > 0$, and $\frac{a}{b} > 1$; may be dimpled or convex; does not pass through the pole

parameter a variable, often representing time, upon which x and y are both dependent

polar axis on the polar grid, the equivalent of the positive x-axis on the rectangular grid

polar coordinates on the polar grid, the coordinates of a point labeled (r, θ), where θ indicates the angle of rotation from the polar axis and r represents the radius, or the distance of the point from the pole in the direction of θ

polar equation an equation describing a curve on the polar grid

polar form of a complex number a complex number expressed in terms of an angle θ and its distance from the origin r; can be found by using conversion formulas $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2 + y^2}$

pole the origin of the polar grid

resultant a vector that results from addition or subtraction of two vectors, or from scalar multiplication

rose curve a polar equation resembling a flower, given by the equations $r = a\cos n\theta$ and $r = a\sin n\theta$; when n is even there are $2n$ petals, and the curve is highly symmetrical; when n is odd there are n petals.

scalar a quantity associated with magnitude but not direction; a constant

scalar multiplication the product of a constant and each component of a vector

standard position the placement of a vector with the initial point at $(0, 0)$ and the terminal point (a, b), represented by the change in the x-coordinates and the change in the y-coordinates of the original vector

terminal point the end point of a vector, usually represented by an arrow indicating its direction

unit vector a vector that begins at the origin and has magnitude of 1; the horizontal unit vector runs along the x-axis and is defined as $v_1 = \langle 1, 0\rangle$ the vertical unit vector runs along the y-axis and is defined as $v_2 = \langle 0, 1\rangle$.

vector a quantity associated with both magnitude and direction, represented as a directed line segment with a starting point (initial point) and an end point (terminal point)

vector addition the sum of two vectors, found by adding corresponding components

Key Equations

Law of Sines	$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$ $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$
Area for oblique triangles	$\text{Area} = \frac{1}{2}bc\sin\alpha$ $= \frac{1}{2}ac\sin\beta$ $= \frac{1}{2}ab\sin\gamma$
Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos\alpha$ $b^2 = a^2 + c^2 - 2ac\cos\beta$ $c^2 = a^2 + b^2 - 2ab\cos\gamma$
Heron's formula	$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{(a+b+c)}{2}$
Conversion formulas	$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$ $\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$ $r^2 = x^2 + y^2$ $\tan\theta = \frac{y}{x}$

Key Concepts

8.1 Non-right Triangles: Law of Sines

- The Law of Sines can be used to solve oblique triangles, which are non-right triangles.
- According to the Law of Sines, the ratio of the measurement of one of the angles to the length of its opposite side equals the other two ratios of angle measure to opposite side.

- There are three possible cases: ASA, AAS, SSA. Depending on the information given, we can choose the appropriate equation to find the requested solution. See **Example 1**.
- The ambiguous case arises when an oblique triangle can have different outcomes.
- There are three possible cases that arise from SSA arrangement—a single solution, two possible solutions, and no solution. See **Example 2** and **Example 3**.
- The Law of Sines can be used to solve triangles with given criteria. See **Example 4**.
- The general area formula for triangles translates to oblique triangles by first finding the appropriate height value. See **Example 5**.
- There are many trigonometric applications. They can often be solved by first drawing a diagram of the given information and then using the appropriate equation. See **Example 6**.

8.2 Non-right Triangles: Law of Cosines

- The Law of Cosines defines the relationship among angle measurements and lengths of sides in oblique triangles.
- The Generalized Pythagorean Theorem is the Law of Cosines for two cases of oblique triangles: SAS and SSS. Dropping an imaginary perpendicular splits the oblique triangle into two right triangles or forms one right triangle, which allows sides to be related and measurements to be calculated. See **Example 1** and **Example 2**.
- The Law of Cosines is useful for many types of applied problems. The first step in solving such problems is generally to draw a sketch of the problem presented. If the information given fits one of the three models (the three equations), then apply the Law of Cosines to find a solution. See **Example 3** and **Example 4**.
- Heron's formula allows the calculation of area in oblique triangles. All three sides must be known to apply Heron's formula. See **Example 5** and See **Example 6**.

8.3 Polar Coordinates

- The polar grid is represented as a series of concentric circles radiating out from the pole, or origin.
- To plot a point in the form (r, θ), $\theta > 0$, move in a counterclockwise direction from the polar axis by an angle of θ, and then extend a directed line segment from the pole the length of r in the direction of θ. If θ is negative, move in a clockwise direction, and extend a directed line segment the length of r in the direction of θ. See **Example 1**.
- If r is negative, extend the directed line segment in the opposite direction of θ. See **Example 2**.
- To convert from polar coordinates to rectangular coordinates, use the formulas $x = r\cos\theta$ and $y = r\sin\theta$. See **Example 3** and **Example 4**.
- To convert from rectangular coordinates to polar coordinates, use one or more of the formulas: $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, $\tan\theta = \frac{y}{x}$, and $r = \sqrt{x^2 + y^2}$. See **Example 5**.
- Transforming equations between polar and rectangular forms means making the appropriate substitutions based on the available formulas, together with algebraic manipulations. See **Example 6**, **Example 7**, and **Example 8**.
- Using the appropriate substitutions makes it possible to rewrite a polar equation as a rectangular equation, and then graph it in the rectangular plane. See **Example 9**, **Example 10**, and **Example 11**.

8.4 Polar Coordinates: Graphs

- It is easier to graph polar equations if we can test the equations for symmetry with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole.
- There are three symmetry tests that indicate whether the graph of a polar equation will exhibit symmetry. If an equation fails a symmetry test, the graph may or may not exhibit symmetry. See **Example 1**.
- Polar equations may be graphed by making a table of values for θ and r.
- The maximum value of a polar equation is found by substituting the value θ that leads to the maximum value of the trigonometric expression.
- The zeros of a polar equation are found by setting $r = 0$ and solving for θ. See **Example 2**.
- Some formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$. See **Example 3**.
- The formulas that produce the graphs of a cardioid are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$, for $a > 0$, $b > 0$, and $\frac{a}{b} = 1$. See **Example 4**.

- The formulas that produce the graphs of a one-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $1 < \frac{a}{b} < 2$. See **Example 5**.
- The formulas that produce the graphs of an inner-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $a > 0$, $b > 0$, and $a < \text{b}$. See **Example 6**.
- The formulas that produce the graphs of a lemniscates are given by $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta$, where $a \neq 0$. See **Example 7**.
- The formulas that produce the graphs of rose curves are given by $r = a\cos n\theta$ and $r = a\sin n\theta$, where $a \neq 0$; if n is even, there are $2n$ petals, and if n is odd, there are n petals. See **Example 8** and **Example 9**.
- The formula that produces the graph of an Archimedes' spiral is given by $r = \theta$, $\theta \geq 0$. See **Example 10**.

8.5 Polar Form of Complex Numbers

- Complex numbers in the form $a + bi$ are plotted in the complex plane similar to the way rectangular coordinates are plotted in the rectangular plane. Label the x-axis as the real axis and the y-axis as the imaginary axis. See **Example 1**.
- The absolute value of a complex number is the same as its magnitude. It is the distance from the origin to the point: $|z| = \sqrt{a^2 + b^2}$. See **Example 2** and **Example 3**.
- To write complex numbers in polar form, we use the formulas $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2 + y^2}$. Then, $z = r(\cos\theta + i\sin\theta)$. See **Example 4** and **Example 5**.
- To convert from polar form to rectangular form, first evaluate the trigonometric functions. Then, multiply through by r. See **Example 6** and **Example 7**.
- To find the product of two complex numbers, multiply the two moduli and add the two angles. Evaluate the trigonometric functions, and multiply using the distributive property. See **Example 8**.
- To find the quotient of two complex numbers in polar form, find the quotient of the two moduli and the difference of the two angles. See **Example 9**.
- To find the power of a complex number z^n, raise r to the power n, and multiply θ by n. See **Example 10**.
- Finding the roots of a complex number is the same as raising a complex number to a power, but using a rational exponent. See **Example 11**.

8.6 Parametric Equations

- Parameterizing a curve involves translating a rectangular equation in two variables, x and y, into two equations in three variables, x, y, and t. Often, more information is obtained from a set of parametric equations. See **Example 1**, **Example 2**, and **Example 3**.
- Sometimes equations are simpler to graph when written in rectangular form. By eliminating t, an equation in x and y is the result.
- To eliminate t, solve one of the equations for t, and substitute the expression into the second equation. See **Example 4**, **Example 5**, **Example 6**, and **Example 7**.
- Finding the rectangular equation for a curve defined parametrically is basically the same as eliminating the parameter. Solve for t in one of the equations, and substitute the expression into the second equation. See **Example 8**.
- There are an infinite number of ways to choose a set of parametric equations for a curve defined as a rectangular equation.
- Find an expression for x such that the domain of the set of parametric equations remains the same as the original rectangular equation. See **Example 9**.

8.7 Parametric Equations: Graphs

- When there is a third variable, a third parameter on which x and y depend, parametric equations can be used.
- To graph parametric equations by plotting points, make a table with three columns labeled t, $x(t)$, and $y(t)$. Choose values for t in increasing order. Plot the last two columns for x and y. See **Example 1** and **Example 2**.
- When graphing a parametric curve by plotting points, note the associated t-values and show arrows on the graph indicating the orientation of the curve. See **Example 3** and **Example 4**.
- Parametric equations allow the direction or the orientation of the curve to be shown on the graph. Equations that are not functions can be graphed and used in many applications involving motion. See **Example 5**.
- Projectile motion depends on two parametric equations: $x = (v_0 \cos\theta)t$ and $y = -16t^2 + (v_0 \sin\theta)t + h$. Initial velocity is symbolized as v_0. θ represents the initial angle of the object when thrown, and h represents the height at which the object is propelled.

8.8 Vectors

- The position vector has its initial point at the origin. See **Example 1**.
- If the position vector is the same for two vectors, they are equal. See **Example 2**. Vectors are defined by their magnitude and direction. See **Example 3**.
- If two vectors have the same magnitude and direction, they are equal. See **Example 4**.
- Vector addition and subtraction result in a new vector found by adding or subtracting corresponding elements. See **Example 5**.
- Scalar multiplication is multiplying a vector by a constant. Only the magnitude changes; the direction stays the same. See **Example 6** and **Example 7**.
- Vectors are comprised of two components: the horizontal component along the positive x-axis, and the vertical component along the positive y-axis. See **Example 8**.
- The unit vector in the same direction of any nonzero vector is found by dividing the vector by its magnitude.
- The magnitude of a vector in the rectangular coordinate system is $|v| = \sqrt{a^2 + b^2}$. See **Example 9**.
- In the rectangular coordinate system, unit vectors may be represented in terms of i and j where i represents the horizontal component and j represents the vertical component. Then, $v = ai + bj$ is a scalar multiple of v by real numbers a and b. See **Example 10** and **Example 11**.
- Adding and subtracting vectors in terms of i and j consists of adding or subtracting corresponding coefficients of i and corresponding coefficients of j. See **Example 12**.
- A vector $v = ai + bj$ is written in terms of magnitude and direction as $v = |v|\cos\theta \mathrm{i} + |v|\sin\theta j$. See **Example 13**.
- The dot product of two vectors is the product of the i terms plus the product of the j terms. See **Example 14**.
- We can use the dot product to find the angle between two vectors. **Example 15** and **Example 16**.
- Dot products are useful for many types of physics applications. See **Example 17**.

CHAPTER 8 REVIEW EXERCISES

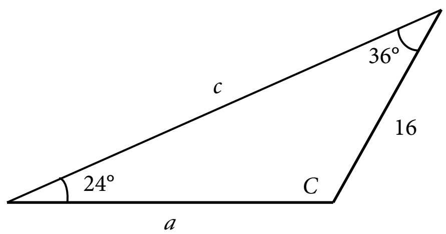
NON-RIGHT TRIANGLES: LAW OF SINES

For the following exercises, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve each triangle, if possible. Round each answer to the nearest tenth.

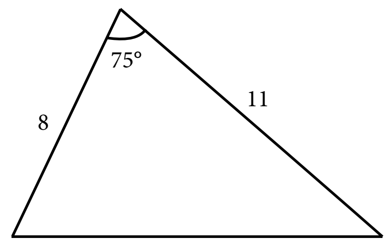
1. $\beta = 50°$, $a = 105$, $b = 45$

2. $\alpha = 43.1°$, $a = 184.2$, $b = 242.8$

3. Solve the triangle.



4. Find the area of the triangle.



5. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 2.1 km apart, to be 25° and 49°, as shown in **Figure 1**. Find the distance of the plane from point A and the elevation of the plane.

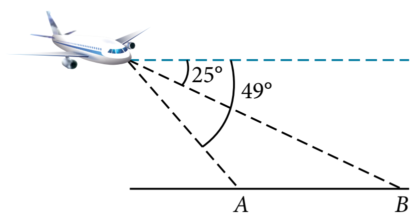


Figure 1

NON-RIGHT TRIANGLES: LAW OF COSINES

6. Solve the triangle, rounding to the nearest tenth, assuming α is opposite side a, β is opposite side b, and γ is opposite side c: $a = 4$, $b = 6$, $c = 8$.

7. Solve the triangle in **Figure 2**, rounding to the nearest tenth.

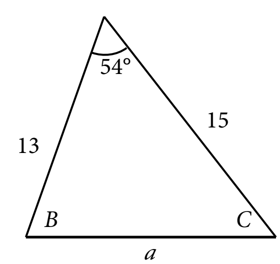


Figure 2

8. Find the area of a triangle with sides of length 8.3, 6.6, and 9.1.

9. To find the distance between two cities, a satellite calculates the distances and angle shown in **Figure 3** (not to scale). Find the distance between the cities. Round answers to the nearest tenth.

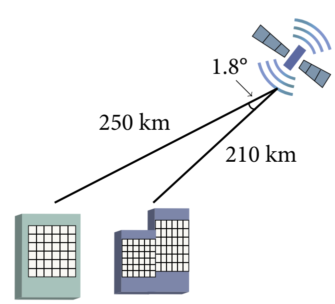


Figure 3

POLAR COORDINATES

10. Plot the point with polar coordinates $\left(3, \frac{\pi}{6}\right)$.

11. Plot the point with polar coordinates $\left(5, -\frac{2\pi}{3}\right)$

12. Convert $\left(6, -\frac{3\pi}{4}\right)$ to rectangular coordinates.

13. Convert $\left(-2, \frac{3\pi}{2}\right)$ to rectangular coordinates.

14. Convert $(7, -2)$ to polar coordinates.

15. Convert $(-9, -4)$ to polar coordinates.

For the following exercises, convert the given Cartesian equation to a polar equation.

16. $x = -2$

17. $x^2 + y^2 = 64$

18. $x^2 + y^2 = -2y$

For the following exercises, convert the given polar equation to a Cartesian equation.

19. $r = 7\cos\theta$

20. $r = \dfrac{-2}{4\cos\theta + \sin\theta}$

For the following exercises, convert to rectangular form and graph.

21. $\theta = \dfrac{3\pi}{4}$

22. $r = 5\sec\theta$

POLAR COORDINATES: GRAPHS

For the following exercises, test each equation for symmetry.

23. $r = 4 + 4\sin\theta$

24. $r = 7$

25. Sketch a graph of the polar equation $r = 1 - 5\sin\theta$. Label the axis intercepts.

26. Sketch a graph of the polar equation $r = 5\sin(7\theta)$.

27. Sketch a graph of the polar equation $r = 3 - 3\cos\theta$

POLAR FORM OF COMPLEX NUMBERS

For the following exercises, find the absolute value of each complex number.

28. $-2 + 6i$

29. $4 - 3i$

Write the complex number in polar form.

30. $5 + 9i$

31. $\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$

For the following exercises, convert the complex number from polar to rectangular form.

32. $z = 5\text{cis}\left(\frac{5\pi}{6}\right)$

33. $z = 3\text{cis}(40°)$

For the following exercises, find the product $z_1 z_2$ in polar form.

34. $z_1 = 2\text{cis}(89°)$, $z_2 = 5\text{cis}(23°)$

35. $z_1 = 10\text{cis}\left(\frac{\pi}{6}\right)$, $z_2 = 6\text{cis}\left(\frac{\pi}{3}\right)$

For the following exercises, find the quotient $\dfrac{z_1}{z_2}$ in polar form.

36. $z_1 = 12\text{cis}(55°)$, $z_2 = 3\text{cis}(18°)$

37. $z_1 = 27\text{cis}\left(\frac{5\pi}{3}\right)$, $z_2 = 9\text{cis}\left(\frac{\pi}{3}\right)$

For the following exercises, find the powers of each complex number in polar form.

38. Find z^4 when $z = 2\text{cis}(70°)$

39. Find z^2 when $z = 5\text{cis}\left(\frac{3\pi}{4}\right)$

For the following exercises, evaluate each root.

40. Evaluate the cube root of z when $z = 64\text{cis}(210°)$.

41. Evaluate the square root of z when $z = 25\text{cis}\left(\frac{3\pi}{2}\right)$.

For the following exercises, plot the complex number in the complex plane.

42. $6 - 2i$

43. $-1 + 3i$

PARAMETRIC EQUATIONS

For the following exercises, eliminate the parameter t to rewrite the parametric equation as a Cartesian equation.

44. $\begin{cases} x(t) = 3t - 1 \\ y(t) = \sqrt{t} \end{cases}$

45. $\begin{cases} x(t) = -\cos t \\ y(t) = 2\sin^2 t \end{cases}$

46. Parameterize (write a parametric equation for) each Cartesian equation by using $x(t) = a\cos t$ and $y(t) = b\sin t$ for $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

47. Parameterize the line from $(-2, 3)$ to $(4, 7)$ so that the line is at $(-2, 3)$ at $t = 0$ and $(4, 7)$ at $t = 1$.

PARAMETRIC EQUATIONS: GRAPHS

For the following exercises, make a table of values for each set of parametric equations, graph the equations, and include an orientation; then write the Cartesian equation.

48. $\begin{cases} x(t) = 3t^2 \\ y(t) = 2t - 1 \end{cases}$

49. $\begin{cases} x(t) = e^t \\ y(t) = -2e^{5t} \end{cases}$

50. $\begin{cases} x(t) = 3\cos t \\ y(t) = 2\sin t \end{cases}$

51. A ball is launched with an initial velocity of 80 feet per second at an angle of 40° to the horizontal. The ball is released at a height of 4 feet above the ground.

a. Find the parametric equations to model the path of the ball.

b. Where is the ball after 3 seconds?

c. How long is the ball in the air?

VECTORS

For the following exercises, determine whether the two vectors, u and v, are equal, where u has an initial point P_1 and a terminal point P_2, and v has an initial point P_3 and a terminal point P_4.

52. $P_1 = (-1, 4)$, $P_2 = (3, 1)$, $P_3 = (5, 5)$ and $P_4 = (9, 2)$

53. $P_1 = (6, 11)$, $P_2 = (-2, 8)$, $P_3 = (0, -1)$ and $P_4 = (-8, 2)$

For the following exercises, use the vectors $u = 2i - j$, $v = 4i - 3j$, and $w = -2i + 5j$ to evaluate the expression.

54. $u - v$

55. $2v - u + w$

For the following exercises, find a unit vector in the same direction as the given vector.

56. $a = 8i - 6j$

57. $b = -3i - j$

For the following exercises, find the magnitude and direction of the vector.

58. $\langle 6, -2 \rangle$

59. $\langle -3, -3 \rangle$

For the following exercises, calculate $u \cdot v$.

60. $u = -2i + j$ and $v = 3i + 7j$

61. $u = i + 4j$ and $v = 4i + 3j$

62. Given $v = \langle -3, 4 \rangle$ draw v, $2v$, and $\frac{1}{2}v$.

63. Given the vectors shown in **Figure 4**, sketch $u + v$, $u - v$ and $3v$.

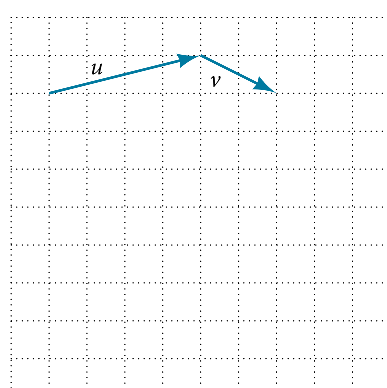


Figure 4

64. Given initial point $P_1 = (3, 2)$ and terminal point $P_2 = (-5, -1)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

CHAPTER 8 PRACTICE TEST

1. Assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve the triangle, if possible, and round each answer to the nearest tenth, given $\beta = 68°$, $b = 21$, $c = 16$.

2. Find the area of the triangle in **Figure 1**. Round each answer to the nearest tenth.

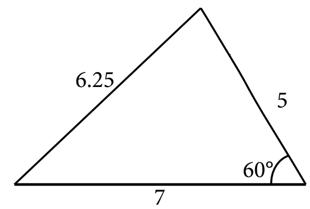


Figure 1

3. A pilot flies in a straight path for 2 hours. He then makes a course correction, heading 15° to the right of his original course, and flies 1 hour in the new direction. If he maintains a constant speed of 575 miles per hour, how far is he from his starting position?

4. Convert (2, 2) to polar coordinates, and then plot the point.

5. Convert $\left(2, \frac{\pi}{3}\right)$ to rectangular coordinates.

6. Convert the polar equation to a Cartesian equation: $x^2 + y^2 = 5y$.

7. Convert to rectangular form and graph: $r = -3\csc\theta$.

8. Test the equation for symmetry: $r = -4\sin(2\theta)$.

9. Graph $r = 3 + 3\cos\theta$.

10. Graph $r = 3 - 5\sin\theta$.

11. Find the absolute value of the complex number $5 - 9i$.

12. Write the complex number in polar form: $4 + i$.

13. Convert the complex number from polar to rectangular form: $z = 5\text{cis}\left(\frac{2\pi}{3}\right)$.

Given $z_1 = 8\text{cis}(36°)$ and $z_2 = 2\text{cis}(15°)$, evaluate each expression.

14. $z_1 z_2$

15. $\frac{z_1}{z_2}$

16. $(z_2)^3$

17. $\sqrt{z_1}$

18. Plot the complex number $-5 - i$ in the complex plane.

19. Eliminate the parameter t to rewrite the following parametric equations as a Cartesian equation:
$$\begin{cases} x(t) = t + 1 \\ y(t) = 2t^2 \end{cases}$$

20. Parameterize (write a parametric equation for) the following Cartesian equation by using $x(t) = a\cos t$ and $y(t) = b\sin t$: $\frac{x^2}{36} + \frac{y^2}{100} = 1$.

21. Graph the set of parametric equations and find the Cartesian equation:
$$\begin{cases} x(t) = -2\sin t \\ y(t) = 5\cos t \end{cases}$$

22. A ball is launched with an initial velocity of 95 feet per second at an angle of 52° to the horizontal. The ball is released at a height of 3.5 feet above the ground.

a. Find the parametric equations to model the path of the ball.

b. Where is the ball after 2 seconds?

c. How long is the ball in the air?

For the following exercises, use the vectors $u = i - 3j$ and $v = 2i + 3j$.

23. Find $2u - 3v$.

24. Calculate $u \cdot v$.

25. Find a unit vector in the same direction as v.

26. Given vector v has an initial point $P_1 = (2, 2)$ and terminal point $P_2 = (-1, 0)$, write the vector v in terms of i and j. On the graph, draw v, and $-v$.

12

Introduction to Calculus

Figure 1 Jamaican sprinter Usain Bolt accelerates out of the blocks. (credit: Nick Webb)

CHAPTER OUTLINE

Introduction

The eight-time world champion and winner of six Olympic gold medals in sprinting, Usain Bolt has truly earned his nickname as the "fastest man on Earth." Also known as the "lightning bolt," he set the track on fire by running at a top speed of 27.79 mph—the fastest time ever recorded by a human runner.

Like the fastest land animal, a cheetah, Bolt does not run at his top speed at every instant. How then, do we approximate his speed at any given instant? We will find the answer to this and many related questions in this chapter.

LEARNING OBJECTIVES

In this section, you will:

- Understand limit notation.
- Find a limit using a graph.
- Find a limit using a table.

12.1 FINDING LIMITS: NUMERICAL AND GRAPHICAL APPROACHES

Intuitively, we know what a limit is. A car can go only so fast and no faster. A trash can might hold 33 gallons and no more. It is natural for measured amounts to have limits. What, for instance, is the limit to the height of a woman? The tallest woman on record was Jinlian Zeng from China, who was 8 ft 1 in.[36] Is this the limit of the height to which women can grow? Perhaps not, but there is likely a limit that we might describe in inches if we were able to determine what it was.

To put it mathematically, the function whose input is a woman and whose output is a measured height in inches has a limit. In this section, we will examine numerical and graphical approaches to identifying limits.

Understanding Limit Notation

We have seen how a sequence can have a limit, a value that the sequence of terms moves toward as the number of terms increases. For example, the terms of the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots$$

gets closer and closer to 0. A sequence is one type of function, but functions that are not sequences can also have limits. We can describe the behavior of the function as the input values get close to a specific value. If the limit of a function $f(x) = L$, then as the input x gets closer and closer to a, the output y-coordinate gets closer and closer to L. We say that the output "approaches" L.

Figure 1 provides a visual representation of the mathematical concept of limit. As the input value x approaches a, the output value $f(x)$ approaches L.

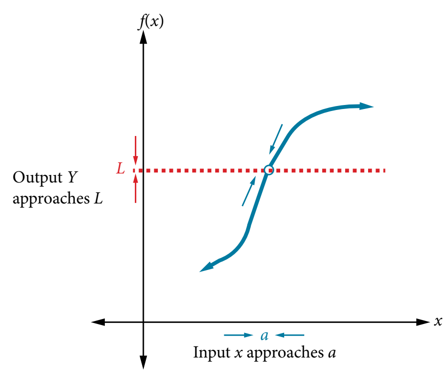


Figure 1 The output (y-coordinate) approaches L as the input (x-coordinate) approaches a.

We write the equation of a limit as

$$\lim_{x \to a} f(x) = L.$$

This notation indicates that as x approaches a both from the left of $x = a$ and the right of $x = a$, the output value approaches L.

Consider the function

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}.$$

We can factor the function as shown.

$$f(x) = \frac{\cancel{(x-7)}(x+1)}{\cancel{x-7}} \qquad \text{Cancel like factors in numerator and denominator.}$$

$$f(x) = x + 1, x \neq 7 \qquad \text{Simplify.}$$

36 https://en.wikipedia.org/wiki/Human_height and http://en.wikipedia.org/wiki/List_of_tallest_people

Notice that x cannot be 7, or we would be dividing by 0, so 7 is not in the domain of the original function. In order to avoid changing the function when we simplify, we set the same condition, $x \neq 7$, for the simplified function. We can represent the function graphically as shown in **Figure 2**.

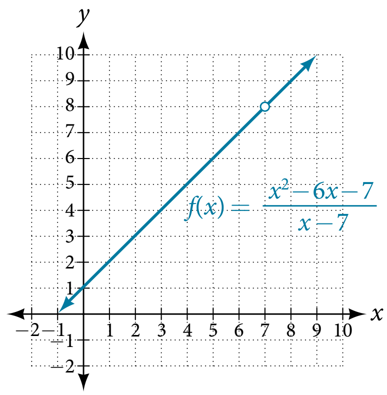


Figure 2 Because 7 is not allowed as an input, there is no point at $x = 7$.

What happens at $x = 7$ is completely different from what happens at points close to $x = 7$ on either side. The notation

$$\lim_{x \to 7} f(x) = 8$$

indicates that as the input x approaches 7 from either the left or the right, the output approaches 8. The output can get as close to 8 as we like if the input is sufficiently near 7.

What happens at $x = 7$? When $x = 7$, there is no corresponding output. We write this as

$$f(7) \text{ does not exist.}$$

This notation indicates that 7 is not in the domain of the function. We had already indicated this when we wrote the function as

$$f(x) = x + 1, x \neq 7.$$

Notice that the limit of a function can exist even when $f(x)$ is not defined at $x = a$. Much of our subsequent work will be determining limits of functions as x nears a, even though the output at $x = a$ does not exist.

the limit of a function

A quantity L is the **limit** of a function $f(x)$ as x approaches a if, as the input values of x approach a (but do not equal a), the corresponding output values of $f(x)$ get closer to L. Note that the value of the limit is not affected by the output value of $f(x)$ at a. Both a and L must be real numbers. We write it as

$$\lim_{x \to a} f(x) = L$$

Example 1 Understanding the Limit of a Function

For the following limit, define a, $f(x)$, and L.

$$\lim_{x \to 2} (3x + 5) = 11$$

Solution First, we recognize the notation of a limit. If the limit exists, as x approaches a, we write

$$\lim_{x \to a} f(x) = L.$$

We are given

$$\lim_{x \to 2} (3x + 5) = 11$$

This means that $a = 2$, $f(x) = 3x + 5$, and $L = 11$.

Analysis *Recall that* $y = 3x + 5$ *is a line with no breaks. As the input values approach* 2*, the output values will get close to* 11*. This may be phrased with the equation* $\lim_{x \to 2} (3x + 5) = 11$*, which means that as x nears* 2 *(but is not exactly* 2*), the output of the function* $f(x) = 3x + 5$ *gets as close as we want to* $3(2) + 5$*, or* 11*, which is the limit L, as we take values of x sufficiently near* 2 *but not at* $x = 2$.

Try It #1

For the following limit, define a, $f(x)$, and L.

$$\lim_{x \to 5} (2x^2 - 4) = 46$$

Understanding Left-Hand Limits and Right-Hand Limits

We can approach the input of a function from either side of a value—from the left or the right. **Figure 3** shows the values of

$$f(x) = x + 1, x \neq 7$$

as described earlier and depicted in **Figure 2**.

	Values of x approach 7 from the left ($x < 7$)			$x = 7$	Values of x approach 7 from the right ($x > 7$)		
x	6.9	6.99	6.999	7	7.001	7.01	7.1
$f(x)$	7.9	7.99	7.999	Undefined	8.001	8.01	8.1

Values of output approach the limit, 8 $\rightarrow$ $\leftarrow$ Values of output approach the limit, 8

Figure 3

Values described as "from the left" are less than the input value 7 and would therefore appear to the left of the value on a number line. The input values that approach 7 from the left in **Figure 3** are 6.9, 6.99, and 6.999. The corresponding outputs are 7.9, 7.99, and 7.999. These values are getting closer to 8. The limit of values of $f(x)$ as x approaches from the left is known as the left-hand limit. For this function, 8 is the left-hand limit of the function $f(x) = x + 1, x \neq 7$ as x approaches 7.

Values described as "from the right" are greater than the input value 7 and would therefore appear to the right of the value on a number line. The input values that approach 7 from the right in **Figure 3** are 7.1, 7.01, and 7.001. The corresponding outputs are 8.1, 8.01, and 8.001. These values are getting closer to 8. The limit of values of $f(x)$ as x approaches from the right is known as the right-hand limit. For this function, 8 is also the right-hand limit of the function $f(x) = x + 1, x \neq 7$ as x approaches 7.

Figure 3 shows that we can get the output of the function within a distance of 0.1 from 8 by using an input within a distance of 0.1 from 7. In other words, we need an input x within the interval $6.9 < x < 7.1$ to produce an output value of $f(x)$ within the interval $7.9 < f(x) < 8.1$.

We also see that we can get output values of $f(x)$ successively closer to 8 by selecting input values closer to 7. In fact, we can obtain output values within any specified interval if we choose appropriate input values.

Figure 4 provides a visual representation of the left- and right-hand limits of the function. From the graph of $f(x)$, we observe the output can get infinitesimally close to $L = 8$ as x approaches 7 from the left and as x approaches 7 from the right.

To indicate the left-hand limit, we write

$$\lim_{x \to 7^-} f(x) = 8.$$

To indicate the right-hand limit, we write

$$\lim_{x \to 7^+} f(x) = 8.$$

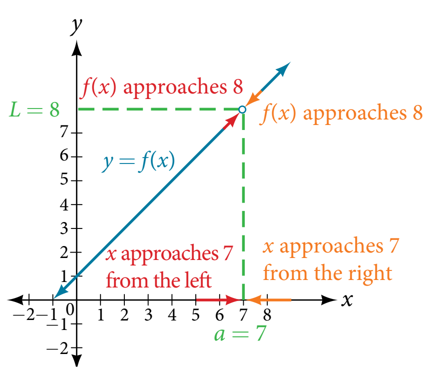


Figure 4 The left- and right-hand limits are the same for this function.

left- and right-hand limits

The **left-hand limit** of a function $f(x)$ as x approaches a from the left is equal to L, denoted by

$$\lim_{x \to a^-} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a such that $x < a$ and $x \neq a$.

The **right-hand limit** of a function $f(x)$, as x approaches a from the right, is equal to L, denoted by

$$\lim_{x \to a^+} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a but greater than a. Both a and L are real numbers.

Understanding Two-Sided Limits

In the previous example, the left-hand limit and right-hand limit as x approaches a are equal. If the left- and right-hand limits are equal, we say that the function $f(x)$ has a **two-sided limit** as x approaches a. More commonly, we simply refer to a two-sided limit as a limit. If the left-hand limit does not equal the right-hand limit, or if one of them does not exist, we say the limit does not exist.

the two-sided limit of function as x approaches a

The limit of a function $f(x)$, as x approaches a, is equal to L, that is,

$$\lim_{x \to a} f(x) = L.$$

if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

In other words, the left-hand limit of a function $f(x)$ as x approaches a is equal to the right-hand limit of the same function as x approaches a. If such a limit exists, we refer to the limit as a two-sided limit. Otherwise we say the limit does not exist.

Finding a Limit Using a Graph

To visually determine if a limit exists as x approaches a, we observe the graph of the function when x is very near to $x = a$. In **Figure 5** we observe the behavior of the graph on both sides of a.

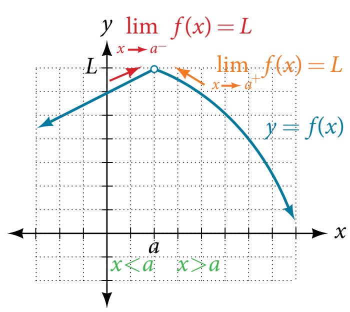


Figure 5

To determine if a left-hand limit exists, we observe the branch of the graph to the left of $x = a$, but near $x = a$. This is where $x < a$. We see that the outputs are getting close to some real number L so there is a left-hand limit.

To determine if a right-hand limit exists, observe the branch of the graph to the right of $x = a$, but near $x = a$. This is where $x > a$. We see that the outputs are getting close to some real number L, so there is a right-hand limit.

If the left-hand limit and the right-hand limit are the same, as they are in **Figure 5**, then we know that the function has a two-sided limit. Normally, when we refer to a “limit,” we mean a two-sided limit, unless we call it a one-sided limit.

Finally, we can look for an output value for the function $f(x)$ when the input value x is equal to a. The coordinate pair of the point would be $(a, f(a))$. If such a point exists, then $f(a)$ has a value. If the point does not exist, as in **Figure 5**, then we say that $f(a)$ does not exist.

How To...

Given a function $f(x)$, use a graph to find the limits and a function value as x approaches a.

1. Examine the graph to determine whether a left-hand limit exists.
2. Examine the graph to determine whether a right-hand limit exists.
3. If the two one-sided limits exist and are equal, then there is a two-sided limit—what we normally call a "limit."
4. If there is a point at $x = a$, then $f(a)$ is the corresponding function value.

Example 2 Finding a Limit Using a Graph

a. Determine the following limits and function value for the function f shown in **Figure 6**.

i. $\lim_{x \to 2^-} f(x)$ **ii.** $\lim_{x \to 2^+} f(x)$ **iii.** $\lim_{x \to 2} f(x)$ **iv.** $f(2)$

b. Determine the following limits and function value for the function f shown in **Figure 7**.

i. $\lim_{x \to 2^-} f(x)$ **ii.** $\lim_{x \to 2^+} f(x)$ **iii.** $\lim_{x \to 2} f(x)$ **iv.** $f(2)$

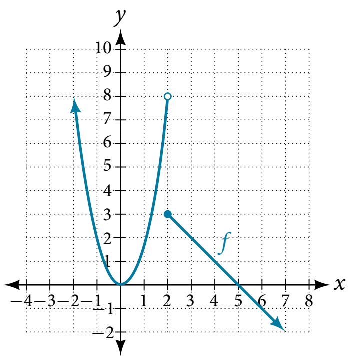


Figure 6

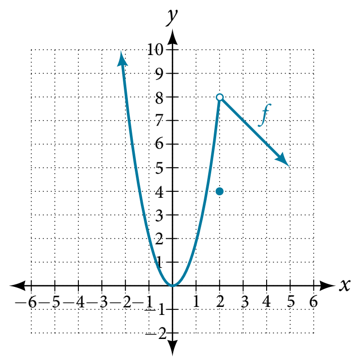


Figure 7

Solution

a. Looking at **Figure 6**:

i. $\lim_{x \to 2^-} f(x) = 8$; when $x < 2$, but infinitesimally close to 2, the output values get close to $y = 8$.

ii. $\lim_{x \to 2^+} f(x) = 3$; when $x > 2$, but infinitesimally close to 2, the output values approach $y = 3$.

iii. $\lim_{x \to 2} f(x)$ does not exist because $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$; the left- and right-hand limits are not equal.

iv. $f(2) = 3$ because the graph of the function f passes through the point $(2, f(2))$ or $(2, 3)$.

b. Looking at **Figure 7**:

i. $\lim_{x \to 2^-} f(x) = 8$; when $x < 2$ but infinitesimally close to 2, the output values approach $y = 8$.

ii. $\lim_{x \to 2^-} f(x) = 8$; when $x > 2$ but infinitesimally close to 2, the output values approach $y = 8$.

iii. $\lim_{x \to 2} f(x) = 8$ because $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 8$; the left and right-hand limits are equal.

iv. $f(2) = 4$ because the graph of the function f passes through the point $(2, f(2))$ or $(2, 4)$.

Try It #2

Using the graph of the function $y = f(x)$ shown in **Figure 8**, estimate the following limits.

a. $\lim_{x \to 0^-} f(x)$ **b.** $\lim_{x \to 0^+} f(x)$ **c.** $\lim_{x \to 0} f(x)$

d. $\lim_{x \to 2^-} f(x)$ **e.** $\lim_{x \to 2^+} f(x)$ **f.** $\lim_{x \to 2} f(x)$

g. $\lim_{x \to 4^-} f(x)$ **h.** $\lim_{x \to 4^+} f(x)$ **i.** $\lim_{x \to 4} f(x)$

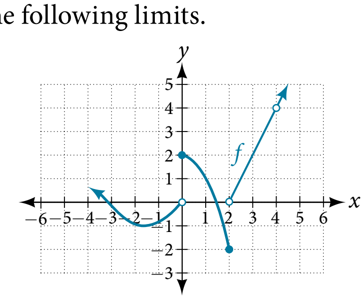


Figure 8

Finding a Limit Using a Table

Creating a table is a way to determine limits using numeric information. We create a table of values in which the input values of x approach a from both sides. Then we determine if the output values get closer and closer to some real value, the limit L.

Let's consider an example using the following function:

$$\lim_{x \to 5} \left(\frac{x^3 - 125}{x - 5} \right)$$

To create the table, we evaluate the function at values close to $x = 5$. We use some input values less than 5 and some values greater than 5 as in **Figure 9**. The table values show that when $x > 5$ but nearing 5, the corresponding output gets close to 75. When $x > 5$ but nearing 5, the corresponding output also gets close to 75.

x	4.9	4.99	4.999	5	5.001	5.01	5.1
$f(x)$	73.51	74.8501	74.985001	Undefined	75.015001	75.1501	76.51

$\lim_{x \to 5^-} f(x) = 75$ $\longrightarrow$ $\longleftarrow$ $\lim_{x \to 5^+} f(x) = 75$

Figure 9

Because

$$\lim_{x \to 5^-} f(x) = 75 = \lim_{x \to 5^+} f(x),$$

then

$$\lim_{x \to 5} f(x) = 75.$$

Remember that $f(5)$ does not exist.

How To...

Given a function f, use a table to find the limit as x approaches a and the value of $f(a)$, if it exists.

1. Choose several input values that approach a from both the left and right. Record them in a table.
2. Evaluate the function at each input value. Record them in the table.
3. Determine if the table values indicate a left-hand limit and a right-hand limit.
4. If the left-hand and right-hand limits exist and are equal, there is a two-sided limit.
5. Replace x with a to find the value of $f(a)$.

Example 3 **Finding a Limit Using a Table**

Numerically estimate the limit of the following expression by setting up a table of values on both sides of the limit.

$$\lim_{x \to 0} \left(\frac{5\sin(x)}{3x} \right)$$

Solution We can estimate the value of a limit, if it exists, by evaluating the function at values near $x = 0$. We cannot find a function value for $x = 0$ directly because the result would have a denominator equal to 0, and thus would be undefined.

$$f(x) = \frac{5\sin(x)}{3x}$$

We create **Figure 10** by choosing several input values close to $x = 0$, with half of them less than $x = 0$ and half of them greater than $x = 0$. Note that we need to be sure we are using radian mode. We evaluate the function at each input value to complete the table.

The table values indicate that when $x < 0$ but approaching 0, the corresponding output nears $\frac{5}{3}$.

When $x > 0$ but approaching 0, the corresponding output also nears $\frac{5}{3}$.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	1.66389	1.666639	1.666666	Undefined	1.666666	1.666639	1.66389

$\lim_{x \to 0^-} f(x) = \frac{5}{3}$ $\qquad$ $\lim_{x \to 0^+} f(x) = \frac{5}{3}$

Figure 10

Because

$$\lim_{x \to 0^-} f(x) = \frac{5}{3} = \lim_{x \to 0^+} f(x),$$

then

$$\lim_{x \to 0^-} f(x) = \frac{5}{3}.$$

Q & A…

Is it possible to check our answer using a graphing utility?

Yes. We previously used a table to find a limit of 75 for the function $f(x) = \frac{x^3 - 125}{x - 5}$ as x approaches 5. To check, we graph the function on a viewing window as shown in **Figure 11**. A graphical check shows both branches of the graph of the function get close to the output 75 as x nears 5. Furthermore, we can use the 'trace' feature of a graphing calculator. By approaching $x = 5$ we may numerically observe the corresponding outputs getting close to 75.

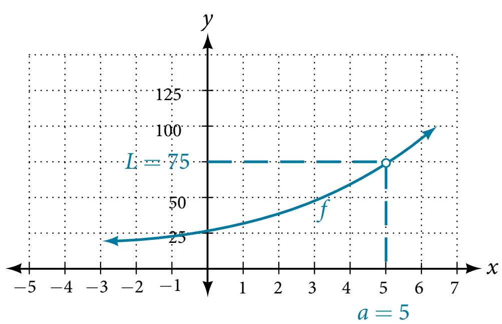


Figure 11

Try It #3

Numerically estimate the limit of the following function by making a table:

$$\lim_{x \to 0} \left(\frac{20\sin(x)}{4x} \right)$$

Q & A…

Is one method for determining a limit better than the other?

No. Both methods have advantages. Graphing allows for quick inspection. Tables can be used when graphical utilities aren't available, and they can be calculated to a higher precision than could be seen with an unaided eye inspecting a graph.

Example 4 Using a Graphing Utility to Determine a Limit

With the use of a graphing utility, if possible, determine the left- and right-hand limits of the following function as x approaches 0. If the function has a limit as x approaches 0, state it. If not, discuss why there is no limit.

$$f(x) = 3\sin\left(\frac{\pi}{x}\right)$$

Solution We can use a graphing utility to investigate the behavior of the graph close to $x = 0$. Centering around $x = 0$, we choose two viewing windows such that the second one is zoomed in closer to $x = 0$ than the first one. The result would resemble **Figure 12** for $[-2, 2]$ by $[-3, 3]$.

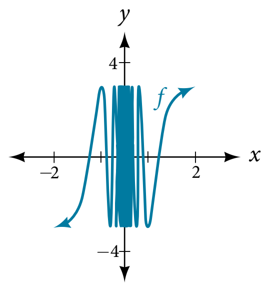


Figure 12

The result would resemble **Figure 13** for $[-0.1, 0.1]$ by $[-3, 3]$.

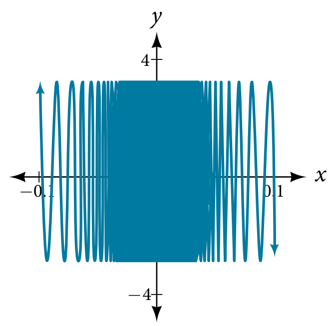


Figure 13 Even closer to zero, we are even less able to distinguish any limits.

The closer we get to 0, the greater the swings in the output values are. That is not the behavior of a function with either a left-hand limit or a right-hand limit. And if there is no left-hand limit or right-hand limit, there certainly is no limit to the function $f(x)$ as x approaches 0.

We write

$$\lim_{x \to 0^-} \left(3\sin\left(\frac{\pi}{x}\right)\right) \text{ does not exist.}$$

$$\lim_{x \to 0^+} \left(3\sin\left(\frac{\pi}{x}\right)\right) \text{ does not exist.}$$

$$\lim_{x \to 0} \left(3\sin\left(\frac{\pi}{x}\right)\right) \text{ does not exist.}$$

Try It #4

Numerically estimate the following limit: $\lim_{x \to 0} \left(\sin\left(\frac{2}{x}\right)\right)$.

Access these online resources for additional instruction and practice with finding limits.

- Introduction to Limits (http://openstaxcollege.org/l/introtolimits)
- Formal Definition of a Limit (http://openstaxcollege.org/l/formaldeflimit)

12.1 SECTION EXERCISES

VERBAL

1. Explain the difference between a value at $x = a$ and the limit as x approaches a.

2. Explain why we say a function does not have a limit as x approaches a if, as x approaches a, the left-hand limit is not equal to the right-hand limit.

GRAPHICAL

For the following exercises, estimate the functional values and the limits from the graph of the function f provided in **Figure 14**.

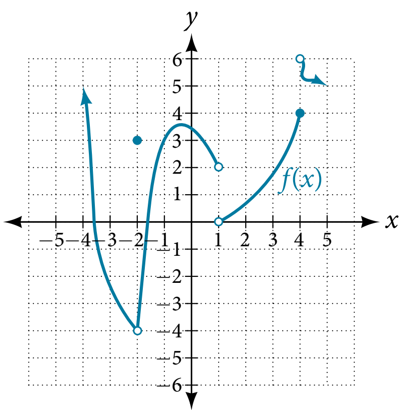


Figure 14

3. $\lim\limits_{x \to -2^-} f(x)$

4. $\lim\limits_{x \to -2^+} f(x)$

5. $\lim\limits_{x \to -2} f(x)$

6. $f(-2)$

7. $\lim\limits_{x \to -1^-} f(x)$

8. $\lim\limits_{x \to 1^+} f(x)$

9. $\lim\limits_{x \to 1} f(x)$

10. $f(1)$

11. $\lim\limits_{x \to 4^-} f(x)$

12. $\lim\limits_{x \to 4^+} f(x)$

13. $\lim\limits_{x \to 4} f(x)$

14. $f(4)$

For the following exercises, draw the graph of a function from the functional values and limits provided.

15. $\lim\limits_{x \to 0^-} f(x) = 2,\ \lim\limits_{x \to 0^+} f(x) = -3,\ \lim\limits_{x \to 2} f(x) = 2,$ $f(0) = 4, f(2) = -1, f(-3)$ does not exist.

16. $\lim\limits_{x \to 2^-} f(x) = 0,\ \lim\limits_{x \to 2^+} = -2,\ \lim\limits_{x \to 0} f(x) = 3,$ $f(2) = 5, f(0)$

17. $\lim\limits_{x \to 2^-} f(x) = 2,\ \lim\limits_{x \to 2^+} = -3,\ \lim\limits_{x \to 0} f(x) = 5, f(0) = 1,$ $f(1) = 0$

18. $\lim\limits_{x \to 3^-} f(x) = 0,\ \lim\limits_{x \to 3^+} f(x) = 5,\ \lim\limits_{x \to 5} f(x) = 0, f(5) = 4,$ $f(3)$ does not exist.

19. $\lim\limits_{x \to 4} f(x)= 6,\ \lim\limits_{x \to 6^+} f(x) = -1,\ \lim\limits_{x \to 0} f(x) = 5, f(4) = 6,$ $f\ (2) = 6$

20. $\lim\limits_{x \to -3} f(x) = 2,\ \lim\limits_{x \to 1^+} f(x) = -2,\ \lim\limits_{x \to 3} f(x) = -\ 4,$ $f(-3) = 0, f(0) = 0$

21. $\lim\limits_{x \to \pi} f(x) = \pi^2,\ \lim\limits_{x \to -\pi} f(x) = \dfrac{\pi}{2},\ \lim\limits_{x \to 1^-} f(x) = 0,$ $f(\pi) = \sqrt{2}, f(0)$ does not exist.

For the following exercises, use a graphing calculator to determine the limit to 5 decimal places as x approaches 0.

22. $f(x) = (1 + x)^{\frac{1}{x}}$

23. $g(x) = (1 + x)^{\frac{2}{x}}$

24. $h(x) = (1 + x)^{\frac{3}{x}}$

25. $i(x) = (1 + x)^{\frac{4}{x}}$

26. $j(x) = (1 + x)^{\frac{5}{x}}$

27. Based on the pattern you observed in the exercises above, make a conjecture as to the limit of $f(x) = (1 + x)^{\frac{6}{x}}, g(x) = (1 + x)^{\frac{7}{x}}$, and $h(x) = (1 + x)^{\frac{n}{x}}$.

For the following exercises, use a graphing utility to find graphical evidence to determine the left- and right-hand limits of the function given as x approaches a. If the function has a limit as x approaches a, state it. If not, discuss why there is no limit.

28. $(x) = \begin{cases} |x| - 1, & \text{if } x \neq 1 \\ x^3, & \text{if } x = 1 \end{cases} \quad a = 1$

29. $(x) = \begin{cases} \dfrac{1}{x + 1}, & \text{if } x = -2 \\ (x + 1)^2, & \text{if } x \neq -2 \end{cases} \quad a = -2$

NUMERIC

For the following exercises, use numerical evidence to determine whether the limit exists at $x = a$. If not, describe the behavior of the graph of the function near $x = a$. Round answers to two decimal places.

30. $f(x) = \dfrac{x^2 - 4x}{16 - x^2}; a = 4$

31. $f(x) = \dfrac{x^2 - x - 6}{x^2 - 9}; a = 3$

32. $f(x) = \dfrac{x^2 - 6x - 7}{x^2 - 7x}; a = 7$

33. $f(x) = \dfrac{x^2 - 1}{x^2 - 3x + 2}; a = 1$

34. $f(x) = \dfrac{1 - x^2}{x^2 - 3x + 2}; a = 1$

35. $f(x) = \dfrac{10 - 10x^2}{x^2 - 3x + 2}; a = 1$

36. $f(x) = \dfrac{x}{6x^2 - 5x - 6}; a = \dfrac{3}{2}$

37. $f(x) = \dfrac{x}{4x^2 + 4x + 1}; a = -\dfrac{1}{2}$

38. $f(x) = \dfrac{2}{x - 4}; a = 4$

For the following exercises, use a calculator to estimate the limit by preparing a table of values. If there is no limit, describe the behavior of the function as x approaches the given value.

39. $\lim\limits_{x \to 0} \dfrac{7\tan x}{3x}$

40. $\lim\limits_{x \to 4} \dfrac{x^2}{x - 4}$

41. $\lim\limits_{x \to 0} \dfrac{2\sin x}{4\tan x}$

For the following exercises, use a graphing utility to find numerical or graphical evidence to determine the left and right-hand limits of the function given as x approaches a. If the function has a limit as x approaches a, state it. If not, discuss why there is no limit.

42. $\lim\limits_{x \to 0} e^{e^{\frac{1}{x}}}$

43. $\lim\limits_{x \to 0} e^{e^{-\frac{1}{x^2}}}$

44. $\lim\limits_{x \to 0} \dfrac{|x|}{x}$

45. $\lim\limits_{x \to -1} \dfrac{|x + 1|}{x + 1}$

46. $\lim\limits_{x \to 5} \dfrac{|x - 5|}{5 - x}$

47. $\lim\limits_{x \to -1} \dfrac{1}{(x + 1)^2}$

48. $\lim\limits_{x \to 1} \dfrac{1}{(x - 1)^3}$

49. $\lim\limits_{x \to 0} \dfrac{5}{1 - e^{\frac{2}{x}}}$

50. Use numerical and graphical evidence to compare and contrast the limits of two functions whose formulas appear similar: $f(x) = \left|\dfrac{1 - x}{x}\right|$ and $g(x) = \left|\dfrac{1 + x}{x}\right|$ as x approaches 0. Use a graphing utility, if possible, to determine the left- and right-hand limits of the functions $f(x)$ and $g(x)$ as x approaches 0. If the functions have a limit as x approaches 0, state it. If not, discuss why there is no limit.

EXTENSIONS

51. According to the Theory of Relativity, the mass m of a particle depends on its velocity v. That is

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_o is the mass when the particle is at rest and c is the speed of light. Find the limit of the mass, m, as v approaches c^-.

52. Allow the speed of light, c, to be equal to 1.0. If the mass, m, is 1, what occurs to m as $v \to c$? Using the values listed in **Table 1**, make a conjecture as to what the mass is as v approaches 1.00.

v	m
0.5	1.15
0.9	2.29
0.95	3.20
0.99	7.09
0.999	22.36
0.99999	223.61

Table 1

LEARNING OBJECTIVES

In this section, you will:

- Find the limit of a sum, a difference, and a product.
- Find the limit of a polynomial.
- Find the limit of a power or a root.
- Find the limit of a quotient.

12.2 FINDING LIMITS: PROPERTIES OF LIMITS

Consider the rational function

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$

The function can be factored as follows:

$$f(x) = \frac{\cancel{(x-7)}(x+1)}{\cancel{x-7}}, \text{ which gives us } f(x) = x + 1, x \neq 7.$$

Does this mean the function f is the same as the function $g(x) = x + 1$?

The answer is no. Function f does not have $x = 7$ in its domain, but g does. Graphically, we observe there is a hole in the graph of $f(x)$ at $x = 7$, as shown in **Figure 1** and no such hole in the graph of $g(x)$, as shown in **Figure 2**.

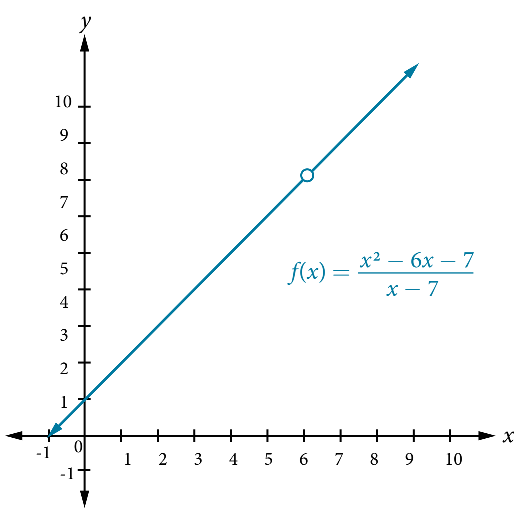


Figure 1 The graph of function f contains a break at $x = 7$ and is therefore not continuous at $x = 7$.

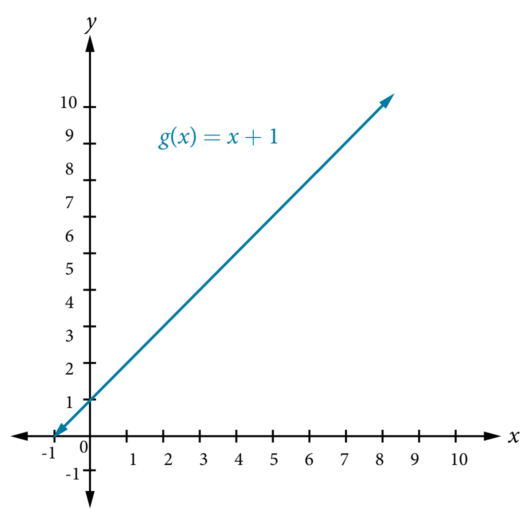


Figure 2 The graph of function g is continuous.

So, do these two different functions also have different limits as x approaches 7?

Not necessarily. Remember, in determining a limit of a function as x approaches a, what matters is whether the output approaches a real number as we get close to $x = a$. The existence of a limit does not depend on what happens when x equals a.

Look again at **Figure 1** and **Figure 2**. Notice that in both graphs, as x approaches 7, the output values approach 8. This means

$$\lim_{x \to 7} f(x) = \lim_{x \to 7} g(x).$$

Remember that when determining a limit, the concern is what occurs near $x = a$, not at $x = a$. In this section, we will use a variety of methods, such as rewriting functions by factoring, to evaluate the limit. These methods will give us formal verification for what we formerly accomplished by intuition.

Finding the Limit of a Sum, a Difference, and a Product

Graphing a function or exploring a table of values to determine a limit can be cumbersome and time-consuming. When possible, it is more efficient to use the **properties of limits**, which is a collection of theorems for finding limits.

Knowing the properties of limits allows us to compute limits directly. We can add, subtract, multiply, and divide the limits of functions as if we were performing the operations on the functions themselves to find the limit of the result. Similarly, we can find the limit of a function raised to a power by raising the limit to that power. We can also find the limit of the root of a function by taking the root of the limit. Using these operations on limits, we can find the limits of more complex functions by finding the limits of their simpler component functions.

properties of limits

Let a, k, A, and B represent real numbers, and f and g be functions, such that $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$. For limits that exist and are finite, the properties of limits are summarized in **Table 1.**

Constant, k	$\lim_{x \to a} k = k$
Constant times a function	$\lim_{x \to a} [k \cdot f(x)] = k \lim_{x \to a} f(x) = kA$
Sum of functions	$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = A + B$
Difference of functions	$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = A - B$
Product of functions	$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = A \cdot B$
Quotient of functions	$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{A}{B}, B \neq 0$
Function raised to an exponent	$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to \infty} f(x)\right]^n = A^n$, where n is a positive integer
nth root of a function, where n is a positive integer	$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} [f(x)]} = \sqrt[n]{A}$
Polynomial function	$\lim_{x \to a} p(x) = p(a)$

Table 1

Example 1 Evaluating the Limit of a Function Algebraically

Evaluate $\lim_{x \to 3} (2x + 5)$.

Solution

$$\begin{aligned} \lim_{x \to 3} (2x + 5) &= \lim_{x \to 3} (2x) + \lim_{x \to 3} (5) && \text{Sum of functions property} \\ &= 2 \lim_{x \to 3} (x) + \lim_{x \to 3} (5) && \text{Constant times a function property} \\ &= 2(3) + 5 && \text{Evaluate} \\ &= 11 \end{aligned}$$

Try It #1

Evaluate the following limit: $\lim_{x \to -12} (-2x + 2)$.

Finding the Limit of a Polynomial

Not all functions or their limits involve simple addition, subtraction, or multiplication. Some may include polynomials. Recall that a polynomial is an expression consisting of the sum of two or more terms, each of which consists of a constant and a variable raised to a nonnegative integral power. To find the limit of a polynomial function, we can find the limits of the individual terms of the function, and then add them together. Also, the limit of a polynomial function as x approaches a is equivalent to simply evaluating the function for a.

How To…

Given a function containing a polynomial, find its limit.

1. Use the properties of limits to break up the polynomial into individual terms.
2. Find the limits of the individual terms.
3. Add the limits together.
4. Alternatively, evaluate the function for a.

Example 2 Evaluating the Limit of a Function Algebraically

Evaluate $\lim\limits_{x \to 3}(5x^2)$.

Solution

$$\begin{aligned} \lim_{x \to 3}(5x^2) &= 5\lim_{x \to 3}(x^2) && \text{Constant times a function property} \\ &= 5(3^2) && \text{Function raised to an exponent property} \\ &= 45 \end{aligned}$$

Try It #2

Evaluate $\lim\limits_{x \to 4}(x^3 - 5)$.

Example 3 Evaluating the Limit of a Polynomial Algebraically

Evaluate $\lim\limits_{x \to 5}(2x^3 - 3x + 1)$.

Solution

$$\begin{aligned} \lim_{x \to 5}(2x^3 - 3x + 1) &= \lim_{x \to 5}(2x^3) - \lim_{x \to 5}(3x) + \lim_{x \to 5}(1) && \text{Sum of functions} \\ &= 2\lim_{x \to 5}(x^3) - 3\lim_{x \to 5}(x) + \lim_{x \to 5}(1) && \text{Constant times a function} \\ &= 2(5^3) - 3(5) + 1 && \text{Function raised to an exponent} \\ &= 236 && \text{Evaluate} \end{aligned}$$

Try It #3

Evaluate the following limit: $\lim\limits_{x \to -1}(x^4 - 4x^3 + 5)$.

Finding the Limit of a Power or a Root

When a limit includes a power or a root, we need another property to help us evaluate it. The square of the limit of a function equals the limit of the square of the function; the same goes for higher powers. Likewise, the square root of the limit of a function equals the limit of the square root of the function; the same holds true for higher roots.

Example 4 Evaluating a Limit of a Power

Evaluate $\lim\limits_{x \to 2}(3x + 1)^5$.

Solution We will take the limit of the function as x approaches 2 and raise the result to the 5th power.

$$\begin{aligned}\lim_{x \to 2}(3x+1)^5 &= \left(\lim_{x \to 2}(3x+1)\right)^5 \\ &= (3(2)+1)^5 \\ &= 7^5 \\ &= 16{,}807\end{aligned}$$

Try It #4

Evaluate the following limit: $\lim\limits_{x \to -4}(10x+36)^3$.

Q & A…

If we can't directly apply the properties of a limit, for example in $\lim\limits_{x \to 2}\left(\dfrac{x^2+6x+8}{x-2}\right)$, can we still determine the limit of the function as x approaches a?

Yes. Some functions may be algebraically rearranged so that one can evaluate the limit of a simplified equivalent form of the function.

Finding the Limit of a Quotient

Finding the limit of a function expressed as a quotient can be more complicated. We often need to rewrite the function algebraically before applying the properties of a limit. If the denominator evaluates to 0 when we apply the properties of a limit directly, we must rewrite the quotient in a different form. One approach is to write the quotient in factored form and simplify.

How To…

Given the limit of a function in quotient form, use factoring to evaluate it.

1. Factor the numerator and denominator completely.
2. Simplify by dividing any factors common to the numerator and denominator.
3. Evaluate the resulting limit, remembering to use the correct domain.

Example 5 Evaluating the Limit of a Quotient by Factoring

Evaluate $\lim\limits_{x \to 2}\left(\dfrac{x^2-6x+8}{x-2}\right)$.

Solution Factor where possible, and simplify.

$$\begin{aligned}\lim_{x \to 2}\left(\frac{x^2-6x+8}{x-2}\right) &= \lim_{x \to 2}\left(\frac{(x-2)(x-4)}{x-2}\right) && \text{Factor the numerator.} \\ &= \lim_{x \to 2}\left(\frac{\cancel{(x-2)}(x-4)}{\cancel{x-2}}\right) && \text{Cancel the common factors.} \\ &= \lim_{x \to 2}(x-4) && \text{Evaluate.} \\ &= 2-4=-2\end{aligned}$$

Analysis *When the limit of a rational function cannot be evaluated directly, factored forms of the numerator and denominator may simplify to a result that can be evaluated.*

Notice, the function

$$f(x)=\frac{x^2-6x+8}{x-2}$$

is equivalent to the function

$$f(x)=x-4,\ x \neq 2.$$

Notice that the limit exists even though the function is not defined at $x=2$.

Try It #5

Evaluate the following limit: $\lim_{x \to 7}\left(\dfrac{x^2 - 11x + 28}{7 - x}\right)$.

Example 6 Evaluating the Limit of a Quotient by Finding the LCD

Evaluate $\lim_{x \to 5}\left(\dfrac{\frac{1}{x} - \frac{1}{5}}{x - 5}\right)$.

Solution Find the LCD for the denominators of the two terms in the numerator, and convert both fractions to have the LCD as their denominator.

$$\lim_{x \to 5}\left(\frac{\frac{1}{x} - \frac{1}{5}}{x - 5}\right) = \lim_{x \to 5}\left(\frac{5x\left(\frac{1}{x} - \frac{1}{5}\right)}{5x(x - 5)}\right) \quad \text{Multiply numerator and denominator by LCD.}$$

$$= \lim_{x \to 5}\left(\frac{5x\left(\frac{1}{x}\right) - 5x\left(\frac{1}{5}\right)}{5x(x - 5)}\right) \quad \text{Apply distributive property.}$$

$$= \lim_{x \to 5}\left(\frac{5 - x}{5x(x - 5)}\right) \quad \text{Simplify.}$$

$$= \lim_{x \to 5}\left(\frac{-1(x - 5)}{5x(x - 5)}\right) \quad \text{Factor the numerator}$$

$$= \lim_{x \to 5} -\frac{1}{5x} \quad \text{Cancel out like fractions}$$

$$= -\frac{1}{5(5)} \quad \text{Evaluate for } x = 5$$

$$= -\frac{1}{25}$$

Analysis *When determining the limit of a rational function that has terms added or subtracted in either the numerator or denominator, the first step is to find the common denominator of the added or subtracted terms; then, convert both terms to have that denominator, or simplify the rational function by multiplying numerator and denominator by the least common denominator. Then check to see if the resulting numerator and denominator have any common factors.*

Try It #6

Evaluate $\lim_{x \to -5}\left(\dfrac{\frac{1}{5} + \frac{1}{x}}{10 + 2x}\right)$.

How To…

Given a limit of a function containing a root, use a conjugate to evaluate.

1. If the quotient as given is not in indeterminate $\left(\frac{0}{0}\right)$ form, evaluate directly.
2. Otherwise, rewrite the sum (or difference) of two quotients as a single quotient, using the least common denominator (LCD).
3. If the numerator includes a root, rationalize the numerator; multiply the numerator and denominator by the conjugate of the numerator. Recall that $a \pm \sqrt{b}$ are conjugates.
4. Simplify.
5. Evaluate the resulting limit.

Example 7 Evaluating a Limit Containing a Root Using a Conjugate

Evaluate $\lim\limits_{x \to 0}\left(\dfrac{\sqrt{25 - x} - 5}{x}\right)$.

Solution

$$\begin{aligned}
\lim_{x \to 0}\left(\frac{\sqrt{25-x}-5}{x}\right) &= \lim_{x \to 0}\left(\frac{\sqrt{25-x}-5}{x} \cdot \frac{\sqrt{25-x}+5}{\sqrt{25-x}+5}\right) && \text{Multiply numerator and denominator by the conjugate.} \\
&= \lim_{x \to 0}\left(\frac{(25-x)-25}{x\left(\sqrt{25-x}+5\right)}\right) && \text{Multiply: } \left(\sqrt{25-x}-5\right)\cdot\left(\sqrt{25-x}+5\right) = (25-x)-25. \\
&= \lim_{x \to 0}\left(\frac{-x}{x\left(\sqrt{25-x}+5\right)}\right) && \text{Combine like terms.} \\
&= \lim_{x \to 0}\left(\frac{-\cancel{x}}{\cancel{x}\left(\sqrt{25-x}+5\right)}\right) && \text{Simplify } \frac{-x}{x} = -1. \\
&= \frac{-1}{\sqrt{25-0}+5} && \text{Evaluate.} \\
&= \frac{-1}{5+5} = -\frac{1}{10}
\end{aligned}$$

Analysis *When determining a* ***limit*** *of a function with a root as one of two terms where we cannot evaluate directly, think about multiplying the numerator and denominator by the conjugate of the terms.*

Try It #7

Evaluate the following limit: $\lim\limits_{h \to 0}\left(\dfrac{\sqrt{16 - h} - 4}{h}\right)$.

Example 8 Evaluating the Limit of a Quotient of a Function by Factoring

Evaluate $\lim\limits_{x \to 4}\left(\dfrac{4 - x}{\sqrt{x} - 2}\right)$.

Solution

$$\begin{aligned}
\lim_{x \to 4}\left(\frac{4-x}{\sqrt{x}-2}\right) &= \lim_{x \to 4}\left(\frac{\left(2+\sqrt{x}\right)\left(2-\sqrt{x}\right)}{\sqrt{x}-2}\right) && \text{Factor.} \\
&= \lim_{x \to 4}\left(\frac{\left(2+\sqrt{x}\right)\cancel{\left(2-\sqrt{x}\right)}}{-\cancel{\left(2-\sqrt{x}\right)}}\right) && \text{Factor } -1 \text{ out of the denominator. Simplify.} \\
&= \lim_{x \to 4} -\left(2+\sqrt{x}\right) && \text{Evaluate.} \\
&= -\left(2+\sqrt{4}\right) \\
&= -4
\end{aligned}$$

Analysis *Multiplying by a conjugate would expand the numerator; look instead for factors in the numerator. Four is a perfect square so that the numerator is in the form*

$$a^2 - b^2$$

and may be factored as

$$(a + b)(a - b).$$

Try It #8

Evaluate the following limit: $\lim_{x \to 3} \left(\dfrac{x - 3}{\sqrt{x} - \sqrt{3}} \right)$.

How To...

Given a quotient with absolute values, evaluate its limit.

1. Try factoring or finding the LCD.
2. If the **limit** cannot be found, choose several values close to and on either side of the input where the function is undefined.
3. Use the numeric evidence to estimate the limits on both sides.

Example 9 Evaluating the Limit of a Quotient with Absolute Values

Evaluate $\lim_{x \to 7} \dfrac{|x - 7|}{x - 7}$.

Solution The function is undefined at $x = 7$, so we will try values close to 7 from the left and the right.

Left-hand limit: $\dfrac{|6.9 - 7|}{6.9 - 7} = \dfrac{|6.99 - 7|}{6.99 - 7} = \dfrac{|6.999 - 7|}{6.999 - 7} = -1$

Right-hand limit: $\dfrac{|7.1 - 7|}{7.1 - 7} = \dfrac{|7.01 - 7|}{7.01 - 7} = \dfrac{|7.001 - 7|}{7.001 - 7} = 1$

Since the left- and right-hand limits are not equal, there is no limit.

Try It #9

Evaluate $\lim_{x \to 6^+} \dfrac{6 - x}{|x - 6|}$.

Access the following online resource for additional instruction and practice with properties of limits.

- Determine a Limit Analytically (http://openstaxcollege.org/l/limitanalytic)

12.2 SECTION EXERCISES

VERBAL

1. Give an example of a type of function f whose limit, as x approaches a, is $f(a)$.

2. When direct substitution is used to evaluate the limit of a rational function as x approaches a and the result is $f(a) = \frac{0}{0}$, does this mean that the limit of f does not exist?

3. What does it mean to say the limit of $f(x)$, as x approaches c, is undefined?

ALGEBRAIC

For the following exercises, evaluate the limits algebraically.

4. $\lim\limits_{x \to 0} (3)$

5. $\lim\limits_{x \to 2} \left(\frac{-5x}{x^2 - 1} \right)$

6. $\lim\limits_{x \to 2} \left(\frac{x^2 - 5x + 6}{x + 2} \right)$

7. $\lim\limits_{x \to 3} \left(\frac{x^2 - 9}{x - 3} \right)$

8. $\lim\limits_{x \to -1} \left(\frac{x^2 - 2x - 3}{x + 1} \right)$

9. $\lim\limits_{x \to \frac{3}{2}} \left(\frac{6x^2 - 17x + 12}{2x - 3} \right)$

10. $\lim\limits_{x \to -\frac{7}{2}} \left(\frac{8x^2 + 18x - 35}{2x + 7} \right)$

11. $\lim\limits_{x \to 3} \left(\frac{x^2 - 9}{x^2 - 5x + 6} \right)$

12. $\lim\limits_{x \to -3} \left(\frac{-7x^4 - 21x^3}{-12x^4 + 108x^2} \right)$

13. $\lim\limits_{x \to 3} \left(\frac{x^2 + 2x - 3}{x - 3} \right)$

14. $\lim\limits_{h \to 0} \left(\frac{(3 + h)^3 - 27}{h} \right)$

15. $\lim\limits_{h \to 0} \left(\frac{(2 - h)^3 - 8}{h} \right)$

16. $\lim\limits_{h \to 0} \left(\frac{(h + 3)^2 - 9}{h} \right)$

17. $\lim\limits_{h \to 0} \left(\frac{\sqrt{5 - h} - \sqrt{5}}{h} \right)$

18. $\lim\limits_{x \to 0} \left(\frac{\sqrt{3 - x} - \sqrt{3}}{x} \right)$

19. $\lim\limits_{x \to 9} \left(\frac{x^2 - 81}{3 - \sqrt{x}} \right)$

20. $\lim\limits_{x \to 1} \left(\frac{\sqrt{x} - x^2}{1 - \sqrt{x}} \right)$

21. $\lim\limits_{x \to 0} \left(\frac{x}{\sqrt{1 + 2x} - 1} \right)$

22. $\lim\limits_{x \to \frac{1}{2}} \left(\frac{x^2 - \frac{1}{4}}{2x - 1} \right)$

23. $\lim\limits_{x \to 4} \left(\frac{x^3 - 64}{x^2 - 16} \right)$

24. $\lim\limits_{x \to 2^-} \left(\frac{|x - 2|}{x - 2} \right)$

25. $\lim\limits_{x \to 2^+} \left(\frac{|x - 2|}{x - 2} \right)$

26. $\lim\limits_{x \to 2} \left(\frac{|x - 2|}{x - 2} \right)$

27. $\lim\limits_{x \to 4^-} \left(\frac{|x - 4|}{4 - x} \right)$

28. $\lim\limits_{x \to 4^+} \left(\frac{|x - 4|}{4 - x} \right)$

29. $\lim\limits_{x \to 4} \left(\frac{|x - 4|}{4 - x} \right)$

30. $\lim\limits_{x \to 2} \left(\frac{-8 + 6x - x^2}{x - 2} \right)$

For the following exercise, use the given information to evaluate the limits: $\lim\limits_{x \to c} f(x) = 3$, $\lim\limits_{x \to c} g(x) = 5$

31. $\lim\limits_{x \to c} \left[2f(x) + \sqrt{g(x)} \right]$

32. $\lim\limits_{x \to c} \left[3f(x) + \sqrt{g(x)} \right]$

33. $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$

For the following exercises, evaluate the following limits.

34. $\lim\limits_{x \to 2} \cos(\pi x)$

35. $\lim\limits_{x \to 2} \sin(\pi x)$

36. $\lim\limits_{x \to 2} \sin\left(\frac{\pi}{x} \right)$

37. $f(x) = \begin{cases} 2x^2 + 2x + 1, & x \leq 0 \\ x - 3, & x > 0 \end{cases}$; $\lim\limits_{x \to 0^+} f(x)$

38. $f(x) = \begin{cases} 2x^2 + 2x + 1, & x \leq 0 \\ x - 3, & x > 0 \end{cases}$; $\lim\limits_{x \to 0^-} f(x)$

39. $f(x) = \begin{cases} 2x^2 + 2x + 1, & x \leq 0 \\ x - 3, & x > 0 \end{cases}$; $\lim\limits_{x \to 0} f(x)$

40. $\lim\limits_{x \to 4} \frac{\sqrt{x + 5} - 3}{x - 4}$

41. $\lim\limits_{x \to 2^+} (2x - [\![x]\!])$

42. $\lim\limits_{x \to 2} \frac{\sqrt{x + 7} - 3}{x^2 - x - 2}$

43. $\lim\limits_{x \to 3^+} \frac{x^2}{x^2 - 9}$

For the following exercises, find the average rate of change $\dfrac{f(x+h)-f(x)}{h}$.

44. $f(x) = x + 1$

45. $f(x) = 2x^2 - 1$

46. $f(x) = x^2 + 3x + 4$

47. $f(x) = x^2 + 4x - 100$

48. $f(x) = 3x^2 + 1$

49. $f(x) = \cos(x)$

50. $f(x) = 2x^3 - 4x$

51. $f(x) = \dfrac{1}{x}$

52. $f(x) = \dfrac{1}{x^2}$

53. $f(x) = \sqrt{x}$

GRAPHICAL

54. Find an equation that could be represented by **Figure 3**.

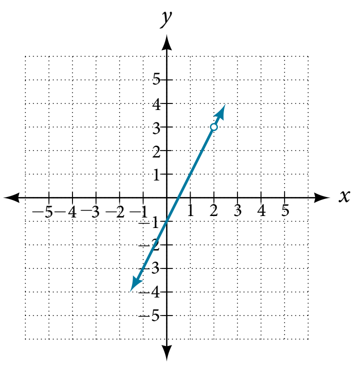


Figure 3

55. Find an equation that could be represented by **Figure 4**.

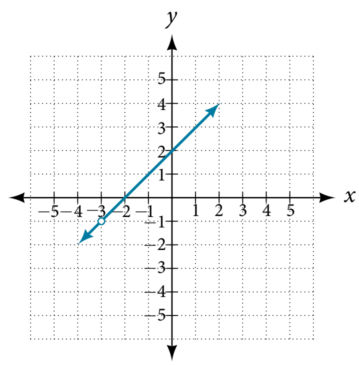


Figure 4

56. What is the right-hand limit of the function as x approaches 0?

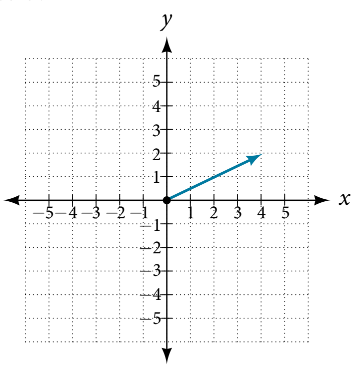


Figure 5

57. What is the left-hand limit of the function as x approaches 0?

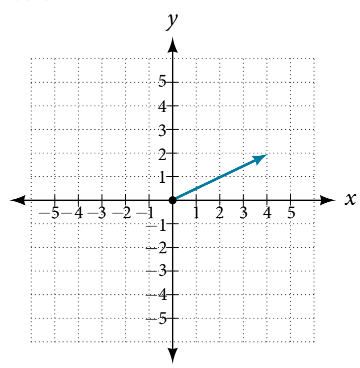


Figure 6

REAL-WORLD APPLICATIONS

58. The position function $s(t) = -16t^2 + 144t$ gives the position of a projectile as a function of time. Find the average velocity (average rate of change) on the interval $[1, 2]$.

59. The height of a projectile is given by $s(t) = -64t^2 + 192t$ Find the average rate of change of the height from $t = 1$ second to $t = 1.5$ seconds.

60. The amount of money in an account after t years compounded continuously at 4.25% interest is given by the formula $A = A_0 e^{0.0425t}$, where A_0 is the initial amount invested. Find the average rate of change of the balance of the account from $t = 1$ year to $t = 2$ years if the initial amount invested is \$1,000.00.

LEARNING OBJECTIVES

In this section, you will:

- Determine whether a function is continuous at a number.
- Determine the numbers for which a function is discontinuous.
- Determine whether a function is continuous.

12.3 CONTINUITY

Arizona is known for its dry heat. On a particular day, the temperature might rise as high as 118° F and drop down only to a brisk 95° F. **Figure 1** shows the function T, where the output of $T(x)$ is the temperature in Fahrenheit degrees and the input x is the time of day, using a 24-hour clock on a particular summer day.

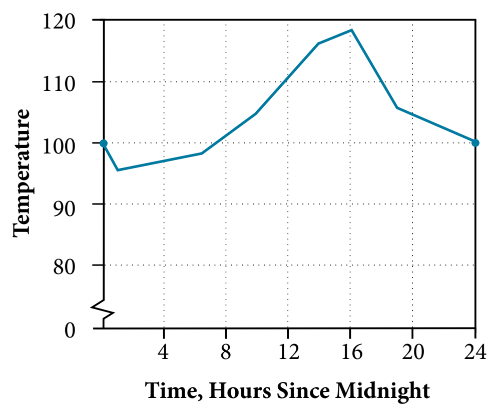


Figure 1 Temperature as a function of time forms a continuous function.

When we analyze this graph, we notice a specific characteristic. There are no breaks in the graph. We could trace the graph without picking up our pencil. This single observation tells us a great deal about the function. In this section, we will investigate functions with and without breaks.

Determining Whether a Function Is Continuous at a Number

Let's consider a specific example of temperature in terms of date and location, such as June 27, 2013, in Phoenix, AZ. The graph in **Figure 1** indicates that, at 2 a.m., the temperature was 96° F. By 2 p.m. the temperature had risen to 116° F, and by 4 p.m. it was 118° F. Sometime between 2 a.m. and 4 p.m., the temperature outside must have been exactly 110.5° F. In fact, any temperature between 96° F and 118° F occurred at some point that day. This means all real numbers in the output between 96° F and 118° F are generated at some point by the function according to the intermediate value theorem,

Look again at **Figure 1**. There are no breaks in the function's graph for this 24-hour period. At no point did the temperature cease to exist, nor was there a point at which the temperature jumped instantaneously by several degrees. A function that has no holes or breaks in its graph is known as a **continuous function**. Temperature as a function of time is an example of a continuous function.

If temperature represents a continuous function, what kind of function would not be continuous? Consider an example of dollars expressed as a function of hours of parking. Let's create the function D, where $D(x)$ is the output representing cost in dollars for parking x number of hours. See **Figure 2**.

Suppose a parking garage charges $4.00 per hour or fraction of an hour, with *a* $25 per day maximum charge. Park for two hours and five minutes and the charge is $12. Park an additional hour and the charge is $16. We can never be charged $13, $14, or $15. There are real numbers between 12 and 16 that the function never outputs. There are breaks in the function's graph for this 24-hour period, points at which the price of parking jumps instantaneously by several dollars.

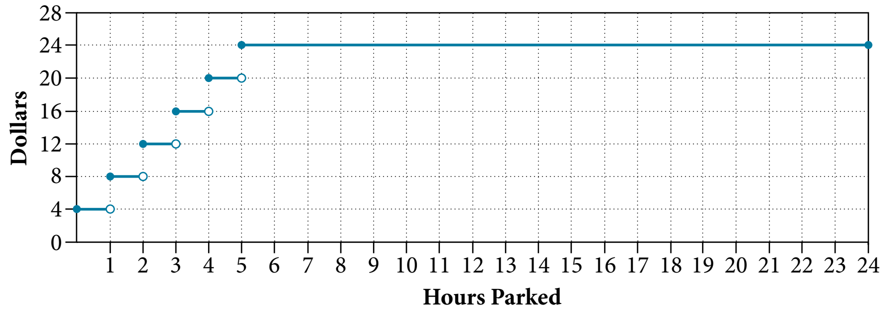


Figure 2 Parking-garage charges form a discontinuous function.

A function that remains level for an interval and then jumps instantaneously to a higher value is called a stepwise function. This function is an example.

A function that has any hole or break in its graph is known as a **discontinuous function**. A stepwise function, such as parking-garage charges as a function of hours parked, is an example of a discontinuous function.

So how can we decide if a function is continuous at a particular number? We can check three different conditions. Let's use the function $y = f(x)$ represented in **Figure 3** as an example.

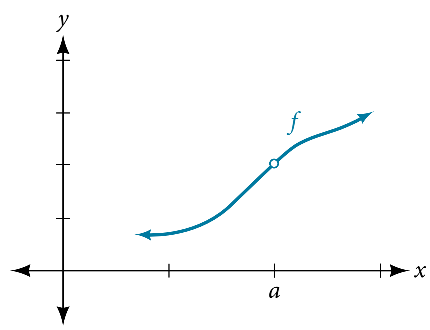


Figure 3

Condition 1 According to Condition 1, the function $f(a)$ defined at $x = a$ must exist. In other words, there is a y-coordinate at $x = a$ as in **Figure 4**.

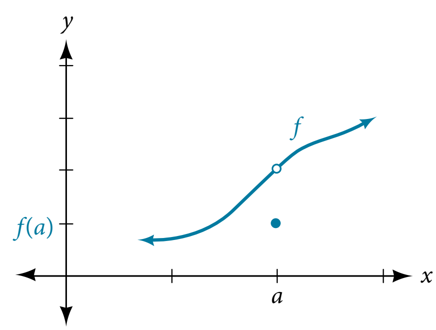


Figure 4

Condition 2 According to Condition 2, at $x = a$ the limit, written $\lim_{x \to a} f(x)$, must exist. This means that at $x = a$ the left-hand limit must equal the right-hand limit. Notice as the graph of f in **Figure 3** approaches $x = a$ from the left and right, the same y-coordinate is approached. Therefore, Condition 2 is satisfied. However, there could still be a hole in the graph at $x = a$.

Condition 3 According to Condition 3, the corresponding y coordinate at $x = a$ fills in the hole in the graph of f. This is written $\lim_{x \to a} f(x) = f(a)$.

Satisfying all three conditions means that the function is continuous. All three conditions are satisfied for the function represented in **Figure 5** so the function is continuous as $x = a$.

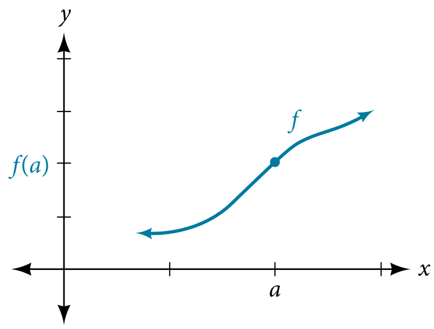


Figure 5 All three conditions are satisfied. The function is continuous at $x = a$.

Figure 6 through **Figure 9** provide several examples of graphs of functions that are not continuous at $x = a$ and the condition or conditions that fail.

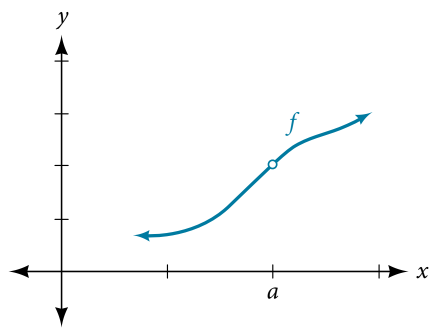


Figure 6 Condition 2 is satisfied. Conditions 1 and 3 both fail.

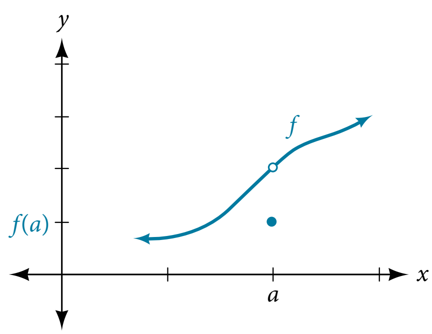


Figure 7 Conditions 1 and 2 are both satisfied. Condition 3 fails.

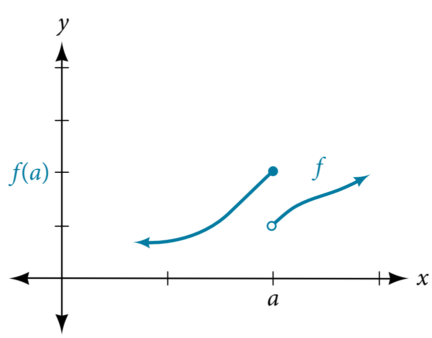


Figure 8 Condition 1 is satisfied. Conditions 2 and 3 fail.

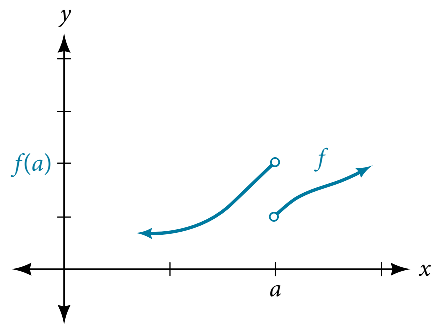


Figure 9 Conditions 1, 2, and 3 all fail.

definition of continuity

A function $f(x)$ is **continuous** at $x = a$ provided all three of the following conditions hold true:

- Condition 1: $f(a)$ exists.
- Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
- Condition 3: $\lim_{x \to a} f(x) = f(a)$.

If a function $f(x)$ is not continuous at $x = a$, the function is **discontinuous** at $x = a$.

Identifying a Jump Discontinuity

Discontinuity can occur in different ways. We saw in the previous section that a function could have a left-hand limit and a right-hand limit even if they are not equal. If the left- and right-hand limits exist but are different, the graph "jumps" at $x = a$. The function is said to have a jump discontinuity.

As an example, look at the graph of the function $y = f(x)$ in **Figure 10**. Notice as x approaches a how the output approaches different values from the left and from the right.

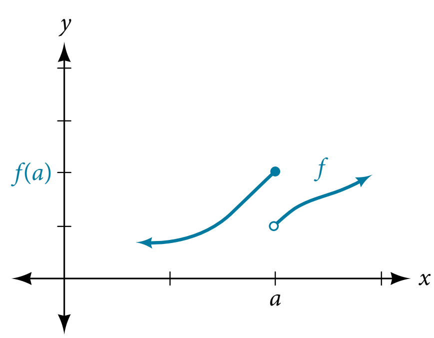


Figure 10 Graph of a function with a jump discontinuity.

jump discontinuity

A function $f(x)$ has a **jump discontinuity** at $x = a$ if the left- and right-hand limits both exist but are not equal:

$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

Identifying Removable Discontinuity

Some functions have a discontinuity, but it is possible to redefine the function at that point to make it continuous. This type of function is said to have a removable discontinuity. Let's look at the function $y = f(x)$ represented by the graph in **Figure 11**. The function has a limit. However, there is a hole at $x = a$. The hole can be filled by extending the domain to include the input $x = a$ and defining the corresponding output of the function at that value as the limit of the function at $x = a$.

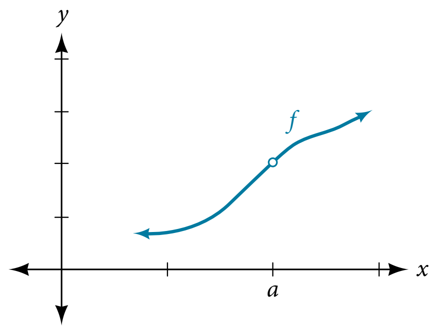


Figure 11 Graph of function f with a removable discontinuity at $x = a$.

> ***removable discontinuity***
>
> A function $f(x)$ has a **removable discontinuity** at $x = a$ if the limit, $\lim_{x \to a} f(x)$, exists, but either
>
> 1. $f(a)$ does not exist or
> 2. $f(a)$, the value of the function at $x = a$ does not equal the limit, $f(a) \neq \lim_{x \to a} f(x)$.

Example 1 **Identifying Discontinuities**

Identify all discontinuities for the following functions as either a jump or a removable discontinuity.

a. $f(x) = \dfrac{x^2 - 2x - 15}{x - 5}$

b. $g(x) = \begin{cases} x + 1, & x < 2 \\ -x, & x \geq 2 \end{cases}$

Solution

a. Notice that the function is defined everywhere except at $x = 5$.

Thus, $f(5)$ does not exist, Condition 2 is not satisfied. Since Condition 1 is satisfied, the limit as x approaches 5 is 8, and Condition 2 is not satisfied. This means there is a removable discontinuity at $x = 5$.

b. Condition 2 is satisfied because $g(2) = -2$.

Notice that the function is a piecewise function, and for each piece, the function is defined everywhere on its domain. Let's examine Condition 1 by determining the left- and right-hand limits as x approaches 2.

Left-hand limit: $\lim_{x \to 2^-} (x + 1) = 2 + 1 = 3$. The left-hand limit exists.

Right-hand limit: $\lim_{x \to 2^+} (-x) = -2$. The right-hand limit exists. But

$$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x).$$

So, $\lim_{x \to 2} f(x)$ does not exist, and Condition 2 fails: There is no removable discontinuity. However, since both left- and right-hand limits exist but are not equal, the conditions are satisfied for a jump discontinuity at $x = 2$.

Try It #1

Identify all discontinuities for the following functions as either a jump or a removable discontinuity.

a. $f(x) = \dfrac{x^2 - 6x}{x - 6}$

b. $g(x) = \begin{cases} \sqrt{x}, & 0 \leq x < 4 \\ 2x, & x \geq 4 \end{cases}$

Recognizing Continuous and Discontinuous Real-Number Functions

Many of the functions we have encountered in earlier chapters are continuous everywhere. They never have a hole in them, and they never jump from one value to the next. For all of these functions, the limit of $f(x)$ as x approaches a is the same as the value of $f(x)$ when $x = a$. So $\lim_{x \to a} f(x) = f(a)$. There are some functions that are continuous everywhere and some that are only continuous where they are defined on their domain because they are not defined for all real numbers.

examples of continuous functions

The following functions are continuous everywhere:

Polynomial functions	Ex: $f(x) = x^4 - 9x^2$
Exponential functions	Ex: $f(x) = 4^{x+2} - 5$
Sine functions	Ex: $f(x) = \sin(2x) - 4$
Cosine functions	Ex: $f(x) = -\cos\left(x + \frac{\pi}{3}\right)$

Table 1

The following functions are continuous everywhere they are defined on their domain:

Logarithmic functions	Ex: $f(x) = 2\ln(x)$, $x > 0$
Tangent functions	Ex: $f(x) = \tan(x) + 2$, $x \neq \frac{\pi}{2} + k\pi$, k is an integer
Rational functions	Ex: $f(x) = \frac{x^2 - 25}{x - 7}$, $x \neq 7$

Table 2

How To…

Given a function $f(x)$, determine if the function is continuous at $x = a$.

1. Check Condition 1: $f(a)$ exists.
2. Check Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
3. Check Condition 3: $\lim_{x \to a} f(x) = f(a)$.
4. If all three conditions are satisfied, the function is continuous at $x = a$. If any one of the conditions is not satisfied, the function is not continuous at $x = a$.

Example 2 Determining Whether a Piecewise Function is Continuous at a Given Number

Determine whether the function $f(x) = \begin{cases} 4x, & x \leq 3 \\ 8 + x, & x > 3 \end{cases}$ is continuous at

a. $x = 3$

b. $x = \frac{8}{3}$

Solution To determine if the function f is continuous at $x = a$, we will determine if the three conditions of continuity are satisfied at $x = a$.

a. Condition 1: Does $f(a)$ exist?

$$f(3) = 4(3) = 12$$

$$\Rightarrow \text{Condition 1 is satisfied.}$$

Condition 2: Does $\lim_{x \to 3} f(x)$ exist?

To the left of $x = 3$, $f(x) = 4x$; to the right of $x = 3$, $f(x) = 8 + x$. We need to evaluate the left- and right-hand limits as x approaches 1.

- Left-hand limit: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 4(3) = 12$
- Right-hand limit: $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (8 + x) = 8 + 3 = 11$

Because $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, $\lim_{x \to 1} f(x)$ does not exist.

$\Rightarrow$ Condition 2 fails.

There is no need to proceed further. Condition 2 fails at $x = 3$. If any of the conditions of continuity are not satisfied at $x = 3$, the function $f(x)$ is not continuous at $x = 3$.

b. $x = \dfrac{8}{3}$

Condition 1: Does $f\left(\dfrac{8}{3}\right)$ exist?

$$f\left(\frac{8}{3}\right) = 4\left(\frac{8}{3}\right) = \frac{32}{3}$$

$\Rightarrow$ Condition 1 is satisfied.

Condition 2: Does $\lim_{x \to \frac{8}{3}} f(x)$ exist?

To the left of $x = \frac{8}{3}$, $f(x) = 4x$; to the right of $x = \frac{8}{3}$, $f(x) = 8 + x$. We need to evaluate the left- and right-hand limits as x approaches $\frac{8}{3}$.

- Left-hand limit: $\lim_{x \to \frac{8}{3}^-} f(x) = \lim_{x \to \frac{8}{3}^-} 4\left(\frac{8}{3}\right) = \frac{32}{3}$
- Right-hand limit: $\lim_{x \to \frac{8}{3}^+} f(x) = \lim_{x \to \frac{8}{3}^+} (8 + x) = 8 + \frac{8}{3} = \frac{32}{3}$

Because $\lim_{x \to \frac{8}{3}} f(x)$ exists,

$\Rightarrow$ Condition 2 is satisfied.

Condition 3: Is $f\left(\dfrac{8}{3}\right) = \lim_{x \to \frac{8}{3}} f(x)$?

$$f\left(\frac{32}{3}\right) = \frac{32}{3} \lim_{x \to \frac{8}{3}} f(x)$$

$\Rightarrow$ Condition 3 is satisfied.

Because all three conditions of continuity are satisfied at $x = \frac{8}{3}$, the function $f(x)$ is continuous at $x = \frac{8}{3}$.

Try It #2

Determine whether the function $f(x) = \begin{cases} \dfrac{1}{x}, & x \leq 2 \\ 9x - 11.5, & x > 2 \end{cases}$ is continuous at $x = 2$.

Example 3 Determining Whether a Rational Function is Continuous at a Given Number

Determine whether the function $f(x) = \dfrac{x^2 - 25}{x - 5}$ is continuous at $x = 5$.

Solution To determine if the function f is continuous at $x = 5$, we will determine if the three conditions of continuity are satisfied at $x = 5$.

Condition 1:

$$f(5) \text{ does not exist.}$$

$$\Rightarrow \text{Condition 1 fails.}$$

There is no need to proceed further. Condition 2 fails at $x = 5$. If any of the conditions of continuity are not satisfied at $x = 5$, the function f is not continuous at $x = 5$.

Analysis *See* ***Figure 12****. Notice that for Condition 2 we have*

$$\begin{aligned}\lim_{x \to 5} \frac{x^2 - 25}{x - 5} &= \lim_{x \to 3} \frac{\cancel{(x - 5)}(x + 5)}{\cancel{x - 5}} \\ &= \lim_{x \to 5}(x + 5) \\ &= 5 + 5 = 10\end{aligned}$$

$$\Rightarrow \text{Condition 2 is satisfied.}$$

At $x = 5$, there exists a removable discontinuity. See ***Figure 12****.*

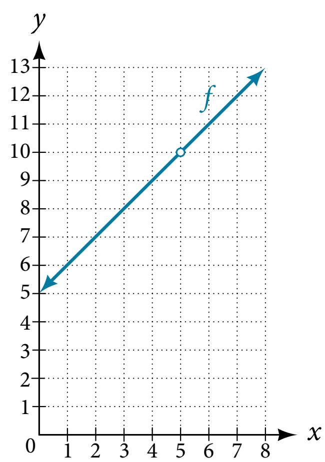


Figure 12

Try It #3

Determine whether the function $f(x) = \dfrac{9 - x^2}{x^2 - 3x}$ is continuous at $x = 3$. If not, state the type of discontinuity.

Determining the Input Values for Which a Function Is Discontinuous

Now that we can identify continuous functions, jump discontinuities, and removable discontinuities, we will look at more complex functions to find discontinuities. Here, we will analyze a piecewise function to determine if any real numbers exist where the function is not continuous. A **piecewise function** may have discontinuities at the boundary points of the function as well as within the functions that make it up.

To determine the real numbers for which a piecewise function composed of polynomial functions is not continuous, recall that polynomial functions themselves are continuous on the set of real numbers. Any discontinuity would be at the boundary points. So we need to explore the three conditions of continuity at the boundary points of the piecewise function.

How To...

Given a piecewise function, determine whether it is continuous at the boundary points.

1. For each boundary point a of the piecewise function, determine the left- and right-hand limits as x approaches a, as well as the function value at a.

2. Check each condition for each value to determine if all three conditions are satisfied.
3. Determine whether each value satisfies condition 1: $f(a)$ exists.
4. Determine whether each value satisfies condition 2: $\lim_{x \to a} f(x)$ exists.
5. Determine whether each value satisfies condition 3: $\lim_{x \to a} f(x) = f(a)$.
6. If all three conditions are satisfied, the function is continuous at $x = a$. If any one of the conditions fails, the function is not continuous at $x = a$.

Example 4 **Determining the Input Values for Which a Piecewise Function Is Discontinuous**

Determine whether the function f is discontinuous for any real numbers.

$$f(x) = \begin{cases} x + 1, & x < 2 \\ 3, & 2 \le x < 4 \\ x^2 - 11, & x \ge 4 \end{cases}$$

Solution The piecewise function is defined by three functions, which are all polynomial functions, $f(x) = x + 1$ on $x < 2$, $f(x) = 3$ on $2 \le x < 4$, and $f(x) = x^2 - 5$ on $x \ge 4$. Polynomial functions are continuous everywhere. Any discontinuities would be at the boundary points, $x = 2$ and $x = 4$.

At $x = 2$, let us check the three conditions of continuity.

Condition 1:

$$f(2) = 3$$
$$\Rightarrow \text{Condition 1 is satisfied.}$$

Condition 2: Because a different function defines the output left and right of $x = 2$, does

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)?$$

- Left-hand limit: $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x + 1) = 2 + 1 = 3$
- Right-hand limit: $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3 = 3$

Because $3 = 3$, $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

$$\Rightarrow \text{Condition 2 is satisfied.}$$

Condition 3:

$$\lim_{x \to 2} f(x) = 3 = f(2)$$

$$\Rightarrow \text{Condition 3 is satisfied.}$$

Because all three conditions are satisfied at $x = 2$, the function $f(x)$ is continuous at $x = 2$.

At $x = 4$, let us check the three conditions of continuity.

Condition 2: Because a different function defines the output left and right of $x = 4$, does $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x)$?

- Left-hand limit: $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} 3 = 3$
- Right-hand limit: $\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} (x^2 - 11) = 4^2 - 11 = 5$

Because $3 \ne 5$, $\lim_{x \to 4^-} f(x) \ne \lim_{x \to 4^+} f(x)$, so $\lim_{x \to 4} f(x)$ does not exist.

$$\Rightarrow \text{Condition 2 fails.}$$

Because one of the three conditions does not hold at $x = 4$, the function $f(x)$ is discontinuous at $x = 4$.

Analysis *See **Figure 13**. At $x = 4$, there exists a jump discontinuity. Notice that the function is continuous at $x = 2$.*

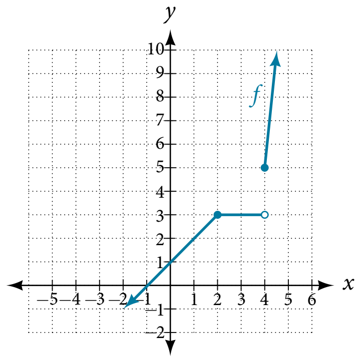


Figure 13 Graph is continuous at $x = 2$ but shows a jump discontinuity at $x = 4$.

Try It #4

Determine where the function $f(x) = \begin{cases} \dfrac{\pi x}{4}, & x < 2 \\ \dfrac{\pi}{x}, & 2 \leq x \leq 6 \\ 2\pi x, & x > 6 \end{cases}$ is discontinuous.

Determining Whether a Function Is Continuous

To determine whether a piecewise function is continuous or discontinuous, in addition to checking the boundary points, we must also check whether each of the functions that make up the piecewise function is continuous.

How To...

Given a piecewise function, determine whether it is continuous.

1. Determine whether each component function of the piecewise function is continuous. If there are discontinuities, do they occur within the domain where that component function is applied?
2. For each boundary point $x = a$ of the piecewise function, determine if each of the three conditions hold.

Example 5 Determining Whether a Piecewise Function Is Continuous

Determine whether the function below is continuous. If it is not, state the location and type of each discontinuity.

$$f(x) = \begin{cases} \sin(x), & x < 0 \\ x^3, & x > 0 \end{cases}$$

Solution The two functions composing this piecewise function are $f(x) = \sin(x)$ on $x < 0$ and $f(x) = x^3$ on $x > 0$. The sine function and all polynomial functions are continuous everywhere. Any discontinuities would be at the boundary point,

At $x = 0$, let us check the three conditions of continuity.

Condition 1:

$$f(0) \text{ does not exist.}$$

$$\Rightarrow \text{Condition 1 fails.}$$

Because all three conditions are not satisfied at $x = 0$, the function $f(x)$ is discontinuous at $x = 0$.

Analysis *See **Figure 14**. There exists a removable discontinuity at $x = 0$; $\lim\limits_{x \to 0} f(x) = 0$, thus the limit exists and is finite, but $f(a)$ does not exist.*

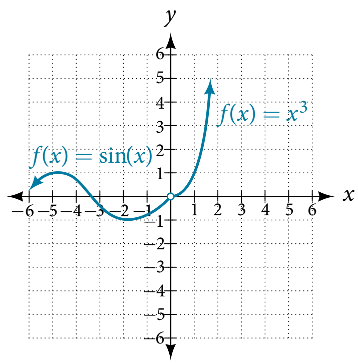


Figure 14 Function has removable discontinuity at 0.

Access these online resources for additional instruction and practice with continuity.

- Continuity at a Point (http://openstaxcollege.org/l/continuitypoint)
- Continuity at a Point: Concept Check (http://openstaxcollege.org/l/contconcept)

12.3 SECTION EXERCISES

VERBAL

1. State in your own words what it means for a function f to be continuous at $x = c$.

2. State in your own words what it means for a function to be continuous on the interval (a, b).

ALGEBRAIC

For the following exercises, determine why the function f is discontinuous at a given point a on the graph. State which condition fails.

3. $f(x) = \ln|x + 3|, a = -3$

4. $f(x) = \ln|5x - 2|, a = \frac{2}{5}$

5. $f(x) = \frac{x^2 - 16}{x + 4}, a = -4$

6. $f(x) = \frac{x^2 - 16x}{x}, a = 0$

7. $f(x) = \begin{cases} x, & x \neq 3 \\ 2x, & x = 3 \end{cases} \quad a = 3$

8. $f(x) = \begin{cases} 5, & x \neq 0 \\ 3, & x = 0 \end{cases} \quad a = 0$

9. $f(x) = \begin{cases} \frac{1}{2 - x}, & x \neq 2 \\ 3, & x = 2 \end{cases} \quad a = 2$

10. $f(x) = \begin{cases} \frac{1}{x + 6}, & x = -6 \\ x^2, & x \neq -6 \end{cases} \quad a = -6$

11. $f(x) = \begin{cases} 3 + x, & x < 1 \\ x, & x = 1 \\ x^2, & x > 1 \end{cases} \quad a = 1$

12. $f(x) = \begin{cases} 3 - x, & x < 1 \\ x, & x = 1 \\ 2x^2, & x > 1 \end{cases} \quad a = 1$

13. $f(x) = \begin{cases} 3 + 2x, & x < 1 \\ x, & x = 1 \\ -x^2, & x > 1 \end{cases} \quad a = 1$

14. $f(x) = \begin{cases} x^2, & x < -2 \\ 2x + 1, & x = -2 \\ x^3, & x > -2 \end{cases} \quad a = -2$

15. $f(x) = \begin{cases} \frac{x^2 - 9}{x + 3}, & x < -3 \\ x - 9, & x = -3 \\ \frac{1}{x}, & x > -3 \end{cases} \quad a = -3$

16. $f(x) = \begin{cases} \frac{x^2 - 9}{x + 3}, & x < -3 \\ x - 9, & x = -3 \\ -6, & x > -3 \end{cases} \quad a = 3$

17. $f(x) = \frac{x^2 - 4}{x - 2}, a = 2$

18. $f(x) = \frac{25 - x^2}{x^2 - 10x + 25}, a = 5$

19. $f(x) = \frac{x^3 - 9x}{x^2 + 11x + 24}, a = -3$

20. $f(x) = \frac{x^3 - 27}{x^2 - 3x}, a = 3$

21. $f(x) = \frac{x}{|x|}, a = 0$

22. $f(x) = \frac{2|x + 2|}{x + 2}, a = -2$

For the following exercises, determine whether or not the given function f is continuous everywhere. If it is continuous everywhere it is defined, state for what range it is continuous. If it is discontinuous, state where it is discontinuous.

23. $f(x) = x^3 - 2x - 15$

24. $f(x) = \frac{x^2 - 2x - 15}{x - 5}$

25. $f(x) = 2 \cdot 3^{x + 4}$

26. $f(x) = -\sin(3x)$

27. $f(x) = \frac{|x - 2|}{x^2 - 2x}$

28. $f(x) = \tan(x) + 2$

29. $f(x) = 2x + \frac{5}{x}$

30. $f(x) = \log_2(x)$

31. $f(x) = \ln x^2$

32. $f(x) = e^{2x}$

33. $f(x) = \sqrt{x-4}$

34. $f(x) = \sec(x) - 3.$

35. $f(x) = x^2 + \sin(x)$

36. Determine the values of b and c such that the following function is continuous on the entire real number line.

$$f(x) = \begin{cases} x+1, & 1 < x < 3 \\ x^2 + bx + c, & |x-2| \geq 1 \end{cases}$$

GRAPHICAL

For the following exercises, refer to **Figure 15**. Each square represents one square unit. For each value of a, determine which of the three conditions of continuity are satisfied at $x = a$ and which are not.

37. $x = -3$

38. $x = 2$

39. $x = 4$

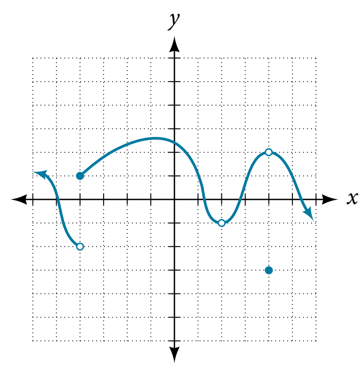


Figure 15

For the following exercises, use a graphing utility to graph the function $f(x) = \sin\left(\frac{12\pi}{x}\right)$ as in **Figure 16**. Set the x-axis a short distance before and after 0 to illustrate the point of discontinuity.

40. Which conditions for continuity fail at the point of discontinuity?

41. Evaluate $f(0)$.

42. Solve for x if $f(x) = 0$.

43. What is the domain of $f(x)$?

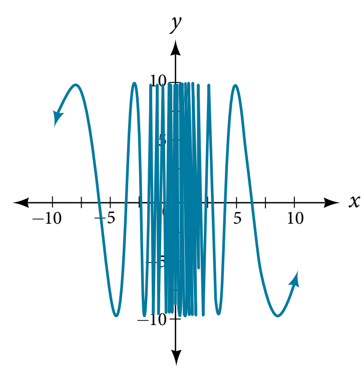


Figure 16

For the following exercises, consider the function shown in **Figure 17**.

44. At what x-coordinates is the function discontinuous?

45. What condition of continuity is violated at these points?

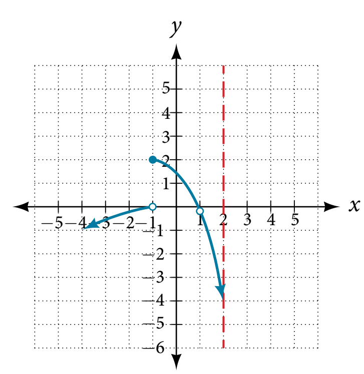


Figure 17

46. Consider the function shown in **Figure 18**. At what x-coordinates is the function discontinuous? What condition(s) of continuity were violated?

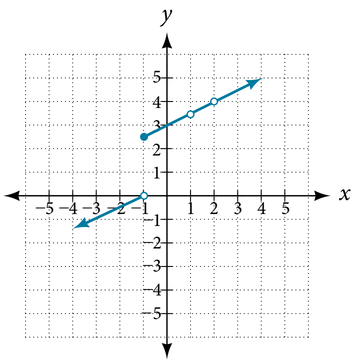


Figure 18

47. Construct a function that passes through the origin with a constant slope of 1, with removable discontinuities at $x = -7$ and $x = 1$.

48. The function $f(x) = \dfrac{x^3 - 1}{x - 1}$ is graphed in **Figure 19**. It appears to be continuous on the interval $[-3, 3]$, but there is an x-value on that interval at which the function is discontinuous. Determine the value of x at which the function is discontinuous, and explain the pitfall of utilizing technology when considering continuity of a function by examining its graph.

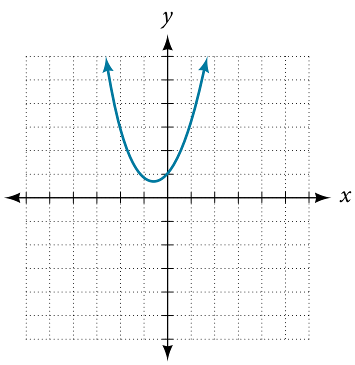


Figure 19

49. Find the limit $\lim\limits_{x \to 1} f(x)$ and determine if the following function is continuous at $x = 1$:

$$f(x) = \begin{cases} x^2 + 4 & x \neq 1 \\ 2 & x = 1 \end{cases}$$

50. The graph of $f(x) = \dfrac{\sin(2x)}{x}$ is shown in **Figure 20**. Is the function $f(x)$ continuous at $x = 0$? Why or why not?

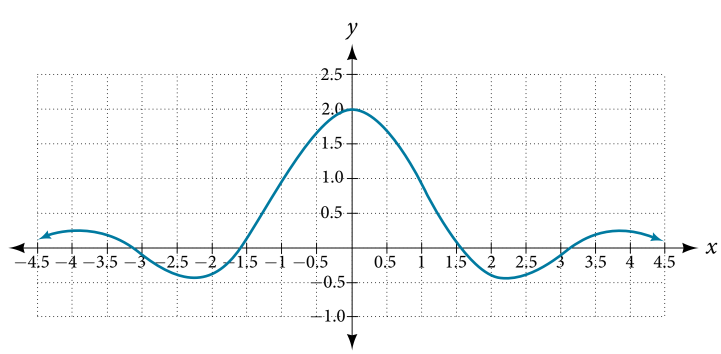


Figure 20

LEARNING OBJECTIVES

In this section, you will:

- Find the derivative of a function.
- Find instantaneous rates of change.
- Find an equation of the tangent line to the graph of a function at a point.
- Find the instantaneous velocity of a particle.

12.4 DERIVATIVES

The average teen in the United States opens a refrigerator door an estimated 25 times per day. Supposedly, this average is up from 10 years ago when the average teenager opened a refrigerator door 20 times per day.[37]

It is estimated that a television is on in a home 6.75 hours per day, whereas parents spend an estimated 5.5 minutes per day having a meaningful conversation with their children. These averages, too, are not the same as they were 10 years ago, when the television was on an estimated 6 hours per day in the typical household, and parents spent 12 minutes per day in meaningful conversation with their kids.

What do these scenarios have in common? The functions representing them have changed over time. In this section, we will consider methods of computing such changes over time.

Finding the Average Rate of Change of a Function

The functions describing the examples above involve a change over time. Change divided by time is one example of a rate. The rates of change in the previous examples are each different. In other words, some changed faster than others. If we were to graph the functions, we could compare the rates by determining the slopes of the graphs.

A **tangent line** to a curve is a line that intersects the curve at only a single point but does not cross it there. (The tangent line may intersect the curve at another point away from the point of interest.) If we zoom in on a curve at that point, the curve appears linear, and the slope of the curve at that point is close to the slope of the tangent line at that point.

Figure 1 represents the function $f(x) = x^3 - 4x$. We can see the slope at various points along the curve.

- slope at $x = -2$ is 8
- slope at $x = -1$ is –1
- slope at $x = 2$ is 8

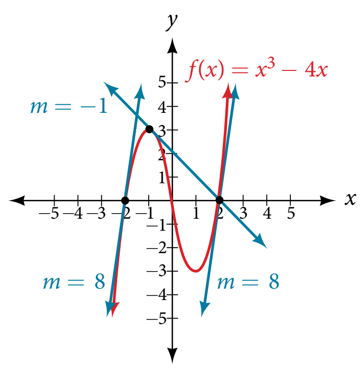


Figure 1 Graph showing tangents to curve at –2, –1, and 2.

Let's imagine a point on the curve of function f at $x = a$ as shown in **Figure 2**. The coordinates of the point are $(a, f(a))$. Connect this point with a second point on the curve a little to the right of $x = a$, with an x-value increased by some small real number h. The coordinates of this second point are $(a + h, f(a + h))$ for some positive-value h.

37 http://www.csun.edu/science/health/docs/tv&health.html Source provided.

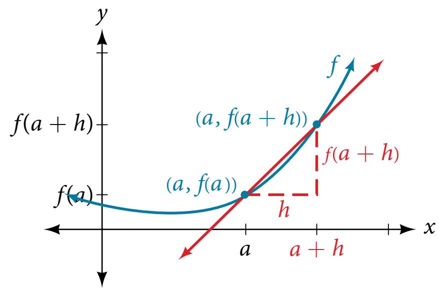


Figure 2 Connecting point a with *a* point just beyond allows us to measure a slope close to that of a tangent line at $x = a$.

We can calculate the slope of the line connecting the two points $(a, f(a))$ and $(a+h, f(a+h))$, called a **secant line**, by applying the slope formula, slope $= \dfrac{\text{change in } y}{\text{change in } x}$.

We use the notation m_{sec} to represent the slope of the secant line connecting two points.

$$\begin{aligned} m_{\text{sec}} &= \frac{f(a+h) - f(a)}{(a+h) - (a)} \\ &= \frac{f(a+h) - f(a)}{\cancel{a} + h - \cancel{a}} \end{aligned}$$

The slope m_{sec} equals the *average rate of change* between two points $(a, f(a))$ and $(a+h, f(a+h))$.

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{h}$$

the average rate of change between two points on a curve

The **average rate of change** (AROC) between two points $(a, f(a))$ and $(a+h, f(a+h))$ on the curve of f is the slope of the line connecting the two points and is given by

$$\text{AROC} = \frac{f(a+h) - f(a)}{h}$$

Example 1 Finding the Average Rate of Change

Find the average rate of change connecting the points $(2, -6)$ and $(-1, 5)$.

Solution We know the average rate of change connecting two points may be given by

$$\text{AROC} = \frac{f(a+h) - f(a)}{h}.$$

If one point is $(2, -6)$, or $(2, f(2))$, then $f(2) = -6$.

The value h is the displacement from 2 to -1, which equals $-1 - 2 = -3$.

For the other point, $f(a+h)$ is the y-coordinate at $a+h$, which is $2 + (-3)$ or -1, so $f(a+h) = f(-1) = 5$.

$$\begin{aligned} \text{AROC} &= \frac{f(a+h) - f(a)}{h} \\ &= \frac{5 - (-6)}{-3} \\ &= \frac{11}{-3} \\ &= -\frac{11}{3} \end{aligned}$$

Try It #1

Find the average rate of change connecting the points $(-5, 1.5)$ and $(-2.5, 9)$.

Understanding the Instantaneous Rate of Change

Now that we can find the average rate of change, suppose we make h in **Figure 2** smaller and smaller. Then $a + h$ will approach a as h gets smaller, getting closer and closer to 0. Likewise, the second point $(a + h, f(a + h))$ will approach the first point, $(a, f(a))$. As a consequence, the connecting line between the two points, called the secant line, will get closer and closer to being a tangent to the function at $x = a$, and the slope of the secant line will get closer and closer to the slope of the tangent at $x = a$. See **Figure 3**.

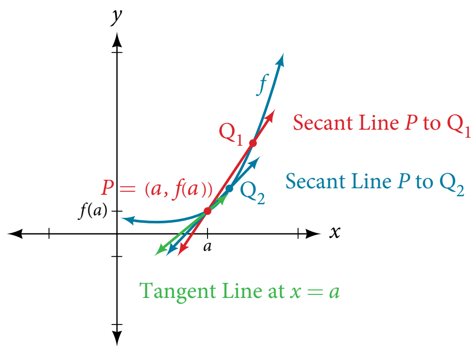


Figure 3 The connecting line between two points moves closer to being a tangent line at $x = a$.

Because we are looking for the slope of the tangent at $x = a$, we can think of the measure of the slope of the curve of a function f at a given point as the rate of change at a particular instant. We call this slope the **instantaneous rate of change**, or the **derivative** of the function at $x = a$. Both can be found by finding the limit of the slope of a line connecting the point at $x = a$ with a second point infinitesimally close along the curve. For a function f both the instantaneous rate of change of the function and the derivative of the function at $x = a$ are written as $f'(a)$, and we can define them as a two-sided limit that has the same value whether approached from the left or the right.

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

The expression by which the limit is found is known as the difference quotient.

definition of instantaneous rate of change and derivative

The **derivative**, or **instantaneous rate of change**, of a function f at $x = a$, is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

The expression $\frac{f(a + h) - f(a)}{h}$ is called the difference quotient.

We use the difference quotient to evaluate the limit of the rate of change of the function as h approaches 0.

Derivatives: Interpretations and Notation

The derivative of a function can be interpreted in different ways. It can be observed as the behavior of a graph of the function or calculated as a numerical rate of change of the function.

- The derivative of a function $f(x)$ at a point $x = a$ is the slope of the tangent line to the curve $f(x)$ at $x = a$. The derivative of $f(x)$ at $x = a$ is written $f'(a)$.
- The derivative $f'(a)$ measures how the curve changes at the point $(a, f(a))$.
- The derivative $f'(a)$ may be thought of as the instantaneous rate of change of the function $f(x)$ at $x = a$.
- If a function measures distance as a function of time, then the derivative measures the instantaneous velocity at time $t = a$.

notations for the derivative

The equation of the derivative of a function $f(x)$ is written as $y' = f'(x)$, where $y = f(x)$. The notation $f'(x)$ is read as "f prime of x." Alternate notations for the derivative include the following:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x)$$

The expression $f'(x)$ is now a function of x; this function gives the slope of the curve $y = f(x)$ at any value of x. The derivative of a function $f(x)$ at a point $x = a$ is denoted $f'(a)$.

How To...

Given a function f, find the derivative by applying the definition of the derivative.

1. Calculate $f(a + h)$.
2. Calculate $f(a)$.
3. Substitute and simplify $\dfrac{f(a+h) - f(a)}{h}$.
4. Evaluate the limit if it exists: $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.

Example 2 Finding the Derivative of a Polynomial Function

Find the derivative of the function $f(x) = x^2 - 3x + 5$ at $x = a$.

Solution We have:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad \text{Definition of a derivative}$$

Substitute $f(a + h) = (a + h)^2 - 3(a + h) + 5$ and $f(a) = a^2 - 3a + 5$.

$$\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{(a+h)(a+h) - 3(a+h) + 5 - (a^2 - 3a + 5)}{h} \\
&= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 3a - 3h + 5 - a^2 + 3a - 5}{h} && \text{Evaluate to remove parentheses.} \\
&= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 \cancel{-3a} - 3h \cancel{+5} \cancel{-a^2} \cancel{+3a} \cancel{-5}}{h} && \text{Simplify.} \\
&= \lim_{h \to 0} \frac{2ah + h^2 - 3h}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(2a + h - 3)}{\cancel{h}} && \text{Factor out an } h. \\
&= 2a + 0 - 3 && \text{Evaluate the limit.} \\
&= 2a - 3
\end{aligned}$$

Try It #2

Find the derivative of the function $f(x) = 3x^2 + 7x$ at $x = a$.

Finding Derivatives of Rational Functions

To find the derivative of a rational function, we will sometimes simplify the expression using algebraic techniques we have already learned.

Example 3 Finding the Derivative of a Rational Function

Find the derivative of the function $f(x) = \dfrac{3+x}{2-x}$ at $x = a$.

Solution

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{3+(a+h)}{2-(a+h)} - \left(\dfrac{3+a}{2-a}\right)}{h} \qquad \text{Substitute } f(a+h) \text{ and } f(a).$$

$$= \lim_{h \to 0} \frac{(2-(a+h))(2-a)\left[\dfrac{3+(a+h)}{2-(a+h)} - \left(\dfrac{3+a}{2-a}\right)\right]}{(2-(a+h))(2-a)(h)} \qquad \text{Multiply numerator and denominator by } (2-(a+h))(2-a).$$

$$= \lim_{h \to 0} \frac{(\cancel{2-(a+h)})(2-a)\left(\dfrac{3+(a+h)}{(\cancel{2-(a+h)})}\right) - (2-(a+h))(\cancel{2-a})\left(\dfrac{3+a}{\cancel{2-a}}\right)}{(2-(a+h))(2-a)(h)} \qquad \text{Distribute.}$$

$$= \lim_{h \to 0} \frac{6 - 3a + 2a - a^2 + 2h - ah - 6 + 3a + 3h - 2a + a2 + ah}{(2-(a+h))(2-a)(h)} \qquad \text{Multiply.}$$

$$= \lim_{h \to 0} \frac{5\cancel{h}}{(2-(a+h))(2-a)(\cancel{h})} \qquad \text{Combine like terms.}$$

$$= \lim_{h \to 0} \frac{5}{(2-(a+h))(2-a)} \qquad \text{Cancel like factors.}$$

$$= \frac{5}{(2-(a+0))(2-a)} = \frac{5}{(2-a)(2-a)} = \frac{5}{(2-a)^2} \qquad \text{Evaluate the limit.}$$

Try It #3

Find the derivative of the function $f(x) = \dfrac{10x+11}{5x+4}$ at $x = a$.

Finding Derivatives of Functions with Roots

To find derivatives of functions with roots, we use the methods we have learned to find limits of functions with roots, including multiplying by a conjugate.

Example 4 Finding the Derivative of a Function with a Root

Find the derivative of the function $f(x) = 4\sqrt{x}$ at $x = 36$.

Solution We have

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{4\sqrt{a+h} - 4\sqrt{a}}{h} \qquad \text{Substitute } f(a+h) \text{ and } f(a)$$

Multiply the numerator and denominator by the conjugate: $\dfrac{4\sqrt{a+h} + 4\sqrt{a}}{4\sqrt{a+h} + 4\sqrt{a}}$.

$$f'(a) = \lim_{h \to 0} \left(\frac{4\sqrt{a+h} - 4\sqrt{a}}{h} \right) \cdot \left(\frac{4\sqrt{a+h} + 4\sqrt{a}}{4\sqrt{a+h} + 4\sqrt{a}} \right)$$

$$= \lim_{h \to 0} \left(\frac{16(a+h) - 16a}{h4(\sqrt{a+h} + 4\sqrt{a})} \right) \quad \text{Multiply.}$$

$$= \lim_{h \to 0} \left(\frac{\cancel{16a} + 16h - \cancel{16a}}{h4(\sqrt{a+h} + 4\sqrt{a})} \right) \quad \text{Distribute and combine like terms.}$$

$$= \lim_{h \to 0} \left(\frac{16\cancel{h}}{\cancel{h}(4\sqrt{a+h} + 4\sqrt{a})} \right) \quad \text{Simplify.}$$

$$= \lim_{h \to 0} \frac{16}{4\sqrt{a+h} + 4\sqrt{a}} \quad \text{Evaluate the limit by letting } h = 0.$$

$$= \frac{16}{8\sqrt{a}} = \frac{2}{\sqrt{a}}$$

$$f'(36) = \frac{2}{\sqrt{36}} \quad \text{Evaluate the derivative at } x = 36.$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

Try It #4

Find the derivative of the function $f(x) = 9\sqrt{x}$ at $x = 9$.

Finding Instantaneous Rates of Change

Many applications of the derivative involve determining the rate of change at a given instant of a function with the independent variable time—which is why the term instantaneous is used. Consider the height of a ball tossed upward with an initial velocity of 64 feet per second, given by $s(t) = -16t^2 + 64t + 6$, where t is measured in seconds and $s(t)$ is measured in feet. We know the path is that of a parabola. The derivative will tell us how the height is changing at any given point in time. The height of the ball is shown in **Figure 4** as a function of time. In physics, we call this the "s-t graph."

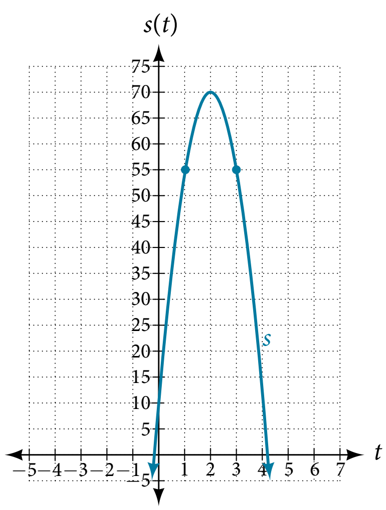


Figure 4

Example 5 Finding the Instantaneous Rate of Change

Using the function above, $s(t) = -16t^2 + 64t + 6$, what is the instantaneous velocity of the ball at 1 second and 3 seconds into its flight?

Solution The velocity at $t = 1$ and $t = 3$ is the instantaneous rate of change of distance per time, or velocity. Notice that the initial height is 6 feet. To find the instantaneous velocity, we find the derivative and evaluate it at $t = 1$ and $t = 3$:

$$
\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} && \\
&= \lim_{h \to 0} \frac{-16(t+h)^2 + 64(t+h) + 6 - (-16t^2 + 64t + 6)}{h} && \text{Substitute } s(t+h) \text{ and } s(t). \\
&= \lim_{h \to 0} \frac{-16t^2 - 32ht - h^2 + 64t + 64h + 6 + 16t^2 - 64t - 6}{h} && \text{Distribute.} \\
&= \lim_{h \to 0} \frac{-32ht - h^2 + 64h}{h} && \text{Simplify.} \\
&= \lim_{h \to 0} \frac{\cancel{h}(-32t - h + 64)}{\cancel{h}} && \text{Factor the numerator.} \\
&= \lim_{h \to 0} -32t - h + 64 && \text{Cancel out the common factor } h. \\
s'(t) &= -32t + 64 && \text{Evaluate the limit by letting } h = 0.
\end{aligned}
$$

For any value of t, $s'(t)$ tells us the velocity at that value of t.

Evaluate $t = 1$ and $t = 3$.

$$s'(1) = -32(1) + 64 = 32$$
$$s'(3) = -32(3) + 64 = -32$$

The velocity of the ball after 1 second is 32 feet per second, as it is on the way up.

The velocity of the ball after 3 seconds is -32 feet per second, as it is on the way down.

Try It #5

The position of the ball is given by $s(t) = -16t^2 + 64t + 6$. What is its velocity 2 seconds into flight?

Using Graphs to Find Instantaneous Rates of Change

We can estimate an instantaneous rate of change at $x = a$ by observing the slope of the curve of the function $f(x)$ at $x = a$. We do this by drawing a line tangent to the function at $x = a$ and finding its slope.

How To...

Given a graph of a function $f(x)$, find the instantaneous rate of change of the function at $x = a$.

1. Locate $x = a$ on the graph of the function $f(x)$.
2. Draw a tangent line, a line that goes through $x = a$ at a and at no other point in that section of the curve. Extend the line far enough to calculate its slope as

$$\frac{\text{change in } y}{\text{change in } x}.$$

Example 6 Estimating the Derivative at a Point on the Graph of a Function

From the graph of the function $y = f(x)$ presented in **Figure 5**, estimate each of the following:

$$f(0);\ f(2);\ f'(0);\ f'(2)$$

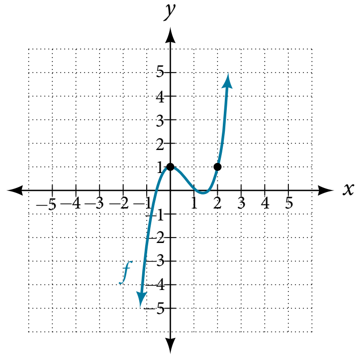


Figure 5

Solution To find the functional value, $f(a)$, find the y-coordinate at $x = a$.

To find the derivative at $x = a$, $f'(a)$, draw a tangent line at $x = a$, and estimate the slope of that tangent line. See **Figure 6**.

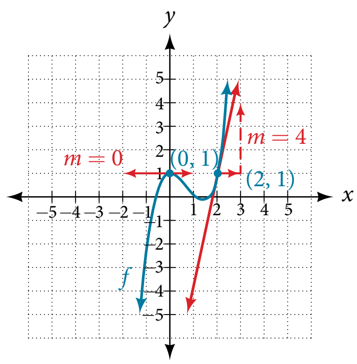


Figure 6

a. $f(0)$ is the y-coordinate at $x = 0$. The point has coordinates (0, 1), thus $f(0) = 1$.

b. $f(2)$ is the y-coordinate at $x = 2$. The point has coordinates (2, 1), thus $f(2) = 1$.

c. $f'(0)$ is found by estimating the slope of the tangent line to the curve at $x = 0$. The tangent line to the curve at $x = 0$ appears horizontal. Horizontal lines have a slope of 0, thus $f'(0) = 0$.

d. $f'(2)$ is found by estimating the slope of the tangent line to the curve at $x = 2$. Observe the path of the tangent line to the curve at $x = 2$. As the x value moves one unit to the right, the y value moves up four units to another point on the line. Thus, the slope is 4, so $f'(2) = 4$.

Try It #6

Using the graph of the function $f(x) = x^3 - 3x$ shown in **Figure 7**, estimate: $f(1)$, $f'(1)$, $f(0)$, and $f'(0)$.

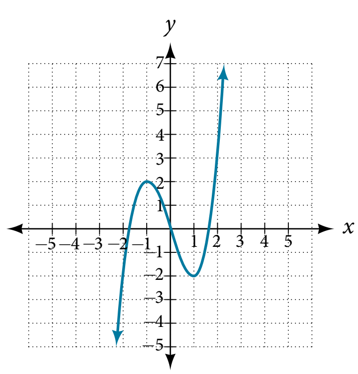


Figure 7

Using Instantaneous Rates of Change to Solve Real-World Problems

Another way to interpret an instantaneous rate of change at $x = a$ is to observe the function in a real-world context. The unit for the derivative of a function $f(x)$ is

$$\frac{\text{output units}}{\text{input unit}}$$

Such a unit shows by how many units the output changes for each one-unit change of input. The instantaneous rate of change at a given instant shows the same thing: the units of change of output per one-unit change of input.

One example of an instantaneous rate of change is a marginal cost. For example, suppose the production cost for a company to produce x items is given by $C(x)$, in thousands of dollars. The derivative function tells us how the cost is changing for any value of x in the domain of the function. In other words, $C'(x)$ is interpreted as a marginal cost, the additional cost in thousands of dollars of producing one more item when x items have been produced. For example, $C'(11)$ is the approximate additional cost in thousands of dollars of producing the 12th item after 11 items have been produced. $C'(11) = 2.50$ means that when 11 items have been produced, producing the 12th item would increase the total cost by approximately \$2,500.00.

Example 7 Finding a Marginal Cost

The cost in dollars of producing x laptop computers is $f(x) = x^2 - 100x$. At the point where 200 computers have been produced, what is the approximate cost of producing the 201st unit?

Solution If $f(x) = x^2 - 100x$ describes the cost of producing x computers, $f'(x)$ will describe the marginal cost. We need to find the derivative. For purposes of calculating the derivative, we can use the following functions:

$$f(a+b) = (x+h)^2 - 100(x+h)$$

$$f(a) = a^2 - 100a$$

$$\begin{aligned}
f'(x) &= \frac{f(a+h) - f(a)}{h} && \text{Formula for a derivative} \\
&= \frac{(x+h)^2 - 100(x+h) - (x^2 - 100x)}{h} && \text{Substitute } f(a+h) \text{ and } f(a). \\
&= \frac{x^2 + 2xh + h^2 - 100x - 100h - x^2 + 100x}{h} && \text{Multiply polynomials, distribute.} \\
&= \frac{2xh + h^2 - 100h}{h} && \text{Collect like terms.} \\
&= \frac{\cancel{h}(2x + h - 100)}{\cancel{h}} && \text{Factor and cancel like terms.} \\
&= 2x + h - 100 && \text{Simplify.} \\
&= 2x - 100 && \text{Evaluate when } h = 0. \\
f'(x) &= 2x - 100 && \text{Formula for marginal cost} \\
f'(200) &= 2(200) - 100 = 300 && \text{Evaluate for 200 units.}
\end{aligned}$$

The marginal cost of producing the 201st unit will be approximately \$300.

Example 8 Interpreting a Derivative in Context

A car leaves an intersection. The distance it travels in miles is given by the function $f(t)$, where t represents hours. Explain the following notations:

$$f(0) = 0\,;\ f'(1) = 60;\ f(1) = 70;\ f(2.5) = 150$$

Solution First we need to evaluate the function $f(t)$ and the derivative of the function $f'(t)$, and distinguish between the two. When we evaluate the function $f(t)$, we are finding the distance the car has traveled in t hours. When we evaluate the derivative $f'(t)$, we are finding the speed of the car after t hours.

a. $f(0) = 0$ means that in zero hours, the car has traveled zero miles.

b. $f'(1) = 60$ means that one hour into the trip, the car is traveling 60 miles per hour.

c. $f(1) = 70$ means that one hour into the trip, the car has traveled 70 miles. At some point during the first hour, then, the car must have been traveling faster than it was at the 1-hour mark.

d. $f(2.5) = 150$ means that two hours and thirty minutes into the trip, the car has traveled 150 miles.

Try It #7

A runner runs along a straight east-west road. The function $f(t)$ gives how many feet eastward of her starting point she is after t seconds. Interpret each of the following as it relates to the runner.

$$f(0) = 0; f(10) = 150; f'(10) = 15; f'(20) = -10; f(40) = -100$$

Finding Points Where a Function's Derivative Does Not Exist

To understand where a function's derivative does not exist, we need to recall what normally happens when a function $f(x)$ has a derivative at $x = a$. Suppose we use a graphing utility to zoom in on $x = a$. If the function $f(x)$ is **differentiable**, that is, if it is a function that can be differentiated, then the closer one zooms in, the more closely the graph approaches a straight line. This characteristic is called *linearity*.

Look at the graph in **Figure 8**. The closer we zoom in on the point, the more linear the curve appears.

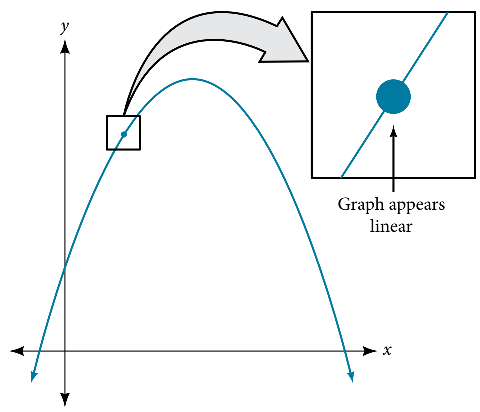


Figure 8

We might presume the same thing would happen with any continuous function, but that is not so. The function $f(x) = |x|$, for example, is continuous at $x = 0$, but not differentiable at $x = 0$. As we zoom in close to 0 in **Figure 9**, the graph does not approach a straight line. No matter how close we zoom in, the graph maintains its sharp corner.

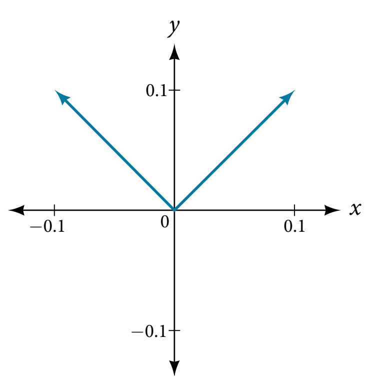


Figure 9 Graph of the function $f(x) = |x|$, with x-axis from −0.1 to 0.1 and y-axis from −0.1 to 0.1.

We zoom in closer by narrowing the range to produce **Figure 10** and continue to observe the same shape. This graph does not appear linear at $x = 0$.

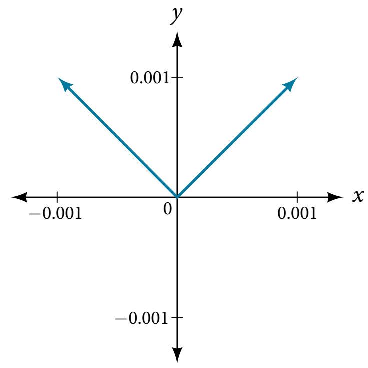


Figure 10 Graph of the function $f(x) = |x|$, with x-axis from -0.001 to 0.001 and y-axis from -0.001 to 0.001.

What are the characteristics of a graph that is not differentiable at a point? Here are some examples in which function $f(x)$ is not differentiable at $x = a$.

In **Figure 11**, we see the graph of

$$f(x) = \begin{cases} x^2, & x \leq 2 \\ 8 - x, & x > 2 \end{cases}.$$

Notice that, as x approaches 2 from the left, the left-hand limit may be observed to be 4, while as x approaches 2 from the right, the right-hand limit may be observed to be 6. We see that it has a discontinuity at $x = 2$.

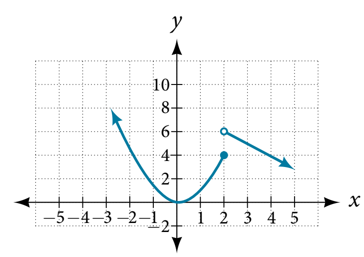


Figure 11 The graph of $f(x)$ has a discontinuity at $x = 2$.

In **Figure 12**, we see the graph of $f(x) = |x|$. We see that the graph has a corner point at $x = 0$.

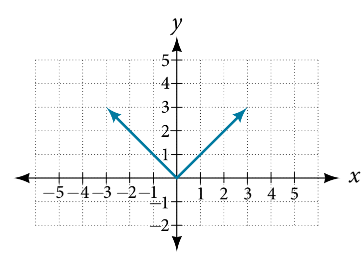


Figure 12 The graph of $f(x) = |x|$ has a corner point at $x = 0$.

In **Figure 13**, we see that the graph of $f(x) = x^{\frac{2}{3}}$ has a cusp at $x = 0$. A cusp has a unique feature. Moving away from the cusp, both the left-hand and right-hand limits approach either infinity or negative infinity. Notice the tangent lines as x approaches 0 from both the left and the right appear to get increasingly steeper, but one has a negative slope, the other has a positive slope.

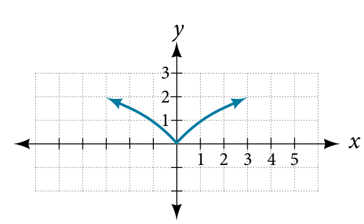


Figure 13 The graph of $f(x) = x^{\frac{2}{3}}$ has a cusp at $x = 0$.

In **Figure 14**, we see that the graph of $f(x) = x^{\frac{1}{3}}$ has a vertical tangent at $x = 0$. Recall that vertical tangents are vertical lines, so where a vertical tangent exists, the slope of the line is undefined. This is why the derivative, which measures the slope, does not exist there.

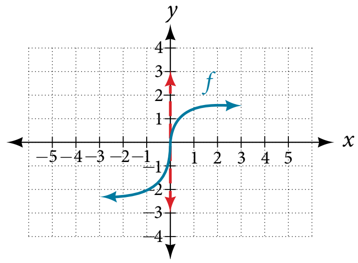


Figure 14 The graph of $f(x) = x^{\frac{1}{3}}$ has a vertical tangent at $x = 0$.

differentiability

A function $f(x)$ is differentiable at $x = a$ if the derivative exists at $x = a$, which means that $f'(a)$ exists. There are four cases for which a function $f(x)$ is not differentiable at a point $x = a$.

1. When there is a discontinuity at $x = a$.
2. When there is a corner point at $x = a$.
3. When there is a cusp at $x = a$.
4. Any other time when there is a vertical tangent at $x = a$.

Example 9 Determining Where a Function Is Continuous and Differentiable from a Graph

Using **Figure 15**, determine where the function is

a. continuous **b.** discontinuous **c.** differentiable **d.** not differentiable

At the points where the graph is discontinuous or not differentiable, state why.

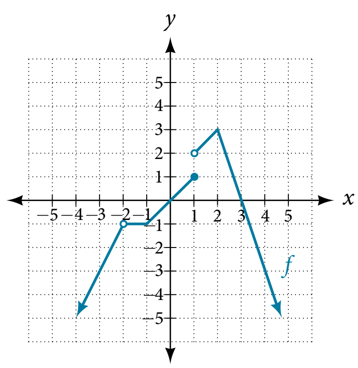


Figure 15

Solution The graph of $f(x)$ is continuous on $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$. The graph of $f(x)$ has a removable discontinuity at $x = -2$ and a jump discontinuity at $x = 1$. See **Figure 16**.

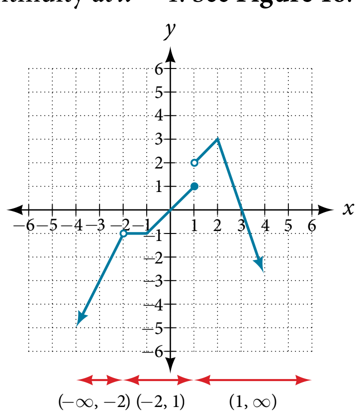


Figure 16 Three intervals where the function is continuous

The graph of $f(x)$ is differentiable on $(-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (1, 2) \cup (2, \infty)$. The graph of $f(x)$ is not differentiable at $x = -2$ because it is a point of discontinuity, at $x = -1$ because of a sharp corner, at $x = 1$ because it is a point of discontinuity, and at $x = 2$ because of a sharp corner. See **Figure 17**.

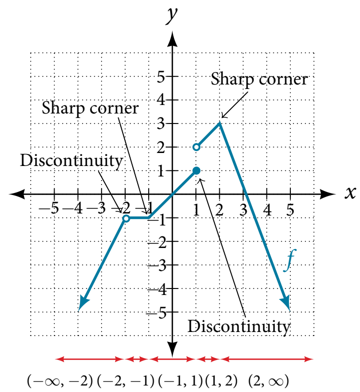


Figure 17 Five intervals where the function is differentiable

Try It #8

Determine where the function $y = f(x)$ shown in **Figure 18** is continuous and differentiable from the graph.

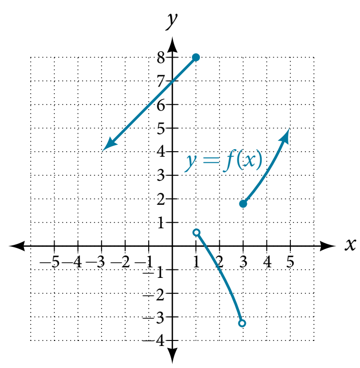


Figure 18

Finding an Equation of a Line Tangent to the Graph of a Function

The equation of a tangent line to a curve of the function $f(x)$ at $x = a$ is derived from the point-slope form of a line, $y = m(x - x_1) + y_1$. The slope of the line is the slope of the curve at $x = a$ and is therefore equal to $f'(a)$, the derivative of $f(x)$ at $x = a$. The coordinate pair of the point on the line at $x = a$ is $(a, f(a))$.

If we substitute into the point-slope form, we have

$$m = f'(a) \qquad x_1 = a \qquad y_1 = f(a)$$

$$y = \underset{\substack{\uparrow \\ f'(a)}}{m}(x - \underset{\substack{\uparrow \\ a}}{x_1}) + \underset{\substack{\uparrow \\ f(a)}}{y_1}$$

The equation of the tangent line is

$$y = f'(a)(x - a) + f(a)$$

the equation of a line tangent to a curve of the function f

The equation of a line tangent to the curve of a function f at a point $x = a$ is

$$y = f'(a)(x - a) + f(a)$$

How To...

Given a function f, find the equation of a line tangent to the function at $x = a$.

1. Find the derivative of $f(x)$ at $x = a$ using $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.
2. Evaluate the function at $x = a$. This is $f(a)$.
3. Substitute $(a, f(a))$ and $f'(a)$ into $y = f'(a)(x - a) + f(a)$.
4. Write the equation of the tangent line in the form $y = mx + b$.

Example 10 Finding the Equation of a Line Tangent to a Function at a Point

Find the equation of a line tangent to the curve $f(x) = x^2 - 4x$ at $x = 3$.

Solution

Using:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Substitute $f(a+h) = (a+h)^2 - 4(a+h)$ and $f(a) = a^2 - 4a$.

$$\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{(a+h)(a+h) - 4(a+h) - (a^2 - 4a)}{h} \\
&= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 4a - 4h - a^2 + 4a}{h} && \text{Remove parentheses.} \\
&= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 \cancel{- 4a} - 4h \cancel{- a^2} \cancel{+ 4a}}{h} && \text{Combine like terms.} \\
&= \lim_{h \to 0} \frac{2ah + h^2 - 4h}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(2a + h - 4)}{\cancel{h}} && \text{Factor out } h. \\
&= 2a + 0 - 4 \\
f'(a) &= 2a - 4 && \text{Evaluate the limit.} \\
f'(3) &= 2(3) - 4 = 2
\end{aligned}$$

Equation of tangent line at $x = 3$:

$$\begin{aligned}
y &= f'(a)(x - a) + f(a) \\
y &= f'(3)(x - 3) + f(3) \\
y &= 2(x - 3) + (-3) \\
y &= 2x - 9
\end{aligned}$$

Analysis *We can use a graphing utility to graph the function and the tangent line. In so doing, we can observe the point of tangency at $x = 3$ as shown in* ***Figure 19****.*

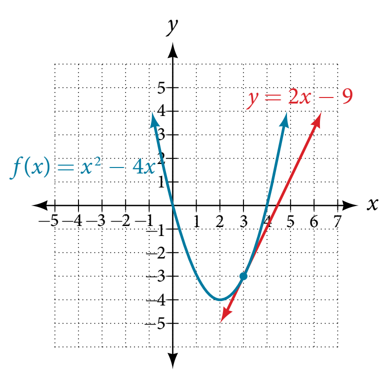


Figure 19 Graph confirms the point of tangency at $x = 3$.

Try It #9

Find the equation of a tangent line to the curve of the function $f(x) = 5x^2 - x + 4$ at $x = 2$.

Finding the Instantaneous Speed of a Particle

If a function measures position versus time, the derivative measures displacement versus time, or the speed of the object. A change in speed or direction relative to a change in time is known as velocity. The velocity at a given instant is known as **instantaneous velocity**.

In trying to find the speed or velocity of an object at a given instant, we seem to encounter a contradiction. We normally define speed as the distance traveled divided by the elapsed time. But in an instant, no distance is traveled, and no time elapses. How will we divide zero by zero? The use of a derivative solves this problem. A derivative allows us to say that even while the object's velocity is constantly changing, it has a certain velocity at a given instant. That means that if the object traveled at that exact velocity for a unit of time, it would travel the specified distance.

instantaneous velocity

Let the function $s(t)$ represent the position of an object at time t. The **instantaneous velocity** or velocity of the object at time $t = a$ is given by

$$s'(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

Example 11 Finding the Instantaneous Velocity

A ball is tossed upward from a height of 200 feet with an initial velocity of 36 ft/sec. If the height of the ball in feet after t seconds is given by $s(t) = -16t^2 + 36t + 200$, find the instantaneous velocity of the ball at $t = 2$.

Solution First, we must find the derivative $s'(t)$. Then we evaluate the derivative at $t = 2$, using

$$s(a+h) = -16(a+h)^2 + 36(a+h) + 200 \text{ and } s(a) = -16a^2 + 36a + 200.$$

$$\begin{aligned}
s'(a) &= \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} \\
&= \lim_{h \to 0} \frac{-16(a+h)^2 + 36(a+h) + 200 - (-16a^2 + 36a + 200)}{h} \\
&= \lim_{h \to 0} \frac{-16(a^2 + 2ah + h^2) + 36(a+h) + 200 - (-16a^2 + 36a + 200)}{h} \\
&= \lim_{h \to 0} \frac{-16a^2 - 32ah - 16h^2 + 36a + 36h + 200 + 16a^2 - 36a - 200}{h} \\
&= \lim_{h \to 0} \frac{\cancel{-16a^2} - 32ah - 16h^2 \cancel{+36a} + 36h \cancel{+200} \cancel{+16a^2} \cancel{-36a} \cancel{-200}}{h} \\
&= \lim_{h \to 0} \frac{-32ah - 16h^2 + 36h}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(-32a - 16h + 36)}{\cancel{h}} \\
&= \lim_{h \to 0} (-32a - 16h + 36) \\
&= -32a - 16 \cdot 0 + 36 \\
s'(a) &= -32a + 36 \\
s'(2) &= -32(2) + 36 \\
&= -28
\end{aligned}$$

Analysis *This result means that at time $t = 2$ seconds, the ball is dropping at a rate of 28 ft/sec.*

Try It #10

A fireworks rocket is shot upward out of a pit 12 ft below the ground at a velocity of 60 ft/sec. Its height in feet after t seconds is given by $s = -16t^2 + 60t - 12$. What is its instantaneous velocity after 4 seconds?

Access these online resources for additional instruction and practice with derivatives.

- Estimate the Derivative (http://openstaxcollege.org/l/estimatederiv)
- Estimate the Derivative Ex. 4 (http://openstaxcollege.org/l/estimatederiv4)

12.4 SECTION EXERCISES

VERBAL

1. How is the slope of a linear function similar to the derivative?

2. What is the difference between the average rate of change of a function on the interval $[x, x + h]$ and the derivative of the function at x?

3. A car traveled 110 miles during the time period from 2:00 P.M. to 4:00 P.M. What was the car's average velocity? At exactly 2:30 P.M., the speed of the car registered exactly 62 miles per hour. What is another name for the speed of the car at 2:30 P.M.? Why does this speed differ from the average velocity?

4. Explain the concept of the slope of a curve at point x.

5. Suppose water is flowing into a tank at an average rate of 45 gallons per minute. Translate this statement into the language of mathematics.

ALGEBRAIC

For the following exercises, use the definition of derivative $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to calculate the derivative of each function.

6. $f(x) = 3x - 4$

7. $f(x) = -2x + 1$

8. $f(x) = x^2 - 2x + 1$

9. $f(x) = 2x^2 + x - 3$

10. $f(x) = 2x^2 + 5$

11. $f(x) = \dfrac{-1}{x-2}$

12. $f(x) = \dfrac{2+x}{1-x}$

13. $f(x) = \dfrac{5-2x}{3+2x}$

14. $f(x) = \sqrt{1+3x}$

15. $f(x) = 3x^3 - x^2 + 2x + 5$

16. $f(x) = 5$

17. $f(x) = 5\pi$

For the following exercises, find the average rate of change between the two points.

18. $(-2, 0)$ and $(-4, 5)$

19. $(4, -3)$ and $(-2, -1)$

20. $(0, 5)$ and $(6, 5)$

21. $(7, -2)$ and $(7, 10)$

For the following polynomial functions, find the derivatives.

22. $f(x) = x^3 + 1$

23. $f(x) = -3x^2 - 7x + 6$

24. $f(x) = 7x^2$

25. $f(x) = 3x^3 + 2x^2 + x - 26$

For the following functions, find the equation of the tangent line to the curve at the given point x on the curve.

26. $f(x) = 2x^2 - 3x \quad x = 3$

27. $f(x) = x^3 + 1 \quad x = 2$

28. $f(x) = \sqrt{x} \quad x = 9$

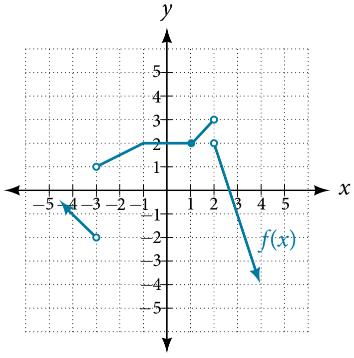
For the following exercise, find k such that the given line is tangent to the graph of the function.

29. $f(x) = x^2 - kx, \quad y = 4x - 9$

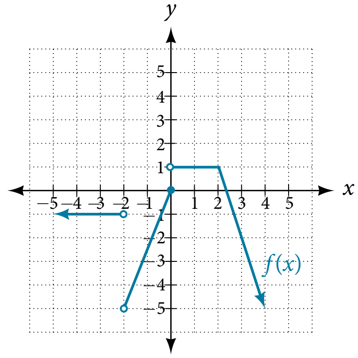
GRAPHICAL

For the following exercises, consider the graph of the function f and determine where the function is continuous/discontinuous and differentiable/not differentiable.

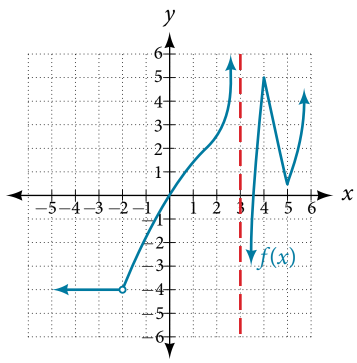
30.



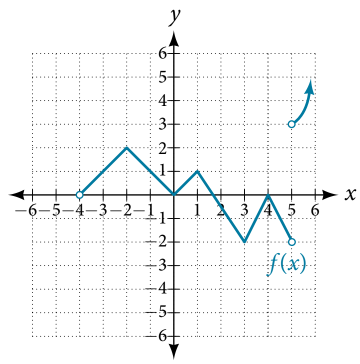
31.



32.



33.



For the following exercises, use **Figure 20** to estimate either the function at a given value of x or the derivative at a given value of x, as indicated.

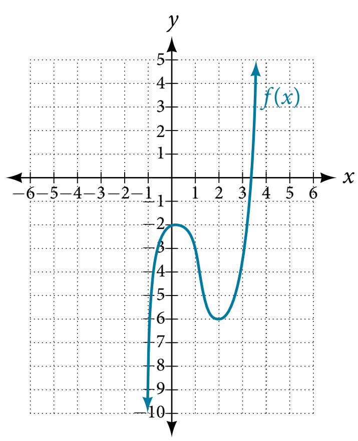


Figure 20

34. $f(-1)$ **35.** $f(0)$ **36.** $f(1)$ **37.** $f(2)$ **38.** $f(3)$ **39.** $f'(-1)$

40. $f'(0)$ **41.** $f'(1)$ **42.** $f'(2)$ **43.** $f'(3)$

44. Sketch the function based on the information below:
$f'(x) = 2x, f(2) = 4$

TECHNOLOGY

45. Numerically evaluate the derivative. Explore the behavior of the graph of $f(x) = x^2$ around $x = 1$ by graphing the function on the following domains: $[0.9, 1.1]$, $[0.99, 1.01]$, $[0.999, 1.001]$, and $[0.9999, 1.0001]$. We can use the feature on our calculator that automatically sets Ymin and Ymax to the Xmin and Xmax values we preset. (On some of the commonly used graphing calculators, this feature may be called **ZOOM FIT** or **ZOOM AUTO**). By examining the corresponding range values for this viewing window, approximate how the curve changes at $x = 1$, that is, approximate the derivative at $x = 1$.

REAL-WORLD APPLICATIONS

For the following exercises, explain the notation in words. The volume $f(t)$ of a tank of gasoline, in gallons, t minutes after noon.

46. $f(0) = 600$ **47.** $f'(30) = -20$ **48.** $f(30) = 0$ **49.** $f'(200) = 30$ **50.** $f(240) = 500$

For the following exercises, explain the functions in words. The height, s, of a projectile after t seconds is given by $s(t) = -16t^2 + 80t$.

51. $s(2) = 96$

52. $s'(2) = 16$

53. $s(3) = 96$

54. $s'(3) = -16$

55. $s(0) = 0, s(5) = 0.$

For the following exercises, the volume V of a sphere with respect to its radius r is given by $V = \frac{4}{3}\pi r^3$.

56. Find the average rate of change of V as r changes from 1 cm to 2 cm.

57. Find the instantaneous rate of change of V when $r = 3$ cm.

For the following exercises, the revenue generated by selling x items is given by $R(x) = 2x^2 + 10x$.

58. Find the average change of the revenue function as x changes from $x = 10$ to $x = 20$.

59. Find $R'(10)$ and interpret.

60. Find $R'(15)$ and interpret. Compare $R'(15)$ to $R'(10)$, and explain the difference.

For the following exercises, the cost of producing x cellphones is described by the function $C(x) = x^2 - 4x + 1000$.

61. Find the average rate of change in the total cost as x changes from $x = 10$ to $x = 15$.

62. Find the approximate marginal cost, when 15 cellphones have been produced, of producing the 16$^{\text{th}}$ cellphone.

63. Find the approximate marginal cost, when 20 cellphones have been produced, of producing the 21$^{\text{st}}$ cellphone.

EXTENSION

For the following exercises, use the definition for the derivative at a point $x = a$, $\lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$, to find the derivative of the functions.

64. $f(x) = \frac{1}{x^2}$

65. $f(x) = 5x^2 - x + 4$

66. $f(x) = -x^2 + 4x + 7$

67. $f(x) = \frac{-4}{3 - x^2}$

CHAPTER 12 REVIEW

Key Terms

average rate of change the slope of the line connecting the two points $(a, f(a))$ and $(a+h, f(a+h))$ on the curve of $f(x)$; it is given by $\text{AROC} = \dfrac{f(a+h) - f(a)}{h}$.

continuous function a function that has no holes or breaks in its graph

derivative the slope of a function at a given point; denoted $f'(a)$, at a point $x = a$ it is $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$, providing the limit exists.

differentiable a function $f(x)$ for which the derivative exists at $x = a$. In other words, if $f'(a)$ exists.

discontinuous function a function that is not continuous at $x = a$

instantaneous rate of change the slope of a function at a given point; at $x = a$ it is given by $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

instantaneous velocity the change in speed or direction at a given instant; a function $s(t)$ represents the position of an object at time t, and the instantaneous velocity or velocity of the object at time $t = a$ is given by $s'(a) = \lim\limits_{h \to 0} \dfrac{s(a+h) - s(a)}{h}$.

jump discontinuity a point of discontinuity in a function $f(x)$ at $x = a$ where both the left and right-hand limits exist, but $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$

left-hand limit the limit of values of $f(x)$ as x approaches a from the left, denoted $\lim\limits_{x \to a^-} f(x) = L$. The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a such that $x < a$ and $x \neq a$. Both a and L are real numbers.

limit when it exists, the value, L, that the output of a function $f(x)$ approaches as the input x gets closer and closer to a but does not equal a. The value of the output, $f(x)$, can get as close to L as we choose to make it by using input values of x sufficiently near to $x = a$, but not necessarily at $x = a$. Both a and L are real numbers, and L is denoted $\lim\limits_{x \to a} f(x) = L$.

properties of limits a collection of theorems for finding limits of functions by performing mathematical operations on the limits

removable discontinuity a point of discontinuity in a function $f(x)$ where the function is discontinuous, but can be redefined to make it continuous

right-hand limit the limit of values of $f(x)$ as x approaches a from the right, denoted $\lim\limits_{x \to a^+} f(x) = L$. The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a where $x > a$, and $x \neq a$. Both a and L are real numbers.

secant line a line that intersects two points on a curve

tangent line a line that intersects a curve at a single point

two-sided limit the limit of a function $f(x)$, as x approaches a, is equal to L, that is, $\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$.

Key Equations

average rate of change	$\text{AROC} = \dfrac{f(a+h) - f(a)}{h}$
derivative of a function	$f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Key Concepts

12.1 Finding Limits: Numerical and Graphical Approaches

- A function has a limit if the output values approach some value L as the input values approach some quantity a. See **Example 1**.
- A shorthand notation is used to describe the limit of a function according to the form $\lim_{x \to a} f(x) = L$, which indicates that as x approaches a, both from the left of $x = a$ and the right of $x = a$, the output value gets close to L.
- A function has a left-hand limit if $f(x)$ approaches L as x approaches a where $x < a$. A function has a right-hand limit if $f(x)$ approaches L as x approaches a where $x > a$.
- A two-sided limit exists if the left-hand limit and the right-hand limit of a function are the same. A function is said to have a limit if it has a two-sided limit.
- A graph provides a visual method of determining the limit of a function.
- If the function has a limit as x approaches a, the branches of the graph will approach the same y- coordinate near $x = a$ from the left and the right. See **Example 2**.
- A table can be used to determine if a function has a limit. The table should show input values that approach a from both directions so that the resulting output values can be evaluated. If the output values approach some number, the function has a limit. See **Example 3**.
- A graphing utility can also be used to find a limit. See **Example 4**.

12.2 Finding Limits: Properties of Limits

- The properties of limits can be used to perform operations on the limits of functions rather than the functions themselves. See **Example 1**.
- The limit of a polynomial function can be found by finding the sum of the limits of the individual terms. See **Example 2** and **Example 3**.
- The limit of a function that has been raised to a power equals the same power of the limit of the function. Another method is direct substitution. See **Example 4**.
- The limit of the root of a function equals the corresponding root of the limit of the function.
- One way to find the limit of a function expressed as a quotient is to write the quotient in factored form and simplify. See **Example 5**.
- Another method of finding the limit of a complex fraction is to find the LCD. See **Example 6**.
- A limit containing a function containing a root may be evaluated using a conjugate. See **Example 7**.
- The limits of some functions expressed as quotients can be found by factoring. See **Example 8**.
- One way to evaluate the limit of a quotient containing absolute values is by using numeric evidence. Setting it up piecewise can also be useful. See **Example 9**.

12.3 Continuity

- A continuous function can be represented by a graph without holes or breaks.
- A function whose graph has holes is a discontinuous function.
- A function is continuous at a particular number if three conditions are met:
  - Condition 1: $f(a)$ exists.
  - Condition 2: $\lim_{x \to a} f(x)$ exists at $x = \text{a}$.
  - Condition 3: $\lim_{x \to a} f(x) = f(a)$.
- A function has a jump discontinuity if the left- and right-hand limits are different, causing the graph to "jump."
- A function has a removable discontinuity if it can be redefined at its discontinuous point to make it continuous. See **Example 1**.

- Some functions, such as polynomial functions, are continuous everywhere. Other functions, such as logarithmic functions, are continuous on their domain. See **Example 2** and **Example 3**.
- For a piecewise function to be continuous each piece must be continuous on its part of the domain and the function as a whole must be continuous at the boundaries. See **Example 4** and **Example 5**.

12.4 Derivatives

- The slope of the secant line connecting two points is the average rate of change of the function between those points. See **Example 1**.
- The derivative, or instantaneous rate of change, is a measure of the slope of the curve of a function at a given point, or the slope of the line tangent to the curve at that point. See **Example 2**, **Example 3**, and **Example 4**.
- The difference quotient is the quotient in the formula for the instantaneous rate of change: $\dfrac{f(a+h)-f(a)}{h}$
- Instantaneous rates of change can be used to find solutions to many real-world problems. See **Example 5**.
- The instantaneous rate of change can be found by observing the slope of a function at a point on a graph by drawing a line tangent to the function at that point. See **Example 6**.
- Instantaneous rates of change can be interpreted to describe real-world situations. See **Example 7** and **Example 8**.
- Some functions are not differentiable at a point or points. See **Example 9**.
- The point-slope form of a line can be used to find the equation of a line tangent to the curve of a function. See **Example 10**.
- Velocity is a change in position relative to time. Instantaneous velocity describes the velocity of an object at a given instant. Average velocity describes the velocity maintained over an interval of time.
- Using the derivative makes it possible to calculate instantaneous velocity even though there is no elapsed time. See **Example 11**.

CHAPTER 12 REVIEW EXERCISES

FINDING LIMITS: A NUMERICAL AND GRAPHICAL APPROACH

For the following exercises, use **Figure 1**.

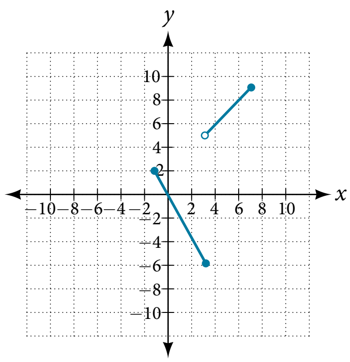


Figure 1

1. $\lim_{x \to -1^+} f(x)$

2. $\lim_{x \to -1^-} f(x)$

3. $\lim_{x \to -1} f(x)$

4. $\lim_{x \to 3} f(x)$

5. At what values of x is the function discontinuous? What condition of continuity is violated?

6. Using **Table 1**, estimate $\lim_{x \to 0} f(x)$.

x	$f(x)$
-0.1	2.875
-0.01	2.92
-0.001	2.998
0	Undefined
0.001	2.9987
0.01	2.865
0.1	2.78145
0.15	2.678

Table 1

For the following exercises, with the use of a graphing utility, use numerical or graphical evidence to determine the left- and right-hand limits of the function given as x approaches a. If the function has limit as x approaches a, state it. If not, discuss why there is no limit.

7. $f(x) = \begin{cases} |x| - 1, & \text{if } x \neq 1 \\ x^3, & \text{if } x = 1 \end{cases} \quad a = 1$

8. $f(x) = \begin{cases} \dfrac{1}{x+1}, & \text{if } x = -2 \\ (x+1)^2, & \text{if } x \neq -2 \end{cases} \quad a = -2$

9. $f(x) = \begin{cases} \sqrt{x+3}, & \text{if } x < 1 \\ -\sqrt[3]{x}, & \text{if } x > 1 \end{cases} \quad a = 1$

FINDING LIMITS: PROPERTIES OF LIMITS

For the following exercises, find the limits if $\lim_{x \to c} f(x) = -3$ and $\lim_{x \to c} g(x) = 5$.

10. $\lim_{x \to c} (f(x) + g(x))$

11. $\lim_{x \to c} \dfrac{f(x)}{g(x)}$

12. $\lim_{x \to c} (f(x) \cdot g(x))$

13. $\lim_{x \to 0^+} f(x), f(x) = \begin{cases} 3x^2 + 2x + 1 & x > 0 \\ 5x + 3 & x < 0 \end{cases}$

14. $\lim_{x \to 0^-} f(x), f(x) = \begin{cases} 3x^2 + 2x + 1 & x > 0 \\ 5x + 3 & x < 0 \end{cases}$

15. $\lim_{x \to 3^+} (3x - \llbracket x \rrbracket)$

For the following exercises, evaluate the limits using algebraic techniques.

16. $\lim_{h \to 0} \left(\frac{(h+6)^2 - 36}{h} \right)$

17. $\lim_{x \to 25} \left(\frac{x^2 - 625}{\sqrt{x} - 5} \right)$

18. $\lim_{x \to 1} \left(\frac{-x^2 - 9x}{x} \right)$

19. $\lim_{x \to 4} \left(\frac{7 - \sqrt{12x + 1}}{x - 4} \right)$

20. $\lim_{x \to -3} \left(\frac{\frac{1}{3} + \frac{1}{x}}{3 + x} \right)$

CONTINUITY

For the following exercises, use numerical evidence to determine whether the limit exists at $x = a$. If not, describe the behavior of the graph of the function at $x = a$.

21. $f(x) = \frac{-2}{x - 4}$; $a = 4$

22. $f(x) = \frac{-2}{(x - 4)^2}$; $a = 4$

23. $f(x) = \frac{-x}{x^2 - x - 6}$; $a = 3$

24. $f(x) = \frac{6x^2 + 23x + 20}{4x^2 - 25}$; $a = -\frac{5}{2}$

25. $f(x) = \frac{\sqrt{x} - 3}{9 - x}$; $a = 9$

For the following exercises, determine where the given function $f(x)$ is continuous. Where it is not continuous, state which conditions fail, and classify any discontinuities.

26. $f(x) = x^2 - 2x - 15$

27. $f(x) = \frac{x^2 - 2x - 15}{x - 5}$

28. $f(x) = \frac{x^2 - 2x}{x^2 - 4x + 4}$

29. $f(x) = \frac{x^3 - 125}{2x^2 - 12x + 10}$

30. $f(x) = \frac{x^2 - \frac{1}{x}}{2 - x}$

31. $f(x) = \frac{x + 2}{x^2 - 3x - 10}$

32. $f(x) = \frac{x + 2}{x^3 + 8}$

DERIVATIVES

For the following exercises, find the average rate of change $\frac{f(x+h) - f(x)}{h}$.

33. $f(x) = 3x + 2$

34. $f(x) = 5$

35. $f(x) = \frac{1}{x + 1}$

36. $f(x) = \ln(x)$

37. $f(x) = e^{2x}$

For the following exercises, find the derivative of the function.

38. $f(x) = 4x - 6$

39. $f(x) = 5x^2 - 3x$

40. Find the equation of the tangent line to the graph of $f(x)$ at the indicated x value.
$f(x) = -x^3 + 4x$; $x = 2$.

For the following exercises, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

41. $f(x) = \frac{x}{|x|}$

42. Given that the volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$ and that a given cone has a fixed height of 9 cm and variable radius length, find the instantaneous rate of change of volume with respect to radius length when the radius is 2 cm. Give an exact answer in terms of π.

CHAPTER 12 PRACTICE TEST

For the following exercises, use the graph of f in **Figure 1**.

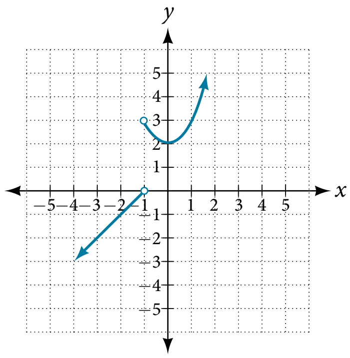


Figure 1

1. $f(1)$

2. $\lim\limits_{x \to -1^+} f(x)$

3. $\lim\limits_{x \to -1^-} f(x)$

4. $\lim\limits_{x \to -1} f(x)$

5. $\lim\limits_{x \to -2} f(x)$

6. At what values of x is f discontinuous? What property of continuity is violated?

For the following exercises, with the use of a graphing utility, use numerical or graphical evidence to determine the left- and right-hand limits of the function given as x approaches a. If the function has a limit as x approaches a, state it. If not, discuss why there is no limit.

7. $f(x) = \begin{cases} \frac{1}{x} - 3, & \text{if } x \le 2 \\ x^3 + 1, & \text{if } x > 2 \end{cases} \quad a = 2$

8. $f(x) = \begin{cases} x^3 + 1, & \text{if } x < 1 \\ 3x^2 - 1, & \text{if } x = 1 \\ -\sqrt{x+3} + 4, & \text{if } x > 1 \end{cases} \quad a = 1$

For the following exercises, evaluate each limit using algebraic techniques.

9. $\lim\limits_{x \to -5} \left(\dfrac{\frac{1}{5} + \frac{1}{x}}{10 + 2x} \right)$

10. $\lim\limits_{h \to 0} \left(\dfrac{\sqrt{h^2 + 25} - 5}{h^2} \right)$

11. $\lim\limits_{h \to 0} \left(\dfrac{1}{h} - \dfrac{1}{h^2 + h} \right)$

For the following exercises, determine whether or not the given function f is continuous. If it is continuous, show why. If it is not continuous, state which conditions fail.

12. $f(x) = \sqrt{x^2 - 4}$

13. $f(x) = \dfrac{x^3 - 4x^2 - 9x + 36}{x^3 - 3x^2 + 2x - 6}$

For the following exercises, use the definition of a derivative to find the derivative of the given function at $x = a$.

14. $f(x) = \dfrac{3}{5 + 2x}$

15. $f(x) = \dfrac{3}{\sqrt{x}}$

16. $f(x) = 2x^2 + 9x$

17. For the graph in **Figure 2**, determine where the function is continuous/discontinuous and differentiable/not differentiable.

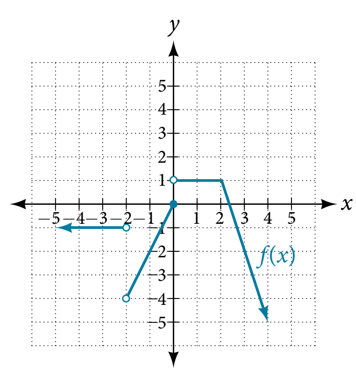


Figure 2

For the following exercises, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

18. $f(x) = |x - 2| - |x + 2|$

19. $f(x) = \dfrac{2}{1 + e^{\frac{2}{x}}}$

For the following exercises, explain the notation in words when the height of a projectile in feet, s, is a function of time t in seconds after launch and is given by the function $s(t)$.

20. $s(0)$

21. $s(2)$

22. $s'(2)$

23. $\dfrac{s(2) - s(1)}{2 - 1}$

24. $s(t) = 0$

For the following exercises, use technology to evaluate the limit.

25. $\lim\limits_{x \to 0} \dfrac{\sin(x)}{3x}$

26. $\lim\limits_{x \to 0} \dfrac{\tan^2(x)}{2x}$

27. $\lim\limits_{x \to 0} \dfrac{\sin(x)(1 - \cos(x))}{2x^2}$

28. Evaluate the limit by hand.

$\lim\limits_{x \to 1} f(x)$, where $f(x) = \begin{cases} 4x - 7 & x \neq 1 \\ x^2 - 4 & x = 1 \end{cases}$

At what value(s) of x is the function discontinuous?

For the following exercises, consider the function whose graph appears in **Figure 3**.

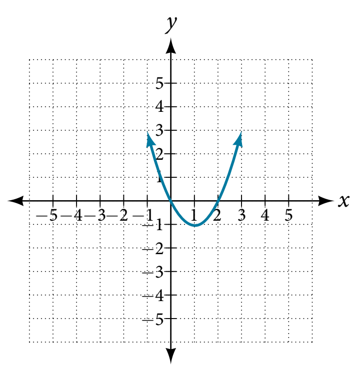


Figure 3

29. Find the average rate of change of the function from $x = 1$ to $x = 3$.

30. Find all values of x at which $f'(x) = 0$.

31. Find all values of x at which $f'(x)$ does not exist.

32. Find an equation of the tangent line to the graph of f the indicated point: $f(x) = 3x^2 - 2x - 6$, $x = -2$

For the following exercises, use the function $f(x) = x(1 - x)^{\frac{2}{5}}$.

33. Graph the function $f(x) = x(1 - x)^{\frac{2}{5}}$ by entering $f(x) = x((1 - x)^2)^{\frac{1}{5}}$ and then by entering $f(x) = x\left((1 - x)^{\frac{1}{5}}\right)^2$.

34. Explore the behavior of the graph of $f(x)$ around $x = 1$ by graphing the function on the following domains, [0.9, 1.1], [0.99, 1.01], [0.999, 1.001], and [0.9999, 1.0001]. Use this information to determine whether the function appears to be differentiable at $x = 1$.

For the following exercises, find the derivative of each of the functions using the definition: $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

35. $f(x) = 2x - 8$

36. $f(x) = 4x^2 - 7$

37. $f(x) = x - \dfrac{1}{2}x^2$

38. $f(x) = \dfrac{1}{x + 2}$

39. $f(x) = \dfrac{3}{x - 1}$

40. $f(x) = -x^3 + 1$

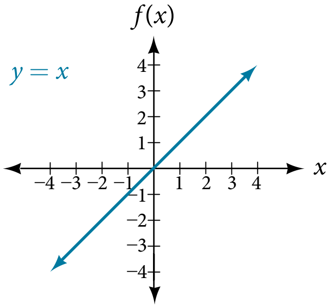
41. $f(x) = x^2 + x^3$

42. $f(x) = \sqrt{x - 1}$

Basic Functions and Identities

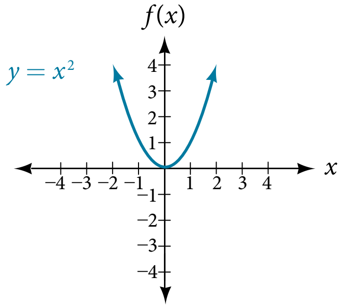
A1 Graphs of the Parent Functions

Identity



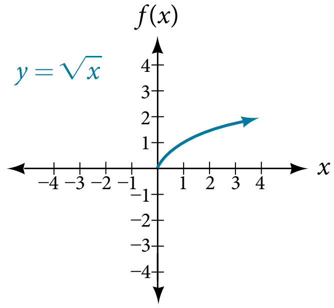
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Square



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

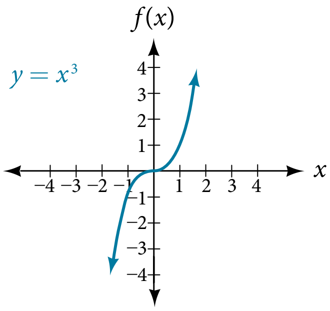
Square Root



Domain: $[0, \infty)$
Range: $[0, \infty)$

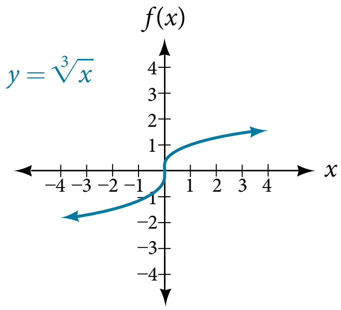
Figure A1

Cubic



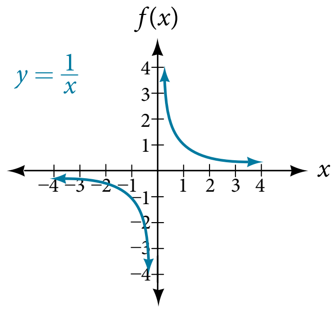
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Cube Root



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

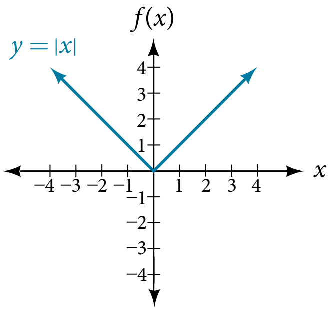
Reciprocal



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

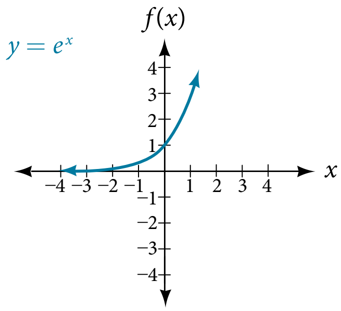
Figure A2

Absolute Value



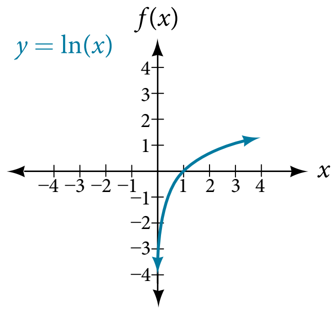
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Exponential



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Natural Logarithm

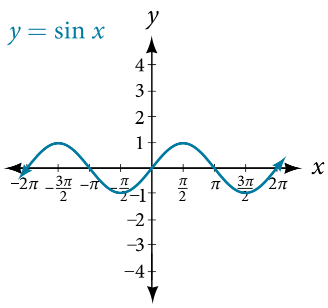


Domain: $(0, \infty)$
Range: $(-\infty, \infty)$

Figure A3

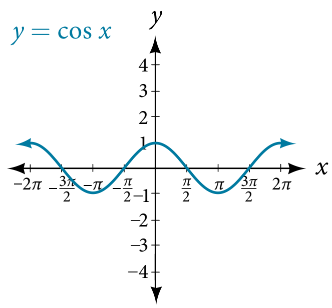
A2 Graphs of the Trigonometric Functions

Sine



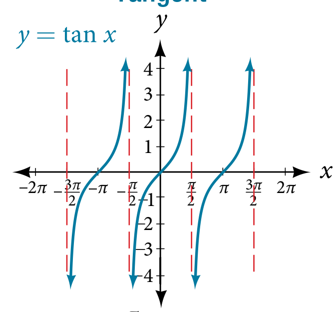
Domain: $(-\infty, \infty)$
Range: $(-1, 1)$

Cosine



Domain: $(-\infty, \infty)$
Range: $(-1, 1)$

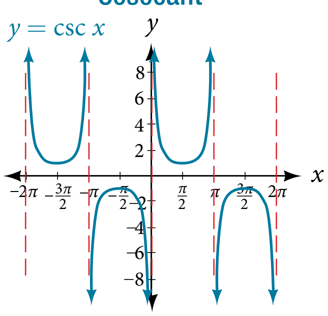
Tangent



Domain: $x \neq \frac{\pi}{2}k$ where k is an odd integer
Range: $(-\infty, -1] \cup [1, \infty)$

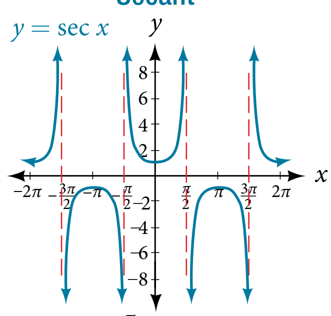
Figure A4

Cosecant



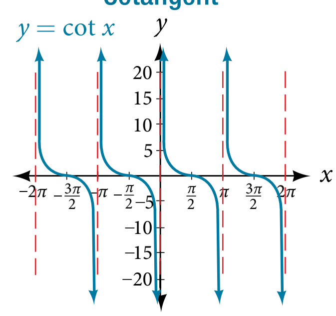
Domain: $x \neq \pi k$ where k is an integer
Range: $(-\infty, -1] \cup [1, \infty)$

Secant



Domain: $x \neq \frac{\pi}{2}k$ where k is an odd integer
Range: $(-\infty, -1] \cup [1, \infty)$

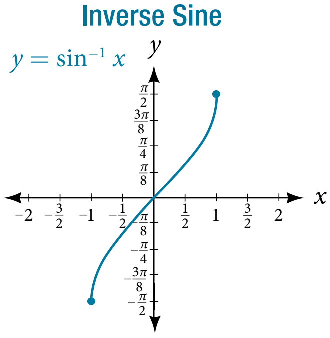
Cotangent



Domain: $x \neq \pi k$ where k is an integer
Range: $(-\infty, \infty)$

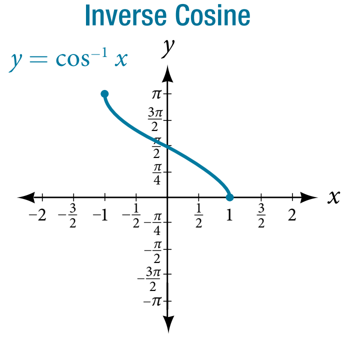
Figure A5

Inverse Sine



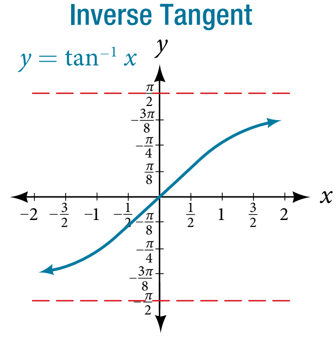
Domain: $[-1, 1]$
Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Inverse Cosine



Domain: $[-1, 1]$
Range: $[0, \pi)$

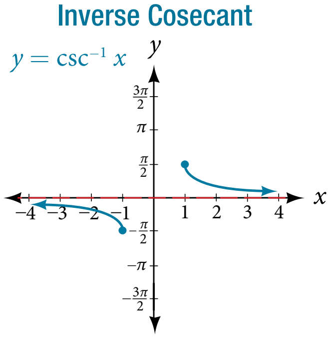
Inverse Tangent



Domain: $(-\infty, \infty)$
Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

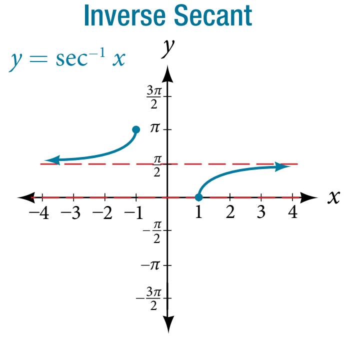
Figure A6

Inverse Cosecant



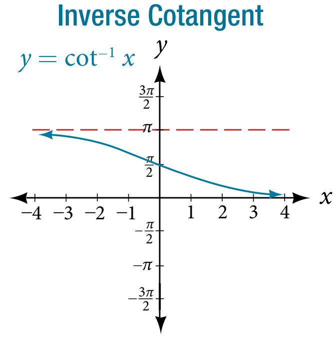
Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Inverse Secant



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Inverse Cotangent



Domain: $(-\infty, \infty)$
Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Figure A7

A3 Trigonometric Identities

Identities	Equations
Pythagorean Identities	$\sin^2\theta + \cos^2\theta = 1$ $1 + \tan^2\theta = \sec^2\theta$ $1 + \cot^2\theta = \csc^2\theta$
Even-odd Identities	$\cos(-\theta) = \cos\theta$ $\sec(-\theta) = \sec\theta$ $\sin(-\theta) = -\sin\theta$ $\tan(-\theta) = -\tan\theta$ $\csc(-\theta) = -\csc\theta$ $\cot(-\theta) = -\cot\theta$
Cofunction identities	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$ $\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$
Fundamental Identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\csc\theta = \frac{1}{\sin\theta}$ $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$
Sum and Difference Identities	$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$
Double-Angle Formulas	$\sin(2\theta) = 2\sin\theta\cos\theta$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $\cos(2\theta) = 1 - 2\sin^2\theta$ $\cos(2\theta) = 2\cos^2\theta - 1$ $\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$

Table A1

Identities	Equations
Half-Angle formulas	$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$ $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$ $\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$ $= \frac{\sin\alpha}{1-\cos\alpha}$ $= \frac{1-\cos\alpha}{\sin\alpha}$
Reduction Formulas	$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$ $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ $\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Product-to-Sum Formulas	$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$ $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$ $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$ $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$
Sum-to-Product Formulas	$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$ $\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ $\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
Law of Sines	$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$ $\frac{\sin a}{\alpha} = \frac{\sin b}{\beta} = \frac{\sin c}{\gamma}$
Law of Cosines	$a^2 = b^2+c^2-2bc\cos\alpha$ $b^2 = a^2+c^2-2ac\cos\beta$ $c^2 = a^2+b^2-2aa\cos\gamma$

Table A1