CONTINUED FRACTIONS AND THEIR APPLICATION TO TOPICS IN NUMBER THEORY

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CHAPTER I

INTRODUCTION

The study of continued fractions has interested mathematicians since the beginning of recorded history. To be sure, the forms in which they were studied in those early days are hardly recognizable today. In fact, the name "continued fraction" was not used until J. Wallis, an English mathematician, first applied the term to the objects he was studying in 1695. The fractions themselves were never the object of study in the earliest days, but rather the properties of number were the center of mathematical investigation. Continued fractions turned out to be a very powerful tool in this investigation.

When Euclid found the greatest common measure of two line segments, or when the same principle was applied to finding the greatest common divisor of the integers a and b, a process was used similar to that of converting a fraction into a continued fraction. An example would be $(48,21) = 3$ since

$$48 = 2\cdot 21 + 6,$$

$$21 = 3\cdot 6 + 3,$$

$$6 = 2\cdot 3,$$

therefore,

$$\frac{48}{21} = 2 + \frac{6}{21} = 2 + \frac{1}{\frac{21}{6}},$$

$$\frac{21}{6} = 3 + \frac{3}{6} = 3 + \cfrac{1}{\frac{6}{3}}$$

$$\frac{6}{3} = 2$$

and hence

$$\frac{48}{21} = 2 + \cfrac{1}{3 + \cfrac{1}{2}} .$$

The above example should serve to alert the reader that the study of continued fractions is actually the study of the Euclidean Algorithm in a very general sense. Some properties developed for the continued fraction, as Theorem 5.11, can be interpreted in terms of the Euclidean Algorithm and reduce the computation involved in applying this algorithm.

It is not known who first proved that the $\sqrt{2}$ is not expressible as the ratio of two integers. It must have been quite a shock to those early Pythagareans who discovered this important fact before the end of the fifth century B.C. In reply to a question by Socrates, Theaetetus says [1]

> Theodorus was writing out for us something about roots, such as the roots of three or five feet, showing that in linear measurement (that is comparing the sides of the squares) they are incommensurable by the unit; he selected the numbers which are roots, up to 17, but he went no further; and as there are innumerable roots, the notion occurred to us of attempting to include them all under one name or class.

With great ingenuity these Pythagareans approximated the $\sqrt{N}$ by successive solutions of the equations $Nx^2 - y^2 = \pm 1$. It will be shown that such equations and their solutions have a key relationship to continued fractions.

The modern era of continued fractions began with the Italian mathematician Bombelli in 1572 when he found that

$$\sqrt{13} = 3 + \cfrac{4}{6 + \cfrac{4}{6 + \dots}} ,$$

and more generally

$$\sqrt{a^2 + b} = a + \cfrac{b}{2a + \cfrac{b}{2a + \dots}} .$$

The next writer to consider these numbers was Cataldi in 1613 when he wrote, substantially in modern form,

$$\sqrt{18} = 4. \ \& \ \cfrac{2}{8. \ \& \ \cfrac{2}{8. \ \& \ \cfrac{2}{8.}}} ,$$

then modified to

$$4 \ \& \ \frac{2}{8.} \ \& \ \frac{2}{8.} \ .$$

Lord Brouncher, the first president of the Royal Mathematics Society, found the following expression for π in 1658,

$$\pi = \cfrac{4}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \text{etc.}}}}}} .$$

This was based on the product discovered by Wallis,

$$\frac{4}{\pi} = \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot} .$$

It was Euler who really started the systematic development of the theory of continued fractions. In 1762 he devised a new notation where

$$a + \cfrac{1}{b + \cfrac{1}{c}} ,$$

is represented by $\frac{(a,b,c)}{(b,c)}$ and infinite continued fractions by $\frac{(a,b,c,d,etc.)}{(b,c,d,etc.)}$. Here, $(a,b,c,d,e) = e(a,b,c,d) + (a,b,c)$ and also $(a,b,c,d,e) = (e,d,c,b,a)$. There exists a similarity between this notation and that which is later defined for P_n and Q_n. Among other things, Euler found that

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \dots}}}}} .$$

Euler gave a tentative method for solving the equation $x^2 - Ay^2 = 1$. He proceeded from the conversion of $\sqrt{A}$ into a continued fraction.

It was Lagrange (1769) who supplied the crucial proof for Euler's method and gave a non-tentative method for obtaining all integer solutions of $x^2 - Ay^2 = B$ where A,B are any given integers. Lagrange's theorem will be examined in this work and applied to the solution of $x^2 - Ay^2 = 1$.

Joseph Liouville in 1844 proved that neither e nor e^2 can be a root of a quadratic equation with rational coefficients. He used the properties of the convergents of a continued fraction representing a root of an algebraic equation with rational coefficients. He established later the existence of numbers -- the so-called transcendental numbers -- which cannot be roots of any such equation. This paper will present Liouville's theorem and use it along with continued fractions to prove this famous existence theorem.

Continued fractions became significant for the theory of functions when E. Laguerre in 1879 converted a divergent power series into a convergent continued fraction. Stieltjes in 1894 established a correspondence between divergent series and convergent continued fractions by which he was able to define integration for the series. The species of integration that he devised has since been named after him.

The list of mathematicians, great and near great, who have made contributions to the theory of continued fractions is rather impressive. To name but a few, who have not already been mentioned, one could list Al-Kalsadi of Granada (15th Century), Jacobi (1850), Leibniz (1697), C. Hygens (1698), E. Stone (1743), A. F. Mobius (1830), G. Lejeune Dirichlet (1854), Pascal, G. Peano (1903), B. Segre (1946), I. Niven (1962), and Daniel Shanks (1954). The dates listed here indicate when their contributions were made to the theory of continued fractions.

With such an imposing list of mathematicians having worked in the area of continued fractions, one might expect the theory to be more highly developed than it is today. A clue to why this is not the case is given by E. Bell when speaking of Lagrange,

> Lagranges experience with continued fractions is typical. Of high theoretical interest, as in proving the irrationality of certain numbers, continued fractions are too cumbersome an algorithm to be of much practicle use.

Recognizing that "practicality" is not always the ultimate criterion by which mathematicians measure their work, this paper will endeavor to present the reader with a basic understanding of the topic of continued fractions. It is intended that anyone with an undergraduate number theory background would be able to understand the majority of this material. A rudimentary understanding of limits and convergent series is required for the remainder of the material.

Specifically, from number theory, one should understand what is meant by greatest common divisor, the basic characterizations of the g.c.d., the greatest integer function $[x]$, and the division algorithm. It is also assumed that the reader is aware of the basic properties of operations on integers and irrational numbers. In this paper Z will be used to represent the set of integers and N the natural numbers (i.e. $\{1,2,3, \ldots\}$).

Even though this paper is intended to be used in a topics or seminar course following a first course in number theory, it certainly could be included with the first course, time permitting. There are parts of this paper, as Chapter II, that could also be used at the high school level by students possessing a few basics in number theory. Most proofs are included here for the sake of completeness. The reader should note that many proofs have a similar flavor and as such can be scanned once he recognizes that similarity. All definitions, theorems, and corollaries are numbered serially with the first digit being the number of the chapter. When equations are denoted by a number for easy reference, the numbers start with (1) in each chapter. When an equation is referred to by number, the chapter will also be given if it is not in the same chapter as the reference.

CHAPTER II

BASIC THEOREMS AND DEFINITIONS

An expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} \tag{1}$$

represents a regular continued fraction. In general, the partial quotients a_i $(i \geq 0)$ can represent any real or complex number. In this paper it will be assumed that they are always real. Later, the discussion will concentrate on simple continued fractions. A simple continued fraction is a regular fraction where $a_0 \in Z$ and $a_i \in N$, $(i > 0)$. The variable a_i $(i \geq 0)$ will be referred to as the ith partial quotient of the continued fraction. The number of these quotients may be infinite (1) or finite as below,

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots\, a_{n-1} + \cfrac{1}{a_n}}}} . \tag{2}$$

The following inductive definitions are made for the sake of preciseness. Also, for notational convenience, the continued fraction of form (1) will be denoted as

$$[a_0;a_1,a_2, \ldots] \tag{3}$$

and form (2) denoted as

$$[a_0;a_1, \ldots, a_n]. \tag{4}$$

<u>Definition</u> 2.1. <u>Finite Continued Fraction</u> (4). For $i \geq 0$, a_i is a real number and $a_n \neq 0$.

i) $[a_0] = a_0$.

ii) $[a_0;a_1] = a_0 + \frac{1}{a_1}$ and if $n \geq 1$

iii) $[a_0;a_1,a_2, \ldots, a_n] = [a_0;[a_1;a_2, \ldots, a_n]]$.

<u>Definition</u> 2.2. <u>Infinite Continued Fraction</u> (3). Same as Definition 2.1 except for (iii) replaced by:

$$[a_0;a_1,a_2, \ldots] = [a_0;[a_1;a_2, \ldots]].$$

<u>Definition</u> 2.3 and 2.4. <u>Simple Continued Fractions</u>. Same as Definitions 2.1 and 2.2 with:

iv) $a_0 \in Z$, and

v) $a_i \in N$ for $i \geq 1$.

Every finite continued fraction is the result of a finite number of operations on real numbers. Since division by zero is expressly excluded, a unique numerical value can always be determined. In the infinite case there are no such assurances. Until some evaluation procedure is adopted, it is only a formal notation similar to that for an infinite series whose convergence or divergence is not brought into question.

Definition 2.5. $S_k = [a_0;a_1,a_2, \ldots, a_k]$ is called a segment of the continued fraction of form (4) for $0 \le k \le n$. Similarly for arbitrary $k \ge 0$, one can call S_k a segment of any continued fraction of form (3).

Definition 2.6. $r_k = [a_k;a_{k+1}, \ldots, a_n]$ is called a remainder of the finite continued fraction (4) and $r_k = [a_k;a_{k+1}, \ldots]$ is a remainder of the infinite continued fraction (3).

Any remainder of a finite continued fraction is finite and for an infinite continued fraction any remainder is infinite. For finite fractions, there exists the following important relationship.

Theorem 2.7. For $0 \le k \le n$,

$$[a_0;a_1, \ldots, a_n] = [a_0;a_1, \ldots, a_{k-1}, r_k].$$

Proof of this statement is shown inductively in [8] on page 99.

The analogous relationship $(k \ge 0)$

$$[a_0;a_1,a_2, \ldots] = [a_0;a_1, \ldots, a_{k-1}, r_k]$$

for infinite continued fractions can be meaningful only as a formal notation since r_k is an infinite continued fraction and has no definite numerical value.

An interesting corollary to Theorem 2.7 is as follows.

Corollary 2.8.

A) If $a_n > 1$ and $n \ge 1$ in the finite simple continued fraction $[a_0;a_1, \ldots, a_n]$, then

$$[a_0;a_1, \ldots, a_n] = [a_0;a_1, \ldots, a_n-1, 1]$$

where $[a_0; \ldots, a_n-1, 1]$ is also a simple continued fraction.

B) If $a_n = 1$, then

$$[a_0;a_1, \ldots, a_{n-1}, 1] = [a_0;a_1, \ldots, a_{n-1}+1].$$

The proof of the above corollary follows from the fact that $[a_n-1, 1] = a_n$. Since uniqueness of representation of real numbers with simple continued fractions is one of the goals of this paper, it will be assumed that the last partial quotient in any finite simple continued fraction is greater than one.

Definition 2.9. Convergents of a Continued Fraction. For all k, $(0 \leq k \leq n)$ and the finite continued fraction (4) and for all k $(k \geq 0)$ and the infinite continued fraction (3), the numerical value of the segment S_k is called the kth convergent to the continued fraction. C_k will be used to represent the numerical value of the segment S_k.

Since the kth convergent is the value of a finite continued fraction with $k + 1$ partial quotients, it always is well defined and may be written as

$$\frac{P_k}{Q_k}$$

where $Q_k \neq 0$. It is also obvious that the value of the last convergent of a finite continued fraction is the numerical value of that fraction. C_n will be used often to represent the numerical value of the finite continued fraction $[a_0;a_1, \ldots, a_n]$.

Theorem 2.10. For all $k \geq 0$,

$$C_k = \frac{P_k}{Q_k}$$

where

A) $P_{-1} = 1$, $P_0 = a_0$, and for $n \geq 1$

$$P_n = a_n P_{n-1} + P_{n-2}$$

and

B) $Q_{-1} = 0$, $Q_0 = 1$, and for $n \geq 1$

$$Q_n = a_n Q_{n-1} + Q_{n-2}.$$

Proof: For $k = 0$ or 1, direct computation verifies the truth of the theorem.

Assume the statement is true for $0 \leq j < k$ where $k \geq 2$. Show that the statement is true for k.

$$C_k = [a_0;a_1, \ldots, a_k]$$

$$= [a_0;a_1;a_2, \ldots, a_{k-2},[a_{k-1};a_k]]$$

$$= [a_0;a_1, \ldots, a_{k-2},a'_{k-1}] \quad \text{where} \quad a'_{k-1} = [a_{k-1};a_k]$$

$$= \frac{a'_{k-1}P_{k-2} + P_{k-3}}{a'_{k-1}Q_{k-2} + Q_{k-3}}$$

$$= \frac{\left(a_{k-1} + \frac{1}{a_k}\right)P_{k-2} + P_{k-3}}{\left(a_{k-1} + \frac{1}{a_k}\right)Q_{k-2} + Q_{k-3}}$$

$$= \frac{a_k(a_{k-1}P_{k-2} + P_{k-3}) + P_{k-2}}{a_k(a_{k-1}Q_{k-2} + Q_{k-3}) + Q_{k-2}}$$

$$= \frac{a_k P_{k-1} + P_{k-2}}{a_k Q_{k-1} + Q_{k-2}}$$

$$= \frac{P_k}{Q_k} .$$

Corollary 2.11. For a simple continued fraction,

$$Q_k \geq \text{Max}\{k,1\} \quad \text{for} \quad k \geq 0.$$

Proof immediate from definitions.

Theorem 2.12. For all $k \geq 0$,

$$Q_k P_{k-1} - P_k Q_{k-1} = (-1)^k.$$

Proof: For $k = 0,1$ direct computation verifies the theorem. For $k \geq 2$, assume the statement is true for $n < k$.

$$\begin{aligned} Q_k P_{k-1} - P_k Q_{k-1} &= (a_k Q_{k-1} + Q_{k-2})P_{k-1} - (a_k P_{k-1} + P_{k-2})Q_{k-1} \\ &= (-1)(-Q_{k-2}P_{k-1} + P_{k-2}Q_{k-1}) \\ &= (-1)(-1)^{k-1} \\ &= (-1)^k. \end{aligned}$$

Corollary 2.13. For all $k \geq 0$,

$$Q_k P_{k-2} - P_k Q_{k-2} = a_k(-1)^{k-1}.$$

Proof:

$$\begin{aligned} Q_k P_{k-2} - P_k Q_{k-2} &= (a_k Q_{k-1} + Q_{k-2})P_{k-2} - (a_k P_{k-1} + P_{k-2})Q_{k-2} \\ &= a_k(Q_{k-1}P_{k-2} - P_{k-1}Q_{k-2}) \end{aligned}$$

$$= a_k(-1)^{k-1}.$$

Two other important results follow immediately from this theorem and are listed without proof below.

Corollary 2.14. For $k \geq 1$,

$$\frac{P_{k-1}}{Q_{k-1}} - \frac{P_k}{Q_k} = \frac{(-1)^k}{Q_{k-1}Q_k} .$$

Corollary 2.15. For $k \geq 2$,

$$\frac{P_{k-2}}{Q_{k-2}} - \frac{P_k}{Q_k} = \frac{a_k(-1)^{k-1}}{Q_{k-2}Q_k} .$$

If one restricts his attention to simple continued fractions, Corollary 2.15 takes on added meaning. In a simple continued fraction $a_k \geq 1$ $(k \geq 2)$, hence

$$\frac{a_k(-1)^{k-1}}{Q_{k-2}Q_k}$$

has the same sign as $(-1)^{k-1}$. If k is even, $(-1)^{k-1}$ is negative, thus the difference

$$\frac{P_{k-2}}{Q_{k-2}} - \frac{P_k}{Q_k} < 0.$$

Or the even ordered convergents form a strictly increasing sequence of real numbers. It could be shown in similar fashion that the odd-ordered convergents form a strictly decreasing sequence of real numbers. These results are listed below as Corollary 2.16.

Corollary 2.16. Let $[a_0;a_1, \ldots, a_n]$ or $[a_0;a_1, \ldots]$ be simple continued fractions; then

A) the sequence

$$\left\{ \frac{P_{2k}}{Q_{2k}} \right\}_{k=0}^{\infty}$$

is strictly increasing (sequence finite for finite fractions).

B) the sequence

$$\left\{ \frac{P_{2k+1}}{Q_{2k+1}} \right\}_{k=0}^{\infty}$$

is strictly decreasing (sequence finite for finite fractions).

In the proof of the above corollary no use was made of the fact that simple continued fractions have integral partial quotients. Hence, the hypothesis could be weakened to include those fractions with positive partial quotients. This latter observation will be known as Corollary 2.16 in its most general form.

Another important corollary of Theorem 2.12 indicates that the convergents, as characterized in Theorem 2.10, are in their lowest terms.

Corollary 2.17. If

$$\frac{P_k}{Q_k}$$

is the kth convergent as defined in Theorem 2.10 of a simple continued fraction, then

$$(P_k,Q_k) = 1.$$

The proof of this follows from the fact that P_k and Q_k will always be integers when the continued fraction is simple. Theorem 2.12 establishes that there always exist integers X and Y such that

$$XQ_k - YP_k = 1.$$

Hence, $(P_k, Q_k) = 1$.

Theorem 2.18. Every odd-ordered convergent of a simple continued fraction is greater than every even-ordered convergent.

Proof: Let k be odd and J be even.

Case I: $k > J$

$$\frac{P_{k-1}}{Q_{k-1}} - \frac{P_k}{Q_k} = \frac{(-1)^k}{Q_{k-1}Q_k} < 0$$

or

$$\frac{P_{k-1}}{Q_{k-1}} < \frac{P_k}{Q_k} .$$

But $k - 1$ is even and greater than or equal to J. By Corollary 2.16 the even convergents form an increasing sequence; hence,

$$\frac{P_J}{Q_J} \le \frac{P_{k-1}}{Q_{k-1}} < \frac{P_k}{Q_k} .$$

Case II: For $k < J$, the proof is similar. Since k and J were arbitrary odd and even integers respectively, the theorem is established.

It is evident that for every finite simple continued fraction, the numerical value α is greater than every even convergent and less than every odd convergent (except, of course, for the last convergent, even or odd, which is equal to α). In the following theorem it is no longer required that the continued fraction be simple.

Theorem 2.19. For arbitrary k $(1 \leq k \leq n)$,

$$[a_0;a_1,a_2, \ldots, a_n] = \frac{P_{k-1}r_k + P_{k-2}}{Q_{k-1}r_k + Q_{k-2}} .$$

P_{k-1}, Q_{k-1}, r_k, etc. refers to the continued fraction in the left member of this equation.

Proof: From Theorem 2.7,

$$[a_0;a_1, \ldots, a_{k-1},a_k, \ldots, a_n] = [a_0;a_1, \ldots, a_{k-1},r_k]$$

$$= \frac{P'_k}{Q'_k} \tag{5}$$

where

$$P'_k = r_kP'_{k-1} + P'_{k-2}$$

and

$$Q'_k = r_kQ'_{k-1} + Q'_{k-2}.$$

But P'_{k-1}, P'_{k-2}, Q'_{k-1} and Q'_{k-2} are computed with the elements $a_0, \ldots, a_{k-1}$ and, hence, are the same as P_{k-1}, P_{k-2}, Q_{k-1} and Q_{k-2} respectively. Substitution in (5) establishes Theorem 2.19.

Theorem 2.20. For arbitrary $k \geq 1$,

$$\frac{Q_k}{Q_{k-1}} = [a_k;a_{k-1}, \ldots, a_1].$$

Proof by induction: The statement is certainly true for $k = 1$. Assume it is true for $u < k$. Consider

$$[a_k;a_{k-1}, \ldots, a_1] = [a_k;[a_{k-1}; \ldots, a_1]]$$

$$= \left[a_k; \frac{Q_{k-1}}{Q_{k-2}}\right]$$

$$= a_k + \frac{Q_{k-2}}{Q_{k-1}} = \frac{a_kQ_{k-1} + Q_{k-2}}{Q_{k-1}} = \frac{Q_k}{Q_{k-1}} .$$

At this point the discussion turns to infinite continued fractions. It has already been observed that every finite continued fraction has a numerical value, namely the value of the last convergent. For infinite fractions no "final" convergent exists. Instead, there exists an infinite sequence of real numbers corresponding to the convergents

$$\frac{P_0}{Q_0}, \frac{P_1}{Q_1}, \ldots, \frac{P_k}{Q_k}, \ldots . \qquad (6)$$

Definition 2.21. The value (or numerical value) of an infinite continued fraction is the limit of its sequence of convergents (6), if that limit exists and is finite. If an infinite continued fraction has a value, it is said to converge; if not, it is said to diverge.

Many properties of infinite continued fractions that converge and finite continued fractions are analogous. The basic property that makes possible the further extension of this analogy is expressed by the following theorem.

Theorem 2.22. If the simple continued fraction $[a_0;a_1,a_2, \ldots]$ converges, then so do all of its remainders; conversely, if at least one of the remainders converges, then the continued fraction itself converges.

Proof: Let

$$\frac{P_k}{Q_k}$$

denote the kth convergent of the given continued fraction as before. The kth convergent of a remainder will be denoted by

$$\frac{P'_k}{Q'_k} .$$

From Theorem 2.19, for $k = 0,1, \ldots,$

$$\frac{P_{n+k}}{Q_{n+k}} = [a_0;a_1, \ldots, a_{n+k}]$$

$$= [a_0:a_1, \ldots, a_{n-1},[a_n;a_{n+1}, \ldots, a_{n+k}]]$$

$$= \frac{P_{n-1}\dfrac{P'_k}{Q'_k} + P_{n-2}}{Q_{n-1}\dfrac{P'_k}{Q'_k} + Q_{n-2}} . \tag{7}$$

It follows immediately that if the remainder r_n converges then so does the continued fraction since if

$$\lim_{k \to \infty} \frac{P'_k}{Q'_k} = L$$

exists, then

$$\lim_{k \to \infty} \frac{P_{n+k}}{Q_{n+k}} = \frac{P_{n-1}L + P_{n-2}}{Q_{n-1}L + Q_{n-2}} = \alpha.$$

To see that the converse of the theorem is true, one needs only to solve (7) for

$$\frac{P'_k}{Q'_k}$$

and apply the same logic.

It should be noted that formula (7) implies the same relationship for infinite continued fractions that was observed in Theorem 2.19 for finite continued fractions. If $\alpha = [a_0;a_1, \ldots]$ is a convergent continued fraction, the fact that for all $n \geq 0$,

$$\alpha = \frac{P_{n-1}r_n + P_{n-2}}{Q_{n-1}r_n + Q_{n-2}},$$

where r_n is the value of the n-th order remainder, will be referred to as Theorem 2.19 in its most generalized form.

<u>Theorem</u> 2.23. The value α of a convergent infinite simple continued fraction is greater than any of its even-order convergents and is less than any of its odd-order convergents.

Since the value α is the limit of the convergents and Theorem 2.18 establishes that every even convergent is less than every odd convergent, α must be the limit of the subsequence of even convergents and that of the odd convergents. Hence, α is between any two consecutive convergents and strictly greater than the even ones and less than the odd ones. As corollaries to the above theorem, the following results.

In each case assume that α, P_k, Q_k, etc. refer to a convergent simple continued fraction.

Corollary 2.24.

$$\left| \alpha - \frac{P_k}{Q_k} \right| < \frac{1}{Q_k Q_{k+1}} .$$

This follows from the fact that α is "strictly" between

$$\frac{P_k}{Q_k} \text{ and } \frac{P_{k+1}}{Q_{k+1}} .$$

Hence, either

$$\frac{P_k}{Q_k} < \alpha < \frac{P_{k+1}}{Q_{k+1}} \tag{8}$$

or

$$\frac{P_{k+1}}{Q_{k+1}} < \alpha < \frac{P_k}{Q_k} . \tag{9}$$

Thus,

$$\left| \alpha - \frac{P_k}{Q_k} \right| < \left| \frac{P_{k+1}}{Q_{k+1}} - \frac{P_k}{Q_k} \right| = \frac{1}{Q_k Q_{k+1}} .$$

The proof of the following corollary also depends on the inequalities (8) and (9).

Corollary 2.25. $Q_{k+1}|Q_k\alpha - P_k| + Q_k|Q_{k+1}\alpha - P_{k+1}| = 1.$

Proof: From (8) and (9), observe that

$$\frac{1}{Q_{k+1}Q_k} = \left| \frac{P_k}{Q_k} - \frac{P_{k+1}}{Q_{k+1}} \right| = \left| \frac{P_k}{Q_k} - \alpha \right| + \left| \alpha - \frac{P_{k+1}}{Q_{k+1}} \right| .$$

Multiplication by $Q_{k+1}Q_k$ completes the proof.

Since the even-order convergents are strictly increasing and the value α of a convergent continued fraction is greater than any even convergent

$$\frac{P_{k-1}}{Q_{k-1}} < \frac{P_{k+1}}{Q_{k+1}} < \alpha < \frac{P_k}{Q_k} \tag{10}$$

where $k - 1$ and $k + 1$ are even. Similarly, for odd order convergents

$$\frac{P_k}{Q_k} < \alpha < \frac{P_{k+1}}{Q_{k+1}} < \frac{P_{k-1}}{Q_{k-1}} \tag{11}$$

where $k - 1$ and $k + 1$ are odd.

<u>Corollary</u> 2.26. For a convergent simple continued fraction $(k > 0)$

$$\frac{1}{Q_{k+1}} < |Q_{k-1}\alpha - P_{k-1}| < \frac{1}{Q_k} .$$

<u>Proof</u>: From Corollary 2.24

$$\left| \alpha - \frac{P_{k-1}}{Q_{k-1}} \right| < \frac{1}{Q_k Q_{k-1}} .$$

From (10) and (11) since $a_k \geq 1$

$$\left| \alpha - \frac{P_{k-1}}{Q_{k-1}} \right| > \left| \frac{P_{k+1}}{Q_{k+1}} - \frac{P_{k-1}}{Q_{k-1}} \right| = \frac{a_k}{Q_{k+1}Q_{k-1}} \geq \frac{1}{Q_{k+1}Q_{k-1}} .$$

Putting these two inequalities together the following results:

$$\frac{1}{Q_{k+1}Q_{k-1}} < \left| \alpha - \frac{P_{k-1}}{Q_{k-1}} \right| < \frac{1}{Q_kQ_{k-1}} .$$

Multiplying both members by Q_{k-1} yields the desired inequality.

Definition 2.27. Intermediate fractions. Let

$$\frac{P_{k-2}}{Q_{k-2}} \text{ and } \frac{P_{k-1}}{Q_{k-1}}$$

be two consecutive convergents, then

$$\frac{aP_{k-1} + P_{k-2}}{aQ_{k-1} + Q_{k-2}}$$

for $a \varepsilon N$ is called an intermediate fraction between

$$\frac{P_{k-2}}{Q_{k-2}} \text{ and } \frac{P_{k-1}}{Q_{k-1}} .$$

Theorem 2.28. Any sequence of intermediate fractions from a simple continued fraction form a strictly increasing sequence or a strictly decreasing sequence depending on whether k is even or odd respectively.

Let i be a positive integer.

$$\frac{(i + 1)P_{k-1} + P_{k-2}}{(i + 1)Q_{k-1} + Q_{k-2}} - \frac{iP_{k-1} + P_{k-2}}{iQ_{k-1} + Q_{k-2}} = \frac{(-1)^k}{[(i + 1)Q_{k-1} + Q_{k-2}][iQ_{k-1} + Q_{k-2}]} .$$

Hence, if k is even,

$$\frac{(i+1)P_{k-1}+P_{k-2}}{(i+1)Q_{k-1}+Q_{k-2}} > \frac{iP_{k-1}+P_{k-2}}{iQ_{k-1}+Q_{k-2}} .$$

The inequality is reversed if k is odd. Since

$$\frac{P_k}{Q_k} = \frac{a_k P_{k-1} + P_{k-2}}{a_k Q_{k-1} + Q_{k-2}}$$

where $a_k \varepsilon N$, it should be noted that

$$\frac{P_k}{Q_k}$$

is an intermediate fraction between

$$\frac{P_{k-2}}{Q_{k-2}} \text{ and } \frac{P_{k-1}}{Q_{k-1}} .$$

If k is even, the smallest intermediate fraction between

$$\frac{P_{k-2}}{Q_{k-2}} \text{ and } \frac{P_{k-1}}{Q_{k-1}}$$

is

$$\frac{P_{k-1}+P_{k-2}}{Q_{k-1}+Q_{k-2}} .$$

Similarly, if k is odd, it is the largest intermediate fraction. The result is that one of the following two inequalities must hold. (Assuming the continued fraction is simple.)

$$\frac{P_{k-2}}{Q_{k-2}} < \frac{P_{k-1}+P_{k-2}}{Q_{k-1}+Q_{k-2}} \le \frac{P_k}{Q_k} < \alpha < \frac{P_{k-1}}{Q_{k-1}} \tag{12}$$

or

$$\frac{P_{k-1}}{Q_{k-1}} < \alpha < \frac{P_k}{Q_k} \leq \frac{P_{k-1} + P_{k-2}}{Q_{k-1} + P_{k-2}} < \frac{P_{k-2}}{Q_{k-2}} \tag{13}$$

Corollary 2.29. For $k \geq 0$ in a simple continued fraction,

$$\frac{1}{Q_k(Q_{k+1} + Q_k)} < \left| \alpha - \frac{P_k}{Q_k} \right| .$$

Proof: By replacing $k - 2$ with k in (12) and (13), the following results:

$$\left| \frac{P_k}{Q_k} - \frac{P_{k+1} + P_k}{Q_{k+1} + Q_k} \right| < \left| \frac{P_k}{Q_k} - \alpha \right| .$$

But

$$\left| \frac{P_k}{Q_k} - \frac{P_{k+1} + P_k}{Q_{k+1} + Q_k} \right| = \frac{1}{Q_k(Q_{k+1} + Q_k)} .$$

The question naturally arises as to whether there are tests for the convergence of continued fractions, just as for infinite series. In the case where $a_i > 0$, for all $i \geq 1$, there is an extremely simple and convenient test for convergence.

Theorem 2.30. For the continued fraction $[a_0;a_1, \ldots]$ to converge, it is necessary and sufficient that the series

$$\sum_{n=1}^{\infty} a_n \tag{14}$$

diverge. ($a_i > 0$, for $i \geq 1$.)

Proof: From Corollary 2.16 in its most general form it is known that the even-order convergents form an increasing sequence of real numbers and the odd-order convergents, a decreasing sequence. Hence, a continued fraction has a limit if and only if the above two sequences have the same limit. They will have the same limit if and only if

$$\lim_{k \to \infty} \left| \frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}} \right| = \lim_{k \to \infty} \frac{1}{Q_k Q_{k-1}} = 0$$

or

$$\lim_{k \to \infty} Q_k Q_{k-1} = \infty. \qquad (15)$$

Thus, condition (15) is necessary and sufficient for the convergence of a continued fraction of this type.

Suppose

$$\sum_{1}^{\infty} a_n$$

converges. By the definition of Q_k

$$Q_k = a_k Q_{k-1} + Q_{k-2} > Q_{k-2}.$$

Hence, either $Q_{k-1} < Q_k$ or $Q_{k-1} > Q_{k-2}$.

In the first case, since (14) converges, there exists N such that if $k \geq N$, then $a_k < 1$. Thus, for all $k \geq N$,

$$Q_k = a_k Q_{k-1} + Q_{k-2}$$

and hence

$$Q_k < a_k Q_k + Q_{k-2}$$

or

$$Q_k < \frac{Q_{k-2}}{1 - a_k} .$$

In the second case

$$Q_k < a_k Q_{k-1} + Q_{k-1} = (a_k + 1) Q_{k-1} .$$

If $a_k < 1$ (as for $k \geq N$)

$$Q_k < \frac{Q_{k-1}}{1 - a_k} , \quad \text{since} \quad a_k + 1 < \frac{1}{1 - a_k} .$$

Thus, for all $k \geq N$, there is $L = k - 1$ or $k - 2$ such that

$$Q_k < \frac{Q_L'}{1 - a_k} .$$

If $L \geq N$, the same inequality may be applied to Q_L. Continuing in this manner, the following inequality results:

$$Q_k < \frac{Q_S}{(1 - a_k)(1 - a_L) \cdots (1 - a_r)} \tag{16}$$

where $k > L > \cdots > r \geq N$ and $S < N$. Because it was assumed that (14) converges, it is known that the following infinite product

$$\prod_{n=N}^{\infty} (1 - a_n) ,$$

also converges and has a positive value λ. Thus,

$$(1 - a_k)(1 - a_L) \cdots (1 - a_r) > \lambda .$$

Let $Q = \max\{Q_0, Q_1, \ldots, Q_{N-1}\}$. From (16) it is noted that

$$Q_k < \frac{Q}{\lambda} \quad (k \geq N)$$

consequently,

$$Q_{k+1}Q_k < \frac{Q^2}{\lambda^2} \quad (k \geq N).$$

This implies that relationship (15) does not hold, and the continued fraction diverges.

Conversely, suppose that (14) diverges. Since $Q_k > Q_{k-2}$ for $k \geq 2$, let $c = \min\{Q_0, Q_1\}$, and thus for all $k \geq 0$, $Q_k \geq c$. Hence,

$$Q_k \geq a_k c + Q_{k-2} \quad (k \geq 2).$$

Successive application of this inequality gives

$$Q_{2k} \geq Q_0 + c \sum_{n=1}^{k} a_{2n}$$

and

$$Q_{2k+1} \geq Q_1 + c \sum_{n=1}^{k} a_{2n} + 1$$

so that

$$Q_{2k} + Q_{2k+1} > Q_0 + Q_1 + c \sum_{n=1}^{2k+1} a_n.$$

In other words, for all odd values of k,

$$Q_k + Q_{k-1} > c \sum_{n=1}^{k} a_n.$$

A similar proof would establish the above inequality for even values of k. Thus, in any product $Q_k Q_{k-1}$ one of the factors must be greater than

$$\frac{c}{2} \sum_{n=1}^{k} a_n .$$

Since the other factor is greater than or equal to c

$$Q_k Q_{k-1} > \frac{c^2}{2} \sum_{n=1}^{k} a_n .$$

Since (14) diverges, then relation (15) must hold; hence, the continued fraction converges. This completes the proof of Theorem 2.30.

The above theorem implies that any infinite simple continued fraction must necessarily converge. This results from the fact that for $i \geq 1$, $a_i \geq 1$, hence

$$\sum_{i=1}^{\infty} a_i = \infty .$$

and $[a_0; a_1, \ldots]$ converges. This observation allows the hypothesis of Theorem 2.23 and Corollaries 2.24, 2.25, and 2.26 to be weakened by replacing "convergent simple continued fraction" with "simple continued fraction." The representation of real numbers with simple continued fractions will now be considered in detail.

<u>Theorem</u> 2.31. Let $\frac{a}{b}$ be a rational number in its lowest terms with $b > 0$. Then there exists a finite simple continued fraction

$[a_0;a_1, \ldots, a_n]$ such that

$$\frac{a}{b} = [a_0;a_1, \ldots, a_n].$$

<u>Proof</u>: Let $a_0 = \left[\frac{a}{b}\right]$ and define $R_1 = \frac{a}{b} - a_0$, then $\frac{a}{b} = a_0 + R_1$ where $0 \leq R_1 < 1$.

Let

$$a_1 = \left[\frac{1}{R_1}\right]$$

and define

$$R_2 = \frac{1}{R_1} - a_1,$$

then

$$\frac{1}{R_1} = a_1 + R_2$$

where $0 \leq R_2 < 1$. Suppose $a_0, a_1, \ldots, a_{j-1}$ and $0 \leq R_j < 1$ have been determined.

Let

$$a_j = \left[\frac{1}{R_j}\right]$$

and define

$$R_{j+1} = \frac{1}{R_j} - a_j$$

then

$$\frac{1}{R_j} = a_j + R_{j+1}$$

where $0 \leq R_{j+1} < 1$.

To complete the theorem it must be shown that,

1) for some k, $R_{k+1} = 0$;

2) $[a_0;a_1, \ldots, a_k]$ is a simple continued fraction; and,

3) $[a_0;a_1, \ldots, a_k] = \frac{a}{b}$.

It can be shown by induction that for all $k \geq 1$, R_k is rational and in its lowest terms when its denominator is taken to be the numerator of R_{k-1}. Simple computation verifies these facts for $k = 1$.

Suppose

$$R_{k-1} = \frac{n_{k-1}}{d_{k-1}}$$

is rational and in its lowest terms (i.e. $(n_{k-1},d_{k-1}) = 1$). By definition,

$$R_k = \frac{1}{R_{k-1}} - a_{k-1}, \quad \text{where} \quad a_{k-1} = \left[\frac{1}{R_{k-1}} \right]$$

$$= \frac{d_{k-1} - a_{k-1}n_{k-1}}{n_{k-1}}$$

$$= \frac{n_k}{d_k},$$

where $d_k = n_{k-1}$ and $n_k = d_{k-1} - a_{k-1}n_{k-1}$. Certainly, R_k is rational and $(d_{k-1} - a_{k-1}n_{k-1},n_{k-1}) = 1$ since $(n_{k-1},d_{k-1}) = 1$. Also,

$$0 \leq R_k = \frac{n_k}{d_k} < 1,$$

which implies

$$n_k < d_k = n_{k-1}.$$

Thus, when expressed in lowest terms the numerators of the remainders R_k form a strictly decreasing sequence of nonnegative numbers. The largest numerator in this sequence is $n_1 = b$. Of necessity, a strictly decreasing sequence of nonnegative integers bounded above (and below) must be finite. The only way for the sequence to terminate is for one of them, say n_{k+1} to be zero. Hence, there is a remainder

$$R_{k+1} = 0$$

and the continued fraction is finite with $k + 1$ terms and is of the form

$$[a_0;a_1, \ldots, a_k].$$

To establish (2) it need only be observed that $a_0 = \left[\frac{a}{b}\right]$ is an integer and for all $j \geq 1$, $0 \leq R_j < 1$ which implies

$$1 < \frac{1}{R_j}$$

and, hence,

$$1 \leq a_j = \left[\frac{1}{R_j}\right]$$

or

$$a_j \in N.$$

Thus, by definition, $[a_0;a_1, \ldots, a_k]$ is a finite simple continued fraction.

It will now be shown by induction that $\frac{a}{b}$ is the numerical value of the continued fraction $[a_0;a_1, \ldots, a_k]$ where a_j is defined as above.

$$\frac{a}{b} = a_0 + R_1 = [a_0;R_1^{-1}] = [a_0;a_1 + R_2]$$

$$= [a_0;[a_1;R_2^{-1}]]$$

$$= [a_0;a_1,R_2^{-1}]$$

Suppose it has been shown that for $0 \leq j < k$

$$\frac{a}{b} = [a_0;a_1, \ldots, a_j,R_{j+1}^{-1}],$$

then

$$\frac{a}{b} = [a_0; \ldots, a_j,a_{j+1} + R_{j+2}]$$

$$= [a_0; \ldots, a_j,[a_{j+1};R_{j+2}^{-1}]]$$

$$= [a_0; \ldots, a_j,a_{j+1},R_{j+2}^{-1}].$$

It has been noted that eventually a zero remainder will be encountered, say R_{j+2}. In that case $R_{j+1}^{-1} \in N$ and $a_{j+1} = R_{j+1}^{-1}$, hence

$$\frac{a}{b} = [a_0;a_1, \ldots, a_j,a_{j+1}].$$

This completes the proof of Theorem 2.31.

The reader should recall that the final partial quotient of any finite simple continued fraction is required to be greater than unity. Suppose now that two simple finite continued fractions whose numerical values are $\frac{a}{b}$ have been computed. That is

$$[a_0;a_1, \ldots, a_k] = \frac{a}{b} = [b_0;b_1, \ldots, b_j]$$

or

$$a_0 + \frac{1}{[a_1; \ldots, a_k]} = b_0 + \frac{1}{[b_1; \ldots, b_j]} .$$

Certainly if either $[a_1; \ldots, a_k]$ or $[b_1; \ldots, b_j]$ has more than one partial quotient, then

$$\frac{1}{[a_1; \ldots, a_k]} \text{ and } \frac{1}{[b_1; \ldots, b_j]}$$

are both less than one. By requiring that the last partial quotient in both continued fractions be greater than one also forces the above expressions to be less than unity even if they consist of a single term, namely,

$$\frac{1}{[a_1]} \text{ or } \frac{1}{[b_1]} .$$

Since a_0 and b_0 are integers less than $\frac{a}{b}$ but less than one unit difference from $\frac{a}{b}$, they must be the same integer. Hence,

$$[a_1; \ldots, a_k] = [b_1; \ldots, b_j]$$

by induction. It follows that

$$a_i = b_i$$

for $0 \leq i \leq k$ and

$$k = j.$$

Thus, the following important theorem has been demonstrated.

Theorem 2.32. The finite simple continued fraction representing a rational number is unique.

Now consider the irrational number α. The process described in Theorem 2.31 is applicable to α. Using such a process, a simple continued fraction could be associated with the irrational number α. This continued fraction would necessarily be infinite since the values R_i $(i \geq 1)$ are all irrational and, therefore,

$$R_{i+1} = \frac{1}{R_i} - a_i, \quad a_i \varepsilon N$$

cannot be zero.

As previously stated, the infinite simple continued fraction $[a_0;a_1, \ldots]$ associated with the irrational number α must of necessity be convergent (see discussion following Theorem 2.30). Conversely, starting with an infinite simple continued fraction, its value may be computed as in Definition 2.21. This value must be irrational since if it was rational, that rational value would have a finite continued fraction coinciding with the infinite fraction, which is impossible. The proof of this last statement is precisely the one used to show the uniqueness of the continued fractions associated with rational numbers.

Theorem 2.33. Every irrational number α has a unique infinite simple continued fraction associated with it (as computed in Theorem 2.31). This fraction is convergent, and its value is α.

Proof: Part of this theorem has already been established in the preceding paragraph. The remainder of the proof will be shown in two parts:

A) If the continued fraction $[a_0;a_1,a_2, \ldots]$ is associated with the irrational number α (i.e. as generated in Theorem 2.31), then its value is α.

B) The infinite continued fraction whose value is α is unique.

A) From Theorem 2.31 note the following construction for the continued fraction,

$$\alpha = a_0 + R_1, \qquad 0 < R_1 < 1$$

$$\frac{1}{R_1} = a_1 + R_2, \qquad 0 < R_2 < 1$$

and

$$\frac{1}{R_n} = a_n + R_{n+1}, \qquad 0 < R_{n+1} < 1.$$

Hence, by induction it could be shown,

$$\alpha = [a_0;a_1,a_2, \ldots, a_n,R_{n+1}^{-1}], \tag{17}$$

where $R_{n+1}^{-1} > 1$ for all n. It is important here to realize that (17) is a finite continued fraction with the same first n elements as the infinite continued fraction associated with α. As such, the first n convergents of the finite fraction will coincide with the first n convergents of the infinite continued fraction. It has been previously noted that for finite continued fractions (of $n + 1$ elements)

$$\alpha = \frac{P'_{n+1}}{Q'_{n+1}}$$

$$= \frac{R_{n+1}^{-1}P_n + P_{n-1}}{R_{n+1}^{-1}Q_n + Q_{n-1}}$$

for all $n \geq 1$.

Consider

$$\left| \alpha - \frac{P_n}{Q_n} \right| = \left| \frac{R_{n+1}^{-1} P_n + P_{n-1}}{R_{n+1}^{-1} Q_n + Q_{n-1}} - \frac{P_n}{Q_n} \right|$$

$$= \left| \frac{Q_n P_{n-1} - P_n Q_{n-1}}{(R_{n+1}^{-1} Q_n + Q_{n-1}) Q_n} \right|$$

$$= \frac{1}{(R_{n+1}^{-1} Q_n + Q_{n-1}) Q_n}$$

$$< \frac{1}{Q_{n-1} Q_n} \,. \tag{18}$$

It is emphasized here again that Q_n and Q_{n-1} are the same for the infinite continued fraction associated with α and the finite continued fraction (17). Since the infinite continued fraction is simple, it is known by Corollary 2.11 that

$$Q_n \geq n$$

and thus

$$\frac{1}{Q_n Q_{n-1}} \leq \frac{1}{n(n-1)} \,.$$

Hence, as $n \to \infty$,

$$\frac{1}{Q_n Q_{n-1}} \to 0.$$

Thus, the conclusion is that

$$\lim_{n \to \infty} \left| \alpha - \frac{P_n}{Q_n} \right| = 0,$$

and

$$\lim_{n \to \infty} \frac{P_n}{Q_n} = \alpha,$$

where

$$\frac{P_n}{Q_n}$$

is the nth convergent of the infinite continued fraction associated with α. This establishes part (A) of the theorem. The proof of part (B) follows exactly the same as that for the uniqueness of finite simple continued fractions.

The preceding arguments have shown that every real number has a unique representation as a simple continued fraction. The importance of this representation lies in the fact that the real number may be approximated to any predetermined degree of accuracy by evaluating the convergents of the continued fraction. Continued fractions are thus similar to decimal representation of real numbers.

Whenever there are dual systems that perform the same task, such as representing real numbers, it is natural to compare the advantages and disadvantages of each. It will be shown in the next chapter that representation by continued fractions is superior to that of decimals in many ways. The nature of the continued fraction is often more descriptive of the nature of the number. It has already been noted that finite continued fractions represent rational numbers while infinite fractions represent irrational numbers. Some rational numbers have finite decimals while others have infinite decimals. It will be shown that algebraic and transcendental numbers have characteristic continued fractions while the same is not true of their decimal representations. One of the disadvantages in using continued fractions to represent real numbers is their unwillingness to combine even under the simplest of arithmetical operations. There is no practical algorithm for the calculations of the sum or product of two continued fractions.

Continued fractions find their primary application in theoretical investigations involving the study of the arithmetic laws of the continuum and the arithmetic properties of individual irrational numbers. The remainder of this paper will be devoted to those ends. The primary tool in this investigation will be that of simple continued fractions. Unless stated otherwise, the term "continued fraction" will now mean simple continued fraction.

CHAPTER III

BEST APPROXIMATIONS AND TRANSCENDENTAL NUMBERS

The representation of a real number α as an ordinary rational fraction is a natural application of the convergents of the continued fraction associated with α. This representation may be made within certain specified margins of accuracy given by Corollaries 2.24 and 2.29,

$$\frac{1}{Q_n(Q_n + Q_{n+1})} < \left| \alpha - \frac{P_n}{Q_n} \right| \leq \frac{1}{Q_n Q_{n+1}} .$$

Equality is necessary for the right-hand symbol since if α is rational, it is equal to its last convergent, say

$$\frac{P_{n+1}}{Q_{n+1}}$$

and

$$\left| \alpha - \frac{P_n}{Q_n} \right| = \left| \frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n} \right| = \frac{1}{Q_n Q_{n+1}} .$$

The problem of approximating irrational (or even rational numbers with large denominators) by rational fractions consists of determining which fraction, within the specified limits of accuracy, has the smallest (positive) denominator. The rational approximation of π within .15 accuracy is 3 since the denominator of 3 is the smallest of all the rational numbers in that range of approximation. Thus, 3 is said to be the best rational approximation of π within .15 units accuracy. This concept is defined formally below.

Definition 3.1. Best Approximation (Type I). The rational number $\frac{a}{b}$ (for $b > 0$) is a best approximation of the first type for a real number α if every other rational fraction with the same or smaller denominator differs from α by a greater amount. To be precise, if $0 < d \leq b$ and $\frac{a}{b} \neq \frac{c}{d}$, then

$$\left| \alpha - \frac{c}{d} \right| > \left| \alpha - \frac{a}{b} \right|.$$

The difference between the real number α and the rational number $\frac{a}{b}$ may also be characterized in another fashion. The expression $|b\alpha - a|$ also is an indication of this difference.

Definition 3.2. Best Approximation (Type II). The rational number $\frac{a}{b}$ (for $b > 0$) is a best approximation for the real number α of the second type if $0 < d \leq b$ and $\frac{a}{b} \neq \frac{c}{d}$, then

$$|d\alpha - c| > |b\alpha - a|.$$

It is easy to show that a best approximation of Type II is also a best approximation of Type I. Consider the contrapositive of the statement. If $\frac{a}{b}$ is not a Type I best approximation, then there is a fraction $\frac{c}{d}$ such that $0 < d \leq b$ and $\frac{a}{b} \neq \frac{c}{d}$ with

$$\left| \alpha - \frac{c}{d} \right| \leq \left| \alpha - \frac{a}{b} \right|. \qquad (1)$$

Multiplying the inequality (1) by the inequality $d \leq b$ the following results

$$|d\alpha - c| \leq |b\alpha - a|.$$

Hence, $\frac{a}{b}$ is not a best approximation of the second type.

The converse, in general, is not true. If α is $\frac{1}{7}$, then $\frac{1}{5}$ is easily seen to be a best Type I approximation. However, that it is not a best approximation of Type II is seen from the inequality

$$\left|1 \cdot \frac{1}{7} - 0\right| < \left|5 \cdot \frac{1}{7} - 1\right|$$

and $1 < 5$.

An important property of Type II approximations is stated in the following theorem.

Theorem 3.3. Every best approximation to the real number α of the second type is a convergent of the continued fraction associated with α.

Proof: Let $[a_0;a_1, \ldots]$ be the continued fraction associated with α. (It should be noted that the following theorem also applies to rational α.) Let $\frac{a}{b}$ be a best approximation to α of the second type. Assume that $\frac{a}{b}$ is not a convergent.

Suppose further that $\frac{a}{b} < a_0$; hence,

$$\left|1 \cdot \alpha - a_0\right| < \left|\alpha - \frac{a}{b}\right| \leq \left|b\alpha - a\right|$$

since $1 \leq b$ and $a_0 \leq \alpha$. Thus, $a_0/1$ would be a "better" approximation of the second type than $\frac{a}{b}$. The conclusion must be that $\frac{a}{b} \geq a_0$.

If $\frac{a}{b}$ is not a convergent to α then either

I) $\frac{P_1}{Q_1} < \frac{a}{b}$, or

II) there is a k such that $\frac{a}{b}$ lies between

$$\frac{P_{k-1}}{Q_{k-1}} \text{ and } \frac{P_{k+1}}{Q_{k+1}} .$$

In the first case, since

$$\alpha \leq \frac{P_1}{Q_1} ,$$

$$\left| \alpha - \frac{a}{b} \right| > \left| \frac{P_1}{Q_1} - \frac{a}{b} \right| \geq \frac{1}{bQ_1} ,$$

and hence

$$|b\alpha - a| > \frac{1}{Q_1} = \frac{1}{a_1} .$$

On the other hand, since $a_1 = \left[\frac{1}{R_1} \right]$ where $R_1 = \alpha - a_0$,

$$|1 \cdot \alpha - a_0| \leq \frac{1}{a_1}$$

so that

$$|b\alpha - a| > |1 \cdot \alpha - a_0|$$

where $1 \leq b$.

This contradicts the hypothesis that $\frac{a}{b}$ was a best Type II approximation.

Case II is the only alternative left. In this case,

$$\left| \frac{a}{b} - \frac{P_{k-1}}{Q_{k-1}} \right| \geq \frac{1}{bQ_{k-1}} \tag{2}$$

since $aQ_{k-1} - bP_{k-1} \neq 0$. Also

$$\left| \frac{a}{b} - \frac{P_{k-1}}{Q_{k-1}} \right| < \left| \frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}} \right| = \frac{1}{Q_k Q_{k-1}} . \tag{3}$$

Combining inequalities (2) and (3) and multiplying both members by bQ_kQ_{k-1} yields $Q_k < b$. On the other hand,

$$\left| \alpha - \frac{a}{b} \right| \geq \left| \frac{P_{k+1}}{Q_{k+1}} - \frac{a}{b} \right| \geq \frac{1}{bQ_{k+1}} ,$$

and hence,

$$|b\alpha - a| \geq \frac{1}{Q_{k+1}} , \tag{4}$$

whereas

$$|Q_k\alpha - P_k| \leq \frac{1}{Q_{k+1}} . \tag{5}$$

Combining inequalities (4) and (5) results in $|b\alpha - a| \geq |Q_k\alpha - P_k|$ where $Q_k < b$ contradicting that $\frac{a}{b}$ is a best approximation of Type II. The conclusion must now be that $\frac{a}{b}$ is a convergent to α.

The converse of Theorem 3.3 is not true. However, in all cases except $\alpha = a_0 + R_1$, $1 > R_1 \geq \frac{1}{2}$, it is true. This case will be referred to as the trivial case.

Theorem 3.4. Every convergent is a best approximation of the second type, the sole exception being the trivial case of

$$\alpha = a_0 + R_1, \quad 1 > R_1 \geq \frac{1}{2}, \quad \text{and} \quad \frac{P_0}{Q_0} = \frac{a_0}{1}.$$

Proof: Case I,

$$\alpha = a_0 + R_1, \quad 0 \leq R_1 < \frac{1}{2} \quad \text{and} \quad \frac{P_0}{Q_0} = \frac{a_0}{1}.$$

Suppose $\frac{a}{b}$ is a "better" approximation of the second type; thus, b must equal one. Since

$$a_0 \leq \alpha < a_0 + \frac{1}{2} < a_0 + 1$$

and

$$\frac{a}{b} = \frac{a}{1} \neq \frac{a_0}{1} = \frac{P_0}{Q_0},$$

it must be true that $a = a_0 + 1$. But in this event $|1 \cdot \alpha - a| > \frac{1}{2}$ whereas $|1 \cdot \alpha - a_0| < \frac{1}{2}$ and $\frac{a}{b}$ is not a "better" approximation of α.

Case II, α is a real number, and the convergent is

$$\frac{P_n}{Q_n}, \quad n \geq 1.$$

The proof of this case is based on Corollaries 2.25 and 2.26. Suppose

$$\frac{a}{b} \neq \frac{P_n}{Q_n}$$

with $1 \leq b \leq Q_n$ and $n \geq 1$. If

$$\frac{a}{b} = \frac{P_{n-1}}{Q_{n-1}},$$

then the latter corollary indicates

$$|Q_n\alpha - P_n| < \frac{1}{Q_{n+1}} < |Q_{n-1}\alpha - P_{n-1}|$$

or

$$|Q_n\alpha - P_n| < |b\alpha - a|.$$

If

$$\frac{a}{b} \neq \frac{P_{n-1}}{Q_{n-1}},$$

then

$$|aQ_{n-1} - bP_{n-1}| \geq 1,$$

and hence,

$$\left|\frac{a}{b} - \alpha\right| + \left|\alpha - \frac{P_{n-1}}{Q_{n-1}}\right| \geq \left|\frac{a}{b} - \frac{P_{n-1}}{Q_{n-1}}\right|$$

$$\geq \frac{1}{bQ_{n-1}}. \tag{6}$$

Multiplying both members of inequality (6) by bQ_{n-1} the following results:

$$Q_{n-1}|b\alpha - a| + b|Q_{n-1}\alpha - P_{n-1}| \geq 1.$$

The assumption that $1 \leq b \leq Q_n$ and Corollary 2.25 leads to

$$1 \geq b|Q_{n-1}\alpha - P_{n-1}| + Q_{n-1}|Q_n\alpha - P_n|$$

whence

$$|Q_n\alpha - P_n| \leq |b\alpha - a|. \tag{7}$$

Equality in (7) is impossible for α irrational since

$$\frac{a}{b} \neq \frac{P_n}{Q_n}.$$

For rational α, excluding equality in (7) requires an additional argument which may run as follows. Assume equality holds in (7) and let

$$\alpha = \frac{P}{Q} = \frac{P_{n-1}r_n + P_{n-2}}{Q_{n-1}r_n + Q_{n-2}}$$

with rational r_n, as in Theorem 2.19, and $Q \geq Q_n$. It is thus the case that

$$|Q_n\alpha - P_n| = |b\alpha - a|,$$

which implies

$$|Q_nP - P_nQ| = |bP - aQ|. \tag{8}$$

From the left member of equality (8), applying Theorem 2.12 and Corollary 2.13, it follows that

$$|Q_n(P_{n-1}r_n + P_{n-2}) - P_n(Q_{n-1}r_n + Q_{n-2})| = |(-1)^n r_n + (-1)^{n-1}a_n|$$
$$= |r_n - a_n|,$$

and hence

$$|bP - aQ| = |r_n - a_n|. \tag{9}$$

Equality (9) indicates that r_n is an integer; thus, by the definition of simple continued fractions, $r_n = a_n$ and

$$\frac{a}{b} = \frac{P}{Q} = \frac{P_n}{Q_n}$$

contrary to the assumption that

$$\frac{a}{b} \neq \frac{P_n}{Q_n} .$$

Thus, equality cannot hold in (7) and $|Q_n\alpha - P_n| < |b\alpha - a|$. Hence,

$$\frac{P_n}{Q_n}$$

is a best approximation of Type II. This completes the argument for Theorem 3.4.

It can be shown that any approximation of the first type will be a convergent or an intermediate fraction. For a proof of this statement, the reader is referred to [10] Theorem 15, page 22. The converse of this theorem is not nearly as complete as that for Theorem 3.3.

The preceding theorems concerning best approximations of the second type have a striking geometric interpretation for an irrational number

α. It was Klein who first observed in 1895 that if one considers all points in the Cartesian Plane with integral coordinates as pegs (or posts) and the equation of $y = \alpha x$, which will pass through only the peg at the origin when α is a positive irrational, then the convergents to α have the following characterization. One is to imagine that two strings have been attached to the graph of $y = \alpha x$ at a point infinitely distant from the origin in the first quadrant. At the origin one string is pulled above the graph and the other string pulled down. The freedom of movement of the strings will be restricted by the pegs, and they will "catch" on certain ones. (See Figure 1.)

Consider the bottom string. The slope of the graph is, of course, α; but the slope of any "segment" of the string is greater than α. By "segment" is meant a part of the string "caught" between two pegs. The result of these conditions require that the bottom string becomes a greater vertical distance from the graph of $y = \alpha x$ as one measures these distances nearer the origin. If one considers the expression

$$|x_0\alpha - y_0|,$$

it is apparent that this is the vertical distance from a point on the graph to point with the coordinates (x_0,y_0).

Let (x_0,y_0) be a peg that "catches" the bottom string. There are no pegs with smaller x coordinates that are closer to the graph (vertically) since the line gets further from the graph as x becomes smaller. There are certainly no pegs between the line and graph. Hence, for all $y \in N$ and $x \leq x_0$, $x \in N$

$$|x\alpha - y| > |x_0\alpha - y_0|.$$

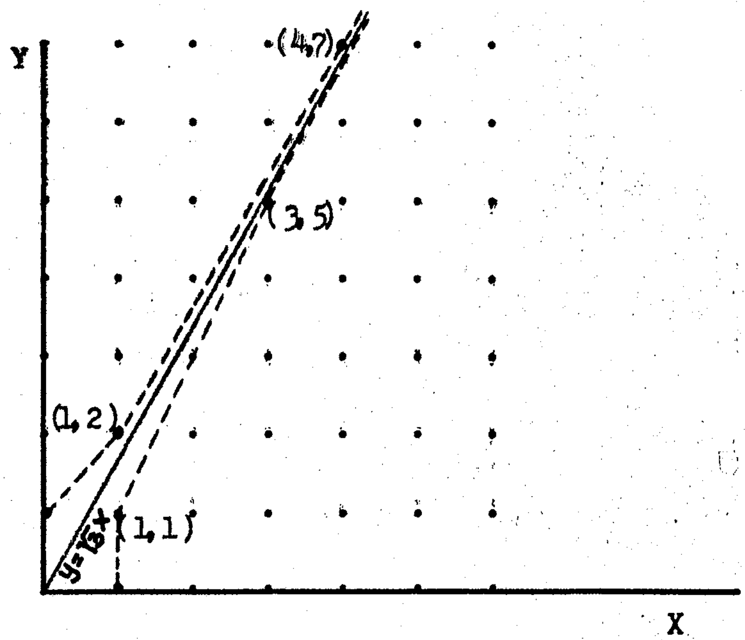

Figure 1

Thus,

$$\frac{y_0}{x_0}$$

is a best approximation of the second type. The "catching" pegs below the line will be even convergents, and those above the line will be odd convergents.

With $\alpha = \sqrt{3} = [1;1,2,1,2,1, \ldots]$, the convergents are:

$$\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \ldots .$$

The pegs below the line are at the points

$$(1,1), (3,5), (11,19), \ldots,$$

and the pegs above the line are at the points

$$(1,2),\ (4,7),\ (15,26),\ \ldots\ .$$

This geometrical interpretation of the convergents of an irrational α leads to many fascinating applications of the foregoing theorems. To suggest but one of many, consider

$$|Q_n P_{n-1} - Q_{n-1} P_n| = |(-1)^n| = 1. \tag{10}$$

If one considers the vectors $[Q_n, P_n]$ and $[Q_{n-1}, P_{n-1}]$, the left-hand member of (10) is just the bar product. The area of the triangle with these vectors as its sides is $\frac{1}{2}$ this product. Hence, the area of any triangle drawn from the origin to points which represent consecutive convergents will have the area $\frac{1}{2}$.

The foregoing material of this chapter concerns itself with minimizing the difference $\left| \alpha - \frac{a}{b} \right|$ for all possible rational numbers $\frac{a}{b}$ where the denominator b has certain restrictions placed on it. The result has been that the convergents

$$\frac{P_n}{Q_n}$$

turn out to be the "best" choice of all the fractions satisfying the given restrictions on the denominator b. Hence, in any examination of rational approximations to the number α, it suffices to examine the properties of the convergents to α. One such property of the convergents was stated in Corollary 2.24 from which the following inequality is derived,

$$\left| \alpha - \frac{P_k}{Q_k} \right| < \frac{1}{Q_k^2} . \tag{11}$$

The question naturally arises as to whether the inequality (11) can be strengthened. Does there exist a rational number $\frac{a}{b}$ $(b > 0)$ such that

$$\left| \alpha - \frac{a}{b} \right| < \frac{1}{f(b)b^2} \tag{12}$$

where $f(b) > 1$? Of course, inequality (11) states that an infinite number of such rationals exist satisfying (12) for $f(b) \equiv 1$ and α irrational. In this light, the following theorems deal with possible values for $f(b)$.

Theorem 3.5. If a number α has a convergent of order $k > 0$, at least one of the following two inequalities must hold:

$$\left| \alpha - \frac{P_k}{Q_k} \right| < \frac{1}{2Q_k^2} , \qquad \left| \alpha - \frac{P_{k-1}}{Q_{k-1}} \right| < \frac{1}{2Q_{k-1}^2} .$$

Proof: Since α lies between

$$\frac{P_{k-1}}{Q_{k-1}} \text{ and } \frac{P_k}{Q_k} ,$$

it follows that

$$\left| \alpha - \frac{P_k}{Q_k} \right| + \left| \alpha - \frac{P_{k-1}}{Q_{k-1}} \right| = \left| \frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}} \right| = \frac{1}{Q_k Q_{k-1}} < \frac{1}{2Q_k^2} + \frac{1}{2Q_{k-1}^2} . \tag{13}$$

This inequality follows from the fact that $(Q_k - Q_{k-1})^2 > 0$; hence, $Q_k^2 + Q_{k-1}^2 > 2Q_kQ_{k-1}$. Divide both members of this inequality by $2Q_k^2Q_{k-1}^2$ and the desired expression results. In order for inequality (13) to be true, at least one of the two inequalities in the conclusion must hold.

Corollary 3.6. If α is irrational, then there are an infinite number of rational numbers $\frac{a}{b}$ satisfying (12) for $f(b) \equiv 2$.

Proof: Since the number of convergents to α is infinite, then so too is the number of pairs of convergents $\{C_{2n}, C_{2n+1}\}$. But Theorem 3.5 states that each pair must contain a rational number satisfying (12) with $f(b) \equiv 2$. Hence, the corollary follows.

The above theorem also has a true converse in a certain sense.

Theorem 3.7. Every irreducible rational fraction $\frac{a}{b}$ that satisfies (12) for $f(b) \equiv 2$ is a convergent of the number α.

Proof: It will be shown that $\frac{a}{b}$ is a best approximation to α of the second kind. Suppose

$$|d\alpha - c| \leq |b\alpha - a| < \frac{1}{2b}$$

for $d > 0$ and $\frac{c}{d} \neq \frac{a}{b}$, then

$$\left| \alpha - \frac{c}{d} \right| < \frac{1}{2bd}$$

and consequently,

$$\left| \frac{c}{d} - \frac{a}{b} \right| \leq \left| \alpha - \frac{c}{d} \right| + \left| \alpha - \frac{a}{b} \right|$$

$$< \frac{1}{2bd} + \frac{1}{2b^2} = \frac{b + d}{2b^2d} .$$

On the other hand, since $\frac{c}{d} \neq \frac{a}{b}$,

$$\left| \frac{c}{d} - \frac{a}{b} \right| \geq \frac{1}{bd} ,$$

thus,

$$\frac{1}{bd} < \frac{b + d}{2b^2 d} ,$$

or

$$2b < b + d;$$

and hence, $b < d$, establishing that $\frac{a}{b}$ is a best Type II approximation. By Theorem 3.3, $\frac{a}{b}$ is a convergent to α.

Corollary 3.8. If α is rational, there are only finitely many rationals $\frac{a}{b}$ in their lowest terms satisfying (12) with $f(b) \equiv 2$.

The proof follows from Theorem 3.7, and the fact that α has only finitely many convergents.

Theorem 3.9. If a number α has a convergent of order $k > 1$, at least one of the following three inequalities must hold:

$$\left| \alpha - \frac{P_k}{Q_k} \right| < \frac{1}{\sqrt{5}\, Q_k^2} , \qquad \left| \alpha - \frac{P_{k-1}}{Q_{k-1}} \right| < \frac{1}{\sqrt{5}\, Q_{k-1}^2} ,$$

$$\left| \alpha - \frac{P_{k-2}}{Q_{k-2}} \right| < \frac{1}{\sqrt{5}\, Q_{k-2}^2} .$$

Proof: Assume the statement is false, then for $k > 1$,

$$\left| \alpha - \frac{P_k}{Q_k} \right| \geq \frac{1}{\sqrt{5}\, Q_k^2}, \qquad \left| \alpha - \frac{P_{k-1}}{Q_{k-1}} \right| \geq \frac{1}{\sqrt{5}\, Q_{k-1}^2}$$

and

$$\left| \alpha - \frac{P_{k-2}}{Q_{k-2}} \right| \geq \frac{1}{\sqrt{5}\, Q_{k-2}^2} .$$

In Chapter II it was observed that for three consecutive convergents to α either,

$$\frac{P_{k-2}}{Q_{k-2}} < \frac{P_k}{Q_k} < \alpha < \frac{P_{k-1}}{Q_{k-1}}$$

or

$$\frac{P_{k-1}}{Q_{k-1}} < \alpha < \frac{P_k}{Q_k} < \frac{P_{k-2}}{Q_{k-2}} .$$

From these inequalities it follows that

$$\frac{1}{\sqrt{5}\, Q_k^2} + \frac{1}{\sqrt{5}\, Q_{k-1}^2} \leq \left| \frac{P_k}{Q_k} - \alpha \right| + \left| \alpha - \frac{P_{k-1}}{Q_{k-1}} \right| = \frac{1}{Q_k Q_{k-1}}$$

and

$$\frac{1}{\sqrt{5}\, Q_{k-2}^2} + \frac{1}{\sqrt{5}\, Q_{k-1}^2} \leq \left| \frac{P_{k-2}}{Q_{k-2}} - \alpha \right| + \left| \alpha - \frac{P_{k-1}}{Q_{k-1}} \right| = \frac{1}{Q_{k-2} Q_{k-1}} .$$

These inequalities may be restated as

$$\left(\frac{Q_k}{Q_{k-1}} \right)^2 - \sqrt{5} \left(\frac{Q_k}{Q_{k-1}} \right) + 1 \leq 0$$

and

$$\left(\frac{Q_{k-1}}{Q_{k-2}}\right)^2 - \sqrt{5}\left(\frac{Q_{k-1}}{Q_{k-2}}\right) + 1 \leq 0.$$

Hence,

$$\left(\frac{Q_k}{Q_{k-1}}\right) \text{ and } \left(\frac{Q_{k-1}}{Q_{k-2}}\right)$$

must both satisfy the inequality

$$x^2 - \sqrt{5}x + 1 \leq 0. \tag{14}$$

Using elementary algebra one discovers the solution set to this last inequality is

$$\frac{\sqrt{5} - 1}{2} \leq x \leq \frac{\sqrt{5} + 1}{2} .$$

Thus, it is apparent that

$$\frac{\sqrt{5} - 1}{2} \leq \frac{Q_k}{Q_{k-1}} , \quad \frac{Q_{k-1}}{Q_{k-2}} \leq \frac{\sqrt{5} + 1}{2} .$$

In Theorem 2.20 it was noted that

$$\frac{Q_k}{Q_{k-1}} = [a_k;a_{k-1}, \ldots, a_1]$$

and hence,

$$\frac{Q_k}{Q_{k-1}} = a_k + \frac{1}{\frac{Q_{k-1}}{Q_{k-2}}} .$$

Let

$$\beta = \frac{Q_{k-1}}{Q_{k-2}} ,$$

then

$$a_k + \frac{1}{\beta} = \frac{Q_k}{Q_{k-1}} .$$

Since β is a solution to (14),

$$\frac{\sqrt{5} - 1}{2} \leq \beta \leq \frac{\sqrt{5} + 1}{2}$$

and

$$\frac{\sqrt{5} + 1}{2} = \frac{2}{\sqrt{5} - 1} \geq \frac{1}{\beta} \geq \frac{2}{\sqrt{5} + 1} = \frac{\sqrt{5} - 1}{2} .$$

Therefore, since $a_k \geq 1$,

$$\frac{Q_k}{Q_{k-1}} = a_k + \frac{1}{\beta} \geq 1 + \frac{1}{\beta} \geq 1 + \frac{\sqrt{5} - 1}{2} = \frac{\sqrt{5} + 1}{2} .$$

Since

$$\frac{Q_k}{Q_{k-1}}$$

is also a solution to (14), equality must be the case and

$$\frac{Q_k}{Q_{k-1}} = \frac{\sqrt{5} + 1}{2} . \tag{15}$$

But

$$\frac{Q_k}{Q_{k-1}}$$

is a finite continued fraction; and hence, it is rational while

$$\frac{\sqrt{5}+1}{2}$$

is clearly irrational. Thus, equality (15) cannot hold, and our original assumption is false.

<u>Corollary</u> 3.10 (Hurwitz). If α is irrational, then there are infinitely many rational numbers $\frac{a}{b}$ satisfying inequality (12) with $f(b) \equiv \sqrt{5}$.

The proof of this theorem follows the same pattern as that of Corollary 3.6.

The reader at this point may suspect that one could continue to strengthen the statements of Theorems 3.5 and 3.9 ad infinitum. However, this suspicion is erroneous as $f(b) \equiv \sqrt{5}$ is the best possible constant value.

<u>Theorem</u> 3.11. There exists irrational α such that inequality (12) holds for only finitely many rationals $\frac{a}{b}$ when $f(b) \equiv c > \sqrt{5}$, c a constant.

Let $\xi = [1;1,1, \ldots]$, then it can be shown that $\xi = \frac{1+\sqrt{5}}{2}$. Since $c > \sqrt{5} > 2$, then

$$\left| \xi - \frac{a}{b} \right| < \frac{1}{cb^2} < \frac{1}{2b^2}$$

implies that $\frac{a}{b}$ is a convergent to ξ by Theorem 3.7. It will be shown that only a finite number of convergents satisfy the stated conditions.

Let $n \geq 1$, then from Theorem 2.20

$$\frac{Q_n}{Q_{n-1}} = [a_n;a_{n-1}, \ldots, a_1]$$

where $a_i \equiv 1$ $(1 \leq i \leq n)$. It can be shown that

$$\lim_{n \to \infty} \frac{Q_n}{Q_{n-1}} = [1;1, \ldots] = \xi.$$

Let r_n be any remainder as defined in Chapter II. It is clear that for all n,

$$r_{n+1} = [1;1, \ldots] = \xi.$$

Hence,

$$\lim_{n \to \infty} \left(r_{n+1} + \frac{Q_{n-1}}{Q_n} \right) = \xi + \frac{1}{\xi} = \sqrt{5}. \tag{16}$$

By the generalized form of Theorem 2.19,

$$\left| \xi - \frac{P_n}{Q_n} \right| = \left| \frac{r_{n+1}P_n + P_{n-1}}{r_{n+1}Q_n + Q_{n-1}} - \frac{P_n}{Q_n} \right|$$

$$= \frac{1}{Q_n^2 \left(r_{n+1} + \dfrac{Q_{n-1}}{Q_n} \right)} = \frac{1}{Q_n^2 \, L(n)} \, .$$

where

$$L(n) = \left(r_{n+1} + \frac{Q_{n-1}}{Q_n} \right).$$

By (16), $L(n)$ can be greater than $c > \sqrt{5}$ for only finitely many n;

hence,

$$\left| \xi - \frac{P_n}{Q_n} \right| = \frac{1}{Q_n^2 L(n)} < \frac{1}{cQ_n^2}$$

for only finitely many n.

It is important to note that this last theorem does not assert that for all irrational α one cannot improve $f(b) \equiv \sqrt{5}$ as in Corollary 3.10, but rather, for at least one, namely $\xi = \frac{\sqrt{5}+1}{2}$, $\sqrt{5}$ is the best (or largest) value. The following theorem asserts that the possibilities for improving the values of $f(b)$ are boundless if the values are not required to work for all irrational numbers.

<u>Theorem</u> 3.12. For any positive function $f(b)$ with natural argument b, there is an irrational number α such that the inequality

$$\left| \alpha - \frac{a}{b} \right| < \frac{1}{f(b)b^2}$$

has an infinite number of solutions in integers a and b $(b > 0)$ with $(a,b) = 1$.

<u>Proof</u>: Define $\alpha = [a_0; a_1, \ldots]$ where for $i \geq 0$, $a_{i+1} \geq f(Q_i)$; thus,

$$\left| \alpha - \frac{P_k}{Q_k} \right| < \frac{1}{Q_k Q_{k+1}} = \frac{1}{Q_k(a_{k+1}Q_k + Q_{k-1})}$$

$$\leq \frac{1}{a_{k+1}Q_k^2}$$

$$\leq \frac{1}{f(Q_k)Q_k^2}.$$

In the most general case it was shown (Corollaries 2.24 and 2.29) that

$$\frac{1}{Q_k(Q_k + Q_{k+1})} < \left| \alpha - \frac{P_k}{Q_k} \right| \leq \frac{1}{Q_k Q_{k+1}}$$

or, equivalently,

$$\frac{1}{Q_k^2\left(a_{k+1} + 1 + \frac{Q_{k-1}}{Q_k}\right)} < \left| \alpha - \frac{P_k}{Q_k} \right| \leq \frac{1}{Q_k^2\left(a_{k+1} + \frac{Q_{k-1}}{Q_k}\right)} .$$

Since

$$\frac{Q_k}{Q_{k-1}} = [a_k; a_{k-1}, \ldots, a_1]$$

and $a_k \geq 1$,

$$0 < \frac{Q_{k-1}}{Q_k} \leq 1.$$

Hence,

$$\frac{1}{Q_k^2(a_{k+1} + 2)} < \left| \alpha - \frac{P_k}{Q_k} \right| \leq \frac{1}{Q_k^2\, a_{k+1}} . \tag{17}$$

Hence, the larger a_{k+1} becomes the more closely

$$\frac{P_k}{Q_k}$$

will approximate α. Thus, irrational numbers with large partial quotients admit good approximations by rational numbers. Numbers with bounded partial quotients must admit poor approximations. In particular, $\xi = [1;1,1, \ldots]$ would have the worst rational approximations of all.

Those approximating properties that are peculiar to numbers with bounded elements are completely expressed in the following proposition.

Theorem 3.13. For every irrational number α with bounded partial quotients, and for sufficiently large c, the inequality

$$\left| \alpha - \frac{a}{b} \right| < \frac{1}{cb^2}$$

has no solution in integers a and b $(b > 0)$. On the other hand, for every number α with an unbounded sequence of elements and arbitrary $c > 0$, the above inequality has an infinite set of such solutions.

Proof: Suppose the partial quotients of irrational α are unbounded, then for all fixed c, there is an infinite set of integers k such that

$$a_{k+1} > c.$$

By the second of inequalities (17)

$$\left| \alpha - \frac{P_k}{Q_k} \right| \leq \frac{1}{Q_k^2 \, a_{k+1}} < \frac{1}{cQ_k^2}$$

for all k in the above infinite set. Thus, the second assertion has been established.

If there exists an $M > 0$ such that $a_i < M$ for $i \geq 0$, then on the basis of the first of the inequalities (17)

$$\frac{1}{Q_k^2(M + 2)} < \left| \alpha - \frac{P_k}{Q_k} \right|$$

for arbitrary $k \geq 0$.

Let a and b be arbitrary integers $(b > 0)$, and let k be chosen by the inequalities $Q_{k-1} < b \leq Q_k$. Since all convergents are best approximations of the first kind,

$$\left| \alpha - \frac{a}{b} \right| \geq \left| \alpha - \frac{P_k}{Q_k} \right| > \frac{1}{Q_k^2(M + 2)}$$

$$= \frac{1}{b^2(M + 2)} \left(\frac{b}{Q_k} \right)^2$$

$$> \frac{1}{b^2(M + 2)} \left(\frac{Q_{k-1}}{Q_k} \right)^2$$

$$= \frac{1}{b^2(M + 2)} \left(\frac{Q_{k-1}}{a_k Q_{k-1} + Q_{k-2}} \right)^2$$

$$> \frac{1}{b^2(M + 2)} \left(\frac{1}{a_k + \frac{Q_{k-2}}{Q_{k-1}}} \right)^2$$

$$> \frac{1}{b^2(M + 2)(a_k + 1)^2} > \frac{1}{b^2(M + 2)(M + 1)^2} .$$

If c is chosen greater than $(M + 2)(M + 1)^2$, then for all k,

$$\left| \alpha - \frac{a}{b} \right| > \frac{1}{b^2(M + 2)(M + 1)^2} > \frac{1}{cb^2} .$$

Thus, Theorem 3.13 has been established.

Based on this previous theorem, one could define an order of approximation for infinite continued fractions with bounded partial quotients. The order could be the minimum value of c such that the inequality

$$\left| \alpha - \frac{a}{b} \right| < \frac{1}{cb^2} \tag{18}$$

has no solution in integers a and b $(b > 0)$. It is apparent that the value c for a continued fraction with unbounded partial quotients would be infinite. Another way one could define an order of approximation for bounded continued fractions is finding the largest value of c such that (18) had an infinite number of solutions. Hurwitz's Theorem would state that $c = \sqrt{5}$ for the irrational ξ. Again, unbounded continued fractions would have infinite orders of approximation.

The following theorem determines the best value of c in inequality (18) such that for any irrational number α the inequality has at least one solution $\frac{a}{b}$, a and b relatively prime, $(b > 0)$, or, more generally, at least m solutions.

Theorem 3.14. Let

$$\frac{P_n}{Q_n}$$

denote the n-th convergent to $\xi_0 = \frac{\sqrt{5} - 1}{2}$, starting with

$$\frac{P_1}{Q_1} = 1.$$

Let

$$k_m = \frac{1}{2}\{\sqrt{5} + 1\} + \frac{P_{2m-1}}{Q_{2m-1}}$$

for $m = 1, 2, \ldots$. Then, for any irrational α, the inequality

$$\left| \alpha - \frac{a}{b} \right| \leq \frac{1}{k_m b^2} \tag{19}$$

has at least m solutions in relatively prime integers a and b, with $b > 0$. Further, if $\alpha = \xi_0$, there are not more than m such solutions.

The proof of this theorem is found in an article by A. V. Prasad [18].

At this point the discussion will turn to the existence of transcendental numbers. In this discussion, the function $f(b)$ in inequality (12) will no longer be a constant function but of the form

$$\frac{1}{f(b)} = \frac{c}{b^k} .$$

In order to make the discussion complete, the following definitions are made.

<u>Definition</u> 3.15. α is an <u>algebraic</u> number if there exists a polynomial of degree n with integral coefficients such that α is a root of that polynomial.

Definition 3.16. α is of degree n, if α is algebraic and satisfies an integral polynomial of degree n and is not the root of any integral polynomial of smaller degree.

Every rational number $\frac{a}{b}$ is algebraic since it is a root of the polynomial $bx - a$. Similarly, for $m \varepsilon z$, $\sqrt[n]{m}$ is algebraic since it is the root of $x^n - m$.

Definition 3.17. α is a transcendental number if it is not algebraic.

Examples of transcendental numbers are e, π, and α^β, where α and β are algebraic, $\alpha \neq 0,1$, and β is irrational. It is not immediately obvious that any of the above examples are transcendental or even that such numbers exist. The first noteworthy step in proving the existence of such numbers is the following theorem known as Liouville's Theorem.

Theorem 3.18 (Liouville). For every real irrational algebraic number of degree n, there exists a positive number c such that, for arbitrary integers a and b $(b > 0)$,

$$\left| \alpha - \frac{a}{b} \right| > \frac{c}{b^n} .$$

The proof of Liouville's Theorem is found many places in the literature. One such reference would be [10], page 45.

Liouville's Theorem shows that algebraic numbers do not admit rational approximations greater than a certain order of accuracy. To establish the existence of nonalgebraic or transcendental numbers, one must only exhibit an irrational number for which rational fractions give extremely close approximations. Theorem 3.12 indicates that the possibilities for this are unlimited.

Corollary 3.19. There exists α that is transcendental.

Proof: Using the apparatus of continued fractions, one may exhibit as many such numbers as he desires. $\alpha = [a_0;a_1, \ldots]$ will be inductively defined as follows:

a_0 is an arbitrary integer and

$$a_{k+1} > Q_k^{k-1},$$

then for all k,

$$\left| \alpha - \frac{P_k}{Q_k} \right| < \frac{1}{Q_k Q_{k+1}} < \frac{1}{Q_k^2 a_{k+1}} < \frac{1}{Q_k^{k+1}} .$$

For c and n fixed, there exists k sufficiently large such that

$$\frac{1}{Q_k^{k+1}} < \frac{c}{Q_k^n}$$

and hence,

$$\left| \alpha - \frac{P_k}{Q_k} \right| < \frac{c}{Q_k^n} .$$

Thus, by Liouville's Theorem, α is not algebraic of degree n. Since n was arbitrary, α is transcendental.

CHAPTER IV

QUADRATIC IRRATIONALS AND PERIODIC CONTINUED FRACTIONS

There are not many irrational numbers whose continued fractions are known to have any features of regularity. Here, the term "regularity," as applied to continued fractions, is used to imply a certain predictability about the partial quotients of the fraction. This predictability could be in the form of an upper (or lower) bound on the partial quotients, or it may be the complete determination of the quotients ad infinitum.

The most prominent of those irrationals whose partial quotients are "regular" are the quadratic irrationals. It will be shown that every continued fraction whose partial quotients are periodic after some point has a value which is a quadratic irrational. The converse of this statement, first shown by Lagrange, is also true.

Theorem 4.1. Let n and s be fixed and $n \geq 0$, $s \in N$, and let α be the value of the continued fraction $[a_0;a_1, \ldots]$ where $a_j = a_{j+s}$ for all $j \geq n$ $(j \in N)$ then α is a quadratic irrational.

Proof: Case 1. $n = 0$ (Purely periodic). As with decimals it is customary to indicate a repeating segment with a bar. With this notation the continued fraction for Case 1 may be written:

$$\alpha = [\overline{a_0;a_1,a_2, \ldots, a_{s-1}}].$$

Using the generalized form of Theorem 2.19,

$$\alpha = \frac{P_{s-1}\alpha + P_{s-2}}{Q_{s-1}\alpha + Q_{s-2}}$$

where P_{s-1}, P_{s-2}, Q_{s-1}, Q_{s-2} are integers and $Q_{s-1} \geq 1$. Thus, α is a solution of

$$Q_{s-1}\alpha^2 + (Q_{s-2} - P_{s-1})\alpha - P_{s-2} = 0$$

and, hence, a quadratic irrational.

Case 2. $n > 0$, $\alpha = [a_0;a_1, \ldots, \overline{a_n, \ldots, a_{n+s-1}}]$. Again using Theorem 2.19,

$$\alpha = \frac{P_{n-1}\beta + P_{n-2}}{Q_{n-1}\beta + Q_{n-2}} \tag{1}$$

where $\beta = [\overline{a_n; \ldots, a_{n+s-1}}]$ and, hence

$$\beta = \frac{A + \sqrt{B}}{C} \tag{2}$$

where A, B, and C are integers with $B > 1$ and $C \neq 0$. By substituting equality (2) into equality (1), Case 2 and Theorem 4.1 are established.

The converse of this theorem is not so simple. Several theorems leading to Lagrange's Theorem will be needed. First, an examination of the construction of the simple continued fraction for the $\sqrt{D}$, where D is a natural number that is not the square of a natural number, will be undertaken. In this discussion and the following theorems when $\sqrt{D}$ is written, it will always be assumed that D is a natural number and not the square of some other natural number. The procedure as outlined in Theorem 2.31 and Theorem 2.33 indicates

$$a_0 = [\sqrt{D}] \quad \text{and} \quad \sqrt{D} = a_0 + R_1$$

where $0 < R_1 < 1$. Thus,

$$\frac{1}{R_1} = \frac{1}{\sqrt{D} - a_0} = \frac{\sqrt{D} + a_0}{D - a_0^2} = \frac{\sqrt{D} + b_1}{c_1}$$

where $b_1 = a_0$, $c_1 = D - a_0^2$ and $c_1 > 0$. Hence,

$$D - b_1^2 = c_1. \tag{3}$$

Continuing,

$$a_1 = \left[\frac{1}{R_1}\right] \quad \text{and} \quad \frac{1}{R_1} = a_1 + R_2,$$

whence, by (3),

$$\frac{1}{R_2} = \frac{1}{\frac{1}{R_1} - a_1} = \frac{1}{\frac{\sqrt{D} + b_1}{c_1} - a_1}$$

$$= \frac{c_1}{\sqrt{D} + b_1 - a_1c_1} = \frac{c_1(\sqrt{D} + a_1c_1 - b_1)}{D - (a_1c_1 - b_1)^2}$$

$$= \frac{c_1(\sqrt{D} + a_1c_1 - b_1)}{D - b_1^2 - a_1^2c_1^2 + 2a_1b_1c_1}$$

$$= \frac{\sqrt{D} + a_1c_1 - b_1}{1 - a_1^2c_1 + 2a_1b_1} = \frac{\sqrt{D} + b_2}{c_2} ,$$

where $b_2 = a_1c_1 - b_1$ and $c_2 = 1 - a_1^2c_1 + 2a_1b_1$. Using the above pattern, the following inductive definitions are made.

Definition 4.2. With b_1, b_2, c_1, and c_2 defined as above, for $n \geq 2$

$$b_{n+1} = a_nc_n - b_n,$$

and

$$c_{n+1} = c_{n-1} - a_n^2c_n + 2a_nb_n.$$

Theorem 4.3. For all $n \geq 2$, $D - b_n^2 = c_{n-1}c_n$.

Proof: (By induction.) For $n = 2$, it has been shown that

$$c_1c_2 = c_1(1 - a_1^2c_1 + 2a_1b_1)$$

and thus,

$$= c_1 - a_1^2c_1^2 + 2c_1a_1b_1$$

$$= D - b_1^2 - a_1^2c_1^2 + 2c_1a_1b_1$$

$$= D - (a_1c_1 - b_1)^2$$

$$= D - b_2^2 .$$

Suppose the statement is true for $n \geq 2$; consider

$$\begin{aligned} D - b_{n+1}^2 &= D - (a_n c_n - b_n)^2 \\ &= D - b_n^2 - a_n^2 c_n^2 + 2a_n b_n c_n \\ &= c_{n-1} c_n - a_n^2 c_n^2 + 2a_n b_n c_n \\ &= c_n (c_{n-1} - a_n^2 c_n + 2a_n b_n) \\ &= c_n c_{n+1}. \end{aligned}$$

Since D is not the square of a natural number, this insures that $c_n \neq 0$ for all n.

Theorem 4.4. For all $n \geq 1$,

$$\frac{1}{R_n} = \frac{\sqrt{D} + b_n}{c_n}.$$

Proof: (By induction.) This has been established for $n = 1$ in the discussion above. Suppose the statement is true for $n \geq 1$, and consider

$$\frac{1}{R_{n+1}} = \frac{1}{\frac{1}{R_n} - a_n} = \frac{1}{\frac{\sqrt{D} + b_n}{c_n} - a_n}$$

$$= \frac{c_n}{\sqrt{D} + b_n - a_n c_n}$$

$$= \frac{c_n(\sqrt{D} + a_n c_n - b_n)}{D - (a_n c_n - b_n)^2}$$

$$= \frac{\sqrt{D} + b_{n+1}}{c_{n+1}} .$$

Theorem 4.5. For all $n \geq 1$,

$$0 < \frac{\sqrt{D} - b_n}{c_n} < 1 < \frac{\sqrt{D} + b_n}{c_n} .$$

Proof: (By induction.) Since $b_1 = a_0 = [\sqrt{D}] < \sqrt{D}$, $0 < \sqrt{D} - b_1 < 1$. But c_1 is a natural number; so,

$$0 < \frac{\sqrt{D} - b_1}{c_1} < 1.$$

Also, since $0 < R_1 < 1$,

$$1 < \frac{1}{R_1}$$

and, hence,

$$1 < \frac{\sqrt{D} + b_1}{c_1} .$$

This establishes the theorem for $n = 1$. Suppose the statement is true for $n \geq 1$. By Theorem 4.4 and the fact that $0 < R_{n+1} < 1$,

$$1 < \frac{1}{R_{n+1}} = \frac{\sqrt{D} + b_{n+1}}{c_{n+1}} .$$

By Definition 4.2 and Theorem 4.3,

$$\frac{\sqrt{D} - b_{n+1}}{c_{n+1}} = \frac{D - b_{n+1}^2}{c_{n+1}(\sqrt{D} + b_{n+1})} = \frac{c_n}{\sqrt{D} + b_{n+1}}$$

$$= \frac{c_n}{\sqrt{D} + a_n c_n - b_n} = \frac{1}{\dfrac{\sqrt{D} - b_n}{c_n} + a_n} .$$

Since

$$0 < \frac{\sqrt{D} - b_n}{c_n} < 1,$$

it follows that

$$\frac{\sqrt{D} - b_n}{c_n} + a_n > a_n \geq 1.$$

Thus,

$$0 < \frac{\sqrt{D} - b_{n+1}}{c_{n+1}} < 1,$$

and the theorem has been established.

If $c_n < 0$ for some n, then by Theorem 4.5, both $\sqrt{D} - b_n$ and $\sqrt{D} + b_n$ are less than zero; hence, $2\sqrt{D} < 0$ which is impossible. Thus, $c_n > 0$ for all n. Consequently, $\sqrt{D} - b_n < c_n < \sqrt{D} + b_n$, whence $\sqrt{D} - b_n < \sqrt{D} + b_n$ and so $b_n > 0$ for all n. Thus, Theorem 4.5 also implies $b_n < \sqrt{D}$ and $c_n < \sqrt{D} + b_n < 2\sqrt{D}$ and the following corollary has been established.

<u>Corollary</u> 4.6. For all $n \geq 1$, $b_n < \sqrt{D}$ and $c_n < 2\sqrt{D}$.

Thus, the possible number of expressions of the form

$$\frac{\sqrt{D} + b_n}{c_n}$$

must be less than 2D. But this implies that among the numbers

$$\frac{1}{R_1}, \frac{1}{R_2}, \ldots, \frac{1}{R_{2D}}$$

at least two are the same. Consequently, there are numbers k and $s < 2D$ such that

$$\frac{1}{R_k} = \frac{1}{R_{k+s}} .$$

Since

$$\frac{1}{R_{n+1}} = \frac{1}{\frac{1}{R_n} - \left[\frac{1}{R_n}\right]}$$

for $n = 1, 2, \ldots,$ it follows that

$$\frac{1}{R_{k+1}} = \frac{1}{R_{k+s+1}}$$

and more generally

$$\frac{1}{R_n} = \frac{1}{R_{n+s}} \tag{4}$$

for $n \geq k$. Consequently, the sequence $a_k, a_{k+1}, \ldots$

$$\left(a_n = \left[\frac{1}{R_n}\right] \quad \text{for} \quad n \geq k \right)$$

is periodic.

<u>Theorem</u> 4.7. For $n \geq 1$,

$$x_{n+1} = a_n + \frac{1}{x_n}$$

where

$$a_n = \left[\frac{1}{R_n}\right] \quad \text{and} \quad \frac{1}{x_n} = \frac{\sqrt{D} - b_n}{c_n} .$$

<u>Proof</u>: Since

$$\frac{1}{R_n} = a_n + R_{n+1},$$

$0 < R_{n+1} < 1$, it follows that

$$a_n = \frac{\sqrt{D} + b_n}{c_n} - \frac{c_{n+1}}{\sqrt{D} + b_{n+1}}$$

$$= \frac{D(b_n + b_{n+1}) - b_n b_{n+1}^2 - b_{n+1}^3}{c_n^2 c_{n+1}}$$

$$= \frac{c_{n+1}}{\sqrt{D} - b_{n+1}} - \frac{\sqrt{D} - b_n}{c_n} .$$

Hence,

$$\frac{c_{n+1}}{\sqrt{D} - b_{n+1}} = a_n + \frac{\sqrt{D} - b_n}{c_n}$$

and

$$x_{n+1} = a_n + \frac{1}{x_n} .$$

By Theorem 4.5,

$$0 < \frac{1}{x_n} < 1,$$

thus,

$$[x_{n+1}] = a_n \quad \text{for} \quad n = 1, 2, \ldots . \tag{5}$$

Furthermore, since equality (4) gives $x_n = x_{n+s}$, for $n = k \geq 2$, it is clear by (5) that

$$a_{k-1} = [x_k] = [x_{k+s}] = a_{k+s-1}.$$

Repeating this argument (provided that $k - 1 \geq 2$) it follows that

$$\frac{1}{R_{k-1}} = a_{k-1} + R_k = a_{k+s-1} + R_{k+s} = \frac{1}{R_{k+s-1}}$$

and, hence, $x_{k-1} = x_{k+s-1}$; thus,

$$a_{k-2} = [x_{k-1}] = [x_{k+s-1}] = a_{k+s-2}.$$

By induction it would follow that $a_{k-j} = a_{k+s-j}$ for $1 \leq k - j \leq k - 1$ and, thus, the sequence $a_1, a_2, a_3, \ldots$ is periodic.

It is thus clear that there is a sequence of formulae

$$\frac{1}{R_1} = a_1 + R_2, \ \frac{1}{R_2} = a_2 + R_3, \ \ldots, \ \frac{1}{R_s} = a_s + R_{s+1} = a_s + R_1 \tag{6}$$

and another related sequence

$$x_2 = a_1 + \frac{1}{x_1}, \ x_3 = a_2 + \frac{1}{x_2}, \ \ldots, \ x_s = a_{s-1} + \frac{1}{x_{s-1}}, \ x_1 = a_s + \frac{1}{x_s}. \tag{7}$$

From (6)

$$\frac{1}{R_1} = \left[a_1;a_2, \ldots, a_s, \frac{1}{R_1} \right]$$

and from (7)

$$x_1 = [a_s;a_{s-1}, \ldots, a_1,x_1]. \tag{8}$$

But $\sqrt{D} = a_0 + R_1$ and $\sqrt{D} = -a_0 + x_1$

$$\left(\text{since } \frac{1}{x_1} = \frac{\sqrt{D} - b_1}{c_1} \right).$$

Hence, the formulae (8) imply the relations

$$\sqrt{D} = \left[a_0;a_1,a_2, \ldots, a_s, \frac{1}{R_1} \right]$$

and

$$\sqrt{D} = [a_s - a_0;a_{s-1},a_{s-2}, \ldots, a_1,x_1].$$

Since

$$\frac{1}{R_1} \text{ and } x_1$$

are both greater than one, the first $s + 1$ partial quotients must be the same. Thus,

$$a_s = 2a_0 = 2[\sqrt{D}],\ a_1 = a_{s-1},\ a_2 = a_{s-2},\ \ldots,\ a_{s-1} = a_1.$$

Thus, it is apparent that the sequence $a_1,a_2, \ldots, a_{s-1}$ is symmetric. These findings are summarized in the following theorem.

Theorem 4.8. If D is a natural number which is not the square of a natural number, then in the representation of $\sqrt{D}$ as a continued fraction,

$$\sqrt{D} = [a_0;a_1,a_2, \ldots],$$

the sequence $a_1,a_2, \ldots$ is periodic. Moreover, the period of the sequence is pure and, if it consists of s terms, $a_1,a_2, \ldots, a_s$, then $s < 2D$, $a_s = 2[\sqrt{D}]$ and the sequence $a_1,a_2, \ldots, a_{s-1}$ is symmetric.

With the appropriate alterations, the foregoing theorem could be changed so that its proof would also establish Lagrange's Theorem. This is extremely complicated and will not be presented here. Instead, another proof which is relatively less complicated will be presented.

Theorem 4.9. (Lagrange) Every periodic continued fraction represents a quadratic irrational number and every quadratic irrational number is represented by a periodic continued fraction.

Proof: The first half of this theorem is actually Theorem 4.1 and has been shown. The converse follows; let α be a quadratic irrational satisfying the quadratic equation

$$a\alpha^2 + b\alpha + c = 0 \tag{9}$$

with a, b, and c integers, $a \neq 0$. Let α be written in terms of its remainders of order n,

$$\alpha = \frac{P_{n-1}r_n + P_{n-2}}{Q_{n-1}r_n + Q_{n-2}}$$

using the generalized form of Theorem 2.19. Thus, r_n satisfies the equation

$$A_n r_n^2 + B_n r_n + C_n = 0 \tag{10}$$

where A_n, B_n, and C_n are integers defined by

$$\begin{aligned} A_n &= aP_{n-1}^2 + bP_{n-1}Q_{n-1} + cQ_{n-1}^2, \\ B_n &= 2aP_{n-1}P_{n-2} + b(P_{n-1}Q_{n-2} + P_{n-2}Q_{n-1}) + 2cQ_{n-1}Q_{n-2}, \\ C_n &= aP_{n-2}^2 + bP_{n-2}Q_{n-2} + cQ_{n-2}^2. \end{aligned} \tag{11}$$

Observe that $C_n = A_{n-1}$ and also

$$\begin{aligned} B_n^2 - 4A_nC_n &= (b^2 - 4ac)(P_{n-1}Q_{n-2} - Q_{n-1}P_{n-2})^2 \\ &= (b^2 - 4ac). \end{aligned} \tag{12}$$

Hence, the discriminant of (10) is the same for all n and is equal to the discriminant of (9). Since

$$\left| \alpha - \frac{P_{n-1}}{Q_{n-1}} \right| < \frac{1}{Q_{n-1}^2},$$

it follows that

$$P_{n-1} = \alpha Q_{n-1} + \frac{\delta_{n-1}}{Q_{n-1}} \quad (|\delta_{n-1}| < 1).$$

Therefore, the first formula of (11) gives us

$$A_n = a\left(\alpha Q_{n-1} + \frac{\delta_{n-1}}{Q_{n-1}}\right)^2 + b\left(\alpha Q_{n-1} + \frac{\delta_{n-1}}{Q_{n-1}}\right)Q_{n-1} + cQ_{n-1}^2$$

$$= (a\alpha^2 + b\alpha + c)Q_{n-1}^2 + 2a\alpha\delta_{n-1} + a\frac{\delta_{n-1}^2}{Q_{n-1}^2} + b\delta_{n-1},$$

from which, using (9),

$$\left| A_n \right| = \left| 2a\alpha\delta_{n-1} + a\frac{\delta_{n-1}^2}{Q_{n-1}^2} + b\delta_{n-1} \right|$$

$$\leq 2|a\alpha| + |a| + |b|,$$

and since $C_n = A_{n-1}$,

$$|C_n| + |A_{n-1}| < 2|a\alpha| + |a| + |b|.$$

Since A_n and C_n are integers whose absolute values are bounded, there is only a finite number of distinct pairs (A_n, C_n). From (12) it is also seen that B_n is a function of the pair (A_n, C_n) and, hence, there is only a finite number of distinct equations of the form (10). Thus, as n is increased from 1 to ∞ there is only a finite number of distinct values r_n and thus there exist $k \geq 0$ and $h \in N$ such that

$$r_k = r_{k+h}.$$

This shows that the continued fraction representing α is periodic and Lagrange's Theorem is established.

It is interesting to note that the continued fractions of algebraic irrationals of degree higher than two are unknown. All that exists at this time are some corollaries to Liouville's Theorem. It is not known, for instance, whether their partial quotients are unbounded or bounded. In general, questions connected with the continued-fraction expansion of algebraic numbers of higher degree than the second are extremely difficult and have hardly been studied.

Before leaving quadratic irrationals, an application of Theorem 4.8 is presented. The problem considered here occupied the skills of many of the great mathematicians until J. L. Lagrange finally settled the matter in 1769-70. An equation of the form

$$x^2 - Dy^2 = 1$$

is known as a Pell equation. Named by Euler who mistakenly attributed a method of solution given by Wallis to John Pell, an English mathematician of that period. A solution in positive integers of the Pell equation is easily obtained in terms of the continued fraction for $\sqrt{D}$ where D is a natural number and not the square of some other natural number. It is of note that such D's provide the only interesting forms of the equation.

<u>Corollary</u> 4.10. If $a_0 = [\sqrt{D}]$, then $a_0 + \sqrt{D}$ is periodic from the 0-th partial quotient. If $\sqrt{D} = [a_0;\overline{a_1, \ldots, a_{s-1}, 2a_0}]$, then $a_0 + \sqrt{D} = [\overline{2a_0;a_1, \ldots, a_{s-1}}]$.

The proof is immediate from Theorem 4.8.

Theorem 4.11. Let D be a natural number which is not the square of a natural number, then if $\sqrt{D} = [a_0; \overline{a_1, \ldots, a_{s-1}, 2a_0}]$ the Pell equation

$$x^2 - Dy^2 = 1 \qquad (13)$$

has the following solutions in positive integers.

A) If the period s of the continued fraction for $\sqrt{D}$ is even, then the numerator and denominator of the (ns-1)-th convergent, $n = 1,2, \ldots,$ form a solution for (13).

B) If the period s of the continued fraction for $\sqrt{D}$ is odd, then the numerator and denominator of the (2ns-1)-th convergent, $n = 1,2, \ldots,$ form a solution of (13).

Moreover, all the solutions are obtained in this way.

Proof: Using the generalized form of Theorem 2.19, the $\sqrt{D}$ may be written

$$\sqrt{D} = \frac{P_{ks-1}(a_0 + \sqrt{D}) + P_{ks-2}}{Q_{ks-1}(a_0 + \sqrt{D}) + Q_{ks-2}} \qquad (14)$$

for $k = 1,2,3, \ldots$. Here, Corollary 4.10 is used to identify the value of any remainder whose first partial quotient has the index ks. Using (14) the following equality results:

$$(a_0Q_{ks-1} - P_{ks-1} + Q_{ks-2})\sqrt{D} + (DQ_{ks-1} - a_0P_{ks-1} - P_{ks-2}) = 0.$$

Hence,

$$a_0Q_{ks-1} - P_{ks-1} = -Q_{ks-2} \quad \text{and} \quad DQ_{ks-1} - a_0P_{ks-1} = P_{ks-2} \qquad (15)$$

since if $A\sqrt{D} + B = 0$, then $A = 0$ and $B = 0$. Multiplying the first equality of (15) by $-P_{ks-1}$ and the second by $-Q_{ks-1}$ and then adding them, the following is obtained:

$$P_{ks-1}^2 - DQ_{ks-1}^2 = Q_{ks-2}P_{ks-1} - P_{ks-2}Q_{ks-1}$$

$$= (-1)^{ks}.$$

If s is even then

$$P_{ks-1}^2 - DQ_{ks-1}^2 = 1,$$

which proves conclusion (A) of the theorem. If s is odd, then

$$P_{ks-1}^2 - DQ_{ks-1}^2 = \begin{cases} -1 & \text{for } k = 1,3,5, \ldots \\ 1 & \text{for } k = 2,4,6, \ldots \end{cases},$$

which proves conclusion (B).

The fact that all solutions are found in this manner is seen by considering the arbitrary solution (t,u), $(t > u)$ of (13). Let

$$\frac{t}{u} = [b_0;b_1, \ldots, b_{k-1}] \tag{16}$$

be the representation of number $\frac{t}{u}$ as a simple continued fraction. As was observed by Corollary 2.8, it is always possible to choose $k - 1$ as an odd number depending on whether or not $b_{k-1} = 1$ or is greater than one. Thus, $\frac{t}{u}$ will have a finite continued fraction with k (even) terms.

Let $\frac{t'}{u'}$ be the $k - 2$ convergent to $\frac{t}{u}$. Thus,

$$\frac{t'}{u'} = [b_0;b_1, \ldots, b_{k-2}]$$

and $u' < u$. Since k is even, by Theorem 2.12, $tu' - ut' = 1$. Now, subtracting the last equality from the equality $t^2 - Du^2 = 1$, the following results:

$$t(u' - t) = u(t' - Du). \tag{17}$$

By (16), $0 < \frac{t}{u} - b_0 \leq 1$; hence,

$$0 < t - b_0 u \leq u. \tag{18}$$

In view of the fact that t and u are relatively prime, there is an integer ℓ with the property that the equalities

$$u' - t = \ell u, \quad t' - Du = \ell t$$

hold. Hence,

$$u' - (t - b_0 u) = (\ell + b_0)u. \tag{19}$$

From the inequalities $0 < u' < u$ and (18), it can be inferred that $|u' - (t - b_0 u)| < u$, which by virtue of (19) gives $\ell + b_0 = 0$, so $\ell = -b_0$. Hence,

$$u' = t - b_0 u, \quad t' = Du - b_0 t,$$

and consequently,

$$\frac{t(b_0 + \sqrt{D}) + t'}{u(b_0 + \sqrt{D}) + u'} = \frac{t\sqrt{D} + Du}{t + u\sqrt{D}} = \sqrt{D}.$$

Hence, $\sqrt{D} = [b_0;b_1, \ldots, b_{k-1},b_0+\sqrt{D}]$ and thus $\sqrt{D} = [b_0;\overline{b_1, \ldots, b_{k-1},2b_0}]$, which is unique by Theorem 2.33. Note that $\frac{t}{u}$ is the k-1-th convergent to $\sqrt{D}$. It should also be noted the period k for the $\sqrt{D}$ may not be the shortest such period. This completes the proof of Theorem 4.11.

CHAPTER V

THE CONTINUED FRACTIONS FOR e^x AND π

The purpose of this chapter is to examine the continued fractions of e^x, for certain values of x, and the continued fraction of π. The derivations of these two continued fractions are representative of two distinct approaches employed by mathematicians to find simple continued fractions whose values are a given irrational number. In the first case, it will be shown that a regular continued fraction exists whose value is e^x. In the second case, the continued fraction will be computed using the decimal representation of π. The regularity, if any, of the resulting continued fraction is unknown.

Theorem 5.1. If $\alpha = [a_0;a_1, \ldots]$ is a continued fraction then for all $n \geq 0$

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \quad (1)$$

where

$$\frac{P_n}{Q_n} \text{ and } \frac{P_{n-1}}{Q_{n-1}}$$

are the nth and n-1-th convergents to α.

Proof: (By induction.) Computation verifies the theorem for $n = 0$. Suppose the theorem is true for $n \geq 0$, then

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} \tag{2}$$

and

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{n+1}P_n + P_{n-1} & P_n \\ a_{n+1}Q_n + Q_{n-1} & Q_n \end{pmatrix}$$

$$= \begin{pmatrix} P_{n+1} & P_{n+1-1} \\ Q_{n+1} & Q_{n+1-1} \end{pmatrix} . \tag{3}$$

Substitution of the left side of (2) in the left side of (3) completes the proof.

It is thus clear that α may be associated with the sequence of matrices

$$\left\{ \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \right\} .$$

To facilitate the relationship between α and the sequence of matrices the following notation is introduced. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a matrix, define

$$K_1(A) = \frac{a}{c} , \quad \text{if} \quad c \neq 0;$$

$$K_2(A) = \frac{b}{d} , \quad \text{if} \quad d \neq 0.$$

If $\{A_n\}$ is a sequence of matrices as described in Theorem 5.1, then

$$K_s(A_1 \cdots A_n) \to \alpha_s \quad (s = 1,2)$$

as $n \to \infty$. But in Theorem 5.1 $\alpha_1 = \alpha_2 = \alpha$, so there is no loss of generality in writing

$$K(A_1 \cdots A_n) \to \alpha$$

as $n \to \infty$. The above limit may also be written

$$K(A_1 \cdot A_2 \cdots) = \alpha.$$

Some important properties of the functions K_1 and K_2 will now be stated. The following theorems, 5.2 to 5.4, are stated in terms of K_1, but apply equally to K_2.

<u>Theorem</u> 5.2. If

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $K_1(A_1 \cdot A_2 \cdots) = \alpha$, where $c\alpha + d \neq 0$, then

$$K_1(BA_1A_2 \cdots) = \frac{a\alpha + b}{c\alpha + d} .$$

<u>Proof</u>: Let

$$A_1A_2 \cdots A_n = \begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix} ,$$

by hypothesis

$$\lim_{n \to \infty} \frac{p_n}{q_n} = \alpha.$$

But

$$BA_1A_2 \cdots A_n = \begin{pmatrix} ap_n + bq_n & ar_n + bs_n \\ cp_n + dq_n & cr_n + ds_n \end{pmatrix}$$

and hence,

$$K_1(BA_1A_2 \cdots A_n) = \frac{ap_n + bq_n}{cp_n + dq_n}$$

$$= \frac{a \dfrac{p_n}{q_n} + b}{c \dfrac{p_n}{q_n} + d} .$$

Thus,

$$\lim_{n \to \infty} K_1(BA_1A_2 \cdots A_n) = \lim_{n \to \infty} \frac{a \dfrac{p_n}{q_n} + b}{c \dfrac{p_n}{q_n} + d} = \frac{a\alpha + b}{c\alpha + d} .$$

<u>Theorem</u> 5.3. If $K_1(A_1A_2 \cdots) = \alpha$, and $\{a_n\}$ is a sequence of nonzero real numbers, then

$$K_1\{(a_1A_1)(a_2A_2) \cdots\} = \alpha.$$

The proof follows from the fact that

$$\begin{pmatrix} a_1 \cdots a_n \end{pmatrix} \begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix} = \begin{pmatrix} (a_1 \cdots a_n)p_n & (a_1 \cdots a_n)r_n \\ (a_1 \cdots a_n)q_n & (a_1 \cdots a_n)s_n \end{pmatrix} ,$$

hence,

$$K_1((a_1A_1 \cdots a_nA_n)) = \frac{(a_1 \cdots a_n)p_n}{(a_1 \cdots a_n)q_n} = \frac{p_n}{q_n} .$$

Theorem 5.4. If $K_1(A_1A_2 \cdots)$ exists and if $\{B_1B_2, \ldots B_n\}$ is a sequence of partial products of $\{A_1A_2 \cdots A_n\}$, then

$$K_1(A_1A_2 \cdots) = K_1(B_1B_2 \cdots);$$

in particular,

$$K_1(A_1A_2 \cdots) = K_1\{(A_1A_2A_3)(A_4A_5A_6) \cdots (A_{3n-2}A_{3n-1}A_{3n}) \cdots\}.$$

The proof follows from the fact that matrix multiplication is associative and that any subsequence of a convergent sequence of real numbers has the same limit point.

Theorem 5.5. If

$$A_1 \cdots A_n = \begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix} = M_n,$$

then

$$|K_1(M_n) - K_2(M_n)| = \frac{1}{|q_ns_n|} \prod_{r=1}^{n} |\det A_r|.$$

Proof:

$$\left| K_1(M_n) - K_2(M_n) \right| = \left| \frac{p_n}{q_n} - \frac{r_n}{s_n} \right|$$

$$= \left| \frac{p_ns_n - r_nq_n}{q_ns_n} \right|$$

$$= \frac{|\det A_1 \cdots A_n|}{|q_n s_n|}$$

$$= \frac{1}{|q_n s_n|} \prod_{r=1}^{n} |\det A_r|.$$

Theorem 5.6. $B, A_1, A_2, \ldots$ are matrices over the ring of integers such that $|\det A_r| = 1$ for $r = 1, 2, \ldots$. $K(A_1, A_2, \ldots) = \alpha$ and if

$$B \neq \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix},$$

then

$$K_1(A_1 \cdots A_n B) \to \alpha$$

as $n \to \infty$; and if

$$B \neq \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix},$$

then

$$K_2(A_1 \cdots A_n B) \to \alpha$$

as $n \to \infty$; and if B has a nonzero element in each column, $K(A_1 \cdots A_n B) \to \alpha$ as $n \to \infty$.

Proof: It suffices to prove the first conclusion of the theorem; the second is similar, and the final result follows from the first two. Let

$$A_1A_2 \cdots A_n = \begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$\alpha_n = K_1(A_1 \cdots A_nB) = \frac{ap_n + cr_n}{aq_n + cs_n}.$$

If a or c is zero, the conclusion follows immediately. Assume both are nonzero.

Since $K(A_1A_2 \cdots)$ exists, by Theorem 5.5 it is apparent that $|q_ns_n| \to \infty$ as $n \to \infty$. Also, $(q_n, s_n) = 1$ because

$$|p_ns_n - q_nr_n| = |\det(A_1A_2 \cdots A_n)| = 1.$$

Hence, $|aq_n + cs_n| \geq 1$ for all large n since $aq_n + cs_n = 0$ implies q_ns_n divides ac, which is impossible for sufficiently large n. Thus,

$$\left| \alpha_n - \frac{p_n}{q_n} \right| \left| \alpha_n - \frac{r_n}{s_n} \right| = \frac{|ac|}{|q_ns_n| \, |aq_n + cs_n|^2} \to 0$$

as $n \to \infty$. Since

$$\frac{p_n}{q_n} \text{ and } \frac{r_n}{s_n}$$

both tend to α as $n \to \infty$, it now follows that $\alpha_n \to \alpha$.

The next result follows directly from the above theorem and is of fundamental importance.

Corollary 5.7. Let $B, A_1, A_2, \ldots$ be matrices over the ring of integers such that $|\det A_r| = 1$, B is nonsingular and for $r = 1, 2, \ldots$, $A_r = BC_rB^{-1}$. If $K(A_1, A_2, \ldots) = \alpha$, then

$$K(BC_1C_2 \cdots) = \alpha.$$

It will be frequently possible to transform the product of matrices $A_1A_2A_3 \cdots$ with integral elements and $|\det A_r| = 1$, $r = 1, 2, \ldots$, into a product of matrices of the form exhibited in (1), so yielding a continued fraction.

This procedure is employed to obtain an expression for e^x in the form $K(A_1A_2A_3 \cdots)$.

Theorem 5.8.

$$e^x = K\left\{\prod_{m=0}^{\infty}\begin{pmatrix}(2m+1)+x & (2m+1)\\ (2m+1) & (2m+1)-x\end{pmatrix}\right\} \tag{4}$$

for all (real) x.

Proof: It will be shown that

$$\prod_{m=1}^{n}\begin{pmatrix}(2m-1)+x & (2m-1)\\ (2m-1) & (2m-1)-x\end{pmatrix} = \begin{pmatrix}f_n(x) & g_n(x)\\ h_n(x) & k_n(x)\end{pmatrix}, \tag{5}$$

where

$$h_n(x) = g_n(-x), \quad k_n(x) = f_n(-x) \tag{6}$$

and

$$f_n(x) = \sum_{K=0}^{n} nC_{n,K}\, x^K,$$

$$g_n(x) = \sum_{K=0}^{n} (n - K)C_{n,K}\, x^K,$$

with

$$C_{n,K} = \frac{(2n - K - 1)!}{(n - K)!K!} .$$

The relations (6) follow immediately from the fact that the left hand side of (5) is unchanged by interchanging rows and then columns of each of the matrices and replacing x by $-x$.

The statement of line (5) is established by induction on n. The result is clearly true for $n = 1$; assume that it is true for $n \geq 1$. To prove it is true for $n + 1$, it suffices in view of (6) to show that

$$(2n + 1)\{f_n(x) + g_n(x)\} + xf_n(x) = f_{n+1}(x)$$

and (7)

$$(2n + 1)\{f_n(x) + g_n(x)\} - xg_n(x) = g_{n+1}(x).$$

The first equation in (7) follows from the fact that $f_{n+1}(x)$ is the upper left hand entry in the matrix formed by

$$\begin{pmatrix} f_n(x) & g_n(x) \\ h_n(x) & k_n(x) \end{pmatrix} \begin{pmatrix} (2n + 1) + x & (2n + 1) \\ (2n + 1) & (2n + 1) - x \end{pmatrix}.$$

The second equality in (7) follows similarly. By using the induction hypothesis and the equations in (7), the required statement for $n + 1$ would follow.

Now, to prove the statement of the theorem, it will be shown that

$$\frac{f_n(x)}{g_n(-x)} \to e^x$$

as $n \to \infty$. This follows from the fact that

$$\frac{f_n(x)}{n(n+1)\cdots(2n-1)}$$

$$= 1 + \sum_{K=1}^{n} \frac{n(n-1)\cdots(n-K+1)}{(2n-1)(2n-2)\cdots(2n-K)} \frac{x^K}{K!}$$

$$= 1 + \sum_{K=1}^{n} \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{K-1}{n}\right)}{\left(1 - \frac{1}{2n}\right)\left(1 - \frac{2}{2n}\right)\cdots\left(1 - \frac{K}{2n}\right)} \frac{\left(\frac{1}{2}x\right)^K}{K!}$$

$$= 1 + \sum_{K=1}^{n} a_{n,K} \frac{\left(\frac{1}{2}x\right)^K}{K!}. \qquad (8)$$

Clearly, $a_{n,K} \to 1$ as $n \to \infty$ for fixed K, and also

$$a_{n,K} < \frac{1}{\left(1 - \frac{K}{2n}\right)^K} \leq \frac{1}{\left(1 - \frac{1}{2}\right)^K} = 2^K.$$

At this point it is necessary to recall Tannery's Theorem [2] for sequences which states:

<u>Hypothesis</u>:

$$F(n) = \sum_{r=0}^{p} V_r(n)$$

where $p \to \infty$ as $n \to \infty$ and $\lim_{n \to \infty} V_r(n) = w_r$ for all fixed r and

$$|V_r(n)| \leq M_r,$$

where M_r is independent of n and

$$\sum_{r=0}^{\infty} M_r$$

is convergent.

Conclusion:

$$\lim_{n \to \infty} F(n) = \sum_{r=0}^{\infty} w_r = W.$$

In applying Tannery's Theorem to (8), the $V_K(n)$ will be

$$a_{n,K} \frac{\left(\frac{1}{2} x\right)^K}{K!}$$

which tends to

$$\frac{\left(\frac{1}{2} x\right)^K}{K!}$$

as $n \to \infty$, the values M_k are

$$\frac{|x|^K}{K!},$$

hence,

$$\lim_{n \to \infty} \frac{f_n(x)}{n(n + 1) \cdots (2n - 1)} = \sum_{K=0}^{\infty} \frac{\left(\frac{1}{2} x\right)^K}{K!} = e^{\frac{1}{2} x}.$$

Similarly,

$$\lim_{n \to \infty} \frac{g_n(x)}{n(n+1)\cdots(2n-1)} = e^{\frac{1}{2}x}.$$

Thus,

$$K_1\left\{\prod_{m=0}^{\infty}\begin{pmatrix}(2m+1)+x & (2m+1)\\ (2m+1) & (2m+1)-x\end{pmatrix}\right\} = \lim_{n\to\infty}\frac{f_n(x)}{h_n(x)}$$

$$= \lim_{n\to\infty}\frac{f_n(x)}{g_n(-x)} = \frac{e^{\frac{1}{2}x}}{e^{-\frac{1}{2}x}} = e^x.$$

Similarly,

$$K_2\left\{\prod_{m=0}^{\infty}\begin{pmatrix}(2m+1)+x & (2m+1)\\ (2m+1) & (2m+1)-x\end{pmatrix}\right\} = e^x$$

and the proof of Theorem 5.8 is established.

It is now possible to deduce some regular continued fractions from the relationship stated in Theorem 5.8.

Corollary 5.9. The following are regular continued fraction expansions for the functions specified where k denotes an integer subject to the restrictions stated.

(i) $e^{1/k} = [\overline{1,(2n+1)k-1,1}]_{n=0}^{\infty}$ $(k > 1)$;

$e = [2,\overline{1,2n,1}]_{n=1}^{\infty}$.

$$\text{(ii)}\quad e^{2/k} = [\overline{1, \tfrac{1}{2}\{(6n+1)k-1\}, 6(2n+1)k, \tfrac{1}{2}\{(6n+5)k-1\}, 1}]_{n=0}^{\infty}$$

(odd $k > 1$)

$$e^2 = [7, \overline{3n+2, 1, 1, 3n+3, 6(2n+3)}]_{n=0}^{\infty}.$$

Using the fact that

$$\begin{pmatrix} a+1 & a \\ a & a-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a-1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad (9)$$

with $a = (2n + 1)k$, the proof of part (i) follows directly from Theorem 5.8. A similar device is employed to prove part (ii) [23]. To make clear the use of the notation in this corollary the first few partial quotients of e are listed:

$$e = [2;1,2,1,1,4,1,1,6,1,1,8,1,1,10, \ldots].$$

The reader should also note that the foregoing theorems could be generalized to the extent that if the matrices are taken over the Gaussian integers, then an additional conclusion could be added to Corollary 5.9,

$$\text{(iii)}\quad \tan\frac{1}{k} = [0, k-1, \overline{1, (2n+1)k-2}]_{n=1}^{\infty} \qquad (k > 1);$$

$$\tan 1 = [\overline{1, 2n-1}]_{n=1}^{\infty}.$$

For an account of this procedure, the reader is referred to [23].

Up to this point in the discussion it may seem that every continued fraction is regular, or at least has some semibalance of regularity. Unfortunately, this illusion cannot be maintained any longer. Most of the known regular continued fractions have now been presented and what remains is a vast infinity of numbers for which nothing is known with regards to regularity. An example of such a number is π. It is mentioned here because it offers the best example of the alternative to discovering regularity in finding the continued fraction expansion for a number. The process that follows is based on an extremely close decimal approximation to π, namely

$$\begin{aligned}\pi = 3.14159\ &26535\ 89793\ 23846\\ 26433\ &83279\ 50288\ 41971\\ 69399\ &37510\ 58209\ 74944\\ 59230\ &78164\ 06286\ 20899\\ 86280\ &34825\ 34211\ 70680.\end{aligned}$$

It will be clear that the method employed here would be applicable to any real number for which a very accurate decimal representation is known.

If α is an irrational number, then the k-th partial quotient is given by the formula

$$a_k = \left[\frac{1}{R_k} \right]$$

where R_k is the remainder given by

$$\frac{1}{R_{k-1}} = a_{k-1} + R_k$$

and $0 < R_k < 1$. It should also be apparent that if

$$\alpha = [a_0;a_1, \ldots, a_{k-1},a_k, \ldots],$$

then

$$\frac{1}{R_k} = [a_k; a_{k-1}, \ldots].$$

In other words

$$\frac{1}{R_k}$$

is just the remainder r_k defined in Chapter II. Hence, the algorithm developed in Theorem 2.33 may be restated in terms of r_k.

$$a_0 = [\alpha], \quad \alpha = a_0 + \frac{1}{r_1},$$

$$a_1 = [r_1], \quad r_1 = a_1 + \frac{1}{r_2},$$

and in general

$$a_k = [r_k], \quad r_k = a_k + \frac{1}{r_{k+1}}.$$

The notation P_k and Q_k have been used to represent the numerator and denominator of the k-th order convergent. It will now be necessary to introduce notation for the k-th order convergent of the remainder $r_j = [a_j; a_{j+1}, \ldots]$. The k-th order convergent of r_j will be

$$\frac{P(k,j)}{Q(k,j)} = [a_j; a_{j+1}, \ldots, a_{j+k}]$$

where

$$\begin{aligned} P(m,j) &= P(m-1,j)a_{j+m} + P(m-2,j), \\ Q(m,j) &= Q(m-1,j)a_{j+m} + Q(m-2,j), \end{aligned} \tag{10}$$

and

$$P(-1,k) = 1, \quad P(0,k) = a_k$$

$$Q(-1,k) = 0, \quad Q(0,k) = 1.$$

The proof that

$$\frac{P(k,j)}{Q(k,j)}$$

is the k-th order convergent to the remainder r_j follows exactly the same pattern as the proof of Theorem 2.10. With this new notation $P_n = P(n,0)$ and $Q_n = Q(n,0)$ for all $n \geq 0$.

The following theorem is important in the sense that it enables one to easily compute a remainder r_{j+n} knowing the remainder r_j and the partial quotients $[a_j; \ldots, a_{j+n-1}]$. This would be possible in any case by applying the appropriate definitions. However, a savings of effort is effected by requiring only the convergents of $[a_j; \ldots, a_{j+n-1}]$ rather than those for $[a_0;a_1, \ldots, a_j, \ldots, a_{j+n-1}]$.

Theorem 5.10. For all $j \geq 0$ and $n \geq 1$,

$$r_{j+n} = -\frac{P(n-2,j) - Q(n-2,j)r_j}{P(n-1,j) - Q(n-1,j)r_j}. \tag{11}$$

Proof: The above theorem follows directly from the generalized form of Theorem 2.19 after the new notation has been employed. Of course, the theorem must be applied to the remainder r_j and solved for r_{j+n}.

Another useful theorem follows. It will give us the same economy in computing convergents that Theorem 5.10 does for remainders.

<u>Theorem</u> 5.11. For $j \geq 0$ and $n \geq 0$,

$$P_{j+n} = P_j P(n-1,j+1) + P_{j-1} Q(n-1,j+1),$$

$$Q_{j+n} = Q_j P(n-1,j+1) + Q_{j-1} Q(n-1,j+1).$$

<u>Proof</u>: (By induction on n.) For $n = 0$ direct computation verifies the theorem. Assume the statement is true for $k \leq n$, show true for $n + 1$.

$$\begin{aligned}
P_{j+n+1} &= a_{j+n+1} P_{j+n} + P_{j+n-1} \\
&= a_{j+n+1}[P_j P(n-1,j+1) + P_{j-1} Q(n-1,j+1)] \\
&\quad + [P_j P(n-2,j+1) + P_{j-1} Q(n-2,j+1)] \\
&= P_j[P(n-1,j+1) a_{j+n+1} + P(n-2,j+1)] \\
&\quad + P_{j-1}[Q(n-1,j+1) a_{j+n+1} + Q(n-2,j+1)] \\
&= P_j P(n,j+1) + P_{j-1} Q(n,j+1) \\
&= P_j P(n+1-1,j+1) + P_{j-1} Q(n+1-1,j+1).
\end{aligned}$$

Now, given an irrational number α with a very large number of significant figures, the first partial quotient $a_0 = [\alpha]$ may be computed by $[\beta]$ where $\beta \approx \alpha$. The advantage here is to choose a β which has few significant figures, but is still very close in its numerical value to α. For any such β, not only will the first partial quotient be the same as that for α, but it could be expected that the first few partial quotients might be the same. The important question to answer is, "How many?" There are two ways this question could be answered. One way is to find $\gamma \approx \alpha$ such that α is between β and γ. The partial quotients of α will be the same as those for

β at least as long as the partial quotients for β coincide with those for γ. The other method of telling how far one might trust the partial quotients of β will be discussed presently.

Having obtained the first few partial quotients, up to say a_{k_1}, now compute the numbers

$$P(k_1,0), \quad Q(k_1,0), \quad P(k_1-1,0), \quad Q(k_1-1,0)$$

using the recurrence formulas (10) with $j = 0$. The relation

$$P(k_1,0)Q(k_1-1,0) - P(k_1-1,0)Q(k_1,0) = (-1)^{k_1-1}$$

affords an almost infallible check on the work.

Thus far the calculation has involved only small numbers. In fact, α has not been used, but merely an approximation of it. By setting $j = 0$ and $n = k_1 + 1$ in (11) the following results:

$$r_{k_1+1} = -\frac{P(k_1-1,0) - Q(k_1-1,0)\alpha}{P(k_1,0) - Q(k_1,0)\alpha}\ . \tag{12}$$

Here for the first time really large numbers are encountered. To be quite certain that the computations thus far have not gone astray, Theorem 3.5 and Theorem 3.7 may be applied. Theorem 3.5 states that either

$$\frac{P_{k_1}}{Q_{k_1}} \quad \text{or} \quad \frac{P_{k_1-1}}{Q_{k_1-1}}$$

must satisfy the condition

$$\left| \alpha - \frac{a}{b} \right| < \frac{1}{2b^2} \tag{13}$$

and Theorem 3.7 says that the one that does satisfy (13) must be a convergent to α. This is the second way the accuracy of the partial quotients provided by $\beta \approx \alpha$ could be checked.

Having computed r_{k_1+1} (on line (12)), this value may now be approximated by β_{k_1} just as β was used to approximate α. Again the partial quotients of β_{k_1} are computed up to that point beyond which they cannot be trusted, say

$$a_{k_1+1},\ a_{k_1+2},\ \ldots,\ a_{k_1+1+k_2}.$$

Using these partial quotients compute the following numbers

$$P(k_2, k_1+1),\quad Q(k_2, k_1+1),\quad P(k_2-1, k_1+1),\quad Q(k_2-1, k_1+1)$$

and check them by means of the relation

$$P(k_2, k_1+1)Q(k_2-1, k_1+1) - P(k_2-1, k_1+1)Q(k_2, k_1+1) = (-1)^{k_2-1}.$$

Using (11) with $j = k_1 + 1$ and $n = k_2 + 1$,

$$r_{k_1+k_2+2} = -\frac{P(k_2-1, k_1+1) - Q(k_2-1, k_1+1)r_{k_1+1}}{P(k_2, k_1+1) - Q(k_2, k_1+1)r_{k_1+1}},$$

which involves for the second time operations with really large numbers. The process may continue by approximating $r_{k_1+k_2+2}$ then finding additional partial quotients and the resulting convergents. At each step, this pair of convergents offers a check to the work as has been outlined above.

In some problems the convergents

$$\frac{P_n}{Q_n}$$

are of no interest. For example, one may want to examine the partial quotients to see if they terminate, become periodic, or obey some law or other. The process as outlined above is adequate to meet these needs.

In other problems, such as finding a best rational approximation to the number α, the convergents are very important. In these cases usually one or two convergents are needed. Sometimes a sequence of convergents starting at some point are required, the earlier convergents being of no use. This happens, for example, when one is looking for a rational approximation to a given real number which not only is sufficiently accurate but whose numerator or denominator has some further property. In each of these cases one would find the formulas in Theorem 5.11 extremely helpful. Whenever two consecutive convergents are found, the equation expressed in Theorem 2.12 would serve as a final check.

The above process was used to compute the continued fraction for π based on the previously mentioned decimal approximation. For a complete development of this application, the reader is referred to the article [11] by D. H. Lehmer. A brief summary of this article follows.

An approximation of π was used to compute the partial quotients $[a_0;a_1, \ldots, a_{18}, \ldots]$ as follows:

$$\pi = [3;7,15,1,292,1,1,1,2,1,3,1,14,2,1,1,2,2,2, \ldots]$$

from which

$$P_{17} = 2549491779, \quad P_{18} = 6167950454,$$

$$Q_{17} = 811528438, \quad Q_{18} = 1963319607.$$

Using the 100 figure accuracy, it was found that

$$r_{19} = -\frac{P_{17} - Q_{17}\pi}{P_{18} - Q_{18}\pi} = 2.989^{*}.$$

An approximation of the value for r_{19} was used to compute the partial quotients $[a_{19}, a_{20}, \ldots, a_{32}, \ldots]$. From this $P(12,19)$, $Q(12,19)$, $P(13,19)$ and $Q(13,19)$ were computed. In this fashion the remainders r_{33}, r_{51}, and r_{77} were determined. Finally all the partial quotients up to a_{90} were found as listed below:

$$\pi = [3;7,15,1,292,1,1,1,2,1,3,1,14,2,1,1,2,2,$$
$$2,2,1,84,2,1,1,15,3,13,1,4,2,6,6,99,1,2,$$
$$2,6,3,5,1,1,6,8,1,7,1,2,3,7,1,2,1,1,12,$$
$$1,1,1,3,1,1,8,1,1,2,1,6,1,1,5,2,2,3,1,2,$$
$$4,4,16,1,161,45,1,22,1,2,2,1,4,1,2,24, \ldots]$$

Using Theorem 5.11 one could compute the convergents up to and including C_{90}. The actual value of C_{90} was found to differ from π by less than $8 \cdot 10^{-97}$.

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*The actual value of r_{19} was suppressed to save space.

The above process has been carried much further than what is indicated here. In 1958, Peder Pedersen [17], using a decimal representation of π with 220 significant figures, was able to compute the first 200 partial quotients for π. Even though this process has been carried to an extreme for the value π, it is important to remember that the process is a very valuable tool when applied to the vast majority of irrational numbers for which there is no knowledge of regularity in their partial quotients.

CHAPTER VI

SUMMARY AND CONCLUSIONS

The first chapter is a brief historical discussion of continued fractions. Also in this chapter are found the prerequisites for understanding the material presented in subsequent chapters. Chapter I contains a few notational conventions that are used throughout the paper. The second chapter contains the basic development of continued fractions. The theorems and definitions in this chapter are necessary before one can understand the remainder of the paper. If one has mastered the concepts presented in Chapter II, he should be able to read most articles in the professional journals related to continued fractions. The representation of rational numbers versus irrational numbers by continued fractions is also found in Chapter II.

Chapter III develops the concept of "best approximation" of a real number by a rational number. The fact that the convergents of continued fractions admit best approximations is the essence of the development in this chapter. Hurwitz's Theorem is presented in Chapter III as well as the definitions of algebraic and transcendental numbers. Finally, in Chapter III, one finds Liouville's Theorem and the proof that transcendental numbers exist. Chapter IV is the introduction to the study of periodic continued fractions. It is shown that such continued fractions always represent quadratic irrationals and conversely. The continued fractions of quadratic irrationals are used to find all the solutions to the Pell equation.

In Chapter V two basic approaches to finding continued fractions are presented. The first approach is based on the use of a product of special matrices to represent continued fractions. This is employed after e^x is associated with such a product. The second approach employs a very accurate decimal representation of a real number to determine the continued fraction. This is then applied to the real number π. Either of the two approaches mentioned in this chapter are applicable to real numbers other than those given.

In conclusion, a few ideas for original research are suggested. Since so little is known about the regularity of continued fractions, it seems as though one might explore this topic. The following is a brief account of the author's endeavors along these lines and is actually a generalization of the method used to find the continued fractions of quadratic irrationals.

If one attempts to find the continued fraction of $n^{1/j}$, where $n,j \in N$ and $j \geq 2$, he soon runs into the problem of rationalizing the denominator of the following expression:

$$\frac{1}{a_{j-1}n^{j-1/j} + a_{j-2}n^{j-2/j} + \cdots + a_1n^{1/j} + a_0} . \qquad (1)$$

What is needed is another expression of the form

$$b_{j-1}n^{j-1/j} + b_{j-2}n^{j-2/j} + \cdots + b_1n^{1/j} + b_0 \neq 0$$

such that

$$(a_{j-1}n^{j-1/j} + \cdots + a_0)(b_{j-1}n^{j-1/j} + \cdots + b_0) \in Z. \tag{2}$$

It can be shown that if $c_{j-1}n^{j-1/j} + \cdots + c_1n^{1/j} + c_0$ is the product of the two expressions on line (2) then

$$\begin{aligned}
c_{j-1} &= a_0b_{j-1} + a_1b_{j-2} + \cdots + a_{j-3}b_2 + a_{j-2}b_1 + a_{j-1}b_0 \\
c_{j-2} &= na_{j-1}b_{j-1} + a_0b_{j-2} + \cdots + a_{j-4}b_2 + a_{j-3}b_1 + a_{j-2}b_0 \\
c_{j-3} &= na_{j-2}b_{j-1} + na_{j-1}b_{j-2} + \cdots + a_{j-5}b_2 + a_{j-4}b_1 + a_{j-3}b_0 \\
&\cdots \\
c_1 &= na_2b_{j-1} + na_3b_{j-2} + \cdots + na_{j-1}b_2 + a_0b_1 + a_1b_0 \\
c_0 &= na_1b_{j-1} + na_2b_{j-2} + \cdots + na_{j-2}b_2 + na_{j-1}b_1 + a_0b_0.
\end{aligned}$$

The above can be expressed in matrix form,

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{j-3} & a_{j-2} & a_{j-1} \\ na_{j-1} & a_0 & a_1 & \cdots & a_{j-4} & a_{j-3} & a_{j-2} \\ na_{j-2} & na_{j-1} & a_0 & \cdots & a_{j-5} & a_{j-4} & a_{j-3} \\ & \cdots & & & & & \\ na_2 & na_3 & na_4 & \cdots & na_{j-1} & a_0 & a_1 \\ na_1 & na_2 & na_3 & \cdots & na_{j-2} & na_{j-1} & a_0 \end{bmatrix} \begin{bmatrix} b_{j-1} \\ b_{j-2} \\ b_{j-3} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} c_{j-1} \\ c_{j-2} \\ c_{j-3} \\ \vdots \\ c_1 \\ c_0 \end{bmatrix}. \tag{3}$$

It is now desirable (to satisfy (2)) that $c_{j-1} = c_{j-2} = \cdots = c_1 = 0$. This is tantamount to finding a vector $[b_{j-1}, b_{j-2}, \ldots, b_1, b_0]$ that is perpendicular to the upper $j - 1$ row vectors of the $j \times j$ matrix in (3).

Theorem 6.1. The vector $[b_{j-1}, b_{j-2}, \ldots, b_1, b_0]$ is perpendicular to each of the upper $j - 1$ row vectors in the $j \times j$ matrix in (3) if b_k is the cofactor computed at $1_{1,k}$ in the following determinant:

$$\begin{vmatrix} 1_{1,j-1} & 1_{1,j-2} & 1_{1,j-3} & \cdots & 1_{1,1} & 1_{1,0} \\ a_0 & a_1 & a_2 & \cdots & a_{j-2} & a_{j-1} \\ na_{j-1} & a_0 & a_1 & \cdots & a_{j-3} & a_{j-2} \\ & \cdots & & & & \\ na_2 & na_3 & na_4 & \cdots & a_0 & a_1 \end{vmatrix} .$$

Proof: Consider the product

$$a_0 b_{j-1} + a_1 b_{j-2} + \cdots + a_{j-1} b_0 = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{j-2} & a_{j-1} \\ a_0 & a_1 & a_2 & \cdots & a_{j-2} & a_{j-1} \\ na_{j-1} & a_0 & a_1 & \cdots & a_{j-3} & a_{j-2} \\ & \cdots & & & & \\ na_2 & na_3 & na_4 & \cdots & a_0 & a_1 \end{vmatrix}$$

$$= 0 \qquad (4)$$

since any determinant with two like rows must be zero. The product on line (4) with any other row would also result in a zero determinant for the same reason; hence, the theorem is established.

When applying Theorem 6.1 to find the continued fraction of $2^{1/3}$ the following results:

$$2^{1/3} = 1 + 2^{1/3} - 1, \quad 0 < 2^{1/3} - 1 < 1,$$

and

$$\frac{1}{2^{1/3} - 1} = 2^{2/3} + 2^{1/3} + 1$$

Here, Theorem 6.1 was employed to rationalize the denominator of

$$\frac{1}{2^{1/3} - 1} .$$

Thus,

$$\frac{1}{2^{1/3} - 1} = 3 + 2^{2/3} + 2^{1/3} - 2, \quad 0 < 2^{2/3} + 2^{1/3} - 2 < 1$$

and

$$\frac{1}{2^{2/3} + 2^{1/3} - 2} = \frac{3 \cdot 2^{2/3} + 4 \cdot 2^{1/3} + 2}{10}$$

$$= 1 + \frac{3 \cdot 2^{2/3} + 4 \cdot 2^{1/3} - 8}{10} .$$

Continuing in this manner, one would soon discover that

$$2^{1/3} = [1;3,1,5,1,1,4, \ldots]$$

where the last partial quotient listed is the greatest integer less than

$$\frac{2044[(3{,}373{,}118)2^{2/3} + (4{,}251{,}390)2^{1/3} + (3{,}636{,}472)]}{6{,}509{,}309{,}244} .$$

It is not known if some pattern evolves in this process as was experienced when it was applied to the quadratic irrationals. Unlike the case involving quadratic irrationals, there is no apparent upper bound to the numbers involved in the process when applied to $2^{1/3}$. A computer would seem to be a very useful tool in any investigation of this method of computing continued fractions.

Another direction in which original research might procede is the discovery of algorithms for the addition or multiplication of continued fractions. While investigating this topic, the following characterization of P_n and Q_n was developed:

$$P_n = \begin{vmatrix} a_0 & -1 & 0 & \cdots & 0 & 0 \\ 1 & a_1 & -1 & \cdots & 0 & 0 \\ 0 & 1 & a_2 & \cdots & 0 & 0 \\ & \cdots & & & & \\ 0 & 0 & 0 & \cdots & a_{n-1} & -1 \\ 0 & 0 & 0 & \cdots & 1 & a_n \end{vmatrix}, \tag{5}$$

and

$$Q_n = \begin{vmatrix} a_1 & -1 & \cdots & 0 & 0 \\ 1 & a_2 & \cdots & 0 & 0 \\ \cdots & & & & \\ 0 & 0 & \cdots & a_{n-1} & -1 \\ 0 & 0 & \cdots & 1 & a_n \end{vmatrix}. \tag{6}$$

<u>Theorem</u> 6.2. For all $n \geq 0$ and the continued fraction $[a_0;a_1, \ldots]$ the value given for P_n on line (5) is correct; and if $n \geq 1$, the value given for Q_n on line (6) is correct.

<u>Proof</u>: (By induction.) Direct computation verifies that (5) is true if $n = 0$. Assume that (5) is true for $0 \leq k \leq n$, show true for

n + 1. Expand the following determinant about the bottom row.

$$\begin{vmatrix} a_0 & -1 & \cdots & 0 & 0 \\ 1 & a_1 & \cdots & 0 & 0 \\ & \cdots & & & \\ 0 & 0 & \cdots & a_n & -1 \\ 0 & 0 & \cdots & 1 & a_{n+1} \end{vmatrix}$$

$$= (-1)^{2(n+1)} a_{n+1} \begin{vmatrix} a_0 & -1 & \cdots & 0 & 0 \\ 1 & a_1 & \cdots & 0 & 0 \\ & \cdots & & & \\ 0 & 0 & \cdots & a_{n-1} & -1 \\ 0 & 0 & \cdots & 1 & a_n \end{vmatrix}$$

$$+ (-1)^{2(n+1)-1}(1) \begin{vmatrix} a_0 & -1 & \cdots & & & 0 \\ 1 & a_1 & \cdots & & & 0 \\ & \cdots & & a_{n-2} & -1 & 0 \\ 0 & 0 & \cdots & 1 & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{vmatrix}$$

$$= a_{n+1}P_n + (-1)^{2(n+1)-1}(-1)^{2n}(-1) \begin{vmatrix} a_0 & -1 & \cdots & 0 & 0 \\ 1 & a_1 & \cdots & 0 & 0 \\ & \cdots & & & \\ 0 & 0 & \cdots & a_{n-2} & -1 \\ 0 & 0 & \cdots & 1 & a_{n-1} \end{vmatrix}$$

$$= a_{n+1}P_n + P_{n-1}$$

$$= P_{n+1}.$$

Hence, by induction (5) has been established. (6) follows similarly and is left to the reader.

Using the fact that determinants are unchanged if multiples of one row or column are added to another row or column respectively, the following corollary was established.

Corollary 6.3. For all $n \geq 0$, if $a_0 \neq 0$,

$$P_n = \begin{vmatrix} [a_0] & 0 & 0 & & \\ 0 & [a_1;a_0] & 0 & & \\ 0 & 0 & [a_2;a_1,a_0] & & \\ & & & \cdots & \\ & & & & [a_n;a_{n-1}, \ldots, a_0] \end{vmatrix}$$

and, hence,

$$P_n = [a_0][a_1;a_0][a_2;a_1,a_0] \cdots [a_n; \ldots, a_0].$$

A similar statement could be shown for Q_n.

The above corollary does suggest the following computational procedure for finding P_n. (Similarly for Q_n.) Consider [2;1,3,4,5];

$$P_0 = 2, \quad P_1 = 2\left(1 + \frac{1}{2}\right) = 2\cdot\frac{3}{2}, \quad P_2 = 2\cdot\frac{3}{2}\cdot\left(3 + \frac{2}{3}\right) = 2\cdot\frac{3}{2}\cdot\frac{11}{3},$$

and

$$P_4 = 2\cdot\frac{3}{2}\cdot\frac{11}{3}\cdot\frac{47}{11}\cdot\frac{246}{47} = 246.$$

The k-th factor in the product is formed by adding the previous factor inverted to the $k - 1$ partial quotient.

Many of the previous theorems concerning P_n and Q_n could be established quite easily using Theorem 6.2 and Corollary 6.3. For example, since $Q_n = [a_1][a_2;a_1] \cdots [a_n; \ldots, a_1]$, it follows that

$$\frac{Q_n}{Q_{n-1}} = \frac{[a_1] \cdots [a_{n-1}; \ldots, a_1][a_n; \ldots, a_1]}{[a_1] \cdots [a_{n-1}; \ldots, a_1]}$$

$$= [a_n; \ldots, a_1],$$

which is Theorem 2.20. Obviously, if $a_0 \neq 0$, then

$$\frac{P_n}{P_{n-1}} = [a_n; \ldots, a_0].$$

When $a_0 \neq 0$, Theorem 5.12 is the result of evaluating the determinant which is equal to P_{j+n} by diagonalizing the matrix for the first j columns and then applying Corollary 6.3; P_{j+n} would be:

$$P_{j+n} = \begin{vmatrix} [a_0] & 0 & & & & \\ 0 & [a_1;a_0] & & & & \\ & & \ddots & [a_j; \ldots, a_0] & -1 & \\ & & & 1 & a_{j+1} & -1 \\ & & & & & \ddots \\ & & & & 1 & \ddots \quad a_{j+n} \end{vmatrix}$$

Further,

$$P_{j+n} = [a_0][a_1;a_0] \cdots [a_{j-1}; \ldots, a_0] \begin{vmatrix} [a_j; \ldots, a_0] & -1 & & \\ 1 & a_{j+1} & & \\ & & \ddots & \\ & & & a_{j+n} \end{vmatrix} \tag{7}$$

$$= P_{j-1} \left\{ [a_j; \ldots, a_0] \begin{vmatrix} a_{j+1} & -1 & \\ 1 & \ddots & -1 \\ & 1 & a_{j+n} \end{vmatrix} \right.$$

$$\left. + (-1) \begin{vmatrix} -1 & 0 & & \\ 1 & a_{j+2} & & -1 \\ & 1 & \ddots & \\ & & & a_{j+n} \end{vmatrix} \right\} \tag{8}$$

$$= P_{j-1}[a_j; \ldots, a_0]P(n-1,j+1)$$

$$+ P_{j-1}(-1)(-1) \begin{vmatrix} a_{j+2} & -1 & \\ 1 & \ddots & \\ & & a_{j+n} \end{vmatrix}$$

$$= P_jP(n-1,j+1) + P_{j-1}Q(n-1,j+1).$$

Here $P(n-1,j+1)$ is the numerator of the n-1st convergent to the continued fraction $[a_{j+1};a_{j+2}, \ldots, a_{j+n}]$ as it was in Chapter V. It should also be noted that line (8) is the result of evaluating the determinant in line (7) by expanding about the nonzero elements of the first column.

This determinant representation of P_n and Q_n appears to be a very powerful tool and might well lead to some new insights into continued fractions. It is interesting to examine determinants of this form when $a_i \equiv 1$ for $i \geq 0$. Such a determinant will always have a numerical value equal to some element in a Fibonacci sequence.

The objective of this paper has been to develop the concept of continued fractions in such a way that any undergraduate student with a number theory course and a few basic concepts in algebra and analysis as background might appreciate its content. The aim has not been to relate everything that is known about the topic, but rather to provide the reader with a framework of knowledge that will permit him to investigate the subject further.

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