FARTHEST-POINT DISTANCE FUNCTION

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PREFACE

This thesis is devoted to a study of the farthest-point distance function. For a subset E of R^n, we define the farthest-point distance function $d_E(x)$ as the supremum of distances between $x \in R^n$ and the points of E. Being in fact the Hausdorff distance between a point and a fixed set, $d_E(x)$ measures the distance from a point to the "whole" E, comparing to the usual (nearest-point) distance function, which takes into account only about the nearest points.

In Chapter I, we make a survey of the history of the question. We present the main results on the farthest-point distance functions. The topic of the farthest points was first studied from the viewpoint of the uniqueness of the farthest point. In Chapter II, we give an elementary proof of the fact that the uniqueness of the farthest point on an open set is equivalent to the differentiability of $d_E(x)$. After that, we deduce a sufficient condition for C^k-smoothness of $d_E(x)$ on an open set.

Recently, Pritsker [16] showed that in R^2 the logarithm of the farthest-point distance function is subharmonic and is equal to the logarithmic potential of a unique probability measure σ_E, and generalized this result to higher dimensions. In the third Chapter, we will talk about the structure of σ_E for polygonal E. In this case, the support of σ_E coincides with the farthest-point Voronoi diagram and is contained in not more than $2n$-3 straight lines.

Pritsker and Laugesen [12] conjecture that $\sigma_E(E) \le \frac{1}{2}$ for any compact $E \subseteq \mathbf{C}$. In Chapter IV, we prove their conjecture in the case E is a rhombus.

Finally, we will consider the conjugate of $d_E(x)$, which is convex as a supremum of a family of convex functions. We will see that the conjugate of $d_E(x)$ is easily constructed from the support function of E and get some consequences of this result.

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LIST OF SYMBOLS

R	Real line
C	Complex plane
E	Compact subset of R^n
$d_E(x)$	Farthest-point distance function of $E \subseteq R^n$
σ_E	Representing measure of $d_E(x)$
$\delta_E(x)$	Distance function of $E \subseteq R^n$
$\lvert x \rvert$	Euclidean norm of the vector x
$B_r(x)$	Closed ball with center x and radius r
B	Unit ball
$S_r(x)$	Boundary of $B_r(x)$
S	Boundary of B
$\overline{X}$	Closure of X
Int (X)	Interior of X
conv X	Convex hull of X
f^*	Convex conjugate to function f
$f0^+$	Recession function of the convex function f
dom f	Domain of function f
conv f	Convex hull of function f

$\mathrm{cl}\, f$	Closure of convex function f
$[x, y]$	Vector product of x and y

CHAPTER I

INTRODUCTION

General Information

Definition 1.1 Let E be a subset of n-dimensional Euclidean space R^n. Function $d_E(x) = \sup_{y \in E} |x - y|$ is called *the farthest-point distance function* of E. We call $z \in E$ *the farthest point of E from* $x \in R^n$ (or antiprojection of x on E) *if* $|x - z| = d_E(x)$.

In case $E \subseteq R^n$ is unbounded, $d_E(x)$ is infinite everywhere. If E is not closed, we can easily see that $d_E = d_{\bar{E}}$. Also, $d_E = d_{\operatorname{conv} E}$ as Laugesen and Pritsker showed in [12]. Hence, it makes sense to think of d_E only for compact convex E.

Laugesen and Pritsker show in [12] that $E = \bigcap_{x \in R^n} B_{d_E(x)}(x)$ as well. This means that the farthest-point distance function completely determines compact convex E. Thus, the mapping $E \to d_E$ gives an interesting one-to-one correspondence between the compact convex sets in R^n and objects of quite a different type, certain functions on R^n. This correspondence is similar to the one between closed convex sets and their support functions, and we will see in Chapter V that this similarity is not casual.

As $|x_1 - z| \le |x_1 - x_2| + |x_1 - z| \le |x_1 - x_2| + d_E(x_2)$, we get $d_E(x_1) \le |x_1 - x_2| + d_E(x_2)$ after maximizing over z. That means d_E is Lipschitzian with constant 1, as we can interchange x_1 and x_2:

$$|d_E(x_1) - d_E(x_2)| \le |x_1 - x_2| \quad \forall x_1, x_2 \in R^n$$

Throughout this paper we will denote the ball with radius r and center x by $B_r(x)$ and its boundary by $S_r(x)$.

Number of Farthest Points

One of the initial steps in the study of the farthest points was done, seemingly, by Motzkin, Straus and Valentine [13]. They considered the number of the farthest points, and tried to answer a question, when there exists a unique farthest point of E from any point of space. Most likely, they were inspired by 1935 Motzkin's work on the nearest points [14]. Motzkin's result states that any point of R^2 has a unique nearest point of closed E if and only if E is convex. As the farthest-point distance function is closely related to the usual ("nearest-point") distance between the point and a set, we start our survey of the history with the description of the results on the usual distance.

Definition 1.2 Let $E \subseteq R^n$. Function $\delta_E(x) = \inf_{y \in E} |x - y|$ is called the *distance function* of E, and we say $z \in E$ is the *nearest point* of E for $x \in R^n$ (or projection of x on E) if $\delta_E(x) = |x - z|$.

Notions of distance functions d_E and δ_E, farthest and nearest points can be naturally defined in arbitrary metric spaces.

Definition 1.3 Let X be a normed linear space. A subset $E \subseteq X$ is called a *Chebyshev set* if for any $x \in X$ there exists unique nearest point of E.

Motzkin's theorem states that a closed set $E \subseteq R^2$ is a Chebyshev set if and only if E is convex. It was extended to R^n by Jessen [9] in 1940. Later such authors as Busemann, Klee, Brönsted and Valentine considered more general spaces and tried to find the relationship between certain geometric properties of the unit ball and Chebyshev sets. Perhaps the nicest generalization of Motzkin's theorem belongs to Efimov and Stechkin [3].

Definition 1.4 A convex body E in the topological linear space X is *smooth* if at each of its boundary points there is a unique support hyperplane to E.

Definition 1.5 A convex body E in the topological linear space X is *strictly convex* if E contains no straight line segments in its boundary.

Theorem 1.1 (Efimov, Stechkin) A finite dimensional Banach space X is strictly convex and smooth if and only if Chebyshev sets are all the closed convex subsets of X.

The infinite dimensional case turns out to be more delicate, and will not be considered in this thesis.

For the farthest point case, the situation seems to be quite simpler. It was stated in [13], but not proved, that if any point of R^n has a unique farthest point of E then E is a singleton. One can find proof of it in Valentine's book "Convex sets" ([19, Theorem 7.11]).

Theorem 1.2 (Motzkin, Strauss and Valentine) Let E be a set in a Minkowski space L_n whose unit ball is strictly convex. Then for each point $x \in L_n$ there exists a unique farthest point of E if and only if E consists of a single point.

If the property of uniqueness of the nearest or farthest point does not hold for the entire space, we may still ask ourselves where it does hold.

Definition 1.6 The set $E \subseteq R^n$ is said to be of positive reach if there is a $\varepsilon > 0$ so that every x with $\delta_E(x) < \varepsilon$ has a unique nearest point in E.

Federer developed quite an accurate treatment of sets of positive reach in his article "Curvature measures" [6]. The following theorem is a consequence of his results.

Theorem 1.3 (Federer) Let $E \subseteq R^{n+1}$ be the n-dimensional C^2 manifold. Then E is of positive reach.

Krantz and Parks [11] give an example of a $C^{2-\varepsilon}$ curve in R^2 which does not have positive reach.

In the farthest point case, such an analytic curve as ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in R^2 with eccentricity $e > \sqrt{2}/2$ does not satisfy the condition of having the unique farthest point at $x=(0,\pm b)$, for example.

Jessen was the first to show the interrelation between the uniqueness of farthest point and the centers of osculating spheres [9].

Theorem 1.4 (Jessen) In a Euclidean space, every point of a complement of a compact convex set E has a unique farthest point of E if and only if E has interior points and contains the centers of all osculating spheres of its boundary.

His results were enhanced by Motzkin, Straus and Valentine [13] in planar case. They extend the generalized notions of curvature described by Bonnesen and Fenchel [2] in the following way. Choose a point $x \in C$, where C is a closed convex curve, together with a line of support *L(x)*. The circle tangent to *L(x)* at x and passing through a point $p \in C \setminus x$ has its center *z(p)* on the normal *N(x)* to *L(x)* at x. Establish an order on *N(x)* in terms of the distance from x and let $E_s(x,\beta(x)) = \sup_{p\in\beta(x)\setminus x} z(p)$, $E_i(x,\beta(x)) = \inf_{p\in\beta(x)\setminus x} z(p)$, where $\beta(x)$ is an arc of C containing x. Four types of centers of curvature are defined as follows: $E_s(x) = E_s(x, C)$, $E_l(x) = E_l(x, C)$, $E_o(x) = \lim_{\beta(x) \to x} E_s(x, C)$,

$E_i(x) = \lim_{\beta(x) \to x} E_l(x, C)$.

Definition 1.7 *The sets*

$$E_s = \bigcup_{x \in C} E_s(x), \quad E_o = \bigcup_{x \in C} E_o(x), \quad E_i = \bigcup_{x \in C} E_i(x) \quad \text{and} \quad E_l = \bigcup_{x \in C} E_l(x)$$

are respectively called *the superior, the outer, the inner* and *the inferior evolutes*.

Theorem 1.4 ([13, Theorem 6]) Let E be a compact convex set in R^2 with boundary C. Then z has a unique farthest point of E if $z \notin E_s(x) \quad \forall x \in C$ and z has more then one farthest points of E if $z = E_s(x), \quad z \neq E_o(x)$ for some $x \in C$.

We should note that a lot of researchers studied the question of the uniqueness of the farthest point on the closed convex surfaces (boundaries of the open bounded convex sets) in R^3 with respect to the intrinsic metric on the surface (see, for example, [22]). Unfortunately, this exciting and fruitful topic is going beyond our studies.

Differentiability of the Distance Function

The study of the differentiability of the nearest-point distance function was initiated by Federer [6]. Again, we pick just one result we are most interested in, from his extensive and thorough work.

Theorem 1.5 ([6, Theorem 4.8]) Let E be a nonempty closed subset of R^n and let U be the set of all points which have the unique nearest point of E. Then $\delta_E(x)$ is continuously differentiable on $\operatorname{Int}(U)\setminus E$ and $\operatorname{grad}\delta_E(x)=\dfrac{x-\xi(x)}{\delta_E(x)}$, where $\xi(x)$ denotes the nearest point of E from x.

In 1980 Fitzpatrick proved very general theorem on the differentiability of nearest and farthest-point distance functions [7, Corollary 3.6].

Theorem 1.6 (Fitzpatrick) Let E be a closed subset of a Banach space X such that the norm of X is both Fréchet differentiable and uniformly Gateaux differentiable and the norm of X^* is Fréchet differentiable.

(a) The following are equivalent for a point x of $X\setminus E$:

(i) δ_E is Fréchet differentiable at x;

(ii) δ_E is Gateaux differentiable at x and $\|d\delta_E(x)\|=1$;

(iii) the metric projection onto E is continuous at x.

(b) If E is bounded and $x\in X$, the following are equivalent:

(i) d_E is Fréchet differentiable at x;

(ii) d_E is Gateaux differentiable at x and $\|d(d_E(x))\|=1$;

(iii) the metric antiprojection onto E is continuous at x.

Fitzpatrick's proof extensively uses the notion of Clarke subgradient. In the next chapter, we will give an elementary proof of a result similar to Federer's theorem 1.5 for $d_E(x)$ in R^n.

Extra differentiability of the distance function was first studied by Serrin. He proved that if $E\subseteq R^{n+1}$ is n-dimensional C^k manifold with $k\geq 2$, then, near E, δ_E is of class C^k [18]. Gilbarg and Trudinger strengthen Serrin's result in [5].

Theorem 1.7 ([5; Lemma 14.16]) Let E be a bounded domain in R^n having non-empty boundary $\partial E \in C^k$, $k \geq 2$. Then there exists $\mu > 0$ such that $\delta_{\partial E}(x) \in C^k$ on $\{x \in \overline{E} \mid \delta_{\partial E}(x)\} < \mu$.

Finally, Krantz and Parks [11], followed by Foote [8], come to the following formulation.

Theorem 1.8 ([8, Theorem 1]) Let $E \subseteq R^n$ be a compact C^k manifold with $k \geq 2$. Then E has a neighborhood U so that $\delta_E \in C^k$ on $U \setminus E$.

Analogous result for the farthest-point distance function in planar case will be proved in Chapter II.

Potential Theory of $d_E(x)$

In 1994, Boyd was studying inequalities connecting a product of uniform norms of polynomials on the unit circle with the norm of their product, and acquired an interesting result about the farthest-point distance function in the polygonal case on the plane [1].

Theorem 1.9 (Boyd) Let $E = \{c_1, ..., c_m\}$, $m > 1$ be a finite set of complex numbers. Then there is a probability measure σ_E whose support is contained in a finite number of straight lines, such that $\log d_E(x) = \int_{\mathbf{C}} \log |z - t| d\sigma_E(t)$ for all $z \in \mathbf{C}$.

In 2001 Pritsker generalized Boyd's result to arbitrary compact subsets of the complex plane. Before we state his result, we need to introduce the notion of subharmonic function.

Definition 1.8 A function $u(x)$ defined in a domain $D \subseteq R^n$ is said to be *subharmonic* if

a) $-\infty \le u(x) < \infty$ in D

b) $u(x)$ is upper semicontinuous in D

c) $\forall x_0 \in D$ there exists $r_0 > 0$ such that for every positive $r < r_0$ we have $u(x_0) \le \frac{1}{c_n r^{n-1}} \int_{S_r(x_0)} u(x) d\sigma(x)$, where $c_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the ball of radius one in R^n.

Definition 1.8 says that subharmonic functions are those functions $u(x)$, which are at most the mean value of $u(x)$ on every small sphere around x_0, at any $x_0 \in D$.

Riesz's Representation Theorem is one of the key facts in the theory of subharmonic functions.

Theorem 1.10 (Riesz) Suppose $u(x)$ is subharmonic and not identically $-\infty$, in a domain $D \subseteq R^n, n \ge 2$. Then there exists a unique Borel measure σ in D such that for any compact subset $F \subseteq D$ $u(x) = \int_E K(x-\xi) d\sigma(\xi) + h(x)$, where *h(x)* is harmonic in the interior of D and $K(x) = \begin{cases} \log|x|, & n = 2 \\ -|x|^{2-m}, & n > 2 \end{cases}$.

Pritsker's theorem [16, Lemma 5.1] asserts that $\log d_E(x)$ has a very specific Riesz representation.

Theorem 1.11 (Pritsker) Let $E \subseteq \mathbf{C}$ be a compact set (not a single point). Then $\log d_E(x)$ is a subharmonic function in $\mathbf{C}$ and $\log d_E(x) = \int_{\mathbf{C}} \log|z-t|\, d\sigma_E(t), \quad z \in \mathbf{C}$, where σ_E is a positive unit Borel measure with unbounded support.

In the rest of this chapter E is a compact subset of a complex plane, different from a single point.

The structure of measure σ_E is closely related to the differentiability properties of d_E. It is known that if a subharmonic function is C^2-smooth, then its representing measure is found as its Laplacian [10]. Precisely, if $d_E \in C^2(U)$ for some domain U, then σ_E can be calculated in U by

$$d\sigma_E(z) = \frac{1}{2\pi} \Delta(\log d_E(x+iy))dxdy, \quad z = x+iy \in U$$

Laugesen and Pritsker [12] explored further properties of the Riesz representation of the distance function above.

Theorem 1.12 [12] The representing measure σ_E is unique. The integral $\int_{\mathbf{C}} \log|z-t|\, d\sigma_E(t)$ converges absolutely.

The uniqueness of σ_E implies that we have one more bijective correspondence: between compact convex subsets of R^2 and certain measures on the complex plane.

The following theorems deal with the support of σ_E: when support does not intersect an open set and when the support equals to the whole plane.

Theorem 1.13 ([12, Theorem 2.1 (a)]) Let G be a domain in the plane. Then $\log d_E$ is harmonic in G, that is $\operatorname{supp}\sigma_E \cap G = \varnothing$, if and only if there exists a point $\varsigma \in \partial E \setminus G$ such that $d_E(z) = |z-\varsigma| \quad \forall z \in G$.

Theorem 1.14 ([12, Theorem 2.1 (b)]) $\partial E \in C^1 \Rightarrow \operatorname{supp}\sigma_E = \mathbf{C}$

Estimates on the distribution of σ_E with respect to small and large disks are given in the next theorem. $B_r(z)$ denotes the closed ball with radius r and center z.

Theorem 1.15 ([12, Theorem 2.2]) For each $z \in \mathbf{C}$

$$\sigma_E(B_r(z)) \le \frac{r}{d_E(z)-r} \qquad \text{for all sufficiently small } r>0,$$

$$\sigma_E(B_r(z)) > 1 - \frac{3\operatorname{diam}(E)}{r+\operatorname{diam}(E)} \qquad \text{for all sufficiently large } r>0.$$

Hence $\sigma_E(B_r(z)) \le O(r)$ as $r \to 0$, and $\sigma_E(B_r(z)) \ge 1 - O(\frac{1}{r})$ as $r \to \infty$.

Laugesen and Pritsker also study how much of σ_E can be captured within E itself, and conjecture that $\sigma_E(E) \le \frac{1}{2}$. This is proved in some special cases.

Theorem 1.16 ([12, Theorems 2.5 and 2.6])

(a) If E is a polygon that can be inscribed in a circle then $\sigma_E(E) \le \frac{1}{2}$ holds, with equality if and only if E is a regular n-gon for some odd n.

(b) If E is a C^2-smooth convex body of constant width, then $\sigma_E(E) = \frac{1}{2}$.

The theorems presented in the preceding pages represent the main results on the farthest-point distance function.

CHAPTER II

DIFFERENTIABILITY OF THE FARTHEST-POINT DISTANCE FUNCTION

In this section we prove results on differentiability of the farthest-point distance function analogous to the results on the usual distance. Everywhere in this chapter E is a compact subset of R^n.

Theorem 2.1 If for some point $x \in R^n$ there exist at least two farthest points of E then $d_E(x)$ is not differentiable at x. Moreover, if E is convex and is not a singleton, there always exists $x \in E$ with such a property.

Proof: Let $a, b \in E$, $d_E(x) = \| x - a \| = \| x - b \| = r$. As rigid motions of R^n do not change the property of differentiability, we may think that $x = 0, a = (r, 0, ..., 0), b = (r\cos\varphi, r\sin\varphi, 0, ..., 0)$ where $\varphi \in (0, 2\pi)$; basically we are working in R^2.

We will prove now that the partial derivative of $d_E(x)$ with respect to the first coordinate does not exist at 0. Consider $c(\Delta x) = (\Delta x, 0, ..., 0)$. If $\Delta x < 0$, then it is clear that a is the farthest point of $B_r(0)$ for $c(\Delta x)$, and as $E \subseteq B_r(0)$, a is the farthest point of E for $c(\Delta x)$. The left partial derivative of $d_E(x)$ with respect to the first coordinate at 0 equals $\lim\limits_{\Delta x \to -0} \dfrac{d_E(c(\Delta x)) - r}{\Delta x} = \lim\limits_{\Delta x \to -0} \dfrac{r - \Delta x - r}{\Delta x} = -1$. The corresponding right derivative satisfies

$$(d_E)_+^{'}(0) = \lim_{\Delta x \to +0} \frac{d_E(c(\Delta x)) - r}{\Delta x} \geq \lim_{\Delta x \to +0} \frac{\| c(\Delta x) - b \| - r}{\Delta x} = \lim_{\Delta x \to +0} \frac{\sqrt{(r\sin\varphi)^2 + (r\cos\varphi - \Delta x)^2} - r}{\Delta x} =$$

$$= \lim_{\Delta x \to -0} \frac{-2r\cos\varphi \Delta x + (\Delta x)^2}{\Delta x(\sqrt{(r\sin\varphi)^2 + (r\cos\varphi - \Delta x)^2} + r)} = \frac{1}{2r} \lim_{\Delta x \to -0} \frac{-2r\cos\varphi \Delta x + (\Delta x)^2}{\Delta x} = -\cos\varphi > -1,$$

which means that d_E is not differentiable at x.

The proof of the rest of the theorem coincides with the proof of Motzkin, Strauss and Valentine's theorem one can find in [20, Theorem 7.5.6]. Let E be a convex set and not a singleton. Suppose that for every point $x \in E$ there is a unique antiprojection $f(x)$ of x on E. We claim that $f(x)$ is continuous on E. Suppose it is not. Let $x_n \in E$, $x \in E$, $x_n \to x$ but $|f(x_n) - f(x)| \geq \varepsilon > 0$. As E is compact, we may assume that $f(x_n) \to y \in E$ without loss of generality. Passing to the limit as $n \to \infty$ in the inequalities $|x - f(x_n)| \leq |x - f(x)|$ and $|x_n - f(x)| \leq |x_n - f(x_n)|$, we get $|x - y| \leq |x - f(x)| \leq |x - y|$, i.e. $|x - y| = |x - f(x)|$. Hence, by uniqueness, $y = f(x)$ and $|f(x_n) - f(x)| \to 0$. Contradiction.

By Brouwer's fixed point theorem, a continuous function $f(x)$ mapping compact convex set E to itself has a fixed point. As E is not a singleton, it is impossible. Contradiction. Q.E.D.

Corollary 2.2 $d_E(x)$ is not differentiable on the whole R^n for any compact E.

Corollary 2.3 The set of points, which have at least two farthest points of E, has Lebesgue measure zero.

Proof: According to Rademacher's theorem, Lipschitzian function d_E is differentiable almost everywhere. By the preceding theorem, it means that for almost all $x \in R^n$, there exists a unique farthest point of E from x. Q.E.D.

A result on the nearest points, similar to Corollary 2.3, was obtained by Erdös [4].

Theorem 2.4 Let U be an open subset of R^n and suppose that every point $x \in U$ has a unique farthest point of E, where E is not a singleton. Then $d_E(x)$ is continuously differentiable on U and $\operatorname{grad} d_E(x) = \dfrac{x - f(x)}{d_E(x)}$, where $f(x)$ denotes the farthest point of E from $x \in U$.

Proof: In the same way as in proof of Theorem 2.1, $f(x)$ is continuous on U. Define a unit vector $a = \dfrac{x - f(x)}{d_E(x)}$ directed from $f(x)$ to x. As for $t > 0$ $E \subseteq B_{d_E(x)}(x) = B_{|x-f(x)|}(x) \subseteq B_{|x+at-f(x)|}(x+at)$ and $S_{d_E(x)}(x) \bigcap S_{|x+at-f(x)|}(x+at) = f(x)$ we conclude that $x + at$ has unique farthest point of E and $f(x+at) = f(x)$. Thus, one-sided directional derivative of $d_E(x)$ with respect to a equals

$$\lim_{t \to +0} \frac{d_E(x+ta) - d_E(x)}{t} = \lim_{t \to +0} \frac{|x+ta-f(x)| - |x-f(x)|}{t} = \lim_{t \to +0} \frac{|ta|}{t} = 1$$

Now suppose that d_E is differentiable at x. As d_E is Lipschitzian with constant 1, any one-sided directional derivative of $d_E(x)$ with respect to a unit vector h has absolute value

$$|(\operatorname{grad} d_E(x), h)| \le \left| \lim_{t \to +0} \frac{d_E(x+th) - d_E(x)}{t} \right| = \lim_{t \to +0} \frac{|d_E(x+th) - d_E(x)|}{t} \le \lim_{t \to +0} \frac{|th|}{t} = 1$$

Hence, $|\operatorname{grad} d_E(x)| \le 1$. But $(\operatorname{grad} d_E(x), a) = 1$, so $\operatorname{grad} d_E(x) = a = \dfrac{x - f(x)}{d_E(x)}$.

Note that all functions on the right-hand side of the preceding equation are continuous on U.

Now, we use Lemma 4.7 from [6], which says that if we have real valued Lipschitzian function $u(x)$ on open $W \subseteq R^n$, and at points where $u(x)$ is differentiable

$\frac{\partial u}{\partial x} = g(x)$ for some $g(x) \in C(W)$, then $\frac{\partial u}{\partial x} = g(x)$ everywhere on W. According to that Lemma, $d_E(x)$ has continuous partial derivatives on U. Thus, $d_E(x)$ is continuously differentiable on U, which finishes the proof of the theorem. Q.E.D.

Corollary 2.5 $d_E(x)$ is differentiable on an open set $U \subseteq R^n$ if and only if every point $x \in U$ has a unique farthest point of E.

Theorem 2.6 Let E be a compact subset of R^2 with C^k boundary, $k \geq 2$, U be an open subset of R^2 and suppose that

(a) Every $x \in U$ has a unique farthest point of E.

(b) There are no centers of curvature of ∂E in U

Then $d_E \in C^k(U)$.

Proof: Let $f(z)$ denote the farthest point of E from $z \in U$. Consider arbitrary point $A \in U$. Choose coordinate system and parameterization of the boundary of E in such a way that $f(A) = 0$ and neighborhood of $f(A)$ in ∂E is given by $\{(t, \varphi(t)) \mid -r < t < r\}$ for some positive r, $\varphi \in C^k$, $\varphi(0) = 0$ and $\varphi'(0) = 0$.

Consider circle $S_{d_E(A)}(A)$ with center A and radius $d_E(A)$. As $\partial E \subseteq E \subseteq B_{d_E(A)}(A)$ and $f(A) \in \partial E \cap S_{d_E(A)}(A)$, $S_{d_E(A)}(A)$ is tangent to ∂E. So, $A - f(A) \perp \partial E$ at $f(A)$, i.e. $A - f(A) \perp (t, \varphi(t))'|_{t=0} = (1, \varphi'(0)) = (1, 0)$ and therefore $A = (0, y_0)$ for some $y_0 \in R$.

Since $S_{d_E(A)}(A)$ and ∂E have contact of the first order at 0 and $\partial E \subseteq B_{d_E(A)}(A)$, the curvature of ∂E at 0 is not greater than the curvature of $S_{d_E(A)}(A)$ at 0, which equals

$\frac{1}{|y_0|}$. Since A is not a center of curvature, we obtain strict inequality $\kappa < \frac{1}{|y_0|}$, where κ denotes the curvature of ∂E at 0.

As we showed in the proof of Theorem 2.1, uniqueness of the farthest point implies continuity of f in U. Therefore, there is some smaller neighborhood $V \subseteq U$ such that $f(A) \in V$, and for every $(x,y) \in V$ $f((x,y))$ is given by $(t(x,y), \varphi(t(x,y)))$ for some $-r < t(x,y) < r$. As $(t(x,y), \varphi(t(x,y)))$ maximizes the function

$$\left[\left(t(x,y)-x\right)^2 + \left(\varphi(t(x,y))-y\right)^2\right]$$

then

$$\frac{\partial}{\partial t}\left[\left(t-x\right)^2 + \left(\varphi(t)-y\right)^2\right] = 2\left(t-x\right) + 2\varphi'(t)\left(\varphi(t)-y\right) = 0$$

at $t = t(x,y)$.

We will show now that the equation

$$F(x,y,t) = 2\left(t-x\right) + 2\varphi'(t)\left(\varphi(t)-y\right) = 0$$

implicitly determines the function $t = t(x,y)$ in some neighborhood of A. In fact, $F(x,y,t)$ is (k-1)-times continuously differentiable on $V \times (-r,r)$ as $\varphi \in C^k(V)$, $F(A,0) = 0$. Thus, we only need to show that $F_t'(x,y,t) \neq 0$ at $(A,0)$. As

$$F_t'(x,y,t) = 2\left(1 + \varphi''(t)\left(\varphi(t)-y\right) + \varphi'^2(t)\right),$$

we get

$$\frac{F_t'(A,0)}{2} = 1 + \varphi''(0)\left(\varphi(t)-y_0\right) + \varphi'^2(0) = 1 - \varphi''(0)y_0$$

By the well-known formula of differential geometry, the curvature of curve $\alpha(t)$ is given by $\dfrac{\|[\alpha',\alpha'']\|}{|\alpha'|^{\frac{3}{2}}}$, $[u,v]$ denoting the vector product of u and v. In our case $\alpha(t)=(t,\varphi(t))$, therefore $\alpha'(t)=(1,\varphi'(t))$, $\alpha''(t)=(0,\varphi''(t))$ and $\alpha'(0)=(1,0)$, $\alpha''(0)=(0,\varphi''(0))$. Hence, the curvature of ∂E at 0 equals $\kappa=\dfrac{\|[(1,0),(0,\varphi''(0)]\|}{|(1,0)|^{\frac{3}{2}}}=|\varphi''(0)|$.

Consequently, $|\varphi''(0)y_0|>\kappa\dfrac{1}{\kappa}=1$, and $F_t'(A,0)\neq 0$.

By the implicit function theorem, there exists unique C^{k-1} function $t=t(x,y)$ in some neighborhood W of A. By theorem 2.4, grad $d_E(z)=\dfrac{z-f(z)}{d_E(z)}=\dfrac{z-t(z)}{d_E(z)}$ is a ratio of two continuously differentiable functions, thereby grad $d_E(z)\in C^1(W)$, i.e. $d_E(z)\in C^2(W)$. Using the same expression for grad $d_E(z)$ again, we see that grad $d_E(z)$ is a ratio of twice continuously differentiable functions, so that grad $d_E(z)\in C^2(W)$ and $d_E(z)\in C^3(W)$. Applying this reasoning inductively, we obtain $d_E(z)\in C^k(W)$. As A is an arbitrary point of U, $d_E(z)\in C^k(U)$. Q.E.D.

Corollary 2.7 Under the conditions of Theorem 2.6, the representing measure σ_E is given in U by $d\sigma_E(z)=\dfrac{1}{2\pi}\Delta(\log d_E(x+iy))dxdy, \quad z=x+iy\in U$.

CHAPTER III

COMPLEXITY OF σ_E

Boyd's Theorem 1.9 states that the support of σ_E consists of the finite number of straight lines in the polygonal case on the plane. Actually, Boyd remarks in his paper [1] that the number of these lines is not greater than $\frac{n(n-1)}{2}$, where n is the number of vertices of polygon E. In this chapter we will show that the growth of the number of straight lines that contain $\operatorname{supp}\sigma_E$ is, in fact, linear. In other terms, we will show that the farthest point Voronoi diagram has linear complexity in the planar case.

Theorem 3.1 Let E be a polygon in $\mathbf{C}$ with vertices $a_1, a_2, .., a_n$, $n > 2$. Then $\operatorname{supp}\sigma_E$ is a union of n rays and not more than $n-3$ line segments. There are at most $n-2$ points in $\mathbf{C}$ which lie in more than one such segment or ray.

Proof: Let us consider the "influence zones" of the vertices of E:

$$A_i = \{z \in \mathbf{C} \mid \ |z-a_i| \geq |z-a_j| \quad \forall j \quad 1 \leq j \leq n\} = \bigcap_{j=1}^{n} \{z \in \mathbf{C} \mid \ |z-a_i| \geq |z-a_j|\}$$

Inasmuch $\{z \in \mathbf{C} \mid |z-a_i| \geq |z-a_j|\}$ is a half-plane when $i \neq j$, all A_i are intersections of finite number of half-planes. Thus, A_i are convex polyhedra. Let us now show by a simple geometric argument that all A_i are nonempty and unbounded. Take any vertex a and find a support line l to E such that $l \cap E = a$. Line l divides complex plane in two parts, and E lies in one of those parts completely. Let m be a line through a, which is perpendicular to l. Moving along m from a deeper to the half-plane containing E, we can clearly see that $|x-z|$ attains its maximum on E at a when we are sufficiently far from a. Thus, A contains some ray. Consequently, all A_i are unbounded convex polygons.

Note, that the set of polygons $\{A_i\}$ is called the farthest point Voronoi diagram of the set $\{a_i\}$.

By Theorem 1.13 the measure σ_E can be supported only on the edges of A_i. Let K be one of the infinite edges of polygon A_1 and b be the vertex of A_1 contained in K. Surely, b is also contained in one of the other sides of A_1, call it L. We claim that b is contained in at least three different edges of some polygons A_i. In fact, one of the angles between K and L is greater than π and thus cannot be an angle between two sides of the convex polygon A_i.

Let V be the set of all vertices of A_i and S be the set of edges of all A_i, $1 \le i \le n$. Given some set of edges $D \subseteq S$, we call vertex $v \in V$ open in D if $\exists s_0 \in D : v \in s_0$ and $\exists s \in S : v \in s \,\&\, s \notin D$, i.e. there is some edge in D containing v, but not all such edges are in D. Also, define $V(D) = \{v \in V \mid \exists s \in D : v \in s\}$ be the set of vertices incident to edges from D, and for set of vertices $W \subseteq V$ let $S(W) = \{s \in S \mid \exists v \in W : v \in s\}$ be the set of edges, incident to the vertices of W.

We will consequently add edges of convex polygons A_i to K, and look at some characteristics of the set we will get.

First, let $D_1 = S(\{b\})$. Suppose D_1 consists of m_1 segments and n_1 rays, besides K. Added rays (together with K) separate complex plane into $n_1 + 1$ connected regions, segments do not change the number of connected regions, so D_1 separates plane into n_1+1 connected regions. The total number of edges in D_1 is m_1+n_1+1, and there are m_1 open vertices in D_1.

Next step, let $D_2 = S(V(D_1))$. Suppose D_2 contains m_2 segments and n_2 rays in addition to those in D_1. Notice that one of the endpoints of any of the added segments should not lie in any of the elements of D_2. If it does, then we have a cycle in D_2 that bounds some polyhedral region, which contradicts unboundedness of A_i. Then D_2 separates the plane in n_1+n_2+1 or more connected regions, the cardinality D_2 of equals $|D_2| = m_2+m_1+n_2+n_1+1$, and there are m_2 open vertices in D_2.

Now, we iterate this procedure till we get $D_k = S$ (it is possible in finite number of steps as $\bigcup_{i=1}^{n} \partial A_i$ is connected). On the last k-th step we add n_k rays and $m_k = 0$ segments to D_{k-1}. The number of connected regions becomes $n \geq \sum_{i=1}^{k} n_i + 1$, the number of open vertices is 0, and $|D_k|$ is equal to $\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} m_i + 1$.

Recall that there are at least three different edges containing each vertex $v \in V$. Hence, for every vertex v open in D_i, we should add at least two edges on step i+1, which implies that $m_{i+1} + n_{i+1} \geq 2m_i$. Summing the latter inequality through $i = 0$ to $k-1$ (here we assume that $m_0 = 1$ represents the open vertex of the original ray K), we get $\sum_{i=1}^{k} n_i \geq \sum_{i=1}^{k} m_i + 2$. Thus, $2n \geq \sum_{i=1}^{k} n_i + \sum_{i=1}^{k} n_i + 2 \geq \sum_{i=1}^{k} n_i + \sum_{i=1}^{k} m_i + 4 = |S| + 3$, i.e. $|S| \leq 2n-3$.

It is easy to see why the number of rays in S equals n: each A_i has two ray sides, and each of them is counted twice.

Now we can estimate $|V|$. In fact, the number of the segments in S should be greater or equal to $\frac{3|V|-n}{2}$ (minimum 3 rays or segments from each vertex minus total number of rays and divided by two as each segment is counted twice). Concequently, $2n-3 \geq \frac{3|V|-n}{2}+n$ and $|V| \leq n-2$. Q.E.D.

CHAPTER IV

REPRESENTING MEASURE OF A RHOMBUS

Laugesen and Pritsker [12] conjecture that $\sigma_E(E) \leq \frac{1}{2}$ for any compact $E \subseteq \mathbf{C}$. They prove it for polygons that can be inscribed in a circle and C^2-smooth bodies of constant width [12, Theorems 2.5 and 2.6]. In this short chapter, their conjecture is proved in the case E is a rhombus.

Theorem 4.1 If E is a rhombus, then $\sigma_E(E) \leq \frac{1}{2}$.

Proof: As we already know from the proof of the theorem 3.1, the representing measure of any polygon is supported on the boundaries of "influence zones" $A_i = \{z \in \mathbf{C} \mid \ |z - a_i| \geq |z - a_j| \quad \forall j \quad 1 \leq j \leq n\}$, a_i denoting vertices of a polygon. Suppose that t is an interior point of a boundary segment separating A_i from A_j (which, of course, is a part of a perpendicular bisector to the segment $a_i a_j$). On this segment, choose local coordinates (ξ, η), with ξ parallel and η perpendicular to the boundary, and so that $\eta = 0$ on that segment of a boundary. Then $(\xi, 0)$ is equidistant from a_i and a_j, so $a_i = (c, d)$ and $a_j = (c, -d)$, say, where $d > 0$ without loss of generality. Boyd [1, p.451] proved that the representing measure on the segment is found by

$$\sigma_E(\xi, \eta) d\xi d\eta = \frac{1}{\pi} \frac{d}{(\xi - c)^2 + d^2} \delta(\eta) d\xi d\eta,$$

δ denoting Dirac measure.

Similarity transformations of the plane do not change $\sigma_E(E)$, as Laugesen and Pritsker point out in [12]. Hence, we can think that vertices of our rhombus are $a_1 = (-b, 0)$,

a_2=(0,1), a_3=(b,0), and a_4=(0,-1) for some b, $0<b\leq 1$. Figure 1 illustrates how three perpendicular bisectors border the "influence zone" of a_2. Figure 2 represents the structure of the support of σ_E and the position of the "influence zones" in rhombic case.

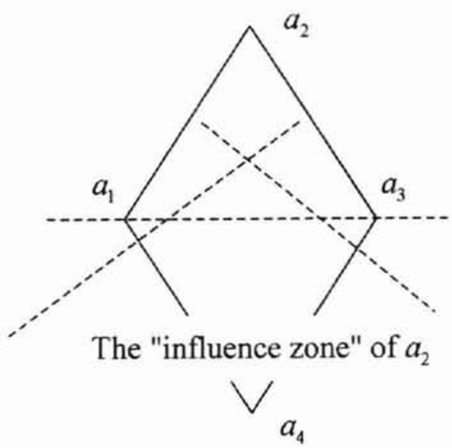


Figure 1

"Influence Zone" of a Vertex of a Rhombus

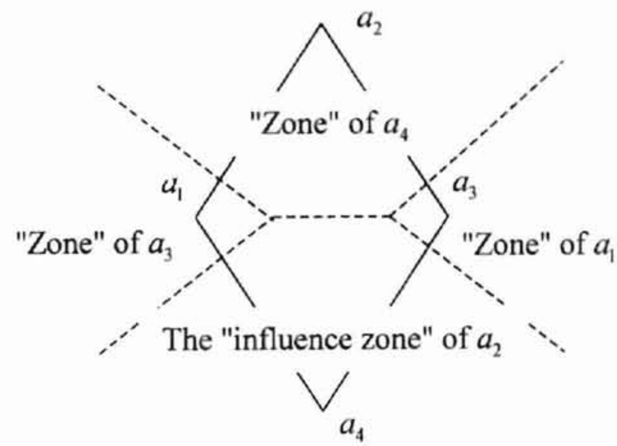


Figure 2

"Influence Zones" in a Rhombic Case

In general, the measure σ_E is supported on the horizontal segment KL and on four rays LS, LR, KP and KQ (see Figure 3). One can analytically find that $K=\left(\frac{b}{2}-\frac{1}{2b},0\right)$ and $L=\left(\frac{1}{2b}-\frac{b}{2},0\right)$, so that

$$\sigma_E(KL)=\frac{1}{\pi}\int_K^L\frac{1}{\xi^2+1}d\xi=\frac{2}{\pi}\arctan\left(\frac{1}{2b}-\frac{b}{2}\right)$$

The diagonal a_1a_3 is contained in KL when $\frac{1}{2b}-\frac{b}{2}\geq b$, i.e. $b\leq\frac{\sqrt{3}}{3}$. In this case, $\operatorname{supp}\sigma_E\cap E=a_1a_3$, so

$$\sigma_E(E)=\sigma_E(a_1a_3)=\frac{1}{\pi}\int_{-b}^{b}\frac{1}{\xi^2+1}d\xi=\frac{2}{\pi}\arctan(b).$$

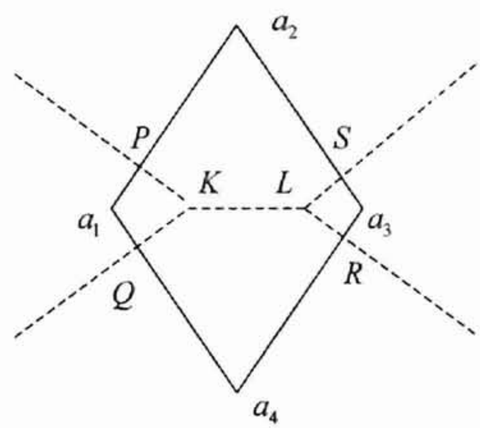


Figure 3

Support of σ_E in a Rhombic Case

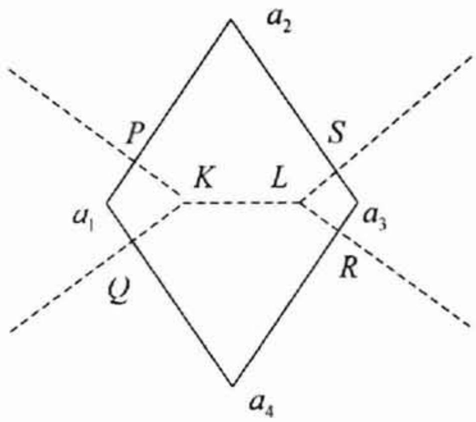
If $b > \frac{\sqrt{3}}{3}$ then, inside E, σ_E is supported on the horizontal segment KL and on four inclined segments LS, LR, KP and KQ (see Figure 3). Due to symmetry, $\sigma_E(LS) = \sigma_E(LR) = \sigma_E(KP) = \sigma_E(KQ)$. By simple calculation one obtains $S = \left(\frac{b(1-b^2)}{1+b^2}, \frac{2b^2}{1+b^2}\right)$. Using the formula given in the beginning of the chapter, we have

$$\sigma_E(LS) = \frac{1}{\pi}\int_L^S \frac{d}{(\xi - c)^2 + d^2} d\xi = \frac{1}{\pi}\int_{\frac{\sqrt{b^2+1}}{2b}}^{\frac{2b}{\sqrt{b^2+1}}} \frac{\sqrt{b^2+1}}{2\left(\xi^2 + \frac{b^2+1}{4}\right)} d\xi = \frac{1}{\pi}\left(\arctan\frac{4b}{b^2+1} - \arctan\frac{1}{b}\right)$$

after an appropriate change of coordinates.

Finally, we obtain

$$\sigma_E(b) = \begin{cases} \frac{2}{\pi} arctg(b), b \le \frac{\sqrt{3}}{3} \\ \frac{4}{\pi}\left(arctg(\frac{4b}{b^2+1}) - arctg(\frac{1}{b})\right) + \frac{2}{\pi} arctg(\frac{1}{2b} - \frac{b}{2}), \frac{\sqrt{3}}{3} < b \le 1 \end{cases}$$



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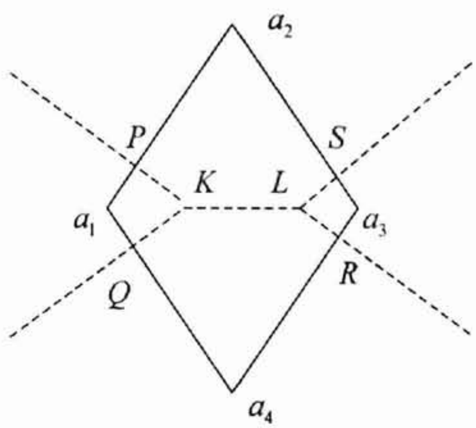

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or, after simplification,

$$\sigma_E(E) = \begin{cases} \dfrac{2}{\pi}\arctan(b), b \le \dfrac{\sqrt{3}}{3} \\ \dfrac{-2}{\pi}\arctan\left(\dfrac{1}{8}\dfrac{b^4 - 14b^2 + 1}{b(b^2+1)}\right), \dfrac{\sqrt{3}}{3} < b \le 1 \end{cases}$$

The derivative of $\sigma_E(E)$ with respect to b equals

$$\sigma_E'(b) = \begin{cases} \dfrac{2}{\pi}\dfrac{1}{b^2+1}, b < \dfrac{\sqrt{3}}{3} \\ 16\dfrac{1-b^2}{\pi\left(b^4+18b^2+1\right)}, \dfrac{\sqrt{3}}{3} < b \le 1 \end{cases}$$

and is clearly positive on $\left[0,\dfrac{\sqrt{3}}{3}\right) \cup \left(\dfrac{\sqrt{3}}{3},1\right]$. Moreover, left and right derivatives coincide at $b = \dfrac{\sqrt{3}}{3}$, so $\sigma_E(b)$ is differentiable on $[0,1]$, with derivative positive everywhere. This means that $\sigma_E(b)$ attains maximum on $[0,1]$ at $b = 1$, i.e. when E is a square. To specify, $\sigma_E(1) = \dfrac{2}{\pi}\arctan(\dfrac{3}{4}) \approx 0.409665...$, which is obviously less than $\dfrac{1}{2}$.

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CHAPTER V

FARTHEST-POINT DISTANCE AND THE SUPPORT FUNCTION

This chapter is devoted to the study of the convex conjugate to the farthest-point distance function. The notion of conjugate function is of great significance in the convex analysis. There are many theorems connecting various properties of a function to the behavior of its conjugate (see, for example, the wonderful book of Rockafellar [17]). In our particular case, we will find out that the conjugate of $d_E(x)$ is somehow formed from the support function of *E*. The support function is one of the basic instruments of the theory of convexity, so that the interplay between the support function and the farthest-point distance function seems to be, at least, interesting.

As usual, E is a compact subset of R^n in this chapter, but we do not demand convexity. We denote the unit ball in R^n by $B = \{x \in R^n \mid |x| \le 1\}$ and a unit sphere by *S*.

We follow Rockafellar's notation. Particularly, we denote the support function of E by $\delta^*(x \mid E)$, the convex conjugate function to f by $f^*(y) = \sup\limits_{x \in R^n}\{\langle x, y\rangle - f(x)\}$ and the recession function of f by $f0^+$. We think of convex functions as of functions defined everywhere on R^n and probably having infinite values at some points. Domain of function $f(x)$ is defined as $\operatorname{dom} f(x) = \{x \in R^n \mid f(x) \ne \infty\}$.

We denote the restriction of function $f(x)$ on a set D as $f|_D(x) = \begin{cases} f(x), & x \in D \\ +\infty, & x \notin D \end{cases}$

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To make the notation shorter, we think that unitary minus has more priority than the operation of restriction, i.e., $-f\,|_D\,(x)$ means $(-f(x))\,|_D$.

We remind that $d_E(x)$ is convex as a supremum of convex functions.

Theorem 5.1 The following statements hold:

a) $d_E(x) = (-\delta^*(x\,|-E)\,|_B)^*$

b) $d_E(x) = (-\delta^*(x\,|-E)\,|_S)^*$

c) $d_E^*(x) = \mathrm{conv}(-\delta^*(x\,|-E)\,|_B)$

d) $d_E 0^+(x) = |x|$

e) The best Lipschitz constant for $d_E(x)$ is 1

Proof: a) The proof follows from the chain of equalities below:

$$d_E(x) = \sup_{y\in E} |\,y - x\,| = \sup_{y\in E}\sup_{z\in B}\langle y-x, z\rangle = \sup_{z\in B}\sup_{y\in E}\langle y-x, z\rangle = \sup_{z\in B}\sup_{y\in E}\left(\langle y,z\rangle - \langle x,z\rangle\right) =$$

$$= \sup_{z\in B}\left[\left(\sup_{y\in E}\langle y,z\rangle\right) - \langle x,z\rangle\right] = \sup_{z\in B}\left[\delta^*(z\,|\,E) - \langle x,z\rangle\right] = \sup_{z\in B}\left[\langle x,z\rangle - \left(-\delta^*(z\,|-E)\right)\right] =$$

$$= \sup_{z\in R^n}\left[\langle x,z\rangle - \left(-\delta^*(z\,|-E)\right)|_B\right] = (-\delta^*(x\,|-E)\,|_B)^*$$

To prove b) we just need to replace B by S in preceding proof.

c) We need to carefully perform conjugation of the equality we obtained in a). It is a well-known fact that the double conjugate of any function f equals to closure of its convex hull cl(conv(f)) (see [17]). It is also known that if f (in our case $f(x) = -\delta^*(x\,|-E)\,|_B$) is continuous on a compact set (B, in our case) and is equal to $+\infty$ at all other points, then conv(f) is closed proper function ([17, Theorem 17.2.1]). Hence, we can delete the operation of taking closure. Q.E.D.

To make the notation shorter, we think that unitary minus has more priority than the operation of restriction, i.e., $-f\,|_D\,(x)$ means $(-f(x))\,|_D$.

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d) This result is a consequence of the fact that $f0^{+}$ is the support function of $\mathrm{dom}(f^{*})$, when f is proper closed convex function [17, Theorem 15.3]. In our case, $f = d_E(x)$, $\mathrm{dom}(f^{*}) = B$, $d_E 0^{+}(x) = \delta^{*}(x \mid B) = |x|$.

e) The last assertion of the theorem may be proved directly, but we deduce it from the nice theorem [17, Theorem 13.3.3], which says that the best Lipshitz constant of a convex function coincides with the maximum norm of the elements of $\mathrm{dom}(f^{*}) = B$. Q.E.D.

So, this is the way to get $d_E(x)$: we take set $-E$, which is symmetrical to original set E, and consider its support function. We turn this support function upside down and take its restriction to the unit ball. Finally, we apply conjugation to get d_E.

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CHAPTER VI

CONCLUSIONS

The basic purpose of this study has been to examine the structure of the farthest point distance function d_E and to determine its properties, which are analogous to those of the "usual" distance function. The sufficient condition of k-differentiability of d_E was found. The complexity of $\operatorname{supp} \sigma_E$ was estimated in the polygonal case. Laugesen and Pritsker's conjecture on $\sigma_E(E)$ was proved in case E is a rhombus. Some convex analytic considerations on d_E led to determining the relation between d_E and the support function of E.

Several problems have been raised by this study, which would be of interest for further consideration. As in the nearest-point distance case, it should be possible to prove extra-smoothness of d_E in case of smooth ∂E in spaces with dimension greater than two. Also, one can hope to obtain estimates on the complexity of $\operatorname{supp} \sigma_E$ in the polyhedral case. The powerful and effective apparatus of convex analysis may as well lead to some surprising results.

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Education: Graduated from the Ural Lyceum in Ekaterinburg, Russia in 1996; received the Bachelor of Science degree from the Ural State University, Ekaterinburg, Russia with a major in mathematics and applied mathematics in June, 2000. Completed the requirements for the Master of Science degree in mathematics at Oklahoma State University in December, 2001.

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